ANALYTICAL EVALUATION OF JET NOISE SOURCES LOCATION TECHNIQUE UTILIZING AN ACOUSTICALLY HARD BAFFLE WITH APERTURE

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## Abstract

An analytical investigation was conducted on the experimental technique that uses shielding by an acoustically hard baffle to determine the location of noise sources in a jet. The jet exhausts through a circular hole in the baffle and the radiated acoustic power is considered to be generated by a distributed source along the jet axis. It is found that the effect of the baffle on the radiated power can be neglected only when the size of the source is on the order of a wavelength of the emitted sound or larger. Since noise sources in a jet are compact, the experimental technique is insufficient to identify these sources.
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SUMMARY

An analytical investigation was conducted on the experimental technique that uses shielding by an acoustically hard baffle to determine the location of noise sources in a jet. The jet exhausts through a circular hole in the baffle and the radiated acoustic power is considered to be generated by a distributed source along the jet axis. It is found that the effect of the baffle on the radiated power can be neglected only when the size of the source is on the order of a wavelength of the emitted sound or larger. Since noise sources in a jet are compact, the experimental technique is insufficient to identify these sources.

INTRODUCTION

In recent years interest has been shown in the use of shielding by acoustically hard baffles to determine the location of noise source regions in air jets. (See ref. 1.) In a typical experiment, an acoustic baffle is used to obscure part of the source region from an observer and the observed acoustic power is measured. By comparing successive measurements as more and more of the source region is revealed to the observer, the spatial distribution of the sound sources is deduced.

There are three hypotheses which must be satisfied before the experimental results can be interpreted in this manner. First, one must assume that the mean square pressure at the observation point is the result of some physically meaningful source density distributed over the source region. Secondly, it is necessary to assume that those areas of the source region obscured by the baffle do not contribute significantly to the observed mean square pressure. Thirdly, it is necessary to assume that the presence of the baffle does not alter the distribution of the acoustic sources.

The purpose of this report is to determine the extent to which these hypotheses are satisfied for the case of a finite-line source protruding through a circular hole in an

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infinite hard baffle. (See fig. 1.) The actual acoustic power radiated on one side of the baffle is evaluated and compared with that obtained when the hypotheses are enforced. From this comparison it is possible to judge the conditions under which the size of the source and its position relative to the aperture can be determined from the measurements.

**SYMBOLS**

- $a$ radius of orifice
- $a_0$ speed of sound in air
- $c$ frequency parameter, $ka$
- $D$ source distribution function
- $F$ time-dependent source function
- $F_n$ function defined by equations (A15)
- $f$ time-independent source function
- $G$ Green's function
- $h$ distance of source centroid from orifice
- $I_{mn}$ function defined by equations (A12)
- $k$ wave number
- $l$ length of line source
- $m,n$ integers
- $N_{mn} = 2 \int_0^1 S_{mn}^2 (-ic, \eta) d\eta$
- $p_d$ time-dependent pressure
- $p_{d,0}$ time-dependent pressure in absence of baffle
\( p_1 \)  
\( p_t \) 
\( \text{time-independent pressure} \)  
\( \text{strength of line source} \)  
\( \text{real and imaginary quantities} \)  
\( R^{(j)}_{mn} \)  
\( \text{oblate spheroidal radial wave functions, } j = 1, 2, 3, 4 \)  
\( r \)  
\( \text{position} \)  
\( r, \nu, \theta \)  
\( \text{spherical coordinates} \)  
\( S \)  
\( \text{surface of baffle} \)  
\( S_{mn} \)  
\( \text{oblate spheroidal angular wave function of first kind} \)  
\( S_0 \)  
\( \text{surface of hemisphere} \)  
\( S_S \)  
\( \text{surface of sphere} \)  
\( T_n = - \frac{R^{(1)}_{0n}(-ic,0)}{R^{(2)}_{0n}(-ic,0)} \)  
\( t \)  
\( \text{time} \)  
\( U(z) \)  
\( \text{unit step function, } 0 \text{ if } z < 0, 1 \text{ if } z > 0 \)  
\( V \)  
\( \text{source volume} \)  
\( W \)  
\( \text{acoustic power radiated above baffle} \)  
\( W_e \)  
\( \text{hypothesized value of acoustic power above baffle} \)  
\( W_0 \)  
\( \text{total acoustic power in absence of baffle} \)  
\( x, y, z \)  
\( \text{Cartesian coordinates} \)  
\( z_1, z_2 \)  
\( \text{position of ends of line source} \)
FORMULATION OF THE PROBLEM

