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DYNAMICS AND CONTROL OF ADVANCED SPACE VEHICLES

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DYNAMICS AND CONTROL OF ADVANCED SPACE VEHICLES

Final Report
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ABSTRACT

The study defined under Contract NAS8-28358 consisted of four parallel efforts: (1) Modal analyses of elastic continua for Liapunov stability analysis of flexible spacecraft; (2) Development of general-purpose simulation equations for arbitrary spacecraft; (3) Evaluation of alternative mathematical models for elastic components of spacecraft; and (4) Examination of the influence of vehicle flexibility on spacecraft attitude control system performance.

This report includes a complete record of achievements under tasks (1) and (3) above, in the form of technical appendices, and a summary description of progress to date under tasks two and four.

Task (1) has provided the basis for the Ph. D. dissertation of André Colin (see Appendix 3, in Volume 2 of this report). This task in itself required two phases of investigation: modal analysis and stability analysis. The modal analysis is accomplished for a range of continuum models (strings, beams and thin plates with various boundary conditions on spinning spacecraft) by means of singular perturbation methods, and the stability analysis is accomplished by using Liapunov theorems with the momentum-constrained Hamiltonian as the testing function.

Task (2) is the basis for the Ph. D. dissertation of Arthur S. Hopkins, which is still in progress.

Task (3) is the subject of two technical papers by the Principal Investigator, included here as Appendices 1 and 2. In these papers the range of applicability of various discrete and continuous models of nonrigid spacecraft is examined. It is concluded that there is a domain of engineering applicability for each of the models considered, but that finite elements models are generally the most valuable for flexible spacecraft simulations.

Task (4) is currently receiving primary attention by the Principal Investigator and a postdoctoral scholar, Dr. Yoshiaki Ohkami. Results will be described in forthcoming documents.

SUMMARY CONCLUSIONS AND RECOMMENDATIONS

The primary qualitative conclusion emerging from this study concerns the relative utility of the several alternative mathematical models of elastic structures that have been advocated and used by various space organizations in recent years. Work emanating from or supported by Goddard Space Flight Center, for example, is very often based on elastic continuum models of flexible spacecraft; the Aerospace Corporation relies almost exclusively on multiple-rigid-body models of nonrigid vehicles; and the Jet Propulsion Laboratory has recently emphasized the use of finite element models of flexible structure. A major objective of the present study has been to weigh the advantages of these several approaches and to establish the range of utility of each.

The paper "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis" (see Appendix 1) presents a comparative evaluation of elastic body models in terms of their suitability for modal analysis. This is a critical consideration, since the use of modal coordinates for elastic body deformations in hybrid coordinate analysis is now almost universally accepted as the optimum procedure for spacecraft simulation. It is the conclusion of this paper that the elastic continuum model is appropriate for a small class of commonly encountered appendages, but in most situations the finite element model is preferable, because the modal analysis equations are linear, constant-coefficient ordinary differential equations whenever any discretized elastic structure is vibrating about a state of rest or constant spin in inertial space. The continuum equations for an appendage in the same motion may be linearized about an equilibrium solution (if such can be found from the nonlinear equations of elasticity), but the equations of motion are then generally linear, variable-coefficient partial differential equations.

Appendix 2 is a paper ("Geometric Stiffness Characteristics of a Rotating Elastic Appendage") in which two features of the geometric stiffness matrix are explored. This matrix defines the stiffness characteristics induced in a finite element model of an elastic structure by preload, such as that due to spin. It is demonstrated by example in this paper that the geometric stiffness matrix can be asymmetric, and that this result can counterbalance a kinematical stiffness matrix which is also asymmetric so as to produce a symmetric total stiffness matrix. This possibility was overlooked in a previous UCLA study for MSFC, so that the present work extends the applicability of the results developed under that previous contract (No. NAS8-26214).

Because the class of situations in which the continuum model is attractive includes many structural appendages found on spacecraft, this model was adopted for one phase of our study, as reflected in the dissertation of André Colin appearing as Appendix 3 in Volume 2 of this report. In this study singular perturbation theory is applied to a series of partial differential equations describing small vibrations of various elastic structures about a steady state of equilibrium deformation induced by spin. Solutions are obtained by the method of matched asymptotic expansions. The elastic bodies accommodated here are taut strings, beams, taut membranes, and thin plates. In each case the small parameter (ϵ) measures a normalized nondimensional ratio of bending stiffness to spin, so the results are applicable to structures with low bending stiffness and/or high spin. The results of the modal analysis are then incorporated in an attitude stability analysis, using the momentum-constrained Hamiltonian as a testing function in two Liapunov-type theorems. Results are then compared with those obtained by Barbera for discretized models of elastic appendages.

A major effort under the present contract has been invested in the development of a general formulation of equations of motion of a spacecraft idealized as an arbitrarily interconnected set of elastic bodies, each of which is modeled by means of finite element techniques. This study will result in the Ph. D. dissertation of Arthur S. Hopkins. The general problem has proven to be quite difficult, but we have preferred to extend the duration of our study rather than compromise scope or analytical integrity. Copies of this dissertation will be provided to MSFC when the work is completed, in late 1973.

The emphasis in this report, as in the report on the preceding contract, has been on problems of dynamics and stability analysis, rather than on problems of active control. This has seemed to be a necessary ordering of priorities in the past, but it has always been understood that once the problems of dynamic analysis were resolved the emphasis in our work would shift to control system analysis and synthesis. This transition has already occurred in the later stages of the present study, although results are still too preliminary to warrant exposition here. In this continuing effort, the principal investigator is working with Dr. Yoshiaki Ohkami, a control system specialist from the Japanese National Aerospace Laboratory who is working as a postdoctoral scholar at UCLA. The results of this promising joint effort will be reported as they emerge. It is the central recommendation of this study that continued support be given to the investigation of the influence of vehicle flexibility on the performance of active attitude control systems.

APPENDIX 1

MATHEMATICAL MODELING OF SPINNING ELASTIC BODIES FOR MODAL ANALYSIS

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ABSTRACT

The problem of modal analysis of an elastic appendage on a rotating base is examined with the following objectives: (a) To establish the relative advantages of various mathematical models of elastic structures, including the elastic continuum model, the distributed-mass finite element model, and the concentrated mass model; and (b) To extract general inferences concerning the magnitude and character of the influence of spin on the natural frequencies and mode shapes of rotating structures. In realization of the first objective, it is concluded that except for a small class of very special cases the elastic continuum model is barren of useful results, while for constant nominal spin rate the distributed-mass finite element model is quite generally tractable, since in the latter case the governing equations are always linear, constant-coefficient, ordinary differential equations. Although with both of these alternatives the details of the formulation generally obscure the essence of the problem and permit very little engineering insight to be gained without extensive computation, this difficulty is not encountered when dealing with simple concentrated mass models, which permit determination of the general inferences sought in objective (b) above.

INTRODUCTION

The literature on flexible spacecraft dynamics is proliferating at a rate which reflects the serious concern of the aerospace community for this problem. Many investigators in this field now employ some system of hybrid coordinates for dynamic analysis, using a combination of discrete coordinates (for the translations and rotations of rigid bodies or reference frames) and distributed or modal coordinates (for the deformations of elastic bodies). Although various idealizations have been adopted for mathematical models of deformable vehicles or vehicle appendages, including elastic continua[1-7], distributed-mass finite element systems[8], and elastically interconnected nodal body systems[9-15], in every case when modal coordinates are employed some rationale for the selection and truncation of these coordinates must be established.

The purpose of this paper is to address the problems of mathematical modeling and modal coordinate selection for an elastic appendage attached to a rigid base which is constrained to rotate with a constant angular speed Ω about a body-axis fixed in inertial space. As shown in many of the references (e.g., Ref. 8), the modal coordinates appropriate for fully constrained base rotation are often also appropriate for a hybrid coordinate representation of deformations of an elastic appendage attached to a rigid base which freely maintains the nominal constant angular velocity when the appendage deformation remains at its constant, steady-state value, but which differs slightly from the nominal constant angular velocity due to appendage deformational perturbations.

Although the restriction to a constant nominal base motion is formally necessary for the development of a rational policy of coordinate selection, it may be expected that experienced engineers will find the results of this special case applicable informally to a wider range of engineering problems than we indicate here.

Modal analysis of an idealized vibrating elastic structure on a rotating base requires the derivation of the linearized equations of small oscillatory deviation of the mathematical model from its constant state of deformation induced by spin, and the transformation of these equations into a system of uncoupled equations of motion in terms of <u>normal mode coordinates</u>. This would be an infinite system for a continuum model, and a finite set for a discretized model, but in either case substantial truncation of the modal coordinates is generally a practical necessity.

The indicated approach to modal analysis is however not always the easiest path, and it is tempting to consider a shortcut, employing for the flexible appendage on a rotating base the coordinates which would be normal mode coordinates if the appendage base were inertially fixed. (This has been the practice in several of the referenced papers). Whether or not this shortcut is acceptable in engineering practice depends of course on the application.

Our purpose here is to try to provide some guide for the analyst who wishes to determine the influence of base rotation on flexible appendage normal mode shapes and natural frequencies, so that engineers responsible for flexible spacecraft simulations will have some basis for modeling decisions and coordinate selection. We shall discuss the <u>elastic continuum model</u>, the <u>distributed-mass finite element model</u>, and the <u>concentrated mass model</u>. The reader interested primarily in results of practical utility will find his greatest reward in the final third of this paper, where a single-particle model provides useful engineering insight into the general problem of rotating flexible appendages.

ELASTIC CONTINUUM MODEL

General Theory. Hamilton's principle permits the construction of equations of motion of any conservative, holonomic system in the form

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0 \tag{1}$$

where T is the kinetic energy, and V the potential energy (which for the free elastic body of interest is the strain energy). The variational symbol implies a path variation in state space from fixed end-points at t_1 and t_2.

