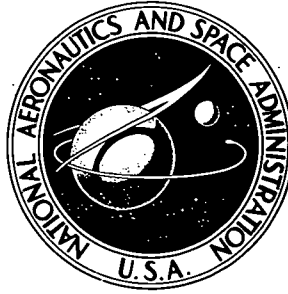


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AXISYMMETRIC OSCILLATION OF  
A PARTIALLY LIQUID-FILLED  
CYLINDRICAL SHELL CONTAINER  
HAVING HEMISPHERICAL BULKHEADS

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## DEFINITION OF SYMBOLS

Symbol	Definition
A	Marks the case in which the free liquid surface is within the upper bulkhead
$A_k(\phi_1, \phi_2)$	Definite integral defined by equation (56)
a	Radius of the cylinder and the hemispherical bulkheads
$a_k$	Coefficient defined by equation (2)
$\begin{bmatrix} a_{j1} \\ a_{j2} \\ . \\ . \\ . \\ a_{jn} \end{bmatrix}$	$j^{\text{th}}$ eigenvector
B	Marks the case in which the free liquid surface is within the cylinder
$B_j(z_1, z_2)$	Definite integral defined by equation (57)
C	Marks the case in which the free liquid surface is within the lower bulkhead
c	Integration constant
$c_j$	Constant defined by equation (63)
$c_{1j}$	Constant defined by equation (70)
$c_{2j}$	Constant defined by equation (71)
$c_{1j}^*$	Constant defined by equation (67)
$c_{2j}^*$	Constant defined by equation (66)

## DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\overline{D}$	Quantity defined by equation (14c)
$E$	Young's modulus of elasticity
$f$	Function symbol defined by equations (62)
$f_j$	Solution of equation (61)
$f_j^{(hom)}$	General solution of the homogeneous equation (61)
$f_j^{(inh)}$	General solution of the inhomogeneous equation (61)
$f_{pj}$	Particular solution of equation (61)
$h$	Increment
$h_i ; i = 1, 2$	Distance of the free liquid surface from the centers of the hemispheres $O_1, O_2$ , respectively (Fig. 2)
$J_0(r)$	Bessel function of the order zero
$k_s$	Constant of the helical spring upon which the container is supported
$l$	Length of the cylindrical part of the container
$N_\phi, N_\theta, N_z$	Membrane forces of the shell
$n$	Outer normal of the container wall
$P$	Legendre's polynomial of the first kind of undefined degree appearing in equation (34b)
$P_j$	Legendre's polynomial of the first kind of degree $j$
$p$	Liquid pressure
$Q_1$	Legendre's polynomial of the second kind of order one

## DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$r$	Cylindrical coordinate
$s_{2k}^{(m)}$	Coefficient defined by equation (29b)
$t$	Time
$v$	Wall displacement; see Figure 1
$v_i ; i = 1, 2$	Displacements of the parallel circles 1, 2, respectively; see Figure 1
$v_{ik} ; i = 1, 2$	Coefficients defined by equations (54) and (55)
$W$	Wetted surface of the container
$w$	Wall displacement; see Figure 1
$w_k$	Coefficients defined by equations (60) and (72)
$x$	Real variable
$z$	Cylindrical coordinate
$z_1, z_2$	Lower and upper limit of the integral (57)
$\alpha$	Constant defined by equation (25)
$\gamma$	Mass density of the liquid
$\delta$	Thickness of the container wall
$\epsilon$	Increment
$\epsilon_\phi, \epsilon_\theta, \epsilon_z$	Strains
$\eta, \eta_k$	Functions defined by equations (44) through (47)
$\theta$	Spherical coordinate

## DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
$\kappa_{jk}$	Matrix coefficient defined by equation (8)
$\lambda$	Parameter defined by equation (4)
$\lambda_j$	Eigenvalue defined by equation (10b)
$\mu_{jk}$	Matrix coefficient defined by equation (9)
$\nu$	Poisson's constant
$\xi$	Function defined by equation (41)
$\xi_k$	Solution of the differential equation (40)
$\bar{\xi}, \bar{\xi}_k$	Functions defined by equations (45) and (47)
$\rho$	Spherical coordinate
$\tau$	Liquid-filled region of the container
$\Phi$	Velocity potential
$\Phi_k$	Potential function
$\phi$	Spherical coordinate
$\phi_1, \phi_2$	Lower and upper limit of the integral (56)
$\phi_s$	Spherical coordinate of the liquid surface
$\Psi$	Function symbol defined by equations (33)
$\Psi_m$	Defined by equations (28) through (30)
$\omega$	Parameter defined by equation (4)
$\omega_i$	Circular frequency defined by equations (10b)



# AXISYMMETRIC OSCILLATION OF A PARTIALLY LIQUID-FILLED CYLINDRICAL SHELL CONTAINER HAVING HEMISPHERICAL BULKHEADS

## INTRODUCTION

The longitudinal oscillations of launch vehicles represent an area of concern. During ignition liftoff or cutoff, longitudinal vibrations occur as transients and may cause considerable dynamic loads. However, longitudinal oscillation can also be excited during powered flight. From experience it is well known that under certain circumstances energy can be tapped by the activity of the system itself and can produce very undesirable self-excited oscillation, the so-called "pogo" oscillation of the vehicle structure. Thus, it is understandable that the longitudinal dynamics of vehicles have received continued attention over the last years.

A simplified vehicle model must be used to analyze the longitudinal dynamic behavior of vehicles or, in other words, to determine the longitudinal frequencies and mode shapes. Usually only a few of the fundamental modes are of interest; thus, only significant dynamic properties must be considered. The liquid propellants constitute a high percentage of the overall vehicle masses throughout much of the powered flight time and, coupled with the solid vehicle masses and springs, may generate the fundamental modes of the entire vehicle. Thus, the correct comprehension of the liquid oscillation inside the elastic container is extremely important.

In this report a spring-supported, cylindrical container of the length  $\ell$  having hemispherical bulkheads is considered. The analysis, however, is also valid for  $\ell = 0$ , which represents the cases of sphere and hemisphere. With regard to these cases, the analysis in Reference 1 is, at least in its characteristics, similar to that in this report. The experimental result of the completely filled hemisphere as reported in Reference 2 agrees well with the analytical result of the present report.

A computer program and numerical evaluation of the analysis contained in this report can be found in the contractor report, Axisymmetric Oscillation of a Partially Liquid-Filled Cylindrical Shell Container Having Hemispherical Bulkheads: Computer Program and Numerical Evaluation, by Dwight Caughfield of the Department of Mathematics, Abilene Christian College, Abilene, Texas. That report was produced under cooperative agreement NCA8-45, Vibration Analysis of Hydro-Elastic Systems, between the Marshall Space Flight Center and Abilene Christian College.

## BASIC ASSUMPTIONS AND THE EIGENVALUE PROBLEM

The container, its geometry, and its wall displacements are shown in Figure 1; the displacements are  $v$  in the direction of the tangent to the meridian and  $w$  in the direction of the normal to the wall.

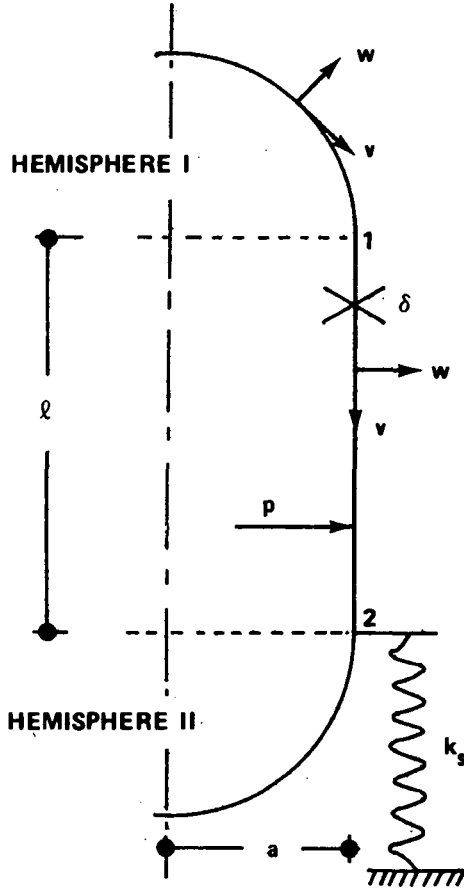


Figure 1. Vibrating system, wall displacements.

$$\frac{\partial \Phi}{\partial n} = \frac{\partial w}{\partial t}$$

At the free surface the pressure caused by the small surface elevation is neglected. Thus,

$$\Phi = 0 \quad (1d)$$

at the undisturbed free surface.

