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ASYMPTOTIC EXPANSION OF AN IMPULSE FOR AN OPTIMAL FINITE BURN

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SUMMARY

This report derives closed form expressions for the position and velocity of a spacecraft during a finite burn using the method involving the theory of asymptotic expansion of the optimal impulsive solution. The small parameter is given by the reciprocal of the mass flow rate. The expansion is given in terms of the impulsive solution and holds up through third-order for the velocity and fourth-order for the position.

DESCRIPTION OF THE OPTIMAL IMPULSIVE SOLUTION

An optimal-impulse transfer from one trajectory to another satisfies certain necessary conditions (Ref. 1). This section will list the pertinent characteristics without derivation. It is assumed, in what follows, that the user has available a method for achieving the optimal impulsive solution which satisfies these conditions.

The coasting trajectories, before and after the impulse burn, satisfy the central force field equations

$$\ddot{R} = -\mu \frac{R}{r^3} \tag{1}$$

where

$$\begin{aligned} R &= f R_o + g \dot{R}_o \\ \dot{R} &= \dot{f} R_o + \dot{g} \dot{R}_o \end{aligned} \tag{1a}$$

and f, g, $\dot{f}$, and $\dot{g}$ are scalar functions of the initial conditions and an anomaly variable which is related to the time by Kepler's equation (Ref. 2).

At the impulse transfer time, we have

$$\begin{aligned} R^{+}(t) &= R^{-}(t) \\ \dot{R}^{+}(t) &= \dot{R}^{-}(t) - c \ln \frac{m^{+}}{m^{-}} \frac{\Lambda}{\lambda} \end{aligned} \tag{2}$$

The vector Λ satisfies the variational equations of Eq. (1)

$$\ddot{\Lambda} = -\mu \frac{\Lambda}{r^3} + 3\mu \frac{R \cdot \Lambda}{r^5} R \tag{3}$$

The equation for the mass is

$$\dot{m} = -\frac{k}{c} \qquad (k, c \text{ constant}) \tag{4}$$

and for the impulse

$$-c \,\ell n \frac{m^+}{m^-} = \left| \dot{R}^+ - \dot{R}^- \right| \tag{5}$$

On each of the coasting arcs, the vector Λ satisfies the transition matrix equations of the variational solution of Eq. (1)

$$\begin{aligned} \Lambda &= \left(\frac{\partial R}{\partial R_o}\right)\Lambda_o + \left(\frac{\partial R}{\partial \dot{R}_o}\right)\dot{\Lambda}_o \\ \dot{\Lambda} &= \left(\frac{\partial \dot{R}}{\partial R_o}\right)\Lambda_o + \left(\frac{\partial \dot{R}}{\partial \dot{R}_o}\right)\dot{\Lambda}_o \end{aligned} \tag{6}$$

The necessary optimality condition is given by the characteristic that the magnitude of Λ, λ is a maximum at an interval impulse, has a non-positive slope at an initial burn, a non-negative slope at a terminal impulse, and that, at all impulse times, λ is equal to the same maximum value. Furthermore, if λ exceeds this maximum constant value anywhere on the coasting arcs, the the solution is not optimal, and a better solution can be found.

The Lagrange multipliers, Λ and $\dot{\Lambda}$, of the position and velocity state must be continuous over the entire arc, except at such times and places where constraints are imposed on the state and other irregular instances (Ref. 3).

We seek a solution of the finite burn solution of the equations

$$\ddot{R} = -\mu \frac{R}{r^3} + \frac{k}{m} \frac{\Lambda}{\lambda}$$

$$\ddot{\Lambda} = -\mu \frac{\Lambda}{r^3} + 3\mu \frac{R \cdot \Lambda}{r^5} R \qquad (7)$$

$$\dot{m} = -\frac{k}{c}$$

in terms of the solution for the impulsive case.

EXPANSION OF THE IMPULSE SOLUTION

Consider the integration of Eq. (7) for a finite time.

$$\dot{R}(t) = \dot{R}(t_o) - \mu\int_{t_o}^{t} \frac{R}{r^3}\, dt + \int_{t_o}^{t} \left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt$$

$$R(t) = R(t_o) + (t-t_o)\dot{R}(t_o) - \mu\iint_{t_o}^{t} \frac{R}{r^3}\, dt\, dt + \iint_{t_o}^{t} \left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt\, dt \tag{8}$$

$$m = m_o + \dot{m}(t-t_o)$$

As a zero-order approximation, consider the coast solution of Eq. (1) from the same initial conditions

$$\dot{\bar{R}}(t) = \dot{R}(t_o) - \mu\int_{t_o}^{t} \frac{\bar{R}}{\bar{r}^3}\, dt$$

$$\bar{R}(t) = R(t_o) + (t-t_o)\dot{R}(t_o) - \mu\iint_{t_o}^{t} \frac{\bar{R}}{\bar{r}^3}\, dt\, dt \tag{9}$$

