X-592-73-164

PREPRINT

NASA TM X-66273

POSITION FROM GRAVITY

R. S. MATHER

JUNE 1973

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

62 p

X-592-73-164

POSITION FROM GRAVITY**

R. S. Mather*
Geodynamics Branch
Geodynamics Program Division

*On leave of absence from the University of New South Wales, Sydney, Australia.

**Invited presentation to the Australian Academy of Science/International Association of Geodesy Symposium on the Earth's Gravitational Field and Secular Variations in Position, 26-30 Nov. 1973, Sydney, Australia.

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

POSITION FROM GRAVITY

R. S. Mather
Geodynamics Branch
Geodynamics Program Division

ABSTRACT

Procedures for obtaining position from surface gravity observations are reviewed and their relevance assessed in the context of the application of modern geodetic techniques to programs of Earth and ocean physics. Solutions based on the use of surface layer techniques, the discrete value approach, and the development from Green's theorem are stated in summary, the latter being extended to order e^3 in the height anomaly.

The representation of the surface gravity field which is required in order that this accuracy may be achieved is discussed. Interim techniques which could be used in the absence of such a representation are also outlined.

The role which can be played by the determination of changes in observed gravity to a few microgal, in the definition of geodetic reference systems for long period studies in Earth physics, is discussed and the consequences of changes of zero degree summarized. The possible use of these techniques in future geodetic practice is also assessed.

CONTENTS

CONTENTS (Continued)

ILLUSTRATIONS

POSITION FROM GRAVITY

1. INTRODUCTION

1.1 PREAMBLE

At first glance, it would appear that geodesists of today should be grateful for the activities of exploration geophysicists which have made significant contribution to all the well known gravity data banks available at present. After all, it is only in the past two decades that the course of events has looked favorably on the collection of gravity information with solely geodetic objectives in view. All the available surface gravity information has still not been able to provide meaningful definitions of position on its own at the time of writing. The techniques of satellite geodesy revolutionized practical determinations related to position from a consideration of the Earth's gravity field and there is little argument that these methods give relevant position-related parameters with a precision of about 2-4%, the higher precision estimate being obtained on the use of combination techniques incorporating surface gravity information with satellite data.

Many factors have changed since the advent of the satellite era less than two decades ago. A technical advance of importance is the development of the surface ship gravimeter which provides a means of defining the surface gravity field in ocean areas with a resolution which if used advantageously, can be shown to be adequate for all present day requirements in Earth and ocean physics. A second development of great significance is the sensational improvement achieved in the precision with which absolute determinations of gravity based on interferometric techniques, can be made (e.g., Cook 1965; Faller 1965). The permanent installation maintained by Bureau International des Poids et Mesures (BIPM) at Sevres, France, has been achieving a measuring precision of ±3 μgal as a matter of routine for some years now (Sakuma 1971), improving the resolution of g from 2 parts in 10^6 to about 3 parts in 10^9 in less than 10 years. To this should be added the capability which has been available for some time, and enables the measurement of differences in gravity with an accuracy of at least 1 part in 10^5 on land without any appreciable measurement time.

These significant improvements in metrology pose a series of interesting problems which must be dealt with before the maximum geodetic information can be obtained from the use of surface gravity measurements. In the first instance, it becomes necessary to review the implications of adopting a rigid body model for the Earth as the basis for computations of position from surface gravity data. A further effect to be considered are changes in the Earth space location of the rotation vector and their influence on the determination of position from gravity. Short period mass changes smaller in magnitude than Earth tide effects, and possibly more difficult to model, may also have to be considered. Into this category fall changes in atmospheric circulation patterns from some model, variations in the water table and similar phenomena. Over a longer time scale, it is necessary to consider the implications of a possible secular change in the gravitational constant G.

As most of these effects are 7-8 orders of magnitude smaller than that of g, it has been accepted practice to convert observed gravity g to the gravity anomaly Δg by differencing g from the value of normal gravity γ for a model of the Earth, afforded by the value of GM, the rate of rotation ω of the Earth, assumed to be constant, together with the equatorial radius a and flattening f of an ellipsoid of revolution which "best fits" the geoid. No allowance is made for the possibility of variations with time, in any of the parameters defining the system of reference. This is not inconsistent with the concept of determinations relevant to a certain epoch, provided

(i) the observations used are all made during the epoch considered; and

(ii) the accuracy sought is less than 1 part in 10^7 in g.

The order of magnitude of gravitational deviations from a solid Earth model are smaller than $o\{10^{-6}\, g\}$. The largest effect is the diurnal Earth tide variation with magnitude $o\{10^{-7}\, g\}$. It has not been considered necessary at the present time, to recommend the adoption of a systematic procedure for modeling and removing the effect of Earth tides from observed gravity except when establishing gravity standardization networks, in view of the limited accuracy of elevation data used in computing the gravity anomaly. This would call for the acceptance of a universally acceptable model for Earth tides, which would be used as a matter of routine to correct observed gravity prior to use in geodetic computations. Such corrections are only necessary at fundamental gravity stations at which determinations are made with the highest possible accuracy for either the definition of the global gravity standardization network (Mather 1973, p. 68) or when attempting to locate changes in the position of the Earth's center of mass (Geocenter) with time (Mather 1972, p. 13). The need for applying such corrections at other stations will depend on the extent of gravity coverage available globally and whether the elevation datums have been unified at the 50 cm level.

Current practice accepts the validity of each individual nation's elevation datum as well as its gravity datum. The continuance of such a practice is unwise if systematic errors at the 50 cm level are not to occur in the final results. The most taxing goal in the definition of position from surface gravity, is the determination of the geoid in ocean areas to the highest possible accuracy, in order that such results could be used with satellite altimeter data to recover the sea surface topography. On present trends, it would appear that 5-10 cm accuracy is desirable in the geoid determinations for a meaningful evaluation of the sea surface topography. The determination of the latter is vital for the study of tides in open oceans, ocean circulation and stationary characteristics of sea surface topography.

It is in this context that the use of gravimetric techniques in the determination of geodetic position should be reviewed. The present development covers

(a) the basic principles underlying the determination of position from gravity;

(b) a review of some of the methods suggested to the present time, for solving the boundary value problem in physical geodesy;

(c) techniques for the preparation of data sets for this task; and

(d) the geodetic interpretation of such solutions in Earth space.

In all sections, an assessment is made of the requirements which will have to be met in order that the independent evaluation of selected geodetic characteristics available from surface gravity determinations, can be used in the resolution of some possible ambiguities from other methods when applied to high precision studies in Earth and ocean physics.

1.2 A GUIDE TO NOTATION

1.2.1 Recurring Symbols

a = equatorial radius of the ellipsoid of reference

$\vec{d}$ = separation vector between equivalent points P on the Earth's surface and Q on the telluroid

dz = increment in orthometric elevation

$d\sigma$ = element of surface area on unit sphere

e = eccentricity of the meridian ellipse; $e^2 = 2f - f^2$

$F(\psi)$ = $f(\psi) \sin \psi$

f = flattening of meridian ellipse

$f(\psi)$ = Stokes function = $\operatorname{cosec} 1/2\psi + 1 - 5 \cos \psi - 6 \sin 1/2\psi - 3 \cos \psi \left[\log \left(\sin 1/2\psi \,(1 + \sin 1/2\psi)\right)\right]$

g = observed gravity at the surface of the Earth

h = ellipsoidal elevation

h_d = height anomaly

h_n = normal height, defined prior to Equation 6

M = mass of the Earth, including the atmosphere

$M\{X\}$ = global mean value of X

m = $a\omega^2/\gamma_e$

m' = $a^3\omega^2/GM = m + o\{f^2\}$

N = elevation of geoid above ellipsoid

$\vec{N}$ = unit vector normal to the surface of the Earth

N_f = Free Air Geoid; the Stokesian contribution to h_d

N_c = Indirect effect to free air geoid; non-Stokesian contribution to h_d

R = geocentric distance

$\bar{R}$ = radius of sphere containing all topography (Brillouin sphere)

R_b = radius of sphere which is internal to the Earth's surface (Bjerhammar sphere)

R_m = mean radius of the Earth

r = distance between the point of computation P and the element of surface area dS

$\bar{r}$ = $2\bar{R}\sin 1/2\,\psi$

r_σ = $2R_m \sin 1/2\,\psi$

S = surface of the Earth

U = spheropotential due to the system of reference

U_0 = spheropotential on the surface of the reference ellipsoid

V_d = disturbing potential

W = geopotential

W_0 = potential of the geoid

X_i = geocentric rectangular Cartesian co-ordinate system $X_1 X_2 X_3$

x_i = local rectangular Cartesian co-ordinate system $x_1 x_2 x_3$ with the x_3 axis along the local normal, the $x_1 x_2$ plane defining the local horizon, with axes oriented north, east respectively.

α = azimuth

β = ground slope; subscripts 1 and 2 refer to components north and east

γ = normal gravity due to the reference system; subscript 0 refer to values on the reference ellipsoid; subscript e refers to equatorial value.

Δg = gravity anomaly at the surface of the Earth, defined by Equation 10.

ΔW = geopotential difference with respect to the geoid

δg = gravity disturbance

λ = longitude, positive east

ξ = components of deflection of the vertical; subscripts 1 and 2 refer to those in north and east directions respectively

ζ = deflection of the vertical, positive if outward vertical lies north, east of normal.

Φ = density of surface layer, except in section 3.4.

ϕ = latitude, positive north; subscripts c, a and g refer to geocentric, astronomically determined and geodetic latitudes respectively

ψ = angular distance at geocenter between the point of computation P and element of surface area dS.