The kinetic energy T may be written as

$$T \triangleq \frac{1}{2} \iiint \dot{\underline{R}} \cdot \dot{\underline{R}} \; \mu' \, dx \, dy \, dz \tag{2}$$

where μ' is mass per unit volume and $\dot{\underline{R}}$ is the inertial time derivative of a position vector $\underline{R}$ locating a differential volume dx dy dz from an inertially fixed reference point 0. For the problem at hand, we can fix point 0 on the spin axis of the rotating base, and replace $\underline{R}$ by its representation in terms of dextral, orthogonal unit vectors $\hat{\underline{e}}_x$, $\hat{\underline{e}}_y$, $\hat{\underline{e}}_z$ fixed in the base, to obtain

$$\underline{R} = (x + u)\hat{\underline{e}}_x + (y + v)\hat{\underline{e}}_y + (z + w)\hat{\underline{e}}_z \tag{3}$$

where x, y, and z are constants which establish the Cartesian coordinates for an origin at 0 of the field point when the continuum is undeformed, and u, v, and w are respectively the $\hat{\underline{e}}_x$, $\hat{\underline{e}}_y$, and $\hat{\underline{e}}_z$ projections of the relative displacement of the material point originally at (x, y, z) to its location in the deformed state. If the inertial angular velocity of the base is given by

$$\underline{\omega} = \Omega \, \hat{\underline{e}}_z \tag{4}$$

then $\dot{\underline{R}}$ becomes

$$\dot{\underline{R}} = \dot{u}\hat{\underline{e}}_x + \dot{v}\hat{\underline{e}}_y + \dot{w}\hat{\underline{e}}_z + \Omega(x+u)\hat{\underline{e}}_y - \Omega(y+v)\hat{\underline{e}}_x \tag{5}$$

and for an arbitrary rotating appendage the kinetic energy may be written as

$$T = \frac{1}{2} \iiint \Big[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \Omega^2 (x+u)^2 + \Omega^2 (y+v)^2$$

$$+ 2\Omega(x+u)\dot{v} - 2\Omega(y+v)\dot{x} \Big] \mu' \, dxdydz \tag{6}$$

Even in the case of large strains, the strain energy of a nondissipative, homogeneous, isotropic body is given by[16]

$$V = \frac{1}{2} \iiint \sigma^{*T} \epsilon \, dxdydz \tag{7}$$

where $\sigma^* \triangleq \left[\sigma^*_{xx} \;\; \sigma^*_{yy} \;\; \sigma^*_{zz} \;\; \sigma^*_{xy} \;\; \sigma^*_{yz} \;\; \sigma^*_{zx} \right]^T$ is the matrix of "generalized stresses", and

$$\epsilon \triangleq \left[\epsilon_{xx} \; \epsilon_{yy} \; \epsilon_{zz} \; \epsilon_{xy} \; \epsilon_{yz} \; \epsilon_{zx} \right]^T \tag{8}$$

is the matrix of strains. The "generalized stresses" in σ^* are related to the actual stresses by a relationship (Ref. 16, p. 79) which we symbolize by the matrix equation

$$\sigma^* = (U + e)\sigma \tag{9}$$

where the matrix of actual stresses is $\sigma \triangleq \left[\sigma_{xx} \; \sigma_{yy} \; \sigma_{zz} \; \sigma_{xy} \; \sigma_{yz} \; \sigma_{zx} \right]^T$, U is the 6x6 unit matrix, and e is a 6x6 matrix which equals zero when all strains are zero.

For arbitrarily small strains (still permitting large relative displacements and angular rotations within the continuum), the distinction between generalized stresses and actual stresses is lost. Moreover we can then assume the validity of Hooke's law, which may for brevity be written in matrix terms as

$$\sigma = S\epsilon \tag{10}$$

where S is a symmetric 6x6 matrix depending on material properties. Under these assumptions, the strain energy integral becomes

$$V = \frac{1}{2} \iiint \epsilon^T S \epsilon \, dxdydz \tag{11}$$

Thus in the general case of small strain vibrations of rotating elastic continua one can combine Equations (11), (6), and (1) in order to obtain equations of motion to be subjected to modal analysis. This combination implies the replacement of ϵ in Equation (11) by functions of u, v, and w, using the strain-displacement equations of elasticity theory. This step introduces a major difficulty because in general (even for the small strain problem) one must use the nonlinear strain-displacement* relationships[16]

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{12}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{13}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \tag{14}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{15}$$

$$\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \tag{16}$$

$$\epsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \tag{17}$$

The requirement for the retention of second degree terms stems from the fact that steady state deformations induced by constant rotation are not arbitrarily small, and cannot be included with the arbitrarily small deviations from the steady state deformations in the linearization process. Perhaps the most convincing way to demonstrate the necessity of retaining nonlinear terms in these equations for the general case is to establish their importance in one specific case for which both linear and nonlinear analyses are available.

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*The shear strains ϵ_{xy}, ϵ_{yz}, ϵ_{zx}, are often defined with an additional factor of 1/2 for convenience in tensorial representation.

Since nothing further can be accomplished in general terms, at this point we focus on special cases, choosing first a problem for which the linear approximation of strain-displacement equations will suffice, and choosing next a problem for which retention of nonlinear terms may be essential.

So great are the difficulties of nonlinear elasticity that applications to rotating vibrating structures in the literature are apparently restricted* to beams, either directed along the spin axis (axial beams) or radiating from the spin axis at right angles (radial beams). References 1 - 7 treat these special cases, as do References 19 and 20, which provide alternatives to a modal description which may be useful for stability analysis of deformable elastic continua. Because of the importance of the rotating axial beam or shaft in machinery dynamics, and the importance of the radial rotating beam in propeller and helicopter rotor dynamics, the behavior of these beams has been examined extensively. In order to illustrate the difficulties of the general problem of rotating elastic continua, we shall briefly examine these special cases, using the general nonlinear strain-displacement equations when appropriate, rather than the ad hoc procedures typically employed in the literature when only the special case is of interest.

Axial beams. When an elastic beam is aligned with the spin axis, it experiences no steady state deformation, and this eliminates the need for the nonlinear terms in Equations (12)-(17). Thus the axial beam uniquely offers a way around the problems of nonlinear elasticity, rather than a special case for which these equations can be solved.

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*We ignore here those applications to vibrating taut strings and membranes which embody the assumption of constant tension, since with this assumption the influence of rotation is discarded. Nor are we concerned here with the applications of Equations (12) - (17) in formulating a "large" static deflection theory for beams or plates[17] or in elastic stability theory for beam-columns or orthogonally loaded plates[18].

The equation of small vibration of a classical beam ** in directions x and y normal to the spin axis (and the beam axis) along z may be shown by the indicated procedure to be[2]

$$\mu(z)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial z^2}\left[EI_x(z)\frac{\partial^2 u}{\partial z^2}\right] - \Omega^2\mu(z)u - 2\mu(z)\,\Omega\frac{\partial v}{\partial t} = 0 \qquad (18)$$

$$\mu(z)\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial z^2}\left[EI_y(z)\frac{\partial^2 u}{\partial z^2}\right] - \Omega^2\mu(z)v + 2\mu(z)\,\Omega\frac{\partial u}{\partial t} = 0 \qquad (19)$$

where u and v are displacements from the undeformed state as previously; $\mu(z)$ is the beam mass per unit length; E is the modulus of elasticity of the beam material; $I_x(z)$ and $I_y(z)$ are the area second moments or "moments of inertia" of the transverse cross-sections of the beam, assumed principal.

Boundary conditions for a cantilevered beam of length L become

$$u(0) = v(0) = \frac{\partial y}{\partial z}(0) = \frac{\partial v}{\partial z}(0) = \frac{\partial^2 u}{\partial z^2}(L) = \frac{\partial^2 v}{\partial z^2}(L) = \frac{\partial^3 u}{\partial z^3}(L) = \frac{\partial^3 v}{\partial z^3}(L) = 0 \quad (20)$$

The influence of the base rotation is clearly manifested in the spin rate Ω, which appears in Equations (18) and (19) in the form of centripetal accelerations and Coriolis accelerations. The latter terms couple the equations, and provide an obstacle to modal analysis except in the special (but common) case for which $\mu(z)$ is the constant μ, and I_x and I_y are the same constant, say I. For the uniform, axisymmetric beam, one can define the complex variable q = u + iv and write Equations (18)-(20) as

$$\ddot{q} + \frac{EI}{\mu}q'''' - \Omega^2 q + 2i\,\Omega\dot{q} = 0 \qquad (21)$$

and

$$q(0) = q'(0) = q''(L) = q'''(L) = 0 \qquad (22)$$

**The classical, or Euler-Bernoulli, beam ignores both shear deformations and rotatory inertia.

where dot denotes temporal differentiation and prime denotes spatial differentiation. Equation (21) admits the separated solution

$$q(z, t) = \phi(z)\eta(t) \tag{23}$$

where the complex functions ϕ (z) and $\eta(t)$ satisfy the ordinary differential equations

$$\phi'''' - \frac{\mu}{EI}(\Omega^2 + \sigma^2)\phi = 0 \tag{24}$$

and

$$\ddot{\eta} + 2i\Omega\dot{\eta} + \sigma^2\eta = 0 \tag{25}$$

obtained by substituting Equation (23) into Equation (21), dividing by $\phi\,\eta$, and equating the resulting separated functions of z and t separately to the constant σ^2. Equation (24) has the boundary conditions

$$\phi(0) = \phi'(0) = \phi''(L) = \phi'''(L) = 0 \tag{26}$$

and Equation (25) is an initial-value problem to be solved only after Equation (24) yields the characteristic values of σ^2 which permit nontrivial solutions ϕ (z).

Equation (24) has precisely the form of the classical result for a fixed-base cantilever beam (see Reference 21, page 162); if ϕ is written as $\phi_1 + i\phi_2$ we can see that actually we have two distinct but identical equations for ϕ_1 and ϕ_2 in the classical form. Familiar beam vibration theory[21] provides a transcendental equation to be solved numerically for the infinity of solutions for $(\Omega^2 + \sigma^2)$, beginning with

$$\Omega^2 + \sigma_1^2 = 12.36\ EI/(\mu L^4) \triangleq \omega_1^2 \tag{27a}$$

$$\Omega^2 + \sigma_2^2 = 485.32\ EI(\mu L^4) \triangleq \omega_2^2 \tag{27b}$$

$$\Omega^2 + \sigma_3^2 = 3806.89\ EI(\mu L^4) \triangleq \omega_3^2 \tag{27c}$$

and continuing with $\omega_j^2 \Delta \ \Omega^2 + \sigma_j^2$ for j ranging to infinity. Literal solutions for the "mode shapes" $\phi_1(z)$ and $\phi_2(z)$ are available in terms of circular and hyperbolic trigonometric functions, with exactly the same functional structure as for fixed base cantilever beams[21], but with the trigonometric function argument dependent upon Ω^2.