Eliminating $\dot{R}(t_o)$ and $R(t_o)$ from Eqs. (8) and (9), we have

$$\dot{R}(t) = \dot{\bar{R}}(t) - \mu\int_{t_o}^{t} \left(\frac{R}{r^3} - \frac{\bar{R}}{\bar{r}^3}\right) dt + \int_{t_o}^{t} \left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt$$

$$R(t) = \bar{R}(t) - \mu\iint_{t_o}^{t} \left(\frac{R}{r^3} - \frac{\bar{R}}{\bar{r}^3}\right) dt\, dt + \iint_{t_o}^{t} \left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt\, dt \tag{10}$$

It is plain from Eq. (10) that, as $t \rightarrow t_o$, the solution of Eq. (10) approaches the impulsive solution given by Eq. (2). We seek a more definitive form of the expansion for short finite time. In what follows, let $t - t_o = \tau$.

Consider the Taylor series expansion of the difference integrand of the gravity term.

$$\frac{R}{r^3} - \frac{\bar{R}}{\bar{r}^3} = \frac{R_o}{r_o^3} - \frac{\bar{R}_o}{\bar{r}_o^3} + \frac{d}{dt}\left(\frac{R_o}{r_o^3} - \frac{\bar{R}_o}{\bar{r}_o^3}\right)\tau + \text{---} \tag{11}$$

Since the initial conditions for both coast and burn solutions are the same, the first non-vanishing term is the τ^2 term. We have

$$\frac{R}{r^3} - \frac{\bar{R}}{\bar{r}^3} = \left(I - 3\frac{R_o R_o^*}{r_o^2}\right)\frac{\{\ddot{R}_o - \ddot{\bar{R}}_o\}}{r_o^3}\frac{\tau^2}{2} + \text{---} \tag{11a}$$

$$= \left(I - 3\frac{R_o R_o^*}{r_o^2}\right)\frac{k}{m_o}\frac{\Lambda_o}{\lambda_o}\frac{\tau^2}{2 r_o^3} + \text{---} \tag{11b}$$

where R_o^* is the transpose of R_o and $I = 3 \times 3$ identity matrix. Examination of Eq. (11b) shows it to be nonlinear in the vector variable R_o. Similarly, the next term, (τ^3), will be nonlinear in R_o and $\dot{R}_o$ and, finally, the τ^4 term will be nonlinear in the variable Λ_o itself. While we could formally carry out this expansion, terms nonlinear in Λ_o and $\dot{\Lambda}_o$ will prove very cumbersome for inversion.

In what follows, we restrict the solution to the expansion which is linear in the vector variable, Λ_o, and its time derivatives.

$$\frac{R}{r^3} - \frac{\bar{R}}{\bar{r}^3} = \frac{1}{r_o^3}\left(I - 3\frac{R_o R_o^*}{r_o^2}\right)\left[\sum_{i=2}^{\infty} \frac{d^i}{d\tau^i}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right)_o \frac{\tau^i}{i!}\right] - \frac{9}{r_o^5}\left(R_o \cdot \dot{R}_o I + \dot{R}_o R_o^* + R_o \dot{R}_o^*\right.$$

$$\left. - 5\,\frac{R_o \cdot \dot{R}_o}{r_o^2} R_o R_o^*\right)\left[\sum_{i=3}^{\infty} \frac{d^i}{d\tau^i}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right)_o \frac{\tau^i}{i!}\right] \tag{12}$$

We have the following identity

$$\iiint_o^t \cdots \int^n f(x)\, dx^n = \sum_{i=n}^{\infty} \left(\frac{d^i}{d\tau^i} f\right)_o \frac{\tau^i}{i!} \tag{13}$$

Equation (10) now becomes

$$\dot{R}(t) = \dot{\bar{R}}(t) + \int_{t_o}^{t} \frac{k}{m}\frac{\Lambda}{\lambda}\, dt - \frac{\mu}{r_o^3}\left(I - 3\frac{R_o R_o^*}{r_o^2}\right)\iiint_{t_o}^{t}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt^3 + \frac{9\mu}{r_o^5}\left(d_o I + \dot{R}_o R_o^*\right.$$

$$\left. + R_o \dot{R}_o^* - 5\, d_o \frac{R_o R_o^*}{r_o^2}\right)\iiiint_{t_o}^{t}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt^4$$

(14)

$$R(t) = \bar{R}(t) + \iint_{t_o}^{t}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt^2 - \frac{\mu}{r_o^3}\left(I - 3\frac{R_o R_o^*}{r_o^2}\right)\iiiint_{t_o}^{t}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt^4 + \frac{9\mu}{r_o^5}\left(d_o I + \dot{R}_o R_o^*\right.$$

$$\left. + R_o \dot{R}_o^* - 5\, d_o \frac{R_o R_o^*}{r_o^2}\right)\int\!\!\iiiint_{t_o}^{t}\left(\frac{k}{m}\frac{\Lambda}{\lambda}\right) dt^5 ,$$

where $d_o = R_o \cdot \dot{R}_o$.