ω = angular velocity of rotation of the Earth

$$\vec{\nabla} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \vec{i}$$

1.2.2 Conventions

$a = b + o\{b^2\} \equiv$ terms whose order of magnitude is equal to or less than b^2 are neglected ($b < 1$)

$x_\alpha y_\alpha = x_1 y_1 + x_2 y_2$

$x_i = a_{ij} b_j \equiv x_i = a_{i1} b_1 + a_{i2} b_2 + \ldots\ldots$, there being as many equations as possible values of i

$a \doteqdot c \equiv$ a is approximately equal to c

2. BASIC PRINCIPLES

2.1 THE SYSTEM OF REFERENCE

The determination of Earth space position from gravity observations at the surface of the Earth, as pointed out in section 1, is implied from deviations of observed gravity from those at an "equivalent" point on some Earth model, whose parameters are completely defined. Current geodetic practice (IAG 1970, p. 12) specifies a rigid body model by the following parameters.

(a) The value μ (= G M) where G is the gravitational constant and M the mass of the Earth.

(b) The constant rate of rotation ω of the rigid Earth model.

(c) The equatorial radius a of an ellipsoid of revolution which presumably is one of best fit to the geoid.

(d) The dynamic form factor J_2 which is equivalent to a value of a flattening f for the reference ellipsoid.

It is conventional to choose a reference ellipsoid whose equatorial radius is such that it has the same volume as the geoid. This is not a necessary condition if zero degree effects are taken into account when formulating a solution for the boundary value problem. What is more important in solutions which aspire to accuracies greater than the order of the flattening (i.e., 30 cm in the height anomaly), is that the ellipsoid lies everywhere within the physical surface of the Earth. This enables the use of Laplace's equation in the representation of the appropriate disturbing potential without approximation.

The adoption of such a procedure without an equivalent adjustment in μ could cause larger values of the gravity anomaly which would in turn, call for greater caution in developing computer algorithms for numerical work.

It can be stated without being contentious that the value adopted for a has to be based on some determination of the scale of Earth space. This would be provided by either the measurement of long arcs at the surface of the Earth by classical techniques (e.g., the Pageos baselines) or else by laser ranges to either satellites or the moon. All determinations of scale are therefore based on the velocity of light.

The value of the flattening f of the reference ellipsoid is best deduced from the second degree zonal harmonic obtained from the secular variations in the right ascension Ω of the node and the argument ω of perigee of near Earth satellites. The precision claimed at the present time (e.g., Lerch et al 1972, p. 27) for this harmonic is 1 part in 10^5, the required relation being (e.g., Mather 1971a, p. 85)

$$C_{20} = \frac{1}{3} m' - \frac{2}{3} f - \frac{3}{7} m' f + \frac{1}{3} f^2 + o\{f^3\}, \tag{1}$$

where

$$m' = \frac{a^3 \omega^2}{GM},$$

ω being the angular velocity of rotation of the Earth.

The exact relation between the observed secular variations $\ddot{\Omega}_{20}$ and C_{20} is (e.g., ibid, p. 151)

$$\dot{\Omega}_{20} = \frac{3\,(GM)^{1/2}}{2\,(1 - e_s^2)^2} \, \frac{a^2}{a_s^{7/2}} \cos i \; C_{20}, \tag{2}$$

a_s being the equatorial radius of the satellite orbit, e_s its eccentricity and i its inclination.

The change df in f due to changes da, $d\omega$ and d(GM) in a, ω and GM are given by

$$df = \frac{3}{2} m' \left(3 \frac{da}{a} + 2 \frac{d\omega}{\omega}\right) - \frac{1}{2} \frac{d(GM)}{GM} (3m' + C_{20}) + o\{f^2 df\}. \tag{3}$$

The ratio df/f is therefore of the same order of magnitude as d(GM)/GM for a specified value of a as the ratio $d\omega/\omega$ is at least an order smaller if these ratios reflect the precision with which each of the quantities are determined.

It is all important that the rotational characteristics assigned to the reference model are exactly equivalent to those influencing gravity as measured at the Earth's surface. This is implicit in deriving Equation 60 from Equations 58 and 59. The rotation vector in Earth space is not fixed and hence deviations from the rigid body model adopted for any system of reference (e.g., IAG 1970, p. 24). The rate of rotation ω is subject to secular variations due to the effects of tidal friction, and certain short period effects which, in theory, have to be accounted for when reducing observed gravity to a rigid body equivalent of the Earth, are not of practical consequence as they have a magnitude of less than 1 part in 10^9 in g. A second factor is the change in position of the instantaneous axis of rotation with respect to the Earth's crust. The total contribution to observed gravity of the rotation is the appropriate resolute of

$$g_r = p\omega^2 \tag{4}$$

directed away from the axis of rotation and perpendicular to it. The changes in g_r due to changes dp in p, which is the distance of the point at which gravity is measured from the axis of rotation, and $d\omega$ in ω is given by

$$g_r = g_r\left(2\,\frac{d\omega}{\omega} + \frac{dp}{p} + o\{10^{-12}\}\right). \tag{5}$$

The effect of the ratio $d\omega/\omega$ will be less than 1 μgal on observed gravity for a 10 msec variation in the length of day and hence this term is not of significance in the reduction of observed gravity, if the latter were restricted to some epoch of observation. The effect of polar motion is o $\{1\ \mu\text{gal}\}$ (Bursa 1971).

More significant short period changes in observed gravity have been reported by Sakuma (1971) after modeling the effect of Earth tides. It should be noted that a 1% change in the local atmospheric density is of o $\{10\ \mu\text{gal}\}$ while quasi-stationary changes in the local geological formations, e.g., in the local water table, could cause gravitational effects of this same magnitude. It is therefore important that both the atmosphere and the local geology be modeled in the vicinity of those gravity stations at which g is to be re-measured at intervals of time with the highest possible precision.

The dominant gravitational variation with time is that due to Earth tides, with magnitude of o $\{10^2\ \mu\text{gal}\}$ and Earth models for this effect are well known in the literature (e.g., Melchoir 1966). It is important that an unambiguous Earth tide model at the 10 μ gal level be uniformly adopted when specifying gravity values at stations comprising the global gravity standardization network described in section 4.

The final parameter defining the reference system is μ(= GM). The commonly accepted values of GM are all based on the analysis of interplanetary space probes. The technique used can be briefly summarized as follows (Esposito 1972).

Doppler data from interplanetary spaceprobes is analyzed using numerical integration procedures for the determination of the motion of the probe with reference to a geocentric inertial co-ordinate system. Perturbations due to the Earth's departure from a sphere, solar radiation pressure, planetary and lunar gravitational effects and spacecraft attitude control forces are modeled when effecting this solution which also provides revised estimates of tracking station co-ordinates as a by product of the solution. The main conclusions of relevance to the present review are the following. Firstly, the value of GM is based on the velocity of light. Secondly, the potential U_0 on the surface of the reference ellipsoid, assumed to be an equipotential is related to the adopted values of GM, a, f and ω by the relation (e.g., Mather 1971a, p. 83)

$$U_0 = \frac{GM}{a} \frac{\sin^{-1} e}{e} + \frac{1}{3} a^2 \omega^2 ,$$

where

$$e^2 = 2f - f^2 .$$

As GM has an uncertainty of 1 part in 10^6 at the present time, it follows that U_0 will differ from the true potential of the geoid consistent with Newtonian gravitation, as scaled by the velocity of light by at least 1 part in 10^6. If U_0 were assumed to be equal to the potential W_0 of the geoid, it would be tantamount to imposing a second scale constraint when using gravitational techniques in geodesy. The only way out of this impasse is to attempt to estimate W_0 by using purely geometrical techniques and thereby make position determinations using gravity data consistent with the scale provided by the velocity of light. In the interim, it must be borne in mind that all position determinations based on gravity may have constant scale error of up to 1 p.p.m. on this account.

In summary, it may be stated that

(a) Only Earth tide effects need to be allowed for in all work except those determinations required for the monitoring of co-ordinate systems;

(b) Position determinations based on gravity are liable to have a constant scale error of up to 1 part in 10^6 due to the uncertainty in GM.

The following conclusions may therefore be drawn about the adoption of a rigid body model of the Earth as an intermediary in the definition of position from gravity, with the highest possible precision as the ultimate goal.

B

(a) The only departures from rigidity which need be considered for solutions of the boundary value problem are the effect of Earth tides on gravity observations constituting gravity standardization networks.

(b) An error in GM will give rise to ambiguities in scale with corresponding magnitude unless independent techniques are developed for using geometrical means for determining the potential W_0 of the geoid. For further discussion, see section 5.1.

2.2 DATA REQUIREMENTS

Observed gravity will be the results of two kinds of determinations. The first type will be absolute determinations with the highest precision possible while the second will be point values established by differential techniques based on the absolute determinations, with a precision which is at least one order of magnitude inferior. The practice adopted in gravimetric determinations is the use of gravity observations and a knowledge of the surface topography of the Earth to determine the separation vector $\vec{d}$ between "equivalent" points on the reference model, whose Earth space position is known, and the physical surface of the Earth, as illustrated in Figure 1. The separation vector can be completely defined by the height anomaly h_d and the angles ξ_α which are more completely defined in the next sub-section.

The most exacting requirements are called for in the definition of position from gravity when determining the geoid for ocean physics applications, where present estimates of requirements call for resolution at the ±10 cm level. The equivalent order of magnitude is e^3 (i.e., 5 parts in 10^4) which can be assessed as ±50 μgal in the gravity anomaly Δg. On the basis of the discussion in the previous sub-section, it would be adequate to maintain a rotating rigid body model as the system of reference and apply the appropriate reductions to observed gravity to make the measurements compatible with the model. The position so defined will be unaffected by short period time variations in the Earth's gravitational field.