Whereas for the fixed-base cantilever beam the natural frequencies are given by the expressions recorded as ω_j above, the twofold influence of spin on the rotating axial beam natural frequency is, firstly for centripetal accelerations, to reduce ω_j to σ_j, and secondly for Coriolis accelerations, to change each σ_j into the two frequencies obtained as the characteristic roots of Equation (25) (with $\sigma^2 = \sigma_j^2$). These roots are the solutions of the characteristic equation.

$$\lambda_j^4 + \lambda_j^2(2\sigma_j^2 + 4\,\Omega^2) + \sigma_j^4 = 0 \tag{28}$$

which are given by

$$\lambda_j^2 = -\sigma_j^2 - 2\,\Omega^2 \pm 2\Omega\sqrt{\sigma_j^2 + \Omega^2} \tag{29}$$

In terms of the fixed base cantilever mode ω_j, these roots are given by

$$\lambda_j^2 = -(\omega_j^2 + \Omega^2) \pm 2\Omega\omega_j = \begin{cases} -(\omega_j + \Omega)^2 \\ -(\omega_j - \Omega)^2 \end{cases} \tag{30}$$

Thus the natural frequencies of vibration of an axisymmetric, axial beam with respect to its rotating base are given by

$$p_{1_j} = \omega_j - \Omega; \quad p_{2_j} = \omega_j + \Omega \tag{31}$$

where ω_j is the j^{th} natural frequency of the same beam on a fixed base and Ω is the spin rate. Evidently a spin rate in excess of the fixed base natural frequency produces instability.

Although Equation (24) indicates that both real and imaginary parts of the spatial function $\phi(z) = \phi_1(z) + i\phi_2(z)$ must satisfy a differential equation of the same structure as that providing the fixed-base cantilever modes[21], these functions can be interpreted as "mode shapes" only with the recognition of phase relationships between u and v associated with modal oscillations. Specifically, if Equation (23) is expanded in terms of its real and imaginary parts, and if an infinite series of such expansions comprises the general solution, we find

$$u(z,t) + iv(z,t) = \sum_{j=1}^{\infty} \phi^j(z)\eta^j(t) = \sum_{j=1}^{\infty}\left\{\left[\phi_1{}^j(z) + i\phi_2{}^j(z)\right]\left[\eta_1{}^j(t) + i\eta_2{}^j(t)\right]\right\}$$

$$= \sum_{j=1}^{\infty}\left\{\left[\phi_1{}^j(z)\eta_1{}^j(t) - \phi_2{}^j(z)\eta_2{}^j(t)\right] + i\left[\phi_2{}^j(z)\eta_1{}^j(t) + \phi_1{}^j(z)\eta_2{}^j(t)\right]\right\} \quad (32)$$

When Equations (24) and (25) are solved for $\phi_1{}^j(z)$, $\phi_2{}^j(z)$, $\eta_1{}^j(t)$, and $\eta_2{}^j(t)$, we find the general free vibration solution

$$u(z,t) = \sum_{j=1}^{\infty}\left\{\left[(\sin\beta_j L - \mathrm{Sinh}\,\beta_j L)(\sin\beta_j z - \mathrm{Sinh}\,\beta_j z) + (\cos\beta_j L + \mathrm{Cosh}\,\beta_j L)(\cos\beta_j z - \mathrm{Cosh}\,\beta_j z)\right]\left[A_j\cos p_{1_j} t + B_j \sin p_{1_j} t + C_j \cos p_{2_j} t + D_j \sin p_{2_j} t\right]\right. \quad (33)$$

$$v(z,t) = \sum_{j=1}^{\infty}\left\{\left[\sin\beta_j L - \mathrm{Sinh}\,\beta_j L)(\sin\beta_j z - \mathrm{Sinh}\,\beta_j z) + (\cos\beta_j L + \mathrm{Cosh}\,\beta_j L)(\cos\beta_j z - \mathrm{Cosh}\,\beta_j z)\right]\left[A_j\sin p_{1_j} t - B_j \cos p_{1_j} t - C_j \sin p_{2_j} t + D_j \cos p_{2_j} t\right]\right. \quad (34)$$

where $\beta_j^4 \triangleq \omega_j^2 \mu/EI$ and the constants A_j, B_j, C_j, and D_j are established by initial conditions. The four possible single-mode oscillations associated with A_j, B_j, C_j, and D_j all involve a circular motion of each point on the beam axis, with u and v always ninety degrees out of phase; the two rotations at frequency p_{1_j} are in the same direction as the vehicle rotation, differing only in phase, while the two rotations at frequency p_{2_j} have the opposite sense. Hence from Equation (31) it is apparent that to an <u>inertially fixed observer</u> the beam axis appears in these two normal modes of vibration to be maintaining a fixed geometry while each point on the axis traverses a circular path in inertial space at the frequency ω_j, with these normal modes differing only in the <u>direction</u> of the indicated circular motion. The striking aspect of this analysis is the degree to which the characteristics of fixed-base vibrations survive the imposition of base spin. In this respect, the axisymmetric uniform beam directed along the spin axis is unique.

If the axial beam is nonuniform or structurally asymmetric, virtually all of the preceding analysis fails (beginning with Equation (21)). To analyze the uniform but asymmetric case, we can rewrite Equations (18) and (19) as the matrix equation.

$$\begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} + \frac{EI}{\mu} \begin{bmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{bmatrix} \begin{bmatrix} u'''' \\ v'''' \end{bmatrix} - \Omega^2 \begin{bmatrix} u \\ v \end{bmatrix} + 2\Omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = 0 \tag{35}$$

where $I \triangleq \frac{1}{2}(I_x + I_y)$ and $\epsilon \triangleq \frac{1}{2I}(I_x - I_y)$. Guided by our experience with the axisymmetric beam, we can assume a solution of Equation (35) in the product form

$$\begin{bmatrix} u \\ \\ v \end{bmatrix} = \begin{bmatrix} \phi_x \\ \\ \phi_y \end{bmatrix} e^{\lambda t} \triangleq \Phi e^{\lambda t} \tag{36}$$

and obtain, by substitution and cancellation of $e^{\lambda t}$,

$$\Phi'''' + \frac{\mu}{EI} \left[(\lambda^2 - \Omega^2) \Phi + 2\Omega\lambda G\Phi \right] + \epsilon J \Phi'''' = 0 \tag{37}$$

where

$$G \triangleq \begin{bmatrix} 0 & -1 \\ \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad J \triangleq \begin{bmatrix} 1 & 0 \\ \\ 0 & -1 \end{bmatrix}$$

For the axisymmetric case $\epsilon \equiv 0$, and the previous solution suggests that Equation (37) will be satisfied by discrete values of λ satisfying Equation (30) augmented by Equation (27), and

$$\Phi \triangleq \begin{bmatrix} \phi_x \\ \\ \phi_y \end{bmatrix} = \phi \begin{bmatrix} 1 \\ \\ \pm i \end{bmatrix} \tag{38}$$

where ϕ is a real scalar satisfying Equation (24) for values of σ^2 given by Equation (27). In the case of the uniform asymmetric axial beam, however, with $\epsilon \neq 0$, one can no longer construct the solutions Φ and λ directly from previously recorded solutions for the fixed base cantilever beam; it becomes necessary to approach Equation (37) directly, either using numerical eigenvalue-eigenvector analysis or (if ϵ is small) a perturbation analysis.

The nonuniform axial beam, with one or more of the quantities $\mu(z)$, $I(z)$, and $\epsilon(z)$ depending upon z, presents even greater challenges to modal analysis, because the linearized equations of motion no longer have constant coefficients.

Radial Beams. When an elastic beam is normal to the spin axis of its inertially rotating base, it does sustain deformations in the steady state configuration in which it remains straight and aligned with a radial line emanating from its base. Therefore one must use nonlinear strain-displacement equations such as Equations (12) - (17) if deformation variables are to be measured from the undeformed state. Although the influence of radial beams on the stability of a spinning satellite has been published since 1969[3], and the numerical modal analysis of such beams has been accomplished many times (see References 6 and 22, for examples of equatorial and meridional vibrations respectively), the underlying equations of motion are typically derived by means of procedures which rely from the outset upon the availability of solutions for the steady-state load distribution and deformation of the elastic continuum, permitting the incorporation into Equation (1) of an expression for work done by an "effective applied load". This is an ideal approach for rotating radial beams (see Reference 21, page 440 ff. for example), but for a general elastic continuum the availability of a steady state solution is not a viable supposition, and the "effective external load" concept is not easily implemented.

In order to demonstrate the relationship between the general theory (employing nonlinear strain-displacement equations) and the special theory repeatedly used in application to the beam in the literature, we shall derive in what follows the equations of meridional vibration of the uniform rotating radial beam, obtaining the classical results (Reference 21, p. 443) in a manner that maintains its validity in the more general case.

For the special case of the uniform radial beam whose axis parallels $\hat{\underline{e}}_y$, vibrating only in the meridional direction (see Figure 1), it is customary to ignore those deformations u which are present only because of the Poisson

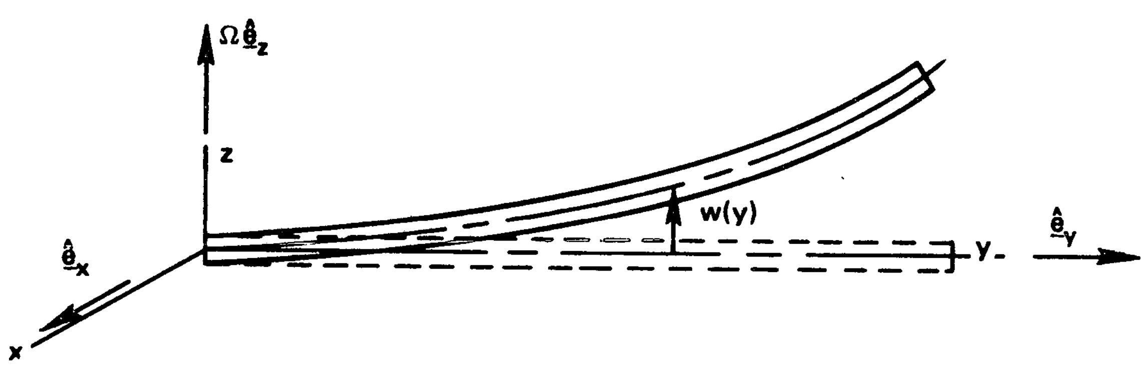

Figure 1. Rotating Uniform Elastic Beam

effect, and to assume that w depends only upon y; moreover, under the assumptions of classical (Euler-Bernoulli) beam theory, the terms in T involving v and $\dot{v}$ are also ignored (see Equation (6)). (The consequences of this latter restriction are explored in a recent paper[23] on the vibrations of a rotating "Rayleigh beam".) Thus for this very special problem the kinetic energy in Equation (6) reduces to

$$T = T_o + \frac{1}{2} \iiint \dot{w}^2 \mu' \, dxdydz$$

$$= T_0 + \frac{1}{2} \int_0^L \mu \dot{w}^2 \, dy = T_0 + \frac{1}{2} \int_0^L \mu \left(\frac{\partial w}{\partial t}\right)^2 dy \qquad (39)$$

where T_0 is a constant and μ is the mass per unit length.