The potential  $\Phi$  will be determined by an approach proposed in Reference 1, which is a Galerkin approach in a generalized sense [3]. Using a complete set of potential functions  $\Phi_j$  ( $j = 1, 2, \dots$ ) which satisfy the free surface condition (1d),  $\Phi$  will be approximated by

The assumptions upon which the following analysis is based are as usual. Because the mass of the wall is small compared with the liquid mass inside the huge container, it will be neglected. Also neglected will be the axial bending as to the small effect on the fundamental frequencies. In summary, the container wall is interpreted to behave like a massless membrane shell.

To describe the liquid motion which is assumed to be small, the existence of a velocity potential  $\Phi$  is assumed. Thus,

$$\nabla^2 \Phi = 0 \quad (1a)$$

within the liquid-filled region  $\tau$  of the container. Then the pressure in  $\tau$  follows as

$$p = -\gamma \frac{\partial \Phi}{\partial t} \quad (1b)$$

The boundary condition at the wetted wall  $W$  is

$$(1c)$$

$$\Phi = a^2 \omega \cos \omega t \sum_{k=1}^n a_k \Phi_k ; \quad (2)$$

thereby the coefficients  $a_k$  and the functions  $\Phi_k$  are assumed to be dimensionless.

It is requested that the error which appears in the boundary condition (1c) by using the approximation (2) is orthogonal to  $\Phi_j$  ( $j = 1, 2, \dots, n$ ), i.e.,

$$\int_W \left( \frac{\partial \Phi}{\partial n} - \frac{\partial w}{\partial t} \right) \Phi_j dW = 0 ; \quad j = 1, 2, \dots, n ,$$

or by the use of Green's theorem [4],

$$\int_W \left( \Phi \frac{\partial \Phi_j}{\partial n} - \Phi_j \frac{\partial \Phi}{\partial n} \right) dW = 0 ; \quad j = 1, 2, \dots, n . \quad (3)$$

From equations (1b) and (2), one obtains

$$p = \gamma a^2 \omega^2 \sin \omega t \sum_{k=1}^n a_k \Phi_k$$

If

$$\lambda = \frac{a^3 \gamma \omega^2}{E \delta} , \quad (4)$$

it follows that

$$p = \frac{E \delta}{a} \lambda \sum_{k=1}^n a_k \Phi_k . \quad (5)$$

Thereby and also in the following equations, the time factor has been omitted.

The displacement  $w$  is caused by  $p$ . Therefore, the following expansion of  $w$ ,

$$w = a\lambda \sum_{k=1}^n a_k w_k, \quad (6)$$

can be concluded;  $w_k$  is a dimensionless quantity which must be determined.

The substitution of equations (2) and (6) into equation (3) results in

$$\sum_{k=1}^n a_k \left( \int_W a \frac{\partial \Phi_j}{\partial n} \Phi_k dW - \lambda \int_W \Phi_j w_k dW \right) = 0 \quad j = 1, 2, \dots \quad (7)$$

If the entire equation (7) is divided by  $2a^2\pi$  and the notations

$$\frac{1}{2a^2\pi} \int_W a \frac{\partial \Phi_j}{\partial n} \Phi_k dW = \kappa_{jk} \quad (8)$$

and

$$\frac{1}{2a^2\pi} \int_W \Phi_j w_k dW = \mu_{jk} \quad (9)$$

are introduced, the equations (7) can be written as

$$\left( \begin{bmatrix} \kappa_{11} & \kappa_{12} & \dots & \kappa_{1n} \\ \kappa_{21} & \kappa_{22} & \dots & \kappa_{2n} \\ & & \dots & \\ \kappa_{n1} & \kappa_{n2} & \dots & \kappa_{nn} \end{bmatrix} - \lambda \begin{bmatrix} \mu_{11} & \mu_{12} & \dots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \dots & \mu_{2n} \\ & & \dots & \\ \mu_{n1} & \mu_{n2} & \dots & \mu_{nn} \end{bmatrix} \right) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0 \quad (10a)$$

In this way the problem at hand is reduced to a matrix eigenvalue problem, the solutions of which are denoted by

$$\lambda_i = \frac{a^3 \gamma \omega_i^2}{E \delta} ; \quad \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} ; \quad i = 1, 2, \dots \quad (10b)$$

Depending on the liquid level, one must consider three basic cases (Fig. 2).

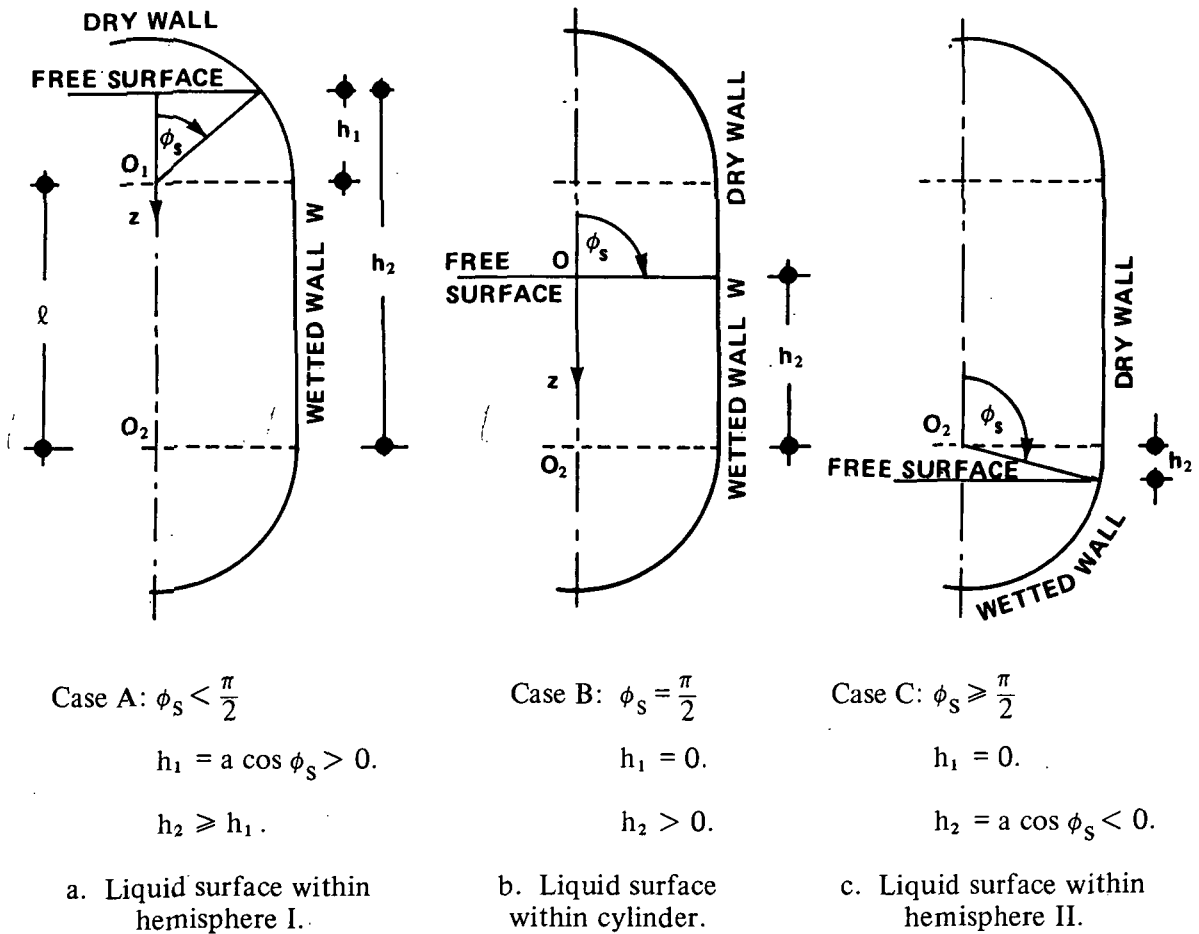


Figure 2. Three basic cases.

Assuming the length  $\ell$  of the cylinder to be zero, the problem is reduced to that of the partially liquid-filled sphere.

As indicated in Figure 2, two different coordinate systems are considered. For the hemispheres, spherical coordinate systems  $(\rho, \phi, \theta)$  having their centers in  $O_1$  or  $O_2$ , respectively, will be used (Fig. 3). The liquid motion within the cylinder is referred to a cylindrical coordinate system  $(r, z, \theta)$  with its origin at the liquid surface (Fig. 4).

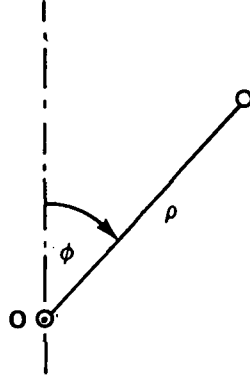


Figure 3. System of spherical coordinates.