The nature of the reductions necessary will depend on the purpose for which the gravity data is required, the accuracy with which it has been established and the nature of the elevation data available for its reduction. Corrections for Earth tides are necessary both for stations monitoring changes in the global reference system (Mather 1972, Sec. 3.3) and to an accuracy which is one order of magnitude less for stations in the global gravity standardization network. The model adopted for Earth tides should therefore be capable of resolution to 1 μgal. Some difficulty may be experienced in removing ocean loading effects in coastal areas which influence the tidal correction in the second significant figure (Hendershott 1972).

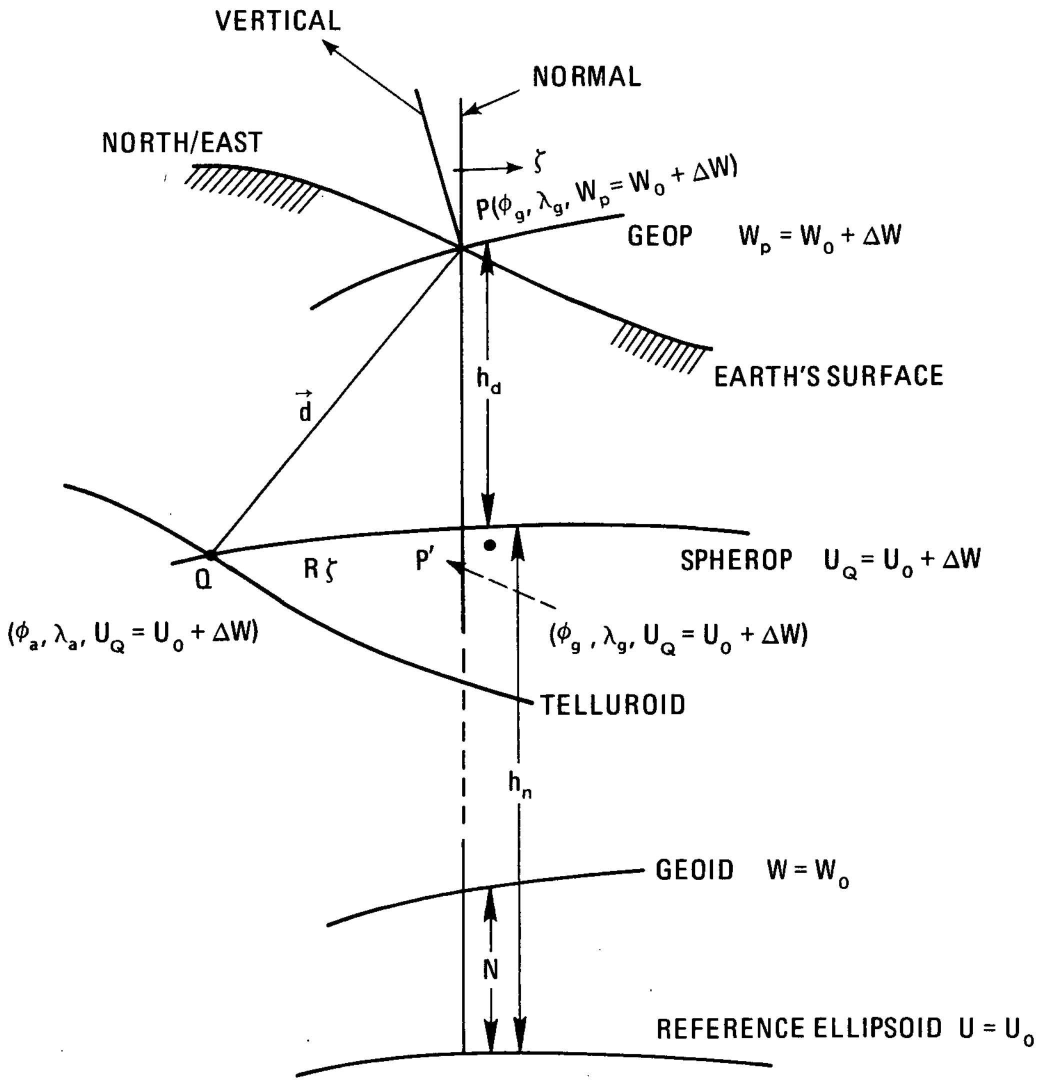

Figure 1. The Separation Vector $\vec{d}$ and the Reference System

Corrections for short period variations in the stratification of the atmosphere and the stratigraphy in the vicinity of gravity stations monitoring the reference system are also necessary. This presupposes the existence of "accepted" models for both the atmosphere and the local stratigraphy, and the effects meteorological changes may have on them. While Earth tide effects should be allowed for when making any gravity determination, the summary in section 4 indicates that the effect of omitting the correction on determinations of position from gravity is likely to be negligible as the tidal effect has the characteristics of a random measurement error. All further discussion will assume observed gravity g to be measured on a solid Earth, rotating with a uniform angular velocity, the gravitational effects of polar motion being allowed for when gravity data is used for the definition of geodetic reference systems.

The formulation of relations at the surface of the Earth is based on the following principles.

(a) An estimate is available of the geocentric co-ordinates of the point P at the surface of the Earth. In classical terms, these surface co-ordinates (ϕ_a, λ_a) are related to the vertical at P by astronomical determinations, and can be determined at best to a factor or two better than 1 part in 10^6 (i.e., ±6 m in position in each co-ordinate). This estimate differs from the true geocentric co-ordinate by amounts up to 10^{-4} radians, depending on the magnitude of the local deflection of the vertical.

(b) The displacement of P above the equivalent point P_0 on the ellipsoid is defined by the normal elevation h_n, which is related to the difference in geopotential ΔW between the equipotential datum for elevations (the geoid) and P as obtained from levelling by the relation

$$\Delta W = -\int_{\text{geoid}}^{P} g\, dz ,$$

g being observed gravity for the section of the line of levelling where the orthometric height difference is dz. The equation defining h_n in terms of ΔW is the relation (e.g., Mather 1971a, p. 100)

$$h_n = \frac{\Delta W}{\gamma_0}\left[1 + \frac{\Delta W}{a\gamma_0}\,(1 + m + f - 2f\,\sin^2\phi) + \left(\frac{\Delta W}{a\gamma_0}\right)^2 + o\{f^3\}\right],$$

where γ_0 is the value of normal gravity on the reference ellipsoid and

$$m = \frac{a\,\omega^2}{\gamma_e} \tag{6}$$

γ_e being the value of normal gravity at the equator, and ΔW is treated without regard to sign for points exterior to the geoid.

It has been shown (Mather 1973, sec. 4.3) that the data requirements for the determination of the height anomaly h_d with a precision equivalent to that possible in establishing h_n are well within the capabilities of measuring and data sampling techniques available at the present time.

Thus position determination from gravity in any absolute sense

(a) requires a knowledge of astronomical co-ordinates; and

(b) a global representation of the gravity anomaly field.

The resolution of the information from positional astronomy at the present time will have to improve by a factor of 50 before the horizontal determinations are of adequate accuracy for the complete determination of position by this method alone. Such a determination will also require the determination of deflections of the vertical ξ_α to o $\{10^{-4}\ \xi_\alpha\}$.

It can therefore be concluded that the determination of geocentric position from positional astronomy and surface gravity to accuracies much in excess of 1 part in 10^6 may not be a practical possibility in the foreseeable future. The determination of the height anomaly, on the other hand, will remain a problem of fundamental interest as it forms an integral part in the definition of sea surface topography from space. The ensuing development will continue to deal with the complete development necessary for the determination of position from gravity, but only in outline. More detailed review will be confined to the techniques for determining the height anomaly.

2.3 BASIC RELATIONS

The formulation of the Molodenskii problem (Heiskanen and Moritz 1967, p. 291) can be treated as one which seeks the determination of the separation vector $\vec{d}$ between "equivalent" points P $(\phi_g,\ \lambda_g,\ W_p\ =\ W_0\ +\ \Delta W)$ on the Earth's surface and Q $(\phi_a,\ \lambda_a,\ U_Q\ =\ U_0\ +\ \Delta W)$ on the associated spherop $U\ =\ U_Q$ of the reference system, as illustrated in Figure 1. If the subscripts a refer to values determined astronomically determined at P, the separation vector is given by

$$\vec{d}\ =\ R_\alpha\,\xi_\alpha\,\vec{\alpha}\ +\ h_d\,\vec{3}, \tag{7}$$

where R_α are the meridian and prime vertical radii of curvature of the associated spherop, $\vec{\alpha}$ are unit vectors oriented along the tangent plane to the spherop at Q in the meridian and prime vertical respectively, while $\vec{3}$ is the unit vector along the outward normal at Q. The subscripts g refer to the surface co-ordinates of the point P' in Figure 1 on the associated spherop $U = U_Q$ whose normal passes through P.

The locus of the point Q is called the Telluroid and mirrors the physical surface of the solid Earth and the oceans to order f^2. The system of reference is based on the family of spherops ($U = U_0 + \Delta W$) exterior to the reference ellipsoid, defined by the geometrical parameters a and f, and the gravitational characteristic μ (= GM), with the constraint that the surface of the reference ellipsoid is the equipotential surface $U = U_0$. As pointed out in section 2.1, there is no necessity for the ellipsoid to be forced to have the same volume as the geoid ($W = W_0$) if terms of zero degree were retained in the solution. In such circumstances, it is easy to show that (e.g., Mather 1973, p. 14) the height anomaly (h_d in Figure 1) is given by

$$h_d = \frac{1}{\gamma}\left[V_d - (W_0 - U_0)\right] + o\{10^{-3}\,m\}, \tag{8}$$

where V_d is the disturbing potential at P given by

$$V_{dp} = W_p - U_p. \tag{9}$$

The quantity W_p is defined by the value W_0 of the geopotential on the equipotential surface used as the datum for geodetic levelling, and the observed difference of geopotential ΔW between this surface and P, by the relation

$$W_p = W_0 + \Delta W.$$

The datum in use at the present time is that afforded by mean sea level derived from tide gauge readings over periods in excess of one year. As solutions of the geodetic boundary value problem require definitions which are applicable globally, it is essential that all regional definitions of mean sea level are correlated on a world wide basis to a common epoch in the first instance before the differences in geopotential (ΔW) can be considered to be referred to an equipotential surface of the Earth's gravitational field. It has been estimated that systematic errors of $o\{\pm 10\text{ cm}\}$ could result in solutions of the boundary value problem if errors on this account were of $o\{\pm 30\text{ cm}\}$ and each datum covered $o\{10^6\text{ km}^2\}$ (Mather 1973, p. 68).