Simplifications in T in this special case are more than matched by simplifications in V, since for the classical beam all strains are ignored except the normal longitudinal strain, which here is ϵ_{yy}. With proper substitutions for S in Equation (11) the strain energy becomes

$$V = \frac{E}{2} \iiint \epsilon_{yy}{}^2 \, dxdydz \qquad (40)$$

According to Equation (13), the nonlinear strain-displacement relationship to be substituted next should be (neglecting u)

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \qquad (41)$$

but at this point yet another simplification is permissible for the radial beam, if we recognize that for this particular problem we can easily separate the deformations from the undeformed state into the steady-state extensional deformation (which we designate $v_0(y)$) and the deviation from that steady state. Then v may be expressed as

$$v = v_0 - z \frac{\partial w}{\partial y} \qquad (42)$$

(assuming as usual that plane sections remain plane) and Equation (41) becomes

$$\epsilon_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left[\left(\frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{43}$$

Although unsatisfactory approximations result from linearization of Equation (43) in all deformation variables, a much improved approximation of Equation (40) can be obtained by retaining quadratic terms in functions of w and assuming that functions of v_0 can be ignored when compared to unity. This approximation preserves in Equation (40) products of $\partial v_0/\partial y$ with second degree terms in the slope $\partial w/\partial y$, while abandoning similar products with second degree terms in $\partial^2 w/\partial y^2$. The result is the approximation

$$V = \frac{E}{2}\iiint\left[-2z\frac{\partial v_o}{\partial y}\frac{\partial^2 w}{\partial y^2} + z^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\partial v_o}{\partial y}\left(\frac{\partial w}{\partial y}\right)^2\right]dxdydz + V_0 \tag{44}$$

where V_0 is the steady-state strain energy, which does not involve w. Upon integrating Equation (44) over x and z, noting that $\int z\,dz = 0$, $\iint dxdz \ \underline{\Delta}\ A$, the cross-sectional area, and $\iint z^2\,dxdz \ \underline{\Delta}\ I$, the area second moment about the x-axis, we find

$$V = \frac{EI}{2}\int_0^L\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dy + \frac{EA}{2}\int_0^L\frac{\partial v_o}{\partial y}\left(\frac{\partial w}{\partial y}\right)^2 dy + V_0 \tag{45}$$

At this point one can recognize the equivalence of this derivation and the familiar textbook derivation for the transverse vibrations of beams subject to an external axial force P(y), since to the first approximation one can substitute the relationship

$$\frac{P(y)}{A} = E\frac{\partial v_o}{\partial y} \tag{46}$$

into Equation (45), and combine the result with Equation (39) into Equation (1) to obtain

$$\delta \int_{t_1}^{t_2} \left[T_o + \frac{1}{2} \int_0^L \mu \left(\frac{\partial w}{\partial t} \right)^2 dy - \frac{EI}{2} \int_0^L \left(\frac{\partial^2 w}{\partial y^2} \right)^2 dy - \frac{1}{2} \int_0^L P(y) \left(\frac{\partial w}{\partial y} \right)^2 dy - V_0 \right] dt = 0 \qquad (47)$$

Routine execution of the variations and integrations implied by Equation (47) produces the equation of motion

$$EI \frac{\partial^4 w}{\partial y^4} - \frac{\partial}{\partial y} \left[P(y) \frac{\partial w}{\partial y} \right] + \mu \frac{\partial^2 w}{\partial t^2} = 0 \qquad (48)$$

where we rely upon the arbitrary magnitude of δw and the absence of such variations at t_1 and t_2, and where the boundary conditions

$$w(0) = \frac{\partial w}{\partial y}(0) = \frac{\partial^2 w}{\partial y^2}(L) = \frac{\partial^3 w}{\partial y^3}(L) = 0 \qquad (49)$$

have been utilized in the course of repeated integrations by parts.

For the rotating uniform radial beam, the steady state axial force is given by

$$P(y) = \int_y^L \mu \, \Omega^2 \eta \, d\eta = \frac{1}{2} \mu \, \Omega^2 \left(L^2 - y^2 \right)$$

so that Equation (48) takes the form

$$EI \frac{\partial^4 w}{\partial y^4} - \frac{1}{2} \mu \, \Omega^2 \left[(L^2 - y^2) \frac{\partial^2 w}{\partial y^2} - 2y \frac{\partial w}{\partial y} \right] + \mu \frac{\partial^2 w}{\partial t^2} = 0 \qquad (50)$$

With the derivation of Equation (50) the immediate objective of this section has been realized; this is a standard first approximation of the rotating beam meridional vibration equation[21, 22], and we have made explicit the oft-disguised necessity of including nonlinear terms in the strain displacement equations when deformations are measured relative to an undeformed state. Now we can safely conclude that any "general theory" of vibration of rotating elastic continua must also meet this stipulation (although this fact is unmentioned in the noted references). When one reviews the long series of cascading approximations that lead from the general equations (1), (6), and (11) for the small strain problem to the equations of motion for the meridional vibrations of the uniform radial beam, it becomes clear that a "general theory" has practical value only for that very small class of problems in elasticity for which one can find steady state solutions about which to linearize the oscillatory deformations, and even in this case implementation of the general theory may be very difficult .

Even for the uniform radial beam, which is next to the uniform axial beam the simplest special case imaginable, the equations of motion do not yield to simple eigenvalue-eigenvector analysis, because the linearized equations in the deformation variable w have nonconstant coefficients (see terms involving y and y^2 in Equation (50)). Even for this relatively simple problem one must turn to more elaborate or very approximate numerical procedures, such as Galerkin's method, Rayleigh's method, or a perturbation approach.

It seems safe to conclude that only for the simplest of configurations is it feasible to extract meaningful conclusions from an elastic continuum model of a rotating flexible appendage. In the following sections we shall explore other options, including the distributed-mass finite element model and the concentrated mass model.

FINITE ELEMENT MODEL

Reference 8 contains a detailed derivation of the equations of vibratory deformation of a distributed-mass finite element model of an elastic appendage attached to a rigid base having arbitrary motion, with particular attention to the present problem of contant base rotation. Our purpose here is to examine those equations in order to assess the relative difficulties of working with continuum models and finite element models, and to explore the consequences of rotation for the latter.

Let it first be understood that the model consists of an arbitrary number of elastic elements, each of which has an arbitrary number of points of contact in common with neighboring elements or the supporting rigid body. Each contact point is called a node and at each of the n nodes there may be located the mass center of a rigid body called a nodal body; the interconnecting elastic elements may however also have distributed mass. It is fundamental to the finite-element "assumed displacement" approach to modeling that the internodal elastic bodies are assigned a pattern of deformation in terms of the deformations at the nodes by means of an interpolation function (called W in Reference 8). Thus the system has a finite number of degrees of freedom established by the number of nodes; in Reference 8 the six independent kinematical coordinates describing deformational displacements from a steady state of the n rigid nodal bodies are accepted as the unknowns characterizing the appendage deformation, so that the appendage has 6n degrees of freedom in deformation. If the 6n by 1 matrix q contains the 6n variables representing the deviations of these nodal bodies from the steady state of deformation, then the equations of vibration may be shown to be (Reference 8, Equation (164), with no damping).

$$M'\ddot{q} + G'\dot{q} + K'q + A'q = 0 \qquad (51)$$

where M', G', K', and A' are 6n by 6n constant matrices, with M' and K' symmetric and G' and A' skew symmetric.

Procedures for coordinate transformation (modal analysis) of Equation (51) are developed in Reference 8, and will not be reviewed here. It is sufficient to note that Equations (51) are constant coefficient, linear, ordinary differential equations to establish the relative simplicity of modal analysis of finite element models in comparison with continuum models. A numerical eigenvalue-eigenvector analysis will always suffice; the major complication introduced by rotation is the introduction of complex eigenvectors. Terms in Equation (51) introduced by spin include centripetal accelerations, coriolis accelerations, and modifications of structural stiffness due to spin-induced loads on the structure in its steady state (the so-called "geometric stiffness"). The last of these influences is developed in Reference 24 more generally than in Reference 8. The physical and mathematical significance of each of these contributions is explored for a simpler model in the section following.

CONCENTRATED MASS MODEL

Because in Reference 8 a rigid body or particle is concentrated at each node, Equation (51) establishes also the structure of the equations of vibratory deformation of a model for which all mass is concentrated in the form of a single particle or rigid body suspended on massless springs. While this is in many cases an unjustifiably coarse approximation, it brings the advantage of equations of motion which are so simple that one can extract from them general conclusions that may serve as a guide to the behavior of more complex continuum or finite element models.

For a single particle on springs, the equations of motion are so simple that their inspection reveals a great deal about the system's vibration characteristics. Three such systems are shown in Figure 2. In each case a particle of mass m is attached by a massless three-degree-of-freedom spring mechanism to a base with prescribed inertial angular velocity $\Omega \hat{\underline{e}}_3$.

