The University of Connecticut
SCHOOL OF ENGINEERING
Storrs, Connecticut 06268

[redacted]

Department of Electrical Engineering

POLE AND ZERO PLACEMENT IN MULTIVARIABLE CONTROL SYSTEMS

by

Banavar Sridhar

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Pages II + III

TABLE OF CONTENTS

Table of Contents (Continued)

LIST OF TABLES

LIST OF ILLUSTRATIONS

I. INTRODUCTION

1.1 Introduction and Historical Background

Multivariable Systems are a characteristic feature of modern industrial and production processes. A Multivariable System, as the name suggests, has several inputs and outputs. The object of design is to control these interdependent inputs and outputs to obtain the desired output from the system.

One design approach is to control the system in such a way that a change in one input variable affects only one output variable. This is the problem of decoupling or noninteraction. Early attempts to solve the decoupling problem by Boksenbom and Hood [1], Freeman [2], and Kavanagh [3] made use of the transfer function approach. These methods made arbitrary assumptions and ran into problems of stability and realizability. There was a need for understanding of system structure, and establishing a compatibility of the design approach with computer methods, both of which are dependent on the state formulation. Morgan [4] formulated the decoupling problem in state space and a complete solution to the design of noninteractive systems was provided by combined efforts of Falb and Wolovich [5], Gilbert [6], and Wonham and Morse [7]. Morse [8] has recently reviewed the status of noninteracting systems.

Approaches to the design of interactive Multivariable Systems were first treated as a logical application of optimal control theory whereby, with a suitable input, the weighting matrices in a cost function are used to achieve a satisfactory solution. The work of Ellert and Merriam [9], and Tyler [10] among others is to be noted.

This method has been found wanting because of problems associated with the solution of the Riccati equation for large systems, the requirement of full state information, and difficulty of summarizing in a quadratic performance criterion the desired response characteristics. No systematic way is known to date for changing the weighting matrices in the multivariable system design problem. Some recent efforts at facing this problem have been reported by Murphy and Narendra [11], Rediess and Whitaker [12], and others. Notably, Ferguson and Rekasius [13] have dealt with the problem of incomplete state measurements.

In an effort to circumvent these difficulties Rosenbrock [14] suggested modal control as a design tool, which is to say, he proposed changing the eigenvalues of the system matrix to achieve the desired control objective. Wonham [15] showed that for a controllable system the eigenvalues of the closed loop system can be assigned arbitrarily using state feedback. Simon and Mitter [16] proposed a theory of modal control.

Much of the work which relates to the concept of modal control has been termed "pole placement". Disregarding work on decoupling, pole placement methods for multivariable systems have been treated in two categories: 1) methods using constant gain feedback 2) methods using dynamic compensators to achieve pole placement.

Retallack and MacFarlane [17] have derived a state-feedback pole-shifting algorithm using the Hsu-Chen theorem [18]. Chidambara [19] has shown that it is possible to solve the pole assignment problem with state feedback for a time-invariant linear system of order (n) having (m) inputs through the solution of the same problem for a similar system of order (n-m-r) [with $r \leq m$] having r_1 [where $r_1 \leq r$] number of inputs. Davison [20] has studied the relationship between controllability, pole assignment and incomplete state feedback.

Brasch and Pearson [21] have shown that for a controllable, observable plant a compensator of order $\beta = \min(\nu_c-1, \nu_0-1)$ is sufficient to obtain arbitrary pole placement in the system consisting of the plant and compensator in cascade feedback configuration. Here $\nu_c(\nu_0)$ is the controllability (observability) index of the plant. Similar results are obtained by Chen and Hsu [22] using a transfer function approach. Ahmari and Vacroux [23] have generalized the theory of pole assignment to include the case in which a controllable observer plant is augmented by a compensator of fixed dimensionality.

Although pole locations are an important element in the specification of satisfactory control, they are by no means sufficient in themselves. The sensitivity of the system to disturbances and to parameter changes is also important, as is the effect of transducer or actuator failure. Overshoot, and the extent of interaction, can also be significant in some applications. Among these various factors which affect the design of the control system the problem of the zeros, i.e. the numerator polynomial roots of the transfer function, is considered in greater detail in a later chapter.

1.2 Problem Statement

The general problem of designing Multivariable Systems can be approached from different design objectives such as decoupling, exact model matching, disturbance rejection and pole placement. The problem investigated in this study may be described as follows. Consider the linear time invariant multivariable deterministic continuous system described by the equation:

$$\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\underline{u} \tag{1.1}$$

$$\underline{y} = \hat{C}\underline{x} \tag{1.2}$$

The object is to use the output feedback $\underline{u} = K\underline{y}$ to place the eigenvalues of the closed loop system

$$\dot{\underline{x}} = (\hat{A} + \hat{B}K\hat{C})\underline{x}$$

in a desired location predetermined by the designer. The case of pole placement with state feedback is considered as a special case of output feedback where $\hat{C} = I$, the identity matrix. In this thesis the design technique is focused on using constant gain feedback matrices of unity rank. The method is applied to the design of a complex system described by the equations (1.1) and (1.2). Further, the effect of feedback on the zeros of the closed loop system, i.e. on the roots of the numerator polynomial $\hat{C}$ Adj $(SI-\hat{A} + \hat{B}K\hat{C})^{-1}B$ is investigated along with the problem of zero-placement.

1.3 Outline of the Thesis

Chapter 2 outlines a design technique to place the poles of the closed-loop system using output feedback. The method results from an alternate derivation of Davison's theorem on controllability, observability, and pole placement using output feedback. Several examples are given to illustrate all the features of the method.

Chapter 3 considers the question of approximate pole placement when it is not possible to place all the poles using output feedback. This problem is approached in two different ways - (i) using the psuedo-inverse to get an approximate solution to a set of inconsistent equations. (ii) Using gradient method to obtain a least square solution to the set of equations.

In Chapter 4 the pole placement design is applied to a complex system. The system chosen is the Boeing-Vertol CH-46 Helicopter. The system is open-loop unstable and has eight states, two inputs, and four outputs. A controller for the Helicopter stabilization is developed using output feedback.

Chapter 5 summarizes some recent contributions to the problem of zero-placement and examines the advantages and limitations of using a unity rank feedback gain matrix for pole zero placement.

Future developments, extensions, and topics for additional research are presented in Chapter 6.

II. POLE PLACEMENT USING OUTPUT FEEDBACK

2.1 Introduction

The design of linear multivariable control systems using output feedback has attracted the attention of several authors [17, 20-22, 24, 25]. There are two ways of approaching this problem. The first method consists of estimating the states of the system using an observer and using these estimated states in the subsequent design. In the second approach, either static or dynamic feedback of the output is used directly in the control problem and this view is adopted here.

Consider a linear time-invariant multivariable system

$$\underline{\dot{x}} = \hat{A}\underline{x} + \hat{B}\underline{u} \tag{2.1}$$

$$\underline{y} = \hat{C}\underline{x} \tag{2.2}$$

where $\underline{x}$ is an n vector of states, $\underline{u}$ is an m vector of inputs and $\underline{y}$ is a p vector of outputs. It is well-known that the problem of pole assignment using state feedback is equivalent to the controllability of the pair $(\hat{A}, \hat{B})$ [15]. Here it is shown as a theorem that for a controllable, observable system $[\hat{A}, \hat{B}, \hat{C}]$ with $\hat{B}$ and $\hat{C}$ full rank max(m,p) poles of the system can be assigned arbitrarily close to desired locations using constant gain output feedback. This theorem, though similar to Davison and Chatterjee [26], leads to a design approach by virtue of the method of derivation. In some cases, more than max(m,p) poles can be assigned arbitrarily. Also, certain pole configurations which cannot be attained by Davison's method can be attained by this method. These advantages are illustrated by means of examples. Assuming the system is output stabilizable, a least square design technique is outlined to approximate the desired pole locations when it is not possible to place all the poles.

2.2 Theorem On Pole-Placement:

Given the system (2.1 and 2.2) with Rank $\hat{B}$ = m $\leq$ n and Rank $\hat{C}$ = p $\leq$ n, then a linear feedback of the output $\underline{u} = K\underline{y}$, where K is a (mxp) constant gain matrix, can always be found such that max(m,p) eigenvalues of the closed loop system are arbitrarily close to pre-assigned (complex eigenvalues occurring in conjugate pairs) values.

Proof

Let $(\lambda_1, \lambda_2, ---, \lambda_n)$ and $(\rho_1, \rho_2, ---, \rho_n)$ be the eigenvalues of the open-loop and closed-loop system respectively.

We have

$$\text{open loop characteristic polynomial} = |sI-\hat{A}| = (s-\lambda_1)(s-\lambda_2)---(s-\lambda_n) \tag{2.3}$$

$$\text{and closed loop characteristic polynomial} = |sI-\hat{A}+\hat{B}K\hat{C}| = (s-\rho_1)(s-\rho_2)---(s-\rho_n) \tag{2.4}$$

Then

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = \det\,(I + \hat{B}K\hat{C}\,(sI-\hat{A})^{-1}]. \tag{2.5}$$

Choosing $K = \underline{f}\,\underline{d}^T$ where $\underline{f}$ is a mx1 (column) vector and $\underline{d}^T$ is a 1xp (row) vector, and using the identity det[I + MN] = det [I + NM], equation (2.5) becomes

$$\begin{aligned}\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} &= 1 + \underline{d}^T\hat{C}\,(sI-\hat{A})^{-1}\,\hat{B}\underline{f} \\ &= 1 + \underline{d}^T\hat{C}T\,(sI-T^{-1}\hat{A}T)^{-1}\,T^{-1}\hat{B}\underline{f} \\ &= 1 + \underline{d}^T C\,(sI-A)^{-1}\,B\underline{f}\end{aligned} \tag{2.6}$$

where $C = \hat{C}T$, $A = T^{-1}\hat{A}T$, $B = T^{-1}\hat{B}$ and T is a nxn nonsingular matrix. For clarity, the theorem is initially proved for the case of distinct eigenvalues of $\hat{A}$ and the multiple eigenvalues of $\hat{A}$ are considered in the latter half of the proof.

<u>Distinct Eigenvalues</u>

In this case equation (2.6) gives

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = 1 + \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \qquad (2.7)$$

The value of α_i depends on the closed loop eigenvalues $(\rho_1, ---, \rho_n)$. From (2.6) amd (2.7)

$$\underline{d}^T C \ (sI-A)^{-1} \ B\underline{f} = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \qquad (2.8)$$

Choosing T as a modal matrix equation (2.8) becomes

$$\underline{d}^T C \ (sI-\Lambda)^{-1} \ B\underline{f} = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \qquad (2.9)$$

where Λ = diag. $(\lambda_1, \lambda_2, ---, \lambda_n)$.

Let $\underline{c}^i$ be the i^{th} column of C and $\underline{b}_i$ be the i^{th} row of B. Then,

$$a_i = \underline{d}^T \underline{c}^i \underline{b}_i \underline{f} \qquad i = 1, 2, ---, n. \qquad (2.10)$$

<u>Case (i)</u>

Let p>m i.e. more outputs than inputs. Choose f_i such that $\underline{b}_i\underline{f} = \delta_i \neq 0$ $i = 1,2,---, n$. This can always be done since $\underline{b}_i \neq 0$, for controllability.

Hence, $\underline{d}^T\underline{c}^i = \alpha_i/\delta_i$ $\qquad i = 1, ---, n. \qquad (2.11)$

This gives $C^T\underline{d} = \underline{\alpha}$ $\qquad (2.12)$

where

$$\underline{\alpha} = \text{col}\ [\alpha_1/\delta_1, \alpha_2/\delta_2, ---, \alpha_n/\delta_n].$$

Now, let C_p be the matrix made of the p independent rows of C^T and $\underline{\alpha}_p$ the corresponding subset of $\underline{\alpha}$. Then

$$\underline{d} = C_p^{-1} \underline{\alpha}_p \tag{2.13}$$

Thus $(d_1, d_2, ---, d_p)$ can be chosen corresponding to the p desired pole locations. Once this is done the remaining (n-p) poles are fixed automatically.

Case (ii)

Let m>p i.e. more inputs than outputs

Choose d_i such that $\underline{d}^T \underline{c}^i = \gamma_i \neq 0 \quad i = 1, 2, ---, n.$

This can always be done since $\underline{c}^i \neq 0$, for observability.

Hence, $\underline{b}_i \underline{f} = \alpha_i / \gamma_i \qquad i = 1, ---, n.$ (2.14)

This gives $B\underline{f} = \underline{\alpha}$ (2.15)

where $\underline{\alpha} = \text{col } [\alpha_1/\delta_1, \alpha_2/\delta_2, ---, \alpha_n/\delta_n]$.

Since the rank of B is m, there are m independent rows of B, B_m, such that

$$B_m \underline{f} = \underline{\alpha}_m$$

where $\underline{\alpha}_m$ is the corresponding subset of $\underline{\alpha}$.

$$\underline{f} = B_m^{-1} \underline{\alpha}_m \tag{2.16}$$

Thus $(f_1, f_2, ---, f_m)$ can be chosen corresponding to the m desired pole locations and the remaining (n-m) poles are located automatically. From case (i) and case (ii) it is evident that at least max (m,p) poles of the system can be assigned arbitrarily.

Multiple Eigenvalues

Let the eigenvalues of matrix $\hat{A}$ be $\lambda_1, \lambda_2, ---, \lambda_\omega$ with multiplicity $n_1, n_2, ---, n_\omega$ respectively. Choose T such that $A = T^{-1}\hat{A}T$ has the Jordan canonical form with ω blocks of respective sizes $n_1, n_2, ---, n_\omega$ and $\lambda_1, \lambda_2, ---, \lambda_\omega$ the corresponding eigenvalues.

Now, we have

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = \frac{(s-\rho_1)(s-\rho_2)\text{ --- }(s-\rho_n)}{(s-\lambda_1)^{n_1}(s-\lambda_2)^{n_2}\text{---}(s-\lambda_\omega)^{n_\omega}} \tag{2.17}$$

where $\sum_{i=1}^{\omega} n_i = n.$

Equation (2.17) can be rewritten as

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = 1 + \sum_{i=1}^{n_1} \frac{\alpha_i^1}{(s-\lambda_1)^i} + \text{---} + \sum_{i=1}^{n_\omega} \frac{\alpha_i^\omega}{(s-\lambda_\omega)^i} \tag{2.18}$$

The value of α_i^j ($i=1$, ---, n_j, $j = 1$, --- ω) depends on the closed loop poles (ρ_1, ρ_2, ---, ρ_n).

From equations (2.6) and (2.18), we get

$$\underline{d}^T C(sI-A)^{-1} B\underline{f} = \sum_{i=1}^{n_1} \frac{\alpha_i^1}{(s-\lambda_1)^i} + \text{---} + \sum_{i=1}^{n_\omega} \frac{\alpha_i^\omega}{(s-\lambda_\omega)^i} \tag{2.19}$$

$(sI-A)^{-1}$ has the quasi-diagonal form diag $[J_1, J_2, \text{---}, J_\omega]$ where J_i is a $n_i \times n_i$ matrix of the form

$$\begin{bmatrix} \frac{1}{(s-\lambda_i)} & \frac{1}{(s-\lambda_i)^2} & \text{---} & \frac{1}{(s-\lambda_i)^{n_i}} \\ 0 & \frac{1}{(s-\lambda_i)} & \text{---} & \frac{1}{(s-\lambda_i)^{n_i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & & \frac{1}{(s-\lambda_i)} & \frac{1}{(s-\lambda_i)^2} \\ 0 & 0 & \text{---} & \frac{1}{(s-\lambda_i)} \end{bmatrix} \tag{2.20}$$

Let $C = [C^1, C^2, ---, C^\omega]$ and $B = [B^1, B^2, --- B^\omega]^T$ where C^j is a pxn_j matrix and B^j is a $n_j xm$ matrix. Then it can be easily seen that

$$\underline{d}^T C^j J_j B^j \underline{f} = \sum_{i=1}^{n_j} \frac{\alpha_i^j}{(s-\lambda_j)^i} \qquad j = 1,2,---,\omega \tag{2.21}$$

Further it can be shown that

$$\alpha_1^j = \underline{d}^T [C_1^j B_1^j + C_2^j B_2^j + --- + C_{n_j}^j B_{n_j}^j] \underline{f}$$

$$\alpha_c^j = \underline{d}^T [C_1^j B_2^j + --- + C_{n_j-1}^j B_{n_j}^j] \underline{f} \tag{2.22}$$

$$\vdots \qquad\qquad j = 1, 2, ---, \omega$$

$$\alpha_{n_j}^j = \underline{d}^T C_1^j B_{n_j}^j \underline{f}.$$

Where C_i^j is the i^{th} column of C^j and B_i^j is the i^{th} row of B^j.