A second problem of consequence and of which little is known at the present time, are the quasi-stationary departures of the sea surface from the equipotential surface defined by the results of geodetic levelling. The phenomenon, known as stationary sea surface topography, has been reported along coastlines in many parts of the world. A summary of results is given by Hamon and Greig (1973), indicating magnitudes of 30-50 cm being commonplace, with one reported as large as 1.7 m. This effect is discussed further in section 4.

The gravity anomaly Δg at the surface of the Earth is defined by

$$\Delta g = g_p - \gamma_{p'}, \tag{10}$$

where g_p is the value of observed gravity at P, corrected for departures of the Earth from the rigid body model, as described in section 2.1, and $\gamma_{p'}$ is normal gravity due to the reference system at P' in Figure 1. $\gamma_{p'}$ is obtained from the equivalent value γ_0 of normal gravity on the reference ellipsoid, given by the commonly used relations of the type (e.g., Heiskanen and Moritz, p. 79)

$$\gamma_0 = \gamma_e \left[1 + \beta \sin^2 \phi_g + \beta_2 \sin^2 2\phi_g + o\{f^3\}\right], \tag{11}$$

where γ_e is equatorial gravity defined by the values adopted for a, f, GM and ω (e.g., Mather 1971, p. 87; IAG 1970, p. 48), $\beta = o\{f\}$ and $\beta_2 = o\{f^2\}$, using the relationship (e.g., Mather 1971a, p. 101)

$$\gamma_{p'} = \gamma_0 - 2\frac{\Delta W}{a}\left[1 + f + m - 2f \sin^2 \phi - \frac{1}{2}\frac{\Delta W}{a\gamma} + o\{f^2\}\right], \tag{12}$$

ΔW having the same significance as in Equation 6.

Possible sources of systematic error in the computed value of $\gamma_{p'}$ and hence Δg arise in the definition of ϕ_g and ΔW. While the effect of errors in the latter have already been described, ϕ_g should be defined to ±0.4 arcsec if $\gamma_{p'}$ is not to have an error of $\pm 10\mu$ gal. It is therefore important to use any of the global solutions available at the present time for the definition of geocentric position, to evaluate geocentric orientation parameters for each of the regional geodetic datums (Mather 1973, p. 16) before computing gravity anomalies for high precision determinations rather than use values referred to regional geodetic datums.

The equation described as the fundamental equation in physical geodesy (Heiskanen and Moritz 1967, p. 86), defines the relationship between the disturbing potential V_d and the gravity anomaly Δg as (Mather 1973, p. 18)

$$\frac{\partial V_d}{\partial h} = -\Delta g + \frac{\partial \gamma}{\partial h} h_d \left(+ \frac{1}{2} g \zeta^2 + o\{1 \mu\text{gal}\} \right) \tag{13}$$

where the terms within the bracket take into account effects smaller than $o\{f \Delta g\}$, ζ being the deflection of the vertical at the point considered.

The philosophy underlying Equation 13 is the contention that the geocentric position of P is not known, though estimates adequate for the linearization of the quantities involved, are available. Circumstances may well arise in the future where accurate horizontal and vertical surveys may be available and the principal practical role of techniques in physical geodesy is the determination of the geoid in ocean areas for studies of sea surface topography. In such a situation, it is envisaged that all the land masses are linked to a geocentric system of reference using laser ranging methods and/or VLBI, giving at least one fundamental station on each geodetic datum. Horizontal survey methods together with geodetic and astrogeodetic levelling will provide data for completely defining geocentric position of points on any regional network which includes at least one fundamental station, with an accuracy of 1 part in 10^6. As surface ship locations can be routinely determined to within one order of magnitude greater, it is of relevance to examine the gravity disturbance δg (e.g., Hotine 1969, p. 312) given by

$$\delta g = g_p - \gamma_p = -\frac{\partial V_d}{\partial h} + \frac{1}{2} g \zeta^2 + o\{1 \mu\text{gal}\}, \tag{14}$$

where the uncertainties in defining the position of P can be estimated as ±0.2 arcsec in horizontal position and ±2 m in normal displacement, if the astrogeodetic levelling is based on an adequate distribution of stations. The effect of errors due to the first source on δg are $o\{1\ \mu\text{gal}\}$ while that of those due to the second are $o\{5 \times 10^2\ \mu\text{gal}\}$. Thus the gravity disturbance, whose order of magnitude is not significantly different from that of the gravity anomaly, is likely to have errors of $o\{5 \times 10^2\ \mu\text{gal}\}$ which are probably correlated with wavelengths in excess of 1000 km (e.g., see Mather, Barlow and Fryer 1971, Fig. 4.2) unless radically new techniques are available for determining

<u>either</u> geocentric position at each gravity station such that the radial component is resolved with systematic biases of wavelengths longer than 1000 km held to below the 20-30 cm level;

<u>or</u> the contribution of astro-geodetic levelling with the same resolution as geodetic levelling.

The projection of present day techniques does not lead to the conclusion that there would be significant advantages in using the gravity disturbance δg in preference to the gravity anomaly Δg in formulating solutions of the boundary value problem.

The separation vector $\vec{d}$, illustrated in Figure 1, can be represented by components along the axes of a local Cartesian co-ordinate system x_i at Q, with the x_3 axis oriented along the spherop normal at Q and the $x_1 x_2$ plane lying in the horizon at Q, as illustrated in Figure 2, in accordance with Equation 7. $\vec{d}$ is of importance in defining the geocentric orientation vector $\vec{0}$ for regional geodetic datums using surface gravity data (Mather 1971b, p. 62). A description of how such information could be used to assemble a world geodetic system linking the

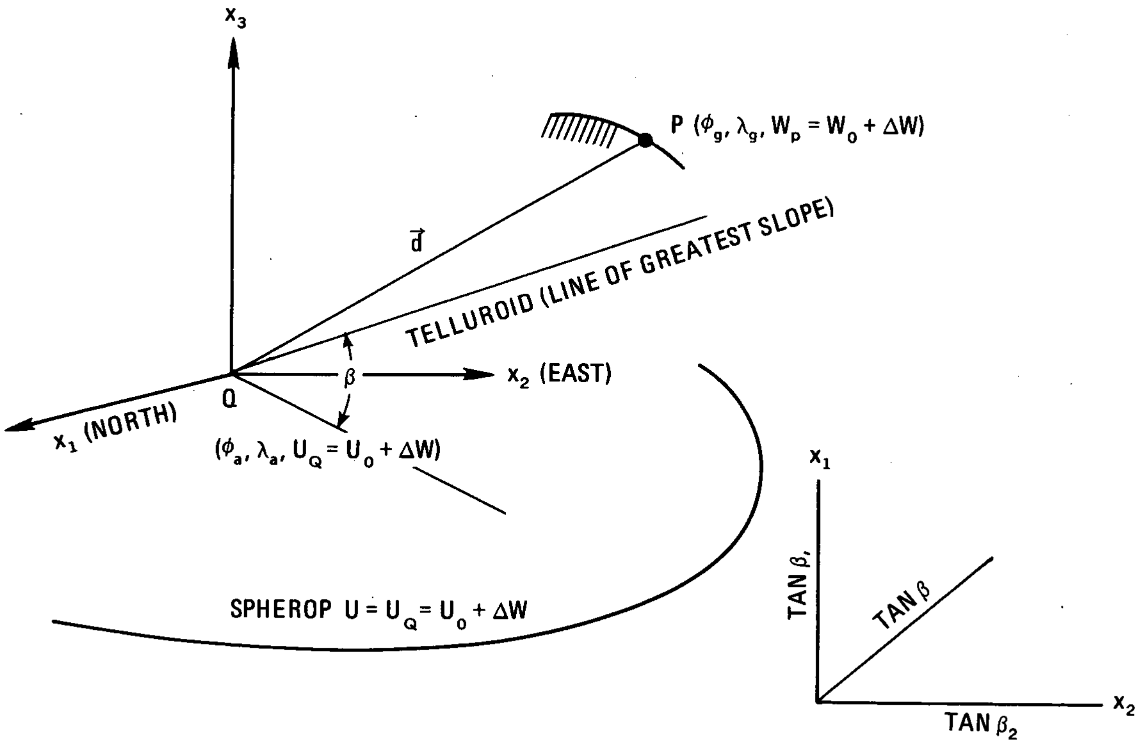

Figure 2. The Separation Vector and a Local Cartesian Co-Ordinate System

major land masses by comparing the separation vectors as obtained from gravimetry and astro-geodesy is given by Mather (1971c).

The mainstream of practical endeavors at the present time is in the determination of the height anomaly h_d. Over 90% of the power in such determinations comes from the "free air geoid" N_f obtained by the use of free air anomalies (i.e., gravity anomalies to the order of the flattening) in Stokes' integral which is set out in Equation 16. The latter is a solution of the boundary value problem for a spherical Earth which is exterior to all matter and whose bounding surface is an equipotential (Stokes 1849).