When $\Omega = 0$, the equations of motion in each case are

$$m\ddot{u}_1 + k_{10}u_1 = 0 \quad \text{or} \quad \ddot{u}_1 + \sigma_{10}^2 u_1 = 0 \tag{52}$$

$$m\ddot{u}_2 + k_{20}u_2 = 0 \quad \text{or} \quad \ddot{u}_2 + \sigma_{20}^2 u_2 = 0 \tag{53}$$

$$m\ddot{u}_3 + k_{30}u_3 = 0 \quad \text{or} \quad \ddot{u}_3 + \sigma_{30}^2 u_3 = 0 \tag{54}$$

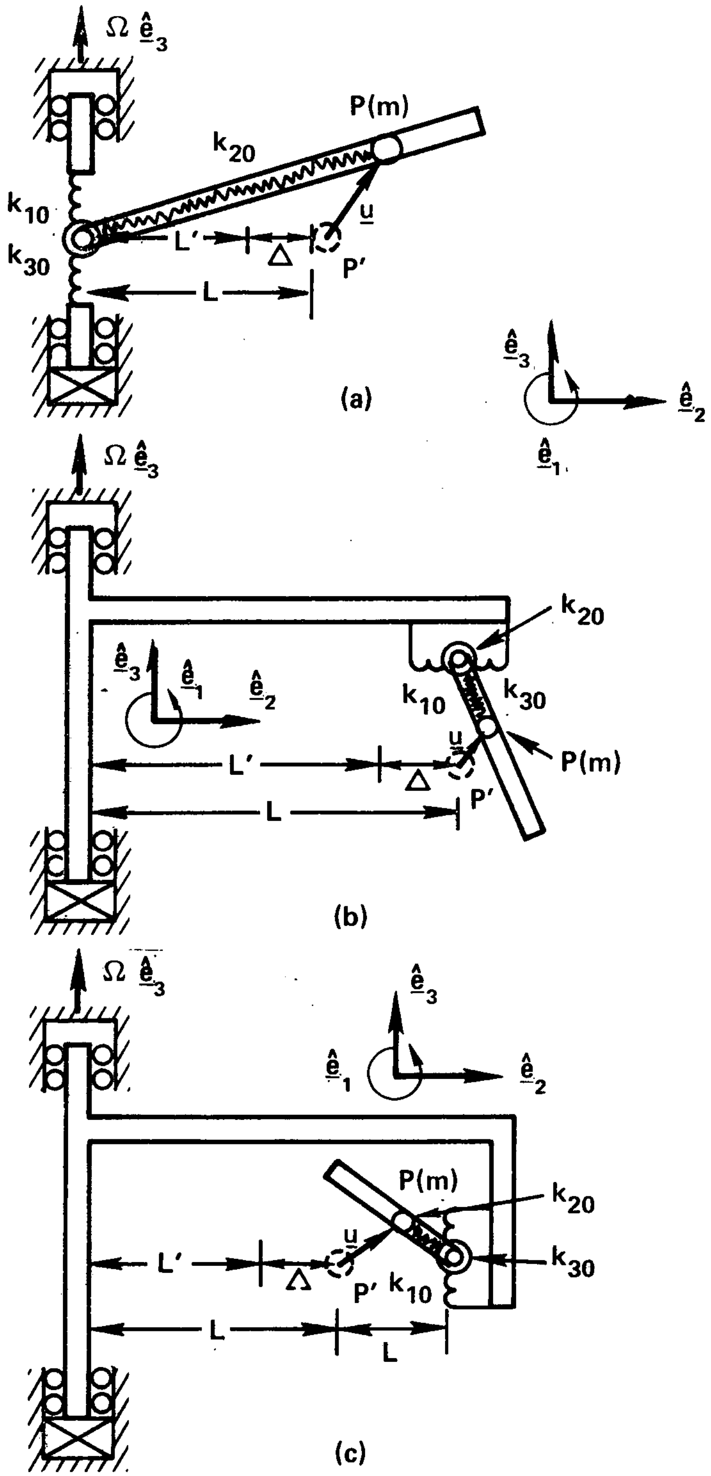

Figure 2. Single Particle Spring - Connected to Rotating Base

where, as shown in Figure 3, u_1, u_2, and u_3 are orthogonal displacements of the particle from the position occupied when the springs are undeformed. We assume that the three spring constants k_{10}, k_{20} and k_{30} are characteristic of each of the mechanisms in Figures 3a, b, and c. When $\Omega \neq 0$, the equations become

$$m\ddot{u}_1 - 2m\,\Omega\,\dot{u}_2 + (k_1 - m\,\Omega^2)u_1 = 0 \tag{55a}$$

$$\left.\begin{aligned} m\ddot{u}_2 + 2m\,\Omega\,\dot{u}_1 + (k_2 - m\,\Omega^2)u_2 &= 0 \\ m\ddot{u}_3 + k_3 u_3 &= 0 \end{aligned}\right. \tag{56}$$

or

$$\ddot{u}_1 - 2\,\Omega\,\dot{u}_2 + (\sigma_1^{\,2} - \Omega^2)u_1 = 0 \tag{57a}$$

$$\ddot{u}_2 + 2\,\Omega\,\dot{u}_1 + (\sigma_2^{\,2} - \Omega^2)u_2 = 0 \tag{57b}$$

$$\ddot{u}_3 + \sigma_3^{\,2}u_3 = 0 \tag{58}$$

where now u_1, u_2, and u_3 represent orthogonal displacements from the position occupied in the steady state in which the particle is located with respect to the inertially fixed point 0 by the vector $L\,\hat{\underline{e}}_2$. Because the spring mechanisms have different stiffness characteristics in the loaded state induced by rotation than in the unloaded state, the spring constants k_1, k_2, and k_3 generally differ from k_{10}, k_{20}, and k_{30}. In addition to this change (identified as the geometric stiffness in the preceding section), the influence of rotation is manifested in Equations (55), (56) in the centripetal accelerations and the coriolis accelerations. In what follows we shall consider these influences individually, and attempt to draw general conclusions about the contributions of each to the natural frequencies of particle oscillations.

As is clear from Equations (55) or (57), the centripetal accelerations reduce the effective stiffness in the equatorial plane (normal to $\hat{\underline{e}}_3$), and have no influence on the meridional vibration (in the $\hat{\underline{e}}_2$, $\hat{\underline{e}}_3$, plane). The coriolis accelerations also make no contribution to the meridional vibration, but these terms couple the two orthogonal vibrations in the equatorial plane, influencing both vibration frequencies. If we define $\omega_1^2 \triangleq \sigma_1^2 - \Omega^2$ and $\omega_2^2 \triangleq \sigma_2^2 - \Omega^2$, and assume for definiteness that $\omega_1 < \omega_2$, then we can artificially isolate the coriolis influence by recording the characteristic equation for Equations (57) as

$$\begin{vmatrix} \lambda^2 + \omega_1^2 & -2\,\Omega\,\lambda \\ 2\,\Omega\,\lambda & \lambda^2 + \omega_2^2 \end{vmatrix} = (\lambda^2+\omega_1^2)\,(\lambda^2+\omega_2^2) + 4\,\Omega^2\lambda^2 = 0$$

If we now formally permit Ω to range from zero to infinity and construct a root locus plot, we find that $\lambda = \pm\ i\,\omega_1$ and $\lambda = \pm\ i\,\omega_2$ for $\Omega = 0$ (corresponding to roots obtained by neglecting the coriolis terms), while for $\Omega \rightarrow \infty$ we have essentially $\lambda^4 + 4\Omega^2\lambda^2 = 0$, providing $\lambda = \pm\ \infty$. Thus we see that the coriolis influence is to elevate the higher frequency in the equatorial plane and to reduce the lower frequency, as compared to those frequencies which would be obtained if centripetal accelerations and geometric stiffness due to spin were accommodated but coriolis accelerations were ignored.

The influences of centripetal and coriolis accelerations on each of the mechanisms shown in Figure 2 are the same, since these are purely kinematic phenomena. The geometric stiffness previously alluded to is a phenomenon of structural mechanics, however, and it differs for the three structures shown.

For the mechanism in Figure 2a, the steady state rotation leaves the spring within the tube with a tensile load of magnitude $mL\Omega^2$. If we imagine a displacement u_3, we can see that this spring force develops a component in the $\hat{\underline{e}}_3$ direction of magnitude $-mL\Omega^2(u_3/L) = -m\Omega^2 u_3$; this is equivalent to the addition of an effective spring constant $m\,\Omega^2$, so that in Equations (56) and (58) we have

$$k_3 = k_{30} + m\,\Omega^2 \quad \text{and} \quad \sigma_3^{\,2} = \sigma_{30}^{\,2} + \Omega^2 \tag{59}$$

Similar arguments provide

$$k_1 = k_{10} + m\,\Omega^2 \quad \text{and} \quad \sigma_1^{\,2} = \sigma_{10}^{\,2} + \Omega^2 \tag{60}$$

but leave $k_2 = k_{20}$ and $\sigma_2^{\,2} = \sigma_{20}^{\,2}$. Thus for the system in Figure 2a, one could replace Equations (57), (58) by

$$\ddot{u}_1 - 2\,\Omega\,\dot{u}_2 + \sigma_{10}^{\,2} u_1 = 0 \tag{61a}$$

$$\ddot{u}_2 + 2\,\Omega\,\dot{u}_1 + (\sigma_{20}^{\,2} - \Omega^2)u_2 = 0 \tag{61b}$$

$$\ddot{u}_3 + (\sigma_{30}^{\,2} + \Omega^2)u_3 = 0 \tag{61c}$$

For the mechanism in Figure 2b, however, the steady state rotation places no load on the spring in the tube, influencing only one of the rotary springs, and there is no geometric stiffness contribution. Thus for this system Equations (57), and (58) are simply

$$\ddot{u}_1 - 2\,\Omega\,\dot{u}_2 + (\sigma_{10}^{\,2} - \Omega^2)u_1 = 0 \tag{62a}$$

$$\ddot{u}_2 + 2\,\Omega\,\dot{u}_1 + (\sigma_{20}^{\,2} - \Omega^2)\,u_2 = 0 \tag{62b}$$

$$\ddot{u}_3 \qquad + \sigma_{30}^{\,2} u_3 = 0 \tag{62c}$$

Finally, for the mechanism in Figure 2c, the steady state rotation places a compressive load of magnitude $m\,\Omega^2 L$ on the spring within the tube, and the result is an effective spring constant for u_3 and u_1 given by $-m\,\Omega^2$. Thus Equations (57), (58) for this system become

$$\ddot{u}_1 - 2\,\Omega\,\dot{u}_2 + (\sigma_{10}^{\,2} - 2\,\Omega^2)\,u_1 = 0 \tag{63a}$$

$$\ddot{u}_2 + 2\,\Omega\,\dot{u}_1 + (\sigma_{20}^{\,2} - \Omega^2)u_2 = 0 \tag{63b}$$

$$\ddot{u}_3 \qquad + (\sigma_{30}^{\,2} - \Omega^2)u_3 = 0 \tag{63c}$$

These three examples illustrate the range of influences of rotation on natural frequencies of vibration. In summary, we can say that centripetal accelerations always reduce oscillation frequencies in the equatorial plane, and coriolis accelerations cause further reduction in the lowest of these frequencies and increase in the highest frequency. Neither centripetal nor coriolis accelerations influence meridional vibrations. The geometric stiffness terms which reflect the change in behavior of the structure due to load may contribute terms of magnitude comparable to centripetal accelerations, but the sense of their influence on frequency cannot be specified generally, since this depends on the structure.

