SOME EFFECTS OF ELASTICITY ON LUNAR ROTATION

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Abstract. A general Hamiltonian for a rotating moon in the field of the earth is expanded in terms of parameters orienting the spin angular momentum relative to the principal axes of the moon and relative to coordinate axes fixed in the orbit plane. The effects of elastic distortion are included as modifications of the moment of inertia tensor, where the magnitude of the distortion is parameterized by the Love number k_2. The principal periodic terms in the longitude of a point on the moon due to variations of the tide caused by the earth are shown to have amplitudes between $3''.9\times10^{-3}$ and $1''.6\times10^{-2}$ with a period of an anomalistic month, $3''.0\times10^{-4}$ and $1''.2\times10^{-3}$ with a period of one-half an anomalistic month and $2''.4\times10^{-4}$ and $9''.6\times10^{-4}$ with a period of one-half of a nodical month. The extremes in the amplitudes correspond to rigidities of $8\times10^{+11}$ cgs and $2\times10^{+11}$ cgs respectively, the former rigidity being comparable to that of the earth. Only the largest amplitude given above is comparable to that detectable by the projected precision of the laser ranging to the lunar retroreflectors, and this amplitude corresponds to an improbably low rigidity for the moon. A detailed derivation of the free wobble of the lunar spin axis about the axis of maximum moment of inertia is given, where it is shown that elasticity can alter the period of the free

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wobble of 75.3 years by only 3×10^{-4} to 10^{-3} of this period. Also, the effect of elasticity on the period of free libration is completely negligible by many orders of magnitude. If the moon's rigidity is close to that of the earth there is no effect of elasticity on the rotation which can be measured with the laser ranging, and therefore no elastic properties of the moon can be determined from variations in the rotation.

1. Introduction

Order of magnitude estimates of the periodic variations in the rotation rate of the moon due to the changing magnitude of the tide induced by the earth indicate marginal detectability of these variations by future laser ranging to the retroreflectors on the lunar surface. It is thus appropriate to determine more accurately the magnitude of the effects of elasticity on lunar rotation to see whether any must be eventually included in the reduction of the laser ranging data. If such effects are in fact measurable, perhaps a more important result will be the determination of the effective elastic properties of the entire moon with possible implications about the nature of the interior.

Three possible effects of elasticity are considered here. The first is the above mentioned variation in the rotation rate due to tidal changes in the inertia tensor. In Section 2 the largest term of this variation is calculated directly from the tidal variation in the moment of inertia about the spin axis. This serves as a check on the general theory described in Section 3 from which all the perturbations of the lunar rotation, including those due to elasticity, can be found to arbitrary order. The amplitudes of the three largest terms of the angular displacement of a point on the lunar surface from its mean sidereal position (corresponding to uniform rotation) are determined to Section 4 for effective rigidities corresponding to those of steel and aluminum.

For the second effect of elasticity to be considered, the theory developed in Section 3 is used in Section 5 to determine the period of the free wobble as modified by the presence of the earth and to demonstrate that the effect of elasticity on the free wobble is negligible. This could be anticipated beforehand from the functional dependence of the modification of the wobble period of the Love number k_2 (see Munk and MacDonald, 1960, for a discussion of Love numbers) and the spin rate $\dot{\psi}$ and a comparison with the modification of the earth's Chandler wobble period.

The period of the free libration in longitude is considered in Section 6, where any alteration of the period by elasticity is quickly dismissed. Even a small change in the free libration period is important because of the existence of forced librations whose periods are very close to the free libration period (Jeffreys, 1957; Eckhardt, 1970; Williams et al. 1973). A few percent change in the free period can lead to a change in the amplitude of the near resonant forced libration of a factor 2 or 3. But no measurable alteration, even with the amplification, is evident.

Section 7 is a summary of the results where it is pointed out that only the largest term in the variation of the rotation rate is possibly of measurable amplitude and that only if the moon has what is perhaps an unusually low rigidity.

2. Tidally Induced Variations in the Spin Rate

If ψ is the angle between the axis of minimum moment of inertia and the ascending node of the lunar equator on the orbit plane then with sufficient accuracy for the short term effects considered here,

$$\frac{d^2\psi}{dt^2} = \frac{-1}{C}\frac{d\psi}{dt}\frac{dC}{dt} \tag{1}$$

for a tidal variation in the moment of inertia C about the spin axis (spin angular momentum is conserved). The time variation in C is determined from the variation in the tidal mass distribution.

The magnitude of the tide at a point $\underset{\sim}{R}$ on the lunar surface is given by

$$\Delta R = h_2 \frac{GM_\oplus}{r^3 g} a_e^2 \; P_2(\cos\theta'') \tag{2}$$

where $\underline{R}$ is measured from the lunar center of mass, P_2 is the Legendre polynomial, $\cos\theta'' = \underline{R}\cdot\underline{r}/(Rr)$, $\underline{r}$ being the position of the earth relative to the moon, G is the gravitational constant, g is the surface gravity on the moon, a_e is the lunar equatorial radius. $M_\oplus$ is the earth mass and h_2 is the Love number defined by (Munk and MacDonald, 1960)

$$h_2 = \frac{5/2}{1 + \frac{19}{2}\frac{\mu}{\rho g a_e}} \tag{3}$$

for a homogeneous sphere, where μ is the coefficient of rigidity and ρ is the lunar mean density. The tidal mass per unit area of the lunar surface $\Delta m/\Delta A = \rho\Delta R$ and

$$C = C_0 + \int_A \frac{\Delta m}{\Delta A} a_e^2 \sin^2\theta dA \tag{4}$$

where θ is the polar angle measured from the spin axis.

Part of the mass in Δm is determined by the fluid Love number ($\mu = 0$) and corresponds to the tide at the mean earth-moon separation. However, this constant tide vanishes in dC/dt, and h_2 defined by Equation (3) is appropriate for the varying tide. Then

$$\frac{dC}{dt} = \int_A \frac{d}{dt}\frac{\Delta m}{\Delta A} a_e^2 \sin^2\theta dA$$
$$= \int_0^\pi \int_0^{2\pi} - 3h_2 \frac{M_\oplus}{M} \frac{a_e^6}{r^4} P_2(\cos\theta'') \frac{dr}{dt} \sin^3\theta d\theta \, d\phi \tag{5}$$

where $M_☽$ is the lunar mass, ϕ is the azimuthal spherical polar coordinate of position on the moon measured from the axis of minimum moment of inertia on the lunar equator. Substitution of $M_☽ = \frac{4}{3}\pi a_e^3\rho$, $dr/dt = nae \sin f/\sqrt{1-e^2}$ with n, a, e, f being respectively the lunar orbital mean motion, semimajor axis, eccentricity and true anomally, $\cos\theta'' = \sin\theta\cos\phi$ and performing the integration in Equation (5) yields

$$\frac{dC}{dt} = -\frac{3}{5}\, h_2 M_\oplus \frac{a_e^5}{a^3}\, e\, n \sin nt \tag{6}$$

where only the first term in the expansion of $(a^4/r^4)\sin f$ in the mean anomaly ($M = nt$) has been kept, higher frequency terms being higher order in e.