In the matrix form equation (2.22) can be written as

$$
\underline{d}^T
\begin{bmatrix}
C_1^1 B_1^1 + C_2^1 B_2^1 + \cdots + C_{n_1}^1 B_{n_1}^1 \\
C_1^1 B_2^1 + C_2^1 B_3^1 + \cdots + C_{n_1-1}^1 B_{n_1} \\
\vdots \\
C_1^1 B_{n_1-1}^1 + C_2^1 B_{n_1} \\
C_1^1 B_{n_1} \\
\hdashline
\vdots \\
\hdashline
C_1^\omega B_1^\omega + C_2^\omega B_2^\omega + \cdots + C_{n_\omega}^\omega B_{n_\omega}^\omega \\
C_1^\omega B_2^\omega + C_2^\omega B_3^\omega + \cdots + C_{n_\omega-1}^\omega B_{n_\omega}^\omega \\
\vdots \\
C_1^\omega B_{n_\omega-1}^\omega + C_2^\omega B_{n_\omega}^\omega \\
C_1^\omega B_{n_\omega}^\omega
\end{bmatrix}
\underline{f} =
\begin{bmatrix}
\alpha_1^1 \\
\alpha_2^1 \\
\vdots \\
\alpha_{n_1-1}^1 \\
\alpha_{n_1}^1 \\
\hdashline
\vdots \\
\hdashline
\alpha_1^\omega \\
\alpha_2^\omega \\
\vdots \\
\alpha_{n_\omega-1}^\omega \\
\alpha_{n_\omega}^\omega
\end{bmatrix}
\qquad (2.23)
$$

<u>Case (i)</u>

Let $p>m$ i.e. more outputs than inputs and $B_i^j\underline{f} = \delta_i^j$, $i=1,\cdots, n_j$ and $j = 1, \cdots, \omega$. For controllability, every row of B corresponding to the last row of each Jordan block of A is linearly independent [27]

i.e., $B^j_{n_j} \neq 0$, $j = 1, 2, --- \omega$, are linearly independent.
Now, we can choose $(f_1, f_2, ---, f_m)$ such that

$$B^j_{n_j}\underline{f} = \delta^j_{n_j} \neq 0, \quad j = 1, ---, \omega \qquad (2.24)$$

Substituting this in equation (2.23) we get

$$\begin{bmatrix} \delta^1_1 C^{1^T}_1 + \delta^1_2 C^{1^T}_2 + --- \quad \delta^1_{n_1} C^{1^T}_{n_1} \\ \delta^1_2 C^{1^T}_1 + \delta^1_3 C^{1^T}_2 + --- + \delta^1_{n_1} C^{1^T}_{n_1-1} \\ \vdots \\ \delta^1_{n_1} C^{1^T}_1 \\ \hdashline \vdots \\ \hdashline \delta^\omega_1 C^{\omega^T}_1 + --- \quad --- + \quad \delta^\omega_{n_\omega} C^{\omega^T}_{n_\omega} \\ \delta^\omega_2 C^{\omega^T}_1 + ------- + \delta^\omega_{n_\omega} C^{\omega^T}_{n_\omega-1} \\ \vdots \\ \delta^\omega_{n_\omega} C^{\omega^T}_{n_\omega} \end{bmatrix} \underline{d} = \underline{\alpha}$$

where $\underline{\alpha} = \text{col}\ [\alpha^1_1, \alpha^1_2 --- \alpha^1_{n_1}, \alpha^2_1 --- \alpha^2_{n_2}, ---, ---, \alpha^\omega_{n_\omega}]$.

Define a quasi-diagonal matrix M,

$$M \triangleq \text{diag}\ [M_1, M_2, ---, M_\omega]$$

where M_j is given by

$$M_j = \begin{bmatrix} \delta^j_1 & \delta^j_2 & \text{------} & \delta^j_{n_j} \\ \delta^j_2 & \delta^j_3 & \delta^j_{n_j} & 0 \\ \vdots & & & \vdots \\ \delta^j_{n_j} & 0 & - - - - & 0 \end{bmatrix}$$

we have $MC^T\underline{d} = \underline{\alpha}$ (2.25)

M is a nxn non-singular matrix since $\det M = \prod_{i=1}^{\omega} (\delta^i_{n_i})^{n_i} \neq 0$ by (2.24).

Hence, Rank M=n and Rank $MC^T = p$. Let C_p be the p independent rows of MC^T and let $\underline{\alpha}_p$ be the corresponding subset of $\underline{\alpha}$. This gives

$$C_p\underline{d} = \underline{\alpha}_p$$

or

$$\underline{d} = C_p^{-1}\underline{\alpha}_p \tag{2.26}$$

Equation (2.26) is similar to equation (2.13) and the rest of the proof follows as in the Case (i) of distinct eigenvalues.

Case (ii) : m>p i.e. more inputs than outputs

Let $\underline{d}^T C^j_i = \delta^j_i \quad i = 1,2,\text{---},n_j, \quad j = 1,2,\text{---}, \omega$

For obervability, every column of C corresponding to the first column of each Jordan block of A is linearly independent [27] i.e.

$C^j_1 \neq 0$, $j = 1,2,\text{---}\ \omega$, are linearly independent.

Now, we can choose $(d_1, d_2, \text{---}, d_p)$ such that $\underline{d}^T C^j_1 = \delta^j_1 \neq 0$,

$j = 1, 2, \text{---}, \omega$ (2.27)

Substituting this in equation (2.23) and defining a quasi-diagonal matrix N,

$$N \triangleq \text{diag}[N_1, N_2, ---, N_\omega]$$

where N_j is given by

$$\begin{bmatrix} \delta_1^j & \delta_2^j & --- & \delta_{n_j}^j \\ 0 & \delta_1^j & & \delta_{n_j-1}^j \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & \delta_1^j \end{bmatrix}$$

It is seen that

$$N B \underline{f} = \underline{\alpha}.$$

N is a nxn non-singular matrix since det N = $\prod_{i=1}^{\omega} (\delta_1^j)^{n_j} \neq 0$ by (2.27). Hence Rank NB=m and let B_m be the m independent rows of NB and let $\underline{\alpha}_m$ be the corresponding subset of $\underline{\alpha}$. This gives

$$B_m \underline{f} = \underline{\alpha}_m$$

or (2.28)

$$\underline{f} = B_m^{-1} \underline{\alpha}_m.$$

Equation (2.28) is similar to equation (2.16) and the rest of the proof follows as in the case (ii) of distinct eigenvalues. This completes the proof in the case of multiple eigenvalues.

2.3 Special Case of State Feedback

The proof of the Theorem 2.2 provides an easy method to verify Wonham's theorem on pole placement. With state feedback $\underline{u} = K\underline{x}$, we have closed loop characteristic polynomial = $|sI-\hat{A}+\hat{B}K|$.

Equation (2.5) and (2.9) reduce to

$$\frac{|sI-\hat{A}+\hat{B}K|}{|sI-\hat{A}|} = \det[I + \hat{B}K\ (sI-\hat{A})^{-1}] \tag{2.29}$$

and

$$\underline{d}^T(sI-\Lambda)^{-1}B\underline{f} = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \tag{2.30}$$

respectively.

From (30), $d_i\underline{b}_i\underline{f} = \alpha_i \qquad i = 1, 2, ---, n$ (2.31)

Now on the assumption of controllability, $\underline{b}_i \neq 0$, and choosing $\underline{f}$ such that $\underline{b}_i\underline{f} \neq 0$, we have

$$d_i = \alpha_i/\underline{b}_i\underline{f} \qquad i = 1, 2, ---, n \tag{2.32}$$

Now d_i can be chosen to satisfy (2.32). Hence, the poles of the closed loop system can be placed arbitrarily using state feedback if the system is controllable.

2.4 Nature of the Design Equation (2.23)

In general, the output feedback gain matrix $K = fd^T$ is obtained by solving the set of n non-linear simultaneous equations in (m+p) variables $(d_1, d_2, ---, d_p, f_1, f_2, --- f_m)$. However, in the proof of the theorem either $(d_1, d_2, ---, d_p)$ or $(f_1, f_2, ---, f_n)$ are selected arbitrarily, thereby reducing (2.23) to a set of linear equations. This assures at least max (m,p) poles can be placed arbitrarily. In certain cases the non-linear nature of (2.23) can be exploited to assign more than max (m,p) poles of the closed loop system, as will be shown.

Complex eigenvalues of the matrix $\hat{A}$ present an interesting situation. The Jordan canonical form $A = T^{-1}\hat{A}T$ and the matrices $B = T^{-1}\hat{B}$ and $C = \hat{C}T$ will then be complex matrices. However, K will be real since the complex columns of T^{-1} and elements of $\underline{\alpha}$ occur in conjugate pairs.

2.5 Examples

Example 1

$$\underline{\dot{x}} = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underline{u}$$

$$\underline{y} = [0 \quad 1 \quad 0]\, \underline{x}$$

The system is controllable and has three inputs. Hence, all the poles can be assigned arbitrarily using output feedback. The open loop poles are at 1, -2, and 3. Let the closed loop poles be at -1, -3, and -4.

The modal matrix T and its inverse are given by

$$T = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix} \text{ and } T^{-1} = 1/30 \begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix}.$$

Then the transformed equations become

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underline{x} + 1/30 \begin{bmatrix} -15 & -10 & 25 \\ 0 & -2 & 2 \\ 15 & 12 & 3 \end{bmatrix} \underline{u}$$

$$y = [1 \quad 1 \quad 1]\, \underline{x}$$

We have, open loop characteristic polynomial = $(s-1)(s+2)(s+3) = s^3 - 2s^2 - 5s + 6$

and closed loop characteristic polynomial = $(s+1)(s+3)(s+4) = s^3 + 8s^2 + 19s + 12$

Now from (2.7)

$$\frac{s^3 + 8s^2 + 19s + 12}{s^3 - 2s^2 - 5s + 6} = 1 + \frac{10s^2 + 24s + 6}{(s-1)(s+2)(s-3)}$$

$$= 1 - \frac{40/6}{s-1} - \frac{2/15}{s+2} + \frac{168/10}{s-3}$$

This gives $\alpha_1 = -40/6$, $\alpha_2 = -2/15$, and $\alpha_3 = 168/10$.

K is given by

$$K = fd^T = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} d$$

and choosing $d = 1$, we get the equations

$$15f_1 + 10f_2 - 25f_3 = 200$$

$$15f_1 + 12f_2 + 3f_3 = 504$$

$$f_2 - f_3 = 2.$$

Solving these equations gives

$$K = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} .$$

With this choice of K the closed loop poles are located at -1, -3, and -4.

<u>Example 2:</u>

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underline{x}$$

This example illustrates the advantage of the design suggested here over Davison's method. The system is controllable and observable with two unstable poles at 1 and 2. Also, m = 3 and p = 2. According to Davison's method three poles can be placed arbitrarily. By Pearson's method, a first order compensator would be needed to place all the poles. Here it will be shown that by solving the equation (2.23) in its non-linear form all the four poles of the system can be placed arbitrarily. We have $d^T \underline{c}^i \underline{b}_i \underline{f} = \alpha_i \quad i = 1,2,3,4.$

So,

$$f_1 d_1 = \alpha_1$$

$$f_2 d_1 = \alpha_2$$

$$f_3 d_2 = \alpha_3$$

$$(f_1 + f_2 + f_3)d_2 = \alpha_4.$$

solving these equations with $d_1 = 1$, we get

$$d_2 = \frac{\alpha_4 - \alpha_3}{\alpha_1 + \alpha_2}, \; f_1 = \alpha_1, \; f_2 = \alpha_2 \text{ and } f_3 = \frac{\alpha_3(\alpha_1 + \alpha_2)}{\alpha_4 - \alpha_3}.$$

$$K = fd^T = \begin{bmatrix} \alpha_1 & (\alpha_4 - \alpha_3)/(\alpha_1 + \alpha_2) \\ \alpha_2 & \alpha_2(\alpha_4 - \alpha_3)/(\alpha_1 + \alpha_2) \\ \alpha_3(\alpha_1 + \alpha_2)/(\alpha_4 - \alpha_3) & \alpha_3 \end{bmatrix}$$

with this choice of K all the closed loop poles can be placed at the desired location.

If the closed loop poles are desired at -1, -2, -3, and -5, then $\alpha_1 = -7.2$, $\alpha_2 = 14$, $\alpha_3 = 0$ and $\alpha_4 = 0.2$. This gives $f_1 = -7.2$, $f_2 = 14$, $f_3 = 0$, $d_1 = 1$, $d_2 = 1/34$.

and

$$K = \begin{bmatrix} -7.2 & -7.2/34 \\ 14 & 14/34 \\ 0 & 0 \end{bmatrix} .$$

Example 3:

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \underline{x}$$

This problem illustrates the nature of (2.23) when A has complex open loop poles. The open loop poles are at 1 and $-\frac{1}{2} \pm 1\sqrt{3}/2$. If the modal matrix T and its inverse are chosen to be

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix} \;\&\; T^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -j\frac{1}{2\sqrt{3}} - \frac{1}{6} & j\frac{1}{2\sqrt{3}} - \frac{1}{6} \\ \frac{1}{3} & j\frac{1}{2\sqrt{3}} - \frac{1}{6} & -j\frac{1}{2\sqrt{3}} - \frac{1}{6} \end{bmatrix}$$

then A, B, and C are given by

$$A = T^{-1}\hat{A}T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} + j\frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}, \quad B = T^{-1}\hat{B} = \frac{1}{3}\begin{bmatrix} 1 \\ -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix}$$

and

$$C = \hat{C}T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2} + j\frac{\sqrt{3}}{2} & \frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix} .$$

If the closed loop poles are chosen to be at $\pm j$ and 1, then we have the set of equations

$$\frac{1}{3}[d_1 + 2d_2]f = \alpha_1 \tag{2.32}$$

$$\frac{1}{6}[-d_1 + d_2] - j\frac{1}{2\sqrt{3}}[d_1 + d_2] = \alpha_2 \tag{2.33}$$

$$\frac{1}{6}[-d_1 + d_2] + j\frac{1}{2\sqrt{3}}[d_1 + d_2] = \alpha_3 \tag{2.34}$$

where $\alpha_1 = 0$, $\alpha_2 = -\frac{1}{2} - \frac{j}{2\sqrt{3}}$ and $\alpha_3 = -\frac{1}{2} + \frac{j}{2\sqrt{3}}$ for the desired pole assignment. Equation (2.33) and (2.34) are complex conjugates and give the same set of equations in (d_1, d_2, f). From equation (2.32) and the real and imaginary parts of (2.33), we get

$$\begin{aligned} (d_1 + 2d_2)f &= 0 \\ (d_1 - d_2)f &= 3 \\ (d_1 + d_2)f &= 1. \end{aligned} \tag{2.35}$$

Solving (2.35) with $f = 1$, gives $d_1 = 2$, $d_2 = -1$ and $K = [2\ -1]$. This choice of K gives the desired pole-placement.

2.6 Remark on Unattainable Poles

Consider a controllable single-input single-output system

$$\dot{\underline{x}} = \hat{A}\underline{x} + \underline{b}u$$
$$y = \underline{c}^T\underline{x}$$

It is well-known that using output feedback the closed loop system $\dot{\underline{x}} = (\hat{A}+\underline{b}\underline{k}^T\underline{c}^T)\underline{x}$ can attain any set of closed loop poles except the zeros of the system. Davison [20] has tried to generalize this result to multivariable systems. He has shown that given a linear time-invariant controllable system $\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\underline{u}$, $\underline{y} = \hat{C}\underline{x}$ with Rank $\hat{C} = \ell$, then ℓ eigenvalues of the closed loop system can be assigned arbitrarily close to ℓ preassigned values. He observes that there are certain values that the preassigned values may not take on, which correspond to the zeros of the various transfer functions existing in the multivariable system. This observation is shown to be incorrect by means of a counter-example and an alternate characterization of the unattainable poles follows.

Consider the system

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x}$$

The system transfer function matrix is

$$H(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{s-3} & \frac{2s-5}{(s-2)(s-3)} \end{bmatrix}$$

Let $(-\rho_1, -\rho_2)$ be preassigned eigenvalues. The matrix s (from equation (31) of Davison's paper with $\theta_2 = 1$) is given by

$$S = \begin{bmatrix} 6 + 5\rho_1 + \rho_1^2 & 6 + 5\rho_2 + \rho_2^2 \\ 7 + 10\rho_1 + 3\rho_1^2 & 7 + 10\rho_2 + 3\rho_2^2 \end{bmatrix}$$

Davison cannot assign poles if this matrix S becomes singular. Notice that det S is not identically equal to zero for $-\rho_1 = 5/2$. Hence $-\rho_1 = 5/2$, which corresponds to a zero of the transfer function, can be a preassigned eigenvalue. However, there are pairs (ρ_1, ρ_2), e.g.

(ρ_1, ρ_2), which make s singular and hence cannot be chosen as closed-loop poles by Davison's method.

An alternate explanation for the set of eigenvalues which the closed-loop system cannot attain is given below.

We have, from (2.13)

$$\underline{d} = C_p^{-1} \underline{\alpha}_p$$

$$= \begin{bmatrix} \psi_1 \; (\rho_1, \rho_2, ---\rho_n) \\ \psi_2 \; (\rho_1, \rho_2, ---\rho_n) \\ \vdots \\ \psi_p \; (\rho_1, \rho_2, ---\rho_n) \end{bmatrix}$$

where $\psi_i \; (\rho_1, ---\rho_n)$, $i = 1, ---p$ are functions of $\rho_1, --- \rho_n$. substituting this value of $\underline{d}$ in the remaining (n-p) equations

$$\underline{d}^T \; \underline{c}^i \underline{b}_i \underline{f} = \alpha_i \qquad i = p+1, ---, n$$

we obtain the functional relation between closed-loop poles as

$$\phi_i \; (\rho_1, \rho_2, --- \rho_n) = \phi_i \; (\rho_1, \rho_2, --- \rho_n) \qquad i = p+1, ---n. \tag{2.36}$$

For any given set of p closed-loop poles $(\rho_1, \rho_2, ---, \rho_p)$, the location of the remaining (n-p) poles $(\rho_{p+1}, ---, \rho_n)$ will be determined by the (n-p) equations (2.36). However, the values of $(\rho_{p+1}, --- \rho_n)$ may become indeterminate for certain configurations of $(\rho_1, \rho_2, --- \rho_p)$, in which case, the p poles $(\rho_1, \rho_2, ---, \rho_p)$ of the closed loop system must be reassigned.

Consider the system

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}$$

The system can be transformed into the form

$$\underline{\dot{x}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \underline{x}$$

The open-loop poles of the system are at -1, -2, and -3. The system transfer function matrix is given by

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} s \\ s^2 \end{bmatrix} .$$

Let the closed loop poles be denoted by $-\rho_1$, $-\rho_2$, and $-\rho_3$. The C.L.C.P. is given by $s^3 + \ell s^2 + ms + n$ where

$$\ell = (\rho_1 + \rho_2 + \rho_3)$$

$$m = \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1$$

$$n = \rho_1\rho_2\rho_3 .$$

we have $\underline{d}^T \underline{c}^i \underline{b}_i \underline{f} = \alpha_i$, $i = 1,2,3$.