For purposes of comparison with the behavior of the idealized system shown in Figure 2a, we next consider the massless radial elastic beam with a tip mass m, shown in Figure 3. It will suffice to consider the geometric stiffness in the direction of u_3.

While we could approach this question with the variational methods of the preceding sections, it may be more instructive to build upon more traditional foundations. According to Timoshenko[25], a uniform cantilever beam of unstretched length L subjected to a tensile axial force of magnitude P develops a lateral (bending) stiffness given (after some manipulation and specialization of the simply supported beam case in the reference) by

$$k_3 = \frac{1}{3} k_{30}(pL_o)^2 \left[1 - \left(\frac{1}{pL_o}\right) \text{Tanh } pL_o\right]^{-1} \tag{64}$$

where the lateral stiffness with no axial load is

$$k_{30} = 3EI/L_o^{\ 3} \tag{65}$$

and where $p \triangleq (P/EI)^{1/2}$, with E designating the elastic modulus of the beam material, and I designating the area second moment of the beam cross-section.

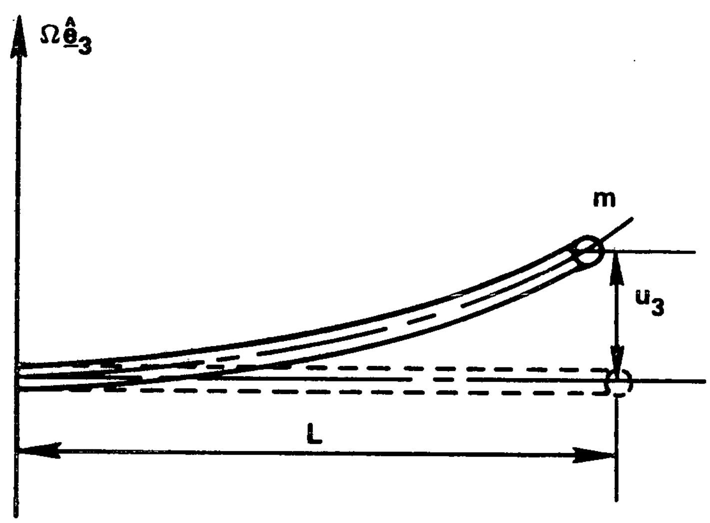

Figure 3. Rotating Elastic Beam with Tip Mass

When P is the consequence of spin rate Ω (see Figure 3), the beam stretches in its steady state to length L, and for a massless beam with a tip mass m we have

$$P = mL\Omega^2 \tag{66}$$

and

$$pL_o = \left(\frac{mL\Omega^2}{EI}\right)^{1/2} L_o = \left(\frac{3mL\Omega^2}{k_{30} L_o}\right)^{1/2} \tag{67}$$

If we now define $\sigma_3{}^2 \triangleq k_3/m$ and $\sigma_{30}{}^2 = k_{30}/m$, and divide Equation (64) by by $m\Omega^2$, we find

$$\left(\frac{\sigma_3}{\Omega}\right)^2 = \left(\frac{L}{L_o}\right)\left(1 - \frac{\text{Tanh } x}{x}\right)^{-1} \tag{68}$$

where

$$x \triangleq \left(\frac{3L}{L_o}\right)^{1/2} \frac{\Omega}{\sigma_{30}} \tag{69}$$

Comparison of the geometric stiffnesses of the massless elastic beam with tip mass shown in Figure 3 and the spring mechanism shown in Figure 2a reduces to comparing Equations (59) and (68). This comparison is particularly convenient when $\Omega \ll \sigma_{30}$, since then $x \ll 1$ and $\text{Tanh } x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$, so that Equation (68) produces

$$\left(\frac{\sigma_3}{\Omega}\right)^2 \approx \frac{3L}{x^2 L_o}\left(1 + \frac{2x^2}{5}\right)$$

With Equation (69), this result provides (after multiplication by Ω^2)

$$\sigma_3{}^2 \approx \sigma_{30}{}^2 + 1.2\,\Omega^2 \tag{70}$$

where the ratio (L/L_o) has been replaced by unity. The similarity of Equations (59) and (70) offers some perspective on the question of physical significance of the mechanism shown in Figure 2a; even for the extreme case of relatively small spin rate, this mechanism is not very different from a beam in its geometric stiffness characteristics. For large values of Ω/σ_{30}, Equation (70) is invalid, and a numerical comparison of Equations (68) and (59) is required. Table 1 shows the results of such a comparison, and establishes the range of validity of Equation (70).

$\left(\frac{\sigma_{30}}{\Omega}\right)$ \ $\left(\frac{\sigma_3}{\Omega}\right)^2$	Eq. (59)	Eq. (68)	Eq. (70)
0	1.00	1.000	1.20
0.1	1.01	1.061	1.21
0.5	1.25	1.405	1.45
1.0	2.00	2.184	2.20
5.0	26.00	26.199	26.20
10.0	101.00	101.200	101.20

Table 1. Comparison of Equation (59) for Figure 2a mechanism and Equation (68) for massless elastic beam and Equation (70) for beam approximation.

CONCLUSIONS

In this paper we have explored the problems of modal analysis of elastic appendages on a rotating base, considering elastic continuum models, distributed-mass finite element models, and concentrated mass models of appendages.

Although the continuum model is ideal for an axial beam, and not infeasible for the radial beam (both within the usual limitations of beam theory), there seems to be a very real practical limit to the class of problems for which meaningful conclusions can be extracted by means of a continuum model. Perhaps a thin circular plate normal to the spin axis would prove tractable (see Reference 26 for a related investigation); elastic membranes can be accommodated; and higher order beam approximations and geometries of motion can be considered with some success (see References 23 and 27); but the limitations in implementation are so severe that there seems little to be gained from a general formulation of the problem.

In contrast, the distributed-mass finite element model always leads to linear, constant-coefficient, ordinary differential equations (see Equation (51)) for the small elastic vibrations of an appendage on a base with an inertially constant body-fixed angular velocity vector. Thus this seems to be the most promising model for most rotating elastic structures. Although one can extract from the detailed structure underlying Equation (51) some general conclusions concerning the influence of spin on mode shapes and natural frequencies (observing for example the "softening" influence of centripetal accelerations on structural stiffness, and noting that coriolis or gyroscopic coupling terms introduce second order differential equation eigenvectors composed of complex numbers)[8] still the algebraic complexities of a distributed-mass finite element model are so great that one obtains little useful insight into system behavior from these equations.

Concentrating all of the mass in a finite element model of a structure into a large number of nodal bodies changes only the detailed structure of the vibration equations; Equations (51) still apply. This step might be taken at the modeling analyst's discretion (perhaps even with restriction to nodal particles), but the vibration equations which emerge are no more or less difficult to subject to modal analysis (except that advantage might be taken numerically of a diagonal or tightly banded mass matrix in some cases).

In order to gain useful insight into system behavior, we have in the penultimate section of this paper considered concentrated mass models, consisting simply of a particle on springs or a particle on a massless beam. For these models we have quantified the several influences of base rotation on modal characteristics, namely centripetal accelerations, coriolis accelerations, and geometric stiffness induced by structural loads in the steady rotation state. It is the conviction of the authors that thorough understanding of these very simple models is a necessary first step for the analyst who seeks to evaluate the influence of spin on the modal characteristics of a structure by means of more elaborate models.

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APPENDIX 2

GEOMETRIC STIFFNESS CHARACTERISTICS OF A ROTATING ELASTIC APPENDAGE*

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Introduction

In Ref. 1 there appear equations of motion which characterize the small time-varying deformations of a distributed-mass finite element model of an elastic appendage attached to a rigid body having arbitrary motions. Ref. 2 provides the equations of motion of a dynamical system of interconnected rigid bodies, each of which has attached to it a nonrigid appendage. In concert these two references establish the basis for a generic digital computer program to be developed for the simulation of nonrigid spacecraft. The purpose of this note is to strengthen Ref. 1 by one subtle but significant generalization and one correction and elaboration.

A Generalization

As shown in Ref. 1, (see Eq. (164), with damping excluded), if one assumes a distributed-mass, finite-element model with mass present also in the form of rigid bodies concentrated at each node, and chooses to characterize the unknowns as the 6n small linear and angular deformational displacements of the n rigid nodal bodies relative to some nominal state, and assembles these in the 6n by 1 column matrix q, then the ordinary differential equations of appendage vibratory deformation have the form

$$M'\ddot{q} + G'\dot{q} + K'q + A'q = L' \tag{1}$$

where M' and K' are symmetric and G' and A' are skew symmetric matrices. If

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the base to which the appendage is attached rotates at a constant rate about an inertially fixed axis, then the coefficient matrices in Eq. (1) are constant, and L' = 0.

It is important in some cases to recognize that the steady state stresses in a rotating elastic system can contribute to the skew-symmetric matrix A' by means of an asymmetric "geometric stiffness matrix", and that the result can be the elimination of the troublesome matrix A'. These possibilities are precluded in Ref. 1 by the seemingly insignificant assumption that nodal body incremental rotations are sequential rotations about permanently orthogonal axes. As a consequence of this assumption, the generalized force Q_α corresponding to a nodal body rotation β_α^j of the j^{th} body is the $\underline{a}_\alpha$ component of the torque $\underline{T}^j$ applied to the j^{th} nodal body, since by first principles

$$Q_\alpha = \underline{T}^j \cdot \frac{\partial \underline{\omega}}{\partial \dot{\beta}_\alpha} = \underline{T}^j \cdot \underline{a}_\alpha = T_\alpha^j \tag{2}$$

In the general case, $\partial \underline{\omega}^j / \partial \dot{\beta}_\alpha \neq \underline{a}_\alpha$, and one must make a distinction between Q_α and T_α^j. (An example of this kind is shown in the following Section). The matrix designated $\overline{L}$ in Ref. 1 can always be interpreted as the matrix of generalized forces; only for the special case treated explicitly in Ref. 1 is the interpretation of $\overline{L}$ as a matrix of scalar components of force and torque for orthogonal axes (as in Eq. (19) of Ref. 1) a valid interpretation. Thus we can broaden the scope of Ref. 1 (to include for example the problem in the following Section) simply by extending the meaning of $\overline{L}$, and establishing for each problem a specific relationship between Q_α and T_α^j. A possible implication of this generalization for the geometric stiffness matrix is established in the example following.