Substitution of Equation (6) into Equation (1), with $d\psi/dt$ and C replaced by their mean values n and $0.4\, M\, a_e^2$ respectively, gives

$$\frac{d^2\psi}{dt^2} = -\frac{3}{2}\, n^2 \frac{a_e^3}{a^3} \frac{M_\oplus}{M}\, h_2\, e \sin nt \tag{7}$$

which upon integration yields an amplitude of variation in ψ from the mean (uniform rotation) value of

$$= \frac{3}{2} \frac{a_e^3}{a^3} \frac{M_\oplus}{M}\, e\, h_2 = \begin{cases} 3''.9\times10^{-3} \\ 1''.6\times10^{-2} \end{cases} \tag{8}$$

with a period of the anomalistic month. The two numerical values correspond respectively to a lunar rigidity like that of steel ($\mu = 8 \times 10''$ dynes/cm^2) and aluminum ($\mu = 2 \times 10''$ dynes/cm^2). The rigidity of the earth is slightly below that of steel (Munk and MacDonald, 1960), so one might expect that of the moon to fall within the above extremes.

The single term in $\Delta\psi$ evaluated above is expected to be the largest effect of elasticity on the lunar rotation. Information about other terms in the rotational variations requires a more general theory, which is described below. This theory is sufficiently general to include rotational distortions and deviations from principal axis rotations and is used to investigate the free wobble in Section 5.

3. General Theory

Parts of the following development were used in earlier investigations of the rotations of the moon, Mercury (Peale, 1969) and Venus (Goldreich and Peale, 1970). In those applications the Hamiltonian did not include the effects of non principal axis rotation which are necessary for an investigation of the free wobble. These effects are added here, and an error in the definition of variables is also corrected here.

A complete development of the variational equations and the form of the Hamiltonian is beyond the scope of this paper. Let it suffice then to outline the procedure and then write the final form of the Hamiltonian and variational equations.

The origin of coordinates is at the center of mass of the rotating body, and the Hamiltonian is the sum of the rotational kinetic energy and the potential energy due to external gravitational fields. The translational kinetic energy and central terms in the potential energy do not contain the coordinates and momenta associated with the spin and orientation and are therefore suppressed. Body fixed axes designated by the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ are the principal inertial axes. A second set of axes designated by ($\underset{\sim}{I}$, $\underset{\sim}{J}$, $\underset{\sim}{K}$), are fixed in the orbit plane of the disturbing body with K being normal to the orbit plane. The earth-moon orbit precesses on the ecliptic plane so $\underset{\sim}{I}$ is chosen to be along the mean ascending node on the ecliptic of the earth's orbit relative to the moon.

The ($\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$) system can be oriented with respect to the ($\underset{\sim}{I}$, $\underset{\sim}{J}$, $\underset{\sim}{K}$) by the ordinary Euler angles (ϕ,θ,ψ). The angular velocities S_x, S_y, S_z can be expressed in terms of the Euler angles (Goldstein, 1950) and generalized momenta p_ϕ, p_θ, p_ψ conjugate to ϕ, θ, ψ can be used to express the kinetic energy part of H with the resulting variational equations being canonical.

The set of variables $(\phi, \theta, \psi, p_\phi, p_\theta, p_\psi)$ is not convenient for expressing the Hamiltonian or for interpreting the variations. The following series of transformations is therefore effected:

$$(\phi, \theta, \psi, p_\phi, p_\theta, p_\psi) \rightarrow (\alpha, \eta, \underline{K}\cdot\underline{a}, \underline{J}\cdot\underline{a}, \underline{k}\cdot\underline{a}, \underline{j}\cdot\underline{a}) \rightarrow$$
$$(\alpha, \eta, \underline{K}\cdot a, \Omega, \underline{k}\cdot a, \omega) ,$$

where $\underline{\alpha} = \alpha\underline{a}$ is the total spin angular momentum, $\underline{a}$ being a unit vector, η is the angle between $\underline{K}\times\underline{a}$ and $\underline{k}\times\underline{a}$ and is thus a measure of the rotation about the spin axis. The variables Ω and ω are defined by

$$\cos\Omega = \frac{-\underline{J}\cdot\underline{a}}{\sqrt{1-\underline{K}\cdot\underline{a}^2}} \qquad \sin\Omega = \frac{\underline{I}\cdot\underline{a}}{\sqrt{1-\underline{K}\cdot\underline{a}^2}}$$

$$\cos \quad = \frac{-\underline{j}\cdot\underline{a}}{\sqrt{1-\underline{k}\cdot\underline{a}^2}} \qquad \sin \quad = \frac{\underline{j}\cdot\underline{a}}{\sqrt{1-\underline{k}\cdot\underline{a}^2}}$$

Ω is thus the angle between $\underline{K}\times\underline{a}$ and $\underline{I}$ and ω is the angle between $\underline{k}\times\underline{a}$ and $\underline{i}$. The above choice of variables conveniently orients the angular momentum relative to space and body axes and describes the rotation. The variational equations in terms of the final set of variables are

$$\begin{aligned}
\frac{d\alpha}{dt} &= -\frac{\partial H}{\partial\eta} \\
\frac{d\eta}{dt} &= \frac{\partial H}{\partial\alpha} - \frac{\underline{K}\cdot\underline{a}}{\alpha}\frac{\partial H}{\partial\underline{K}\cdot\underline{a}} - \frac{\underline{k}\cdot\underline{a}}{\alpha}\frac{\partial H}{\partial\underline{k}\cdot\underline{a}} \\
\frac{d\underline{K}\cdot\underline{a}}{dt} &= -\frac{1}{\alpha}\frac{\partial H}{\partial\Omega} + \frac{\underline{K}\cdot a}{\alpha}\frac{\partial H}{\partial\eta} \\
\frac{d\Omega}{dt} &= \frac{1}{\alpha}\frac{\partial H}{\partial\underline{K}\cdot\underline{a}} \\
\frac{d\underline{k}\cdot\underline{a}}{dt} &= -\frac{1}{\alpha}\frac{\partial H}{\partial\omega} + \frac{\underline{k}\cdot\underline{a}}{\alpha}\frac{\partial H}{\partial\eta}
\end{aligned} \tag{10}$$

$$\frac{d\omega}{dt} = \frac{1}{\alpha}\frac{\partial H}{\partial \mathbf{k}\cdot\mathbf{a}} \tag{10}$$

