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Dynamical Models for a Spacecraft Idealized as a Set of Multi-Hinged Rigid Bodies

V. Larson

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PASADENA, CALIFORNIA

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PREFACE

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.

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ABSTRACT

This report provides a brief description of a canonical set of equations which governs the behavior of an n-body spacecraft. General results are given for the case in which the spacecraft is modeled in terms of n rigid bodies connected by dissipative elastic joints. The final equations are free from constraint torques and involve only r variables (r is the number of degrees of freedom of the system). An advantage which accompanies the elimination of the constraint torques is a decrease in the computer run time (especially when n is large).

Linearized models are obtained and are recast in the familiar form

$$\dot{x}(t) = F\,x(t) + G(x,\ t)\,u(t)$$

where x is the state vector, u is the control vector, F is a constant matrix, and the matrix G depends on x and t. This form for the equations is particularly useful when modern control theory is used to arrive at a stochastic controller for a multi-hinged rigid-body spacecraft.

The models provided in this report will be used in analyzing the cruise, the thrust vector control (TVC), and the articulation control (ARTC) modes associated with the Mariner Jupiter Mars (MJS'77) spacecraft. Due to their generality, the models can also be conveniently used for analyzing a spacecraft appropriate for missions subsequent to the MJS'77 mission.

1. INTRODUCTION

This report provides a brief description of a particularly elegant formulation characterizing the rotational motion of a spacecraft idealized as a set of multi-hinged rigid bodies. Assumptions made in this development include the following:

(1) The spacecraft (S/C) can be adequately modeled as n hinged-rigid bodies connected by dissipative elastic joints.

(2) Chains of connected bodies do not form closed loops.

(3) Only rotational motion is allowed at a joint.

(4) There is a vector constraint torque orthogonal to the axis of rotation at a joint whenever the rotational motion has only one or two degrees of freedom.

Considerable effort has been focused on the problem of obtaining the dynamical equations for an n-hinged rigid body spacecraft (e.g., see Refs. [1] thru [5]). The approach used in this analysis is based primarily on Refs. [1] and [2]. Likins (see Refs [4] and [5]) recently extended the method discussed in Ref. [2] to appropriately account for flexible appendages.

In this analysis, the appendages are considered rigid but viscoelastic joints are allowed. Effectively, the interaction torque existing at a joint connecting a purely rigid body and a flexible appendage is modeled as a spring-damper torque (that is, it is specified in terms of torsional stiffness and damping coefficients). This model is especially useful in the preliminary design and analysis of the spacecraft.

The main objectives of this work are:

(1) To obtain a linearized dynamical model of the n-hinged rigid-body spacecraft.

(2) To provide a brief description of some of the dynamical principles involved in the development of the S/C model.

The linearized dynamical model of the multi-hinged rigid-body spacecraft can be:

(1) Used in a simulation to study the effects of interactions of the hinged members on thrust vector control (TVC) and articulation control (ARTC) performance.

(2) Used in the development of a general stochastic controller (see Refs. [7] and [8]) for a multi-hinged rigid-body spacecraft.

1.1 Compendium of Report Contents

In this section, a compendium of the section by section contents of the report are given.

Section 2 gives the general nonlinear dynamical equations for an n-hinged rigid body spacecraft with r degrees-of-freedom. The main result of this section is given by the vector-matrix equation

$$A \dot{\omega} = L \tag{1-1}$$

where $\dot{\omega}$ consists of the angular acceleration of the base body $\dot{\omega}_0$ and the relative angular acceleration components $\dot{\omega}_R$ and the vector L consists of the forcing functions for the base body (L_0) and for the n - 1 remaining bodies (L_R).

Section 3 provides a set of linearized dynamical equations for the n-hinged rigid-body spacecraft. An intermediate result of this section is given by the partitioned vector-matrix equation

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} \dot{\omega}_0 \\ \hline \dot{\omega}_R \end{array}\right] = \left[\begin{array}{c} G_0(\gamma, t) \\ \hline G_R(\gamma, t) \end{array}\right] F_0 - \left[\begin{array}{c} 0 \\ \hline K \end{array}\right] \gamma - \left[\begin{array}{c} 0 \\ \hline B \end{array}\right] \dot{\gamma} \tag{1-2}$$

where the matrices A, B, K are constants, 0 is an appropriately dimensioned null matrix, and the matrices G_0 and G_R depend on γ (the relative angles) and t. The primary result of this section is the recasting of Eq. (1-2) in the form

$$\dot{x}(t) = F\,x(t) + G(x, t)\,u(t) \tag{1-3}$$

where the state vector x consists of the attitude and angular velocity of the base body (θ and ω_0) and the relative angles (γ) and angular velocities (ω_R) associated with the remaining n - 1 bodies. The control vector u(t) is related to the force F_0 applied to the base body.

Section 4 provides a linearized dynamical model for a specific topological arrangement of 5-hinged rigid bodies. Expressions for the matrices G and F involved in the equation

$$\dot{x} = F x + G(x, t) u$$

are obtained.

Section 5 provides a linearized dynamical model for a 3-hinged rigid-body spacecraft. This result is a special case of that given in Section 4. Expressions for the linearized elements of the matrix A and the vector L appearing in the equation

$$A \dot{\omega} = L$$

are given. In addition, the expressions for the matrices F, G appearing in

$$\dot{x} = F x + G(x, t) u$$

are given.

Section 6 provides a simple linearized model for a single rigid body. A comparison of this result with the model for the n-hinged rigid-body spacecraft is made.

Appendix A provides a discussion of the development of the dynamical equations for a multi-hinged rigid-body spacecraft (in terms of the constraint torques). The development as presented here is based on Ref. [1]. Intermediate results are tabulated to allow the reader to be cognizant of the origin of the various terms that are introduced.

Appendix B provides a discussion of the development of a set of dynamical equations (free from constraint torques) for a multi-hinged rigid-body spacecraft (see Ref. [2]). Intermediate results are tabulated for future reference.

Appendix C provides a detailed description of an application of the general results of Appendix B. The explicit dynamical equations are obtained for a specific topological arrangement of 5-hinged rigid bodies. In Appendix C, the coordinate systems involved in the evaluation of the various terms are discussed. In addition, expressions for the elements of the matrix A and the vector L appearing in the equation

$$A \dot{\omega} = L$$

are tabulated.

2. DYNAMICAL EQUATIONS FOR AN n-HINGED RIGID-BODY SPACECRAFT

In this section, the dynamical equations for an n-hinged rigid-body spacecraft are provided. Emphasis is placed on the procedure used to obtain the results rather than on a detailed and lengthy derivation of the results.

Consider an n-hinged rigid-body spacecraft having r degrees of freedom. The number of scalar constraint torques for such a system is $n_c = 3n - r$. In Ref. [1], Hooker and Margulies showed how to eliminate the constraint torque components so that 3n differential equations for the angular velocity components could be integrated (with the constraints being satisfied). In that technique, the calculation of the right hand sides (RHS) of these differential equations (DE) required solving a system of $3n + n_c = 6n - r$ linear algebraic equations for the 3n angular velocity rates and the n_c constraint torques.*

In the present analysis, a set of r dynamical equations in which the constraint torques <u>do</u> <u>not</u> <u>appear</u> is given. The pivotal steps involved in obtaining this canonical set of equations involves (see Ref. [2]):

(1) Recognizing that if the vector dynamical equations of <u>all</u> the bodies are summed, then the constraint torques cancel in pairs.

(2) Noting that a vector constraint torque at a typical joint j can be isolated by summing the vector dynamical equations over all bodies that lie to one side of joint j (the constraint torques on this set of bodies all cancel in pairs, except for the one at joint j).

(3) Observing that the constraint torque (isolated in step 2) at joint j is orthogonal to the gimbal axis at joint j.

Effectively, 3 scalar equations result from the projection of the vector equations summed over <u>all</u> the bodies on to a suitable reference frame. Moreover, r - 3 additional scalar equations result from the dot products of the r - 3 gimbal axes and the constraint torques associated with these axes. The salient advantage associated with the elimination of the constraint torques is the accompanying reduction of the computer time required for integrating the equations (this is especially true for large n)!

---

*In Ref. [3], Fleischer describes a general computer simulation based on this technique.

Although more than a modicum of labor and more than a soupçon of effort were involved in the verification of the treatment given very succinctly by Hooker in Ref. [2], nevertheless, it is appropriate to keep the present discussion brief. This brief treatment will allow attention to be focused on the underlying assumptions, on the interpretation of the results rather than on the detailed derivations of the multitude of lengthy equations!

The procedure used to arrive at the r scalar equations entails the following steps* (see Table A-1):

(1) Writing Newton's and Euler's equations for each body λ.

(2) Eliminating the unknown interaction force F_{λ_j}.

(3) Evaluating the term

$$\sum_{j \epsilon J_\lambda} C_{\lambda_j} \times F_{\lambda_j}$$

which represents the sum of the moments about the center of mass of body λ due to interaction forces F_{λ_j} existing at joints j.

(4) Interpreting Euler's equations for body λ (after using the results of step (3)) as the equations for the augmented body λ relative to its barycenter B_λ.

(5) Expressing the interaction moment M_{λ_j} acting at joint j on body λ as a sum of a constraint torque $M^C_{\lambda_j}$ and a spring-damper torque $M^{SD}_{\lambda_j}$, i.e.,

$$M_{\lambda_j} = M^C_{\lambda_j} + M^{SD}_{\lambda_j}$$

---

*The terms and symbols are defined in Table A-1 as they are needed in the development.

(6) Recognizing that if the vector dynamical equations for the augmented bodies λ are summed over all λ, then the constraint torques cancel in pairs and consequently disappear, i.e.,

$$\sum_{\lambda \epsilon S} \sum_{j \epsilon J_\lambda} M^C_{\lambda_j} = 0$$

(7) Recognizing that the constraint torque at joint j acting on body λ can be isolated by summing over all bodies to one side of joint j.

(8) Recognizing that the gimbal axis g_j is orthogonal to the constraint torque $M^C_{\lambda_j}$ at joint j.

Steps (1) thru (5) are discussed in turn in Appendix A; steps (6) thru (8) are discussed in Appendix B.

2.1 Compact Form for Dynamical Equations for an n-Hinged Rigid-Body Spacecraft

In this section, the set of dynamical equations derived in Appendix B for an n-hinged rigid-body spacecraft are presented. In vector-matrix notation, the equations are:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{pmatrix} \dot{\omega}_0 \\ \hline \dot{\omega}_R \end{pmatrix} = \begin{pmatrix} L_0 \\ \hline L_R \end{pmatrix} \tag{2-1}$$

or

$$A \dot{\omega} = L$$

In scalar form the equations are (see Table 2-1 for definitions of terms):

$$a_{00} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{0k} \ddot{\gamma}_k = L_0 = \sum_{\lambda=0}^{n-1} E_\lambda \tag{2-2}$$

$$a_{i0} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{ik} \ddot{\gamma}_k = L_i = g_i \cdot \sum_{\lambda=0}^{n-1} \epsilon_{i\lambda} E^*_\lambda , \quad i = 1, 2, \ldots r-3$$

where

$$a_{00} = \sum_{\lambda=0}^{n-1} \sum_{\mu=0}^{n-1} \Phi_{\lambda\mu}$$

$$a_{0k} = \sum_{\lambda} \sum_{\mu} \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot g_k$$

$$a_{i0} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu}$$

$$a_{ik} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu} \cdot g_k$$

$$E_\lambda = M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C^\lambda_\mu \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu\lambda}) \right]$$

$$- \omega_\lambda \times \Phi_{\lambda\lambda} \cdot \omega_\lambda + \sum_{j \in J_\lambda} \tau_{\lambda_j}$$

$$E^*_\lambda = E_\lambda - \sum_{\mu} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} \cdot \dot{g}_k \dot{\gamma}_k$$

A comparison of Eqs. (2-1) and (2-2) reveals that:

(1) A_{11} is the 3 x 3 matrix representation of the operator $a_{00} \cdot$ (where a_{00} is a dyadic, and $\cdot$ represents the dot product).

(2) A_{12} is the 3 x r - 3 matrix representation of the vectors a_{0k}.

(3) A_{22} is the r - 3 x r - 3 matrix representation of the scalars a_{ik} (with i, k = 1, 2, ... r - 3)

Moreover, in Eqs. (2-1) and (2-2), $\dot{\omega}_0$ represents the angular acceleration of the base body, $\dot{\omega}_R$ represents the relative angular accelerations ($\ddot{\gamma}_k$) of the remaining n - 1 bodies, L_0 is a 3 x 1 matrix; and L_R is a r - 3 x 1 matrix.

Table 2-1. Definitions of Terms

Item	Definitions and Equations
n	The number of rigid bodies involved in spacecraft model.
r	The number of degrees-of-freedom of the system.
$D_{\lambda j}$	The vector from the barycenter of body λ to the joint j of body λ.
$D_{\lambda\mu}$	$D_{\lambda\mu} = D_{\lambda j}$ for all bodies μ belonging to $S_{\lambda j}$ (the set of bodies connected to body λ via joint j).
D_{λ}	Vector from barycenter of body λ to c.m. of body λ.
$\Box_{\lambda}$	Inertia matrix relative to c.m. of body λ.
m_{λ}	Mass of body λ
$\Phi_{\lambda\lambda}$	Augmented inertia matrix for body λ relative to barycenter B_{λ}; $\Phi_{\lambda\lambda} = \Box_{\lambda} + \left[m_{\lambda}\left(D_{\lambda}^2 U - D_{\lambda} D_{\lambda}\right) + \sum_{\mu \neq \lambda} m_{\mu}\left(D_{\lambda\mu}^2 U - D_{\lambda\mu} D_{\lambda\mu}\right)\right]$ where U is unit dyadic
$\Phi_{\lambda\mu}$	$\Phi_{\lambda\mu} = - m\left[D_{\lambda\mu} \cdot D_{\mu\lambda} U - D_{\mu\lambda} D_{\lambda\mu}\right]$
g_k	Gimbal axes, $k = 1, 2, \ldots r - 3$; g_k is a unit vector
$\epsilon_{k\mu}$	$\epsilon_{k\mu} = 1$ if gimbal axis g_k is between body μ and body o, otherwise $\epsilon_{k\mu} = 0$; $\epsilon_{k\mu}$ specifies bodies μ which sense the rotation $\dot{\gamma}_k g_k$.
F_{λ}	Vector representing external force applied to body λ.
M_{λ}	Vector representing external moment applied to body λ.
C_{μ}^{λ}	Direction cosine matrix transforming coordinates of body μ to coordinates of body λ.
$\tau_{\lambda j} = M_{\lambda_j}^{SD}$	Vector representing spring-damper interaction torque on body λ at joint j.
J_{λ}	Set of labels for joints j belonging to body λ.

3. LINEARIZED SET OF r DYNAMICAL EQUATIONS FOR AN n-HINGED RIGID-BODY SPACECRAFT

In this section, a linearized set of r dynamical equations for an n-hinged rigid-body spacecraft is provided. Linearization* is accomplished by retaining only terms of first order in ω_0, γ_k and their derivatives in the solution (i.e., products of ω_0 and γ_k with $k = 1, 2, \ldots r-3$ and their derivatives are neglected). In addition, it is assumed that γ_k (with $k = 1, 2, \ldots r-3$) and θ_i (with $i = 1, 2, 3$) are small angles -- hence the direction cosine matrices take a particularly simple form.

3.1 Direction Cosine Matrices

Typical direction cosine matrices for the linearized case become (see, e.g., Fig. 4-1)

$$
\begin{aligned}
C_1^2 &= E - \gamma_2 \tilde{g}_2 \\
C_3^4 &= E - \gamma_4 \tilde{g}_4 \\
C_0^1 &= E - \gamma_1 \tilde{g}_1 \\
C_0^2 &= C_1^2 C_0^1 = [E - \gamma_2 \tilde{g}_2][E - \gamma_1 \tilde{g}_1] \cong E - \gamma_1 \tilde{g}_1 - \gamma_2 \tilde{g}_2 \\
C_0^3 &= [E - \gamma_3 \tilde{g}_3] \\
C_0^4 &= C_3^4 C_0^3 = [E - \gamma_4 \tilde{g}_4][E - \gamma_3 \tilde{g}_3] \cong E - \gamma_3 \tilde{g}_3 - \gamma_4 \tilde{g}_4 \\
C_N^0 &= E - \tilde{\theta} = E - \begin{bmatrix} 0 & -\theta_1 & \theta_2 \\ \theta_1 & 0 & -\theta_3 \\ -\theta_2 & \theta_3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \theta_1 & -\theta_2 \\ -\theta_1 & 1 & \theta_3 \\ \theta_2 & -\theta_3 & 1 \end{bmatrix}
\end{aligned}
\qquad (3\text{-}1)
$$

*Products such as $\dot{\omega}_0\gamma$ and $\ddot{\gamma}\gamma$ are neglected in the linearization in this report; such terms can be retained and included in the forcing function L if it is desirable!

where the vector θ consists of ordered rotations θ_3, θ_2, θ_1, E is a 3 x 3 identity matrix, and ~ over a vector represents the matrix representation of the cross-product operation.