The general technique for locating acoustic source regions with shielding by the hard baffle shown in figure 1 is outlined briefly. One assumes that the pressure field $p_{d,o}$ in the absence of the baffle satisfies the wave equation

$$\left( \nabla^2 - \frac{1}{a_o^2} \frac{\partial^2}{\partial t^2} \right) p_{d,o}(\vec{r},t) = -4\pi F(\vec{r},t) \quad (1)$$

where the source term $F$ is nonzero only in some finite region $V$. The pressure field then satisfies the relationship

$$p_{d,o}(\vec{r},t) = \int_V \frac{F(\vec{r}',t - \frac{1}{a_o} |\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} dV(\vec{r}') \quad (2)$$
The mean square pressure at an observer point \( \mathbf{r} \) then becomes

\[
\left\langle \left| p_{d,o}(\mathbf{r},t) \right|^2 \right\rangle = \int \int_{VV} \frac{\langle F(\mathbf{r'},t - \frac{1}{a_o} | \mathbf{r} - \mathbf{r'} |)F(\mathbf{r''},t - \frac{1}{a_o} | \mathbf{r} - \mathbf{r''} |) \rangle}{| \mathbf{r} - \mathbf{r'} || \mathbf{r} - \mathbf{r''} |} d\mathbf{v}(\mathbf{r'}) d\mathbf{v}(\mathbf{r''})
\]

(3)

where the brackets \( \langle \rangle \) denote a time average. By choosing the origin of the coordinate system near the source region \( V \), the total acoustic power radiated by the source distribution in the absence of the baffle can then be expressed as

\[
W_o = \frac{1}{\rho a_o} \int_{S_S} \left\langle \left| p_{d,o}(\mathbf{r},t) \right|^2 \right\rangle d\mathbf{s}(\mathbf{r})
\]

(4)

where \( S_S \) is the surface of a large sphere of radius \( | \mathbf{r} | \) centered at the origin.

It is desired to express \( W_o \) in terms of a distribution over \( V \) of a function \( D \) which can be interpreted as that part of the acoustic power coming from the corresponding point in \( V \); that is,

\[
W_o = \int_{V} D(\mathbf{r'}) d\mathbf{v}(\mathbf{r'})
\]

(5)

This distribution function \( D \) is thus

\[
D(\mathbf{r'}) = \frac{1}{\rho a_o} \int_{S_S} \int_{V} \frac{\langle F(\mathbf{r'},t - \frac{1}{a_o} | \mathbf{r} - \mathbf{r'} |)F(\mathbf{r''},t - \frac{1}{a_o} | \mathbf{r} - \mathbf{r''} |) \rangle}{| \mathbf{r} - \mathbf{r'} || \mathbf{r} - \mathbf{r''} |} d\mathbf{v}(\mathbf{r''}) d\mathbf{s}(\mathbf{r})
\]

(6)

In this paper it will be determined to what extent the corresponding integral of \( D(\mathbf{r'}) \) can be used to approximate the power radiated above the baffle

\[
W = \frac{1}{\rho a_o} \int_{S_o} \left\langle \left| p_d(\mathbf{r},t) \right|^2 \right\rangle d\mathbf{s}(\mathbf{r})
\]

(7)

where \( p_d \) is the pressure field generated by certain particular source distributions \( F \) in the presence of the baffle \( S \), and the surface \( S_o \) is the hemisphere of large radius centered at the origin in figure 1 and lying entirely in the upper half space.