The deflections of the vertical ξ_α are usually obtained using the principles generally attributed to Vening Meinesz (1928). Working on a spherical reference system, he showed that if the separation N_f between the physical and reference surfaces were given by Stokes' integral

$$N_{fp} = \frac{R}{4\pi\gamma} \iint f(\psi)\, \Delta g\, d\sigma, \tag{15}$$

where Δg is the value of the gravity anomaly at the element of surface area $d\sigma$ on unit sphere which is at an angular distance ψ from the point of computation P, $f(\psi)$ being Stokes' function (e.g., Heiskanen and Moritz, 1967, p. 94), then

$$\xi_\alpha = -\frac{\partial N}{\partial x_\alpha} \tag{16}$$

as illustrated in Figure 3, where x_α is a two-dimensional Cartesian system in the horizon plane at the point of computation P, with the x_1 axis oriented north and the x_2 axis east. It is not difficult to show in the case of Stokes' problem that

$$\xi_\alpha = \frac{1}{4\pi\gamma} \iint \frac{\partial\,[f(\psi)]}{\partial\psi} \cos A_\alpha\, \Delta g\, d\sigma, \tag{17}$$

as

$$\frac{\partial}{\partial x_\alpha} = -\frac{1}{R}\frac{\partial}{\partial\psi} \cos A_\alpha, \tag{18}$$

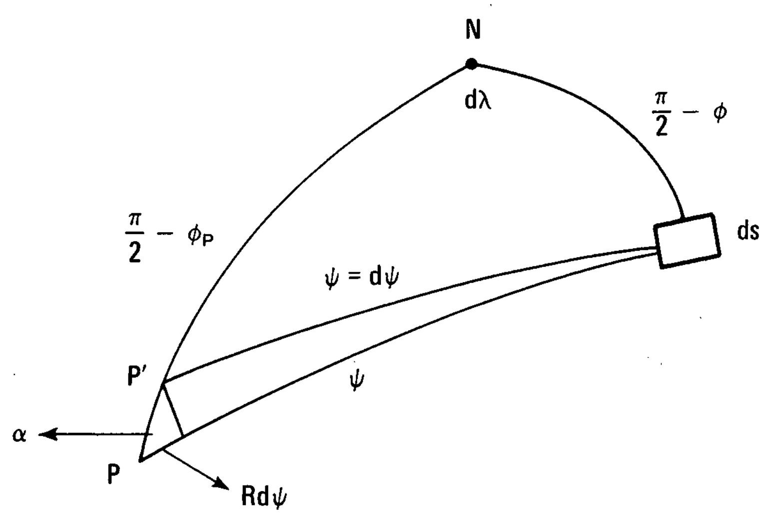

Figure 3. The Vening Meinesz Problem

where

$$A_1 = \alpha \qquad \text{and} \qquad A_2 = \frac{1}{2}\pi - \alpha, \tag{19}$$

α being the azimuth of $d\sigma$ from P. This follows as only ψ in the kernel of the integral at (15) changes as the point of computation changes from P to some adjacent point Q in the case of Stokes' problem.

The required expression for the Molodenskii problem is not the same as the elevation h_p of P appears in the kernel of the integral. As the deflection of the vertical at the surface of the Earth is obtained from the height anomaly h_d (ibid, p. 312), which is given by

$$h_d = h_d(\phi, \lambda, h_p) = h_d(\psi, \alpha_\sigma, h_p), \tag{20}$$

where α_σ is the azimuth of P from the element of surface area $d\sigma$, it can be shown that (Mather 1971c, p. 88)

$$\xi_\alpha = \frac{1}{h_\alpha}\left[\frac{\partial h_d}{\partial \psi}\frac{\partial \psi}{\partial u_\alpha} + \frac{\partial h_d}{\partial \alpha_\sigma}\frac{\partial \alpha_\sigma}{\partial u_\alpha}\right], \tag{21}$$

u_α being a set of curvilinear surface co-ordinates on the reference surface, and h_α the associated linearization parameters. For the latitude-longitude system

$$u_1 = \phi \quad , \quad u_2 = \lambda , \tag{22}$$

while

$$h_1 = R \quad \text{and} \quad h_2 = R \cos \phi \tag{23}$$

for a spherical approximation of the Earth. In the case of solutions to order e^3,

$$h_1 = (\rho + h) \quad ; \quad h_2 = (\nu + h) \cos \phi , \tag{24}$$

ρ and ν being equivalent to R_a defined in Equation 7.

The detailed development of solutions of the boundary value problem in this case use the geocentric latitude ϕ_c instead of the geodetic latitude ϕ, all parameters referring to quantities relating angular displacements between the geocentric radii to the pole, P and $d\sigma$.

As explained above, the principal task in determining position from gravity, is the definition of the height anomaly h_d, which is equal to the geoid height N in ocean areas, where $\Delta W = 0$. The next section deals in summary with some of the methods which have been proposed for obtaining the height anomaly.

3. TECHNIQUES FOR THE SOLUTION OF THE BOUNDARY VALUE PROBLEM

3.1 INTRODUCTION

Attention has been confined to three techniques whose use to obtain solutions to the boundary value problem has been extensively reported in the literature. The methods considered deal with formulations of solutions to what is known as Molodenskii's problem, at the physical surface of the Earth. It is not intended to formulate solutions for surfaces other than that of the Earth, e.g., the geoid, obtained by defining N instead of h_d. Neither is any attempt being made to discuss the merits of regularization (e.g., Molodenskii et al, p. 45 et seq.), where the conditions applicable to Stokes' problem are artificially created by the transfer of mass to within the geoid. The main advantage claimed for such techniques is a utility which is desirable when the surface gravity coverage is poor, as the adoption of certain types of mass transfers enables more reliable predictions of gravity anomalies for the chosen model. The validity of such claims is open to question if the end-product of the calculations is to be a meaningful determination of positional parameters, which can only be as good as the available data.

The three techniques which will be covered, ostensibly do not require a knowledge of the stratification of matter within the Earth, defining solutions in terms of an "adequate" sampling of gravity at the surface of the Earth, in conjunction with a complete definition of the associated topography. They can be classified as

(1) Surface layer solutions;

(2) Solutions from data sampled at discrete points on the Earth's surface; and

(3) Solutions from Green's third identity.

It is of interest to summarize the basis of each of these methods.

3.2 THE SURFACE LAYER TECHNIQUE

This method, initially developed by Molodenskii (Molodenskii et al 1962, p. 118 et seq.) was first published in 1949. Considerable material is available on the problems associated with the practical use of this technique by Moritz (1966; 1970; 1972) and members of the Soviet school (e.g., Brovar 1964; Marych 1969; Yeremeyev 1969; Pellinen 1972). The derivation calls for the representation of the disturbing potential V_d at the surface of the Earth by a surface layer of density Φ such that the former can be represented at any point P either on the surface of the Earth or exterior to it by the relation

$$V_{dp} = \iint_S \frac{\Phi}{r}\, dS, \tag{25}$$

where there is no restriction on the shape of the surface S. It can be shown that

$$\left(\frac{\partial V_d}{\partial h}\right)_p = -2\pi\Phi_p \cos\beta_p + \frac{\partial}{\partial h_p}\left(\iint \frac{\Phi}{r}\, dS\right), \tag{26}$$

where the subscript p refers to evaluation at P, β_p being the ground slope at P.

The first term on the right appears due to the indeterminance at P itself. The inner zone in this region is treated as a disk (e.g., Heiskanen and Moritz 1967, p. 129), the negative sign being introduced as the outward derivative is required, while the attraction of the disk is toward the geocenter. The cos β term allows for slope of the surface of the disk with respect to the vertical. No approximations are involved in the derivation of Equation 26.

The ensuing development, which is well documented by Heiskanen and Moritz (ibid, p. 300) can be summarized as follows, retaining those terms whose contributions are greater than o $\{f h_d\}$. On using Equations 8, 13, 25 and 26,

$$\Delta g = 2\pi \Phi_p \cos \beta_p - \frac{W_0 - U_0}{\gamma_p} \left(\frac{\partial \gamma}{\partial h}\right)_p \tag{27}$$

$$- \iint \left[\frac{\partial}{\partial h_p} \left(\frac{1}{r}\right) - \frac{1}{\gamma_p} \left(\frac{\partial \gamma}{\partial h}\right)_p \frac{1}{r}\right] \Phi \, dS + o\{f \Delta g\} \cdot$$