Illustration of Asymmetric Geometric Stiffness Matrix

Consider the rigid body B supported in a rotating housing body A by means of spring-mounted massless gimbals B' and A', as shown in Fig. 1. Note the dextral orthogonal sets of unit vectors of corresponding labels in the figure (e.g., $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ and $\underline{b}_1'$, $\underline{b}_2'$, $\underline{b}_3'$). Imagine that there exists a steady-state motion for which B maintains a fixed relationship to A, while the mass center C of B remains fixed in inertial space and A maintains the constant inertial angular velocity $\underline{\Omega}$, fixed somewhere in A but not parallel to $\underline{a}_1$, $\underline{a}_2$, or $\underline{a}_3$. Imagine further that in this steady state all unit vectors of like index are aligned, so that the gimbal hinge axes are orthogonal. In this state B is rotating at a constant rate about a nonprincipal axis, so that a body-fixed torque must be applied to B by means of the elastic springs at the three gimbal hinge axes parallel to $\underline{a}_1 \equiv \underline{a}_1'$, $\underline{a}_2' \equiv \underline{b}_2'$, and $\underline{b}_3' \equiv \underline{b}_3$. Rotations of the corresponding angles from the unstressed state to the proposed steady state are designated Δ_1, Δ_2, and Δ_3, and the corresponding spring constants are k_1, k_2, and k_3, so that in the steady state the torque applied to B is given by

$$\begin{aligned} \underline{T}_0 &= -k_1\Delta_1\underline{b}_1 - k_2\Delta_2\underline{b}_2 - k_3\Delta_3\underline{b}_3 \\ &= -k_1\Delta_1\underline{a}_1 - k_2\Delta_2\underline{a}_2 - k_3\Delta_3\underline{a}_3 \end{aligned} \tag{3}$$

When body B is perturbed from its steady state orientation relative to A, the expression for the torque $\underline{T}$ applied to B becomes perhaps surprisingly complicated. If θ_1, θ_2, θ_3 are gimbal rotations from the steady state corresponding to axes parallel to $\underline{a}_1 \equiv \underline{a}_1'$, $\underline{a}_2' \equiv \underline{b}_2'$, and $\underline{b}_3' \equiv \underline{b}_3$ respectively, then the inertial angular velocity of B becomes

$$\underline{\omega} = \underline{\Omega} + \dot{\theta}_1\underline{a}_1 + \dot{\theta}_2\underline{a}_2' + \dot{\theta}_3\underline{b}_3' \tag{4}$$

and our immediate knowledge of $\underline{T}$ is limited to the observations

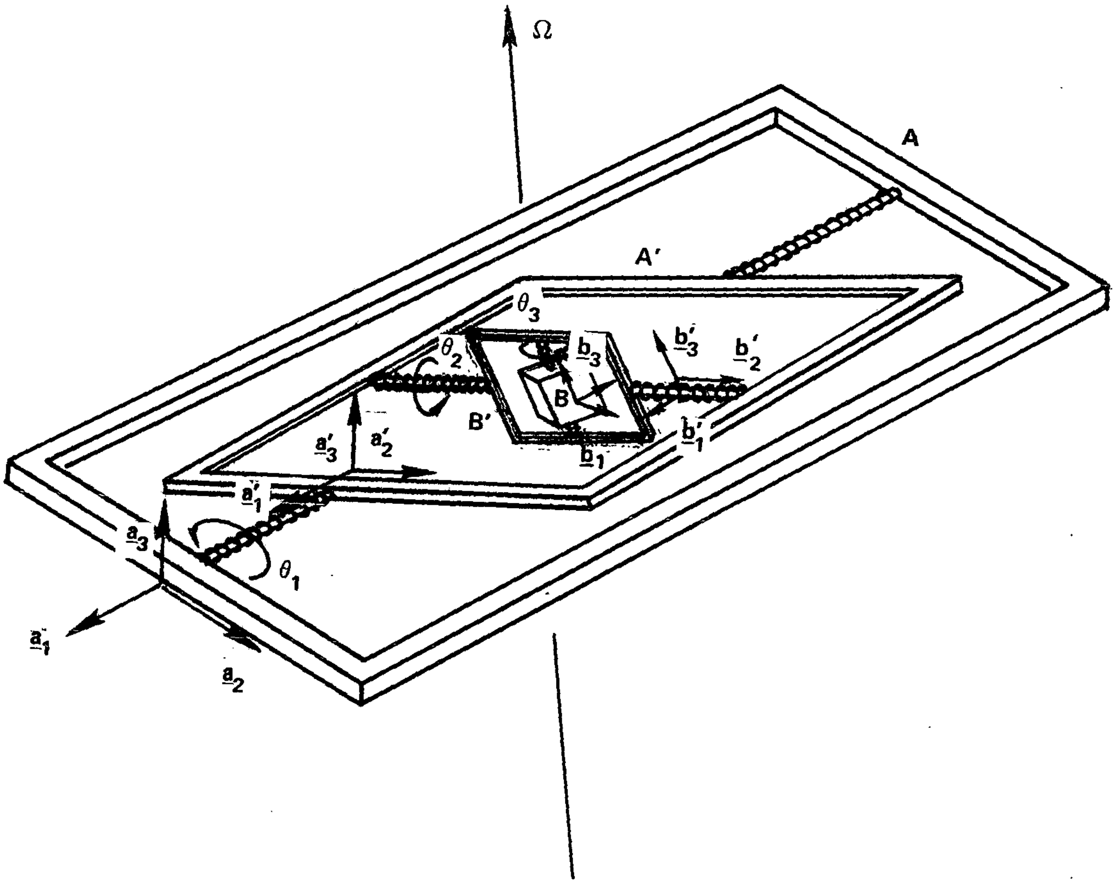

Figure 1. Rotating Rigid Body on Spring-Mounted Gimbals

$$\underline{T} \cdot \underline{a}_1 = - k_1(\Delta_1 + \theta_1)$$

$$\underline{T} \cdot \underline{a}_2' = - k_2(\Delta_2 + \theta_2) \tag{5}$$

$$\underline{T} \cdot \underline{b}_3' = - k_3(\Delta_3 + \theta_3)$$

Although one can manipulate these expressions algebraically to obtain $\underline{T}$ in any vector basis, such as $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, present purposes are best served by calculating first the generalized forces

$$Q_1 \triangleq \underline{T} \cdot \frac{\partial \underline{\omega}}{\partial \dot{\theta}_1} = \underline{T} \cdot \underline{a}_1 = - k_1(\Delta_1 + \theta_1)$$

$$Q_2 \triangleq \underline{T} \cdot \frac{\partial \underline{\omega}}{\partial \dot{\theta}_2} = \underline{T} \cdot \underline{a}_2' = - k_2(\Delta_2 + \theta_2) \tag{6}$$

$$Q_3 \triangleq \underline{T} \cdot \frac{\partial \underline{\omega}}{\partial \dot{\theta}_3} = \underline{T} \cdot \underline{b}_3' = - k_3(\Delta_3 + \theta_3)$$

To obtain the matrix T representing $\underline{T}$ in vector basis $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, we can define the matrices

$$T \triangleq \underline{T} \cdot \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix}; \qquad \omega \triangleq \underline{\omega} \cdot \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix};$$

$$Q \triangleq \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}; \qquad \theta \triangleq \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

and write

$$Q = \left(\frac{\partial \omega^T}{\partial \dot{\theta}}\right) T \tag{7}$$

finally inverting to obtain

$$T = \left(\frac{\partial \omega^T}{\partial \dot{\theta}}\right)^{-1} Q \equiv \begin{bmatrix} \frac{\partial \omega_1}{\partial \dot{\theta}_1} & \frac{\partial \omega_2}{\partial \dot{\theta}_1} & \frac{\partial \omega_3}{\partial \dot{\theta}_1} \\ \frac{\partial \omega_1}{\partial \dot{\theta}_2} & \frac{\partial \omega_2}{\partial \dot{\theta}_2} & \frac{\partial \omega_3}{\partial \dot{\theta}_2} \\ \frac{\partial \omega_1}{\partial \dot{\theta}_3} & \frac{\partial \omega_2}{\partial \dot{\theta}_3} & \frac{\partial \omega_3}{\partial \dot{\theta}_3} \end{bmatrix}^{-1} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \tag{8}$$

In this case a little algebra provides

$$\begin{aligned} \underline{\omega} = \underline{\Omega} &+ \underline{a}_1(\dot{\theta}_1 + \dot{\theta}_3 \sin \theta_2) \\ &+ \underline{a}_2(\dot{\theta}_2 \cos \theta_1 - \dot{\theta}_3 \sin \theta_1 \cos \theta_2) \\ &+ \underline{a}_3(\dot{\theta}_3 \cos \theta_1 \cos \theta_2 + \dot{\theta}_2 \sin \theta_1) \end{aligned} \tag{9}$$

so that in the linear approximation

$$T \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}^{-1} Q \tag{10}$$

or

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} \begin{bmatrix} -k_1(\Delta_1+\theta_1) \\ -k_2(\Delta_2+\theta_2) \\ -k_3(\Delta_3+\theta_3) \end{bmatrix} \cong \begin{bmatrix} -k_1(\Delta_1+\theta_1) \\ -k_2(\Delta_2+\theta_2) + k_3\Delta_3\theta_1 \\ k_1\Delta_1\theta_2 - k_2\Delta_2\theta_1 \; -k_3(\Delta_3+\theta_3) \end{bmatrix} \tag{11}$$

It is perhaps more illuminating to record this result in the form

$$T = -k_0\Delta - k_0\theta - k_\Delta\theta \tag{12}$$

where $\Delta \triangleq [\Delta_1 \quad \Delta_2 \quad \Delta_3]^T$ and

$$k_0 \triangleq \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}; \qquad k_\Delta \triangleq \begin{bmatrix} 0 & 0 & 0 \\ -k_3\Delta_3 & 0 & 0 \\ +k_2\Delta_2 & -k_1\Delta_1 & 0 \end{bmatrix} \tag{13}$$

thereby revealing the asymmetric character of the "geometric stiffness matrix" k_Δ induced by the load existing in the springs in the steady state.