The development of the potential part of the Hamiltonian follows that of Kaula (1961). The terms in the expansion used by Kaula are each expanded to second order in $\sqrt{1-\mathbf{k}\cdot\mathbf{a}^2}$ such that the variational equations are correct at least to first order in this quantity. This is sufficient accuracy, since all sizable solar system bodies are expected to be rotating nearly about a principal axis ($\mathbf{k}\cdot\mathbf{a} \approx 1$), driven there by energy dissipation. The development of the kinetic energy part of the Hamiltonian follows that of Peale (1969) except now the allowed distortion of the elastic body introduces increments in the components of the inertia tensor which are determined by rotational and tidal distortions. As such, these distortions will depend on the variables used in Equations (10), but this functional dependence of the distortions is considered only after the differentiations have been performed on the right hand sides of Equations (10). The Hamiltonian is

$$H = \frac{1}{2}\,\boldsymbol{\alpha} \cdot [I]^{-1}\boldsymbol{\alpha} - \boldsymbol{\mu}\cdot\boldsymbol{\alpha} + V \tag{11}$$

where $[I]^{-1}$ is the inverse of the inertia tensor. $\boldsymbol{\mu}$ is the precessional angular velocity of the orbit plane and V is the potential.

$$\boldsymbol{\alpha} = \alpha\left[(\mathbf{i}\cdot\mathbf{a})i + (\mathbf{j}\cdot\mathbf{a})j + (\mathbf{k}\cdot\mathbf{a})k\right] \tag{12}$$

$$\boldsymbol{\mu} = -\mu\left[\sin\iota\,\mathbf{J} + \cos\iota\,\mathbf{K}\right] \tag{13}$$

where ι is the inclination of the lunar orbit to the ecliptic.

Substitution of Equation (12) and (13) into Equation (11), use of Equations (9) and expansion of the terms in V gives the final general form of H expressed in the variables of Equation (10).

$$H = \frac{\alpha^2}{2|I|}\Bigg\{\frac{(1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)}{2}\left[I_{22}I_{33} - I_{23}^{\;2} + I_{11}I_{33} - I_{13}^{\;2}\right] + \underset{\sim}{k}\cdot\underset{\sim}{a}^2\left[I_{11}I_{22} - I_{12}^{\;2}\right]$$

$$+ \frac{(1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)}{2}\cos 2\omega\left[I_{11}I_{33} - I_{13}^{\;2} - I_{22}I_{33} + I_{23}^{\;2}\right]$$

$$+ (1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)\sin 2\omega\left[I_{23}I_{13} - I_{12}I_{33}\right]$$

$$- 2\sqrt{1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2}\sin\omega\left[I_{12}I_{23} - I_{22}I_{13}\right]$$

$$- 2\sqrt{1-k\cdot a^2}\cos\omega\left[I_{13}I_{12} - I_{11}I_{23}\right]$$

$$+ \mu\alpha\left[-\sin\iota\sqrt{1-\underset{\sim}{K}\cdot\underset{\sim}{a}^2}\cos\Omega + \cos\iota\;\;\underset{\sim}{K}\cdot\underset{\sim}{a}\right]$$

$$\frac{-GM_{\oplus}M_{☽}}{a}\sum_{\ell=2}^{\infty}\left(\frac{a_e}{a}\right)^{\ell}\sum_{m=0}^{\ell}\sum_{p=0}^{\ell}\sum_{q=-\infty}^{\infty}G_{\ell pq}(e)\;.$$

$$\Bigg\{\left[F_{\ell mp}(\underset{\sim}{K}\cdot\underset{\sim}{a}) + E_{\ell mp}(1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)\right]\left[C_{\ell m}\begin{Bmatrix}\cos v_{\ell mpq}\\ \\ \sin v_{\ell mpq}\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}} + S_{\ell m}\begin{Bmatrix}\sin v_{\ell mpq}\\ \\ -\cos v_{\ell mpq}\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}}\right]$$

$$+ \sqrt{1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2}\,A_{\ell mp}(\underset{\sim}{K}\cdot\underset{\sim}{a})\left[C_{\ell m}\begin{Bmatrix}\cos(v_{\ell mpq}+\eta)\\ \\ \sin(v_{\ell mpq}+\eta)\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}} + S_{\ell m}\begin{Bmatrix}\sin(v_{\ell mpq}+\eta)\\ \\ -\cos(v_{\ell mpq}+\eta)\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}}\right]$$

$$+ \sqrt{1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2}\,B_{\ell mp}(\underset{\sim}{K}\cdot\underset{\sim}{a})\left[C_{\ell m}\begin{Bmatrix}\cos(v_{\ell mpq}-\eta)\\ \\ \sin(v_{\ell mpq}-\eta)\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}} + S_{\ell m}\begin{Bmatrix}\sin(v_{\ell mpq}-\eta)\\ \\ -\cos(v_{\ell mpq}-\eta)\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}}\right]$$

$$+ (1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)C_{\ell mp}(\underset{\sim}{K}\cdot\underset{\sim}{a})\left[C_{\ell m}\begin{Bmatrix}\cos(v_{\ell mpq}+2\eta)\\ \\ \sin(v_{\ell mpq}+2\eta)\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}} + S_{\ell m}\begin{Bmatrix}\sin(v_{\ell mpq}+2\eta)\\ \\ -\cos(v_{\ell mpq}+2\eta)\end{Bmatrix}_{\ell-m\text{ odd}}^{\ell-m\text{ even}}\right]$$

$$+ (1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)\, D_{\ell mp}(\underset{\sim}{K}\cdot\underset{\sim}{a}) \left[C_{\ell m} \begin{Bmatrix} \cos(v_{\ell mpq}-2\eta) \\ \sin(v_{\ell mpq}-2\eta) \end{Bmatrix}_{\ell-m\ \text{odd}}^{\ell-m\ \text{even}} + S_{\ell m} \begin{Bmatrix} \sin(v_{\ell mpq}-2\eta) \\ \cos(v_{\ell mpq}-2\eta) \end{Bmatrix}_{\ell-m\ \text{odd}}^{\ell-m\ \text{even}} \right] \Bigg\}$$

(14) con't.