3.2 Relationship Between Attitude and Angular Velocity of Base Body 0

The relationship between the attitude rate and the angular velocity of the base body 0 becomes (for the linearized case)

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & \theta_2 \\ 0 & 1 & -\theta_1 \\ 0 & \theta_1 & 1 \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_0 \cong \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{3-2}$$

when the small angle assumption is used and in addition products of θ_i and ω_i are neglected.

3.3 Evaluation of Elements $a_{\ell m}$ for Linear Case

In this section the elements $a_{\ell m}$ are evaluated for the linear case. Recognizing that products of γ_k and $\ddot{\gamma}_k$ and γ_k and $\dot{\omega}_0$ can be frequently neglected (for $k = 1, 2, \ldots r-3$), it is clear that only those portions of $a_{\ell m}$ that are not functions of γ_k are to be retained. Recall from Eqs. (2-1) and (2-2) that the $a_{\ell m}$'s are the multipliers of $\dot{\omega}_0$ and $\dot{\omega}_R$. Effectively, this implies that the direction cosine matrices C_k^0 (with $k = 1, 2, \ldots r-3$) appearing in the expressions for the $a_{\ell m}$'s can be approximated by identity matrices. The matrix A which is composed of the elements $a_{\ell m}$ then becomes a constant.

3.4 Evaluation of Forcing Function L for Linearized Case

In this section, the terms involved in the evaluation of the forcing function L are provided for the linearized case. Recall that L is defined according to (see Eq. (2-1))

$$A\,\dot{\omega} = L \tag{3-3}$$

where $\dot{\omega}$ consists of $\dot{\omega}_0$ and $\dot{\omega}_R$ and L consists of L_0 and L_R.

Recall, too, that L_0 and L_R, the components of L, are given by

$$L_0 = \sum_{\lambda} E_{\lambda}$$

$$L_R = \begin{pmatrix} g_1 \cdot \sum_{\lambda} \epsilon_{1\lambda} E_{\lambda}^{*} \\ g_2 \cdot \sum_{\lambda} \epsilon_{2\lambda} E_{\lambda}^{*} \\ \vdots \\ g_{r-3} \cdot \sum_{\lambda} \epsilon_{r-3,\lambda} E_{\lambda}^{*} \end{pmatrix} \tag{3-4}$$

The linearized versions of E_{λ} and E_{λ}^{*} reduce to

$$E_{\lambda} = M_{\lambda} + D_{\lambda} \times F_{\lambda} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_{\mu}^{\lambda} F_{\mu} + \sum_{j \epsilon J_{\lambda}} \tau_{\lambda j}$$

$$E_{\lambda}^{*} = E_{\lambda} \tag{3-5}$$

and consequently, the linearized versions of L_0 and L_R reduce to

$$L_0 = \sum_{\lambda} \left\{ M_{\lambda} + D_{\lambda} \times F_{\lambda} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_{\mu}^{\lambda} F_{\mu} + \sum_{j \epsilon J_{\lambda}} \tau_{\lambda j} \right\} \tag{3-6}$$

$$L_R = \begin{pmatrix} g_1 \cdot \sum_{\lambda} \epsilon_{1\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \in J_\lambda} \tau_{\lambda j} \right\} \\ g_2 \cdot \sum_{\lambda} \epsilon_{2\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \in J_\lambda} \tau_{\lambda j} \right\} \\ \vdots \\ g_{r-3} \cdot \sum_{\lambda} \epsilon_{r-3\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \in J_\lambda} \tau_{\lambda j} \right\} \end{pmatrix}$$

Note that the term

$$\sum_{\lambda} \sum_{j \in J_\lambda} \tau_{\lambda j}$$

is identically zero in the equation for L_0 (interaction moments cancel in pairs).

3.4.1 Evaluation of L for the Specialized Case in Which External Forces and Moments are Applied Only to Base Body

For the specialized case in which external forces and moments are applied solely to the base body, the equations for L_0 and L_R reduce to

$$L_0 = M_0 + D_0 \times F_0 + \sum_{\lambda=1}^{n-1} D_{\lambda 0} \times C_0^\lambda F_0 \tag{3-7}$$

$$L_R = \begin{pmatrix} g_1 \cdot \sum_{\lambda=1}^{n-1} \epsilon_{1\lambda} D_{\lambda 0} \times C_0^{\lambda} F_0 + g_1 \cdot \sum_{\lambda} \epsilon_{1\lambda} \sum_{j \epsilon J_\lambda} \tau_{\lambda_j} \\ g_2 \cdot \sum_{\lambda=1}^{n-1} \epsilon_{2\lambda} D_{\lambda 0} \times C_0^{\lambda} F_0 + g_2 \cdot \sum_{\lambda} \epsilon_{2\lambda} \sum_{j \epsilon J_\lambda} \tau_{\lambda_j} \\ \vdots \\ g_{r-3} \cdot \sum_{\lambda=1}^{n-1} \epsilon_{r-3,\lambda} D_{\lambda 0} \times C_0^{\lambda} F_0 + g_{r-3} \cdot \sum_{\lambda} \epsilon_{r-3,\lambda} \sum_{j \epsilon J_\lambda} \tau_{\lambda j} \end{pmatrix}$$

3.5 Compact Form the for Linearized Set of r Dynamical Equations

In this section, the linearized set of r dynamical equations for an n-hinged rigid-body spacecraft are expressed in compact form.

First, the term

$$g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \sum_{j \epsilon J_\lambda} \tau_{\lambda_j} \qquad (3\text{-}8)$$

is examined. As pointed out in Appendix B, summing the dynamical equations over bodies λ, which are connected beyond gimbal axis g_i relative to the base body, isolates the interaction moment at joint i on body λ. This implies that

$$g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \sum_{j \epsilon J_\lambda} \tau_{\lambda_j} \equiv g_i \cdot \tau_{ii} = g_i \cdot \left[- K_i \gamma_i - B_i \dot{\gamma}_i \right] g_i \qquad (3\text{-}9)$$

where

$$\tau_{ii} = -K_i \gamma_i g_i - B_i \dot{\gamma}_i g_i$$

In Eq. (3-9), K_i and B_i are the stiffness and damping coefficients associated with joint i. Substitution of Eq. (3-9) into Eq. (3-7) yields

$$L_0 = M_0 + D_0 \times F_0 + \sum_{\lambda=1}^{n-1} D_{\lambda 0} \times C_0^{\lambda} F_0 \tag{3-10}$$

$$L_R = \begin{pmatrix} g_1 \cdot \sum_{\lambda} \epsilon_{1\lambda} D_{\lambda 0} \times C_0^{\lambda} F_0 - (K_1 \gamma_1 + B_1 \dot{\gamma}_1) \\ g_2 \cdot \sum_{\lambda} \epsilon_{2\lambda} D_{\lambda 0} \times C_0^{\lambda} F_0 - (K_2 \gamma_2 + B_2 \dot{\gamma}_2) \\ \vdots \\ g_{r-3} \cdot \sum_{\lambda} \epsilon_{r-3,\lambda} D_{\lambda 0} \times C_0^{\lambda} F_0 - (K_{r-3} \gamma_{r-3} + B_{r-3} \dot{\gamma}_{r-3}) \end{pmatrix}$$

Equation (3-10) can be written as

$$L_0 = \left[\tilde{\ell}_0 + \tilde{D}_0 + \sum_{\lambda} \tilde{D}_{\lambda 0} \; C_0^{\lambda} \right] F_0 = G_0(\gamma, t) \, F_0 \tag{3-11}$$

$$L_R = G_R(\gamma, t) \, F_0 - K \gamma - B \dot{\gamma} = G_R(\gamma, t) \, F_0 - K \gamma - B \dot{\gamma}$$

where

$G_0(\gamma, t)$ is a 3 x 3 matrix

$G_R(\gamma, t)$ is a r-3 x 3 matrix

K, B are r - 3 x r - 3 diagonal matrices

~ over a term represents the skew symmetric matrix representation of the cross product (e.g., $D_{\lambda 0} \times = \tilde{D}_{\lambda 0}$)

γ, $\dot{\gamma}$ are r - 3 x 1 matrices consisting of elements γ_k, $\dot{\gamma}_k$ for k = 1, 2, ... r - 3

Equation (3-11) can thus be written as

$$L = \begin{pmatrix} L_0 \\ \hline L_R \end{pmatrix} = \begin{bmatrix} G_0 \\ \hline G_R \end{bmatrix} F_0 + \begin{bmatrix} 0 \\ \hline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hline -B \end{bmatrix} \dot{\gamma} \tag{3-12}$$

or

$$L = G\, F_0 + \begin{bmatrix} 0 \\ \hline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hline -B \end{bmatrix} \dot{\gamma}$$

where 0 is an appropriately dimensioned null matrix.

Collecting the results of Sections 3-3, 3-4 and the results of Eq. (3-12), it follows that a set of r linearized dynamical equations for an n-hinged rigid-body spacecraft is given by

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{pmatrix} \dot{\omega}_0 \\ \hline \dot{\omega}_R \end{pmatrix} = \begin{bmatrix} G_0 \\ \hline G_R \end{bmatrix} F_0 + \begin{bmatrix} 0 \\ \hline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hline -B \end{bmatrix} \dot{\gamma}$$

or

$$A\, \dot{\omega} = G(\gamma, t)\, F_0 + \begin{bmatrix} 0 \\ \hline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hline -B \end{bmatrix} \dot{\gamma} \tag{3-13}$$

where A, K, B are <u>constant</u> matrices and G depends on γ and t.

3.6 Linear Dynamical Model for n-Hinged Rigid-Body Spacecraft (State Equations)

In this section, the linear model developed in Section 3.5 is cast in a form suitable for use in modern control theory. Essentially, the state equations are sought. As seen in Section 3.2,

$$\dot{\theta} \cong \omega_0$$

and

$$\dot{\gamma} = \omega_R$$

where the vectors θ and ω_0 are the attitude and angular velocity of the base body relative to an inertial frame and the vectors γ and ω_R represent the relative attitude and angular velocities γ_k, $\dot{\gamma}_k$ for k = 1, 2, ... r - 3. The state can thus be defined as the 2r x 1 vector

$$x = \begin{pmatrix} \theta \\ \omega_0 \\ \gamma \\ \omega_R \end{pmatrix}$$

The differential equations for θ and γ are given above and those for ω_0 and ω_R can be obtained from Eq. (3-13).

Manipulation of Eq. (3-13) yields

$$A_{11}\,\dot{\omega}_0 + A_{12}\,\dot{\omega}_R = G_0(\gamma, t)\,F_0 \qquad (3\text{-}14)$$

$$A_{21}\,\dot{\omega}_0 + A_{22}\,\dot{\omega}_R = G_R(\gamma, t)\,F_0 - K\gamma - B\dot{\gamma}$$

As discussed in Appendix C, Eq. (3-14) can be written as

$$\dot{\omega}_0 = \left[A_{11} - A_{12} A_{22}^{-1} A_{21}\right]^{-1} G_0(\gamma, t) F_0 \tag{3-15}$$

$$- \left[A_{11} - A_{12} A_{22}^{-1} A_{21}\right]^{-1} A_{12} A_{22}^{-1} \left(G_R(\gamma, t) F_0 - K\gamma - B\dot{\gamma}\right)$$

$$\dot{\omega}_R = A_{22}^{-1} \left\{G_R(\gamma, t) F_0 - K\gamma - B\dot{\gamma} - A_{21} \dot{\omega}_0\right\}$$

Redefining the bracketed matrix as $\boldsymbol{a}$, it follows that

$$\dot{\omega}_0 = \boldsymbol{a}^{-1} G_0(\gamma, t) F_0 - \boldsymbol{a}^{-1} A_{12} A_{22}^{-1} \left\{G_R(\gamma, t) F_0 - K\gamma - B\dot{\gamma}\right\}$$

$$\dot{\omega}_R = A_{22}^{-1} \left\{G_R(\gamma, t) F_0 - K\gamma - B\dot{\gamma}\right\} - A_{21} \omega_0 \tag{3-16}$$

In vector-matrix rotation, the state equations become

$$\begin{pmatrix} \dot{\theta} \\ \hline \dot{\omega}_0 \\ \hline \dot{\gamma} \\ \hline \dot{\omega}_R \end{pmatrix} = \left[\begin{array}{c|c|c|c} 0 & E & 0 & 0 \\ \hline 0 & 0 & \boldsymbol{a}^{-1} A_{12} A_{22}^{-1} K & \boldsymbol{a}^{-1} A_{12} A_{22}^{-1} B \\ \hline 0 & 0 & 0 & E \\ \hline 0 & 0 & -A_{22}^{-1}\left(E + A_{21}\boldsymbol{a}^{-1} A_{12} A_{22}^{-1}\right) K & -A_{22}^{-1}\left(E + A_{21}\boldsymbol{a}^{-1} A_{12} A_{22}^{-1}\right) B \end{array}\right] \begin{pmatrix} \theta \\ \hline \omega_0 \\ \hline \gamma \\ \hline \omega_R \end{pmatrix}$$

$$+ \left[\begin{array}{c} 0 \\ \hline \boldsymbol{a}^{-1} G_0 - \boldsymbol{a}^{-1} A_{12} A_{22}^{-1} G_R \\ \hline 0 \\ \hline A_{22}^{-1}\left(E + A_{21}\boldsymbol{a}^{-1} A_{12} A_{22}^{-1}\right) G_R - A_{22}^{-1} A_{21} \boldsymbol{a}^{-1} G_0 \end{array}\right] F_0 \tag{3-17}$$

where E is a 3 x 3 identity matrix and 0 is an appropriately dimensioned null matrix. Note that Eq. (3-17) has the same form as

$$\dot{x} = F x + G(x, t) u \tag{3-18}$$

where x is the state and u is the control variables. It is important to note that F is a constant and G depends on x and t! Equation (3-17) represents the primary result of this report.

It is immediately recognized that the solution to Eq. (3-18) can be written as

$$x(t) = \phi(t, 0) x(0) + \int_0^t \phi(t) \phi^{-1}(\tau) G(x, \tau) u(\tau) d\tau \tag{3-19}$$

where x(0) is the initial state and $\phi(t, \tau)$ is the transition matrix. The transition matrix can be computed from the matrix differential equation

$$\dot{\phi} = F \phi \tag{3-20}$$

with

$$\phi(0) = I$$

or it can be computed analytically if n is small.

4. LINEARIZED DYNAMICAL MODEL FOR A 5-HINGED RIGID-BODY SPACECRAFT

In this section, the general linearized dynamical model developed in Section 3 is used to obtain expressions for the elements of the constant matrix A and the vector L for the case in which n = 5 (see Fig. 4-1). The expressions for the elements of the matrix A are summarized in Table 4-1 and those for the elements of the vector L are summarized in Table 4-2. Note that in Table 4-1 the subscripts appearing to the right of a term in parentheses refer to the coordinate systems in which the terms are computed or expressed.

The 7 x 7 matrix A is given by

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ & a_{11} & a_{12} & a_{13} & a_{14} \\ & & a_{22} & a_{23} & a_{24} \\ \text{symmetric} & & & a_{33} & a_{34} \\ & & & & a_{44} \end{bmatrix} \quad ; \qquad (4\text{-}1$$

The partitioned matrix A (involving the 3 x 3 matrix A_{11}, the 3 x 4 matrix A_{12}, the 4 x 3 matrix A_{21}, and the 4 x 4 matrix A_{22}) is given by

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|cccc} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ \hline a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad ; \qquad (4\text{-}2)$$

The 3 x 3 time-varying matrix G_0 is given by

$$G_0 = \tilde{\ell}_0 + \tilde{D}_0 + \left(C_1^0 D_{11} + C_2^0 D_{22} + C_3^0 D_{33} + C_4^0 D_{44}\right)^{\sim} ; \qquad (4\text{-}3)$$

The 4 x 3 time-varying matrix G_R is given by

$$G_R = \begin{bmatrix} g_1^T \left(D_{11} + C_2^1 D_{22}\right)^{\sim} C_0^1 \\ g_2^T \tilde{D}_{22} C_0^2 \\ g_3^T \left(D_{33} + C_4^3 D_{44}\right)^{\sim} C_0^3 \\ g_4^T \tilde{D}_{44} C_0^4 \end{bmatrix} ; \qquad (4\text{-}4)$$

The constant diagonal matrices K and B are given by

$$K = \begin{bmatrix} K_1 & & & \\ & K_2 & & \\ & & K_3 & \\ & & & K_4 \end{bmatrix} ; B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & B_3 & \\ & & & B_4 \end{bmatrix} ; \qquad (4\text{-}5)$$

and the constant 3 x 3 matrix $\boldsymbol{a}$ is given by

$$\boldsymbol{a} = \left[A_{11} - A_{12} A_{22}^{-1} A_{21}\right] \qquad (4\text{-}6)$$

Hence, all matrices involved in the generic form [see Eqs. (3-17) and (3-18)]

$$\dot{x} = Fx + G(x,t)\,u$$

have been defined! Note that the symbol ~ appearing as

$$(\ )^{\sim}$$

indicates that the tilde is to be applied to the resultant expression within the parentheses.