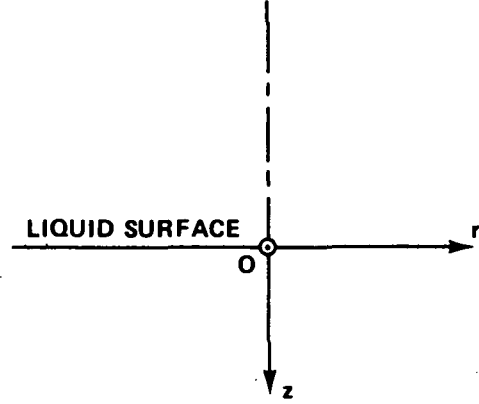


Figure 4. System of cylindrical coordinates.

The surface element of the wetted wall in both of the systems, respectively, is

$$dW = a^2 \sin \phi \, d\phi \, d\theta$$

and

$$dW = a \, dz \, d\theta$$

The introduction of these coordinates into the integrals (8) and (9) for the most comprehensive case  $A(\phi_s < \pi/2; h_1 > 0; h_2 \geq h_1)$  yields

$$\kappa_{jk} = \int_{\phi_s}^{\pi} a \frac{\partial \Phi_j}{\partial \rho} \Phi_k \sin \phi \, d\phi + \int_{h_1}^{h_2} a \frac{\partial \Phi_j}{\partial r} \Phi_k \frac{dz}{a} \quad (11)$$

and

$$\mu_{jk} = \int_{\phi_s}^{\pi} \Phi_j w_k \sin \phi \, d\phi + \int_{h_1}^{h_2} \Phi_j w_k \frac{dz}{a} , \quad (12)$$

where  $\rho = r = a$ .

Equations (11) and (12) comprise also the cases B and C. In case B,  $h_1$  equals zero and  $\phi_s$  equals  $\pi/2$ , as indicated by Figure 2. In case C, the second integral of equations (11) and (12) must be omitted.

### THE WALL DISPLACEMENTS

The wall displacements depend on the pressure exerted by the liquid at the wetted container wall. As mentioned previously, the wall is assumed to behave like a membrane shell. Hence, to obtain the equations that link the displacements to the pressure, the basic equations of the membrane shell theory must be brought up. First, the hemispheres will be considered [5].

The equilibrium conditions are

$$N_\phi + N_\theta = ap \quad (13a)$$

and

$$\frac{d}{d\phi} (N_\phi \sin \phi) - N_\theta \cos \phi = 0 \quad (13b)$$

Hooke's law is

$$N_\theta = \bar{D}(\nu \epsilon_\phi + \epsilon_\theta) \quad (14a)$$

and

$$N_{\phi} = \bar{D}(\epsilon_{\phi} + \nu\epsilon_{\theta}) \quad , \quad (14b)$$

where

$$\bar{D} = \frac{E\delta}{1-\nu^2} \quad . \quad (14c)$$

The strain-displacement relations are

$$v \cotan \phi + w = a\epsilon_{\theta} \quad (15a)$$

and

$$\frac{dv}{d\phi} + w = a\epsilon_{\phi} \quad . \quad (15b)$$

The positive directions of  $v$  and  $w$  can be seen from Figure 1.

By substituting equations (14a) through (15b) into equations (13) (see Appendix), one obtains

$$(\nabla^2 + 2) \xi = p \quad (16a)$$

and

$$w = \frac{a^2}{E\delta} [p - (1 + \nu) \xi] \quad (16b)$$

where, according to equations (A-3) and (A-4) in the Appendix,

$$\frac{d\xi}{d\phi} = - \frac{E\delta}{(1 + \nu) a^2} v \quad (16c)$$



and

$$\nabla^2 \xi = \frac{1}{\sin \phi} \frac{d}{d\phi} \left( \frac{d\xi}{d\phi} \sin \phi \right) \quad (16d)$$

The liquid pressure  $p$  is acting on the wetted container wall. Hence, one has to assume that  $p$  equals zero in the preceding equations where the wall is dry.

Equation (16a) is a differential equation of the second order, in the general solution of which two arbitrary constants occur. Hence, two boundary conditions must be assigned. These conditions are, for hemisphere I,

$$\left. \frac{d\xi}{d\phi} \right|_{\phi=0} = 0 \quad (17a)$$

and

$$\left. \frac{d\xi}{d\phi} \right|_{\phi=\frac{\pi}{2}} = - \frac{E\delta}{(1+\nu) a^2} v_1 \quad (17b)$$

and for hemisphere II,

$$\left. \frac{d\xi}{d\phi} \right|_{\phi=\frac{\pi}{2}} = - \frac{E\delta}{(1+\nu) a^2} v_2 \quad (18a)$$

and

$$\left. \frac{d\xi}{d\phi} \right|_{\phi=\pi} = 0 \quad (18b)$$

The conditions (17a) and (18b) express the fact that the problem is an axisymmetric one while the remaining two conditions (17b) and (18a) follow from equation (16c).

There are, however, additional so-called compatibility conditions [1] which assure that the membrane at the liquid surface remains continuous after deformation, i.e., that it has no gap:

$$\xi \Big|_{\phi = \phi_s - \epsilon} = \xi \Big|_{\phi = \phi_s + \epsilon} \quad (19a)$$

and

$$\frac{d\xi}{d\phi} \Big|_{\phi = \phi_s - \epsilon} = \frac{d\xi}{d\phi} \Big|_{\phi = \phi_s + \epsilon} \quad (19b)$$

For the axisymmetrical load  $p$  to determine the displacements  $v$  and  $w$ , the differential equation (16a) under consideration of the boundary and compatibility conditions (17a) through (19b) must be solved. The displacements then follow from equations (16b) and (16c).

Now the corresponding equations of the cylindrical part will be considered.

The equilibrium conditions are

$$N_\theta = ap$$

and

$$N_z = a \int_{\phi_s}^{\frac{\pi}{2}} p \cos \phi \sin \phi \, d\phi$$

These conditions can be verified easily by separating the appropriate parts from the shell by means of cuts.

Obviously,  $N_\theta$  equals zero where the wall is dry, whereas  $N_z$  equals zero in cases B and C where hemisphere I is empty.

Hooke's law is

$$N_z = \bar{D} (\epsilon_z + \nu \epsilon_\theta)$$

and

$$N_\theta = \bar{D} (\nu \epsilon_z + \epsilon_\theta)$$

The strain-displacement relations are

$$\frac{w}{a} = \epsilon_\theta$$

and

$$\frac{\partial v}{\partial z} = \epsilon_z$$

Once again, the elimination of the strains and membrane forces from these equations results in

$$\frac{E\delta}{a^2} w = p - \nu \int_{\phi_s}^{\frac{\pi}{2}} p \cos \phi \sin \phi d\phi \quad (20)$$

and

$$\frac{E\delta}{a} \frac{\partial v}{\partial z} = -\nu p + \int_{\phi_s}^{\frac{\pi}{2}} p \cos \phi \sin \phi d\phi \quad (21)$$

Equations (20) and (21) represent the most comprehensive case A ( $\phi_s < \pi/2$ ). In cases B and C ( $\phi_s \geq \pi/2$ ) the integrals on the right side of equations (20) and (21) equal zero. For dry regions of the cylinder the entire right sides of equations (20) and (21) equal zero.

To determine  $v - v_1$  for case A ( $\phi_s < \pi/2$ ;  $h_1 > 0$ ;  $h_2 \geq h_1$ ), equation (21) must be integrated over the cylinder from  $h_1$  to the variable coordinate  $z$ . This results in

$$v = v_1 + \frac{a^2}{E\delta} \left( \frac{z - h_1}{a} \int_{\phi_s}^{\frac{\pi}{2}} p \cos \phi \sin \phi d\phi - \nu \int_{h_1}^z p \frac{d\xi}{a} \right) \quad (22a)$$

In case B ( $\phi_s = \pi/2$ ;  $h_1 = 0$ ;  $h_2 > 0$ ),

$$v = v_1 - \frac{a^2}{E\delta} \nu \int_0^z p \frac{d\xi}{a} \quad (22b)$$

In case C ( $\phi_s \geq \pi/2$ ;  $h_1 = 0$ ;  $h_2 \leq 0$ ),

$$v = v_1 \quad (22c)$$

If  $z = h_2$ , it follows from equation (22a) that

$$v_2 = v_1 + \frac{a^2}{E\delta} \left( \frac{h_2}{a} \int_{\phi_s}^{\frac{\pi}{2}} p \cos \phi \sin \phi d\phi - \nu \int_{h_1}^{h_2} p \frac{d\xi}{a} \right) \quad (23)$$