where I_{ij} are the components of the inertia tensor, $|I|$ is the determinant of the inertia tensor, G, $M_{\oplus}$, $M_{☽}$, a, a_e e have been defined previously, $C_{\ell m}$, $S_{\ell m}$ are the ordinary harmonic coefficients in the expansion of the lunar gravitational field defined by

$$\begin{Bmatrix} C_{\ell m} \\ S_{\ell m} \end{Bmatrix} = \frac{(\ell-m)!\,(2-\delta_{\ell m})}{M\, a_e^{\ell}\,(\ell+m)!} \int\int\int \rho(r',\theta',\phi')\, r'^{\ell+2} P_{\ell m}(\cos\theta')\cdot \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \sin\phi'\, d\theta'\, d\phi'\, dr' \qquad (15)$$

the integration being over the entire volume of the moon. $G_{\ell pq}(e)$ are expansions in the eccentricity and $F_{\ell mp}$ $(\underset{\sim}{K}\cdot\underset{\sim}{a})$ are functions which are both defined and tabulated by Kaula (1966) with $(\underset{\sim}{K}\cdot\underset{\sim}{a})$ replacing cos i. $A_{\ell mp}$, $B_{\ell mp}$, $C_{\ell mp}$, $D_{\ell mp}$, $E_{\ell mp}$ are defined in terms of the $F_{\ell mp}$, and are given in Table I for $\ell = 2$. Finally

$$v_{\ell mpq} = (\ell-2p)(\pi-\Omega+\gamma) + (\ell-2p+q)M + m(\pi-\eta-\omega)$$

is equivalent to Kaula's definition (1964) of this angle, but the variables except for the mean anomaly M, are not those used in that work. We have used the orbit plane as a reference whereas Kaula has used the instantaneous equator plane. In addition, non-rincipal axis rotation forces us to use the vector $\underset{\sim}{a}$ parallel to the spin angular momentum for body orientation rather than the principal axis coincident with the spin vector. This introduces η and ω into $v_{\ell mpq}$, where in the limit of principal axis rotation $(k\cdot a = 1)$ $\eta + \omega = \psi$ locates the axis of minimum moment of inertia from the node of the equator on the orbit plane. The angle γ locates the pericenter of the earth's apparent orbit about the moon relative to the X axis, which is along the orbit node on the ecliptic.

In the moment of inertia tensor, the products of inertia come from tidal and rotational distortion of an elastic moon and are therefore small compared to the diagonal terms. The diagonal terms have small increments as well due to elastic distortions. This suggests that we write the diagonal terms

$$\begin{aligned} I_{11} &= A + \Delta A \\ I_{22} &= B + \Delta B \\ I_{33} &= C + \Delta C \end{aligned} \tag{16}$$

where A, B, C are the principal moments of inertia in the order of increasing magnitude and ΔA, ΔB, ΔC are the contributions from elastic distortion of the same order as the products of inertia. The determinant $|I|$ can then be expanded in the equations (after partial derivatives of H are taken) and only first order terms in $\Delta I_{ii}/I_{ii}$ and I_{ij}/I_{ii} kept. This simplifies the equations of motion considerably and sufficient accuracy is ensured by the smallness of the increments. It will also be noticed that elastic distortion introduces a time varying increment into the harmonic coefficients $C_{\ell m}$ and $S_{\ell m}$. However, since these coefficients are themselves very small and are preceded by another small coefficient, the effect of the small elastic variations in $C_{\ell m}$ and $S_{\ell m}$ are second order or higher in small quantities and can be ignored. Elastic distortion has its major influence in the kinetic energy terms except perhaps when the secular effects of dissipation are considered.

The increments in the moment of inertia tensor are evaluated by comparing the potential from the elastic redistribution of mass with the terms in the expansion of the moon's gravitational field depending on the second moments of the mass distribution. Both the rotational and tidal distortions are caused by harmonic disturbing potentials of the second degree (e.g., see Munk and MacDonald, 1960). For rotation, the disturbing potential per unit mass is

$$V_R' = \frac{R^2S^2}{3} P_2(\cos\theta') - \frac{R^2S^2}{3} \tag{17}$$

where $\underset{\sim}{S}$ is the spin angular velocity, $\underset{\sim}{R}$ is the position of the field point relative to the lunar center of mass and $\cos\theta' = \underset{\sim}{R}\cdot\underset{\sim}{S}/(RS)$. The central term in Equation (17) can be absorbed by the central term of the general lunar field and will not be considered further. The tide raising potential is given by

$$V_T' = \frac{-GM_{\oplus}R^2}{r^3} P_2(\cos\theta'') \tag{18}$$

where $\cos\theta'' = \underset{\sim}{R}\cdot\underset{\sim}{r}/(Rr)$. Love (1944) has shown that the increment in the potential at the surface of a spherical body distorted by a spherical harmonic potential is proportional to that distorting potential and falls off exterior to the body as $r^{-(n+1)}$ where n is the degree of the disturbing potential. The external potentials due to the lunar mass redistributed by rotation and tides are thus

$$V_R = \frac{k_2}{3}\frac{a_e^5}{R^5}\left(\frac{3}{2}(\underset{\sim}{R}\cdot\underset{\sim}{S})^2 - \frac{1}{2}R^2S^2\right) \tag{19}$$

$$V_T = -k_2 \frac{GM_{\oplus}}{r^3}\frac{a_e^5}{R^5}\left[\frac{3}{2}\frac{(\underset{\sim}{R}\cdot\underset{\sim}{r})^2}{r^2R^2} - \frac{1}{2}\right] \tag{20}$$

where k_2 is the Love number defined by

$$k_2 = \frac{3/2}{1 + \frac{19}{2}\frac{\mu}{\rho g a}} , \tag{21}$$

for a homogeneous sphere. If $\underset{\sim}{R}$, $\underset{\sim}{S}$, $\underset{\sim}{r}$ are written in terms of their components in Equations (19) and (20) and compared with a similar development of the second order term in the lunar gravitational field, the two representations of the field are equivalent and the trace of the inertia tensor invariant before and after distortion only if the increments in the tensor components are given by

$$\Delta I_{ii} = \frac{k_2 a_e^5 S_i^2}{3G}\left(1 - \frac{1}{3}\frac{S^2}{S_i^2}\right) \frac{-k_2 a_e^5 M_\oplus r_i^2}{r^5}\left(1 - \frac{r^2}{3r_i^2}\right)$$

$$I_{ij} = \frac{k_2 a_e^5}{3G} S_i S_j - \frac{k_2 a_e^5}{r^5} M_\oplus r_i r_j \qquad . \qquad i \neq j \; . \tag{22}$$

We can separate out that part of the rotational increment which appropriate to the fluid Love number by assuming principal axis rotation ($S_1 = S_2 = 0$, $S_3 = S$). These we shall include in the permanent moments A, B, C and write

$$I_{ii} = \frac{k_2 a_e^5}{3G} S_i^2 - \frac{k_2 a_e^5}{r^5} M_\oplus r_i^2 \left(1 - \frac{1}{3}\frac{r^2}{r_i^2}\right) \qquad i = 1,2$$

$$I_{33} = \frac{-k_2 a_e^5}{3G} (S_1^2 + S_2^2) \; \frac{-k_2 a_e^5}{r^5} M_\oplus r_3^2 \left(1 - \frac{1}{3}\frac{r^2}{r_3^2}\right) \tag{23}$$

where ΔI_{ij} ($i \neq j$) remains unchanged. The tidal increments will also have a component appropriate to the fluid Love number. However, since these involve the coordinates of the earth, the tidal terms can be expanded in terms of the variables and functions used in the Hamiltonian and it is a simple matter to remove the constant terms from these expansions such that the appropriate k_2 for the remaining terms is that defined by Equation (21). The above development can be applied in a straightforward manner to the effects of elasticity on the lunar rotation.