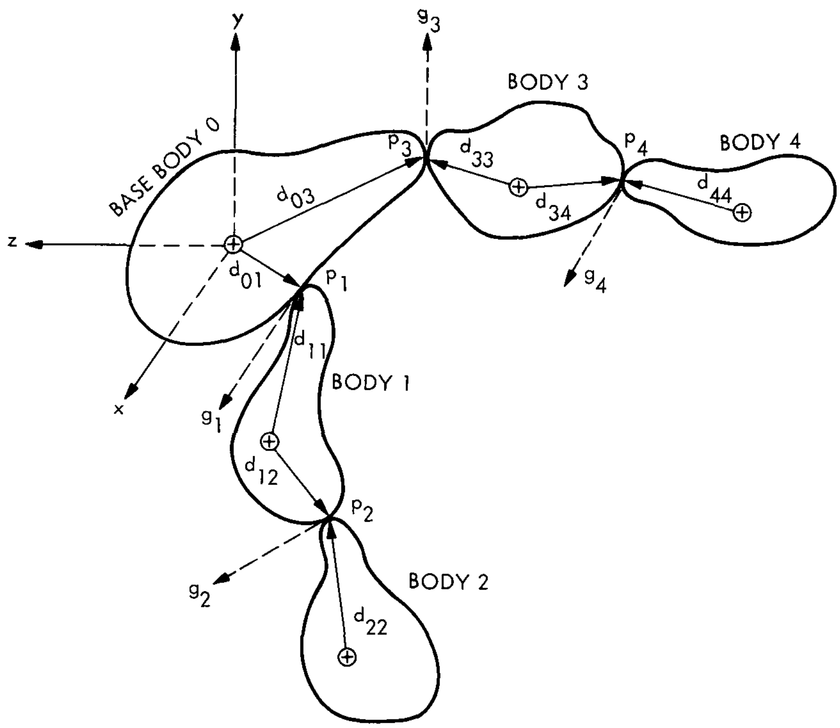

Figure 4-1. Pictorial Sketch of a 5-Hinged Rigid-Body Spacecraft

Table 4-1. Expressions for Linearized Elements $a_{\ell m}$ ($\ell, m = 0, 1, \ldots 4$) for a 5-Hinged Rigid-Body Spacecraft

Item	Equations
a_{00}	$[\Phi_{00} + \Phi_{01} + \Phi_{02} + \Phi_{03} + \Phi_{04}]_0$ $+ [\Phi_{10} + \Phi_{11} + \Phi_{12} + \Phi_{13} + \Phi_{14}]_1$ $+ [\Phi_{20} + \Phi_{21} + \Phi_{22} + \Phi_{23} + \Phi_{24}]_2$ $+ [\Phi_{30} + \Phi_{31} + \Phi_{32} + \Phi_{33} + \Phi_{34}]_3$ $+ [\Phi_{40} + \Phi_{41} + \Phi_{42} + \Phi_{43} + \Phi_{44}]_4$
a_{01}	$\Big[[\Phi_{01} + \Phi_{02}]_0 + [\Phi_{11} + \Phi_{12}]_1 + [\Phi_{21} + \Phi_{22}]_2$ $+ [\Phi_{31} + \Phi_{32}]_3 + [\Phi_{41} + \Phi_{42}]_4\Big] \cdot (g_1)_1$
a_{02}	$\Big\{[\Phi_{02}]_0 + [\Phi_{12}]_1 + [\Phi_{22}]_2 + [\Phi_{32}]_3 + [\Phi_{42}]_4\Big\} \cdot (g_2)_2$
a_{03}	$\Big\{[\Phi_{03} + \Phi_{04}]_0 + [\Phi_{13} + \Phi_{14}]_1 + [\Phi_{23} + \Phi_{24}]_2$ $+ [\Phi_{33} + \Phi_{34}]_3 + [\Phi_{43} + \Phi_{44}]_4\Big\} \cdot (g_3)_3$
a_{04}	$\Big\{[\Phi_{04}]_0 + [\Phi_{14}]_1 + [\Phi_{24}]_2 + [\Phi_{34}]_3 + [\Phi_{44}]_4\Big\} \cdot (g_4)_4$
a_{11}	$(g_1)_1 \cdot \Big\{[\Phi_{11} + \Phi_{12}]_1 + [\Phi_{21} + \Phi_{22}]_2\Big\} \cdot (g_1)_1$
a_{12}	$(g_1)_1 \cdot \Big\{[\Phi_{12}]_1 + [\Phi_{22}]_2\Big\} \cdot (g_2)_2$
a_{13}	$(g_1)_1 \cdot \Big\{[\Phi_{13} + \Phi_{14}]_1 + [\Phi_{23} + \Phi_{24}]_2\Big\} \cdot (g_3)_3$

Table 4-1. Expressions for Linearized Elements $a_{\ell m}$ (ℓ, m = 0, 1, ... 4) for a 5-Hinged Rigid-Body Spacecraft (contd)

Item	Equations
a_{14}	$(g_1)_1 \cdot \left\{ [\Phi_{14}]_1 + [\Phi_{24}]_2 \right\} \cdot (g_4)_4$
a_{22}	$(g_2)_2 \cdot [\Phi_{22}]_2 \cdot (g_2)_2$
a_{23}	$(g_2)_2 \cdot [\Phi_{23} + \Phi_{24}]_2 \cdot (g_3)_3$
a_{24}	$(g_2)_2 \cdot [\Phi_{24}]_2 \cdot (g_4)_4$
a_{33}	$(g_3)_3 \cdot \left\{ [\Phi_{33} + \Phi_{34}]_3 + [\Phi_{43} + \Phi_{44}]_4 \right\} \cdot (g_3)_3$
a_{34}	$(g_3)_3 \cdot \left\{ [\Phi_{34}]_3 + [\Phi_{44}]_4 \right\} \cdot (g_4)_4$
a_{44}	$(g_4)_4 \cdot [\Phi_{44}]_4 \cdot (g_4)_4$
a_{k0}	$(a_{0k})^T$, k = 1, 2, ... 4
$a_{\ell m}$	$(a_{m\ell})^T$, i, k = 1, 2, ... 4, i $\neq$ k

Table 4-2. Linearized Forcing Function L for a 5-Hinged Rigid-Body Spacecraft (External Forces and Moments Applied Only to Base Body)

Item	Equations
L_0	$M_0 + D_0 \times F_0$ $+ \left(C_1^0 D_{11} + C_2^0 D_{22} + C_3^0 D_{33} + C_4^0 D_{44}\right) \times F_0$
L_1	$g_1 \cdot \left(D_{11} + C_2^1 D_{22}\right) \times C_0^1 F_0 - K_1 \gamma_1 - B_1 \dot{\gamma}_1$
L_2	$g_2 \cdot \left(D_{22} \times C_0^2 F_0\right) - K_2 \gamma_2 - B_2 \dot{\gamma}_2$
L_3	$g_3 \cdot \left(D_{33} + C_4^3 D_{44}\right) \times C_0^3 F_0 - K_3 \gamma_3 - B_3 \dot{\gamma}_3$
L_4	$g_4 \cdot \left(D_{44} \times C_0^4 F_0\right) - K_4 \gamma_4 - B_4 \dot{\gamma}_4$

5. LINEARIZED DYNAMICAL MODEL FOR A 3-HINGED RIGID-BODY SPACECRAFT

In this section, the linearized dynamical model for a 3-hinged rigid-body spacecraft is given.† This result is a special case of that given in Section 4 and is obtained by defining the new base body to include bodies 0, 3, 4 of Fig. 4-1. Figure 5-1 shows a pictorial sketch of the resulting 3-body system.

The expressions for the elements of the 5 x 5 constant matrix A and the vector L are given in Tables 5-1 and 5-2, respectively.

The 5 x 5 constant matrix A is given by

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \tag{5-1}$$

The partitioned matrix A (involving the 3 x 3 matrix A_{11}, the 3 x 2 matrix A_{12}, the 2 x 3 matrix A_{21} and the 2 x 2 matrix A_{22}) is given by

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|cc} a_{00} & a_{01} & a_{02} \\ \hline a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{array}\right] \tag{5-2}$$

The 3 x 3 time-varying matrix G_0 is given by

$$G_0 = \tilde{\mathcal{I}}_0 + \tilde{D}_0 + \left(C_1^0 D_{11} + C_2^0 D_{22}\right)^{\sim} \quad , \tag{5-3}$$

†This model was used to obtain the results given in Ref (6).

The 2 x 3 time-varying matrix (G_R is given by

$$G_R = \begin{bmatrix} g_1^T \left(D_{11} + C_2^1 D_{22}\right)^{\sim} C_0^1 \\ g_2^T \tilde{D}_{22} C_0^2 \end{bmatrix} \tag{5-4}$$

The constant 2 x 2 diagonal matrices K and B are given by

$$K = \begin{bmatrix} K_1 & \\ & K_2 \end{bmatrix}$$
$$B = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} , \tag{5-5}$$

and the constant 3 x 3 matrix $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha} = \left[A_{11} - A_{12} A_{22}^{-1} A_{21}\right] \tag{5-6}$$

Hence, all the terms needed in the generic form of the state equations given in Eq. (3-17) and (3-18) have been specified.

Table 5-1. Expressions for Linearized Elements $a_{\ell m}$ (ℓ, m = 0, 1, 2) for a 3-Hinged Rigid-Body Spacecraft

Item	Equations
a_{00}	$[\Phi_{00} + \Phi_{01} + \Phi_{02}]_0 + [\Phi_{10} + \Phi_{11} + \Phi_{12}]_1 + [\Phi_{20} + \Phi_{21} + \Phi_{22}]_2$
a_{01}	$\left\{ [\Phi_{01} + \Phi_{02}]_0 + [\Phi_{11} + \Phi_{12}]_1 + [\Phi_{21} + \Phi_{22}]_2 \right\} \cdot (g_1)_1$
a_{02}	$\left\{ [\Phi_{02}]_0 + [\Phi_{12}]_1 + [\Phi_{22}]_2 \right\} \cdot (g_2)_2$
a_{11}	$(g_1)_1 \cdot \left\{ [\Phi_{11} + \Phi_{12}]_1 + [\Phi_{21} + \Phi_{22}]_2 \right\} \cdot (g_1)_1$
a_{12}	$(g_1)_1 \cdot \left\{ [\Phi_{12}]_1 + [\Phi_{22}]_2 \right\} \cdot (g_2)_2$
a_{22}	$(g_2)_2 \cdot \left\{ [\Phi_{22}]_2 \right\} \cdot (g_2)_2$
a_{10}	$(a_{01})^T$
a_{20}	$(a_{02})^T$
a_{21}	$(a_{12})^T$

Table 5-2. Linearized Forcing Function L for a 3-Hinged Rigid-Body Spacecraft (External Forces and Moments Applied Only to Base Body)

Item	Equations
L_0	$M_0 + D_0 \times F_0 + \left(C_1^0 D_{11} + C_2^0 D_{22}\right) \times F_0$
L_1	$g_1 \cdot \left(D_{11} + C_2^1 D_{22}\right) \times C_0^1 F_0 - K_1 \gamma_1 - B_1 \gamma_1$
L_2	$g_2 \cdot \left(D_{22} \times C_0^2 F_0\right) - K_2 \gamma_2 - B_2 \gamma_2$

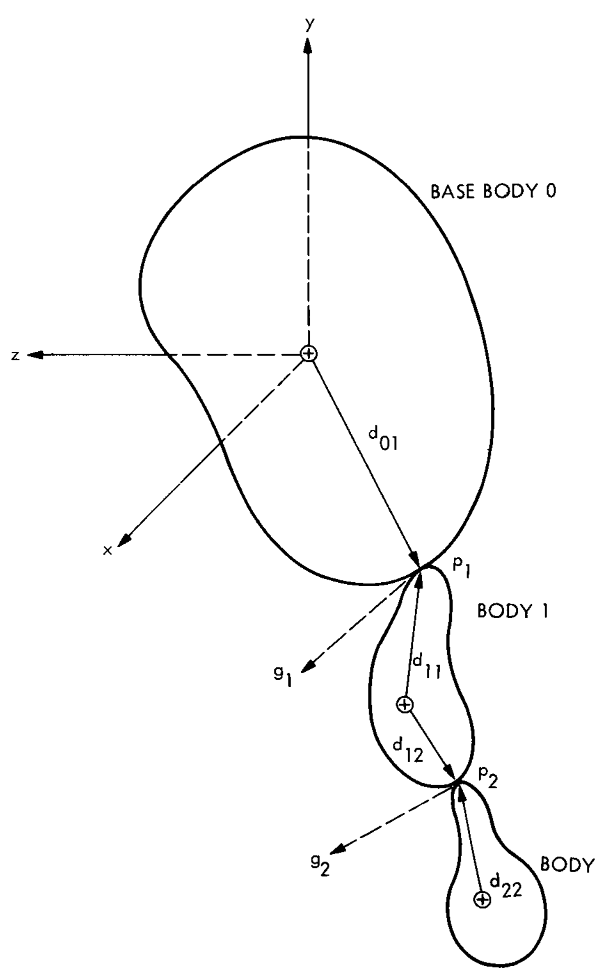

Figure 5-1. Pictorial Sketch of a 3-Hinged Rigid-Body Spacecraft

6. LINEARIZED SINGLE RIGID-BODY SPACECRAFT MODEL

In this section, the linearized model for a single rigid-body model of a S/C is given. In this case, the generalized results for a n-hinged rigid body are not directly applicable. Instead, the linearized version of Euler's equations for a rigid body are used. That is, the state equations are

$$\begin{pmatrix} \dot{\theta} \\ \hline \dot{\omega}_0 \end{pmatrix} = \left[\begin{array}{c|c} 0 & E \\ \hline 0 & 0 \end{array}\right] \begin{pmatrix} \theta \\ \hline \omega_0 \end{pmatrix} + \left[\begin{array}{c} 0 \\ \hline I^{-1} \end{array}\right] M_0 \tag{6-1}$$

where

E is a 3 x 3 identity matrix

M_0 is the external moment applied to the S/C

I is the 3 x 3 inertia matrix of the S/C

0 is a 3 x 3 null matrix

It is interesting to note that the linearized form for an n-hinged rigid-body spacecraft can still be used for a single rigid body model of the S/C. This is achieved by eliminating the vectors γ and ω_R from the state and appropriately interpreting the results. Comparing Eqs. (3-17) and (6-1), it follows that for the single rigid body model

$\mathbf{a} \equiv I$ (the inertia matrix)

$G_0 = \tilde{\ell}_0$

and γ, ω_R, G_R, K, B do not appear (they are deleted from the general result).

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APPENDIX A

DYNAMICAL EQUATIONS (IN TERMS OF CONSTRAINT TORQUES) FOR A SPACECRAFT IDEALIZED AS n-HINGED RIGID BODIES

The primary result of this appendix is a compact version of the dynamical equations for a spacecraft idealized as n-hinged rigid bodies; the equations are written in terms of the constraint torques. The pivotal steps involved in the derivation are briefly described and the intermediate results are recorded* (see Table A-1).

The steps involved in obtaining the desired result include:

(1) Writing Newton's and Euler's equations for each body λ.

(2) Eliminating the unknown interaction force F_{λ_j}

(3) Evaluating the term

$$\sum_{j \epsilon J_\lambda} C_{\lambda_j} \times F_{\lambda_j}$$

(The sum of the moments about the center of mass of body λ due to interaction forces F_{λ_j} existing at joints j).

(4) Interpreting the results as the equations describing the motion of the augmented body λ relative to its barycenter.

A-1 Newton's and Euler's Equations

The development of the equations of motion for an n-hinged rigid-body spacecraft begins with Newton's and Euler's equations written for each body λ. That is, for all $\lambda \epsilon S$:

*The intermediate results are important since they indicate the origin of various terms that appear in the final result.

$$F_\lambda + \sum_{j \epsilon J_\lambda} F_{\lambda_j} = m_\lambda (\ddot{R} + \ddot{\rho}_\lambda) = m_\lambda \ddot{R}_{c_\lambda}$$

$$\square_\lambda \cdot \dot{\omega}_\lambda + \omega_\lambda \times \square_\lambda \cdot \omega_\lambda = M_\lambda + \sum_{j \epsilon J_\lambda} M_{\lambda_j} + \sum_{j \epsilon J_\lambda} C_{\lambda_j} \times F_{\lambda_j} \tag{A-1}$$

where

$\square_\lambda$ is the inertia dyadic for body λ relative to its center of mass

m_λ is the mass of body λ

F_λ is the external force applied to body λ

ω_λ, $\dot{\omega}_\lambda$ are the angular velocity and angular acceleration of body λ relative to an inertial frame

F_{λ_j} is the interaction force acting on body λ at joint j

R is the position vector from the origin of an inertial reference frame to the point "0" ("0" is the origin of the S/C coordinate system in the undeformed state)

ρ_λ is the position vector from the point "0" to the center of mass of body λ

R_{c_λ} is the position vector from the origin of the inertial reference frame to the center of mass of body λ

R_c is the position vector from the origin of the inertial reference frame to the composite mass center of the system

M_λ is the external moment applied to body λ relative to its center of mass

M_{λ_j} is the interaction moment acting on body λ at joint j

$c_{\lambda_j} = d_{\lambda_j}$ is the position vector from the center of mass of body λ to joint j

S is the set of all labels for bodies λ $\left(\text{that is, } S = \{0, 1, \ldots n-1\}\right)$

J_λ is the set of labels for all joints associated with body λ

A-2 Elimination of the Interaction Forces

In order to obtain a solution for ω_λ from Eq. (A-1) the interaction forces F_{λ_j} are first eliminated. This is accomplished by recognizing that

$$\sum_{\mu \in S_{\lambda_j}} \sum_{j \in J_\lambda} F_{\lambda_j} = F_{\lambda_j} \tag{A-2}$$

where S_{λ_j} is the set of those bodies connected to body λ at joint j.