Choosing $f = 1$, this reduces to

$$-d_1 + d_2 = 2\alpha_1$$
$$-2d_1 + 4d_2 = -\alpha_2$$
$$-3d_1 + 9d_2 = 2\alpha_3$$

where $\alpha_1 = \frac{1}{2}(\ell - m + n - 1)$

$$\alpha_2 = - (4\ell - 2m + n - 8)$$

$$\alpha_3 = \frac{1}{2} (9\ell - 3m + n - 27).$$

From (2.13)

$$\underline{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = C_p^{-1} \underline{\alpha}_p$$

$$= -\frac{1}{2} \begin{bmatrix} -2m + 3n + 4 \\ -2 \quad + n + 6 \end{bmatrix}$$

This value of $\underline{d}$ should satisfy the equation (2.36). Hence, the closed loop poles should satisfy the relation

$$-3d_1 + 9d_2 = 2\alpha_3$$

i.e. $\quad 3/2\ (-2m + 3n + 4) - 9/2\ (-2\ell + n + 6) = 9\ell - 3m + n - 27$

or $\quad (n-6) = (\rho_1 \rho_2 \rho_3 - 6) = 0.$

According to the theorem two poles can be assigned arbitrarily close to desired values. Let us assign the poles $-\rho_1$ and $-\rho_2$. This results in

$$-\rho_3 = -6/\rho_1\rho_2 = -6((-\rho_1)(-\rho_2).$$

However, if either $-\rho_1$ or $-\rho_2$ is choosen to be equal to zero then $-\rho_3$ becomes infinite and $K = \underline{f}^T\underline{d} = \underline{d}$ becomes infinite.

Seraji [28] has shown that in the single input (output) multiple output (input) case the unattainable poles correspond to zeros which are common to all the transfer functions of the open loop system.

III. APPROXIMATE POLE PLACEMENT

3.1 Introduction

In Chapter 2 it was shown that for a controllable, observable system $\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\underline{u}$, $\underline{y} = \hat{C}\underline{x}$ with m inputs and p output at least max(m,p) poles can be assigned arbitrarily close to desired locations using constant gain output feedback. The choice of max(m,p) poles automatically fixes the location of the remaining [n-max(m,p)] poles of the system. Let us call these poles the "dependent poles", $\underline{\beta}$, where $\underline{\beta}=[\rho_{max(m,p)+1}, \ldots, \rho_{n-1}, \rho_n]$. In some cases, by taking advantage of the non-linear nature of equation (2.23) more than max(m,p) poles can be arbitrarily assigned and this reduces the number of dependent poles. However, nothing can be said a priori about the location of these dependent poles. The problem of the dependent poles can be handled in two different ways. The first method, due to Brasch and Pearson [21], uses a dynamic compensator and in the second method a constant gain feedback controller is realized which positions all the poles approximately.

3.2 Pole Placement Using Dynamic Compensator

Brasch and Pearson [21] have considered the problem of designing a compensator to obtain arbitrary pole placement in the system consisting of the plant and compensator in cascade. The design uses only those state variables which can be measured.

Consider the controllable and observable system defined by equations (2.1) and (2.2). Let E be the co-ordinate space and {B} denote the subspace of E spanned by the column vectors of B. If E is cyclic, then there exists an n vector $b \in \{B\}$ such that (A,b) is controllable. It can be shown that the matrix $\hat{A}+\hat{B}K\hat{C}$ can be made cyclic using output feedback. From this it follows that any multi-input multi-output linear time-invariant system may be made controllable (observable) from a single input (output) using only output feedback. This result is useful in arriving at the following theorem.

Let $\nu_c(\nu_0)$ be the controllability (observability) index of the plant. Define $\beta = \min(\nu_c-1, \nu_0-1)$. Let

$$A_\ell = \begin{bmatrix} \overset{n\times n}{\hat{A}} & 0 \\ 0 & \underset{\ell\times\ell}{0} \end{bmatrix} \tag{3.1}$$

represent the plant dynamics plus ℓ additional integrators. It is assumed that every state in the compensator can be directly measured and directly controlled. Thus

$$B_\ell = \begin{bmatrix} \hat{B} & 0 \\ 0 & I_\ell \end{bmatrix}, \quad C_\ell \begin{bmatrix} \hat{C} & 0 \\ 0 & I_\ell \end{bmatrix} \tag{3.2}$$

where I_ℓ is an $\ell x \ell$ identity matrix. Let $\Lambda_\ell = \{\lambda_1, \lambda_2, --- \lambda_{n+\ell}\}$ be a set of arbitrary numbers subject only to the condition that complex numbers occur in conjugate pairs.

<u>Theorem (Brasch and Pearson)</u>

Let $(\hat{A},\hat{B},\hat{C})$ be controllable observable system and let A_β, B_β, C_β be as defined in (3.1) and (3.2) where $\beta = \min(\nu_c-1, \nu_0-1)$. Given any set Λ_β, there exists a matrix K such that the eigenvalues of

$A_\beta + B_\beta K C_\beta$ are precisely the elements of the set Λ_β.

Thus a compensator of order β is sufficient to place all the poles of the plant and the compensator in cascade. It should be pointed out that the order of the compensator is not necessarily minimum. Similar results have been obtained by Chen and Hsu [22] using a transfer function approach.

3.3 Approximate Pole Placement Using Psuedoinverse

The problem of pole placement is reduced to the problem of solving the n non-linear equations in (m+p) variables (d_1, d_2, ..., d_p, f_1, f_2, ...f_m) and we can place max(m,p) poles. When max $(m,p)<n$, we want to place the n poles approximately by finding a least square error solution to the n equations.

Consider the linear case and let $p>m$. Write $M = C^T$ and recall that C^T has full rank. From (2.12)

$$M\underline{d} = \underline{\alpha} \tag{3.3}$$

Since $p<n$, the system of equations is inconsistent and there is no solution vector $\underline{d}$ which satisfies (3.3). Now, the question to be answered is, "does there exist a vector $\underline{d}$ so that equation (3.3) is approximately satisfied for a suitable definition of approximate?" We can write (3.3) as

$$M\underline{d} - \underline{\alpha} = \underline{e}(\underline{d}) \tag{3.4}$$

Since there is no vector $\underline{d}$ such that $\underline{e}(\underline{d}) = 0$, it is desirable to find a d* which produces a "smaller" $\underline{e}(\underline{d})$ than any other vector $\underline{d}$. $\underline{d}$* is the best approximate solution (BAS) to the system of equations (3.4).

Definition:

Best Approximate Solution:

The vector $\underline{d}^*$ is defined to be the best approximate solution (BAS) to the system of equations (M is an nxp)

$$M\underline{d} - \underline{\alpha} = \underline{e}(\underline{d})$$

if and only if

(i) for all $\underline{d}$, $[M\underline{d} - \underline{\alpha}]^T[M\underline{d} - \underline{\alpha}] \geq [M\underline{d}^* - \underline{\alpha}]^T[M\underline{d}^* - \underline{\alpha}]$

(ii) and for those $\underline{d} \neq \underline{d}^*$ such that $[M\underline{d} - \underline{\alpha}]^T[M\underline{d} - \underline{\alpha}] = [M\underline{d}^* - \underline{\alpha}]^T$ $[M\underline{d}^* - \underline{\alpha}]$ the relation $\underline{d}^T d > \underline{d}^{*T}\underline{d}^*$. holds.

The definition essentially states that $\underline{d}^*$ minimizes the sum of squares of deviations; and if there is a set Φ of vectors such that each member in the set gives the minimum sum of squares of deviations, then the vector $\underline{d}^*$ in Φ is chosen as BAS if for all other vectors in Φ the sum of the squares $\underline{d}^T\underline{d}$ is larger than $\underline{d}^{*T}\underline{d}^*$.

The following theorem by Penrose [29] shows that the BAS exists and the generalized inverse of the coefficient matrix can be used to find it.

<u>Theorem</u>:

The best approximate solution to the system of equations $M\underline{d} = \underline{\alpha}$ is given uniquely by

$$d^* = M^{\dagger}\underline{\alpha} \tag{3.5}$$

where $M^{\dagger} = (M^TM)^{-1}M^T$ is the generalized inverse of M.

<u>Proof</u>:

We have to show that for $\underline{d}^* = M^{\dagger}\underline{\alpha}$

$$[M\underline{d} - \underline{\alpha}]^T[M\underline{d} - \underline{\alpha}] \geq [M\underline{d}^* - \underline{\alpha}]^T[M\underline{d}^* - \underline{\alpha}]$$

for all $\underline{d}$ and for those vectors such that the equality holds, we have $\underline{d}^T\underline{d} > \underline{d}^{*T}\underline{d}^*$ if $d \neq d^*$.

Now,

$$[M\underline{d} - \underline{\alpha}]^T[M\underline{d} - \underline{\alpha}]$$

$$= [M\underline{d} - MM^{\dagger}\underline{\alpha} + MM^{\dagger}\underline{\alpha} - \underline{\alpha}]^T[M\underline{d} - MM^{\dagger}\underline{\alpha} + MM^{\dagger}\underline{\alpha} - \underline{\alpha}]$$

$$= [M(\underline{d} - M^{\dagger}\underline{\alpha}) + (MM^{\dagger} - I)\underline{\alpha}]^T\ [M(\underline{d} - M^{\dagger}\underline{\alpha}) + (MM^{\dagger} - I)\underline{\alpha}]$$

$$= [M(\underline{d} - M^{\dagger}\underline{\alpha})]^T[M(\underline{d} - M^{\dagger}\underline{\alpha})] + [(MM^{\dagger} - I)\underline{\alpha}]^T\ [(MM^{\dagger} - I)\underline{\alpha}]$$

$$\geq [(MM^{\dagger} - I)\underline{\alpha}]^T[(MM^{\dagger} - I)\underline{\alpha}]$$

$$= \underline{\alpha}^T\ [MM^{\dagger} - I]^T[MM^{\dagger} - I]\ \underline{\alpha}$$

The inequality

$$[M\underline{d} - \underline{\alpha}]^T\ [M\underline{d} - \underline{\alpha}] \geq \underline{\alpha}^T[(MM^{\dagger} - I)^T\ (MM^{\dagger} - I)]\ \underline{\alpha}$$

holds for all $\underline{d}$.

If $\underline{d}^* = M^{\dagger}\underline{\alpha}$, then

$$[M\underline{d} - \underline{\alpha}]^T\ [M\underline{d} - \underline{\alpha}] \geq [MM^{\dagger}\underline{\alpha} - \underline{\alpha}]^T\ [MM^{\dagger}\underline{\alpha} - \underline{\alpha}]$$

$$= [M\underline{d}^* - \underline{\alpha}]^T\ [M\underline{d}^* - \underline{\alpha}] \quad \text{for all } \underline{d}.$$

The equality holds if and only if $[M(\underline{d} - M^{\dagger}\underline{\alpha})]^T\ [M(\underline{d} - M^{\dagger}\underline{\alpha})] = 0$

i.e. iff $M\underline{d} = MM^{\dagger}\underline{\alpha}$.

Next we have to show that, for the set of $\underline{d}$'s for which $M\underline{d} = MM^{T}\underline{\alpha}$

the inequality

$$\underline{d}^T\underline{d} \geq (M^{\dagger}\underline{\alpha})^T\ (M^{\dagger}\underline{\alpha}) = \underline{d}^{*T}\underline{d}^*.$$

For all $\underline{d}$,

$$[M^{\dagger}\underline{\alpha} + (I - M^{\dagger}M)\underline{d}]^T\ [M^{\dagger}\underline{\alpha} + (I - M^{\dagger}M)\underline{d}]$$

$$= (M^{\dagger}\underline{\alpha})^T\ (M^{\dagger}\underline{\alpha}) + [(I - M^{\dagger}M)\underline{d}]^T[(I - M^{\dagger}M)\underline{d}] \tag{3.6}$$

substituting $M\underline{d} = MM^{\dagger}\underline{\alpha}$ or equivently $M^{\dagger}\underline{\alpha}$ for M $M\underline{d}$ equation (3.6)

becomes

$$\underline{d}^T\underline{d} = (M^{\dagger}\underline{\alpha})^T\ (M^{\dagger}\underline{\alpha}) + (\underline{d} - M^{\dagger}\underline{\alpha})^T\ (\underline{d} - M^{\dagger}\underline{\alpha})$$

$$= \underline{d}^{*T}\underline{d}^* + (\underline{d} - M^{\dagger}\underline{\alpha})^T\ (\underline{d} - M^{\dagger}\underline{\alpha})$$

This implies that $\underline{d}^T\underline{d} > d^{*T}\underline{d}$ if $\underline{d} \neq \underline{d}^*$.

Hence the BAS always exists and is unique.

Least Square Solution:

The vector $\underline{d}^*$ is defined to be a least square solution (LSS) of the system $M\underline{d} - \underline{\alpha} = \underline{e}(\underline{d})$ (where M is an nxp matrix of rank p<n) if and only if for all $\underline{d}$ the following relationship holds:

$$[M\underline{d} - \underline{\alpha}]^T [M\underline{d} - \underline{\alpha}] \geq [M\underline{d}^* - \underline{\alpha}]^T [M\underline{d}^* - \underline{\alpha}] \tag{3.7}$$

Remark;

A LSS must satisfy equation (3.7). A BAS in addition to (3.7) must satisfy the condition $\underline{d}^T\underline{d} > \underline{d}^{*T}\underline{d}^*$. Thus there may be several least square solutions to a linear system while the best approximate solution is unique. Thus the approximate pole placement can be done either by obtaining the BAS or LSS $\underline{d}^*$. Then the output feedback gains are given by $K = f(d^*)^T$.

3.3.1 Computation of the Generalized Inverse

There are several methods available for the computation of the generalized inverse [30, 31]. Peters and Wilkinson [32] have developed them from a uniform standpoint. In addition, the methods are shown to be natural extensions of the several methods available to find the inverse of a matrix.

If the mxn matrix M is of rank r then it can be factorized in the form

$$M = LN$$

where L is an mxr matrix and N is an rxn matrix and both are of rank r. The matrix

$$Z = N^T(NN^T)^{-1} (L^TL)^{-1}L^T \tag{3.8}$$

is the psuedoinverse of M. It is easy to see that it is independent of the particular LN factorization chosen and this can be verified by replacing L by LY^{-1} and N by YN, where Y is any non-singular rxr matrix.

Most algorithms to invert a nxn matrix are based on factorizations M = LN of M where L and N are easily invertible non-singular matrices - e.g. L and N could be upper and lower triangular matrices and unitary (orthogonal) matrices. Each well-known method for inverting a matrix has an analogous method for computing the psuedoinverse.

When solving linear equations it is more economical to work directly with the factors L and N by solving

$$L\underline{g} = \underline{\alpha}, \quad N\underline{d} = \underline{g}$$

rather than computing $N^{-1}L^{-1}$ explicitly. Similarly, when solving the least squares problem it is uneconomical to compute the psuedoinverse directly. Hence we compute $\underline{d}$ indirectly as follows. From (3.5) and (3.8) we have

$$N^T(NN^T)^{-1} (LL^T)^{-1} L^T = N^T (L^TLNN^T)^{-1}L^T$$

and $$\underline{d} = N^T (NN^T)^{-1} (LL^T)^{-1} L^T \underline{\alpha} \qquad (3.9)$$

If we compute $\underline{d}$ given by (3.9) with

$$\underline{w} = L^T\underline{\alpha}, \quad (L^TLNN^T)\underline{v} = \underline{w}, \quad \underline{d} = N^T\underline{v}$$

then the solution of $(L^TLNN^T)\underline{v} = \underline{w}$ requires only some factorization of the matrix pre-multiplying $\underline{v}$, not its explicit inverse.

When solving squations some pivoting strategy is usually employed to achieve greater numerical stability. This has the effect that we determine a factorization of a matrix $\tilde{M}$, rather than M itself, where

$\tilde{M}$ is obtained from M by suitably permuting its rows and/or columns. The factorization results in matrices

$$\tilde{M} = P_1MP_2 = LN \quad \text{or } M = P_1^T LNP_2^T$$

$$\tilde{M}^{-1} = (P_1MP_2)^{-1} = P_2^{-1}MP_1^{-1} = P_2^T M^{-1} P_1^T$$

where P_1 and P_2 are permutation matrices. Hence, the inverses derived via L and N merely give the required matrix with its rows and columns permuted. We can derive a similar result for the psuedoinverse using equation (3.8). Therefore, row and column interchanges can be freely used in factoring the matrix M to find its psuedoinverse.

All the three common methods - (i) methods related to Gaussian elimination, (ii) Householder and Givens method [33], (iii) modified Gram-Schmidt factorization - of finding the inverse of a matrix can be extended to find the psuedoinverse. The Householder and Givens method is slightly better than Gauss elimination methods regarding numerical stability. However, Householder's method and Given's method require two times and four times more work respectively. The modified version of Gram-Schmidt factorization gives better results than Householder's method.

The most difficult practical problem associated with the computation of the psuedoinverse is the determination of the rank. Round-off errors are involved in the factorization and a decision has to be made as to when the 'remaining' elements can be regarded as zero during the course of the reduction. Golub and Kahan [29] have described an effective algorithm for determining the rank of a matrix. The requirement that the residual vector $\underline{e}(\underline{d})$ should be a minimum while $\underline{d}$ itself should be minimal often conflict to some extent.