4. Application of General Theory to Fluctuations in the Rotation Rate

For our purposes here, it is sufficient to assume principal axis rotation and to ignore the effects of the precessing coordinate system, since these can be added at any time. Principal axis rotation eliminates the rotational increments in the inertia tensor, and since we have neglected second order effects, we need consider only the kinetic energy term of the Hamiltonian. Both η and ω are undefined when $\underset{\sim}{k} = \underset{\sim}{a}$, but their sum locates the axis of minimum moment of inertia from the node. Hence, we wish to determine

$$\frac{d\psi}{dt} = \lim_{\underset{\sim}{k}\cdot\underset{\sim}{a}\to 1} \frac{d}{dt}(\eta+\omega) = \lim_{(\underset{\sim}{k}\cdot\underset{\sim}{a}\to 1)} \left[\frac{\partial H}{\partial \alpha} + \frac{1}{\alpha}(1-\underset{\sim}{k}\cdot\underset{\sim}{a}) \frac{\partial H}{\partial(\underset{\sim}{k}\cdot\underset{\sim}{a})}\right] \tag{24}$$

where $\partial H/\partial(K\cdot a) = 0$ since only the kinetic energy terms are involved. The limit is taken after the differentiations of H. From Equations (14 and (24)

$$\frac{d\psi}{dt} = \frac{\alpha}{|I|}\left[I_{11}I_{22} - I_{12}^{2}\right] \quad . \tag{25}$$

Using Equation (16) and keeping only the first order terms in the expansion of both the numerator and denominator, we find

$$\frac{d\psi}{dt} = \frac{\alpha}{C}\left(1 - \frac{\Delta C}{C}\right) \tag{26}$$

where ΔC is here the tidal increment. The angular part of ΔC(tide) is the same as that in the coefficient of C_{20} in the general potential of the moon. Hence, we can lift the coefficient of C_{20} from Equation (14) and write

$$\frac{d\psi}{dt} = \frac{\alpha}{C}\left(1 + \frac{2}{3}\frac{k_2 M_{\oplus} a_e^{5}}{Ca^{3}} \sum_{p=0}^{2} \cdot \sum_{q} G_{2pq}(e) F_{20p}(K\cdot a) \cdot \cos\left[(2-2p)(\Pi-\Omega+\gamma) + (2-2p+q)M - \varepsilon_{20pq}\right]\right) \tag{27}$$

where ε_{20pq} is a phase lag due to dissipation (Kaula, 1964) and is included here only for completeness. If we combine terms of the same frequency and use Cassini's laws to set $\Omega = \pi$, we have

$$\frac{d\psi}{dt} = \frac{\alpha}{C} + \frac{2}{3}\frac{\alpha}{C^2}\frac{k_2 M_\oplus}{a^3} a_e^5 \times$$

$$\begin{Bmatrix} 2F_{200}\, G_{20-1} \cos[2\gamma+M] \\ + 2F_{200}\, G_{200} \cos[2\gamma+2M] \\ + 2F_{200}\, G_{201} \cos[2\gamma+3M] \\ + 2F_{200}\, G_{202} \cos[2\gamma+4M] \\ + 2F_{201}\, G_{212} \cos[2M] \\ + 2F_{201}\, G_{211} \cos[M] \end{Bmatrix} . \tag{28}$$

The last two terms depend on orbital position relative to the perigee and are hence related to the changing magnitude of the tide. The first four terms depend on position of the moon relative to the node and are related to the latitude of the tidal maximum. With $F_{200} = -3(1-\underset{\approx}{K}\cdot\underset{\approx}{a}^2)/8$, $F_{201} = \frac{1}{4} - \frac{3}{4} K\cdot a^2$, $G_{200} = 1 - \frac{5}{2} e^2$, $G_{212} = \frac{9}{4} e^2$, $G_{211} = \frac{3}{2} e$, $\underset{\approx}{K}\cdot\underset{\approx}{a} = \cos(6°41')$, $e = 0.0549$, integration of Equation (28) yields the following amplitudes in $\Delta\psi$ for the three largest terms:

Amplitude	Period
$3''.9\times10^{-3}$ $1''.6\times10^{-2}$	Anomalistic month
$3''.0\times10^{-4}$ $1''.2\times10^{-3}$	half of anomalistic month
$2''.4\times10^{-4}$ $9''.6\times10^{-4}$	nodical month

The values in each pair correspond respectively to lunar rigidities like that of steel and aluminum. The term with the period of the anomalistic month is just that determined in Section 2, and its amplitude agrees with the one determined there.

5. Free Wobble

The wobble of the lunar spin axis about the axis of maximum moment is determined by the last two of Equations (10). In addition to the contributions to $d(\underset{\sim}{k}\cdot\underset{\sim}{a})/dt$ and $d\omega/dt$ by the kinetic energy terms, those terms from the potential part of H which are constant or depend only on ω must also be retained. All other terms in the variation of ω and $\underset{\sim}{k}\cdot\underset{\sim}{a}$ will have phases and amplitudes determined by the forcing term and therefore do not contribute to the "free" wobble. Let us first select the terms which must be retained in the potential part of H.