As

$$\frac{1}{\gamma_p} \left(\frac{\partial \gamma}{\partial h}\right)_p = - \frac{2}{R_p} + o\left\{f \frac{1}{\gamma} \frac{\partial \gamma}{\partial h}\right\}, \tag{28}$$

and

$$r = (R_p^2 + R^2 - 2 R R_p \cos \psi)^{1/2}, \tag{29}$$

it follows from Figure 4 that

$$\frac{\partial}{\partial h_p} \left(\frac{1}{r}\right) = \frac{\partial}{\partial R_p} \left(\frac{1}{r}\right) + o\left\{f^2 \frac{\partial}{\partial h_p} \left(\frac{1}{r}\right)\right\} = - \frac{1}{r^3} [R_p - R \cos \psi] =$$

$$- \frac{R_p}{2 r^3} + \frac{R^2}{2 R_o r^3} - \frac{1}{2 R_p r}$$

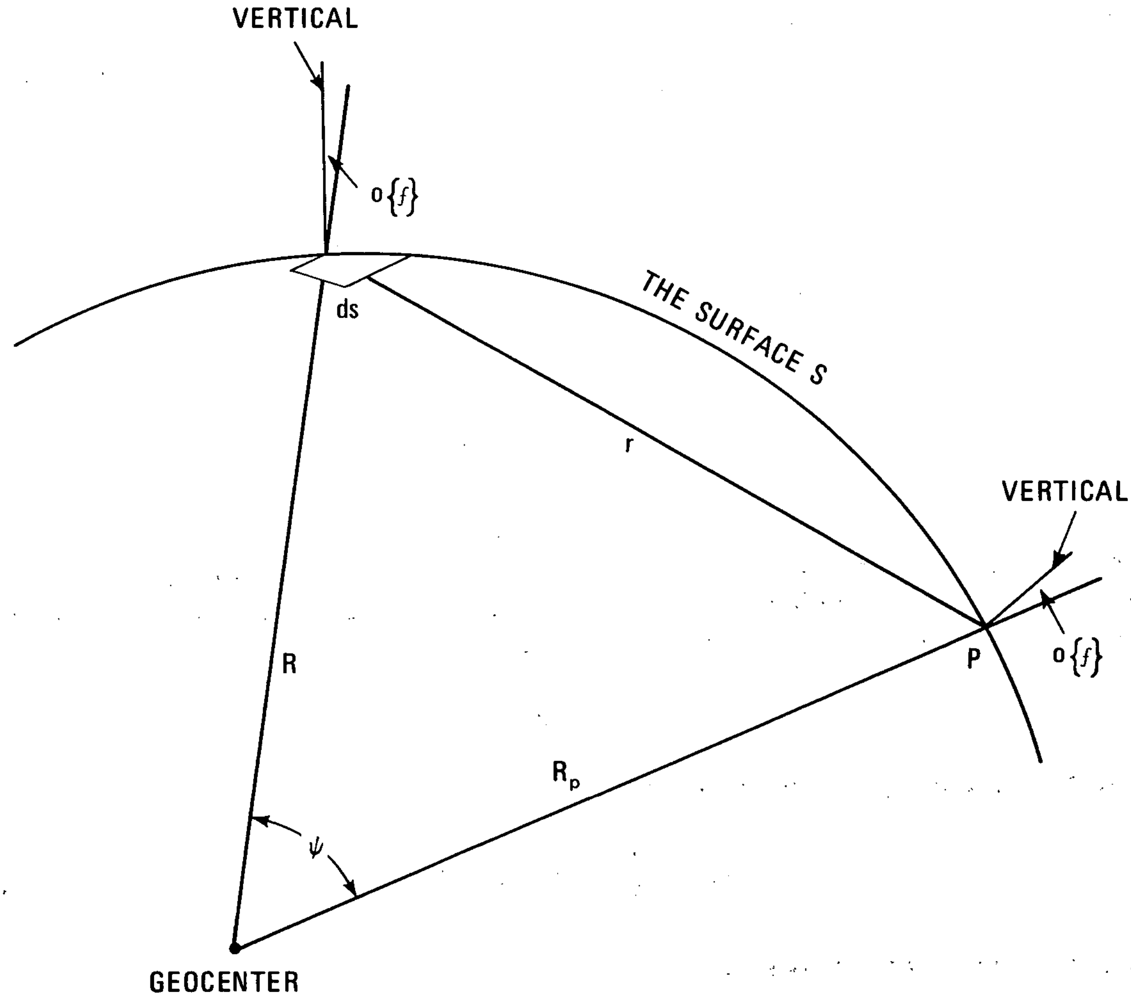

Figure 4. Relations for r

Thus

$$\frac{\partial}{\partial h_p}\left(\frac{1}{r}\right) - \frac{1}{\gamma_p}\left(\frac{\partial \gamma}{\partial h}\right)_p \frac{1}{r} = \frac{3}{2R_p r} + \frac{R^2 - R_p^2}{2R_p r^3} + o\left\{f \frac{1}{R_p r}\right\}. \tag{30}$$

Equation 27 can therefore be written as

$$\Delta g = 2\pi \Phi_p \cos \beta_p - \frac{W_0 - U_0}{\gamma_p} \left(\frac{\partial \gamma}{\partial h}\right)_p - \iint \left[\frac{3}{2R_p r} + \frac{R^2 - R_p^2}{2R_p r^3} \right] \Phi \, dS + o\{f \Delta g\} . \tag{31}$$

The solution suggested by Molodenskii (1962, p. 120) for Equation 31 is based on the method of successive approximations where the surface S of the Earth is transformed into the surface $\bar{S}$ using a parameter k which specifies the relationship between the geocentric radii R and $\bar{R}$ to equivalent points on S and $\bar{S}$ by the relation

$$\bar{R} = R_m + k(R - R_m), \tag{32}$$

where R_m is the mean radius of the Earth, and $0 \leq k \leq 1$. Thus S and $\bar{S}$ coincide when $k = 1$, while the classical Stokesian case in which no topography exists exterior to the geoid, is obtained when $k = 0$. This is equivalent to scaling all elevations and grades by k from h and tan β to $\bar{h}$ and k tan β where

$$\bar{h} = k h$$

and the related angle $\bar{\beta}$ is given by

$$\bar{\beta} = \cos^{-1} \left([1 + k^2 \tan^2 \beta]^{-1/2} \right) . \tag{33}$$

Other relevant conversions are

$$\bar{r} = \left[r_0^2 + k^2 (h - h_p)^2 \right]^{1/2} + o\{f \bar{r}\} , \tag{34}$$

where r_0 is the expression for the spherical case, given by

$$r_0 = 2R_m \sin \frac{1}{2} \psi . \tag{35}$$

Molodenskii simplifies the solution by introducing the parameter χ defined by the equation

$$\chi = \frac{R^2}{R_m^2} \Phi \sec \beta , \tag{36}$$

which, when taken in conjunction with the relation

$$dS = R^2 \, d\sigma \sec\beta, \tag{37}$$

where $d\sigma$ is the element of surface area on unit sphere, enables Equation 31 to be written as

$$\begin{aligned} \Delta g = {} & \frac{R_m^2}{R_p^2} 2\pi \chi_p \cos^2\beta_p - \frac{W_0 - U_0}{\gamma_p} \left(\frac{\partial \gamma}{\partial h}\right)_p \\ & - \frac{3R_m^2}{2R_p} \iint \frac{\chi}{r} \, d\sigma - \frac{R_m^2}{2R_p} \iint \frac{R^2 - R_p^2}{r^3} \chi \, d\sigma + o\{f \Delta g\} . \end{aligned} \tag{38}$$

It can be shown (ibid, p. 121) that if χ were expanded in a power series of the form

$$\chi = \sum_{i=0}^{\infty} \chi_i k^i , \tag{39}$$

and on introducing a set of functions G_i, the use of Equations 33, 34 and 39 in Equation 38 gives a system of integral equations

$$G_i = 2\pi \chi_i + 2 \frac{W_0 - U_0}{R_m} - \frac{3}{2} R_m \iint \frac{\chi_i}{r_0} \, d\sigma \quad , \quad i = 0, \infty, \tag{40}$$

on equating the coefficients of k^i, and as $R^2 - R_p^2 = 2R^2 (h - h_p) + o\{fR^2\}$. The quantities G_i are obtained in this manipulation as

$$G_0 = \Delta g \tag{41}$$

$$G_1 = R_m^2 \iint \frac{h - h_p}{r_0^3} \chi_0 \, d\sigma , \tag{42}$$

with more complex expressions for higher values of i (ibid, p. 122), being of the form

$$G_i = G_i (h, h_p, \chi_1, \chi_2, \ldots\ldots \chi_{(i-1)}) . \tag{43}$$

Equation 38 reduces to Equation 40 when $h = h_p = 0$ and $\beta = 0$. The Molodenskii problem is equivalent to Stokes' problem in such a case, the solution of which is Equation 15, which, on taking zero degree effects into account (Mather 1971c, p. 85) can be written as

$$N_f = \frac{1}{\gamma} [V_d + W_0 - U_0] = \frac{W_0 - U_0}{\gamma} - \frac{R_m M\{\Delta g\}}{\gamma} + \frac{R_m}{4\pi\gamma} \iint f(\psi) \, \Delta g \, d\sigma , \tag{44}$$

where $M\{\Delta g\}$ is the global mean value of Δg.

The substitution of Equation 25 in its appropriately modified form, in Equation 40 gives

$$\chi_i = \frac{1}{2\pi} \left[G_i + \frac{3V_{di}}{2R_m} \right] , \tag{45}$$

on adoption of the representation

$$V_d = \sum_{i=0}^{\infty} k^i V_{di} = \sum_{i=0}^{\infty} \iint k^i \frac{\chi_i}{r_0} \, d\sigma + o\{f V_d\} . \tag{46}$$

The second equality in Equation 46 would be consistent with Equation 25 only if there were no topography. If this inconsistency were removed (Molodenskii et al 1962, p. 123), the final expression for the height anomaly would be a series in G_i embedded in the form set out at 44 with some topographic correction terms whose effects are purely local in character and need only be considered in areas of rugged topography, being functions of r_0^{-3}, the series being obtained when k = 1. In this case,

$$h_d = \frac{W_0 - U_0}{\gamma} - R\frac{M\{G\}}{\gamma} + \frac{R_m}{4\pi\gamma}\iint f(\psi)\, G\, d\sigma + T, \tag{47}$$

where

$$G = \sum_{i=0}^{\infty} G_i \tag{48}$$

and T are the series of topographic correction terms whose form is given by Molodenskii (ibid). Moritz (1966, p. 91) has given alternative forms for G_1, and shows that if gravity anomalies are linearly correlated with elevation, G_1 reduces to the terrain correction. Thus the combination of Equations 42 and 45 gives

$$G_1 = \frac{R_m^2}{2\pi}\iint \frac{h - h_p}{r_0^3}\,\Delta g\, d\sigma = \frac{1}{2}\pi\rho R_m^2 \iint \frac{(h - h_p)^2}{r_0^3}\, d\sigma, \tag{49}$$

the second equality being based on the assumption of linear height correlation of gravity anomalies (ibid, p. 88).