3.4 Approximate Pole Placement by Minimizing Least Square Error Criterion:

It has been shown that pole placement is reduced to the problem of solving n non-linear equations in (m+p) variables. The nonlinear equations can be reduced to linear equations by choosing $\underline{d}$ or $\underline{f}$ arbitrarily. When the number of poles that can be placed is less than n, the psuedoinverse can be used to position all the poles approximately. However, a larger class of feedback matrices can be obtained by solving the n non-linear equations

$$\underline{d}^T \underline{c}^i \underline{b}^i \underline{f} = \alpha_i \qquad i = 1,2,---, n$$

in (m+p) variables $(d_1, d_2, ---, d_p, f_1, f_2, ---, f_m)$.

It should be recalled that α_i is a function of the closed loop poles $(\rho_1, \rho_2, ---, \rho_n)$. By minimizing a least square error criterion of the form

$$J = \sum_{i=1}^{n} q_i(\underline{d}^T \underline{c}^i \underline{b}^i \underline{f} - \alpha_i(\underline{\rho})) \qquad (3.10)$$

subject to the constraints $\underline{g}(\underline{\rho}) \geq 0$ an approximate set of desired closed loop poles can be realized. The weighting coefficients q_i can be used to control the error between a pole in the desired set and its corresponding pole in the approximate set. The constraint equations

$$q_i(\rho_1, \rho_2, ---, \rho_n) \geq 0 \qquad i = 1,2, ---, \ell$$

depends on the individual problem. The minimization procedure can be easily carried out using one of the standard static optimization techniques like the conjugate gradient method.

3.5 Example

Consider the controllable and observable system

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underline{x}$$

This system has two inputs and two outputs. As a result only two poles can be placed arbitrarily close to desired locations. Here all the poles will be positioned approximately using the two methods suggested in this section.

We have $\underline{d}^T \underline{c}^i \underline{b}^i \underline{f} = \alpha_i \qquad i = 1,2,3,4.$

where $\underline{d}^T = [d_1, d_2]$ and $\underline{f} = [f_1, f_2]^T$. The non-linear equations are given by

$$d_1 f_1 = \alpha_1$$

$$d_1 f_2 = \alpha_2$$

$$d_2 f_1 = \alpha_3$$

$$d_2 (f_1 + f_2) = \alpha_4.$$

Let the desired closed loop poles be at $-\rho_1$, $-\rho_2$, $-\rho_3$ and $-\rho_4$. The open loop poles are at 1,2,-3, and -4. Then,

closed loop characterictic polynomial $= s^4 + \ell_3 s^3 + \ell_2 s^2 + \ell_1 s + \ell_0$

where

$$\ell_3 = \rho_1 + \rho_2 + \rho_3 + \rho_4$$

$$\ell_2 = \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_4 + \rho_4\rho_1 + \rho_1\rho_3 + \rho_2\rho_4$$

$$\ell_1 = \rho_1\rho_2\ (\rho_3 + \rho_4) + \rho_3\rho_4\ (\rho_1 + \rho_2)$$

$$\ell_0 = \rho_1\rho_2\rho_3\rho_4$$

and open loop characteristic polynomial = $s^4 + 4s^3 - 7s^2 - 22s + 24$.

The values of α_i are given by

$$\alpha_1 = -(\ell_0 + \ell_1 + \ell_2 + \ell_3 + 1)/20$$

$$\alpha_2 = (\ell_0 + 2\ell_1 + 4\ell_2 + 8\ell_3 + 16)/30$$

$$\alpha_3 = (\ell_0 - 3\ell_1 + 9\ell_2 - 27\ell_3 + 81)/20$$

$$\alpha_4 = -(\ell_0 - 4\ell_1 + 16\ell_2 - 64\ell_3 + 256)/30$$

Let the design requirements be such that the closed loop poles are at -1, -2, -3, and -5 resulting in $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 3$, and $\rho_4 = 5$. Corresponding to these pole locations $\alpha_1 = -7.2$, $\alpha_2 = 14$, $\alpha_3 = 0$, and $\alpha_4 = 0.2$. Equations (3.11) can be rewritten as

$$d_1 = \bar{\alpha}_1 \tag{3.12.1}$$

$$d_1 = \bar{\alpha}_2 \tag{3.12.2}$$

$$d_2 = \bar{\alpha}_3 \tag{3.12.3}$$

$$d_2 = \bar{\alpha}_4 \tag{3.12.4}$$

where $\bar{\alpha}_1 = \alpha_1/f_1$, $\bar{\alpha}_2 = \alpha_2/f_2$, $\bar{\alpha}_3 = \alpha_3/f_1$ and $\bar{\alpha}_4 = \alpha_4/(f_1+f_2)$.

In a matrix form (3.12) becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_3 \\ \bar{\alpha}_4 \end{bmatrix} \tag{3.13}$$

Since (3.13) is inconsistent, the best approximate solution to (3.13) is given by

$$d^* = M^{\dagger}\underline{\alpha} = (M^TM)^{-1}M^T\underline{\alpha}.$$

Thus

$$d^* = \left[\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}\right]^{-1}\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\underline{\alpha}$$

$$= \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}^{-1}\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\underline{\alpha}$$

$$= \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\underline{\alpha}$$

and

$$d_1^* = \frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_2)$$

$$= \frac{1}{2}\left[\frac{-7.2}{f_1} + \frac{14}{f_2}\right]$$

$$d_2^* = \frac{1}{2}(\bar{\alpha}_3 + \bar{\alpha}_4) = \frac{1}{2}\left[\frac{1}{5(f_1+f_2)}\right]$$

Notice that the d_1^* and d_2^* obtained by using the pseudoinverse is the same value one would pick intuitively for d_1 and d_2 to satisfy (3.13) approximately. d_1 has to satisfy the equations (3.12.1) and (3.12.2) and the best one could do is to pick $d_1 = \frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_2)$.

Similarly, $d_2 = \frac{1}{2}(\bar{\alpha}_3 + \bar{\alpha}_4)$. Then output feedback gain matrix K is given by

$$K = fd^{*T} = \begin{bmatrix} f_1 d_1^* & f_1 d_2^* \\ f_2 d_1^* & f_2 d_2^* \end{bmatrix}$$

$$= \begin{bmatrix} -3.6 + 7f_1/f_2 & f_1/10(f_1 + f_2) \\ 7 - 3.6\ f_2/f_1 & f_2/10(f_1 + f_2) \end{bmatrix}$$

with $r = f_1/f_2$,

$$BKC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3.6 + 7r & r/10(1+r) \\ 7-3.6/r & 1/10(1+r) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7r-3.6 & 7r-3.6 & r/10(1+r) & r/10(1+r) \\ 7-3.6/r & 7-3.6/r & 1/10(1+r) & 1/10(1+r) \\ 7/r-3.6 & 7/r-3.6 & r/10(1+r) & r/10(1+r) \\ 3.4(1+1/r) & 3.4(1+1/r) & 1/10 & 1/10 \end{bmatrix}$$

and

$$A-BKC = \begin{bmatrix} 4.6-7r & 3.6-7r & -r/10(1+r) & -r/10(1+r) \\ 3.6/r-7 & -5+3.6/r & -1/10(1+r) & -1/10(1+r) \\ 3.6-7/r & -7/r+3.6 & -3-r/10(1+r) & -r/10(1+r) \\ -3.4(1+1/r) & -3.4(1+1/r) & -1/10 & -41/10 \end{bmatrix}$$

The closed loop poles of the system are the eigenvalues of A-BKC and these are tabulated for different values of r in Table 1.

r	Gains		Closed Loop Poles
-0.8	-9.2	-0.4	0.36 ± j 3.19
	11.5	0.5	-2.60, -4.12
-0.5	-7.1	-0.1	-3.31 ± j 1.5
	14.2	0.2	-4.45, -4.024
-0.44	-6.7	-0.078	-4.35, - 4.35
	15.2	0.170	-3.80, -0.0012
-0.3	-5.7	-0.042	-10.3, -3.31
	19.0	0.141	-4.23, 0.521
1.0	3.4	0.05	-5.84, -3.76
	3.4	0.05	-2.87, 1.53

TABLE 1 Approximate Pole Placement Using Psuedoinverse

The closed loop poles are functions of $r=f_1/f_2$. The desired closed loop poles are at -1, -2, -3, and -5. The open loop system has two unstable poles at 1 and 2. For r in the range -0.4 to -0.5 the system can be stabilized with the closed loop poles and feedback gains as shown in Table 1.

The problem is reconsidered by minimizing the least square error criterion. Let the design requirements be such that $\rho_1 \simeq 1$, $\rho_2 \simeq 2$, $\rho_3 \geq 3$ and $\rho_4 \geq 5.0$. This can be met by minimizing the performance index of the form

$$J_1 = (d_1f_1-\alpha_1)^2 + (d_1f_2-\alpha_2)^2 + (d_2f_1-\alpha_3)^2 + [d_2(f_1 + f_2) -\alpha_4]^2$$

subject to the constraints

$$|\rho_1 - 1| \leq \varepsilon_1$$
$$|\rho_2 - 2| \leq \varepsilon_2$$
$$\rho_3 \geq 3$$
and
$$\rho_5 \geq 5.$$

ε_1 and ε_2 are small positive numbers. Redefine the performance index to include the constraints as

$$J = (d_1f_1 - \alpha_1)^2 + (d_1f_2 - \alpha_2)^2 + (d_2f_1 - \alpha_3)^2 + (d_2(f_1+f_2)-\alpha_4)^2$$
$$+ K_1(\rho_1-1)^2U_1 + K_2(\rho_2-2)^2U_2 + K_3(\rho_3-3)^2U_3 + K_4(\rho_4-5)^2U_4.$$

where K_1, K_2, K_3, K_4 are constants and the functions U_1, U_2, U_3 and U_4 are defined below.

$$U_1 = \begin{cases} 0 & |\rho_1-1| \leq \varepsilon_1 \\ 1 & \text{Otherwise} \end{cases}$$

$$U_2 = \begin{cases} 0 & |\rho_2-2| \leq \varepsilon_2 \\ 1 & \text{Otherwise} \end{cases}$$

$$U_3 = \begin{cases} 0 & \rho_3 \geq 3 \\ 1 & \text{Otherwise} \end{cases}$$

$$U_4 = \begin{cases} 0 & \rho_4 \geq 5 \\ 1 & \text{Otherwise} \end{cases}$$

J was minimized using the conjugate gradient technique and the values of the K matrix and the corresponding closed loop pole locations for different initial values and different weighting coefficients K_1, K_2, K_3, K_4 are shown in Table 2. ε_1 and ε_2 were chosen to be 0.05.

Conditions of the Run	K	Pole Position
Initial values: $\rho_1=6$, $\rho_2=8$, $\rho_3=1$, $\rho_4=3$, $d_1=5$, $d_2=5$, $f_1=5$ $f_2 = 5$; $K_1=1$, $K_2=1$, $K_3=10$, $K_4=10$	-9.0 -0.195 16.8 0.365	-1.0343, -2.015 -3.0702, -6.2226
Initial values: $\rho_1=6$, $\rho_2=8$ $\rho_3=1$, $\rho_4=3$, $d_1=5$, $d_2=5$, $f_1=5$, $f_2=5$; $K_1=10$, $K_2=10$, $K_3=1$, $K_4=1$	-8.5 0.067 16.05 -0.125	-0.97, -1.9858 -3.8041, -5.0143
Initial values: $\rho_1=1$, $\rho_2=1.0$, $\rho_3=3.1$, $\rho_4=5.1$, $d_1=1$, $d_2=1$, $f_1=1$, $f_2=1$; $K_1=1$, $K_2=1$, $K_3=1$, $K_4=1$	-7.8 -0.036 15.0 0.070	-0.0907, -2.05 -3.3142, -4.9951

TABLE 2

Approximate Pole Placement Using Gradient Method

The example shows that the solution of the non linear equations by minimizing a least square error criterion provides a more desirable approximate pole configuration than the one obtained by using the psuedoinverse. This is because the psuedo-inverse method is equivalent to minimizing a cost function of the form

$$J = \sum_{i=1}^{4} [d_i - \alpha_i(\rho_1, \rho_2, \rho_3, \rho_4, r)]^2$$

with no penalty on the error between the desired poles $-\rho_{1d}$, $-\rho_{2d}$, $-\rho_{3d}$, $-\rho_{4d}$ and the actual pole positions at $-\rho_1$, $-\rho_2$, $-\rho_3$, and $-\rho_4$. This penalty is included in the method using optimization and hence we get a closer pole configuration to the desired set of poles.

IV. APPLICATION OF POLE PLACEMENT THEORY TO HELICOPTER STABILIZATION SYSTEMS

In this chapter the results of Chapters II and III are used to design a controller for a complex dynamical system using output feedback. The system selected for study is the Boeing-Vertol CH-46 tandem rotor helicopter. The output feedback gains are obtained by a least square solution of the nonlinear equations to achieve a satisfactory set of poles for the closed loop system.

4.1 Boeing-Vertol CH-46 Helicopter

The dynamics of the helicopter are characterized by linear pertubation equations written about steady flight conditions. Further, it is assumed that the dynamics could be separated into the standard aircraft longitudinal and lateral directional modes thereby reducing the equation to two independent sets. Although not always a valid assumption for helicopters, it is believed a valid assumption for the CH-46 due to the hinged rotor blades. The equations for the longitudinal dynamics under level flight at 110 Kilometers/hr. is given by Gray, Rempter and Stevenson [34]. The helicopter instability is most pronounced at this flight condition. In [34], the feedback gains were obtained by minimizing a quadratic performance index and then a suboptimal system was obtained by (i) feeding back the available states and (ii) estimating the unavilable states by using filters.

The outputs of the system are pitch attitude (θ), rate of descent (V_z), pitch attitude rate ($\dot{\theta}$), and forward velocity (V_x). The attitude rate ($\dot{\theta}$) is provided by a rate gyro, attitude by an inertial sensing unit and it is assumed that the rate of descent (V_z) and forward velocity (V_x) is of a quality suitable for use in the flight control system. There are two control inputs to the system. The rotor blade angle of attack on both rotors can be varied together to vary lift (collective input) or varied in opposition to produce a pitching moment (differential collective input). Electro-hydraulic servo actuators accept electrical signals and drive the rotor blades in the appropriate manner. Both the actuator and the rotor blades exhibit dynamics when excited.

The helicopter, including the rotors and the actuators, has twelve states. These are the four outputs and the eight unavailable states of the actuators and rotors. In this report the actuator dynamics are ignored resulting in a system with eight states, two inputs and four outputs. The linearized equations of motion together with the rotor and actuator dynamics are given below.

$$\Delta\dot{V}_x = \left(\frac{X_u}{m} + \frac{X_w}{m}\tan\theta_0\right)\Delta V_x + (\tan\theta_0)\,\Delta\dot{V}_z - \left(\frac{X_u}{m}\tan\theta_0 - \frac{X_w}{m}\right)\Delta V_z$$

$$+ \frac{1}{\cos\theta_0}\left[\frac{X_q}{m}\,\Delta\dot{\theta} - \left(\frac{X_u}{m}W_0 - \frac{X_w}{m}U_0 + g\cos\theta_0\right)\Delta\theta\right] + \frac{X_{\delta_e}}{m}\delta_e + \frac{X_{\delta_c}}{m}\delta_c$$

$$\Delta\dot{V}_z = -(\tan\theta_0)\Delta\dot{V}_x + \left(\frac{Z_u}{m} + \frac{Z_w}{m}\tan\theta_0\right)\Delta V_x - \left(\frac{Z_u}{m}\tan\theta_0 - \frac{Z_w}{m}\right)\Delta V_z$$

$$+ \frac{1}{\cos\theta_0}\left[\frac{Z_q}{m}\,\Delta\dot{\theta} - \left(\frac{Z_u}{m}W_0 - \frac{Z_w}{m}U_0 + g\sin\theta_0\right)\Delta\theta\right] + \frac{Z_{\delta_e}}{m}\delta_e + \frac{Z_{\delta_c}}{m}\delta_c$$

$$\Delta\ddot{\theta} = \left(\frac{M_u}{I_{yy}} \cos\theta_0 + \frac{M_w}{I_{yy}} \sin\theta_0\right) \Delta V_x - \left(\frac{M_u}{I_{yy}} \sin\theta_0 - \frac{M_w}{I_{yy}} \cos\theta_0\right) \Delta V_z$$

$$+ \frac{M_q}{I_{yy}} \Delta\dot{\theta} - \left(\frac{M_u}{I_{yy}} W_0 - \frac{M_w}{I_{yy}} U_0\right) \Delta\theta + \frac{M_{\delta_e}}{I_{yy}} \delta_e + \frac{M_{\delta_c}}{I_{yy}} \delta_c$$

where at 110 kilometers/hr.

θ_0 = 4.75 deg, m = 416.0 slugs, I_{yy} = 76000.0 slug ft, u_0 = 100.6 ft^2/sec., W_0 - 8.4 ft/sec., g = 32.2 ft/sec^2.