In the problem being considered, it is of interest to compute the acoustic power radiated to one side of the baffle by a line source of length \( l \) located near the orifice of radius \( a \). The coordinate system is so chosen that the line source lies along the \( Z \)-axis, the baffle lies in the \( XY \)-plane, and the origin of the coordinate system lies at the center of the orifice. That part of the source lying in the upper half space is enclosed, except at the orifice, by the baffle \( S \) and hemisphere \( S_o \). The source function \( F \) is assumed to have the sinusoidal time dependence.
\[ F(x,y,z,t) = f(x,y,z) e^{-i\omega t} \] (8)

which yields a solution for the pressure of the form

\[ p_d(x,y,z,t) = p_i(x,y,z) e^{-i\omega t} \] (9)

The spatial dependence of the pressure is then the solution of

\[ (\nabla^2 + k^2)p_i(x,y,z) = -4\pi f(x,y,z) \] (10)

subject to the boundary condition of zero acoustic velocity normal to the baffle,

\[ \frac{\partial p_i}{\partial z} = 0 \] (On S) (11)

and the radiation condition for outgoing waves,

\[ \lim_{r \to \infty} r i k G = 0 \] (12)

where \( k = \omega/a_o \) and \( r = \sqrt{x^2 + y^2 + z^2} \).

The pressure \( p_i \) is evaluated by using the Green's function technique. The appropriate Green's function will be the pressure at the field point \( x,y,z \) caused by a point source at \( x',y',z' \) and hence will satisfy

\[ (\nabla^2 + k^2)G(x,y,z;x',y',z') = -4\pi \delta(x - x') \delta(y - y') \delta(z - z') \] (13)

\[ \frac{\partial G}{\partial z} = 0 \] (On S) (14)

\[ \lim_{r \to \infty} r (i k G - \frac{\partial G}{\partial r}) = 0 \] (15)

The pressure at the point \( x,y,z \) is then given by

\[ p_i(x,y,z) = \int_{V'} f(x',y',z') G(x,y,z;x',y',z') \, dV(x',y',z') \] (16)

where \( V' \) is the source region.

For the case being considered, that of a line source distributed along the Z-axis from \( z_1 \) to \( z_2 \), the source function can be represented by

\[ f(x,y,z) = p_l \delta(x) \delta(y) \left[ U(z - z_1) - U(z - z_2) \right] \] (17)
where \( U \) is the unit step function and \( p_z \) is the source strength per unit length. The resulting pressure is then given by

\[
p_1(x, y, z) = p_z \int_{z_1}^{z_2} G(x, y, z; 0, 0, z') \, dz'
\]  

(18)

The Green's function is evaluated in the oblate spheroidal coordinate system, the set of coordinates natural for this problem. (See fig. 2.) Of the different transformations between the Cartesian and oblate spheroidal systems, the one chosen here is that of references 2 and 3; that is,

\[
\begin{align*}
x &= a\left(1 + \xi^2\right)^{1/2} \left(1 - \eta^2\right)^{1/2} \cos \phi \\
y &= a\left(1 + \xi^2\right)^{1/2} \left(1 - \eta^2\right)^{1/2} \sin \phi \\
z &= a\xi \eta
\end{align*}
\]  

(19)

The range of the oblate spheroidal coordinates can also be chosen in different ways. It is convenient for this problem to use the ranges

\[
\begin{align*}
-\infty < \xi < \infty \\
0 \leq \eta \leq 1 \\
0 \leq \phi \leq 2\pi
\end{align*}
\]  

(20)

Under this transformation, the baffle \( S \) is given by \( \eta = 0 \), the circular hole by \( \xi = 0 \), and the Z-axis by \( \eta = 1 \).

The Green's function is derived in the appendix. For the line source being considered, the resulting pressure is independent of the angular coordinate \( \phi \) and equation (18) becomes

\[
p_1(\xi, \eta) = p_z a \int_{\xi_1}^{\xi_2} \bar{G}(\xi, \eta; \xi') \, d\xi'
\]  

(21)

where \( \bar{G} \) is given by equation (A14).