According to Eq. (60) of Ref. 1, the equation of rotational motion of B must be

$$T = \tilde{\Omega} I \Omega + I \ddot{\theta} + [\tilde{\Omega} I - (I\Omega)^{\sim} + I\tilde{\Omega}]\dot{\theta}$$
$$+ \{\tilde{\Omega} I \tilde{\Omega} - \frac{1}{2} [\tilde{\Omega}(I\Omega)^{\sim} + (I\Omega)^{\sim}\tilde{\Omega}] - \frac{1}{2} (\tilde{\Omega} I\Omega)^{\sim}\}\theta \tag{14}$$

where I is the inertia matrix of B in its own vector basis, $\Omega \triangleq [\underline{\Omega} \cdot \underline{a}_1 \quad \underline{\Omega} \cdot \underline{a}_2 \quad \underline{\Omega} \cdot \underline{a}_3]^T$, and the tilde operator has a significance illustrated by

$$\tilde{\Omega} \triangleq \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \tag{15}$$

Thus it follows from the existence of $\theta \equiv 0$ as a steady state solution for Eq. (14) that

$$-k_0\Delta = \tilde{\Omega} I \Omega \tag{16}$$

By scalar expansion of the expressions in Eqs. (8) and (13), noting Eq. (10), we find the linear approximation

$$k_\Delta \theta \cong \left[\left(\frac{\partial \omega^T}{\partial \dot{\theta}}\right)^{-1} - U\right] k_0 \Delta$$

which with Eq. (16) becomes

$$k_\Delta \theta = \left[\left(\frac{\partial \omega^T}{\partial \dot{\theta}}\right)^{-1} - U\right] \tilde{\Omega} I \Omega \tag{17}$$

It is with this interpretation that one must consider the final equations of vibration in the form

$$I\ddot{\theta} + [\tilde{\Omega} I - (I\Omega)^{\sim} + I\tilde{\Omega}]\dot{\theta}$$
$$+ \{\tilde{\Omega} I \tilde{\Omega} - \frac{1}{2} [\tilde{\Omega}(I\Omega)^{\sim} + (I\Omega)^{\sim}\tilde{\Omega}] + k_0 - \frac{1}{2} (\tilde{\Omega} I \Omega)^{\sim} + k_\Delta\}\theta = 0 \tag{18}$$

recognizing that the asymmetric form of k_Δ retrieves the possibility that the matrix coefficient of θ may be symmetric. For this illustrative example, one can extract from Eq. (16) the expression

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \triangleq \Delta = - k_0^{-1}\tilde{\Omega} I \Omega = \begin{bmatrix} (I_2 - I_3)\Omega_2\Omega_3/k_1 \\ (I_3 - I_1)\Omega_3\Omega_1/k_2 \\ (I_1 - I_2)\Omega_1\Omega_2/k_3 \end{bmatrix} \tag{19}$$

and combine this result with Eq. (13) to find the geometric stiffness matrix

$$k_\Delta = \begin{bmatrix} 0 & 0 & 0 \\ (I_2 - I_1)\Omega_1\Omega_2 & 0 & 0 \\ -(I_1 - I_3)\Omega_3\Omega_1 & (I_3 - I_2)\Omega_2\Omega_3 & 0 \end{bmatrix} \tag{20}$$

By expanding other terms in the coefficient matrix of θ in Eq. (18), one finds dramatic simplification, and Eq. (18) reduces to the form

$$I\ddot{\theta} + [\tilde{\Omega} I - (I\Omega)^{\sim} + I\tilde{\Omega}]\dot{\theta} + [\tilde{\Omega} I \tilde{\Omega} - \tilde{\Omega}(I\Omega)^{\sim} - (I\Omega)^{\sim}\tilde{\Omega}]\theta = 0 \tag{21}$$

Equation (21) has the classical form adopted by vibrating rotating systems, with the coefficients of θ and $\ddot{\theta}$ symmetric and the coefficient of $\dot{\theta}$ skew symmetric.

The importance of this example stems from its demonstration of the possibility of retrieving the symmetric form of the overall "stiffness matrix" in the final equation of vibration. This result is reassuring, since it conforms with the fact well-known in Lagrangian mechanics that it must be possible to structure the equations of motion of any linearized, conservative, holonomic system so as to obtain a symmetric coefficient-matrix for the generalized coordinates.

A Correction for Nonlinearities

Reference 3 indicates the importance of retaining certain nonlinear terms in the strain-displacement equations for the determination of the stiffness characteristics of an elastic continuum vibrating relative to a deformed state.

The second purpose of this addendum to Ref. 1 is to indicate that these nonlinear terms were incorrectly omitted in that development, and to show how these nonlinearities can in some cases contribute to the geometric stiffness matrix of the finite element model.

In Ref. 1 the 6 by 1 matrix of the element stresses induced by steady state rotation is denoted $\bar{\sigma}'$, and the corresponding strain matrix is called $\bar{\varepsilon}'$. The incremental (variational) stress and strain matrices are designated $\bar{\sigma}$ and $\bar{\varepsilon}$ respectively. Under the restriction to small strain (but without further restriction on deformational displacements), we can record the element strain energy $\mathcal{U}$ as

$$\begin{aligned}\mathcal{U} &= \frac{1}{2}\iiint(\bar{\sigma}^T + \bar{\sigma}'^T)(\bar{\varepsilon} + \bar{\varepsilon}')dxdydz \\ &= \frac{1}{2}\iiint(\bar{\varepsilon}^T + \bar{\varepsilon}'^T)(\bar{\sigma} + \bar{\sigma}')dxdydz\end{aligned} \tag{22}$$

and the variational strain energy $\mathcal{U}^*$ (Eq. (21) of Ref. 1) would be

$$\mathcal{U}^* = \frac{1}{2}\iiint\left[\delta\bar{\varepsilon}^T(\bar{\sigma} + \bar{\sigma}') + (\bar{\varepsilon}^T + \bar{\varepsilon}'^T)\delta\bar{\sigma}\right] dxdydz \tag{23}$$

If now we record Hooke's law in the matrix form

$$\sigma = S\varepsilon \tag{24}$$

where σ and ε are total stress and strain matrices, and S is symmetric, then Eq. (23) becomes

$$\mathcal{U}^* = \iiint \delta\bar{\varepsilon}^T(\bar{\sigma} + \bar{\sigma}')dxdydz \tag{25}$$

in conformity with Eq. (21) of Ref. 1. However, in Ref. 1 only the linear approximations of the strain-displacement equations are substituted for $\bar{\varepsilon}$ into the variational strain energy, and this we can now recognize from Ref. 3 to be insufficient if the influences of steady state stress on structural stiffness are to be fully accommodated. Accordingly, we now consider the appropriate additional terms to be added to Ref. 1.

In terms of the matrix notation of Ref. 1, the strain displacement equations analogous to Eqs. (12) - (17) of Ref. 3 but descriptive of the relationship between incremental strain matrix $\bar{\varepsilon}$ and the matrix $\bar{w}$ of incremental displacements $\bar{w}_1$, $\bar{w}_2$ and $\bar{w}_3$ can be written as

$$\bar{\varepsilon} = D\bar{w} + \frac{1}{2}(\bar{w}\ \Delta)^T\bar{w} \tag{26}$$

where the operators* D and Δ are defined in terms of local orthogonal coordinates ξ, η, and ζ by

$$D \triangleq \begin{bmatrix} \partial/\partial\xi & 0 & 0 \\ 0 & \partial/\partial\eta & 0 \\ 0 & 0 & \partial/\partial\zeta \\ \partial/\partial\eta & \partial/\partial\xi & 0 \\ 0 & \partial/\partial\zeta & \partial/\partial\eta \\ \partial/\partial\zeta & 0 & \partial/\partial\xi \end{bmatrix}$$

and

$$\Delta \triangleq \begin{bmatrix} \frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta} & \frac{\partial}{\partial\zeta}\frac{\partial}{\partial\zeta} & 2\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} & 2\frac{\partial}{\partial\eta}\frac{\partial}{\partial\zeta} & 2\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\xi} \end{bmatrix}$$

Equation (26) is a nonlinear generalization[4] of Eq. (14) of Ref. 1. When this result is substituted into Eq. (25), and second degree terms in $\bar{w}$ are preserved when multiplied by the steady-state stress matrix $\bar{\sigma}'$, the result is the addition of the new term

$$\bar{y}^{*T}\bar{k}_\Delta\bar{y} \triangleq \bar{y}^{*T}\int W^T W\Delta\bar{\sigma}'\,dv\ \bar{y} \tag{27}$$

to the variational strain energy in Eq. (21) of Ref. 1, and correspondingly the new term $\bar{k}_\Delta\bar{y}$ to the expression for interaction force and torque in Eq. (34)

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*These operators will be treated as matrices, but caution must be exercised in preserving a meaningful sequence of operations; in Eq. (26), for example, the operation $\bar{w}\ \Delta$ precedes the transposition, and such "products" as $\bar{w}_1\ 2\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}$ are understood to mean the operator $2\frac{\partial\bar{w}_1}{\partial\xi}\frac{\partial}{\partial\eta}$.

of that paper. Here $\bar{y}$ is the $6\mathcal{N}$ by 1 matrix of incremental displacements of the $\mathcal{N}$ nodes of the finite element, and $\bar{k}_{\Delta}$ is the element geometric stiffness matrix. (The existence of this matrix is noted in Ref. 1, but no specific instructions for its construction are provided there.)

Summary and Conclusions

This addendum has had the objectives of expanding the scope of Ref. 1 and correcting a deficiency in that work which resulted from the neglect of certain potentially significant nonlinear terms in the strain-displacement equations. Even with the deeper appreciation of the subtleties of the mechanics of rotating finite elements reflected in this addendum, there remain many unanswered questions relating to the suitability of specific finite element models. The next step should be the detailed evaluation of the behavior of various finite element models of simple rotating structures, with the objective of evaluating the consequences of modeling decisions which are routine for nonrotating systems but potentially critical for rotating structures.

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