Notes:

(1) This technique will be practically effective only if the contributions of the higher G_i are significantly smaller than those obtained for i = 0 and 1. The evaluation of any particular G_i presupposes a knowledge of all χ_j (j < i), which in turn, are defined through Equation 45.

The solution is therefore iterative, and as the series in χ_i is theoretically infinite, it is desirable that

$$\chi_i = o\{10^{-1}\}\,\chi_{i-1} \tag{50}$$

for efficient practical evaluation. As G_0 is the gravity anomaly, the first iteration is the free air geoid, which contains over 90% of the power in the solution. It should therefore require only three iterations to obtain a solution to order e^3 in h_d, if Equation 50 were satisfied.

(2) There would be little difficulty in meeting this criterion if the ratio $(h - h_p)/r_0 = o\{10^{-1}\}$. As oceanic regions comprising 70% of the Earth's surface area and non-mountainous regions make little or no significant contribution to topographical effects, the magnitude of the effects would be small if the above criterion were satisfied. All topography with grades in excess of 5° pose problems in this respect when they occur within a few km of the point of computation, distant zone effects being rapidly submerged by the r_0^{-3} term. Also see section 3.5.

(3) Serious embarrassment is caused when slopes exceed $1/4\pi$. Divergent series are obtained, making an iterative approach unstable. Discussions on the problems of convergence are available in the literature (Moritz 1970; Moritz 1972; Krarup 1972). For a further discussion, see section 3.5.

(4) The quantity G can have no first degree harmonic, as the solution of Stokes problem forbids the existence of such harmonics. Consequently, the reference ellipsoid used for computing normal gravity is situated at the center of mass of the mass distribution needed to produce values of gravity at the surface of the Earth which would give rise to a gravity anomaly distribution equivalent to that of G. The writer is not aware of a detailed investigation of this problem but it is unlikely that the net effect would exceed $o\{5 \times 10 \text{ cm}\}$.

(5) The extension of this technique to orders of accuracy greater than that of the flattening, is possible in theory. Such a solution could be obtained on including all effects of relevant magnitude in Equations 27 and 28, and on allowing for the existence of the atmosphere, noting that Stokes' integral is strictly valid only if there is no mass exterior to the physical surface. In addition it is necessary to take into account the Earth's ellipticity, especially when utilizing the orthogonal properties of surface harmonics.

3.3 SOLUTION FROM DISCRETE VALUES

This technique was originally proposed by Bjerhammar who summarizes the problem as follows (Bjerhammar 1964, p. 14).

"A finite number of gravity data (gravity anomalies) is given for a non spherical surface, and it is required to find such a solution that the boundary values for the gravity data (gravity anomalies) are satisfied in all given points."

The Bjerhammar problem is differently posed to that of Molodenskii and a different approach is used for the representation of the gravity anomalies at the surface of the Earth. Working on the basis that the representation of the surface gravity field can only be in terms of samples taken at discrete points at the Earth's surface, Bjerhammar proposes the interpretation of such data in terms of a set of model anomalies Δg^* on the surface of a sphere of radius R_b which is less than or equal to the polar radius of the best fitting ellipsoid. The appropriate requirement in Earth space is that any point on the Earth's surface lies exterior to the sphere of radius R_b (the Bjerhammar sphere), whose center is collocated with the geocenter.

The technique of solution can be summarized as follows. The surface of the Bjerhammar sphere is partitioned into a grid, each element of which has a surface area $R_b^2\, d\sigma$, and is represented by the model gravity anomaly Δg^*, assumed constant over the area. The disturbing potential V_{dp} at any exterior point P, whose geocentric distance, as illustrated in Figure 5, is R_p, is given by

$$V_{dp} = \frac{R_b^2}{4\pi R_p} \iint \Delta g^* \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R_b}{R_p}\right)^n P_{n0}(\cos\psi)\, d\sigma \tag{51}$$

under conditions applicable to Stokes' problem. Δg^* obviously cannot have a first degree harmonic and the possibility of satisfying this condition in conjunction with the geocentric collocation of the Bjerhammar sphere is subject to the same arguments as outlined in note 4 to section 3.2.

The observational data is in the form of gravity anomalies Δg as determined at discrete points at the surface of the Earth. Using the fact that Poisson's integral

$$H_p = \frac{R_b(R_p^2 - R_b^2)}{4\pi} \iint \frac{H}{r^3}\, d\sigma, \tag{52}$$

applies without approximation to any function H which is harmonic exterior to the Bjerhammar sphere, it is possible to define gravity anomalies Δg at all exterior points, if a surface distribution of the data set Δg^* were available on the sphere. Alternately, if Δg_i are the gravity anomalies measured at the surface of the Earth, the equation defining the Δg_i in terms of the Δg^* is obtained from Equation 52 as

$$\Delta g_i = \frac{R_b^2 (R_{pi}^2 - R_b^2)}{4\pi R_p} \sum_j \frac{\Delta g_j^*}{r_{ij}^3} d\sigma_j , \tag{53}$$

where r_{ij} is the distance between P_i on the earth's surface, with geocentric distance R_{pi}, and surface element $d\sigma_j$ on the Bjerhammar sphere. Equation 53, called a discrete integral equation by Bjerhammar (1968, p. 6), can be treated as a set of observation equations which can be solved by standard techniques for the elements Δg^*. The technique is subject to certain practical difficulties when tested on models with heavy point masses between the sphere and the Earth's surface (ibid, p. 67). In such cases, Bjerhammar advocates the use of the disturbing potential rather than the gravity anomaly in Equation 53, presumably by recourse to an iterative procedure. The validity of the technique hinges on whether the resulting disturbing potential at points P_i at the Earth's surface due to the Bjerhammar system is identical with that due to the Earth. For a summary of the proof of this condition, see (Bjerhammar 1969, pp. 452-6).

The instability of the inversion procedure due to the nature of gravitational attraction and its susceptibility to large masses locally (e.g., mountainous regions) led Bjerhammar to suggest that the more stable disturbing potential V_d^* on the Bjerhammar sphere be used as an intermediary in the solution on the following lines (ibid, p. 498 et seq.).

The disturbing potential V_{di} at P_i on the Earth's surface is given by Equation 52 as

$$V_{di} = \frac{R_b (R_{pi}^2 - R_b^2)}{4\pi} \iint \frac{V_d^*}{r_i^3} d\sigma . \tag{54}$$

As

$$\left(\frac{\partial V_d}{\partial h}\right)_i = \frac{R_b}{8\pi R_{pi}} \iint \frac{4 R_{pi}^2 r_i^3 - 3 r_i (R^2 - R_b^2 + r^2)(R_p^2 - R_b^2)}{r_i^6} V_d^* d\sigma \tag{54}$$

$$+ o\left\{f^2 \frac{\partial V_d}{\partial h}\right\},$$

Δg is obtained from Equation 13 on considering terms greater than o $\{f \Delta g\}$, as

$$\Delta g_i = -\frac{R_b}{8\pi R_{pi}} \iint \left[\frac{4R_{pi}^2}{r_i^3} - \frac{3(R_{pi}^2 - R_b^2)^2}{r_i^5} + \frac{R_{pi}^2 - R_b^2}{r_i^3} \right] V_d^* \, d\sigma + o\{f \Delta g\} . \tag{56}$$

Equation 58 is simplified by differencing V_d^* from the value V_{d0}^*, which is the value of V_d^* at the point on the Bjerhammar sphere corresponding to P. It can be shown (ibid, p. 499) that Equation 56 can be transformed to

$$\Delta g_i = -\frac{R_b V_{d0}^*}{R_p^2} - \frac{R_b}{8\pi R_{pi}} \iint \left[\frac{5R_{pi}^2 - R_b^2}{r_i^3} - \frac{3(R_{pi}^2 - R_b^2)}{r_i^5} \right] (V_d^* - V_{d0}^*) \, d\sigma \tag{57}$$

$$+ o\{f \Delta g_p\} ,$$

which is a generalized version of the Molodenskii inverse of Stokes' integral (Molodenskii et al. 1962, p. 50).

Notes:

(1) The use of this system would, at first glance appear to be a prohibitive task. This is not the case as the terms being integrated are scaled by r^{-3}, and hence only limited regions need be considered around each primary point at which evaluations are made. Details of test in the West Alps using a 5° × 5° area with a 15° × 15° buffer zone, with basic subdivisions of 5' × 5', are given by Bjerhammar (ibid, p. 508), an iterative procedure being used to recover the Δg^*.

(2) The intellectual elegance of the method is enhanced by its ability to combine all manifestations of the Earth's gravitational field into a single solution entity. It must be added that this same end can be achieved by using the methods proposed by Krarup (1969), though the problems associated with practical implementation have yet to be tackled in the case of high precision determinations.

(3) The factors which have to be taken into account to extend the solution to orders smaller than that of the flattening are similar to those outlined in section 3.2(5).

(4) The solution, like that from Krarup's method, is unique for a given distribution of data. This, of course, does not mean that the answer obtained is correct to the order of accuracy with which the problem is formulated. Data requirements for solutions of the boundary value problem are dealt with in section 4.

(5) For completeness, the solution should incorporate terms of zero degree as in section 3.2.