From Equation (14) a general term has arguments

$$\begin{aligned} &V_{\ell mpq} \\ &V_{\ell mpq} \pm \eta \\ &V_{\ell mpq} \pm 2\eta \quad . \end{aligned} \tag{29}$$

But from Cassini's laws $\Omega = \pi$ and $\eta + \omega \approx \psi = M + \psi_o$ where ψ_o is the value of ψ at perigee. For completely damped librations, $\psi_o = \gamma$ completes the law of stable synchronous rotation. These conditions can be applied only after the partial differentiation of H in the equations of motion, but can be used beforehand to select those terms which will depend only on ω. Eliminating η, Ω and ψ_o from the arguments by Cassini's laws leaves

$$\begin{aligned} &(\ell-2p-m)\gamma + (\ell-2p+q-m)M + m\pi \\ &(\ell-2p-m\pm 1)\gamma + (\ell-2p+q-m\pm 1)M + m\pi \mp \omega \\ &(\ell-2p-m\pm 2)\gamma + (\ell-2p+q-m\pm 2)M + m\pi \mp 2\omega \end{aligned}$$

corresponding to the five arguments in (29). The only permitted arguments for $\ell = 2$ which are constant or depend only on ω are

$$v_{2010} \longrightarrow \text{constant}$$
$$v_{2200} \longrightarrow \text{constant}$$
$$v_{2110} + \eta \longrightarrow -\omega$$
$$v_{2100} - \eta \longrightarrow +\omega$$
$$v_{2210} + 2\eta \rightarrow -2\omega$$
$$v_{2020} + 2\eta \rightarrow -2\omega$$
$$v_{2000} - 2\eta \rightarrow +2\omega$$

Selecting only these arguments from the Hamiltonian, the variations of ω and $\underset{\sim}{k}\cdot\underset{\sim}{a}$ assume the forms

$$\frac{d\psi}{dt} = \frac{\alpha}{2}\left\{\frac{-\underset{\sim}{k}\cdot\underset{\sim}{a}}{A}\left(1 - \frac{\Delta A}{A}\right) - \frac{\underset{\sim}{k}\cdot\underset{\sim}{a}}{B}\left(1 - \frac{\Delta B}{B}\right) + \frac{2\underset{\sim}{k}\cdot\underset{\sim}{a}}{C}\left(1 - \frac{\Delta C}{C}\right)\right\}$$
$$+ \frac{2GM_{\oplus}M_{\mathbb{D}}}{\alpha a^3} a_e^{\,2}\ \underset{\sim}{k}\cdot\underset{\sim}{a}(E_{201}G_{210}C_{20} + E_{220}G_{200}C_{22})$$
$$+ \frac{\alpha(1-2\underset{\sim}{k}\cdot\underset{\sim}{a}^2)}{\sqrt{1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2}}\frac{I_{23}}{BC}\cos\omega + \frac{\alpha(1-2\underset{\sim}{k}\cdot\underset{\sim}{a}^2)}{\sqrt{1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2}}\frac{I_{13}}{AC}\sin\omega \qquad (30)$$
$$+ \left\{\frac{\alpha}{2}\ \underset{\sim}{k}\cdot\underset{\sim}{a}\left[\frac{-1}{B}\left(1 - \frac{\Delta B}{B}\right) + \frac{1}{A}\left(1 - \frac{\Delta A}{A}\right)\right] + \frac{2GM_{\oplus}M_{\mathbb{D}}}{\alpha a^3} a_e^{\,2}\ \underset{\sim}{k}\cdot\underset{\sim}{a}\left[C_{221}G_{210}C_{22} + \right.\right.$$
$$\left.\left. + C_{202}G_{220}C_{20} + D_{200}G_{200}C_{20}\right]\right\}\cos 2\omega + \alpha\underset{\sim}{k}\cdot\underset{\sim}{a}\ \frac{I_{12}}{AB}\sin 2\omega$$

$$\frac{d\underset{\sim}{k}\cdot\underset{\sim}{a}}{dt} = -\alpha\underset{\sim}{k}\cdot\underset{\sim}{a}\ \sqrt{1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2}\left[\frac{I_{13}}{AC}\cos\omega - \frac{I_{23}}{BC}\sin\omega\right]$$
$$+ \alpha(1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)\ \frac{I_{12}}{AB}\cos 2\omega \qquad (31)$$
$$+ \left\{\frac{\alpha}{2}\ (1-\underset{\sim}{k}\cdot\underset{\sim}{a}^2)\left[\frac{1}{B}\left(1 - \frac{\Delta B}{B}\right) - \frac{1}{A}\left(1 - \frac{\Delta A}{A}\right)\right]\right.$$

$$- \frac{2GM_{\oplus}M_{\rangle}}{a^3_{\alpha}} a_e^2 (1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2)\left[C_{221}G_{210}C_{22} + \left(C_{202}G_{220} + D_{200}G_{200}\right)C_{20}\ \underset{\approx}{k}\cdot\underset{\approx}{a}\right]\Big\}\sin 2\omega .$$
(31) cont'd.

In Equations (30) and (31) the terms involving C_{21}, S_{21} and S_{22} have been omitted since they involve products of inertia which are only induced by elastic deformation and are preceded by a small factor. Also the ΔI_{ij} will involve only the rotational deformation since the tidal increments involve terms which contain ω only in the sum $\eta + \omega$ which is transformed to $M + \psi_o$ by Cassini's laws. Hence, there is no way to isolate a term depending only on ω in the ΔI_{ij} due to tides and the tidal distortion terms in the wobble classify as forced terms. For the rotational increments it is sufficient for first order accuracy to write

$$S_i = \frac{\alpha_i}{I} \qquad (32)$$

and to set $I = A = B = C = 0.4\ M\ a_e^2$ in the denominators of Equations (32). Substitution of the above form for S_i into Equations (22) (minus the tidal contributions) and use of Equations (9) gives

$$\begin{aligned}
\frac{\Delta A}{I} &= \zeta(\underset{\approx}{i}\cdot\underset{\approx}{a})^2 = \zeta(1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2)\sin^2\omega \\
\frac{\Delta B}{I} &= \zeta(\underset{\approx}{j}\cdot\underset{\approx}{a})^2 = \zeta(1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2)\cos^2\omega \\
\frac{\Delta C}{I} &= \zeta(i\cdot a^2+j\cdot a^2) = -\zeta(1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2) \\
\frac{I_{12}}{I} &= \zeta(i\cdot a)(j\cdot a) = \zeta(1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2)\sin 2\omega/2 \qquad (33) \\
\frac{I_{13}}{I} &= \zeta(i\cdot a)(k\cdot a) = -\zeta\sqrt{1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2}\ k\cdot a\ \sin\omega \\
\frac{I_{23}}{I} &= \zeta(\underset{\approx}{j}\cdot\underset{\approx}{a})(\underset{\approx}{k}\cdot\underset{\approx}{a}) = -\zeta\sqrt{1-\underset{\approx}{k}\cdot\underset{\approx}{a}^2}\ k\cdot a\ \cos\omega
\end{aligned}$$

where

$$\zeta = \frac{5}{6}\,\frac{k_2 a_e^3}{GM_☽}\,\dot{\psi}^2 \tag{34}$$

with $\alpha = I\dot{\psi}$ being used in Equation (34).