That is, by summing the interaction forces acting on body λ over all bodies μ belonging to the chain of bodies connected to λ at joint j, the interaction force F_{λ_j} can be isolated. Using Eq. (A-1), it follows that

$$F_{\lambda_j} = \sum_{\mu \in S_{\lambda_j}} (F_\mu - m_\mu \ddot{R}_{c_\mu}) \tag{A-3}$$

The form of F_{λ_j} given by Eq. (A-3) is next substituted into Eq. (A-1) to eliminate F_{λ_j} from the equations, that is

$$\mathbb{I}_\lambda \cdot \dot{\omega}_\lambda + \omega_\lambda \times \mathbb{I}_\lambda \cdot \omega_\lambda = M_\lambda + \sum_{j \in J_\lambda} M_{\lambda_j} + \sum_{j \in J_\lambda} C_{\lambda_j} \times \sum_{\mu \in S_{\lambda_j}} (F_\mu - m_\mu \ddot{R}_{c_\mu}) \tag{A-4}$$

A-3 Evaluation of the term $\sum_{j \in J_\lambda} C_{\lambda_j} \times F_{\lambda_j}$

In this section, attention is focused on the term

$$\sum_{j \in J_\lambda} C_{\lambda_j} \times F_{\lambda_j} = \sum_{j \in J_\lambda} C_{\lambda_j} \times \sum (F_\mu - m_\mu \ddot{R}_{c_\mu})$$

It can be easily shown that

$$\sum_{j \in J_\lambda} C_{\lambda_j} \times F_{\lambda_j} = D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_\mu - \sum_{\mu \neq \lambda} m_\mu C_{\lambda\mu} \times (\ddot{R}_{c_\mu} - \ddot{R}_c) \tag{A-5}$$

where

$$D_\lambda \equiv - \sum_{\mu \neq \lambda} m_\mu m^{-1} C_{\lambda\mu}$$

$$D_{\lambda\mu} = D_\lambda + C_{\lambda\mu}$$

If the point masses m_{λ_j} (the mass of all bodies attached to λ via joint j) located at joints j are augmented to the mass of body λ, the augmented body λ results. The barycenter B_λ is defined as the new center of mass of the augmented body λ. Physically, D_λ is the vector from the barycenter B_λ of the augmented body λ to the original center of mass of body λ, D_{λ_j} is the vector from B_λ to joint j. Note that

$$D_{\lambda_j} = D_{\lambda\mu} \quad \text{for all } \mu \in S_{\lambda_j}$$

$$C_{\lambda_j} = C_{\lambda\mu}$$

Next the term

$$- \sum_{\mu \neq \lambda} m_\mu C_{\lambda\mu} \times (\ddot{R}_{c_\mu} - \ddot{R}_c)$$

is examined. Using the facts that

$$R_{c_\mu} - R_c = D_\mu + \sum_{\nu \neq \mu} D_{\nu\mu}$$

$$D_{\mu} = D_{\mu\nu} - C_{\mu\nu}$$

$$m_{\lambda} D_{\lambda} + \sum_{\mu \neq \lambda} m_{\mu} D_{\lambda\mu} = 0$$

it can be shown that

$$- \sum_{\mu \neq \lambda} m_{\mu} C_{\lambda\mu} \times (\ddot{R}_{c_{\mu}} - \ddot{R}_{c}) = - X_{\lambda} \cdot \dot{\omega}_{\lambda} - \omega_{\lambda} \times X_{\lambda} \cdot \omega_{\lambda} \qquad \text{(A-6)}$$

$$+ m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda}$$

where

$$X \equiv m_{\lambda} D_{\lambda}^{2} U - m_{\lambda} D_{\lambda} D_{\lambda} + \sum_{\mu \neq \lambda} m_{\mu} \left(D_{\lambda\mu}^{2} U - D_{\lambda\mu} D_{\lambda\mu} \right)$$

U is an identity dyadic

$D_{\lambda} D_{\lambda}$ and $D_{\lambda\mu} D_{\lambda\mu}$ are dyadics.

Note that X_{λ} physically represents the inertia dyadic which must be added to $\mathbb{I}_{\lambda}$ to yield the inertia dyadic of the augmented body λ relative to its barycenter B_{λ}.

The term

$$m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda}$$

can be expanded to yield

$$m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda} = m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[\dot{\omega}_{\mu} \times D_{\mu\lambda} + \omega_{\mu} \times (\omega_{\mu} \times D_{\mu\lambda})\right] \quad \text{(A-7)}$$

since $D_{\mu\lambda}$ is assumed fixed in body μ (the rigid body assumption). Collecting the results of Eqs. (A-5) thru (A-7), it follows that

$$\sum_{j \in J_{\lambda}} C_{\lambda_j} \times F_{\lambda_j} = D_{\lambda} \times F_{\lambda} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_{\mu} - X_{\lambda} \cdot \dot{\omega}_{\lambda} - \omega_{\lambda} \times X_{\lambda} \cdot \omega_{\lambda}$$

$$+ m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[\dot{\omega}_{\mu} \times D_{\mu\lambda} + \omega_{\mu} \times (\omega_{\mu} \times D_{\mu\lambda})\right] \quad \text{(A-8)}$$

If Eq. (A-8) is substituted in Eq. (A-1), an elegant result is obtained -- the equations governing the behavior of the augmented body λ realtive to its barycenter B_{λ}!

A-4 Compact Form for the Rotational Equations Characterizing Body λ Written Relative to its Barycenter

Use of the results of Eqs (A-1) and (A-8) allows an interesting interpretation of the equations of motion for body λ to be made. Substitition of Eq. (A-8) into Eq. (A-1) yields

$$\boxed{}_{\lambda} \cdot \dot{\omega}_{\lambda} + \omega_{\lambda} \times \boxed{}_{\lambda} \cdot \omega_{\lambda} = M_{\lambda} + \sum_{j \in J_{\lambda}} M_{\lambda_j}$$

$$+ D_{\lambda} \times F_{\lambda} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_{\mu} - X_{\lambda} \cdot \dot{\omega}_{\lambda}$$

$$- \omega_{\lambda} \times X_{\lambda} \cdot \omega_{\lambda} + m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times (\dot{\omega}_{\mu} \times D_{\mu\lambda})$$

$$+ m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[\omega_{\mu} \times (\omega_{\mu} \times D_{\mu\lambda})\right] \quad \text{(A-9)}$$

Manipulation of Eq. (A-9) yields

$$\Phi_{\lambda\lambda} \cdot \dot{\omega}_\lambda + \omega_\lambda \times \Phi_{\lambda\lambda} \cdot \omega_\lambda = M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_\mu$$

$$+ m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[\dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$$

$$+ \sum_{j \epsilon J_\lambda} M_{\lambda_j} \qquad \text{(A-10)}$$

where $\Phi_{\lambda\lambda}$ is the inertia dyadic of the augmented body relative to the barycenter B_λ.

Equation (A-10) is the main result given in Ref [1]. Briefly, Eq. (A-10) implies that the rotational motion of body λ of an n-body system can be obtained by

(1) First forming the augmented body λ by adjoining the masses m_{λ_j} occurring at joints j belonging to J_λ to the mass of body λ (m_λ).

(2) Determining the inertia dyadic of the augmented body λ according to

$$\Phi_{\lambda\lambda} = \mathbb{I}_\lambda + m_\lambda D_\lambda^2 U - m_\lambda D_\lambda D_\lambda + \sum_{\mu \neq \lambda} m_\mu \left[D_{\lambda\mu}^2 U - D_{\lambda\mu} D_{\lambda\mu}\right]$$

(3) Considering the terms $M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_\mu$ as the external moment applied to the augmented body λ relative to its barycenter.

(4) Considering the terms

$$m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times (\dot{\omega}_\mu \times D_{\mu\lambda}) + m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[\omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$$

to be due to "inertial" forces.

Consequently, a particularly elegant and appealing result is obtained! In fact, the form of Eq. (A-10) is reminiscent of that appropriate for a purely rigid body relative to an arbitrary point P, viz.,

$$\mathcal{J}^P \cdot \dot{\omega} + \omega \times \mathcal{J}^P \cdot \omega = M^P - m\, d_{pc} \times \ddot{R}_P \tag{A-11}$$

where

$\mathcal{J}^P$ is the inertia dyadic relative to the point P

R_P is the vector from the inertial frame to the point P

d_{pc} is the vector originating at P and terminating at the center of mass c

ω, $\dot{\omega}$ are the angular velocity and angular acceleration relative to an inertial frame.

A compact form for Eq. (A-10) can be obtained by manipulating its terms. First, the term

$$m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times (\dot{\omega}_\mu \times D_{\mu\lambda})$$

is examined. It is clear that

$$D_{\lambda\mu} \times (\dot{\omega}_\mu \times D_{\mu\lambda}) = - D_{\lambda\mu} \times (D_{\mu\lambda} \times \dot{\omega}_\mu) \tag{A-12}$$

and that

$$D_{\lambda\mu} \times (D_{\mu\lambda} \times \dot{\omega}_\mu) = - \left[D_{\mu\lambda} \cdot D_{\lambda\mu} U - D_{\mu\lambda} D_{\lambda\mu} \right] \cdot \dot{\omega}_\mu$$

where U is the unit dyadic and $D_{\lambda\mu}\, D_{\mu\lambda}$ is a dyadic (note that the vectors $D_{\lambda\mu}$ and $D_{\mu\lambda}$ are not fixed in the same bodies).

Substitution of Eq. (A-12) into Eq. (A-10) yields

$$\Phi_{\lambda\lambda} \cdot \dot{\omega}_{\lambda} + \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = E_{\lambda} + \sum_{j \in J_{\lambda}} M^{c}_{\lambda_j}$$

or

$$\sum_{\mu \in S} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = E_{\lambda} + \sum_{j \in J_{\lambda}} M^{c}_{\lambda_j} \tag{A-13}$$

where

$$\Phi_{\lambda\mu} = - m \left[D_{\mu\lambda} \cdot D_{\lambda\mu} U - D_{\mu\lambda} D_{\lambda\mu} \right] , \quad \mu \neq \lambda$$

$$\Phi_{\lambda\lambda} = [\,]_{\lambda} + m_{\lambda} (D_{\lambda} \cdot D_{\lambda} U - D_{\lambda} D_{\lambda}) + \sum_{\mu \neq \lambda} m_{\mu} (D_{\lambda\mu} \cdot D_{\lambda\mu} U - D_{\lambda\mu} D_{\lambda\mu})$$

$$E_{\lambda} = M_{\lambda} + D_{\lambda} \times F_{\lambda} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_{\mu} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[m\omega_{\mu} \times (\omega_{\mu} \times D_{\mu\lambda}) \right]$$

$$- \omega_{\lambda} \times \Phi_{\lambda\lambda} \cdot \omega_{\lambda} + \sum_{j \in J_{\lambda}} M^{SD}_{\lambda_j}$$

$$M_{\lambda_j} = M^{c}_{\lambda_j} + M^{SD}_{\lambda_j}$$

In Eq. (A-13), note that the interaction moment M_{λ_j} is assumed to consist of a constraint moment and a spring-damper moment. Recall that modeling the interaction moment in terms of a spring-damper is a consequence of the assumption that the joint is dissipative and elastic.

Table A-1. Rotational Equations for an n-Hinged Rigid-Body Spacecraft (In Terms of Constraint Torques)

Item	Equations	Remarks
Newton's and Euler's equations for body λ	$F_\lambda + \sum_{j\in J_\lambda} F_{\lambda_j} = m_\lambda(\ddot{R} + \ddot{\rho}_\lambda) = m_\lambda \ddot{R}_{c_\lambda}$ $\mathbf{I}_\lambda \cdot \dot{\omega}_\lambda + \omega_\lambda \times \mathbf{I}_\lambda \cdot \omega_\lambda = M_\lambda + \sum_{j\in J_\lambda} M_{\lambda_j} + \sum_{j\in J_\lambda} c_{\lambda_j} \times F_{\lambda_j}$	m_λ is mass of body λ c_{λ_j} is vector from c.m. of body λ to joint j; J_λ is set of labels for joints on body λ; S_{λ_j} is set of these bodies connected to body λ at joint j; S is set of labels of the ensemble of bodies; J_λ is set of labels for joints on body λ; F_λ is external force on body λ; F_{λ_j} is the interaction force on body λ due to joint j; M_λ is external torque on body λ M_{λ_j} is interaction torque on body λ due to joint j; $\mathbf{I}_\lambda$ is the inertia dyadic for body λ
Elimination of unknown interaction forces F_{λ_j}	$F_{\lambda_j} = \sum_{\mu\in S_{\lambda_j}} (F_\mu - m_\mu \ddot{R}_{c_\mu})$	Summing the interaction forces F_{λ_j} over all bodies belonging to S_{λ_j} <u>isolates</u> F_{λ_j}
Evaluation of $\sum_{j\in J_\lambda} c_{\lambda_j} \times F_{\lambda_j}$	$\sum_{j\in J_\lambda} c_{\lambda_j} \times F_{\lambda_j} = D_\lambda \times F_\lambda + \sum_{\mu\neq\lambda} D_{\lambda\mu} \times F_\mu$ $- \sum_{\mu\neq\lambda} m_\mu c_{\lambda\mu} \times (\ddot{R}_{c_\mu} - \ddot{R}_c)$ where $D_\lambda = - \sum_{\mu\neq\lambda} m_\mu m^{-1} c_{\lambda\mu}$ $D_{\lambda\mu} = D_\lambda + c_{\lambda\mu}$	R_c is vector from origin of Newtonian frame to composite mass center; barycenter of body λ is the new center of mass B_λ obtained by augmenting the point masses m_{λ_j} (the mass of all bodies attached to λ via joint j) located at joints j to the mass of body λ; D_λ is the vector from barycenter B_λ to c.m. (c_λ) of body λ; D_{λ_j} is vector from B_λ to joint j; $D_{\lambda_j} = D_{\lambda\mu}$ for all $\mu \in S_{\lambda_j}$

Table A-1. Rotational Equations for an n-Hinged Rigid-Body Spacecraft (In Terms) of Constraint Torques) (contd)

Item	Equations	Remarks
Evaluation (contd) Examination of term $-\sum_{\mu\neq\lambda} m_\mu c_{\lambda\mu} \times (\ddot{R}_{c_\mu} - \ddot{R}_c)$	$-\sum_{\mu\neq\lambda} m_\mu c_{\lambda\mu} \times (\ddot{R}_{c_\mu} - \ddot{R}_c) =$ $- X_\lambda \cdot \dot{\omega}_\lambda - \omega_\lambda \times X_\lambda \cdot \omega_\lambda + m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda}$ where $X_\lambda = m_\lambda D_\lambda^2 U - m_\lambda D_\lambda D_\lambda + \sum_{\mu\neq\lambda} m_\mu \left(D_{\lambda\mu}^2 U - D_{\lambda\mu} D_{\lambda\mu}\right)$	U is the identity dyadic; $D_\lambda D_\lambda$ and $D_{\lambda\mu} D_{\lambda\mu}$ are dyadics; X_λ is the inertia dyadic which when added to $\mathbb{I}_\lambda$ results in the inertia dyadic relative to B_λ of the augmented body λ
Examination of term $m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda}$	$\ddot{D}_{\mu\lambda} = \dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})$ $m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda} =$ $m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \left[\dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$	$D_{\mu\lambda}$ is fixed in body μ (rigid body assumption);
Expression for $-\sum_{\mu\neq\lambda} m_\mu c_{\lambda\mu} \times (\ddot{R}_{c_\lambda} - \ddot{R}_c)$	$-\sum_{\mu\neq\lambda} m_\mu c_{\lambda\mu} \times (\ddot{R}_{c_\lambda} - \ddot{R}_c) = - X_\lambda \cdot \dot{\omega}_\lambda - \omega_\lambda \times X_\lambda \cdot \omega_\lambda$ $+ m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \left[\dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$	
Expression for $\sum_{j\in J_\lambda} c_{\lambda_j} \times F_{\lambda_j}$	$\sum_{j\in J_\lambda} c_{\lambda_j} \times F_{\lambda_j} = D_\lambda \times F_\lambda + \sum_{\mu\neq\lambda} D_{\lambda\mu} \times F_\mu - X_\lambda \cdot \dot{\omega}_\lambda$ $- \omega_\lambda \times X_\lambda \cdot \omega_\lambda + m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \left[\dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$	

Table A-1. Rotational Equations for an n-Hinged Rigid-Body Spacecraft (In Terms) of Constraint Torques) (contd)