Now, if the equilibrium of the spring force and the resulting force stemming from the pressure inside the container is considered, one obtains

$$v_2 = - \frac{a^2}{E\delta} \alpha \int_{\phi_s}^{\pi} p \cos \phi \sin \phi d\phi \quad (24)$$

where

$$\alpha = \frac{2\pi E\delta}{k_s} \quad (25)$$

From equations (23) and (24) and the remarks after equations (21), the following can be concluded:

In case A ( $\phi_s < \pi/2$  ;  $h_1 > 0$  ;  $h_2 \geq h_1$  ) ,

$$v_1 = \frac{a^2}{E\delta} \left( -\alpha \int_{\phi_s}^{\pi} p \cos \phi \sin \phi d\phi - \frac{\ell}{a} \int_{\phi_s}^{\frac{\pi}{2}} p \cos \phi \sin \phi d\phi + \nu \int_{h_1}^{h_2} p \frac{d\xi}{a} \right) \quad (26a)$$

In Case B ( $\phi_s = \pi/2$  ;  $h_1 = 0$  ;  $h_2 > 0$  ) ,

$$v_1 = \frac{a^2}{E\delta} \left( -\alpha \int_{\frac{\pi}{2}}^{\pi} p \cos \phi \sin \phi d\phi + \nu \int_0^{h_2} p \frac{d\xi}{a} \right) \quad (26b)$$

In case C ( $\phi_s \geq \pi/2$  ;  $h_1 = 0$  ;  $h_2 \leq 0$  ) ,

$$v_1 = - \frac{a^2}{E\delta} \alpha \int_{\phi_s}^{\pi} p \cos \phi \sin \phi d\phi = v_2 \quad (26c)$$

## THE COORDINATE FUNCTIONS

Considered herein is a complete set of polynomials

$$\Phi_m(r, z) ; \quad m = 1, 2, \dots ,$$

which fulfill equation (1a). In the cylindrical coordinates  $r, z$  equation (1a) is

$$\nabla^2 \Phi(r, z) \equiv r^2 \Phi_{rr} + r \Phi_r + r^2 \Phi_{zz} = 0 \quad (27)$$

As is easy to prove,  $e^z J_0(r)$  is a solution of equation (27). If one now multiplies the series

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

and

$$J_0(r) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{r}{2}\right)^{2j}}{(j!)^2}$$

term by term and collects the terms having equal degrees, one obtains [6]

$$e^z J_0(r) = \sum_{m=0}^{\infty} \frac{\Psi_m(r, z)}{m!} , \quad (28)$$

where

$$\Psi_m(r, z) = \sum_{k=0}^{\left[\frac{m}{2}\right]} s_{2k}^{(m)} r^{2k} z^{m-2k} \quad (29a)$$

( $\left[\frac{m}{2}\right]$  is the next smaller integer of  $m/2$ ), and

$$s_{2k}^{(m)} = \frac{(-1)^k}{2^{2k}} \binom{2k}{k} \binom{m}{2k} \quad (29b)$$

Because the operator of the left side of equation (27) leaves the degree of these polynomials unchanged, each of the polynomials (29) must be a solution of Laplace's equation, i.e., a potential function. The polynomials (29) of the degree 0 through 7 are

$$\Psi_0 = 1 \quad ,$$

$$\Psi_1 = z \quad ,$$

$$\Psi_2 = z^2 - \frac{1}{2} r^2 \quad ,$$

$$\Psi_3 = z^3 - \frac{3}{2} r^2 z \quad ,$$

$$\Psi_4 = z^4 - 3 r^2 z^2 + \frac{3}{8} r^4 \quad , \quad (30)$$

$$\Psi_5 = z^5 - 5 r^2 z^3 + \frac{15}{8} r^4 z \quad ,$$

$$\Psi_6 = z^6 - \frac{15}{2} r^2 z^4 + \frac{45}{8} r^4 z^2 - \frac{5}{16} r^6 \quad ,$$

and

$$\Psi_7 = z^7 - \frac{21}{2} r^2 z^5 + \frac{105}{8} r^4 z^3 - \frac{35}{16} r^6 z \quad .$$

If the cylindrical coordinates are transformed into spherical coordinates according to

$$z = -\rho \cos \phi$$

and

$$r = \rho \sin \phi$$

(see Figs. 3 and 4), one obtains [6, 7]

$$\Psi_m(r, z) = (-\rho)^m P_m(\cos \phi) \quad , \quad (31)$$

where  $P_m(x)$  is Legendre's polynomial or Legendre's function of the zero<sup>th</sup> order of the degree  $m$ . For  $m = 0, 1, 2 \dots$  Legendre's polynomials are [6]

$$P_0 = 1 \quad ,$$

$$P_1 = x \quad ,$$

$$P_2 = \frac{1}{2} (3x^2 - 1) \quad ,$$

$$P_3 = \frac{1}{2} (5x^2 - 3x)$$

and so forth.

Differentiation of the polynomials (29) with respect to  $z$  results in

$$\frac{\partial \Psi_m}{\partial z} = m \Psi_{m-1} \quad , \quad (32a)$$



as can easily be concluded from equation (28). Thus,

$$\int \Psi_m(r, z) dz = \frac{1}{m+1} \Psi_{m+1}(r, z) + c \quad (32b)$$

Another relation follows by multiplying equation (28) by

$$e^h = \sum_{j=1}^{\infty} \frac{h^j}{j!} ;$$

one obtains

$$\Psi_m(r, z+h) = \sum_{j=0}^m \binom{m}{j} \Psi_{m-j} h^j \quad (33a)$$

or by symbolic operations,

$$\Psi_m(r, z+h) = (\Psi + h)^m ; \quad (33b)$$

thereby,

$$\Psi^j = \Psi_j(r, z) .$$

If the polynomials of equations (33) are replaced by Legendre's functions according to equation (31), the result is that

$$\Psi_m(r, z+h) = (-\rho)^m \sum_{j=0}^m \binom{m}{j} P_{m-j} \left( -\frac{h}{\rho} \right)^j , \quad (34a)$$

or symbolically

$$\Psi_m(r, z+h) = (-\rho)^m \left[ P(\cos \phi) - \frac{h}{\rho} \right]^m \quad (34b)$$

The dimensionless polynomials

$$\Phi_j(r, z) = \Psi_{2j-1}\left(\frac{r}{a}, \frac{z}{a}\right) ; j = 1, 2, \dots \quad (35)$$

may serve as coordinate functions. The polynomials satisfy equation (1a) and the free surface condition (1d); hence, all requirements are fulfilled. Some of these polynomials are

$$\begin{aligned} \Phi_1 &= \frac{z}{a} , \\ \Phi_2 &= \left(\frac{z}{a}\right)^3 - \frac{3}{2}\left(\frac{r}{a}\right)^2 \frac{z}{a} , \\ \Phi_3 &= \left(\frac{z}{a}\right)^5 - 5\left(\frac{r}{a}\right)^2 \left(\frac{z}{a}\right)^3 + \frac{15}{8}\left(\frac{r}{a}\right)^4 \frac{z}{a} , \\ \Phi_4 &= \left(\frac{z}{a}\right)^7 - \frac{21}{2}\left(\frac{r}{a}\right)^2 \left(\frac{z}{a}\right)^5 + \frac{105}{8}\left(\frac{r}{a}\right)^4 \left(\frac{z}{a}\right)^3 - \frac{35}{16}\left(\frac{r}{a}\right)^6 \frac{z}{a} \end{aligned} \quad (36)$$

and so forth.

As far as the cylindrical part of the container is concerned, the representation (35) and (36) of the coordinate functions is appropriate. The velocity potential inside the hemispheres, however, must be referred to the centers  $O_1, O_2$  having the distances  $h_1, h_2$ , respectively, from the free surface. Following equations (34) and (35) the coordinate functions referred to the respective centers are

$$\Phi_j = -\left(\frac{\rho}{a}\right)^{2j-1} \left[ P(\cos \phi) - \frac{h_i}{\rho} \right]^{2j-1} ; \quad i = 1, 2 ; \quad j = 1, 2, \dots \quad (37)$$

Thereby,  $i$  equals one or two accordingly as hemisphere I or hemisphere II is concerned.

For the latter use the derivatives of the coordinate functions at the wall in the direction of the outer normal

$$a \frac{\partial \Phi_j}{\partial n} ; \quad j = 1, 2, \dots$$

will be determined. For the cylindrical wall, one concludes from equation (35) that

$$a \frac{\partial \Phi_j}{\partial r} \bigg|_{r=a} = \frac{\partial^{2j-1} \Psi \left( \frac{r}{a}, \frac{z}{a} \right)}{\partial \left( \frac{r}{a} \right)} \bigg|_{r=a} \quad (38)$$

The normal derivative of the hemispheres follows from equation (37) as (symbolically)

$$a \frac{\partial \Phi_j}{\partial \rho} \bigg|_{\rho=a} = - (2j-1) P \left( P - \frac{h_i}{a} \right)^{2j-2} ; \quad i = 1, 2 \quad (39)$$

where

$$P P^k = P_{k+1}$$

The coordinate functions (35) and (37) and their derivatives (38) and (39) at the container wall are shown in Table 1.