For convenience in determining the power radiated above the baffle, the hemisphere \( S_o \) is replaced by the half ellipsoid \( \xi = \xi_o \) = Constant. An element of surface area of this coordinate surface is given by

\[
dS = a^2 \left(1 + \xi_o^2\right)^{1/2} \left(\eta^2 + \xi_o^2\right)^{1/2} \, d\eta \, d\phi
\]  

(22)

The power is evaluated from equation (7) by letting \( r \) (or \( \xi_o \)) \( \rightarrow \infty \) and reduces to
\[ W = \frac{2\pi a^2 \xi_0^2}{\rho a_o} \int_{\eta=0}^{\eta=1} \left| p_i(\xi_o, \eta) \right|^2 d\eta \]  

As derived in the appendix (eq. (A14))

\[ \overline{G}(\xi, \eta; \xi') = 2k \sum_{n'=0}^{\infty} \frac{S_{0n}(-ic, \eta)}{N_{0n}(-ic)} S_{0n}(-ic, \eta) R_{0n}(3)(-ic, i\xi) F_n(-ic, \xi') \]  

where the prime denotes even values of \( n \) only, \( S \) and \( R \) are angular and radial oblate spheroidal wave functions, respectively, \( N \) is a normalization factor, and \( F \) is a combination of radial functions evaluated at the source coordinate and at the origin. By utilizing these equations and the fact that the angular wave functions are orthogonal,

\[ \int_{\eta=0}^{1} S_{0n}(-ic, \eta) S_{0m}(-ic, \eta) d\eta = \frac{1}{2} N_{0n}(-ic) \delta_{mn} \]  

(ref. 2, p. 22), and that

\[ \lim_{\xi_o \to -\infty} k^2 a^2 \xi_o^2 \left| F_{0n}(3)(-ic, i\xi_o) \right|^2 = 1 \]  

(ref. 2, p. 32), the acoustic power becomes

\[ W = \frac{4\pi a^2 p_l^2}{\rho a_o} \sum_{n'=0}^{\infty} \frac{S_{0n}(-ic, 1)}{N_{0n}(-ic)} \left[ \left( \int_{\xi_1}^{\xi_2} \text{Re} \left( F_n(-ic, \xi') \right) d\xi' \right)^2 + \left( \int_{\xi_1}^{\xi_2} \text{Im} \left( F_n(-ic, \xi') \right) d\xi' \right)^2 \right] \]  

where \( \xi_1 = z_1/a \) and \( \xi_2 = z_2/a \).

It is now useful to derive an expression for the distribution function \( D \) associated with the line source. By expressing the total acoustic power emitted by the source in free space as

\[ W_o = \int_{z_1}^{z_2} D(z') dz' \]  

the distribution function will be determined from evaluation of \( W_o \).

The acoustic pressure caused by the line source in the absence of the baffle is given by equation (18) where now \( G \) is the free-space Green's function

\[ G_{fs}(x,y,z;0,0,z') = \frac{e^{ik\sqrt{x^2+y^2+(z-z')^2}}}{\sqrt{x^2+y^2+(z-z')^2}} \]
The total power radiated by the source is obtained by substituting this pressure into equation (4). Introducing spherical coordinates by the transformation

\[
\begin{align*}
x &= r \sin \nu \cos \theta \\
y &= r \sin \nu \sin \theta \\
z &= r \cos \nu 
\end{align*}
\] (30)

results in

\[
W_o = \frac{p \ell^2}{\rho a_o} \int_0^{2\pi} d\theta \int_0^\pi d\nu \int_{z_1}^{z_2} dz' \int_{z_1}^{z_2} dz'' \sin \nu e^{ik(z''-z')} \cos \nu 
\] (31)

Introducing \( \zeta = \cos \nu \) and comparing equations (28) and (31) yields

\[
D(z') = \frac{2\pi p \ell^2}{\rho a_o} \int_{-1}^{1} d\zeta \int_{z_1}^{z_2} dz'' e^{ik\zeta(z''-z')} 
\] (32)

for the distribution function.

The total acoustic power radiated by the line source in free space is obtained from equation (31) as

\[
W_o = \frac{8\pi p \ell^2}{\rho a_0 k^2} \int_0^1 \frac{1 - \cos[k(z_2 - z_1)\zeta]}{\zeta^2} d\zeta 
\] (33)

This free space power is used to normalize \( W \), the power obtained above the baffle,

\[
\frac{W}{W_o} = \frac{2}{c^2} \sum_{n'=0}^{\infty} \frac{S_{0n}(-ic,1)}{N_{0n}(-ic)} \left[ \left( \int_{\xi_1}^{\xi_2} \Re \left\{ F_n(-ic,\xi') \right\} d\xi' \right)^2 + \left( \int_{\xi_1}^{\xi_2} \Im \left\{ F_n(-ic,\xi') \right\} d\xi' \right)^2 \right] \\
\int_0^1 \frac{1 - \cos[k(z_2 - z_1)\zeta]}{\zeta^2} d\zeta 
\] (34)