$\frac{X_u}{m} = -.036 \frac{ft/sec^2}{ft/sec}$ $\qquad$ $\frac{Z_u}{m} = -.022 \frac{ft/sec^2}{ft/sec}$

$\frac{X_w}{m} = .089 \frac{ft/sec^2}{ft/sec}$ $\qquad$ $\frac{Z_w}{m} = -.802 \frac{ft/sec^2}{ft/sec}$

$\frac{X_q}{m} = .850 \frac{ft/sec^2}{rad.sec}$ $\qquad$ $\frac{Z_q}{m} = -1.814 \frac{ft/sec^2}{rad/sec}$

$\frac{X_{\delta_e}}{m} = .142 \frac{ft/sec^2}{in.}$ $\qquad$ $\frac{Z_{\delta_e}}{m} = .568 \frac{ft/sec^2}{in.}$

$\frac{X_{\delta_c}}{m} = .803 \frac{ft/sec^2}{in.}$ $\qquad$ $\frac{Z_{\delta_c}}{m} = -8.524 \frac{ft/sec^2}{in.}$

$\frac{M_u}{I_{yy}} = -.007 \frac{rad/sec^2}{ft/sec}$ $\qquad$ $\frac{M_w}{I_{yy}} = .014 \frac{rad/sec^2}{ft/sec}$

$\frac{M_q}{I_{yy}} = -1.460 \frac{rad/sec^2}{rad/sec}$ $\qquad$ $\frac{M_{\delta_e}}{I_{yy}} = .450 \frac{rad/sec^2}{in.}$

$\frac{M_{\delta_c}}{I_{yy}} = .068 \frac{rad/sec^2}{in.}$

Servo Actuator Dynamics

$$\ddot{\partial}_c + 2\zeta_A\omega_A\dot{\partial}_c + \omega_A^2\partial_c = \omega_A^2\delta_{cc}$$

$$\ddot{\partial}_e + 2\zeta_A\omega_A\dot{\partial}_e + \omega_A^2\partial_e = \delta_A^2\delta_{ec}$$

Rotor Dynamics

$$\ddot{\delta}_c + (\gamma\Omega/8)\ \dot{\delta}_c + \Omega^2\delta_c = \Omega^2\partial_c$$

$$\ddot{\delta}_e + (\gamma\Omega/8)\ \dot{\delta}_e + \Omega^2\delta_e = \Omega^2\partial_e$$

where

$\zeta_A = .60$ $\qquad\qquad \gamma = 10.0$

$\omega_A = 15.0$ rad/sec $\qquad\qquad \Omega = 28.0$ rad/sec

The equations can be rewritten as

$$\Delta\dot{V}_x = 0.02109\ \Delta V_x + 0.02352\ \Delta V_z + 0.69686\ \Delta\dot{\theta} - 29.6417\ \Delta\theta + 0.1879\ \delta_e + 0.09406\ \delta_c.$$

$$\Delta\dot{V}_z = -0.090393\ \Delta V_x - 0.802275\ \Delta V_z - 1.87830\ \Delta\dot{\theta} - 80.98\ \Delta\theta + 0.5524\ \delta_e - 8.5172\ \delta_c.$$

$$\Delta\ddot{\theta} = -\ 0.0058169\ \Delta V_x + 0.014531\ \Delta V_z - 1.46\ \Delta\dot{\theta} + 1.4672\ \Delta\theta + 0.450\ \delta_e + 0.068\ \delta_c.$$

$$\ddot{\delta}_c = -\ 35\ \dot{\delta}_c - 784\ \delta_c + 784\ \partial_c$$

$$\ddot{\delta}_e = -\ 35\ \dot{\delta}_e - 784\ \delta_e + 784\ \partial_e$$

Expressing these in the matrix form, the helicopter + rotor dynamics are described by the state equations

$$\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\ \underline{u}$$

$$\underline{y} = \hat{C}\underline{x}$$

where

$$\underline{x}^T = [\Delta\dot{V}_x \ \Delta\dot{V}_z \ \ \Delta\theta \ \ \Delta\ddot{\theta} \ \ \dot{\delta}_e \ \ \ddot{\delta}_e \ \ \dot{\delta}_c \ \ \ddot{\delta}_c]$$

$$\underline{y}^T = [\Delta V_x \ \ \Delta V_z \ \ \Delta\theta \ \ \ \Delta\dot{\theta}]$$

and the matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ are given by

$$\hat{A} = \begin{bmatrix} 0.0210 & 0.025 & -29.64 & 0.6968 & .1879 & 0 & -.0941 & 0 \\ -0.0903 & -0.802 & -80.98 & -1.878 & .5524 & 0 & -8.517 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.0058 & -0.0145 & 1.4672 & -1.460 & .45 & 0 & 0.068 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -784 & -35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -784 & -35 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 784 & 0 \\ 0 & 0 \\ 0 & 784 \end{bmatrix} \quad \text{and} \quad \hat{C} = \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \bigcirc \\ 0 & 0 & 0 & 1 & \end{array}\right]$$

The eigenvalues of the system computed by using Francis' [35] method are -2.3585084, 0.50432908, -0.19350035 ± j 0.35283477 and -17.5 ± j 21.857493 (double roots).

4.2 Helicopter Stabilization Using Output Feedback

The open loop of the CH-46 helicopter system has a pole, 0.50432908, in the right half s-plane and is unstable. The major part of the design is to stabilize the closed loop system. Let the desired closed loop poles be at -0.1, -0.2 + j 0.4, -0.2 - j 0.4, - 2.5, $-\rho_5 + j\,\rho_6$, $-\rho_5 - j\,\rho_6$, $-\rho_7 + j\,\rho_8$ and $\rho_7 - j\,\rho_8$ with $\rho_5, \rho_6, \rho_7, \rho_8 > 15.0$. The open loop poles and the desired closed loop poles are shown in Fig (1).

The first step in the design is to reduce the given system $\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\underline{u}$, $\underline{y} = \hat{C}\underline{x}$ to its diagonal or Jordan canonical form depending on the eigenvalues of $\hat{A}$. The transformation matrix T is made up of eigenvectors and generalized eigenvectors. An algorithm to compute T and the Jordan canonical form $T^{-1}AT$ is given in Appendix A. The Computation of the eigenvectors and the generalized eigenvectors depend on the accuracy with which the eigenvalues of $\hat{A}$ are computed. Francis algorithm is suggested for computing the eigenvalues.

The eigenvectors corresponding to the distinct roots are

$$\begin{bmatrix} 0.25710 \\ 1.00000 \\ 0.02003 \\ -0.04724 \\ 0.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \end{bmatrix}, \quad \begin{bmatrix} 1.00000 \\ 0.91564 \\ -0.01571 \\ -0.00792 \\ 0.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \end{bmatrix}, \quad \begin{bmatrix} -0.0956 \mp j\,0.6460 \\ 1.0000 + j\,0.0000 \\ -0.0074 \mp j\,0.0035 \\ -0.0027 \mp j\,0.0019 \\ 0.0000 + j\,0.0000 \\ 0.0000 + j\,0.0000 \\ 0.0000 + j\,0.0000 \\ 0.0000 + j\,0.0000 \end{bmatrix}$$

respectively. Each of the double roots has two eigenvectors associated with it. These are

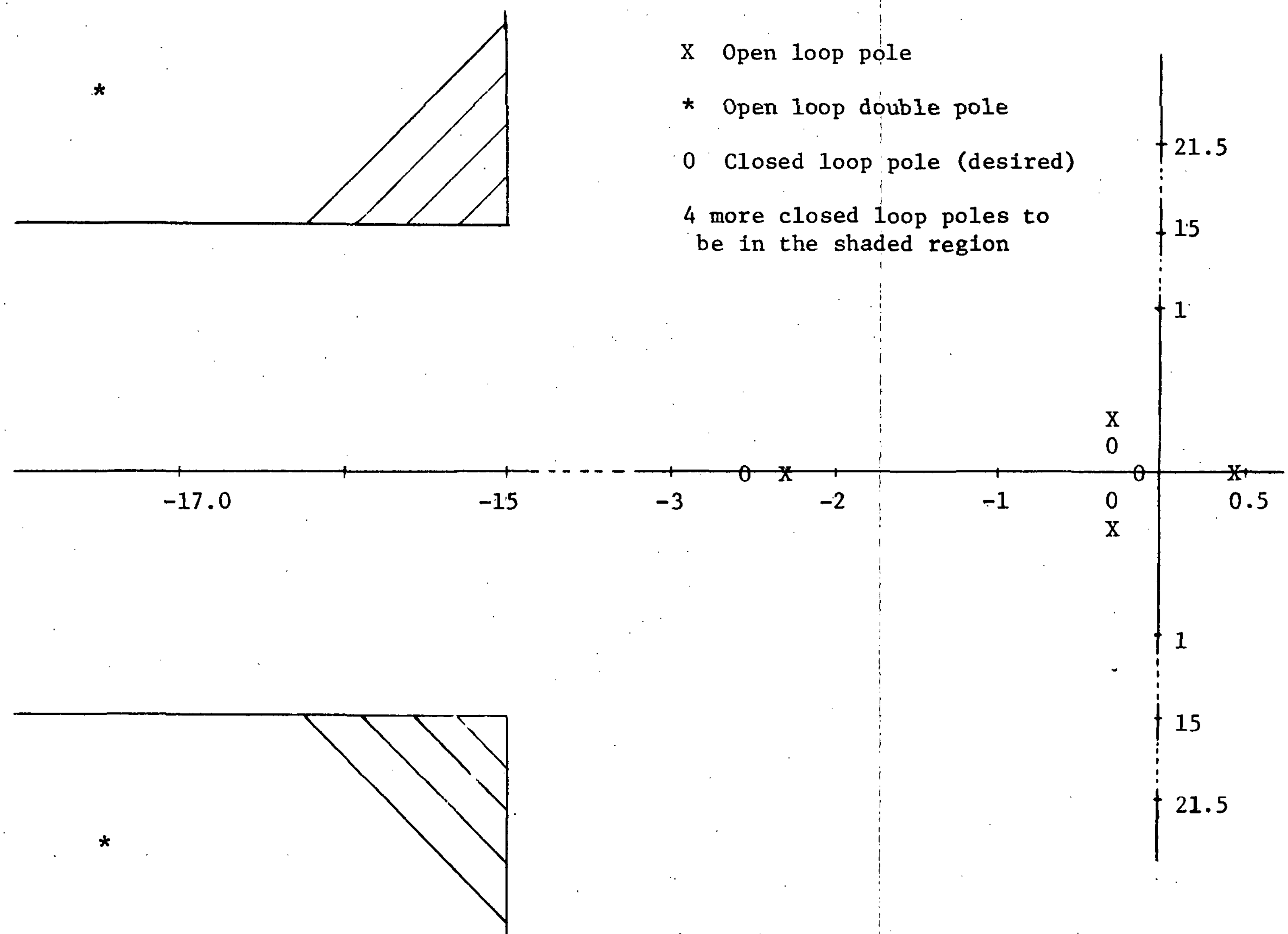

Figure 1

Pole Configuration of CH-46 Helicopter

$$\begin{bmatrix} -0.00488 \mp j\,0.0045 \\ -0.01348 \mp j\,0.0164 \\ -0.00015 \pm j\,0.0005 \\ -0.00979 \mp j\,0.0133 \\ 1.00000 + j\,0.0000 \\ -17.5000 \pm j\,21.8574 \\ 0.00000 + j\,0.00000 \\ 0.00000 + j\,0.00000 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.00205 \pm j\,0.00244 \\ 0.18777 \pm j\,0.24599 \\ -0.000028 \pm j\,0.00008 \\ -0.001432 \mp j\,0.0021821 \\ 0.000000 + j\,0.0000000 \\ 0.000000 + j\,0.0000000 \\ 1.000000 + j\,0.0000000 \\ -17.5000 \pm j\,21.8574 \end{bmatrix}$$

since the multiple eigenvalues have as many eigenvectors as their multiplicity, the Jordan canonical form for this matrix is diagonal and is given by

$$A = \text{diag}\,[-2.3585 \quad 0.5043 \quad -0.1935 + j\,0.3528 \quad -0.1935 - j\,0.3528 \quad -17.5 + j\,21.8574 \quad -17.5 + j\,21.8574 \quad -17.5 - j\,21.8574 \quad -17.5 - j\,21.8574]$$

The $B = T^{-1}\hat{B}$ and $\hat{C} = CT$ matrices are given by

$$B = \begin{bmatrix} -6.629 & -2.1981 \\ -10.215 & 0.5197 \\ 8.286 + j\,10.54 & -3.55 - j\,0.565 \\ 8.286 - j\,10.54 & -3.55 + j\,0.565 \\ -j\,17.93 & 0.0 \\ 0.0 & -j\,17.93 \\ j\,17.93 & 0.0 \\ 0.0 & j\,17.93 \end{bmatrix} \quad \text{and}$$

$$C = \left[\begin{array}{cccc:c} 0.2751 & 1.0 & -0.0956 - j\,0.646 & -0.0956 + j\,0.646 & \\ 1.0 & 0.9156 & 1 + j\,0 & 1 + j\,0 & \bigcirc \\ 0.02003 & -0.0157 & -0.0074 - j\,0.0035 & -0.0074 + j\,0.0035 & \\ 0.0472 & -0.0079 & -0.0027 - j\,0.0019 & -0.0027 + j\,0.0019 & \end{array}\right]$$

The open loop characteristic polynomial is

$$(s+2.358)(s-0.504)(s+0.193 - j\,0.352)(s+0.193 + j\,0.352)$$
$$(s+ 17.54 - j\,21.85)^2\,(s+17.54 + j\,21.85)^2$$
$$= s^8 + 72.24s^7 + 2949.57s^6 + 61117.75s^5 + 736774.84s^4 + 1360083.54s^3$$
$$-199841.81s^2 - 108957.17s - 118392.35$$

Let the desired closed loop poles be at $-\rho_1$, $-\rho_2$, $-\rho_3 + j\,\rho_4$, $-\rho_3 - j\,\rho_4$, $-\rho_5 + j\rho_6$, $-\rho_5 - j\,\rho_6$, $-\rho_7 + j\,\rho_8$ and $-\rho_7 - j\,\rho_8$. The requirements are

$$|\rho_1-2.5| \leq \varepsilon_1$$
$$|\rho_2-0.1| \leq \varepsilon_2$$
$$|\rho_3-0.2| \leq \varepsilon_3$$
$$|\rho_4-0.4| \leq \varepsilon_4$$
$$\rho_5,\rho_6,\rho_7,\rho_8 > 15.0$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ are small positive numbers.

The closed loop characteristic polynomial

$$= (s+\rho_1)(s+\rho_2)(s+\rho_3 - j\,\rho_4)(s+\rho_3 + j\,\rho_4)(s+\rho_5 - j\,\rho_6)(s+\rho_5 + j\,\rho_6)$$
$$(s+\rho_7 - j\,\rho_8)(s+\rho_7 + j\,\rho_8)$$
$$= s^8 + \ell_7 s^7 + \ell_6 s^6 + \text{----} + \ell_1 s^1 + \ell_0$$

where

$$\ell_7 = (a_3 + b_3)$$
$$\ell_6 = (a_2 + a_3 b_3 + b_2)$$
$$\ell_5 = (a_1 + a_2 b_3 + a_3 b_2 + b_1)$$

$$\ell_4 = (a_0 + a_1 b_3 + a_2 b_2 + a_3 b_1 + b_0)$$
$$\ell_3 = (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)$$
$$\ell_2 = (a_0 b_2 + a_1 b_1 + a_2 b_0)$$
$$\ell_1 = (a_0 b_1 + a_1 b_0)$$
$$\ell_0 = a_0 b_0$$

and

$$a_3 = 2(\rho_5 + \rho_7)$$
$$a_2 = \rho_5^2 + \rho_6^2 + \rho_7^2 + \rho_8^2 + 4\rho_5\rho_7$$
$$a_1 = 2\ (\rho_7\ (\rho_5^2 + \rho_6^2) + \rho_5(\rho_7^2 + \rho_8^2))$$
$$a_0 = (\rho_5^2 + \rho_6^2)(\rho_7^2 + \rho_8^2)$$
$$b_3 = \rho_1 + \rho_2 + 2\rho_3$$
$$b_2 = \rho_1\rho_2 + 2\rho_3\ (\rho_1 + \rho_2) + (\rho_3^2 + \rho_4^2)$$
$$b_1 = 2\rho_1\rho_2\rho_3 + (\rho_1 + \rho_2)(\rho_3^2 + \rho_4^2)$$
$$b_0 = \rho_1\rho_2\ (\rho_3^2 + \rho_4^2).$$

Let $f(\lambda) = \dfrac{\text{C.L.C.P.} - \text{O.L.C.P.}}{\text{O.L.C.P.}}$

$$= \sum_{i=1}^{4} \frac{\alpha_i}{(s-\lambda_i)} + \frac{\alpha_5^1}{(s-\lambda_5)} + \frac{\alpha_5^2}{(s-\lambda_5)^2} + \frac{\alpha_6^1}{(s-\lambda_6)} + \frac{\alpha_6^2}{(s-\lambda_6)^2}\ .$$

The coefficients in the partial fraction expansion can be evaluated by

$$\alpha_i = (s-\lambda_i)\ f(\lambda)\ \Big|_{s=\lambda_i} \qquad i = 1,2,3,4$$

$$\alpha_i^2 = (s-\lambda_i)^2\ f(\lambda)\ \Big|_{s=\lambda_i} \qquad i = 5,6$$

$${}^{1}_{i} = \frac{d}{ds}\ (s-\lambda_i)^2\ f(\lambda)\ \Big|_{s=\lambda_i} \quad i = 5,6$$

and they are found to be

$$\begin{aligned}\alpha_1 = {} & (7.3720\text{ E-09})\ell_7 + (1.4617\text{ E-08})\ell_6 + (2.8984\text{ E-08})\ell_5 \\ & + (5.7470\text{ E-08})\ell_4 + (1.1395\text{ E-07})\ell_3 + (2.2595\text{ E-07})\ell_2 \\ & + (4.4802\text{ E-07})\ell_1 + (8.8836\text{ E-07})\ell_0 + (1.4064\text{ E-09}).\end{aligned}$$

$$\begin{aligned}\alpha_2 = {} & (5.895\text{ E-05})\ell_7 - (2.4995\text{ E-05})\ell_6 + (1.0598\text{ E-05})\ell_5 \\ & - (4.4935\text{ E-06})\ell_4 + (1.9052\text{ E-06})\ell_3 - (8.0782\text{ E-07})\ell_2 \\ & + (3.4251\text{ E-07})\ell_1 - (1.4522\text{ E-07})\ell_0 - 1.3901\text{ E-04}.\end{aligned}$$

$$\begin{aligned}\alpha_3 = {} & (-1.8675\text{ E-09} - 1.4048\text{ E-09})\ell_7 + (-8.2945\text{ E-10} + 5.7478\text{ E-09})\ell_6 \\ & + (1.3514\text{ E-08} - 5.0609\text{ E-09})\ell_5 + (-2.7176\text{ E-08} - 2.3399\text{ E-08})\ell_4 \\ & + (-1.8511\text{ E-08} + 8.7174\text{ E-08})\ell_3 + (2.1206\text{ E-07} - 6.3834\text{ E-08})\ell_2 \\ & + (-3.9248\text{ E-07} - 3.8577\text{ E-07})\ell_1 + (-3.7156\text{ E-07} + 1.3161\text{ E-06})\ell_0 \\ & + (7.3906\text{ E-10} - 1.7790\text{ E-09})\end{aligned}$$

$\alpha_4 = \alpha_3^*$ (complex conjugate of α_3).

$$\begin{aligned}\alpha_5^1 = {} & (1.2734\text{ E+00})\ell_7 + (1.9839\text{ E-02})\ell_6 + (4.7370\text{ E-04})\ell_5 \\ & + (-2.4398\text{ E-05})\ell_4 + (2.1340\text{ E-07})\ell_3 + (7.0130\text{ E-08})\ell_2 \\ & + (-5.2240\text{ E-09})\ell_1 + (2.1875\text{ E+01}).\end{aligned}$$

$$\begin{aligned}\alpha_5^2 = {} & (-1.0465\text{ E-01} - 6.0017\text{ E-00})\ell_7 + (6.6289\text{ E-02} + 4.2575\text{ E-01})\ell_6 \\ & + (1.0390\text{ E-02} - 1.1351\text{ E-02})\ell_5 + (-5.4839\text{ E-04} - 3.6287\text{ E-05})\ell_4 \\ & + (1.1229\text{ E-05} + 1.6098\text{ E-05})\ell_3 + (1.9817\text{ E-07} - 6.7241\text{ E-07})\ell_2 \\ & + (-2.3170\text{ E-08} + 9.4842\text{ E-09})\ell_1 + (7.8160\text{ E-10} + 4.3427\text{ E-10})\ell_0 \\ & + (3.1423\text{ E-02} - 1.2372\text{ E-02})\end{aligned}$$

α_6^1 and α_6^2 are complex conjugates of α_5^1 and α_5^2 respectively.