Item	Equations	Remarks
Resulting rotational equations for body λ after evaluating $\sum_{j\in J_\lambda} c_{\lambda_j} \times F_{\lambda_j}$	$\mathbb{I}_\lambda \cdot \dot{\omega}_\lambda + \omega_\lambda \times \mathbb{I}_\lambda \cdot \omega_\lambda = M_\lambda + \sum_{j\in J_\lambda} M_{\lambda_j} + D_\lambda \times F_\lambda$ $+ \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_\mu - X_\lambda \cdot \dot{\omega}_\lambda - \omega_\lambda \times X_\lambda \cdot \omega_\lambda$ $+ m \sum_{\mu \neq \lambda} D_{\lambda\mu} \times \left[\dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$ or $(\mathbb{I}_\lambda + X_\lambda) \cdot \dot{\omega}_\lambda + \omega_\lambda \times (\mathbb{I}_\lambda + X_\lambda) \cdot \omega_\lambda$ $= M_\lambda + D_\lambda \times F_\lambda + \sum_{j\in J_\lambda} M_{\lambda_j} + \sum_{\mu \neq \lambda} D_{\lambda\mu}$ $\times \left[F_\mu + m\left(\dot{\omega}_\mu \times D_{\mu\lambda} + \omega_\mu \times (\omega_\mu \times D_{\mu\lambda})\right)\right]$	Obtained by substituting expression for $\sum_{j\ J_\lambda} c_{\lambda_j} \times F_{\lambda_j}$ into Euler's equation for body λ; the term $M_\lambda + D_\lambda \times F_\lambda$ is the moment relative to the barycenter B_λ due to external force F_λ; the term $\mathbb{I}_\lambda + X_\lambda$ is the inertia dyadic of the augmented body λ relative to the barycenter B_λ
Compact form	$\sum_{\mu\in S} \Phi_{\lambda\mu} \cdot \dot{\omega}_\mu = E_\lambda + \sum_{j\in J_\lambda} M^c_{\lambda_j}$ where $E_\lambda = M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \left[F_\mu + m \;{}_\mu \times (\omega_\mu \times D_{\mu\lambda})\right]$ $- \omega_\lambda \times \Phi_{\lambda\lambda} \cdot \omega_\lambda + \sum_{j\in J_\lambda} M^{SD}_{\lambda_j}$ $\Phi_{\lambda\mu} = -m\left[D_{\mu\lambda} \cdot D_{\lambda\mu} U - D_{\mu\lambda} D_{\lambda\mu}\right]$ for $\mu \neq \lambda$ $\Phi_{\lambda\lambda} = \mathbb{I}_\lambda + X_\lambda = \mathbb{I}_\lambda + \left[m_\lambda (D_\lambda^2 U - D_\lambda D_\lambda) + \sum_{\mu\neq\lambda} m_\mu (D^2_{\lambda\mu} U - D_{\lambda\mu} D_{\lambda\mu})\right]$	M_{λ_j} is considered the sum of an unknown constraint torque and a spring-damper torque ($M_{\lambda_j} = M^c_{\lambda_j} + M^{SD}_{\lambda_j}$); note that the vectors used in defining $\Phi_{\lambda\mu}$ are not fixed in the same bodies; the following relationship involving a cross product and the dot product of a dyadic and a vector is used in obtaining $\Phi_{\lambda\mu}$ $A \times (B \times C) = A \cdot C B - A \cdot B C = -[A \cdot B U - B A] \cdot C$

APPENDIX B

A SET OF r DYNAMICAL EQUATIONS (FREE FROM CONSTRAINT TORQUES) FOR A SPACECRAFT IDEALIZED AS n-HINGED RIGID BODIES

In this Appendix, a set of r dynamical equations free from constraint torques is given (see Table B-1). As noted in Section 1, the procedure for eliminating the constraint torque involves:

(1) Recognizing that the sum of the vector dynamical equations for each λ over all bodies contains no constraint torques (they cancel in pairs according to Newton's Third Law).

(2) Recognizing that a vector constraint torque at a typical joint j can be isolated by summing the vector dynamical equations over all bodies that lie to one side of joint j (the constraint torques on this set of bodies all cancel in pairs, except for the one at joint j).

(3) Recognizing that the constraint torque at joint j is normal to the gimbal axis (axis of rotation) at joint j (this follows from the definition of a constraint).

B-1 Summing the Dynamical Equations for Body λ Over All Bodies

In this section, the equation obtained by summing the dynamical equations for body λ over all the bodies of the system is examined. This result is given by (using Eq. (A-13))

$$\sum_{\lambda}\sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = \sum_{\lambda} E_{\lambda} + \sum_{\lambda}\sum_{j \epsilon J_{\lambda}} M^{c}_{\lambda_j} \tag{B-1}$$

As noted previously, however, the constraint torques vanish when they are summed over all $j \epsilon J_{\lambda}$ and all $\lambda \epsilon S$, i.e.,

$$\sum_{\lambda}\sum_{j \epsilon J_{\lambda}} M_{\lambda_j} = 0 \tag{B-2}$$

Furthermore, if there is a chain of bodies connecting a particular body μ to the so-called base body designated by 0, then

$$\omega_\mu = \omega_0 + \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_k g_k \qquad \text{(B-3)}$$

where

ω_0 is the angular velocity of the base body

g_k is the unit vector representing the gimbal axis at joint k (there are r-3 gimbal axes in all)

$\dot{\gamma}_k$ is the relative angular rotation of the two bodies connected at joint k

$\epsilon_{k\mu}$ is 1 if body μ senses the relative rate $\dot{\gamma}_k g_k$ and is 0 if body μ does not sense it.

It is clear that if joint k lies between body 0 and body μ then $\epsilon_{k\mu} = 1$, otherwise $\epsilon_{k\mu} = 0$. Using the result of Eq. (B-3) it follows that

$$\dot{\omega}_\mu = \dot{\omega}_0 + \sum_{k=1}^{r-3} \epsilon_{k\mu} \ddot{\gamma}_k g_k + \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_k \dot{g}_k \qquad \text{(B-4)}$$

Substitution of Eq. (B-4) into Eq. (B-1), yields

$$\sum_\lambda \sum_\mu \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + \sum_\lambda \sum_\mu \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} g_k \ddot{\gamma}_k$$

$$= \sum_\lambda \left[E_\lambda - \sum_\mu \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_k \dot{g}_k \right] \qquad \text{(B-5)}$$

Manipulation of Eq. (B-5) yields

$$\sum_{\lambda}\sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3}\left(\sum_{\lambda}\sum_{\mu} \Phi_{\lambda\mu}\, \epsilon_{k\mu} \cdot g_k\right)\ddot{\gamma}_k$$

$$= \sum_{\lambda}\left[E_{\lambda} - \sum_{\mu} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu}\, \dot{\gamma}_k\, \dot{g}_k\right] \qquad \text{(B-6)}$$

In compact form Eq. (B-6) can be written as

$$a_{00} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{0k}\, \ddot{\gamma}_k = \sum_{\lambda} E_{\lambda}^{*} \qquad \text{(B-7)}$$

where

$$a_{00} = \sum_{\lambda}\sum_{\mu} \Phi_{\lambda\mu}$$

$$a_{0k} = \sum_{\lambda}\sum_{\mu} \Phi_{\lambda\mu}\, \epsilon_{k\mu} \cdot g_k$$

$$E_{\lambda}^{*} = E_{\lambda} - \sum_{\mu} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu}\, \dot{g}_k\, \dot{\gamma}_k = E_{\lambda} - \sum_{k=1}^{r-3}\left(\sum_{\mu} \epsilon_{k\mu}\, \Phi_{\lambda\mu} \cdot \dot{g}_k\right)\dot{\gamma}_k$$

In Eq. (B-7), a_{00} is a dyadic, a_{0k} is a vector, and E_{λ} and E_{λ}^{*} are vectors. Note too that the term $\epsilon_{k\mu}$ picks out those particular bodies μ that sense the rotation $\dot{\gamma}_k\, g_k$. It should also be observed that the right hand side of Eq. (B-7) can be written as

$$\sum_{\lambda} E_{\lambda}^{*} = \sum_{\lambda} \left[E_{\lambda} - \sum_{k=1}^{r-3} \left(\sum_{\mu} \epsilon_{k\mu} \Phi_{k\mu} \cdot \dot{g}_k \right) \dot{\gamma}_k \right]$$

$$= \sum_{\lambda} E_{\lambda} - \sum_{k=1}^{r-3} \left(\sum_{\lambda} \sum_{\mu} \Phi_{\lambda\mu} \epsilon_{k\mu} \cdot \dot{g}_k \right) \dot{\gamma}_k \qquad \text{(B-8)}$$

In particular, the operator (dyadic)

$$\sum_{\lambda} \sum_{\mu} \Phi_{\lambda\mu} \epsilon_{k\mu}$$

occurring in the RHS of Eq. (B-8) is the same as that used in defining the term a_{0k}, as expected.

B-2 Obtaining r-3 Additional Scalar Equations from the Orthogonality of the Gimbal Axis and Constraint Moment at Each Joint

In this section, r-3 additional scalar equations are obtained by using the facts that:

(1) A vector constraint torque at a typical joint can be isolated by summing the vector dynamical equations over all bodies that lie to one side of joint j

(2) The constraint torque at joint j is orthogonal to the gimbal axis at joint j (i.e., the dot product of the gimbal axis and the constraint torque at joint j vanishes).

That is, for each gimbal axis i, if the equation

$$\sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = E_{\lambda} + \sum_{j \in J_{\lambda}} M_{\lambda_j}^{c}$$

is summed over all bodies that lie to one side of joint i (that is, beyond gimbal axis g_i relative to body 0), then

$$\sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = \sum_{\lambda} \epsilon_{i\lambda} E_{\lambda} + \sum_{\lambda} \epsilon_{i\lambda} \sum_{j \in J_{\lambda}} M^{c}_{\lambda_j} \tag{B-9}$$

where

$\epsilon_{i\lambda} = 1$ for all those bodies that lie to one side of joint i and is 0, otherwise.

Note that the term

$$\sum_{\lambda} \epsilon_{i\lambda} \sum_{j \in J_{\lambda}} M^{c}_{\lambda_j} = M^{c}_{\lambda_j}$$

since the constraint torques cancel in pairs except at the joint j (when the constraint torques at joint j are summed over all bodies lying to one side of the joint relative to body 0).

Using the fact that the constraint torque at the joint j is orthogonal to the gimbal axis at that joint, it follows that

$$g_i \cdot \left[\sum_{\lambda} \epsilon_{i\lambda} \left(\sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} - E_{\lambda}\right)\right] = 0 \qquad i = 1, 2, \ldots r-3 \tag{B-10}$$

Manipulation of Eq. (B-10) yields

$$g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu} \Phi_{\lambda\mu} \cdot \left(\dot{\omega}_0 + \sum_{k=1}^{r-3} \epsilon_{k\mu}\, g_k\, \ddot{\gamma}_k\right)$$

$$= g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \left(E_{\lambda} - \sum_{\mu} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu}\, \dot{g}_k\, \dot{\gamma}_k\right) = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E^{*}_{\lambda} \tag{B-11}$$

In compact form, Eq. (B-11) can be written as

$$a_{i0} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{ik} \ddot{\gamma}_k = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E_{\lambda}^{*} \tag{B-12}$$

where

$$a_{i0} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu}$$

$$a_{ik} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot g_k$$

In Eq. (B-12), a_{ik} is a scalar and a_{i0} is a vector.

B-3 Set of r Dynamical Equations (Free From Constraint Torques)

In this section, the set of r dynamical equations obtained in Section B-1 and B-2 is given. Collecting the results of Eq. (B-7) and (B-12), it follows that

$$a_{00} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{0k} \ddot{\gamma}_k = \sum_{\lambda} E_{\lambda}^{*}, \text{ 3 scalar equations}$$

$$a_{i0} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{ik} \ddot{\gamma}_k = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E_{\lambda}^{*}, \text{ r-3 equations with } i = 1, 2, \ldots r-3 \tag{B-13}$$

where

$$a_{00} = \sum_\lambda \sum_\mu \Phi_{\lambda\mu}, \text{ a dyadic}$$

$$a_{0k} = \sum_\lambda \sum_\mu \Phi_{\lambda\mu}\, \epsilon_{k\mu} \cdot g_k, \text{ a vector}$$

$$E_\lambda^* = E_\lambda - \sum_\mu \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu}\, \dot{g}_k\, \dot{\gamma}_k = E_\lambda - \sum_{k=1}^{r-3} \left(\sum_\mu \epsilon_{k\mu}\, \Phi_{\lambda\mu} \cdot \dot{g}_k \right) \dot{\gamma}_k$$

$$a_{i0} = g_i \cdot \sum_\lambda \sum_\mu \epsilon_{i\lambda}\, \Phi_{\lambda\mu}, \text{ a vector}$$

$$a_{ik} = g_i \cdot \sum_\lambda \sum_\mu \epsilon_{i\lambda}\, \epsilon_{k\mu}\, \Phi_{\lambda\mu} \cdot g_k, \text{ a scalar}$$

$\epsilon_{k\mu}$ identifies those bodies μ which sense the relative rate $\dot{\gamma}_k\, g_k$

$\epsilon_{i\lambda}$ identifies the bodies λ over which the dynamical equations are summed to isolate the constraint torque at joint i

In matrix notation, Eq. (B-13) can be written as

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \cdots & a_{0,r-3} \\ a_{10} & a_{11} & \cdots\cdots & a_{1,r-3} \\ \vdots & & & \\ a_{r-3,0} & a_{r-3,1} & \cdots\cdots & a_{r-3,r-3} \end{bmatrix} \begin{pmatrix} \dot{\omega}_0 \\ \ddot{\gamma}_1 \\ \vdots \\ \ddot{\gamma}_{r-3} \end{pmatrix} = \begin{pmatrix} \sum_\lambda E_\lambda^* \\ g_1 \cdot \sum_\lambda \epsilon_{1\lambda} E_\lambda^* \\ \vdots \\ g_{r-3} \cdot \sum_\lambda \epsilon_{r-3} E_\lambda^* \end{pmatrix}$$

It is understood that the dyadics and vectors appearing in Eqs. (B-7) and (B-12) are replaced by matrices of their components in Eq. (B-14). It is also noted from the definitions of a_{00}, a_{0k}, a_{i0}, a_{ik} that the matrix involving these elements in Eq. (B-14) is symmetric. The dimensions of the matrix elements are as follows: a_{00} -- 3 x 3, a_{0k} -- 3 x 1, a_{k0} -- 1 x 3, and a_{ik} is a scalar for i, k = 1, 2, ... r-3.

Equation (B-14) represents the desired set of r dynamical equations governing the motion of an n-hinged rigid-body spacecraft.