Before proceeding with the solution, it is timely to point out the time-scale difference in the variation of the Lagrange multipliers, Λ, and the mass flow rate. We note that the unit vector in the direction of the thrust rotates relatively slowly as compared to the time-variation of the mass. Thus, we take a linear variation in thrust direction as a good enough approximation during the burn. We have

$$\frac{d}{dt}\frac{\Lambda}{\lambda} = \frac{(\Lambda \times \dot{\Lambda}) \times \Lambda}{\lambda^3} \tag{15}$$

Finally, from the definition of the mass flow rate, we have

$$dt = \frac{dm}{\dot{m}}$$

and (16)

$$\frac{k}{m} dt^n = - \frac{c}{\dot{m}^{n-1}} dm^n$$

We define the i^{th} integral of the reciprocal of the mass

$$m_i(m, m_o) = \overbrace{\iiint \cdots \int_{m_o}^{m}}^{i} \frac{dm^i}{m}$$

$$m_1 = \ln \frac{m}{m_o}$$

$$m_2 = m \ln \frac{m}{m_o} - (m - m_o) \tag{17}$$

$$m_3 = \frac{m^2}{2} \ln \frac{m}{m_o} - \frac{m^2 - m_o^2}{4} - \frac{(m - m_o)^2}{2}$$

etc.

In addition, we define the i^{th} integral of the first moment of the reciprocal of the mass

$$s_i(m, m_o) = \int\int\int_{m_o}^{m} \cdots \int^{i} \frac{(m - m_o)dm^i}{m}$$

$$s_1 = -m_o m_1 + (m - m_o) \tag{17a}$$

$$s_2 = -m_o m_2 + \frac{(m - m_o)^2}{2}$$

etc.

The required asymptotic expansion in terms of the reciprocal of the constant mass flow rate is given by

$$\dot{R}(t) = \dot{\bar{R}}(t) - c\left[m_1 \frac{\Lambda_o}{\lambda_o} + \frac{s_1}{\dot{m}} \frac{(\Lambda_o \times \dot{\Lambda}_o) \times \Lambda_o}{\lambda_o^3}\right] + \frac{c\mu}{r_o^3 \dot{m}^2}\left[I - 3\frac{R_o R_o^*}{r_o^2}\right]$$

$$\left[m_3 \frac{\Lambda_o}{\lambda_o} + \frac{s_3}{\dot{m}} \frac{(\Lambda_o \times \dot{\Lambda}_o) \times \Lambda_o}{\lambda_o^3}\right] - \frac{9c\mu}{r_o^5 \dot{m}^3}\left[d_o I + \dot{R}_o R_o^* + R_o \dot{R}_o^* - 5 \frac{R_o R_o^*}{r_o^2}\right]$$

$$\left[m_4 \frac{\Lambda_o}{\lambda_o} + \frac{s_4}{\dot{m}} \frac{(\Lambda_o \times \dot{\Lambda}_o) \times \Lambda_o}{\lambda_o^3}\right]$$

(18)

$$R(t) = \bar{R}(t) - \frac{c}{\dot{m}}\left[m_2 \frac{\Lambda_o}{\lambda_o} + \frac{s_2}{\dot{m}} \frac{(\Lambda_o \times \dot{\Lambda}_o) \times \Lambda_o}{\lambda_o^3}\right] + \frac{c\mu}{r_o^3 \dot{m}^3}\left[I - 3\frac{R_o R_o^*}{r_o^2}\right]$$

$$\left[m_4 \frac{\Lambda_o}{\lambda_o} + \frac{s_4}{\dot{m}} \frac{(\Lambda_o \times \dot{\Lambda}_o) \times \Lambda_o}{\lambda_o^3}\right] - \frac{9c\mu}{r_o^5 \dot{m}^4}\left[d_o I + \dot{R}_o R_o^* + R_o \dot{R}_o^* - 5\, d_o \frac{R_o R_o^*}{r_o^2}\right]$$

$$\left[m_5 \frac{\Lambda_o}{\lambda_o} + \frac{s_5}{\dot{m}} \frac{(\Lambda_o \times \dot{\Lambda}_o) \times \Lambda_o}{\lambda_o^3}\right]$$

REFERENCES

1. Pines, S.; "Constants of the Motion for Optimum Thrust Trajectories in a Central Force Field," AIAA Journal, Vol. 2, No. 11, November 1964.

2. Pines, S. and Fang, T. C.; "A Uniform Closed Solution of the Variational Equations for Optimal Trajectories During Coast," in Advanced Problems and Methods for Space Flight Optimization, F. de Veubeke, Ed., Pergamon Press, Oxford & New York, 1969.

3. Bryson, A. E. and Ho, Y. C.; Applied Optimal Control, Blaisdell Publishing Co., Massachusetts, 1969.