Using the expressions in Equations (33) in Equations (30) and (31) we arrive at the expressions

$$\frac{d\omega}{dt} = \left[K_1 + K_2 \cos 2\omega\right]\underset{\sim}{k}\cdot\underset{\approx}{a}$$

$$\frac{d(\underset{\sim}{k}\cdot\underset{\approx}{a})}{dt} = -K_2'(1-\underset{\sim}{k}\cdot\underset{\approx}{a}^2)\sin 2\omega \tag{35}$$

where

$$K_1 = \frac{\alpha}{2A}\left[\frac{B-A}{B} - \frac{2(C-A)}{C} + 2\zeta\right]$$

$$+ \frac{2GM_{\oplus}M_☽}{\alpha a^3}\, a_e^2 \left[E_{201}G_{210}C_{20} + E_{220}G_{200}C_{22}\right]$$

$$K_2 = \frac{\alpha}{2}\,\frac{(B-A)}{AB} + \frac{2GM_{\oplus}M_☽}{\alpha a^3}\, a_e^2\left[C_{221}G_{210}C_{22} + (C_{202}G_{220} + D_{200}G_{200})C_{20}\right] \tag{36}$$

$$K_2' = \frac{\alpha}{2}\,\frac{(B-A)}{AB} + \frac{2GM_{\oplus}M_☽}{\alpha a^3}\, a_e^2\left[C_{221}G_{210}C_{22} + (C_{202}G_{220} + D_{200}G_{200})C_{20}(k\cdot a)\right]$$

In equations (36), $\zeta/A = \zeta/B = \zeta/C$ was assumed for first order accuracy. The expressions for K_2 and K_2' differ only by the factor $\underset{\sim}{k}\cdot\underset{\approx}{a}$ multiplying some of the terms. However, $d(\underset{\sim}{k}\cdot\underset{\approx}{a})/dt$ contains $1-\underset{\sim}{k}\cdot\underset{\approx}{a}^2$ as a factor so it is consistent with the first order accuracy to set $\underset{\sim}{k}\cdot\underset{\approx}{a} = 1$ in K_2' in which case $K_2 = K_2'$.

Equations (35) are more easily solved in terms of the variables $(\underset{\sim}{i}\cdot\underset{\approx}{a})$, $(\underset{\sim}{j}\cdot\underset{\approx}{a})$ which are related to $\underset{\sim}{k}\cdot\underset{\approx}{a}$ and ω by Equations (9). This transformation (with $K_2 = K_2'$) yields

$$\frac{d(\underset{\sim}{i}\cdot\underset{\approx}{a})}{dt} = (K_1+K_2)(\underset{\sim}{j}\cdot\underset{\approx}{a})(\underset{\sim}{k}\cdot\underset{\approx}{a})$$

$$\frac{d(\underset{\sim}{j}\cdot\underset{\sim}{a})}{dt} = (K_1-K_2)(\underset{\sim}{i}\cdot\underset{\sim}{a})(\underset{\sim}{k}\cdot\underset{\sim}{a})$$

$$\frac{d(\underset{\sim}{k}\cdot\underset{\sim}{a})}{dt} = 2K_2(\underset{\sim}{i}\cdot\underset{\sim}{a})(\underset{\sim}{j}\cdot\underset{\sim}{a}) \qquad (37)\text{ cont'd.}$$

which are just a form of Euler's equations for force free rigid body motion. For small deviations from principal axis rotation ($i\cdot a$, $j\cdot a \ll 1$, $\underset{\sim}{k}\cdot\underset{\sim}{a} \approx 1$), the solution is immediate with

$$\underset{\sim}{i}\cdot\underset{\sim}{a} = A_1\cos\left(\sqrt{(K_1+K_2)(K_1-K_2)}\,t + \phi_1\right)$$

$$\underset{\sim}{j}\cdot\underset{\sim}{a} = \sqrt{\frac{K_1-K_2}{K_1+K_2}}\,A_1\sin\left(\sqrt{(K_1+K_2)(K_1-K_2)}\,t + \phi_1\right) \qquad (38)$$

giving the expected elliptical motion of $\underset{\sim}{a}$ about $\underset{\sim}{k}$. The shape of the ellipse and the wobble frequency follow from the values of K_1 and K_2 which are obtained with

$$C_{202} = -\frac{1}{8}F_{220} = -\frac{3}{32}(1+\underset{\sim}{K}\cdot\underset{\sim}{a})^2$$

$$C_{221} = -3F_{212} = -\frac{3}{4}(1-3\underset{\sim}{K}\cdot\underset{\sim}{a}^2)$$

$$D_{200} = -\frac{1}{8}F_{220} = -\frac{3}{32}(1+\underset{\sim}{K}\cdot\underset{\sim}{a})^2 \qquad (39)$$

$$E_{201} = -\frac{3}{2}F_{201} = -\frac{3}{8}(1-3\underset{\sim}{K}\cdot\underset{\sim}{a}^2)$$

$$E_{220} = -\frac{1}{2}F_{220} = -\frac{3}{8}(1+\underset{\sim}{K}\cdot\underset{\sim}{a})^2$$

where $\underset{\sim}{K}\cdot\underset{\sim}{a} = \cos(6°41')$. All of the G functions in K_1 and K_2 differ from 1 only by terms of order e^2 so will not be listed explicitly. Placing numerical values into Equations (39) and expressing moment of inertia differences in terms of α', β', γ' (correct to first order) where $\alpha' = \frac{C-B}{A}$, $\beta' = (C-A)/B$, $\gamma' = (B-A)/C$, we find

$$K_1+K_2 \approx \dot{\psi}\{\gamma'-\beta'+\zeta\} - .001\dot{\psi}\{\alpha'+\beta'+\gamma'\}$$

$$K_1-K_2 \approx \dot{\psi}\{-\beta+\zeta\} - 1.48\dot{\psi}\{\alpha'+\beta'+\gamma'\} \qquad (40)$$

where the second terms on the right hand sides of Equations (40) explicitly demonstrate the influence of the earth (with $n = \dot{\psi}$). The effect of elasticity is contained in the parameter ζ. That elasticity has essentially no effect on the wobble is seen by comparing the value of ζ with α', β', γ'. With

$$\begin{aligned} \alpha' &= 3.97\times10^{-4} \\ \beta' &= 6.27\times10^{-4} \\ \gamma' &= 2.30\times10^{-4} \\ \zeta &= 1.5\times10^{-7} \text{ to } 6\times10^{-7} \end{aligned} \qquad (41)$$

we see that ζ influences the motion by only a few parts in 10^4 to a part in 10^3 and is likely to be comparable to or smaller than some of the neglected contributions to the free wobble from $\ell = 3$ terms in the potential.