The 8 non-linear equations are

$$\underline{d}^T\,\underline{c}_1\underline{b}_1\underline{f} = \alpha_1$$

$$\underline{d}^T\,\underline{c}_2\,\underline{b}_2\underline{f} = \alpha_2$$

$$\underline{d}^T\,\underline{c}_3\underline{b}_3\underline{f} = \alpha_3$$

$$\underline{d}^T\,\underline{c}_4\underline{b}_4\underline{f} = \alpha_3^*$$

$$\underline{d}^T\ (\underline{c}_5\underline{b}_5 + \underline{c}_6\underline{b}_6)\underline{f} = \alpha_5^1$$

$$\underline{d}^T\ (\underline{c}_7\underline{b}_7 + \underline{c}_8\underline{b}_8)\underline{f} = \alpha_6^1$$

$$0 = \alpha_5^2$$

$$0 = \alpha_6^2$$

Also notice that $(\underline{c}_5\underline{b}_5 + \underline{c}_6\underline{b}_6)$ and $(\underline{c}_7\underline{b}_7 + \underline{c}_8\underline{b}_8)$ should be complex conjugates.

The nonlinear equations are solved by minimizing the cost function

$$J_1 = q_1(\underline{d}^T\underline{c}_1\underline{b}_1\underline{f} - \alpha_1)^2 + q_2(\underline{d}^T\underline{c}_2\underline{b}_2\underline{f} - \alpha_2)^2 + q_3(\text{Real } \underline{d}^T\underline{c}_3\underline{b}_3\underline{f} - \text{Real } \alpha_3)^2 + q_4\ (\text{Im } \underline{d}^T\underline{c}_3\underline{b}_3\underline{f} - \text{Im } \alpha_3)^2 + q_5(\text{Real } \underline{d}^T(\underline{c}_5\underline{b}_5 + \underline{c}_6\underline{b}_6)\underline{f} - \text{Real } \alpha_5^1)^2 + q_6\ (\text{Im } \underline{d}^T(\underline{c}_5\underline{b}_5 + \underline{c}_6\underline{b}_6)\underline{f} - \text{Im } \alpha_5^1)^2$$

Subject to

$$\text{Real } \alpha_5^2 = 0$$

$$\text{Im } \alpha_5^2 = 0$$

$$|\rho_1 - 2.5| \leq \varepsilon_1,\ |\rho_2 - 0.1| \leq \varepsilon_2,\ |\rho_3 - 0.2| \leq \varepsilon_3,\ |\rho_4 - 0.4| \leq \varepsilon_4$$

and $\rho_5, \rho_6, \rho_7, \rho_8 \geq 15.0$. q_i, i=1,2,---6 are the weighting coefficients. This equivalent to minimizing the cost function.

$$J = J_1 + K_1\ (\rho_1 - 2.5)^2\ U_1 + K_2\ (\rho_2 - 0.1)^2\ U_2 + K_3\ (\rho_3 - 0.2)^2 + K_4\ (\rho_4 - 0.4)^2 + K_5\ (\rho_5 - 15)^2\ U_5 + K_6\ (\rho_6 - 15)^2 U_6 + K_7\ (\rho_7 - 15)^2\ U_7 + K_8\ (\rho_8 - 15)^2\ U_8 + q_7(\text{Real } \alpha_5^2)^2 + q_8\ (\text{Im } \alpha_5^2)^2$$

where

$U_1 = 1$ if $|\rho_1 - 2.5| > 0.1$

$= 0$ otherwise

$U_2 = 1$ if $|\rho_2 - 0.1| > 0.01$

$= 0$ otherwise

$U_3 = 1$ if $|\rho_3 - 0.2| > 0.01$

$= 0$ otherwise

$U_4 = 1$ if $|\rho_4 - 0.4| > 0.02$

$= 0$ otherwise

$U_i = 1$ if $\rho_i - 15 < 0 \quad i = 5,6,7,8$

$= 0$ otherwise

q_7, q_8 and K_i, $i = 1,2,$---8 are the weighting coefficients

The results of the optimization using the conjugate gradient method with different set of weighting coefficients are shown in Table 3 .

Weighting Coefficients	Gain Matrix				Closed Loop Poles
$q_1, --- q_8 = 0.01$	0.154	-0.407	0.207	0.169	-2.45, -0.093
$K_1, --- K_8 = 10.0$	1.281	-3.15	1.605	1.309	$-0.181 \pm j\ 0.418$
					$-20.74 \pm j\ 18.9$ (double pole)
$q_1, q_2, --- q_8 = 0.01$	0.165	-0.407	0.207	0.169	-2.45, -0.0896
$K_1, K_2, --- K_8 = 100$	1.281	-3.15	1.605	1.309	$-0.187 \pm j\ 0.399$
					$-20.77 \pm j\ 18.18$ (double pole)
$q_1, --- q_8 = 0.01$	0.135	-.316	0.296	0.217	-2.50 -0.116
$K_1 = 100$, $K_2 = 1$,					
$K_3 = 100$, $K_4 = 1$,					$-0.180 \pm j\ 0.442$
$K_5, --- K_8 = 10$	1.425	-3.35	3.13	2.3	$-19.91 \pm j\ 17.51$ (double pole

TABLE 3

Closed Loop Poles and Feedback

Gains For The Helicopter

4.3 Discussion

This chapter has demonstrated that approximate pole placement can be achieved in complex systems using output feedback. It is assumed that the systems are output stabilizable. The different weighting coefficients give rise to sets of acceptable pole configurations and the corresponding output feedback gains.

V. ZEROS IN MULTIVARIABLE SYSTEMS

5.1 Introduction

The previous chapters have focussed on the design of multivariable systems with the location of the closed loop poles as the design criterion. Although, pole locations are an important element in the specification of satisfactory control, they are by no means sufficient in themselves. The dynamic response of the system also depends on the zeros of the system. This chapter reviews the different types of zeros in multivariable systems and their significance to multivariable system design. Further, the advantages and limitations of using a unity rank feedback matrix to provide a total design which includes both poles and zeros are examined.

5.2 Zeros of the Numerator Polynomial

The zero problem is well defined and clearly understood in the single-input single-output (SISO) case. The zeros are the roots of the numerator polynomial of the transfer function and affect the transient behaviour of the system. The zeros of the SISO system are invariant under state feedback. Brockett [36] has shown that the zeros are the poles of the inverse system. Loscutoff, Schenz and Beyer [37] have shown that the zeros of any system with either a single input or a single output are invariant under any feedback policy, i.e. either state or output feedback.

Two types of zeros are defined in the literature on multivariable systems - (i) the roots of the numerator polynomials of the transfer function matrix. These will be referred to as zeros in subsequent discussion. (ii) the roots of the numerator polynomials of the Smith McMillan canonical form of the transfer function matrix. These will be referred to as McMillan Zeros in subsequent discussion.

There are certain difficulties in extending the known results about the zeros in SISO systems to multivariable systems. There may be a large number of zeros, as many as mp(n-1) in an n^{th} order system with m inputs and p outputs, in a multivariable system. Furthermore, the movement of the zeros in the s-plane with feedback cannot be as readily predicted as the movement of the poles of the system. In the multivariable case the eigenvalues of the inverse system to not correspond to the zeros of the particular transfer function G(s), but rather they correspond to the zeros of $|G(s)|$ which, it will be seen, bears a relation to the McMillan Zeros. Simon and Mitter [38] have generalized the results on [36] to that for a special class of systems the poles of which can be moved arbitrarily using state feedback while the zeros are invariant. This class of systems called "systems with disjoint control" have distinct eigenvalues and are completely controllable. Further, each actuating vector influences a different set of eigenvalues. The conditions which guarantee zero invariance are very restrictive, and the general problem of identifying invariant zeros remains unsolved. Chen [39] has attempted to place certain zeros and poles by using a sequential design approach that takes advantage of the invariance of zeros under single input feedback. However, his design is limited to placing zeros in only one component of the transfer function matrix, and seems to have exploited the possibilities of using zero invariance in the designing technique.

It is apparent that most of the problems regarding zero placement are due to two inherent facts - (i) considering the number of zeros in the system, exact zero placement would be too demanding for the design freedom available. (ii) There is no readily available method of predicting zero movements with feedback as compared to the movement of poles with feedback [40]. (iii) Even if zeros can be placed at will the problem remains of where indeed they should be placed. Hence some simplifications of the problem statement is necessary and some of these are considered in Section 5.4.

5.3 McMillan Zeros of a System

Let $G(s) = C(sI-A)^{-1}B$ be the (pxm) transfer function matrix of the system. Then, G(s) can be expressed as

$$G(s) = P(s)M(s)Q(s) \tag{5.1}$$

where

(a) P(s) and Q(s) are pxp and mxm polynomial matrices, respectively, which have constant, nonzero determinants, i.e. P(s) and Q(s) are unimodular.

(b) M(s) is a pxm Smith-McMillan canonical matrix [41] of G(s)

$$M(s) = \left[\begin{array}{ccc|c} \frac{e_1(s)}{\psi_1(s)} & & & \\ & \ddots & & 0 \\ & & \frac{e_k(s)}{\psi_k(s)} & \\ \hline & 0 & & 0 \end{array}\right] \tag{5.2}$$

satisfying the conditions (i) $e_i(s)$ and $\psi_i(s)$ are relatively prime monic polynomials for $i=1,2,---k$. k denotes the rank of $G(s)$. (ii) each $e_i(s)$ is a factor of $e_{i+1}(s)$ and each $\psi_{i+1}(s)$ is a factor of $\psi_i(s)$ for $i = 1,2,---,k-1$. The factors $r_i(s) = e_i(s)/\psi_i(s)$, $i = 1,2,---k$ are referred to as the invariant factors of $G(s)$ [43].

The McMillan zeros of the system are the roots of $e_k(s) = 0$. The roots of $\psi_1(s) = 0$ give all the poles of the system. Although, the McMillan zeros have certain system significant properties they do not have any direct relation to the zeros of the system as discussed in 5.2.

Given $G(s)$, the Smith-McMillan form can be obtained in the following manner. Let $d(s)$ be the monic least common denominator of the elements of G, and write $G = N(s)/d(s)$. Now, the polynomial matrix $N(s)$ can be brought to Smith form by the transformation

$$L(s)N(s)R(s) = S(s) \tag{5.3}$$

where $L(s)$ and $R(s)$ are unimodular matrices. Recall that in the Smith form of $N(s)$,

$$s(s) = \left[\begin{array}{cccc:c} \varepsilon_1(s) & & & & \\ & \varepsilon_2(s) & & & \bigcirc \\ & & \ddots & & \\ & & & \varepsilon_k(s) & \\ \hdashline & \bigcirc & & & \bigcirc \end{array}\right] \tag{5.4}$$

the polynomial $\varepsilon_i(s)$ is the greatest common divisor of all minors of order i of the matrix $S(s)$. In the rational matrix $S(s)/d(s)$, there may be common factors between numerator and denominator in the elements on the leading diagonal. On cancelling these common factors, we get the Smith-McMillan form of G(s). This procedure is illustrated in Example 5.3.1.

5.3.1 <u>Example:</u>

Let

$$G(s)=\begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+2)^2} \end{bmatrix}$$

Then, $d(s) = (s+1)^2(s+2)^2$

$$N(s) = \begin{bmatrix} (s+2)^2 & (s+1)(s+2) \\ (s+1)(s+2) & (s+1)^2(s+3) \end{bmatrix} ,$$

$$S(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)^2(s+2)^3 \end{bmatrix}$$

and the Smith-McMillan form

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & (s+2) \end{bmatrix}$$

Thus, $\varepsilon_1(s) = 1$, $\varepsilon_2(s) = (s+2)$, $\psi_2(s) = 1$ and $\psi_1(s) = (s+1)^2(s+2)^2$. The system has poles at $s=-1$ and $s=-2$, a zero at $s=-3$ and a McMillan zero at $s=-2$.

5.3.2 Effect of Feedback on the McMillan Zeros

Consider the system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad y = C\underline{x} \tag{5.5}$$

with n states, m number of inputs and outputs and (A,B) controllable. The transfer function matrix $G(s) = C(sI-A)^{-1}B$ has the McMillan form $\mathrm{Diag}[e_1(s)/\psi_1(s) \text{ --- } e_m(s)/\psi_m(s)]$. If feedback is applied to the system according to the rule

$$\underline{u} = v-\underline{y} \tag{5.6}$$

let the closed loop transfer function matrix be

$$H(s) = [I_m + G(s)]^{-1}\, G(s)$$

with McMillan form $\mathrm{Diag}\,[e_1(s)/\psi_1'(s) \text{ ---} e_m(s)/\psi_m'(s)]$.

Rosenbrock [42] has shown that C can be chosen such that

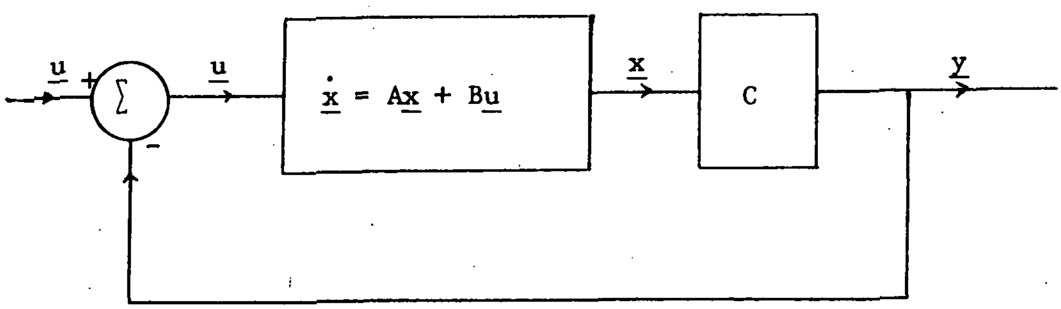

Figure 2 System In Rosenbrock's Pole-Zero Allocation Problem

(i) $e_i(s)$ in the McMillan form of G(s) are arbitrary monic polynomials satisfying the necessary conditions.

(a) e_i divides e_{i+1}

(b) $\delta(e_1) + \delta(e_2) + \cdots + (e_m) \leq n-m$

where $\delta(e_i)$ is the degree of the polynomial $\varepsilon_i(s)$

(ii) $\psi_i'(s)$ in the McMillan form of H(s) are arbitrary monic polynomials satisfying the necessary conditions

(a) ψ_i' divides ψ_{i-1}'

(b) $\delta(\psi_1' \psi_2' \cdots \psi_m') = n.$

5.3.3 Role of McMillan Zeros in Multivariable Systems

The McMillan Zeros are the roots of the polynomial $e_k(s)$. The polynomial $e_k(s)$ plays an important role in certain aspects of multivariable system design, and in some respects are similar to that of the numerator polynomial in SISO systems. In the SISO case, $e_k(s)$ is a scalar multiple of the numerator polynomial.

A system $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x} + D\underline{u}$ is defined by some authors to be minimum phase if all the roots of the polynomial $e_k(s)$ are in the left half-plane. Moore and Silverman [43] have shown that a stable psuedoinverse of G(s) exists if and only if (A,B,C,D) is, according to the above definition, minimum phase. For a system with an equal number of inputs and outputs (=k)

$$|G(s)| = e_1(s)\, e_2(s) \cdots e_k(s)/\psi_1(s) \cdots \psi_k(s).$$

Minimum phase properties of a system arise in connection with the linear regulator problem. Consider the minimal system

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

$$y(t) = C\underline{x}(t)\ ,\ \underline{x}(t_0) = \underline{x}_0$$

with quadratic cost function

$$V(Q,R,\underline{x}_0,\rho) = \int_0^\infty [\underline{y}^T(t)\; Q\; \underline{y}(t) + \underline{u}^T(t)\; \rho R\underline{u}(t)]\; dt$$

where Q and R are positive definite matrices and ρ is a scalar. In designing a control system, it is usually necessary to make a tradeoff between achieving better performance and using smaller control forces. By increasing the amplitudes of the control variables it is possible to achieve smaller deviations of the controlled variable from its desired trajectory. Systems with unlimited accuracy are those for which the performance index can be reduced to zero (i.e. the deviation is instantaneously reduced to zero) if the amplitudes of the inputs are allowed to be arbitrarily large.