This result is to be compared with the power \( W_e \) obtained under the three working hypotheses discussed in the Introduction. Under these assumptions the power radiated by the line source to the upper half space in the presence of the baffle at \( z = 0 \) is

\[
W_e = \begin{cases} 
W_o & (z_1 \geq 0) \\
\int_0^{z_1} D(z')dz' & (z_1 < 0 \text{ and } z_2 > 0) \\
0 & (z_2 \leq 0) 
\end{cases} 
\] (35)

Utilizing the expression for \( D \) obtained in equation (32) and normalizing by \( W_o \) yields
\[ W_{e} \over W_{o} = \begin{cases} 1 \\ \int_{0}^{1} \frac{d\xi}{\xi^{2}} \cos \left( \frac{k\xi z_1}{2} \right) \left[ \cos \left( \frac{k\xi z_1}{2} \right) - \cos \left( \frac{k\xi (2z_2 - z_1)}{2} \right) \right] \\ \int_{0}^{1} \frac{d\xi}{\xi^{2}} \left\{ 1 - \cos \left[ k\xi (z_2 - z_1) \right] \right\} \end{cases} \]

\begin{align*}
&= \begin{cases} 0 \\
\left( z_1 < 0 \text{ and } z_2 > 0 \right) \end{cases} \\
\left( z_2 \leq 0 \right) 
\end{align*}

A similar analysis for the case of the degenerate line source (a point source located at \((0,0,z_p)\)) yields

\[ W \over W_{o} = \sum_{n'=0}^{\infty} \frac{S_{0n}(-ic,1)}{N_{0n}(-ic)} \left| F_{n}(-ic, \frac{z_p}{a}) \right|^{2} \]

\[ \left( z_p > 0 \right) \]

\[ \left( z_p < 0 \right) \]

NUMERICAL CALCULATIONS AND ERROR ANALYSIS

The functions \( F_{n} \) used in calculating the radiated power are simple combinations of the radial oblate spheroidal wave functions evaluated at the source points (eqs. (A15)). The values of these oblate functions were obtained from a modification of the computer program described in reference 3. The number of points required for the numerical integrations was determined from observation of the behavior of the \( F_{n} \) and depends on the length of the source distribution. The number of terms of the infinite sum necessary for any chosen accuracy criterion depends mainly on the distance from the source element to the orifice and slightly on the frequency parameter \( c \). The first 25 terms were found to be sufficient to give better than six figure accuracy in the computed power for the data presented.

A check on the numerical results is provided for the limiting case \( c \rightarrow 0 \). This case results in a vanishing orifice and corresponds to the classical image problem of a source located above an infinite hard plane. For the case of a point source, the power is easily calculated (ref. 4, p. 373) as
\[
\frac{W}{W_0} = 1 + \frac{\sin\left(\frac{4\pi z_p}{\lambda}\right)}{4\pi z_p / \lambda}
\]  

(39)

where \( z_p \) is the distance from the source to the plane and \( \lambda \) is the wavelength of the emitted sound. This result is plotted in figure 3 as the \( c = 0 \) curve, along with results obtained from equation (37). It is seen that this curve is indeed being approached as \( c \) gets small. Similar agreement is obtained between the image problem of a line source and the values obtained from equation (34) as \( c \to 0 \).

RESULTS

Representative results are given in figures 4 and 5. The calculated power \( W/W_0 \) is plotted against the distance from the center of the source distribution to the orifice. The dashed curves represent the hypothesized values of the acoustic power \( W_e/W_0 \).