TABLE 1. COORDINATE FUNCTIONS AND THEIR NORMAL DERIVATIVES AT THE CONTAINER WALL

	$\Phi_j \Big _{\rho = r = a}$	$a \frac{\partial \Phi_j}{\partial n} \Big _{\rho = r = a}$
Hemisphere I $\phi_s \leq \phi \leq \frac{\pi}{2}$	$- \left[ P(\cos \phi) - \frac{h_1}{a} \right]^{2j-1}$	$- (2j - 1) P(\cos \phi) \left[ P(\cos \phi) - \frac{h_1}{a} \right]^{2j-2}$
Cylinder $h_1 \leq z \leq h_2$	$\Psi_{2j-1} \left( 1, \frac{z}{a} \right)$	$\frac{\partial \Psi_{2j-1} \left( \frac{r}{a}, \frac{z}{a} \right)}{\partial \left( \frac{r}{a} \right)} \Big _{r = a}$
Hemisphere II $\frac{\pi}{2} \leq \phi \leq \pi$	$- \left[ P(\cos \phi) - \frac{h_2}{a} \right]^{2j-1}$	$- (2j - 1) P(\cos \phi) \left[ P(\cos \phi) - \frac{h_2}{a} \right]^{2j-2}$

### THE MATRIX COEFFICIENTS

The matrix coefficients  $\kappa_{jk}$  as given by equation (11) depend on the coordinate functions and their derivatives only. By use of Table 1 and equations (29),  $\kappa_{jk}$  can easily be evaluated. To cover cases B and C, the remark following equation (12) must be considered. In the case of  $\mu_{jk}$  given by equation (12), the situation is different. For  $\mu_{jk}$  to be determined,  $w_k$  must be known. As far as the cylindrical part of the container is concerned,  $w_k$  follows easily from equations (5) and (20). The determination of  $w$  and hence of  $w_k$  for the hemispheres, however, is laborious, as can be seen from equations (16).

First, the hemispheres are considered. From equations (5) and (16a) it can be concluded that if

$$(\nabla^2 + 2) \xi_k = \begin{cases} 0 & \text{dry wall} \\ \Phi_k & \text{wetted wall} \end{cases} \quad (40)$$

where  $\xi_k$  satisfies the boundary conditions (17a) and (18b) and the compatibility conditions (19); then, for the wetted wall,

$$\xi = \frac{E\delta}{a} \lambda \sum_1^n a_k \xi_k, \quad (41)$$

provided that the series on the right side of this equation satisfies the boundary conditions (17b) and (18a). If this is the case, then from equations (5), (16b), and (41) it follows that

$$w = a\lambda \sum_1^n a_k [\Phi_k(a, \phi) - (1 + \nu) \xi_k] \quad (42)$$

A comparison of equations (6) and (42) results in

$$w_k = \Phi_k(a, \phi) - (1 + \nu) \xi_k \quad (43)$$

Now the problem is how to obtain the proper solution of equation (40). For convenience  $\xi$  will be composed of two terms, i.e.,

$$\xi = \bar{\xi} + \eta \quad (44)$$

where, in accordance with equation (41),

$$\bar{\xi} = \frac{E\delta}{a} \lambda \sum_1^n a_k \bar{\xi}_k, \quad (45)$$

$$\eta = \frac{E\delta}{a} \lambda \sum_1^n a_k \eta_k, \quad (46)$$

and

$$\xi_k = \bar{\xi}_k + \eta_k \quad (47)$$

To assure the fulfillment of equation (16a) and of the boundary and compatibility conditions (17) through (19) by  $\xi$  as presented by equations (44) through (47), the following assumptions are imposed:

$$(\nabla^2 + 2) \bar{\xi} = 0 \quad (48)$$

at the dry and wetted wall. Furthermore,

$$(\nabla^2 + 2) \eta = \begin{cases} 0 & \text{at the dry wall} \\ p & \text{at the wetted wall} \end{cases} \quad (49)$$

Thereby,  $\bar{\xi}$  will satisfy the boundary conditions (17) and (18). Because equation (48) is homogeneous, no compatibility problem exists. The second term  $\eta$  will satisfy the homogeneous boundary conditions (17a) and (18b). The conditions (17b) and (18a) will be replaced by

$$\left. \frac{d\eta}{d\phi} \right|_{\phi = \frac{\pi}{2}} = 0 \quad (50)$$

Furthermore,  $\eta$  must satisfy the compatibility conditions (19). As indicated by equation (49),  $\eta$  consists of a homogeneous solution at the dry wall and of an inhomogeneous one at the wetted wall. The fulfillment of the compatibility conditions joins these solutions.

According to equations (16d) and (48),  $\bar{\xi}$  is a solution of Legendre's equation having the degree one; hence, it is linear, composed of  $\cos \phi$  and

$$Q_1(\cos \phi) = -1 - \cos \phi \ln \tan \frac{\phi}{2} \quad (51)$$

(see Reference 7). However, the coefficient of  $Q_1$  equals zero, as follows from the boundary conditions (17a) and (18b) and the fact that  $dQ_1/d\phi$  is singular at  $\phi = 0$  and  $\phi = \pi$ . Thus, regarding the boundary conditions (17b) and (18a), one obtains

$$\bar{\xi} = \frac{E\delta}{(1+\nu)a^2} v_i \cos \phi ; \quad i = 1, 2 \quad . \quad (52)$$

Thereby,  $i$  equals one or two accordingly as hemisphere I or hemisphere II is concerned.

From equations (5) and (24) through (26c), it follows that

$$v_i = a\lambda \sum_1^n a_k v_{ik} ; \quad i = 1, 2 \quad , \quad (53)$$

$$v_{1k} = -\alpha A_k(\phi_s, \pi) - \frac{\varrho}{a} A_k(\phi_s, \frac{\pi}{2}) + \nu B_k(h_1, h_2) \quad \text{in case A} \quad , \quad (54a)$$

$$v_{1k} = -\alpha A_k(\frac{\pi}{2}, \pi) + \nu B_k(0, h_2) \quad \text{in case B} \quad , \quad (54b)$$

$$v_{1k} = -\alpha A_k(\phi_s, \pi) \quad \text{in case C} \quad , \quad (54c)$$

and

$$v_{2k} = -\alpha A_k(\phi_s, \pi) \quad (\phi_s = \frac{\pi}{2} \quad \text{in case B}) \quad , \quad (55)$$

where the notations

$$\int_{\phi_1}^{\phi_2} \Phi_k(a, \phi) \cos \phi \sin \phi \, d\phi = A_k(\phi_1, \phi_2) ; \quad \phi_1 \leq \phi_2 \quad (56)$$

and

$$\int_{z_1}^{z_2} \Phi_j(a, z) \frac{dz}{a} = B_j(z_1, z_2) ; \quad z_1 \leq z_2 \quad (57)$$

have been used.

Combining equations (45), (52), and (53) results in

$$\bar{\xi}_k = \frac{v_{ik}}{1+\nu} \cos \phi ; \quad i = 1, 2 \quad (58)$$

Substituting equation (58) into equation (47) yields

$$\xi_k = \eta_k + \frac{v_{ik}}{1+\nu} \cos \phi ; \quad i = 1, 2 \quad (59)$$

where  $v_{ik}$  is given by equations (54) and (55). Finally, from equations (43) and (59) it follows that

$$w_k = \Phi_k(a, \phi) - (1+\nu)\eta_k - v_{ik} \cos \phi ; \quad i = 1, 2 \quad (60)$$

for the wetted wall.

Now  $\eta_k$  as defined by equations (5), (46), and (49) and the boundary conditions (17a), (18b), and (50) and the compatibility conditions (19) must be determined. From equations (5), (37), (46), and (49) one concludes that if

$$(\nabla^2 + 2) f_j = \begin{cases} 0 & \text{at the dry wall} \\ P_j & \text{at the wetted wall} \end{cases} \quad (61)$$

$j = 0, 1, 2, \dots$



and  $f_j$  satisfies the above-mentioned boundary and compatibility conditions, then

$$\eta_k = -\left(f - \frac{h_i}{a}\right)^{2k-1} ; \quad i = 1, 2 \quad (62a)$$

where (symbolic operations)

$$f^j = f_j ; \quad j = 0, 1, 2, \dots \quad (62b)$$

The subscript  $i$  equals one or two accordingly as hemisphere I or hemisphere II is considered.