Table B-1. Set of r Dynamical Equations for an n-Hinged Rigid-Body Spacecraft (Free from Constraint Torques)

Item	Equations	Remarks
Starting point for obtaining set of n dynamical equations (free from constraints)	$\sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = E_{\lambda} + \sum_{j \in J_{\lambda}} M^{c}_{\lambda_j}$ where $E_{\lambda} = M_{\lambda} + D_{\lambda} \times F_{\lambda} + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times F_{\mu} + \sum_{\mu \neq \lambda} D_{\lambda\mu}$ $\times \left[m\, \omega_{\mu} \times (\omega_{\mu} \times D_{\mu\lambda}) \right] - \omega_{\lambda} \times \Phi_{\lambda\lambda} \cdot \omega_{\lambda} + \sum_{j \in J_{\lambda}} M^{SD}_{\lambda_j}$ $\Phi_{\lambda\lambda} = \mathbb{I}_{\lambda} + X_{\lambda} - \mathbb{I}_{\lambda} + \left[m_{\lambda} \left(D^{2}_{\lambda} U - D_{\lambda} D_{\lambda} \right) \right.$ $\left. + \sum_{\mu \neq \lambda} m_{\mu} \left(D^{2}_{\lambda\mu} U - D_{\lambda\mu} D_{\lambda\mu} \right) \right]$ $\Phi_{\lambda\mu} = -m \left[D_{\mu\lambda} \cdot D_{\lambda\mu} U - D_{\mu\lambda} D_{\lambda\mu} \right], \; \mu \neq \lambda$	Set of equations for n-hinged rigid-body spacecraft (involving constraint moments) that were derived in Appendix A; $M^{c}_{\lambda_j}$ is constraint moment at joint j on body λ; all other terms are defined in Table A-1.
Elimination of constraint moments	$\sum_{\lambda} \sum_{\mu} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} = \sum_{\lambda} E_{\lambda}$ since $\sum_{\lambda} \sum_{j \in J_{\lambda}} M^{c}_{\lambda_j} \equiv 0$	Summing the constraint moments on body λ over all joints $j \in J_{\lambda}$ and over all bodies removes the constraint torques since they cancel in pairs (Newton s third law)
Simplification of result obtained by summing over all bodies	$\sum_{\lambda} \sum_{\mu} \Phi_{\lambda\mu} \cdot \left[\dot{\omega}_{0} + \sum_{k=1}^{r-3} \epsilon_{k\mu} \ddot{\gamma}_{k} g_{k} + \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_{k} \dot{g}_{k} \right]$ $= \sum_{\lambda} E_{\lambda}$	g_k is the unit vector representing the gimbal axis at joint k; $\epsilon_{k\mu}$ specifies the bodies μ which sense the rotation $\dot{\gamma}_k g_k$; $\epsilon_{k\mu}$ is 1 if the gimbal axis is between body μ and body 0, otherwise $\epsilon_{k\mu} = 0$;

Table B-1. Set of r Dynamical Equations for an n-Hinged Rigid-Body Spacecraft (Free from Constraint Torques) (contd)

Item	Equations	Remarks
Elimination of constraint moments (contd.) Simplification of result obtained by summarizing over all bodies λ	where $\omega_\mu = \omega_0 + \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_k g_k$ Rearranging, $\sum_\lambda \sum_\mu \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} \left(\sum_\lambda \sum_\mu \Phi_{\lambda\mu} \epsilon_{k\mu} \cdot g_k \right) \ddot{\gamma}_k$ $= \sum_\lambda \left[E_\lambda - \sum_\mu \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} \cdot \dot{g}_k \dot{\gamma}_k \right] = \sum_\lambda E^*_\lambda$	
Compact form	$a_{00} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{0k} \ddot{\gamma}_k = \sum_\lambda E^*_\lambda$ where $E^*_\lambda = E_\lambda - \sum_\mu \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} \cdot \dot{g}_k \dot{\gamma}_k$ $a_{00} = \sum_\lambda \sum_\mu \Phi_{\lambda\mu}$ $a_{0k} = \sum_\lambda \sum_\mu \Phi_{\lambda\mu} \epsilon_{k\mu} \cdot g_k$	a_{00} is a dyadic; a_{0k}, E_λ, E^*_λ are vectors; result represents 3 scalar equations.
Determination of additional r-3 scalar equations Isolating constraint moment at joint j	$\sum_\lambda \epsilon_{i\lambda} \left\{ \sum_\mu \Phi_{\lambda\mu} \cdot \left[\dot{\omega}_0 + \sum_{k=1}^{r-3} \epsilon_{k\mu} \ddot{\gamma}_k g_k + \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_k \dot{g}_k \right] \right\} = \sum_\mu \epsilon_{i\lambda} E_\lambda + \sum_\mu \epsilon_{i\lambda} \sum_{j \epsilon J_\lambda} M^c_{\lambda_j}$ where $\sum_\lambda \epsilon_{i\lambda} \sum_{j \epsilon J} M^c_{\lambda_j} = M^c_{\lambda j}$ for j = i	Summing the vector dynamical equations over all bodies that lie to one side of joint i isolates the constraint torque $M^c_{\lambda_j}$; $\epsilon_{i\lambda}$ identifies the bodies λ involved in sum

Table B-1. Set of r Dynamical Equations for an n-Hinged Rigid-Body Spacecraft (Free from Constraint Torques) (contd)

Item	Equations	Results
Determination of additional r-3 scalar equations Orthogonality of gimbal axis i and constraint moment at joint i	$g_i \cdot M^c_{\lambda_i} = 0$ $g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \left\{ \sum_{\mu} \Phi_{\lambda\mu} \cdot \left[\dot{\omega}_0 + \sum_{k=1}^{r-3} \epsilon_{k\mu} \ddot{\gamma}_k g_k \right] + \sum_{\mu} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} \dot{\gamma}_k \dot{g}_k - E_{\lambda} \right\} \quad 0$ $g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + g_i \cdot \sum_{k=1}^{r-3} \left(\sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot g_k \right) \ddot{\gamma}_k = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E^*_{\lambda}$	By definition of a constraint moment, it is orthogonal to the gimbal axis g_i
Compact form for orthogonality conditions	$a_{i0} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{ik} \ddot{\gamma}_k = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E^*_{\lambda}$ where $a_{i0} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu}$ $a_{ik} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot g_k$	a_{i0} is a vector, a_{ik} is a scalar; with i = 1, 2, ... r-3, it follows that r-3 scalar equations result

Table B-1. Set of r Dynamical Equations for an n-Hinged Rigid-Body Spacecraft (Free from Constraint Torques) (contd)

Item	Equations	Results
Set of r dynamical equations for n-hinged rigid-body spacecraft (free from constraint moments)	$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0,r-3} \\ a_{10} & a_{11} & \cdots & a_{1,r-3} \\ \vdots & & & \\ a_{r-3,0} & & \cdots & a_{r-3,r-3} \end{bmatrix} \begin{pmatrix} \dot{\omega}_0 \\ \ddot{\gamma}_1 \\ \vdots \\ \ddot{\gamma}_{r-3} \end{pmatrix} = \begin{pmatrix} \sum_\lambda E_\lambda^* \\ g_1 \cdot \sum_\lambda \epsilon_{i\lambda} E_\lambda^* \\ \vdots \\ g_{r-3} \cdot \sum_\lambda \epsilon_{r-3} E_\lambda^* \end{pmatrix}$$	Matrix components of dyadics and vectors are used in this equation; a_{00} is 3 x 3, a_{0k} is 3 x 1, a_{i0} is 1 x 3, a_{ik} is a scalar for i, k = 1, 2, ... r - 3; matrix involving a_{00}, a_{0k}, a_{i0}, a_{ik} is symmetric.

APPENDIX C

DETERMINATION OF EXPLICIT FORM FOR DYNAMICAL EQUATIONS FOR A SPECIFIC TOPOLOGICAL ARRANGEMENT OF 5-HINGED RIGID BODIES

In this section, the general results described in Appendix B are applied to a specific topological arrangement of 5 hinged bodies. In particular, results are obtained in this section for the arrangement of bodies shown in Fig. 5-1. Hence, the explicit form of the dynamical equations for the case in which n = 5 or less is obtained.

C-1 3 Scalar Equations Obtained by Summing Dynamical Equations Over All 5 Bodies

Three scalar equations are obtained by summing the dynamical results for each body λ over all 5 bodies. This result is written as

$$\sum_{\lambda=0}^{4} \left[\sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_{\mu} \right] = \sum_{\lambda=0}^{4} \left[E_{\lambda} + \sum_{j \epsilon J_{\lambda}} M^{c}_{\lambda_j} \right] \tag{C-1}$$

Since the constraint torques cancel in pairs, the term

$$\sum_{\lambda=0}^{4} \sum_{j \epsilon J_{\lambda}} M^{c}_{\lambda_j} \equiv 0 \tag{C-2}$$

It can be seen from Fig. 5-1 that the angular velocities of the bodies are

$$\begin{aligned}
\omega_0 &= \omega_0 \\
\omega_1 &= \omega_0 + \dot{\gamma}_1 g_1 \\
\omega_2 &= \omega_0 + \dot{\gamma}_1 g_1 + \dot{\gamma}_2 g_2 \\
\omega_3 &= \omega_0 + \dot{\gamma}_3 g_3 \\
\omega_4 &= \omega_0 + \dot{\gamma}_3 g_3 + \dot{\gamma}_4 g_4
\end{aligned} \tag{C-3}$$

or, in general,

$$\omega_\mu = \omega_0 + \sum_{k=1}^{4} \epsilon_{k\mu}\, \dot{\gamma}_k\, g_k \tag{C-4}$$

where $\epsilon_{k\mu}$ identifies which angular rates $\dot{\gamma}_k\, g_k$ are involved in the angular velocity of body μ. Effectively, $\epsilon_{k\mu} = 1$ if the gimbal axis g_k lies <u>between</u> body μ and body 0 and is 0 otherwise. Using Eq. (C-4) in Eq. (C-1), it follows that

$$\sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + \sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{4} \epsilon_{k\mu}\, g_k\, \ddot{\gamma}_k$$

$$= \sum_{\lambda=0}^{4} \left(E_\lambda - \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{4} \epsilon_{k\mu}\, \dot{g}_k\, \dot{\gamma}_k \right) \tag{C-5}$$

Equation (C-5) can be rewritten as

$$\sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + \sum_{k=1}^{4} \left(\sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \epsilon_{k\mu}\, \Phi_{\lambda\mu} \right) \cdot g_k\, \ddot{\gamma}_k$$

$$= \sum_{\lambda=0}^{4} E_\lambda - \sum_{k=1}^{4} \left(\sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \epsilon_{k\mu}\, \Phi_{\lambda\mu} \right) \cdot \dot{g}_k\, \dot{\gamma}_k \tag{C-6}$$

In compact form, Eq. (C-6) becomes

$$a_{00} \cdot \dot{\omega}_0 + \sum_{k=1}^{4} (b_{0k} \cdot g_k)\, \ddot{\gamma}_k = \sum_{\lambda=0}^{4} E_\lambda - \sum_{k=1}^{4} b_{0k} \cdot \dot{g}_k\, \dot{\gamma}_k$$

where

$$a_{00} = \sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} = b_{00} \tag{C-7}$$

$$b_{0k} = \sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \epsilon_{k\mu} \Phi_{\lambda\mu}$$

$$a_{0k} = b_{0k} \cdot g_k$$

The dyadics b_{0k} for k = 1, 2, 3, 4 can be evaluated by inspection of Fig. 5-1 if the meaning of $\epsilon_{k\mu}$ is kept in mind (recall that $\epsilon_{k\mu}$ identifies those bodies μ which sense the angular rate $\dot{\gamma}_k g_k$).

It follows that

$$b_{01} = \sum_{\lambda=0}^{4} \sum_{\mu=1}^{2} \Phi_{\lambda\mu} \qquad \text{(since only bodies 1 and 2 sense } \dot{\gamma}_1 g_1\text{)}$$

$$b_{02} = \sum_{\lambda=0}^{4} \Phi_{\lambda 2} \qquad \text{(since only body 2 senses } \dot{\gamma}_2 g_2\text{)}$$

(C-8)

$$b_{03} = \sum_{\lambda=0}^{4} \sum_{\mu=3}^{4} \Phi_{\lambda\mu} \qquad \text{(since bodies 3 and 4 sense } \dot{\gamma}_3 g_3\text{)}$$

$$b_{04} = \sum_{\lambda=0}^{4} \Phi_{\lambda 4} \qquad \text{(since only body 4 senses } \dot{\gamma}_4 g_4\text{)}$$

C-2 Equations Resulting from Orthogonality of Constraint Moments and Gimbal Axes

In this section, the additional four equations resulting from the orthogonality of the constraint moment and the gimbal axis for each of the four joints are obtained. First, it is seen from Fig. 4-1 that the constraint moment at joint 1 is isolated by summing over bodies $\lambda = 1, 2$; that the constraint moment at joint 2 is isolated by considering body 2 only; that the constraint moment at joint 3 is isolated by summing over bodies 3, 4; and that the constraint moment at joint 4 is isolated by considering body 4 only. From the orthogonality of the constraint moment and the gimbal axis at each joint, it follows that

$$
\begin{aligned}
g_1 \cdot M^c_{1_1} &= 0 \\
g_2 \cdot M^c_{2_2} &= 0 \\
g_3 \cdot M^c_{3_3} &= 0 \\
g_4 \cdot M^c_{4_4} &= 0
\end{aligned}
\qquad \text{(C-9)}
$$

where a typical constraint moment $M^c_{\lambda_j} = M^c_{1_1}$ for $\lambda = 1$, $j = 1$.

For a fixed i, it follows that

$$
\begin{aligned}
M^c_{\lambda_j} &= \sum_{\lambda} \epsilon_{i\lambda} \sum_{j \epsilon J_\lambda} M^c_{\lambda_j} \qquad \text{with } j = i \\
g_i \cdot M^c_{\lambda_j} &= 0
\end{aligned}
\qquad \text{(C-10)}
$$

where $\epsilon_{i\lambda}$ identifies those bodies λ over which the sum is taken. It has already been observed that for $i = 1$ the sum is taken over $\lambda = 1, 2$; for $i = 2$, only $\lambda = 2$ is involved; for $i = 3$, the sum is taken over $\lambda = 3, 4$; and for $i = 4$, only $\lambda = 4$ is involved. Using Eq. (C-10), it follows that

$$\sum_{\lambda} \epsilon_{i\lambda} \sum_{j \epsilon J_\lambda} M^c_{\lambda_j} = \sum_{\lambda} \epsilon_{i\lambda} \left(\sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_\mu - E_\lambda \right)$$

$$g_i \cdot M^c_{\lambda_j} = 0 = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \left(\sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_\mu - E_\lambda \right) \tag{C-11}$$

Expansion of Eq. (C-11) yields

$$0 = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \left(\sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{4} \epsilon_{k\mu}\, g_k\, \ddot{\gamma}_k \right.$$

$$\left. + \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{4} \epsilon_{k\mu}\, \dot{g}_k\, \dot{\gamma}_k - E_\lambda \right) \tag{C-12}$$

Simplification of Eq. (C-12) yields

$$g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \cdot \dot{\omega}_0 + g_i \cdot \sum_{k=1}^{4} \left(\sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu=0}^{4} \epsilon_{k\mu}\, \Phi_{\lambda\mu} \right) \cdot g_k\, \ddot{\gamma}_k$$

$$= g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E_\lambda - g_i \cdot \sum_{k=1}^{4} \left(\sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu=0}^{4} \epsilon_{k\mu}\, \Phi_{\lambda\mu} \right) \cdot \dot{g}_k\, \ddot{\gamma}_k \tag{C-13}$$

In compact form, Eq. (C-13) can be written as

$$g_i \cdot b_{i0} \cdot \dot{\omega}_0 + g_i \cdot \sum_{k=1}^{4} (b_{ik} \cdot g_k)\, \ddot{\gamma}_k$$

$$= g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E_\lambda - g_i \cdot \sum_{k=1}^{4} (b_{ik} \cdot \dot{g}_k)\, \dot{\gamma}_k \tag{C-14}$$

where

$$b_{i0} = \sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu=0}^{4} \Phi_{\lambda\mu}$$

$$b_{ik} = \sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu=0}^{4} \epsilon_{k\mu} \Phi_{\lambda\mu}$$

In the notation of Appendix B, Eq. (C-14) can be written as

$$a_{i0} \cdot \dot{\omega}_0 + \sum_{k=1}^{4} a_{ik} \ddot{\gamma}_k = g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} E_{\lambda} - \sum_{k=1}^{4} (g_i \cdot b_{ik} \cdot \dot{g}_k) \dot{\gamma}_k \tag{C-15}$$

where

$$\begin{aligned} a_{i0} &= g_i \cdot b_{i0} \\ a_{ik} &= g_i \cdot b_{ik} \cdot g_k \end{aligned} \tag{C-16}$$

The dyadics b_{i0} can be written as (see Table C-1)

$$b_{10} = \sum_{\lambda=1}^{2} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \qquad \text{(since } \lambda = 1, 2 \text{ are involved in sum over } \lambda\text{)}$$

$$b_{20} = \sum_{\mu=0}^{4} \Phi_{2\mu} \qquad \text{(since only } \lambda = 2 \text{ is involved in sum over } \lambda\text{)}$$

$$b_{30} = \sum_{\lambda=3}^{4} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} \qquad \text{(since } \lambda = 3, 4 \text{ are involved in sum over } \lambda\text{)}$$

$$b_{40} = \sum_{\mu=0}^{4} \Phi_{4\mu} \qquad \text{(since only } \lambda = 4 \text{ is involved in sum over } \lambda) \tag{C-17}$$

Note that the bodies μ involved in the sum

$$\sum_{\mu=0}^{4} \epsilon_{k\mu} \Phi_{\lambda\mu}$$

for k = 1, 2, 3, 4 have already been identified ($\epsilon_{k\mu}$ identifies the bodies μ which sense the rotation $\dot{\gamma}_k g_k$). Now the bodies λ which are picked out by the sum

$$\sum_{\lambda} \epsilon_{i\lambda}$$

have also been identified for each i. Consequently, the bodies μ, λ involved in the sum

$$b_{ik} = \sum_{\lambda} \epsilon_{i\lambda} \sum_{\mu=0}^{4} \epsilon_{k\mu} \Phi_{\lambda\mu}$$

can be written down by inspection. Written in terms of an array, elements b_{00}, b_{0k} (derived in Section C-1), and b_{k0}, b_{ik} become

$$
\begin{bmatrix}
\sum\limits_{\lambda=0}^{4}\sum\limits_{\mu=0}^{4}\Phi_{\lambda\mu} & \sum\limits_{\lambda=0}^{4}\sum\limits_{\mu=1}^{2}\Phi_{\lambda\mu} & \sum\limits_{\lambda=0}^{4}\Phi_{\lambda 2} & \sum\limits_{\lambda=0}^{4}\sum\limits_{\mu=3}^{4}\Phi_{\lambda\mu} & \sum\limits_{\lambda=0}^{4}\Phi_{\lambda 4} \\
\sum\limits_{\lambda=1}^{2}\sum\limits_{\mu=0}^{4}\Phi_{\lambda\mu} & \sum\limits_{\lambda=1}^{2}\sum\limits_{\mu=1}^{2}\Phi_{\lambda\mu} & \sum\limits_{\lambda=1}^{2}\Phi_{\lambda 2} & \sum\limits_{\lambda=1}^{2}\sum\limits_{\mu=3}^{4}\Phi_{\lambda\mu} & \sum\limits_{\lambda=1}^{2}\Phi_{\lambda 4} \\
\sum\limits_{\mu=0}^{4}\Phi_{2\mu} & \sum\limits_{\mu=1}^{2}\Phi_{2\mu} & \Phi_{22} & \sum\limits_{\mu=3}^{4}\Phi_{2\mu} & \Phi_{24} \\
\sum\limits_{\lambda=3}^{4}\sum\limits_{\mu=0}^{4}\Phi_{\lambda\mu} & \sum\limits_{\lambda=3}^{4}\sum\limits_{\mu=1}^{2}\Phi_{\lambda\mu} & \sum\limits_{\lambda=3}^{4}\Phi_{\lambda 2} & \sum\limits_{\lambda=3}^{4}\sum\limits_{\mu=3}^{4}\Phi_{\lambda\mu} & \sum\limits_{\lambda=3}^{4}\Phi_{\lambda 4} \\
\sum\limits_{\mu=0}^{4}\Phi_{\lambda\mu} & \sum\limits_{\mu=1}^{2}\Phi_{4\mu} & \Phi_{42} & \sum\limits_{\mu=3}^{4}\Phi_{4\mu} & \Phi_{44}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
b_{00} & b_{01} & b_{02} & b_{03} & b_{04} \\
b_{10} & b_{11} & b_{12} & b_{13} & b_{14} \\
b_{20} & b_{21} & b_{22} & b_{23} & b_{24} \\
b_{30} & b_{31} & b_{32} & b_{33} & b_{34} \\
b_{40} & b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix} \qquad \text{(C-18)}
$$