Figure 4 shows the effect of the source frequency on the calculated power. Results for a point source are shown in figure 4(a). It is seen that for low frequency (large \( \lambda \)), the calculated power increases far beyond the hypothesized value as the source is moved higher. At higher frequency the calculated power increases at a slower rate and reaches the hypothesized value at a further distance from the orifice. The oscillations apparent in the calculated result have a period approaching one-half wavelength and decay in amplitude as the frequency is increased. The same two frequencies are shown in figures 4(b) and 4(c) for a source length of 0.5\( \lambda \). This value corresponds to 0.08\( \lambda \) for the low-frequency case and 2.0\( \lambda \) at the high frequency. It can be seen that the hypothesized curve gives a reasonable representation for the actual calculated power at the higher frequency, whereas at the lower frequency the discrepancy mentioned is still present. This result means that for a given size source, the higher the frequency the better the actual values will agree with the hypothesized values.

In figure 5 the effects of varying the size of the orifice and the length of the source at a given frequency are shown. As the length of the source is increased, the hypothesized power becomes a better approximation of the actual power for both values of the hole radius. Also note that the smaller hole radius gives better results except for the oscillation present when the source length is smaller than on the order of a wavelength.

CONCLUDING REMARKS

The evaluation of the location of noise sources by the use of shielding by an acoustically hard baffle when the source region is represented as a distribution and the effects of the baffle on the sources are neglected appears to yield reasonable results under
certain conditions. The main condition is that the size of the source be on the order of a wavelength of the sound or larger. Hence for a given size source region, the source location will be more easily determined when the frequency is higher. Also, the smaller the orifice size, the better the representation, although the diameter of the orifice is probably limited by the size of the source region. However, since it is generally considered (ref. 5) that aerodynamic sources of sound are invariably compact \((\ell \ll \lambda)\), an attempt to discriminate noise sources in a jet by this method fails.

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APPENDIX

CONSTRUCTION OF THE FUNCTION \( \overline{G}(\xi, \eta; \xi', \eta') \)

The function \( \overline{G} \) will be obtained from the derivation of the full Green's function for the problem

\[
\left( \nabla^2 + k^2 \right) G(x,y,z;x',y',z') = -4\pi \delta(x - x') \delta(y - y') \delta(z - z') \quad (A1)
\]

where the condition of zero acoustic velocity normal to the baffle yields the boundary condition

\[
\frac{\partial G}{\partial z} = 0 \quad \text{(On S)} \quad (A2)
\]

and \( G \) satisfies the radiation condition for outgoing waves

\[
\lim_{r \to \infty} r \left( i k G - \frac{\partial G}{\partial r} \right) = 0 \quad (A3)
\]

In the oblate spheroidal coordinate system of figure 2, equation (A1) becomes

\[
\frac{1}{a^3(\xi'^2 + \eta'^2)} \left\{ \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial}{\partial \eta} \right] + \frac{\partial}{\partial \xi} \left[ (1 + \xi^2) \frac{\partial}{\partial \xi} \right] + \frac{\xi^2 + \eta^2}{(1 + \xi^2)(1 - \eta^2)} \frac{\partial^2}{\partial \phi^2} + c^2(\xi^2 + \eta^2) \right\} G
\]

\[
= -\frac{4\pi}{a} \frac{\delta(\xi - \xi') \delta(\eta - \eta') \delta(\phi - \phi')}{a^3(\xi^2 + \eta^2)} \quad (A4)
\]

with the boundary condition

\[
\frac{\partial G}{\partial \eta} = 0 \quad (\eta = 0) \quad (A5)
\]

where \( c = ka \) and \( G \) satisfies the radiation condition as \( |\xi| \to \infty \).

Since the angular coordinate \( \phi \) is periodic in \( 2\pi \), the Green's function must satisfy

\[
G(\xi, \eta, \phi; \xi', \eta', \phi') = G(\xi, \eta, \phi + 2\pi; \xi', \eta', \phi') \quad (A6)
\]

\[
\frac{\partial}{\partial \phi} G(\xi, \eta, \phi; \xi', \eta', \phi') = \frac{\partial}{\partial \phi} G(\xi, \eta, \phi + 2\pi; \xi', \eta', \phi') \quad (A7)
\]

One can then expand \( G \) as the following Fourier series:
These functions satisfy the following equations:

\[
G(\xi, \eta, \phi; \xi', \eta', \phi') = \sum_{m=0}^{\infty} \left[ G_m^{(1)}(\xi, \eta; \xi', \eta', \phi') \cos m\phi + G_m^{(2)}(\xi, \eta; \xi', \eta', \phi') \sin m\phi \right]
\]