The homogeneous solution  $f_j^{(hom)}$  is composed of  $\cos \phi$  and  $Q_1(\cos \phi)$  as given by equation (51). At the wetted wall a particular solution  $f_{pj}$  must be added. Hence,

$$f_j^{(hom)} = c_{1j} \cos \phi - c_{2j} Q_1(\cos \phi)$$

and

$$f_j^{(inh)} = f_{pj}(\phi) + c_{1j} \cos \phi + c_{2j} Q_1(\cos \phi)$$

Table 2 is a compilation of  $\eta_k$  as given by equation (62a) and  $f_j$ ;  $j = 0, 1, 2, \dots$ . According to the boundary conditions (17a), (18b), and (50) and to the behavior of  $dQ_1/d\phi$ , which equals zero at  $\phi = \pi/2$  and is singular at  $\phi = 0$  and  $\phi = \pi$ ,  $f_j^{(hom)}$  and  $f_j^{(inh)}$  follow as presented in Table 2. In cases A and B,

$$c_{1j} = c_j = f_{pj} \bigg|_{\phi = \frac{\pi}{2}} \quad (63)$$

TABLE 2.  $\eta_k$  AND  $f_j$  FOR CASES A, B, AND C (SEE FIG. 2)

	$\eta_k$	Case A	Case B	Case C
		$f_j$	$f_j$	$f_j$
Hemisphere I $0 \leq \phi \leq \frac{\pi}{2}$	$-\left(f - \frac{h_1}{a}\right)^{2k-1}$	Dry: $0 \leq \phi \leq \phi_s$  $c_{1j}^* \cos \phi$		
		Wet: $\phi_s \leq \phi \leq \frac{\pi}{2}$  $f_{pj} + c_j \cos \phi + c_{2j} Q_1$		
Hemisphere II $\frac{\pi}{2} \leq \phi \leq \pi$	$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	Wet: $\frac{\pi}{2} \leq \phi \leq \pi$  $f_{pj} + c_j \cos \phi$	Wet: $\frac{\pi}{2} \leq \phi \leq \pi$  $f_{pj} + c_j \cos \phi$	Dry: $\frac{\pi}{2} \leq \phi \leq \phi_s$  $-c_{2j}^* Q_1$
				Wet: $\phi_s \leq \phi \leq \pi$  $f_{pj} + c_{1j} \cos \phi$

The compatibility conditions serve to determine the remaining constants. These conditions, following Table 2, take the form ( $\phi = \phi_s$ ):

$$-(c_j - c_{1j}^*) \cos \phi - c_{2j} Q_1 = f_{pj}$$

and

$$(c_j - c_{1j}^*) \sin \phi - c_{2j} \frac{dQ_1}{d\phi} = f_{pj}'$$

for hemisphere I (case A) and

$$-c_{1j} \cos \phi - c_{2j}^* Q_1 = f_{pj}$$

and

$$c_{1j} \sin \phi - c_{2j}^* \frac{dQ_1}{d\phi} = f_{pj}'$$

for hemisphere II (case C).

Because

$$Q_1 \sin \phi + \frac{dQ_1}{d\phi} \cos \phi = -\frac{1}{\sin \phi},$$

one obtains from the second pair of equations,

$$c_{2j}^* = (\sin^2 \phi f_{pj} + \sin \phi \cos \phi f_{pj}') \Big|_{\phi = \phi_s} \quad (64)$$

and

$$c_{1j} = - \left( \frac{1 + Q_1 \sin^2 \phi}{\cos \phi} f_{pj} + Q_1 \sin \phi f_{pj}' \right) \Big|_{\phi = \phi_s} \quad (65)$$

Comparing the structure of both systems of equations, one may write the solution of the first pair as

$$c_{2j} = c_{2j}^* \quad (66)$$

and

$$c_{1j}^* = c_j - c_{1j} \quad (67)$$

However, it must be realized that  $\phi_s$  of equations (64) and (65) lies in the second quadrant whereas  $\phi_s$  of equations (66) and (67) lies in the first quadrant.

The particular solutions of the differential equation (61) are

$$f_{p1} = -\frac{1}{3} \left[ 1 + \cos \phi \ln(1 - \cos \phi) \right] \quad (68a)$$

and

$$f_{pj} = -\frac{P_j}{(j-1)(j+2)} ; \quad j = 0, 2, 3, \dots \quad (68b)$$

This can be proved by substitution into equation (61). The derivatives with respect to  $\phi$  are

$$f_{p1}' = -\frac{\sin \phi}{3} \left[ \frac{\cos \phi}{1 - \cos \phi} - \ln(1 - \cos \phi) \right] \quad (69a)$$

and

$$f_{pj}' = -\frac{j}{(j-1)(j+2)} \cdot \frac{\cos \phi P_j - P_{j-1}}{\sin \phi} ; \quad j = 0, 2, 3, \dots \quad (69b)$$

Substituting equations (68) and (69) into equations (64) through (66) results in

$$c_{11} = \frac{1}{3} (\cos^2 \phi Q_1 + \ln \sin \phi) \Big|_{\phi = \phi_s}, \quad (70a)$$

$$c_{1j} = \frac{1}{(j-1)(j+2)} \left[ \frac{(\sin^2 \phi + j \cos^2 \phi) Q_1 + 1}{\cos \phi} P_j - j Q_1 P_{j-1} \right] \Big|_{\phi = \phi_s}, \quad (70b)$$

$j = 0, 2, 3, \dots$

$$c_{21} = -\frac{1}{3} (1 + \cos^3 \phi_s), \quad (71a)$$

and

$$c_{2j} = -\frac{1}{(j-1)(j+2)} \left[ (\sin^2 \phi + j \cos^2 \phi) P_j - j \cos \phi P_{j-1} \right] \Big|_{\phi = \phi_s}, \quad (71b)$$

$j = 0, 2, 3, \dots$

Regarding the cylinder,  $w_k$  is given by equations (5), (6), and (20). It follows that

$$w_k = \Phi_k(a, z) - \nu \int_{\phi_s}^{\pi} \Phi_k(a, z) \cos \phi \sin \phi d\phi,$$

or under consideration of equation (56)

$$w_k = \Phi_k(a, z) - \nu A_k(\phi_s, \frac{\pi}{2}), \quad (72a)$$

where

$$A_k(\phi_s, \frac{\pi}{2}) = 0 \text{ if } \phi_s \geq \frac{\pi}{2} \quad (72b)$$

(cases B and C) as can be concluded from equation (56) and the remark following equation (21).

Substituting equations (60) and (72a) into equation (12) under consideration of equations (54) through (57) results in

(case A:  $\phi_s < \pi/2$  ;  $h_1 > 0$  ;  $h_2 \geq h_1$  )

$$\begin{aligned}
 \mu_{jk} = & \int_{\phi_s}^{\pi} \Phi_j(a, \phi) \Phi_k(a, \phi) \sin \phi \, d\phi - (1 + \nu) \int_{\phi_s}^{\pi} \Phi_j(a, \phi) \eta_k \sin \phi \, d\phi \\
 & + \int_{h_1}^{h_2} \Phi_j(a, z) \Phi_k(a, z) \frac{dz}{a} + \alpha A_j(\phi_s, \pi) A_k(\phi_s, \pi) \\
 & + \frac{\ell}{a} A_j(\phi_s, \frac{\pi}{2}) A_k(\phi_s, \frac{\pi}{2}) - \nu \left[ A_j(\phi_s, \frac{\pi}{2}) B_k(h_1, h_2) \right. \\
 & \left. + B_j(h_1, h_2) A_k(\phi_s, \frac{\pi}{2}) \right] \quad (73a)
 \end{aligned}$$

Equation (73a) also comprises cases B and C. Under consideration of equation (72b) one obtains in case B ( $\phi_s = \pi/2$  ;  $h_1 = 0$  ;  $h_2 > 0$ )

$$\begin{aligned}
 \mu_{jk} = & \int_{\frac{\pi}{2}}^{\pi} \Phi_j(a, \phi) \Phi_k(a, \phi) \sin \phi \, d\phi - (1 + \nu) \int_{\frac{\pi}{2}}^{\pi} \Phi_j(a, \phi) \eta_k \sin \phi \, d\phi \\
 & + \int_0^{h_2} \Phi_j(a, z) \Phi_k(a, z) \frac{dz}{a} + \alpha A_j(\frac{\pi}{2}, \pi) A_k(\frac{\pi}{2}, \pi) \quad (73b)
 \end{aligned}$$

and in case C ( $\phi_s \geq \pi/2$  ;  $h_1 = 0$  ;  $h_2 \leq 0$ ) ,

$$\begin{aligned} \mu_{jk} = & \int_{\phi_s}^{\pi} \Phi_j(a, \phi) \Phi_k(a, \phi) \sin \phi \, d\phi - (1 + \nu) \int_{\phi_s}^{\pi} \Phi_j(a, \phi) \eta_k \sin \phi \, d\phi \\ & + \alpha A_j(\phi_s, \pi) A_k(\phi_s, \pi) \quad . \end{aligned} \quad (73c)$$

From equation (56) it follows that for  $\phi_s < \pi/2$  ,

$$A_j(\phi_s, \pi) = A_j(\phi_s, \frac{\pi}{2}) + A_j(\frac{\pi}{2}, \pi) \quad . \quad (74)$$

Table 3 summarizes the functions involved in evaluating the integrals of equations (11) and (73). It is composed of Tables 1 and 2 under consideration of equations (66) and (67). In addition to the table, equations (29) and (35) and equations (51), (63), and (68) through (71), respectively, must be considered to determine  $\Phi_j$  and  $\eta_k$ .