This lack of influence of elasticity could have been anticipated by the functional dependence of ζ and a knowledge of the change in the period of the Earth's Chandler wobble by elasticity. The ratio of ζ/β' is the important parameter as seen from Equations (40).

$$\frac{\zeta}{\beta'} \approx \frac{k_2}{\beta'} \frac{\dot{\psi}^2}{n_o^2} \qquad (42)$$

where n_o is the orbital angular velocity of a satellite near the surface of the moon (or earth). The ratio in Equation (42) is about 0.35 for the earth with $\dot{\psi}/n_o = 1.5/24$ and $k_2 = 0.3$, and one observes about this fractional increase in the period of the Chandler wobble over that for a rigid earth. For the moon this ratio is seen from Equations (41) to be 2.5×10^{-4} to 1×10^{-3}.

The parameter ζ depends on the Love number and on the square of the rotation period. The former quantity drops for the moon by an order of magnitude from that of the earth and the latter drops by three orders of magnitude. These two effects combine to reduce the influence of elasticity on the lunar wobble by a comparable factor from the influence on the Chandler wobble.

Finally with $K_1+K_2 = -3.98\times10^{-4}\dot{\psi}$ and $K_1-K_2 = -2.48\times10^{-3}\dot{\psi}$, the ratio of the major and minor axes of the elliptical path of $\underset{\sim}{a}$ relative to $\underset{\sim}{k}$ is 2.57 with the long axis perpendicular to the axis of minimum moment of inertia. The wobble period is 75.3 years, which had been obtained earlier (Sekeguchi, 1970). This period would be increased by a factor of 2 if the earth were removed.

6. Free Libration

The existence of a forced libration whose frequency is very near the 3 year period of the free libration motivates a check on possible alternations of the free libration period. However, we can quickly dismiss the effects of elasticity on this period. The major effect will be the tidal distortion, but the variations in the rotation rate discussed in Sections 2 and 4 are high frequency and will not disturb the libration. We are thus left only with the tidal torque arising from a dissipation caused phase lag. In the *limit of small tides the net torque on a librating moon is just the sum of* that on the permanent lunar bulge and the tidal torque. The latter is given approximately by

$$T = -3k_2 \frac{GM_\oplus^2}{2a^6} \frac{a_e^5}{Q} \tag{43}$$

where Q is the specific dissipation function (MacDonald, 1964). The torque on the permanent deformation is determined from the ℓmpq = 2200 term in the

Hamiltonian with $2(\psi_o-\gamma) = 2\delta$ being the libration angle. With $\psi = M+\psi_o$, we have $C\ddot{\psi} = C(\ddot{\psi}_o) = C(\ddot{\psi}_o-\ddot{\gamma})$ since $\ddot{\gamma} \approx 0$, or

$$C\ddot{\delta} = \frac{-GM_\oplus}{a^3} G_{200}\ F_{220}\ \frac{(B-A)}{C}\ \delta - 3\frac{k_2 GM_\oplus^{\ 2}}{2a^6 Q}\ a_e^{\ 5}\ .$$

If we make Q inversely proportional to frequency, then

$$\ddot{\delta} + \frac{3}{2}\frac{k_2 GM_\oplus^{\ 2}}{Ca^6}\frac{a_e^{\ 5}\dot{\delta}}{Q_o\dot{\delta}_o} + \frac{GM_\oplus}{a^3}\ G_{200}F_{220}\ \frac{(B-A)}{C}\ \delta = 0 \tag{45}$$

where Q_o is the value of Q at a reference angular velocity $\dot{\delta}_o$. With the coefficients of δ and $\dot{\delta}$ being ω_o^2 and 2ξ respectively, the libration frequency is just $\sqrt{\omega_o^{\ 2}-\xi^2} = \omega_o[1-\xi^2/(2\omega_o^2)]$. For $F_{220} \approx 3$ and $G_{200} \approx 1$, we have that

$$\frac{\xi^2}{\omega_o^{\ 2}} \approx 10^{-21} \tag{46}$$

does not affect the period. A value of $Q_o = 100$ with $\dot{\delta}_o$ corresponding to a 3 year period was used in evaluating ξ. That the only effect of tides is the ordinary torque due to a phase lag of the lunar response is verified by evaluating $d\alpha/dt$ from Equations (10) and selecting those terms in ΔI_{ij} with arguments which are integer multiples of $(\psi_o-\gamma)$. All terms with these arguments cancel exactly to first order except for the effects of phase lags considered above.

7. Discussion

The results of the previous sections imply that the only possibly important perturbations of the lunar rotation due to elasticity are the periodic fluctuations in the spin angular velocity. Even these are so small that their measurement cannot be expected in the forseeable future. The ultimate range accuracy of the laser radar to the moon now anticipated is on the order of a few centimeters. (P. Bender private communication, 1972) which is comparable to the linear displacement on the lunar equator corresponding to 0".01 shift in longitude. The results of Section 4 show that the amplitude of the longitude variation from the mean is comparable to this value only for the low rigidity limit. Since siesmic wave velocities on the moon are comparable to those in the earth's upper mantle (Toksöz, et al., 1972) and the density of the mantle and moon are comparable, one infers comparable rigidities of the moon and earth. Since the smaller amplitude of Equation (8) is thus more likely appropriate to the moon, measurement of the effects of elasticity on rotation must await the development of the next generation of instrumentation perhaps requiring placement on the moon itself.

On the brighter side these results indicate that elasticity can most probably be safely neglected in the reduction of the laser ranging data.

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TABLE I

Inclination Functions

m_1p	A_{2mp}	B_{2mp}	C_{2mp}	D_{2mp}	E_{2mp}
00	F_{212}	F_{210}	$-\frac{1}{8}F_{222}$	$-\frac{1}{8}F_{220}$	$-\frac{3}{2}F_{200}$
01	F_{211}	F_{211}	$-\frac{1}{8}F_{221}$	$-\frac{1}{8}F_{221}$	$-\frac{3}{2}F_{201}$
02	F_{210}	F_{212}	$-\frac{1}{8}F_{220}$	$-\frac{1}{8}F_{222}$	$-\frac{3}{2}F_{202}$
10	$-6F_{200}$	$-F_{220}$	$-\frac{3}{4}F_{212}$	0	$-\frac{5}{4}F_{210}$
11	$-6F_{201}$	$-F_{221}$	$-\frac{3}{4}F_{211}$	0	$-\frac{5}{4}F_{211}$
12	$-6F_{202}$	$-F_{222}$	$-\frac{3}{4}F_{210}$	0	$-\frac{5}{4}F_{212}$
20	$4F_{210}$	0	$-3F_{202}$	0	$-\frac{1}{2}F_{220}$
21	$4F_{211}$	0	$-3F_{212}$	0	$-\frac{1}{2}F_{221}$
22	$4F_{212}$	0	$-3F_{200}$	0	$-\frac{1}{2}F_{222}$