(A8)

These functions satisfy the following equations:

\[
\frac{1}{a^3(\xi^2 + \eta^2)} \left\{ \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial}{\partial \eta} \right] + \frac{\partial}{\partial \xi} \left[ (1 + \xi^2) \frac{\partial}{\partial \xi} \right] - \frac{(\xi^2 + \eta^2)m^2}{(1 + \xi^2)(1 - \eta^2)} + \epsilon^2(\xi^2 + \eta^2) \right\} \left\{ \begin{array}{l}
G_m^{(1)} \\
G_m^{(2)}
\end{array} \right\}
\]

\[
= -\frac{2\epsilon_m}{a} \frac{\delta(\xi - \xi') \delta(\eta - \eta')}{a^3(\xi^2 + \eta^2)} \left\{ \begin{array}{l}
\cos m\phi' \\
\sin m\phi'
\end{array} \right\}
\]

(A9)

where \( \epsilon_m = 1 \) if \( m = 0 \), \( \epsilon_m = 2 \) otherwise.

By considering first the case \( \xi' \geq 0 \), the functions \( G_m^{(1)} \) and \( G_m^{(2)} \) are expanded in the set of analytic angular oblate spheroidal wave functions of the first kind, \( \{ S_{mn}(-ic, \eta) \} \), where \( m - n \) is even. (The notation used for the spheroidal wave functions is that of ref. 2.) This set is complete in the region \( 0 \leq \eta \leq 1 \) and every element satisfies the condition that its derivative vanishes at \( \eta = 0 \).

The coefficients in the expansions for the angular functions satisfy the differential equation defining the radial oblate spheroidal wave functions. Hence they are expanded as a linear combination of the radial functions \( R_{mn}^{(3)}(-ic, i\xi) \) and \( R_{mn}^{(4)}(-ic, i\xi) \), representing outward and inward traveling waves, respectively, in each of the three regions \( \xi < 0, \ 0 \leq \xi \leq \xi' \), and \( \xi > \xi' \).

The coefficients in these expansions are determined from

1. The radiation condition as \( \xi \to -\infty \)
2. The radiation condition as \( \xi \to -\infty \)
3. Continuity of \( G \) at \( \xi = 0 \)
4. Continuity of \( \frac{\partial G}{\partial \xi} \) at \( \xi = 0 \)
5. Continuity of \( G \) at \( \xi = \xi' \), and
6. The value of the discontinuity in \( \frac{\partial G}{\partial \xi} \) at \( \xi = \xi' \) as determined from the differential equation.
The solution for the Green's function then becomes

\[
G(\xi, \eta, \phi; \xi', \eta', \phi') = ik \sum_{m=0}^{\infty} \sum_{n'=\left\lfloor \frac{m+1}{2} \right\rfloor}^{\infty} \frac{e^{im}}{N_{mn}(-ic)} \cos m(\phi - \phi') \, S_{mn}(-ic, \eta) \, S_{mn}(-ic, \eta') \, I_{mn}
\]

(A10)

where the prime refers to even values of \( n \) only, \( \left\lfloor \frac{m+1}{2} \right\rfloor \) represents the integer part of \( \frac{m+1}{2} \),

\[
N_{mn}(-ic) = 2 \int_{0}^{1} S_{mn}(-ic, \eta) \, d\eta
\]

(A11)

and

\[
I_{mn} = R_{mn}^{(3)}(-ic, i\xi) \left\{ 2R_{mn}^{(4)}(-ic, i\xi') - R_{mn}^{(3)}(-ic, i\xi') \frac{R_{mn}^{(4)}(-ic, 0)}{R_{mn}^{(3)}(-ic, 0)} + \frac{R_{mn}'(-ic, 0)}{R_{mn}'(-ic, 0)} \right\}
\]

\( (\xi > \xi') \) (A12a)

\[
I_{mn} = R_{mn}^{(3)}(-ic, i\xi') \left\{ 2R_{mn}^{(4)}(-ic, i\xi) - R_{mn}^{(3)}(-ic, i\xi) \frac{R_{mn}^{(4)}(-ic, 0)}{R_{mn}^{(3)}(-ic, 0)} + \frac{R_{mn}'(-ic, 0)}{R_{mn}'(-ic, 0)} \right\}
\]