Now the remaining quantities of equations (73) must be determined. The constant  $\alpha$  is given by equation (25). The integrals (56) and (57) must be evaluated to obtain the quantities  $A_k$  ,  $B_j$  . If equation (37) ( $\rho = a$ ) is substituted into equation (56), one may write (symbolic operations)

$$A_k(\phi_1, \phi_2) = - \int_{\phi_1}^{\phi_2} \left( P - \frac{h_1}{a} \right)^{2k-1} \cos \phi \sin \phi \, d\phi \quad ; \quad i = 1, 2 \quad (75a)$$

where

$$\int P_j \cos \phi \sin \phi \, d\phi = \sin^2 \phi \, f_{jp} + \sin \phi \cos \phi \, \frac{df_{jp}}{d\phi} + c \quad (75b)$$

and  $f_{jp}$  is given by equations (68). Equation (75b) can be proved by differentiation. In doing so, equations (16d) and (61) may be used.

TABLE 3.  $\Phi_j$ ,  $a \frac{\partial \Phi_j}{\partial n}$ , AND  $\eta_k$  (WETTED WALL) FOR CASES A, B, AND C

	$\Phi_j \Big _{\rho=r=a}$	$a \frac{\partial \Phi_j}{\partial n} \Big _{\rho=r=a}$	$\eta_k$	$f_i; i=0, 1, 2, \dots$
Case A				
Hemisphere I $\phi_s \leq \phi \leq \frac{\pi}{2}$	$-\left(P - \frac{h_1}{a}\right)^{2j-1}$	$-(2j-1)P\left(P - \frac{h_1}{a}\right)^{2j-2}$	$-\left(f - \frac{h_1}{a}\right)^{2k-1}$	$f_{pi} + c_i \cos \phi + c_{2i} Q_1$
Cylinder $h_1 \leq z \leq h_2$	$\Psi_{2j-1}\left(1, \frac{z}{a}\right)$	$\frac{\partial \Psi_{2j-1}}{\partial \left(\frac{r}{a}\right)} \Big _{r=a}$		
Hemisphere II $\frac{\pi}{2} \leq \phi \leq \pi$	$-\left(P - \frac{h_2}{a}\right)^{2j-1}$	$-(2j-1)P\left(P - \frac{h_2}{a}\right)^{2j-2}$	$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	$f_{pi} + c_i \cos \phi$
Case B				
Cylinder $0 \leq z \leq h_2$	$\Psi_{2j-1}\left(1, \frac{z}{a}\right)$	$\frac{\partial \Psi_{2j-1}}{\partial \left(\frac{r}{a}\right)} \Big _{r=a}$		
Hemisphere II $\frac{\pi}{2} \leq \phi \leq \pi$	$-\left(P - \frac{h_2}{a}\right)^{2j-1}$	$-(2j-1)P\left(P - \frac{h_2}{a}\right)^{2j-2}$	$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	$f_{pi} + c_i \cos \phi$
Case C				
Hemisphere II $\phi_s \leq \phi \leq \pi$	$-\left(P - \frac{h_2}{a}\right)^{2j-1}$	$-(2j-1)P\left(P - \frac{h_2}{a}\right)^{2j-2}$	$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	$f_{pi} + c_{1i} \cos \phi$



Considering equation (68), one obtains for the cases at hand,  $\phi_s < \pi/2$ ,  $\phi_s = \pi/2$ , and  $\phi_s > \pi/2$ , respectively,

$$A_k(\phi_s, \frac{\pi}{2}) = - \int_{\phi_s}^{\frac{\pi}{2}} \left( P - \frac{h_1}{a} \right)^{2k-1} \cos \phi \sin \phi d\phi \quad (76a)$$

and

$$\int_{\phi_s}^{\frac{\pi}{2}} P_j \cos \phi \sin \phi d\phi = f_{pj} \Big|_{\phi=\frac{\pi}{2}} - c_{2j} \quad ; \quad j=0, 1, 2, \dots \quad (76b)$$

$$A_k(\frac{\pi}{2}, \pi) = - \int_{\frac{\pi}{2}}^{\pi} \left( P - \frac{h_2}{a} \right)^{2k-1} \cos \phi \sin \phi d\phi \quad (77a)$$

and

$$\int_{\frac{\pi}{2}}^{\pi} P_j \cos \phi \sin \phi d\phi = -f_{pj} \Big|_{\phi=\frac{\pi}{2}} \quad ; \quad (77b)$$

and

$$A_k(\phi_s, \pi) = - \int_{\phi_s}^{\pi} \left( P - \frac{h_2}{a} \right)^{2k-1} \cos \phi \sin \phi d\phi \quad (78a)$$

and

$$\int_{\phi_s}^{\pi} P_j \cos \phi \sin \phi d\phi = -c_{2j} \quad , \quad (78b)$$

where  $c_{2j}$  is given by equations (71).

The integral (57) can be evaluated with the aid of equations (32) and (35). It follows ( $z_2 \geq z_1$ ) that

$$B_j(z_1, z_2) = \frac{1}{2j} \left[ \Psi_{2j}(1, \frac{z_2}{a}) - \Psi_{2j}(1, \frac{z_1}{a}) \right] \quad . \quad (79)$$

### MODES OF VIBRATION

The characteristic patterns assumed by the free vibrating system according to the eigenvalues and eigenvectors (10b) are the so-called modes of vibration. First, the liquid pressure and the displacements may be considered.

Without the time factor, the pressure is

$$p^{(j)} = \frac{E\delta}{a} \lambda_j \sum_1^n a_{jk} \Phi_k$$

as it follows from equations (5) and (10b).

Equations (6) and (10b) result in

$$w^{(j)} = a \lambda_j \sum_1^n a_{jk} w_k \quad ,$$

where  $w_k$  is given by equations (60) and (72) for the hemispheres and for the cylinder, respectively. Table 4 may serve to ease the computation of  $w_k$ . This table contains the equations (54), (55), (60), and (72) and the functions  $\Phi_k$  and  $\eta_k$  as given in Tables 1

TABLE 4.  $w_k$  FOR CASES A, B, AND C (SEE FIG. 2)

	$w_k$	$\Phi_k$	$\eta_k$	$f_j, j=0, 1, 2, \dots$	$v_{ik}, i=1, 2$
Case A					
Hemisphere I $0 < \phi < \phi_s$ Dry $\phi_s < \phi < \frac{\pi}{2}$ Wet	$-(1+\nu)\eta_k - v_{1k} \cos \phi$		$-\left(f - \frac{h_1}{a}\right)^{2j-1}$	$(c_j - c_{1j}) \cos \phi$	$-aA_k(\phi_s, \pi) - \frac{a}{2}A_k(\phi_s, \frac{\pi}{2})$ $+ \nu B_k(h_1, h_2)$
	$\Phi_k - (1+\nu)\eta_k - v_{1k} \cos \phi$	$-\left(p - \frac{h_1}{a}\right)^{2k-1}$		$f_{pj} + c_j \cos \phi + c_{2j} Q_1$	
Cylinder $h_1 < z < h_2$ Wet	$\Phi_k - \nu A_k(\phi_s, \frac{\pi}{2})$	$\Psi_{2k-1}\left(1, \frac{z}{a}\right)$			
Hemisphere II $\frac{\pi}{2} < \phi < \pi$ Wet	$\Phi_k - (1+\nu)\eta_k - v_{2k} \cos \phi$	$-\left(p - \frac{h_2}{a}\right)^{2k-1}$	$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	$f_{pj} + c_j \cos \phi$	$-aA_k\left(\frac{\pi}{2}, \pi\right)$
Case B					
Hemisphere I $0 < \phi < \frac{\pi}{2}$ Dry	$-v_{k1} \cos \phi$				$-aA_k\left(\frac{\pi}{2}, \pi\right)$ $+ \nu B_k(0, h_2)$
Cylinder $0 < z < h_2$ Wet	$\Phi_k$	$\Psi_{2k-1}\left(1, \frac{z}{a}\right)$			
Hemisphere II $\frac{\pi}{2} < \phi < \pi$ Wet	$\Phi_k - (1+\nu)\eta_k - v_{2k} \cos \phi$	$-\left(p - \frac{h_2}{a}\right)^{2k-1}$	$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	$f_{pj} + c_j \cos \phi$	$-aA_k\left(\frac{\pi}{2}, \pi\right)$
Case C					
Hemisphere I $0 < \phi < \frac{\pi}{2}$ Dry	$-v_{1k} \cos \phi$				$-aA_k(\phi_s, \pi)$
Cylinder					
Hemisphere II $\frac{\pi}{2} < \phi < \phi$ Dry $\phi_s < \phi < \pi$ Wet	$-(1+\nu)\eta_k - v_{2k} \cos \phi$		$-\left(f - \frac{h_2}{a}\right)^{2k-1}$	$-c_{2j} Q_1$	
	$\Phi_k - (1+\nu)\eta_k - v_{2k} \cos \phi$	$-\left(p - \frac{h_2}{a}\right)^{2k-1}$		$f_{pj} + c_{1j} \cos \phi$	$-aA_k(\phi_s, \pi)$