Note that Eq. (C-18) could have been written immediately by inspection of Fig. 5-1! In row 1 of the array, the summation over λ is from $\lambda = 0$ to $\lambda = 4$; in row 2 (corresponding to the dot product of g_1 and the constraint moment at joint 1) the summation over λ is from $\lambda = 1$ to 2; in row 3 (corresponding to the dot product of g_2 and the constraint moment at joint 2), only $\lambda = 2$ is involved; in row 4 (corresponding to the dot product of g_3 and the constraint moment at joint 3), the summation over λ is from $\lambda = 3$ to $\lambda = 4$; in row 5 (corresponding to the dot product of g_4 and the constraint vector at joint 4), only $\lambda = 4$ is involved. Similarly, in column 1, the summation over μ is from 0 to 4 (all bodies sense ω_0); in column 2, the summation over μ is from

1 to 2 (only bodies 1 and 2 sense $\dot{\gamma}_1\, g_1$); in column 3, the summation over μ involves only $\mu = 2$ (only body 2 senses $\dot{\gamma}_2\, g_2$ -- it is the end of a chain!); in column 4, the summation over μ is from 3 to 4 (only bodies 3 and 4 sense $\dot{\gamma}_3\, g_3$); in column 5, the summation over μ involves only body 4 -- it, too, is the end of a chain!).

C-3 Set of r (r = 7) Dynamical Equations for a 5-Hinged Rigid-Body Spacecraft

In the section, the set of 7 dynamical equations governing the behavior of the 5-hinged rigid-body spacecraft shown in Fig. 5-1 are given. These equations are obtained by collecting the results generated in Sections C-1 and C-2.

In the following equation, it is understood that the matrix components of operations involving dyadics and vectors are used (recall, that the dyadics $b_{\ell m}$ with ℓ, m = 0, 1, ..4 are defined in Eq. (C-18))

$$
\begin{bmatrix}
b_{00} & b_{01}\cdot g_1 & b_{02}\cdot g_2 & b_{03}\cdot g_3 & b_{04}\cdot g_4 \\
g_1\cdot b_{10} & g_1\cdot b_{11}\cdot g_1 & g_1\cdot b_{12}\cdot g_2 & g_1\cdot b_{13}\cdot g_3 & g_1\cdot b_{14}\cdot g_4 \\
g_2\cdot b_{20} & g_2\cdot b_{21}\cdot g_1 & g_2\cdot b_{22}\cdot g_2 & g_2\cdot b_{23}\cdot g_3 & g_2\cdot b_{24}\cdot g_4 \\
g_3\cdot b_{30} & g_3\cdot b_{31}\cdot g_1 & g_3\cdot b_{32}\cdot g_2 & g_3\cdot b_{33}\cdot g_3 & g_3\cdot b_{34}\cdot g_4 \\
g_4\cdot b_{40} & g_4\cdot b_{41}\cdot g_1 & g_4\cdot b_{42}\cdot g_2 & g_4\cdot b_{43}\cdot g_3 & g_4\cdot b_{44}\cdot g_4
\end{bmatrix}
\begin{pmatrix} \dot{\omega}_0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \\ \ddot{\gamma}_3 \\ \ddot{\gamma}_4 \end{pmatrix}
=
\begin{pmatrix}
\sum_{\lambda=0}^{4} E_\lambda - \sum_{k=1}^{4} (b_{0k}\cdot \dot{g}_k)\,\dot{\gamma}_k \\
g_1\cdot \sum_{\lambda=1}^{2} E_\lambda - \sum_{k=1}^{4} g_1\cdot b_{1k}\cdot \dot{g}_k\,\dot{\gamma}_k \\
g_2\cdot E_2 - \sum_{k=1}^{4} g_2\cdot b_{2k}\cdot \dot{g}_k\,\dot{\gamma}_k \\
g_3\cdot \sum_{\lambda=3}^{4} E_\lambda - \sum_{k=1}^{4} g_3\cdot b_{3k}\cdot \dot{g}_k\,\dot{\gamma}_k \\
g_4\cdot E_4 - \sum_{k=1}^{4} g_4\cdot b_{4k}\cdot \dot{g}_k\,\dot{\gamma}_k
\end{pmatrix}
\qquad \text{(C-19)}
$$

It is worth repeating the fact that the dimensions of the elements are as follows:

b_{00} -- 3 x 3, $b_{0k} \cdot g_k$ -- 3 x 1, $g_k \cdot b_{k0}$ -- 1 x 3, $g_i \cdot b_{ik} \cdot g_k$ -- 1 x 1

for i, k = 1, ... 4.

As mentioned previously, the square matrix on the LHS of Eq. (3-19) is symmetric so only

$$s = \frac{n}{2}(a + \ell) = \frac{5}{2}(5 + 1) = 15$$

of the 25 elements must be evaluated (note, however, that b_{00} is a symmetric matrix, too!). Typical elements of this array are the 3 x 3 matrix b_{00}, the 3 x 1 vector $b_{01} \cdot g_1$ and the scalar $g_1 \cdot b_{12} \cdot g_2$. Note, too, that the elements of this array were previously, defined as (see Eq. B-13 and Eq. B-14).

$$\begin{aligned} a_{00} &= b_{00} & & \\ a_{0k} &= b_{0k} \cdot g_k, & & k = 1, 2, \ldots 4 \\ a_{k0} &= g_k \cdot b_{k0}, & & k = 1, 2, \ldots 4 \\ a_{ik} &= g_i \cdot b_{ik} \cdot g_k, & & i, k = 1, 2, \ldots 4 \end{aligned} \tag{C-20}$$

In compact form, Eq. (C-19) can be written as

$$A \dot{\omega} = L \tag{C-21}$$

where A is a 7 x 7 matrix, $\dot{\omega}$ is a 7 x 1 vector (consisting of $\dot{\omega}_0$ and the relative angular accelerations $\ddot{\gamma}_k$), and L is a 7 x 1 vector (which can be considering the forcing function of the matrix differential equations).

C-3.1 Transformation of Elements into Appropriate Coordinate System

In this section, the vector basis in which the elements must be expressed in order for the operations to be performed is selected. A logical basis is that associated with body 0 (the base body). For the attitude control problem, it is usually necessary to control only the attitude of the base body (the attitude being kinematically related to the angular velocity vector ω_0). Hence, the transformations from bodies 1, 2, 3, 4 to body 0 are required.

C-3.1.1 Coordinate Transformations

The transformations required in performing the operations depicted in Eq. (C-19) are (note these are not restricted to small angles)

$$
\begin{aligned}
C_0^1 &= \cos \gamma_1 E + (1 - \cos \gamma_1) g_1 g_1^T - S \gamma_1 \tilde{g}_1 \\
C_1^2 &= \cos \gamma_2 E + (1 - \cos \gamma_2) g_2 g_2^T - S \gamma_2 \tilde{g}_2 \\
C_0^3 &= \cos \gamma_3 E + (1 - \cos \gamma_3) g_3 g_3^T - S \gamma_3 \tilde{g}_3 \\
C_3^4 &= \cos \gamma_4 E + (1 - \cos \gamma_3) g_4 g_4^T - S \gamma_4 \tilde{g}_4
\end{aligned}
\qquad (C\text{-}22)
$$

where

E is the 3 x 3 identity matrix

$g_k g_k^T$ is an outer product, $k = 1, 2, \ldots 4$

$\tilde{g}_k$ is a 3 x 3 skew symmetric matrix representing the cross product operator g_k x, $k = 1, 2, \ldots 4$

C_{k-1}^k is the transformation from the coordinate frame of body k-1 to that for body k

The results given in Eq. (C-22) were obtained in this analysis by manipulation of the relationship between two coordinate systems in terms of Euler's parameters (for a different approach see Ref. [5]). Consequently, the transformations relating bodies 1, 2, 3, 4 to body 0 can be expressed as (see Fig. 4-1)

$$
\begin{aligned}
C_0^1 &= C_0^1 \\
C_0^2 &= C_1^2\, C_0^1 \\
C_0^3 &= C_0^3 \\
C_0^4 &= C_3^4\, C_0^3
\end{aligned}
\qquad \text{(C-23)}
$$

Note that in Eq. (C-22), it is assumed that the components of g_1 are known in the coordinates of body 0, the components of g_2 are known in the coordinates of body 1, the components of g_3 are known in the coordinates of body 2; and the components of g_4 are known in the coordinates of body 3.

Moreover, the direction cosine matrix relating the coordinate system for body 0 and the coordinate system of the interial frame N for a 3-2-1 sequence of rotations is

$$
C_N^0 = \begin{bmatrix} C_1C_2 & S_1C_2 & -S_2 \\ -S_1C_3 + C_1S_2S_3 & C_1C_3 + S_1S_2S_3 & C_2S_3 \\ C_1S_2C_3 + S_1S_3 & S_1S_2C_3 - C_1S_3 & C_2C_3 \end{bmatrix} \qquad \text{(C-24)}
$$

where $S_i = \sin\theta_i$, $C_i = \cos\theta_i$ (with i = 1, 2, 3)

The attitude angles θ_1, θ_2, θ_3 of body 0 are related to the angular velocity ω_0 according to

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \begin{bmatrix} 1 & \frac{S_1 S_2}{C_2} & \frac{C_1 S_2}{C_2} \\ 0 & C_1 & -S_1 \\ 0 & \frac{S_1}{C_2} & \frac{C_1}{C_2} \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad \text{(C-25)}$$

C-3.2 Evaluation of Elements $b_{\ell m}$ and $a_{\ell m}$

The following conventions are used in evaluating the dyadics $b_{\ell m}$ and $a_{\ell m}$, ℓ, m = 0, 1, ... 4:

(1) A typical element $\Phi_{\lambda\mu}$ (λ, μ = 0, 1, ... 4; $\lambda \neq \mu$) is evaluated by expressing all vectors involved in the coordinate system of body λ (i.e., the basis of the first subscript of $\Phi_{\lambda\mu}$); e.g.,

$$\Phi_{21} = - m \left[D_{12} \cdot D_{21} - D_{12} D_{21} \right]$$

where D_{12} is computed in the coordinates of body 1 (the first subscript of $D_{\lambda\mu}$) and then transformed to the coordinates of body 2 by

$$(D_{12})_2 = C_1^2 \, (D_{12})_1$$

and D_{21} is already known in the coordinates of body 2; consequently

$$[\Phi_{21}]_2 = - m \left[\left(C_1^2 D_{12}\right) \cdot D_{21} - \left(C_1^2 D_{12}\right) D_{21} \right]$$

(2) A typical element b_{0k}, k = 0, 1, ... 4 is evaluated by transforming all $\Phi_{\lambda\mu}$'s from the coordinate system to body λ to that of body 0 according to

$$[\Phi_{\lambda\mu}]_0 = C_\lambda^0 \, [\Phi_{\lambda\mu}]_\lambda \, C_0^\lambda , \quad \text{with } \lambda \neq \mu , \; \lambda, \mu = 0, 1, \ldots 4,$$

e.g.,

$$b_{01} = [\Phi_{01} + \Phi_{02}]_0 + C_1^0\ [\Phi_{11} + \Phi_{12}]_1\ C_0^1 + C_2^0\ [\Phi_{21} + \Phi_{22}]_2\ C_0^2$$

$$+ C_3^0\ [\Phi_{31} + \Phi_{32}]_3\ C_0^3 + C_4^0\ [\Phi_{41} + \Phi_{42}]_2\ C_0^4$$

(3) A typical element b_{ik} ($i, k = 1, 2, \ldots 4$) is evaluated by transforming all $\Phi_{\lambda\mu}$'s involved to the coordinate system of body i, e.g.

$$[b_{13}]_1 = [\Phi_{13} + \Phi_{14}]_1 + C_2^1\ [\Phi_{23} + \Phi_{24}]_2\ C_1^2$$

$$[b_{33}]_3 = [\Phi_{33} + \Phi_{34}]_3 + C_4^3\ [\Phi_{43} + \Phi_{44}]_4\ C_3^4$$

Similarly, the operations $b_{0k} \cdot g_k$ and $g_i \cdot b_{ik} \cdot g_k$ involved in the $a_{\ell m}$'s are evaluated according to

(1) $b_{0k} \cdot g_k = [b_{0k}]_0 \cdot \left(C_k^0\, g_k\right)$

(2) $g_i \cdot b_{ik} \cdot g_k = (g_i)_i \cdot [b_{ik}]_i \cdot \left(C_k^i\, g_k\right)$

These results are summarized in Tables C-1 and C-2.

C-3.3 Evaluation of Forcing Function L

In this section, the terms involved in the evaluation of the 7 x 1 forcing function L (the RHS of Eq. C-19) are given (see Table C-3). The column matrix L can be partioned as

$$L = \begin{pmatrix} L_0 \\ \hline L_R \end{pmatrix}$$

where L_0 is a 3 x 1 vector and is associated with body 0 and L_R is a 4 x 1 vector associated with bodies 1 thru 4.

For notational convenience, the terms $M_{\lambda_j}^{SD}$ are redefined as $\tau_{\lambda j}$ and represent the spring-damper moment acting on body λ at joint j.

C-3.3.1 Viscoelastic Moments and Joints

In this section, the assumed form of the spring-damper interaction moments acting on body λ at joint j is given. Equations representing the interaction moments are given by

$$
\begin{aligned}
\tau_{22} &= - K_2 \gamma_2 - B_2 \dot{\gamma}_2 \\
\tau_{12} &= - \tau_{22} \\
\tau_{11} &= - K_1 \gamma_1 - B_1 \dot{\gamma}_1 \\
\tau_{01} &= - \tau_{11} \\
\tau_{44} &= - K_4 \gamma_4 - B_4 \dot{\gamma}_4 \\
\tau_{34} &= - \tau_{44} \\
\tau_{33} &= - K_3 \gamma_3 - B_3 \dot{\gamma}_3 \\
\tau_{03} &= - \tau_{33}
\end{aligned}
\tag{C-26}
$$

Equation (C-26) implies that the total interaction spring-damper torques acting on bodies 0 thru 4 are

$$\tau_{01} + \tau_{03} = K_1 \gamma_1 + B_1 \dot{\gamma}_1 + K_3 \gamma_3 + B_3 \dot{\gamma}_3$$

$$\tau_{11} + \tau_{12} = - K_1 \gamma_1 - B_1 \dot{\gamma}_1 + K_2 \gamma_2 + B_2 \dot{\gamma}_2$$

$$\tau_{22} = - K_2 \gamma_2 - B_2 \dot{\gamma}_2$$

$$\tau_{33} + \tau_{34} = - K_3 \gamma_3 - B_3 \dot{\gamma}_3 + K_4 \gamma_4 + B_4 \dot{\gamma}_4$$

$$\tau_{44} = - K_4 \gamma_4 - B_4 \dot{\gamma}_4 \qquad \text{(C-27)}$$

C-4 Explicit Equation for ω_0 (The Angular Velocity of the Base Body 0)

In this section, the explicit form for the equation governing ω_0 is given. This can be obtained by partitioning the equation

$$A \dot{\omega} = L$$

and eliminating the relative angular velocities from the equation. In partitioned form, Eq. (C-21) becomes

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left(\begin{array}{c} \dot{\omega}_0 \\ \hline \dot{\omega}_R \end{array}\right) = \left(\begin{array}{c} L_0 \\ \hline L_R \end{array}\right) \qquad \text{(C-28)}$$

where the dimensions of A_{11}, A_{12}, A_{21}, A_{22}, $\dot{\omega}_0$, $\dot{\omega}_R$, L_0, and L_R are 3 x 3, 3 x 4, 4 x 3, 4 x 4, 3 x 1, 4 x 1, 3 x 1, 4 x 1, respectively. Manipulation of Eq. (C-28) yields

$$A_{11} \dot{\omega}_0 + A_{12} \dot{\omega}_R = L_0 \qquad \text{(C-29a)}$$

$$A_{21} \dot{\omega}_0 + A_{22} \dot{\omega}_R = L_R \qquad \text{(C-29b)}$$

Solving Eq. (C-29b) for $\dot{\omega}_R$ and substituting the result into Eq. (C-29a), it follows that

$$\dot{\omega}_0 = [A_{11} - A_{12} A_{22}^{-1} A_{21}]^{-1} L_0 - [A_{11} - A_{12} A_{22}^{-1} A_{21}]^{-1} A_{12} A_{22}^{-1} L_R$$

$$\dot{\omega}_R = A_{22}^{-1} L_R - A_{22}^{-1} A_{21} \dot{\omega}_0 \qquad \text{(C-30)}$$

Equation (C-30) suggests that the RHS of the equation for $\dot{\omega}_0$ could first be evaluated; next, the expresion for $\dot{\omega}_0$ can be used in evaluating the RHS of the equation for $\dot{\omega}_R$; then, ω_0 and ω_R can be obtaining by integrating $\dot{\omega}_0$ and $\dot{\omega}_R$, respectively. The fact that $\dot{\omega}_R$ can be expressed in terms of $\dot{\omega}_0$ is significant when one is interested in arriving at a suboptimal Kalman filter in which the dimension of the state to be estimated is kept as low as possible.