\( (0 \leq \xi \leq \xi') \) (A12b)

\[
I_{mn} = R_{mn}^{(3)}(-ic, i\xi) \, R_{mn}^{(3)}(-ic, i\xi') \left\{ \frac{R_{mn}'(-ic, 0)}{R_{mn}^{(3)}(-ic, 0)} - \frac{R_{mn}^{(4)}(-ic, 0)}{R_{mn}'(-ic, 0)} \right\}
\]

\( (\xi < 0) \) (A12c)

For the case \( \xi' < 0 \), the Green's function is obtained from symmetry conditions.

In the problem to be solved, the Green's function is required only for \( \eta' = 1 \) and large \( \xi \). Since \( S_{mn}(-ic, 1) = 0 \) when \( m > 0 \), one can write

\[
G(\xi, \eta, \phi; \xi', 1, \phi') = \overline{G}(\xi, \eta; \xi')
\]

The radial functions of the third and fourth kinds evaluated at the orifice and the source point are replaced by those of the first and second kinds, and upon introducing the notation
\( T_{n}(-ic) = \frac{R^{(1)}_{0n}(-ic,0)}{R^{(2)}_{0n}(-ic,0)} \)  

one obtains

\[
G(\xi, \eta, \xi') = 2k \sum_{n'=0}^{\infty} \frac{S_{0n}(-ic,1)}{N_{0n}(-ic)} S_{0n}(-ic,\eta) R^{(3)}_{0n}(-ic,i\xi) \ F_{n}(-ic,\xi')
\]

(A14)

where

\[
F_{n}(-ic,\xi') = \left[ \frac{T_{n}(-ic) R^{(1)}_{0n}(-ic,i\xi') + T^{2}_{n}(-ic) R^{(2)}_{0n}(-ic,i\xi')}{1 + T^{2}_{n}(-ic)} \right] + i \left[ \frac{2 + T^{2}_{n}(-ic) R^{(1)}_{0n}(-ic,i\xi') + T_{n}(-ic) R^{(2)}_{0n}(-ic,i\xi')}{1 + T^{2}_{n}(-ic)} \right] \hspace{1cm} (\xi' \geq 0) \]  

(A15a)

and

\[
F_{n}(-ic,\xi') = -\left[ \frac{T_{n}(-ic) R^{(1)}_{0n}(-ic,-i\xi') + T^{2}_{n}(-ic) R^{(2)}_{0n}(-ic,-i\xi')}{1 + T^{2}_{n}(-ic)} \right] + i \left[ \frac{T^{2}_{n}(-ic) R^{(1)}_{0n}(-ic,-i\xi') - T_{n}(-ic) R^{(2)}_{0n}(-ic,-i\xi')}{1 + T^{2}_{n}(-ic)} \right] \hspace{1cm} (\xi' < 0) \]  

(A15b)

The Green's function for this problem is given in reference 6 (p. 1520) in cylindrical coordinates. However, since the computer program of reference 3 enables one to evaluate the oblate spheroidal radial wave functions with relative ease, the formulation of equation (A14) is much easier to work with than this cylindrical representation.
REFERENCES


Figure 1.- Line source on the axis of a circular hole in an infinite hard baffle.
Figure 2.- Oblate spheroidal coordinate system.

\[
x = a \left[ 1 - \eta^2 (\xi^2 + 1) \right]^{1/2} \cos \phi \\
y = a \left[ 1 - \eta^2 (\xi^2 + 1) \right]^{1/2} \sin \phi \\
z = a \eta \xi
\]

\((0 < \eta < 1, \ -\infty < \xi < \infty, \ 0 < \phi < 2\pi)\)
Figure 3.- Convergence of acoustic power radiated above baffle by a point source to results of image problem.
Figure 4.- Comparison of actual and estimated values of radiated power at two frequencies.
(a) Point source ($l/\lambda = 0$).

(b) Line source of one half wavelength ($l/\lambda = 0.5$).

(c) Line source of two wavelengths ($l/\lambda = 2.0$).

Figure 5.- Comparison of actual and estimated values of radiated power for different size sources at a given frequency.
"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."
—National Aeronautics and Space Act of 1958

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