Let $\underline{u}^*(\rho,t)$ denote the control which minimizes the cost function and let $\underline{y}^*(\rho,t)$ be the output of the system. Kwakernaak and Sivan [44] define a system to be of unlimited accuracy if and only if

$$\lim_{\rho \to 0} \int_0^\infty \underline{y}^{*T}(\rho,t)\; Q\; \underline{y}^*(\rho,t)\; dt = 0$$

for all $\underline{x}_0$. They have shown that the necessary and sufficient condition for achieving unlimited accuracy is that (i) the number of inputs be at least as large as the number of controlled variables, and (ii) the system should be minimum phase.

At present it is not known to what extent the McMillan zeros would be useful in developing algorithms to design multivariable systems. Apart from [42-44] very little has been done about using McMillan zeros in system design. The McMillan zeros appear to have no bearing upon design for conventional pole zero placement. This is a potential area for future research.

5.4 Design Freedom Using Unity Rank Feedback For Pole Zero Placement

So far, unity rank feedback has been used to place the poles of the closed loop system. Consider the system defined by the equations (2.1) and (2.2). Here, they are repeated for convenience.

$$\underline{\dot{x}} = \hat{A}\underline{x} + \hat{B}\underline{u} \qquad (5.6)$$

$$\underline{y} = \hat{C}\underline{x} \qquad (5.7)$$

Let the state feedback

$$\underline{u} = -K\underline{x} = -\underline{f}\underline{d}^T\underline{x} \qquad (5.8)$$

be used to place the closed loop poles. From (2.32). for pole placement with state feedback we have to satisfy the equation

$$d_i = \alpha_i/\underline{b}_i\underline{f} \qquad i = 1,2,\text{---}\ n \qquad (5.9)$$

$\underline{d}$ is chosen to satisfy (5.9) and we are free to choose $\underline{f}$ to satisfy some other design requirement in addition to pole placement. The feedback gain matrix K is given by

$$K=\underline{f}\underline{d}^T=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} [d_1\ d_2\ \text{---}\ d_n]$$

$$= \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \left[\frac{\alpha_1}{\underline{b}_1\underline{f}}\quad \frac{\alpha_2}{\underline{b}_2\underline{f}} \cdots \frac{\alpha_n}{\underline{b}_n\underline{f}}\right] \qquad (5.10)$$

$$= \begin{bmatrix} \dfrac{\alpha_1 f_1}{\underline{b}_1 \underline{f}} & \cdots\cdots\cdots & \dfrac{\alpha_n f_1}{\underline{b}_n \underline{f}} \\ \vdots & & \vdots \\ \dfrac{\alpha_1 f_m}{\underline{b}_1 \underline{f}} & & \dfrac{\alpha_n f_m}{\underline{b}_n \underline{f}} \end{bmatrix}$$

Notice that we can normalize $\underline{f}$ by writing

$$\underline{f} = f_1 \cdot \underline{f}_n$$

where $\underline{f}_n = [1 \; f_2/f_1 \; \text{----} \; f_m/f_1]^T$. Now,

$$K = \underline{f}\underline{d}^T = f_1 \underline{f}_n \left[\frac{\alpha_1}{f_1 \underline{b}_1 \underline{f}_n} \; - - - - \; \frac{\alpha_n}{f_1 \underline{b}_n \underline{f}_n} \right]$$

$$= \underline{f}_n \left[\frac{\alpha_1}{\underline{b}_1 \underline{f}_n} \; - - - - - \; \frac{\alpha_n}{\underline{b}_n \underline{f}_n} \right]$$

Thus normalizing does not affect K.

The ijth element of K has the form

$$[K]_{ij} = \frac{\alpha_j f_i}{b_j \underline{f}}$$

$$= \frac{\alpha_j f_i}{(b_{j1} f_1 + b_{j2} f_2 + \text{---} + b_{jm} f_m)}$$

$$= \frac{\alpha_j f_i / f_k}{(b_{j1} \frac{f_1}{f_k} + b_{j2} \frac{f_2}{f_k} + \text{---} + b_{jm} \frac{f_m}{f_k})} \tag{5.11}$$

This shows that we are free to choose the quantities

$$f_j/f_k, \quad j = 1,\text{---},m, \quad j \neq k$$

Thus, there are (m-1) degrees of freedom left after pole placement. This freedom in choosing K can be used to (a) restrict magnitude of feedback gains (b) design for acceptable steady state behavior (c) zero-placement.

5.4.1 Example:

This example illustrates the freedom in design using unity rank feedback. The simplified model of a d.c. to a.c. rotary converter [53] can be described by the state equation

$$\dot{\underline{x}} \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} -4 & 0 \\ -4 & -2 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} \tag{5.12}$$

The transfer function matrix $G(s) = C(sI-A)^{-1}B$ is given by

$$\begin{bmatrix} \frac{-4}{(s+6)} & \frac{4}{(s+6)(s+2)} \\ \frac{-4}{s+6} & -\frac{2(s+4)}{(s+6)(x+2)} \end{bmatrix}$$

The system has two poles at S= -6 and S= 12 and a zero at S= -4. State feedback is used to place the closed loop poles at -5 and -1. There is one degree (m=2 in this case) of freedom left after placing the poles.

Choosing

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

system (5.12) can be transformed into

$$\dot{\underline{x}} = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix} \underline{x}$$

The C.L.C.P. = $s^2 + 6s + 5$ and the O.L.C.P. = $s^2 + 8s + 12$. Also

$$\frac{\text{C.L.C.P.}}{\text{O.L.C.P.}} - 1 = \frac{-(2s+7)}{(s+6)(s+2)} = \frac{-3/4}{(s+2)} + \frac{-5/4}{(s+6)} .$$

Hence, $\alpha_1 = -3/4$ and $\alpha_2 = -5/4$

From (5.9),

$$d_1 = \alpha_1/\underline{b}_1\underline{f} = -3/4f_2$$

$$d_2 = \alpha_2/\underline{b}_2\underline{f} = 5/4\ (4f_1 + f_2).$$

Now, $BK = B\underline{fd}^T = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} [d_1 \quad d_2]$

$$= \begin{bmatrix} f_2 d_1 & f_2 d_2 \\ -4f_1 d_1 - f_2 d_1 & -4f_1 d_2 - f_2 d_2 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 & \frac{5}{4} \dfrac{f_2}{4f_1 + f_2} \\ \frac{3}{4} \dfrac{4f_1 + f_2}{f_2} & -5/4 \end{bmatrix}$$

The closed loop transfer function matrix $H(s) = C(sI-A + BK)^{-1}B$ is equal to

$$H(s) = \frac{1}{s^2 + 6s + 5} \begin{bmatrix} -4(s-a+5/4 & 14/4+a-b \\ -4(s+a+5/4) & -2(s+3+\frac{a+b}{2}) \end{bmatrix}$$

where

$$a = 5/4 \frac{f_2}{4f_1+f_2} \quad \text{and } b = 3/4 \frac{(4f_1+f_2)}{f_2} .$$

The zeros of the system are at

$$s= a-5/4, \quad s= -a-5/4, \text{ and } \quad s= -3- \frac{1}{2}(a+b).$$

Notice, that although we can choose f_1 and f_2 independently the zero location is affected only by the ratio f_1/f_2. Due to this only one zero can be placed.

Let us position the zero corresponding to first input and first output at -2.

$$\text{i.e.} \quad a - 5/4 = -2$$

$$\frac{5}{4} \left[\frac{f_2}{4f_1+f_2} -1\right] = -2.$$

This gives $f_1/f_2 = - 2/3$. Choosing $f_2 = 1$ gives $f_1 = -2/3$

$$d_1 = -3/4, \ d_2 = -3/4 \text{ and } K = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} .$$

With this value of K the closed loop poles are at -1 and -5 and one of the zeros at -2.

Next, the system is re-designed for approximate pole-zero placement and diagonal dominance at steady state using state feedback.

Let the closed loop poles be at $-\rho_1$ and $-\rho_2$ and the zeros be at $-\xi_1$, $-\xi_2$ and $-\xi_3$. α_1 and α_2 are functions of ρ_1 and ρ_2 and are equal to

$$\frac{1}{4}[-2(\rho_1 + \rho_2 - 8) + (\rho_1\rho_2 - 12)]$$

and

$$\frac{1}{4}[6(\rho_1 + \rho_2 - 8) - (\rho_1\rho_2 - 12)] \quad \text{respectively.}$$

This results in

$$(sI-A+BK) = \begin{bmatrix} s+2+\alpha_1 & -\alpha_2 f_2/(4f_1 + f_2) \\ -\alpha_1 \dfrac{(4f_1+f_2)}{f_2} & s+6+\alpha_2 \end{bmatrix}$$

and

$$(sI-A+BK)^{-1} = \frac{1}{\Delta} \begin{bmatrix} (s+6+\alpha_2) & \dfrac{\alpha_2 f_2}{4f_1 + f_2} \\ \dfrac{\alpha_1(4f_1+f_2)}{f_2} & s+2+\alpha_1 \end{bmatrix}$$

Where $\Delta = s^2 + (\rho_1 + \rho_2)\, s + \rho_1\rho_2$.

The closed loop transfer function matrix

$$H(s) = \frac{1}{\Delta} \begin{bmatrix} -4(s+2+\alpha_1+\alpha_2\ell) & 4-(\alpha_2\ell+\alpha_1) + (\alpha_2 - \alpha_1/\ell) \\ -4(s+2+\alpha_1 - \alpha_2\ell) & -2[s+4+\frac{1}{2}(\alpha_2\ell + \frac{\alpha_1}{\ell}) - \frac{1}{2}(\alpha_1 + \alpha_2)] \end{bmatrix}$$

where

$$\ell = \frac{f_2}{4f_1 + f_2}.$$

The poles of the system are at $-\rho_1$, $-\rho_2$ and the zeros at $-\xi_1 = -(2+\alpha_1 + \alpha_2 \ell)$, $-\xi_2 = -(2 + \alpha_1 - \alpha_2 \ell)$ and $-\xi_3 = -(4 + \frac{1}{2}(\alpha_2 \ell + \frac{\alpha_1}{\ell}), - \frac{1}{2}(\alpha_1 + \alpha_2)]$. Let the desired closed loop poles and zeros be at $-\rho_{1d}$, $-\rho_{2d}$, $-\xi_{1d}$, $-\xi_{2d}$ and $-\xi_{3d}$ respectively. Thus, we have to satisfy the equations

$$d_1 = \alpha_1 / f_2$$

$$d_2 = -\alpha_2/(4f_1 + f_2)$$

$$\alpha_1 = \frac{1}{4}[-2(\rho_1 + \rho_2 - 8) + (\rho_1 \rho_2 - 12)]$$

$$\alpha_2 = \frac{1}{4}[6(\rho_1 + \rho_2 - 8) - (\rho_1 \rho_2 - 12)]$$

$$\xi_1 = 2 + \alpha_1 + \alpha_2 \ell$$

$$\xi_2 = 2 + \alpha_1 - \alpha_2 \ell$$

$$\xi_3 = 4 + \frac{1}{2}(\alpha_2 \ell + \frac{\alpha_1}{\ell}) - \frac{1}{2}(\alpha_1 + \alpha_2)$$

$$\ell = f_2/(4f_1 + f_2)$$

Let $\xi_4 = 4 - (\alpha_2 \ell + \alpha_1) + (\alpha_2 - \alpha_1/\ell)$ and $\alpha_3 = 4f_1 + f_2$. For diagonal dominance with pole zero placement we have to satisfy (5.13) subject to the constraint $\xi_1 \xi_3 \geq 5 \xi_2 \xi_4$. This can be done by minimizing the cost function

$$\begin{aligned} J = {} & K_1(f_2 d_1 - \alpha_1)^2 + K_2(d_2\alpha_3 + \alpha_2)^2 + K_3(4\xi_1\alpha_3 - 8\alpha_3 - 4\alpha_1\alpha_3 - 4f_2\alpha_2)^2 \\ & + K_4(8\xi_3\alpha_3 f_2 - 32\alpha_3 f_2 - 4\alpha_2 f_2^2 - 4\alpha_1\alpha_3^2 + 4\alpha_3(\rho_1 + \rho_2 - 8))^2 \\ & + K_5(4\xi_2\alpha_3 - 8\alpha_3 - 4\alpha_1\alpha_3 + 4\alpha_2 f_2)^2 + K_6(4\xi_4 f_2\alpha_3 - 16\alpha_3 f_2 - 4f_2\alpha_2\alpha_3 \\ & + 4\alpha_2 f_2^2 + 4\alpha_1\alpha_3 f_2 + 4\alpha_1\alpha_3^2)^2 + (\rho_1 - \rho_{1d})^2 U_1 + (\rho_2 - \rho_{2d})^2 U_2 \\ & + (\xi_1 - \xi_{1d})^2 U_3 + (\xi_2 - \xi_{2d})^2 U_4 + (\xi_3 - \xi_{3d})^2 U_5 + (|\xi_1\xi_3| - 5|\xi_2\xi_4|)^2 U_6. \end{aligned}$$

Desired Pole-Zero Configuration	Closed Loop Poles and Zeros By Minimization	Gain Matrix	Weighting Coefficients	Remark
Poles: -5,-1 Zeros: -2,	Poles: -5.01,-1.08 Zeros: -1.966	0.502 0.454 -0.698 -0.631	$K_1,K_2,K_3=1$, $K_4,K_5,K_6=0$ $U_4,U_5,U_6=0$	Poles and a zero placed satisfactory
Poles: -4.5,-3.0 Zeros: -2.0	Poles: -4.37, -2.90 Zeros: -2.0, -2.0, -3.0; $\xi_4=0.311$	0.420 0.239 0.533 0.303	$K_1,K_2=1$, K_3,K_4, $K_5=0$, $K_6=1$ $U_3,U_4,U_5=0$	Pole placement and diagonal dominance
Poles: 04.5 -3.0 Zeros: -2.0,-1.8, -3.0	Poles: -4.66,-2.56 Zeros: -2.09, -2.62, -3.0, $\xi_4=-0.3$	0.299 0.220 0.366 0.269	K_1, --- $K_6=1$	Pole-zero placement with diagonal dominance

Table 4 Pole Zero Placement Using Unity Rank Feedback

U_1, U_2, ... U_6 are functions similar to those defined in Chapter III.

The results of the minimization are shown in Table 4.

Table 4 shows that there is a degradation in performance as more and more requirements are placed on the design. This is to be expected because of the limited amount of design freedom. Thus the (m-1) degrees of freedom can be used to satisfy other design requirements in addition to placing poles of the system.

CHAPTER VI. CONCLUDING REMARKS

6.1 Conclusions

A new method is proposed for designing multivariable systems. The design is primarily based on an alternate derivation of Davison's theorem on pole placement and the solution of the nonlinear equations for the feedback gains by the least square error method. Output feedback is used to control a complex dynamical system. The freedom in design, after allocating poles, is used to place zeros and/or satisfy other design objectives. Throughout, the design is carried out using unity rank feedback gain matrices. This has a number of consequences. On the one hand it results in algorithms which are computationally attractive. However, this is done at a considerable sacrifice in terms of the design freedom available. For a system with m inputs and p outputs we can choose only (m+p) variables instead of mp variables.

6.2 Areas for Further Investigation

There are several natural extensions and areas for further investigation which follow from the work reported here. Some of these are (i) study of pole-zero placement using feedback gains of rank greater than one (ii) use of dunamic compensator for pole zero placement and (iii) pole and McMillan Zero placement using feedback.

The design procedure can be logically extended to feedback matrices of higher rank than one. The procedure is illustrated for a third order system.

Consider the system

$$\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\underline{u}$$

$$\underline{y} = \hat{C}\underline{x} \qquad (6.1)$$

where $\hat{A}$ is a 3x3 matrix and has distinct eigenvalues λ_1, λ_2, and λ_3. The system (6.1) can be transformed into

$$\dot{\underline{x}} = \Lambda\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x} \qquad (6.2)$$

by a similarity transformation.

The O.L.C.P. = $|sI-A| = \Delta = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)$, and the C.L.C.P. = $|sI- \Lambda + BKC|$. Let m_{ij} be the ij^{th} element of matrix M where M = BKC. Also, $m_{ij} = \underline{b}_i K\underline{c}_j$ where $\underline{b}_i = i^{th}$ row of B and $\underline{c}^j$ is the j^{th} column of C. The C.L.C.P. can be expressed as

$$|sI - \Lambda + BKC| = \Delta + \frac{m_{11}\cdot\Delta}{sI-\lambda_1} + \frac{m_{22}\cdot\Delta}{sI-\lambda_2} + \frac{m_{33}\cdot\Delta}{sI-\lambda_3}$$

$$+ \frac{n_1\cdot\Delta}{(sI-\lambda_2)(sI-\lambda_3)} + \frac{n_2\cdot\Delta}{(sI-\lambda_3)(sI-\lambda_1)}$$

$$+ \frac{n_3\cdot\Delta}{(sI-\lambda_1)(sI-\lambda_2)} + |M|. \qquad (6.3)$$

where n_i is the co-factor of m_{ii}. If K has rank one then equation (6.3) reduces to

$$|sI- \Lambda + BKC| = \Delta + \frac{m_{11}\cdot\Delta}{(sI-\lambda_1)} + \frac{m_{22}\cdot\Delta}{(sI-\lambda_2)} + \frac{m_{33}\cdot\Delta}{sI-\lambda_3} \qquad (6.4)$$

Notice that if $K = \underline{f}\underline{d}^T$, $m_{ii} = \underline{b}_i KC = \underline{b}_i\ \underline{f}\underline{d}^T\underline{c}^i = \underline{d}^T\underline{c}^i\underline{b}_i\underline{f}$.