Table C-1. Expression for Dyadics $b_{\ell m}$, ℓ, m = 0, 1, ... 4

Item	Equations
b_{00}	$\sum_{\lambda=0}^{4} \sum_{\mu=0}^{4} \Phi_{\lambda\mu} = [\Phi_{00} + \Phi_{01} + \Phi_{02} + \Phi_{03} + \Phi_{04}]_0$ $+ C_1^0 [\Phi_{10} + \Phi_{11} + \Phi_{12} + \Phi_{13} + \Phi_{14}]_1 \; C_0^1$ $+ C_2^0 [\Phi_{20} + \Phi_{21} + \Phi_{22} + \Phi_{23} + \Phi_{24}]_2 \; C_0^2$ $+ C_3^0 [\Phi_{30} + \Phi_{31} + \Phi_{32} + \Phi_{33} + \Phi_{34}]_3 \; C_0^3$ $+ C_4^0 [\Phi_{40} + \Phi_{41} + \Phi_{42} + \Phi_{43} + \Phi_{44}]_4 \; C_0^4$
b_{01}	$\sum_{\lambda=0}^{4} \sum_{\mu=1}^{4} \Phi_{\lambda\mu} = [\Phi_{01} + \Phi_{02}]_0 + C_1^0 [\Phi_{11} + \Phi_{12}]_1 \, C_0^1$ $+ C_2^0 [\Phi_{21} + \Phi_{22}]_2 \, C_0^2 + C_3^0 [\Phi_{31} + \Phi_{32}]_3 \, C_0^3$ $+ C_4^0 [\Phi_{41}^{0} + \Phi_{42}]_4 \; C_0^4$
b_{02}	$\sum_{\lambda=0}^{4} \Phi_{\lambda 2} = \Phi_{02} + C_1^0 [\Phi_{12}]_1 \, C_0^1 + C_2^0 [\Phi_{22}]_2 \; C_0^2$ $+ C_3^0 [\Phi_{32}]_3 \, C_0^3 + C_4^0 [\Phi_{42}]_4 \; C_0^4$

Table C-1. Expression for Dyadics $b_{\ell m}$, ℓ, m = 0, 1, ... 4 (contd)

Item	Equations
b_{03}	$\sum_{\lambda=0}^{4} \sum_{\mu=3}^{4} \Phi_{\lambda\mu} = [\Phi_{03} + \Phi_{04}]_0 + C_1^0 [\Phi_{13} + \Phi_{14}]_1 C_0^1 + C_2^0 [\Phi_{23} + \Phi_{24}]_2 C_0^2 + C_3^0 [\Phi_{33} + \Phi_{34}]_3 C_0^3 + C_4^0 [\Phi_{43} + \Phi_{44}]_4 C_0^4$
b_{04}	$\sum_{\lambda=0}^{4} \Phi_{\lambda 4} = [\Phi_{04}]_0 + C_1^0 [\Phi_{14}]_1 C_0^1 + C_2^0 [\Phi_{24}]_2 C_0^2 + C_3^0 [\Phi_{34}]_3 C_0^3 + C_4^0 [\Phi_{44}]_4 C_0^4$
b_{11}	$\sum_{\lambda=1}^{2} \sum_{\mu=1}^{2} \Phi_{\lambda\mu} = [\Phi_{11} + \Phi_{12}]_1 + C_2^1 [\Phi_{21} + \Phi_{22}]_2 C_1^2$
b_{12}	$\sum_{\lambda=1}^{2} \Phi_{\lambda 2} = [\Phi_{12}]_1 + C_2^1 [\Phi_{22}]_2 C_1^2$
b_{13}	$\sum_{\mu=1}^{2} \sum_{\mu=3}^{4} \Phi_{\lambda\mu} = [\Phi_{13} + \Phi_{14}]_1 + C_2^1 [\Phi_{23} + \Phi_{24}]_2 C_1^2$
b_{14}	$\sum_{\lambda=1}^{2} \Phi_{\lambda 4} = [\Phi_{14}]_1 + C_2^1 [\Phi_{24}]_2 C_1^2$

Table C-1. Expression for Dyadics $b_{\ell m}$, ℓ, m = 0, 1, ... 4 (contd)

Item	Equations
b_{22}	$[\Phi_{22}]_2$
b_{23}	$\sum_{\mu=3}^{4} \Phi_{2\mu} = [\Phi_{23} + \Phi_{24}]_2$
b_{24}	$[\Phi_{24}]_2$
b_{33}	$\sum_{\lambda=3}^{4} \sum_{\mu=3}^{4} \Phi_{\lambda\mu} = [\Phi_{33} + \Phi_{34}]_3 + C_4^3 \, [\Phi_{43} + \Phi_{44}]_4 \, C_3^4$
b_{34}	$\sum_{\lambda=3}^{4} \Phi_{\lambda 4} = [\Phi_{34}]_3 + C_4^3 \, [\Phi_{44}]_4 \, C_3^4$
b_{44}	$[\Phi_{44}]_4$
$b_{\ell m}$	$(b_{m\ell})^T$, ℓ, m = 0, 1, ... 4; $\ell \neq m$

Table C-2. Expressions for Elements $a_{\ell m}$ (ℓ, m = 0, 1, ... 4)

Item	Equations
$a_{00} = b_{00}$	$[\Phi_{00} + \Phi_{01} + \Phi_{02} + \Phi_{03} + \Phi_{04}]_0$ $+ C_1^0 \; [\Phi_{10} + \Phi_{11} + \Phi_{12} + \Phi_{13} + \Phi_{14}]_1 \; C_0^1$ $+ C_2^0 \; [\Phi_{20} + \Phi_{21} + \Phi_{22} + \Phi_{23} + \Phi_{24}]_2 \; C_0^2$ $+ C_3^0 \; [\Phi_{30} + \Phi_{31} + \Phi_{32} + \Phi_{33} + \Phi_{34}]_3 \; C_0^3$ $+ C_4^0 \; [\Phi_{40} + \Phi_{41} + \Phi_{42} + \Phi_{43} + \Phi_{44}]_4 \; C_0^4$
a_{01}	$\Big[[\Phi_{01} + \Phi_{02}]_0 + C_1^0 \; [\Phi_{11} + \Phi_{12}]_1 \; C_0^1 + C_2^0 \; [\Phi_{21} + \Phi_{22}]_2 \; C_0^2$ $+ C_3^0 \; [\Phi_{31} + \Phi_{32}]_3 \; C_0^3 + C_4^0 \; [\Phi_{41} + \Phi_{42}]_4 \; C_0^4 \Big] \cdot C_1^0 \; (g_1)_1$
a_{02}	$\Big[[\Phi_{02}]_0 + C_1^0 \; [\Phi_{12}]_1 \; C_0^1 + C_2^0 \; [\Phi_{22}]_2 \; C_0^2 + C_3^0 \; [\Phi_{32}]_3 \; C_0^3$ $+ C_4^0 \; [\Phi_{42}]_4 \; C_0^4 \Big] \cdot C_2^0 \; (g_2)_2$
a_{03}	$\Big[[\Phi_{03} + \Phi_{04}]_0 + C_1^0 \; [\Phi_{13} + \Phi_{14}]_1 \; C_0^1 + C_2^0 \; [\Phi_{23} + \Phi_{24}]_2 \; C_0^2$ $+ C_3^0 \; [\Phi_{33} + \Phi_{34}]_3 \; C_0^3 + C_4^0 \; [\Phi_{43} + \Phi_{44}]_4 \; C_0^4 \Big] \cdot C_3^0 \; (g_3)_3$
a_{04}	$\Big[[\Phi_{04}]_0 + C_1^0 \; [\Phi_{14}]_1 \; C_0^1 + C_2^0 \; [\Phi_{24}]_2 \; C_0^2 + C_3^0 \; [\Phi_{34}]_3 \; C_0^3$ $+ C_4^0 \; [\Phi_{44}]_4 \; C_0^4 \Big] \cdot C_4^0 \; (g_4)_4$

Table C-2. Expressions for Elements $a_{\ell m}$ (ℓ, m = 0, 1, ... 4) (contd)

Item	Equations
a_{11}	$(g_1)_1 \cdot \left[[\Phi_{11} + \Phi_{12}]_1 + C_2^1 [\Phi_{21} + \Phi_{22}]_2 C_1^2 \right] \cdot (g_1)_1$
a_{12}	$(g_1)_1 \cdot \left[[\Phi_{12}]_1 + C_2^1 [\Phi_{22}]_2 C_1^2 \right] \cdot C_2^1 (g_2)_2$
a_{13}	$(g_1)_1 \cdot \left[[\Phi_{13} + \Phi_{14}]_1 + C_2^1 [\Phi_{23} + \Phi_{24}]_2 C_1^2 \right] \cdot C_3^1 (g_3)_3$
a_{14}	$(g_1)_1 \cdot \left[[\Phi_{14}]_1 + C_2^1 [\Phi_{24}]_2 C_1^2 \right] \cdot C_4^1 (g_4)_4$
a_{22}	$(g_2)_2 \cdot [\Phi_{22}]_2 \cdot (g_2)_2$
a_{23}	$(g_2)_2 \cdot [\Phi_{23} + \Phi_{24}]_2 \cdot C_3^2 (g_3)_3$
a_{24}	$(g_2)_2 \cdot [\Phi_{24}]_2 \cdot C_4^2 (g_4)_4$
a_{33}	$(g_3)_3 \cdot \left[[\Phi_{33} + \Phi_{34}]_3 + C_4^3 [\Phi_{43} + \Phi_{44}]_4 C_3^4 \right] \cdot (g_3)_3$
a_{34}	$(g_3)_3 \cdot \left[[\Phi_{34}]_3 + C_4^3 [\Phi_{44}]_4 C_3^4 \right] \cdot C_4^3 (g_4)_4$
a_{44}	$(g_4)_4 \cdot [\Phi_{44}]_4 \cdot (g_4)_4$
a_{k0}	$(a_{0k})^T$, k = 1, 2, ... 4
$a_{\ell m}$	$(a_{m\ell})^T$, i, k = 1, 2, ... 4, i ≠ k

Table C-3. Evaluation of Forcing Function L (Appearing in $A\,\dot{\omega} = L$)

Item	Equations
E_0	$M_0 + D_0 \times F_0 + \sum_{\mu \neq \lambda} D_{0\mu} \times C^0_\mu \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu 0})\right]$ $- \omega_0 \times \Phi_{00} \cdot \omega_0 + K_1 \gamma_1 + B_1 \dot{\gamma}_1 + K_3 \gamma_3 + B_3 \dot{\gamma}_3$
E_1	$M_1 + D_1 \times F_1 + \sum_{\mu \neq \lambda} D_{1\mu} \times C^1_\mu \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu 1})\right]$ $- \left[\left(C^1_0\, \omega_0 + \dot{\gamma}_1\, g_1\right) \times [\Phi_{11}]_1 \cdot \left(C^1_0\, \omega_0 + \dot{\gamma}_1\, g_1\right)\right]$ $+ K_2 \gamma_2 + B_2 \dot{\gamma}_2 - K_1 \gamma_1 - B_1 \dot{\gamma}_1$
E_2	$M_2 + D_2 \times F_2 + \sum_{\mu \neq \lambda} D_{2\mu} \times C^2_\mu \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu 2})\right]$ $- \left[C^2_0\, \omega_0 + C^2_1\, \dot{\gamma}_1\, g_1 + \dot{\gamma}_2\, g_2\right] \times [\Phi_{22}]_2 \cdot \left[C^2_0\, \omega_0 + C^2_1 \dot{\gamma}_1 g_1 + \dot{\gamma}_2 g_2\right]$ $- K_2 \gamma_2 - B_2 \dot{\gamma}_2$
E_3	$M_3 + D_3 \times F_3 + \sum_{\mu \neq \lambda} D_{3\mu} \times C^3_\mu \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu 3})\right]$ $- \left[C^3_0\, \omega_0 + \dot{\gamma}_3\, g_3\right] \times [\Phi_{33}]_3 \cdot \left[C^3_0\, \omega_0 + \dot{\gamma}_3\, g_3\right]$ $+ K_4 \gamma_4 + B_4 \dot{\gamma}_4 - K_3 \gamma_3 - B_3 \dot{\gamma}_3$

Table C-3. Evaluation of Forcing Function L
(Appearing in $A \dot{\omega} = L$) (contd)

Item	Equations
E_4	$M_4 + D_4 \times F_4 + \sum_{\mu \neq \lambda} D_{4\mu} \times C_\mu^4 \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu 4})\right]$ $- \left[C_0^4 \omega_0 + C_3^4 \dot{\gamma}_3 g_3 + \dot{\gamma}_4 g_4\right] \times [\Phi_{44}]_4 \cdot \left[C_0^4 \omega_0 + C_3^4 \dot{\gamma}_3 g_3\right.$ $\left. + \dot{\gamma}_4 g_4\right] - K_4 \gamma_4 - B_4 \dot{\gamma}_4$
$\dot{g}_1$	$C_0^1 \omega_0 \times g_1$
$\dot{g}_2$	$\left(C_0^2 \omega_0 + C_1^2 \dot{\gamma}_1 g_1\right) \times g_2$
$\dot{g}_3$	$C_0^3 \omega_0 \times g_3$
$\dot{g}_4$	$\left(C_0^4 \omega_0 + C_3^4 \dot{\gamma}_3 g_3\right) \times g_4$
L_0	$\sum_{\lambda=0}^{4} E_\lambda - \sum_{k=1}^{4} (b_{0k} \cdot \dot{g}_k) \dot{\gamma}_k = E_0 + C_1^0 E_1 + C_2^0 E_2 + C_3^0 E_3$ $+ C_4^0 E_4 - \left(b_{01} \cdot C_1^0 \dot{g}_1\right) \dot{\gamma}_1 - \left(b_{02} \cdot C_2^0 \dot{g}_2\right) \dot{\gamma}_2 - \left(b_{03} \cdot C_3^0 \dot{g}_3\right) \dot{\gamma}_3$ $- \left(b_{04} \cdot C_4^0 \dot{g}_4\right) \dot{\gamma}_4$

Table C-3. Evaluation of Forcing Function L (Appearing in $A\ \dot{\omega} = L$) (contd)

Item	Equations
L_1	$g_1 \cdot \left(E_1 + C_2^1 E_2\right)$ $- g_1 \cdot [b_{11} \cdot \dot{g}_1 \dot{\gamma}_1 + b_{12} \cdot \dot{g}_2 \dot{\gamma}_2 + b_{13} \cdot \dot{g}_3 \dot{\gamma}_3 + b_{14} \cdot \dot{g}_4 \dot{\gamma}_4]$
L_2	$g_2 \cdot E_2 - g_2 \cdot [b_{21} \cdot \dot{g}_1 \dot{\gamma}_1 + b_{22} \cdot \dot{g}_2 \dot{\gamma}_2 + b_{23} \cdot \dot{g}_3 \dot{\gamma}_3$ $+ b_{24} \cdot \dot{g}_4 \dot{\gamma}_4]$
L_3	$g_3 \cdot \left(E_3 + C_4^3 E_4\right) - g_3 \cdot [b_{31} \cdot \dot{g}_1 \dot{\gamma}_1 + b_{32} \cdot \dot{g}_2 \dot{\gamma}_2$ $+ b_{33} \cdot \dot{g}_3 \dot{\gamma}_3 + b_{34} \cdot \dot{g}_4 \dot{\gamma}_4]$
L_4	$g_4 \cdot E_4 - g_4 \cdot [b_{41} \cdot \dot{g}_1 \dot{\gamma}_1 + b_{42} \cdot \dot{g}_2 \dot{\gamma}_2 + b_{43} \cdot \dot{g}_3 \dot{\gamma}_3$ $+ b_{44} \cdot \dot{g}_4 \dot{\gamma}_4]$

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