and 2 under consideration of equations (66) and (67). To supplement Table 4, the defining equations (25), (29), (35), (51), (63), (68) through (71), and (74) through (79) must be considered.

For the mode  $v^{(j)}$  an expansion similar to that of  $w^{(j)}$  can be concluded, namely,

$$v^{(j)} = a \lambda_j \sum_{k=1}^n a_{jk} v_k$$

As far as the hemispheres are considered,  $v_k$  follows from equations (16c), (41), and (59) and  $v_k$  of the cylindrical part is covered by equations (5), (22), and (53) through (57). The most important equations and formula are compiled in Table 5 for cases A, B, and C to outline the computational procedure. In addition to the expressions  $v_k$  of the hemispheres and the cylinder and equations (54) and (55), the tables contain the derivatives of the functions of Table 2. In addition to Table 5, the equations (25), (29), (51), (63), (69) through (71), and (74) through (79) must be considered.

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National Aeronautics and Space Administration

Marshall Space Flight Center, Alabama 35812, May 1, 1972

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TABLE 5.  $v_k$  FOR CASES A, B, and C (SEE FIG. 2)

	$v_k$	$\eta_k'$	$f_j'; j = 0, 1, 2, \dots$	$v_{ik}; i = 1, 2$
Case A				
Hemisphere I $0 < \phi < \phi_s$ Dry $\phi_s < \phi < \frac{\pi}{2}$ Wet	$-(1 + \nu) \eta_k' + v_{1k} \sin \phi$	$-\left(f' - \frac{h_1}{a}\right)^{2k-1}$	$-(c_j - c_{1j}) \sin \phi$ $f_{pj}' - c_j \sin \phi + c_{2j} \frac{dQ_1}{d\phi}$	$-\alpha A_k(\phi_s, \pi)$ $-\frac{q}{a} A_k(\phi_s, \frac{\pi}{2})$ $+ \nu B_k(h_1, h_2)$
Cylinder $h_1 < z < h_2$ Wet	$v_{1k} + \frac{z - h_1}{a} A_k(\phi_s, \frac{\pi}{2}) - \nu B_k(h_1, z)$			
Hemisphere II $\frac{\pi}{2} < \phi < \pi$ Wet	$-(1 + \nu) \eta_k' + v_{2k} \sin \phi$	$-\left(f' - \frac{h_2}{a}\right)^{2k-1}$	$f_{pj}' - c_j \sin \phi$	$-\alpha A_k(\phi_s, \pi)$
Case B				
Hemisphere I $0 < \phi < \frac{\pi}{2}$ Dry	$v_{1k} \sin \phi$			$-\alpha A_k(\frac{\pi}{2}, \pi)$ $+ \nu B_k(0, h_2)$
Cylinder $0 < z < h_2$ Dry Wet	$v_{1k}$ $v_{1k} - \nu B_k(0, z)$			
Hemisphere II $\frac{\pi}{2} < \phi < \pi$ Wet	$-(1 + \nu) \eta_k' + v_{2k} \sin \phi$	$-\left(f' - \frac{h_2}{a}\right)^{2k-1}$	$f_{pj}' - c_j \sin \phi$	$-\alpha A_k(\frac{\pi}{2}, \pi)$
Case C				
Hemisphere I $0 < \phi < \frac{\pi}{2}$ Dry	$v_{1k} \sin \phi$			$-\alpha A_k(\phi_s, \pi)$
Cylinder Dry	$v_{1k}$			
Hemisphere II $\frac{\pi}{2} < \phi < \phi_s$ Dry $\phi_s < \phi < \pi$ Wet	$-(1 + \nu) \eta_k' + v_{2k} \sin \phi$	$-\left(f' - \frac{h_2}{a}\right)^{2k-1}$	$-c_{2j} \frac{dQ_1}{d\phi}$ $f_{pj}' - c_{1j} \sin \phi$	$-\alpha A_k(\phi_s, \pi)$

## APPENDIX

### SUBSTITUTION OF THE DISPLACEMENTS INTO THE DYNAMIC EQUILIBRIUM CONDITIONS OF THE SPHERICAL SHELL

Substituting equations (15) into equations (14) yields

$$N_{\theta} = \frac{E\delta}{a(1-\nu^2)} \left[ \nu \left( \frac{dv}{d\phi} + w \right) + v \cotan \phi + w \right] \quad (A-1a)$$

and

$$N_{\phi} = \frac{E\delta}{a(1-\nu^2)} \left[ \frac{dv}{d\phi} + w + \nu(v \cotan \phi + w) \right] \quad (A-1b)$$

By substituting equations (A-1) into equation (13a), one obtains

$$\frac{dv}{d\phi} + v \cotan \phi + 2w = (1-\nu) \frac{a^2}{E\delta} p \quad (A-2)$$

Using the notation

$$v = -(1+\nu) \frac{a^2}{E\delta} \frac{d\xi}{d\phi}, \quad (A-3)$$

it is realized that

$$\frac{dv}{d\phi} + v \cotan \phi = -(1+\nu) \frac{a^2}{E\delta} \frac{1}{\sin \phi} \frac{d}{d\phi} \left( \frac{d\xi}{d\phi} \sin \phi \right)$$

It is natural to denote [1, 2] that

$$\frac{1}{\sin \phi} \frac{d}{d\phi} \left( \sin \phi \frac{d}{d\phi} \right) = \nabla^2 \quad (\text{A-4})$$

because the left side is Laplace's operator of functions independent of  $\rho$  and  $\theta$ . Using this notation,

$$\frac{dv}{d\phi} + v \cotan \phi = -(1 + \nu) \frac{a^2}{E\delta} \nabla^2 \xi \quad (\text{A-5})$$

is obtained. Thus, from equation (A-2) it follows that

$$-(1 + \nu) \frac{a^2}{E\delta} \nabla^2 \xi + 2w = (1 - \nu) \frac{a^2}{E\delta} p \quad (\text{A-6})$$

Substituting equations (A-1) into equation (13b), one obtains (after a little algebra)

$$\begin{aligned} (1 - \nu) \left( \frac{dv}{d\phi} - v \cotan \phi \right) \cos \phi \\ + \sin \phi \frac{d}{d\phi} \left[ \frac{dv}{d\phi} + w + \nu(v \cotan \phi + w) \right] = 0 \end{aligned} \quad (\text{A-7})$$

Because

$$\left( \frac{dv}{d\phi} - v \cotan \phi \right) \cos \phi = \sin \phi \left[ v + \frac{d}{d\phi} (v \cotan \phi) \right],$$

as is easy to prove by differentiation, equation (A-7) can be written as

$$(1 - \nu)v + \frac{d}{d\phi} \left[ (1 - \nu)v \cotan \phi + \frac{dv}{d\phi} + w + \nu(v \cotan \phi + w) \right] = 0,$$

or under consideration of equations (A-3) and (A-5),

$$\frac{d}{d\phi} \left[ -\frac{a^2}{E\delta} \nabla^2 \xi - \frac{a^2}{E\delta} (1 - \nu)\xi + w \right] = 0$$

Following equations (A-3),  $\xi$  is determined only up to an arbitrary constant. Thus, it is concluded that

$$\frac{a^2}{E\delta} (\nabla^2 + 1 - \nu)\xi - w = 0 \quad (A-8)$$

(According to Reference 2, equations (A-6) and (A-8) are given by V.Z. Vlassow, Selected Works, Volume 1, Moscow, 1926.)

Now the elimination of  $w$  and  $\nabla^2 \xi$  from equations (A-6) and (A-8), respectively, results in equations (16a) and (16b).



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