Now

$$\frac{\text{C.L.C.P.} - \text{O.L.C.P.}}{\text{O.L.C.P.}} = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \qquad (6.5)$$

From (6.3),

$$\frac{|sI - \Lambda + BKC| - \Delta}{\Delta} = \sum_{i=1}^{3} \frac{m_{ii}}{(s-\lambda_i)} + \frac{n_1}{(s-\lambda_2)(s-\lambda_3)}$$

$$+ \frac{n_2}{(s-\lambda_3)(s-\lambda_1)} + \frac{n_3}{(s-\lambda_1)(s-\lambda_2)} + \frac{|M|}{\Delta}.$$

$$= \sum_{i=1}^{3} \frac{m_{ii}}{(s-\lambda_i)} + \left[\frac{n_2}{(\lambda_1-\lambda_3)} + \frac{n_3}{(\lambda_1-\lambda_2)} + \frac{|M|}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right]\frac{1}{(s-\lambda_1)}$$

$$+\left[\frac{n_1}{(\lambda_2-\lambda_3)} + \frac{n_3}{(\lambda_2-\lambda_1)} + \frac{|M|}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}\right]\frac{1}{(s-\lambda_2)}$$

$$+\left[\frac{n_1}{(\lambda_3-\lambda_2)} + \frac{n_2}{(\lambda_3-\lambda_1)} + \frac{|M|}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]\frac{1}{(s-\lambda_3)} \qquad (6.6)$$

Comparing (6.5) and (6.6), for pole placement we have to satisfy the n equations

$$m_{11} + \frac{n_2}{(\lambda_1-\lambda_3)} + \frac{n_3}{(\lambda_1-\lambda_2)} + \frac{|M|}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = 1\cdot$$

$$m_{22} + \frac{n_1}{(\lambda_2-\lambda_3)} + \frac{n_3}{(\lambda_2-\lambda_1)} + \frac{|M|}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} = \alpha_2 \qquad (6.7)$$

$$m_{33} + \frac{n_1}{(\lambda_3-\lambda_2)} + \frac{n_2}{(\lambda_3-\lambda_1)} + \frac{|M|}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = \alpha_3$$

in mp variables. It is interesting to note that for $K = fd^T$,

(i) $n_1, n_2, n_3, |M| = 0$

(ii) $m_{ii} = \underline{d}^T \underline{c}^i \underline{b}^i \underline{f} \qquad i = 1,2,3$

and (6.7) reduces to (2.10).

This is not intended to be a complete treatment of the pole placement problem using output feedback matrix gains of rank greater than one. It is introduced to show that the techniques discussed in the previous chapters can be easily modified to take advantage of the increased design freedom afforded by K of rank greater than unity. A comparison between equations (6.7) and (2.10) shows the increased amount of computation and complexity.

Brasch and Pearson [21] have used a dynamic compensator to place all the poles of the system using output feedback. The feedback gains are not unique and this design freedom can be used to place zeros or satisfy other systems requirements.

It has been pointed out in Chapter 5 that the McMillian Zeros have certain important properties related to the behavior of the multivariable system. Apart from Rosenbrock's [42] work very little has been done about using McMillan Zeros in system design. One important problem is to find the conditions under which a feedback gain matrix K exists such that

(i) given the system $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x}$ with transfer function matrix G(s).

(ii) a feedback law $\underline{u} = \underline{v} + K\underline{y}$

the closed loop transfer function matrix $H(s) = G(s)\ (I + G(s))^{-1}$ has a desired McMillan form.

The solution to some of these problems should provide more effective ways of designing multivariable systems.

APPENDIX A

AN ALGORITHM FOR CALCULATION OF THE JORDAN CANONICAL FORM OF A MATRIX

Introduction

It is well-known that any matrix may be brought into the Jordan canonical form by a similarity transformation [45]. There are several methods available to compute the eigenvectors of a matrix when the eigenvalues are distinct [46-47]. Some of these could be used to compute the eigenvectors for matrices with multiple roots. In Varah's method [48] multiple eigenvalues are handled by perturbing the multiple eigenvalue to produce distinct eigenvalues. Eberlin and Boothroyd [49] also compute eigenvectors for multiple eigenvalues. However, none of these methods generate the basis vectors necessary to transform the given matrix into it's Jordan canonical form. Chen [27] has suggested a procedure for computing the Jordan canonical form. Here, a simple and efficient algorithm, based on the notion of a generalized eigenvector, and using Gauss elimination techniques is given to compute the Jordan form of an nxn matrix.

BACKGROUND

Given the nxn matrix A, we want to find the matrix T such that $T^{-1}AT$ *is a Jordan matrix* J. Let $(\lambda_1, \lambda_2, ---, \lambda_m)$ be the eigenvalues of A with multiplicity $(n_1, n_2, ---, n_m)$ respectively. The number of eigenvectors associated with the eigenvalue λ_1 is given by α_i = n-Rank $(A-\lambda_i I)$. The Jordan matrix, J, has the form

$$J = \text{diag}\,[J_{11}, J_{12}, ---, J_{1\alpha_1} \vdots J_{21}, J_{22}, ---, J_{2\alpha_2} \vdots --- \vdots J_{m1}, J_{m2}, ---, J_{m\alpha_m}] \qquad \text{(A.1)}$$

with

$$J_{ik} = \begin{bmatrix} \lambda_i & 1 & & \bigcirc \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ \bigcirc & & & \lambda_i \end{bmatrix} \quad \begin{array}{l} i = 1,2,---,m \\ k = 1,2,---,\alpha_i \end{array} \qquad \text{(A-2)}$$

Let β_{ik} be the dimension of the block J_{ik} and define

$$\sigma_{ik} = \sum_{\ell=1}^{i-1} \sum_{j=1}^{\alpha_\ell} \beta_{\ell j} + \sum_{j=1}^{k} \beta_{ij}. \quad \text{with } \sigma_{10} = 0 \qquad \text{(A-3)}$$

Let the generalized eigenvectors and the eigenvector corresponding to J_{ik} be $\underline{t}_{\sigma_{i(k-1)}+1}$, $\underline{t}_{\sigma_{i(k-1)}+2}$, ..., $\underline{t}_{\sigma_{i(k-1)}+\beta_{ik}-1}$ and $\underline{t}_{\sigma_{ik}}$ respectively. The transformation matrix T is made up of the n columns $(\underline{t}_1, \underline{t}_2, ---, \underline{t}_{\sigma_{11}}, \underline{t}_{\sigma_{11}+1}, \ldots, \underline{t}_{\sigma_{12}}, \ldots, \underline{t}_{\sigma_{1(\alpha_1-1)}+1}, \ldots, \underline{t}_{\sigma_{1\alpha_1}}, \ldots, \underline{t}_{\sigma_{m\alpha_m}}$. The similarity transformation satisfies the relation

$$AT = TJ \qquad \text{(A-4)}$$

i.e. $\quad A[\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_n] = [\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_n]J$

Then, the eigenvectors of λ_r satisfy the relation

$$(A-\lambda_r I)\underline{t}_\ell = \underline{0} \qquad \ell = \sigma_{r1}, \sigma_{r2}, \ldots, \sigma_{r\alpha_r} \qquad \text{(A-5)}$$

Given an eigenvector of λ_r the corresponding generalized eigenvectors satisfy the recursive relationship

$$(A-\lambda_r I)\, \underline{t}_{\ell-1} = \underline{t}_\ell \qquad \ell = \sigma_{rk}, \sigma_{rk}-1, \ldots, \sigma_{rk}-\beta_{rk}+1$$

$$k = 1, 2, \ldots, \alpha_k. \tag{A-6}$$

The solution of equations (A-5) and (A-6) yields the transformation matrix T.

Computation of the Eigenvectors

Let $\bar{A} = (A - \lambda_r I)$. We can choose non-singular matrices P_r and Q_r such that $P_r \bar{A} Q_r = U_r$, where, U_r has the form

$$U_r = \left[\begin{array}{cc} U_{11} & A_{12} \\ \hline \multicolumn{2}{c}{0} \end{array}\right] \Big\} \alpha_r \text{ rows}$$

Here U_{11} is an $(n-\alpha_r)x(n-\alpha_r)$ upper triangular matrix with $|U_{11}| \neq 0$ and A_{12} is an $(n-\alpha_r)x\alpha_r$ matrix. Given $(A-\lambda_r I)$, P_r, Q_r and U_r can be obtained by Gauss elimination with full pivoting [50]. The α_r eigenvectors corresponding to the eigenvalue λ_r are obtained by solving the equation

$$U_r \underline{t}_\ell = \underline{0} \tag{A-7}$$

using a back substitution scheme employing α_r independent selections of the last α_r components of $\underline{t}_\ell$. Full pivoting guarantees that this will result in α_r linearly independent solutions which become the α_r independent eigenvectors corresponding to λ_r. Substitution of these eigenvectors in equation (A-6) yields the set of generalized eigenvectors.

<u>Algorithm</u>:

1. Find the eigenvalues of A. Label them $\lambda_1, \lambda_2, \ldots, \lambda_m$.

2. Solve the equation $U_r \underline{t}_\ell = \underline{0}$ for all eigenvectors corresponding to λ_r using independent selection of undetermined constants. The solution involves undefined variables $v_r, w_r, \ldots$. Generate an independent set of eigenvectors for λ_r by setting each undefined variable in turn equal to 1 while holding all other variables equal to 0. Denote the eigenvectors by $\underline{t}_{\sigma_{r1}}, \underline{t}_{\sigma_{r2}}, \ldots, \underline{t}_{\sigma_{r\alpha_r}}$.

3. For each eigenvector $\underline{t}_{\sigma_{ri}}$, $i = 1,2,\ldots, \alpha_r$ form $P_r Q_r \underline{t}_{\sigma_{r1}}$ and solve

$$U_r \underline{t}_{\sigma_{ri}-1} = P_r Q_r \underline{t}_{\sigma_{ri}}$$

for generalized eigenvector corresponding to eigenvector $\underline{t}_{\sigma_{ri}}$ with the undetermined constants taking values given to them while evaluating $\underline{t}_{\sigma_{ri}}$.

4. Repeat step 3 by forming $P_r Q_r \underline{t}_{\sigma_{ri}-1}$ and solve $U_r \underline{t}_{\sigma_{ri}-2} = P_r Q_r \underline{t}_{\sigma_{ri}-1}$.

5. Continue to generate generalized eigenvectors as in step 4 until the equation $U_r \underline{t}_{\sigma_{ri}-j-1} = P_r Q_r \underline{t}_{\sigma_{ri}-j}$ becomes inconsistent i.e. when a non-zero quantity appears on the right hand side corresponding to zero rows of U_r. This gives the basis vectors corresponding to the eigenvalue λ_r.

6. Repeat step 2 thru 5 for r = 1, 2, ..., m. to obtain all the basis vectors and hence the matrix T.

7. Obtain the Jordan canonical form from $J = T^{-1}AT$. Note that J need not be calculated directly since the block structure of (A-2)

is determined by the number of generalized eigenvectors that are generated for each eigenvector.

Computational Discussion:

The computation of the eigenvectors and the generalized eigenvectors depend on the accuracy with which the eigenvalues of A are computed. Francis' [35] algorithm is suggested for computing the eigenvalues. When the eigenvalues are approximate the calculation of the eigenvector can be refined as suggested by Wilkinson [51].

The algorithm suggested in this paper results in a large reduction in the amount of computation necessary to obtain the Jordan canonical form. The number of computations necessary for an n^{th} order system with m distinct eigenvalues is shown in Table A-1.

A similar analysis of Chen's algorithm [27] shows that the number of computations are of the order $0(\frac{5}{3} n^4)$. Thus the algorithm suggested here results in at least a fivefold saving in the number of computations. The method does not require the evaluation of the rank of matrices of powers of $(A-\lambda_r I)$ as in Chen's method.

TABLE A-1

STEP	NUMBER OF COMPUTATIONS
$P_i(A-\lambda_i I)Q$	$\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n} i = \frac{1}{3}(n^3-n)$
Total elimination for m eigenvalues	$m(n^3-n)/3$
Back substitution	$\leq \sum_{i=1}^{n-1} i = \frac{n^2-n}{2}$
Total for n back substitution	$\leq \frac{n^3-n^2}{2}$
To construct a right hand side ($P_i Q_i \underline{x}$)	$\sum_{i=1}^{n} i = \frac{n^2-n}{2}$
Total R.H.S.	$n(n^2-n)/2 = \frac{n^3-n^2}{2}$
Total	$\frac{mn^3}{3} - \frac{mn}{3} + n^3-n^2 = 0(\frac{m+1}{3} n^3)$

Examples:

The algorithm is applied to find the eigenvectors and the Jordan canonical form of two different matrices.

A. Fourth order matrix:

$$\begin{bmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{bmatrix}$$

This matrix is taken from Eberlin and Boothroyd [49]. The eigenvalues of the matrix are 5.23606797749979 (double root) and 0.763932022500210 (double root).

The eigenvector and the generalized eigenvector associated with the double root 5.23606797749979 are

$$\begin{bmatrix} 0.4270509831 \\ 1.0000000000 \\ 0.3819660113 \\ 1.1458980340 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.5868810394 \\ 1.0000000000 \\ 0.4721359550 \\ 1.0901699410 \end{bmatrix} \text{ respectively.}$$

For the double root 0.763932022500210 the corresponding vectors are given by

$$\begin{bmatrix} -0.3726779962 \\ 0.1273220038 \\ 0.3333333333 \\ 1.0000000000 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.2197175016 \\ 0.4182146692 \\ -0.3171224407 \\ 1.0000000000 \end{bmatrix}$$

Notice that the two eigenvectors and the two generalized eigenvectors are all independent unlike in [49]. The Jordan canonical form can be readily written as

$$\begin{bmatrix} 5.2360 & 1 & 0 & 0 \\ 0 & 5.2360 & 0 & 0 \\ 0 & 0 & 0.7639 & 1 \\ 0 & 0 & 0 & 0.7639 \end{bmatrix}$$

The execution time was 1.57 secs with a WATFIV (Univ. of Waterloo - Fast Fortran) compiler.

B. System Matrix of Boeing Helicopter

The following 8x8 matrix arises in the design of the helicopter stabilization system used in Chapter IV.

$$\begin{bmatrix} .021 & .025 & -29.64 & .6968 & .1879 & 0 & -.0941 & 0 \\ -.0903 & -.802 & -80.98 & -1.878 & .5524 & 0 & -8.517 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -.0058 & .0145 & 1.4672 & -1.460 & .45 & 0 & .068 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -784 & -35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -784 & -35 \end{bmatrix}$$

The eigenvalues of the system computed by using Francis' method are 0.50432908, -2,3585084, $-0.19350035 \pm j\ 0.35283477$ and $-17.5 \pm j\ 21.857493$ (double root). The eigenvectors corresponding to the distinct root are

$$\begin{bmatrix} 1.0000000000 \\ 0.9167473189 \\ -0.0157197678 \\ -0.0079269851 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{bmatrix}, \begin{bmatrix} 0.2528902161 \\ 1.0000000000 \\ 0.0200347219 \\ -0.0472520599 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{bmatrix}, \begin{bmatrix} -0.0949009676 \mp j\ 0.6460398691 \\ 1.0000000000 + j\ 0.0000000000 \\ -0.0074706563 \mp j\ 0.0035914411 \\ 0.0026856551 \pm j\ 0.0019501968 \\ 0.0000000000 + j\ 0.0000000000 \\ 0.0000000000 + j\ 0.0000000000 \\ 0.0000000000 + j\ 0.0000000000 \\ 0.0000000000 + j\ 0.0000000000 \end{bmatrix}$$

respectively. Each of the double roots has two eigenvectors associated with it. These are

$$\begin{bmatrix} -0.0000183498 \pm j\ 0.0002379966 \\ -0.0001564383 \pm j\ 0.0007421192 \\ 0.0000193897 \mp j\ 0.0000084539 \\ -0.0001545381 \pm j\ 0.0005715717 \\ -0.0223214285 \mp j\ 0.0278794553 \\ 1.0000000000 + j\ 0.0000000000 \\ 0.0000000000 + j\ 0.0000000000 \\ 0.0000000000 + j\ 0.0000000000 \end{bmatrix} \text{ and } \begin{bmatrix} 0.0000224177 \mp j\ 0.0001119734 \\ 0.0026667158 \mp j\ 0.0107258554 \\ 0.0000031152 \mp j\ 0.0000011743 \\ -0.0000288496 \pm j\ 0.0000886422 \\ 0.0000000000 + j\ 0.0000000000 \\ 0.0000000000 + j\ 0.0000000000 \\ -0.0223214285 \mp j\ 0.0278794553 \\ 1.0000000000 + j\ 0.0000000000 \end{bmatrix}$$

Since the multiple eigenvalues have as many eigenvectors as their multiplicity, the Jordan canonical form for this matrix is diagonal and is given by

$$\text{diag}\ [.50432908,\ -2.3585084,\ -0.19350035 + j\ 0.35283477,\ -0.19350035 - j\ 0.35283477,\ -17.5 + j\ 21.857493,\ -17.5 + j\ 21.857493,\ -17.5 - j\ 21.857493,\ -17.5 - j\ 21.857493]$$

The execution time using a WATFIV compiler was 8.69 secs.

Flowchart and Computer Program:

These are given in [52].

Conclusion:

A method has been outlined to find the basis vectors to transform a given nxn matrix to its Jordan canonical form. The method is simple and efficient. It does not require the evaluation of the rank of matrices of powers of $(A-\lambda_i I)$ as in Chen's method [27]. There is at least a fivefold reduction in the number of computations. Two examples are given to illustrate this method.

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