3.4 SOLUTIONS FROM GREEN'S THIRD IDENTITY

Considerable work has been done in this field (e.g., Arnold 1959; Koch 1965; Moritz 1965; Mather 1971c). The basic integral used is Green's third identity which is obtained by the application of Green's theorem to two scalars r^{-1} and W which is harmonic in the volume V_e exterior to a surface S. On combining the gravitational and rotational effects (e.g., Heiskanen and Moritz 1967, p. 15), the final expression obtained for the gravitational potential (W_p) at a point P on the surface S, on the assumption that all matter is contained within S and rotates with constant angular velocity ω, is

$$W_p = \frac{1}{2\pi} \iint \left(\frac{1}{r} \vec{\nabla} \cdot \vec{N} W - W \vec{\nabla} \cdot \vec{N} \frac{1}{r} \right) dS - 2\omega^2 \iiint \frac{1}{r} \, dV_i \, , \qquad (58)$$

where $\vec{\nabla} = \frac{\partial}{\partial x_i} \vec{i}$, $\vec{i}$ being unit vectors along the axes x_i of a Cartesian coordinate system, and r the distance of the elements of surface area dS and volume dV_i interior to S, from P. A similar expression is obtained for the potential U_p due to a gravitating reference ellipsoid which has the same rotational characteristics of the Earth, on considering the identical surface S, which is that of the Earth, when

$$U_p = \frac{1}{2\pi} \iint \left(\frac{1}{r} \vec{\nabla} \cdot \vec{N} U - U \vec{\nabla} \cdot \vec{N} \frac{1}{r} \right) dS - 2\omega^2 \iiint \frac{1}{r} \, dV_i \, , \qquad (59)$$

$\vec{N}$ in both Equations 58 and 59 being the unit normal vector to S at dS.

Both equations hold exactly if U and W are harmonic exterior to S. This condition requires that no matter exists on either system exterior to the Earth's surface. The practical consequences, which are of significance when resolution approaching the order of the flattening is sought as the end result of computations, are the following.

1. The reference ellipsoid must always lie within S. As S coincides with the ocean surface over 70% of the Earth, the reference ellipsoid must be smaller than the ellipsoid which best fits the geoid by an amount greater than the largest negative geoid undulation (o $\{10^2 \text{ m}\}$).

2. There should be no atmosphere exterior to S if Equation 60 is to hold to accuracies in excess of the order of $10^{-2} V_d$.

3. Both the reference ellipsoid and the Earth are assumed to rotate with the same constant angular velocity ω. Irregularities in the Earth's rotation have to be allowed for as corrections to observations in instances where such magnitudes are of significance. For details, see sections 2.2 and 4.

The practice to date has been to treat atmospheric effects as those which should be modeled and allowed for as corrections to observations prior to use in computations (e.g., IAG 1970, p. 18). In a recent solution Mather (1973, p. 28 et seq.) formulated a solution of the boundary value problem to o $\{e^3 h_d\}$ by separating the gravitational effects of the atmosphere from those of the solid Earth and oceans.

In conventional solutions, the disturbing potential V_d is obtained on differencing Equations 58 and 59, when

$$V_{dp} = W_p - U_p = \frac{1}{2\pi} \iint \left(V_d \vec{\nabla} \cdot \vec{N} \frac{1}{r} - \frac{1}{r} \vec{\nabla} \cdot \vec{N} V_d \right) dS. \tag{60}$$

This equation is not valid to orders smaller than o $\{10^{-2} V_d\}$ as it assumes the geopotential W to be harmonic outside S. A function which does satisfy Laplace's equation exterior to S is the potential W' due to the solid Earth and oceans, which is related to W by the relation

$$W' = W - V_a, \tag{61}$$

where V_a is the potential of the atmosphere, which is of order 10^{-6} W, and more significantly, $V_a = o\{10^{-2} V_d\}$. As such, it is desirable to construct a theory which allows for its existence in the course of the derivation.

The final solution using this technique can only be obtained by iteration, the number of iterations required as in the surface layer method, being a function of the accuracy sought. Favorable conditions for the adoption of an iterative procedure are the following.

a. A significant amount (>90%) of the power should be generated in the first iteration.

b. The iterative procedure should have the ability to converge to the correct result.

c. The number of iterations necessary for achieving the desired degree of resolution should be as small as possible.

When surface gravity is the sole source of information, the only procedure available for obtaining an adequate first approximation to the height anomaly h_d is the use of Stokes' approach. Fundamental to this technique is the assumption that the disturbing potential is harmonic exterior to and on the surface S, and therefore can be expressed in the form

$$V_d = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}} \quad , \quad n \neq 1, \tag{62}$$

where

$$A_n = \sum_{m=0}^{n} A_{nm}, \tag{63}$$

and

$$A_{nm} = P_{nm}(\sin \phi_c) [C_{nm} \cos m\lambda + S_{nm} \sin m\lambda], \tag{64}$$

the last equation being the standard expression for a surface harmonic.

The adoption of this model enables the combination of the effects of the disturbing potential V_d and its vertical gradient $\partial V_d / \partial h$ on using Equation 13, thereby transforming this formulation to a representation of the observed quantity, the gravity anomaly Δg. Details of the problems involved in obtaining a solution of the boundary value problem to order $e^3 h_d$ are dealt with by Mather (1973, p. 31 et seq.). To preserve flexibility in the formulation of results to any required order of accuracy, it is desirable to retain physical relevance in the derivation by constructing the integral at 60 such that the disturbing potential V_d is replaced by the quantity

$$V_d' = V_d - V_a , \tag{65}$$

where V_a is the potential of the atmosphere.

A generalized solution which did not consider either the existence of the atmosphere or the fact that the potential of the geoid was not known, the latter being disregarded after due consideration as a quantity which correctly cannot be determined from gravimetric methods alone (Molodenskii et al 1962, p. 104), was formulated by Molodenskii in 1945 (ibid, p. 93). A specific solution was given by Arnold (1959), which could be written as

$$\begin{aligned} h_{dp} &= \frac{R_m}{4\pi\gamma} \iint (\Delta g - \gamma \xi_\alpha \tan \beta_\alpha) f(\psi) \, d\sigma \\ &\quad + \frac{R_m^2}{2\pi\gamma} \iint \frac{1}{r_0^3} \left[(h_p - h) + R \sin \psi \frac{dh}{dr} \right] V_d - d\sigma , \end{aligned} \tag{66}$$

where ξ_α are the components of the deflection of the vertical, $f(\psi)$ is Stokes' function, $\tan \beta_\alpha$ being the components of the gradient of the ground slope in the north and east directions,

$$\frac{dh}{dr} = \cos A_\alpha' \tan \beta_\alpha , \tag{67}$$

where

$$A_1' = \alpha_\sigma \quad ; \quad A_2' = \frac{1}{2}\pi - \alpha_\sigma \tag{68}$$

and r_0 is given by Equation 35. A revision of the derivation to the order of the flattening showed that (Mather 1971c, p. 85)

$$h_{dp} = \frac{W_0 - U_0}{\gamma} - R_m \frac{M\{\Delta g\}}{\gamma} + \frac{R_m}{4\pi\gamma}\iint \Delta g \; f(\psi) \; d\sigma$$

(69)

$$+ \frac{R_m^2}{2\pi\gamma}\iint \frac{1}{r_0}\left[\left((h_p - h) + R_m \sin\psi \frac{dh}{dr}\right)\frac{V_d}{r_0^2} - \gamma\xi_\alpha \tan\beta_\alpha\right] d\sigma + o\{f\, h_d\},$$

if

$$\left[\frac{(h_p - h)}{r_0}\right]^2 = o\{f\}, \tag{70}$$

and Laplace's equation were satisfied to o $\{f\, \vec{\nabla}^2 V_d\}$ at all points exterior to, and on S. Equations (66) and (69) would be equivalent if

$$\frac{1}{2}\iint \gamma\, \xi_\alpha \tan\beta_\alpha \, f(\psi)\, d\sigma = R_m \iint \frac{1}{r_0}\, \xi_\alpha \tan\beta_\alpha \, d\sigma. \tag{71}$$

The effect of the terms common to the kernels of the integrals on either side of the equality in Equation 71 can be expected to arise from only 30% of the surface area of the globe. Significant contributions to h_d will be restricted to only about 5% of the surface area, being about one order of magnitude smaller than Δg if ξ_α = o $\{10^{-4}\}$ and tan β = o $\{10^{-1}\}$. The high probability of correlation between the signs of ξ_α and β_α in regions where the latter has significant magnitude, indicate that this effect is likely to be always positive. As h_d = o $\{10^2 \text{ m}\}$, it is realistic to estimate the effect of the above term as o $\{5 \times 10 \text{ cm}\}$, the effect being consequential if regions of mountainous topography occur near the point of computation.

The first term in the second integral at 69 converges much more quickly with increase in r_0 and can be treated as a purely local effect. The advantage of the solution at 69 over that at 66 is the fact that all terms due to the interaction between the ground slope and the slope of the equipotential can be treated as purely local effects, giving solutions where the neglected effects do not have magnitudes much in excess of the order of the flattening. Another advantage of the solution at 69 is its unambiguous definition in Earth space as the Stokesian term defined a contribution with reference to an ellipsoid whose center is at the geocenter of an Earth which had no atmosphere.

The generalization to order e^3 in h_d (i.e., ±5 cm) deals not only with the effect of the topography, the interactions between the slopes of the topography and the equipotential surfaces of the Earth's gravitational field, as well as the atmosphere, but is also in keeping with the physical characteristics of the scalar potential (Mather 1973). It also establishes the nature of the relationship between the Stokesian term and the indirect effect without limitations imposed by the simplistic approximations permitted by the adoption of a lower order of accuracy and identifies the anomaly to be used in Stokes integral. The geometry of the solution is also specified in Earth space, as the center of the reference ellipsoid is located at the center of mass G' of the solid Earth and oceans, whose co-ordinates $\bar{X}_{ei}$ with respect to a geocentric Cartesian co-ordinate system are given by (ibid, p. 26)

$$\bar{X}_{ei} = -\frac{M_a}{M_e} \bar{X}_{ai} , \tag{72}$$

where $\bar{X}_{ai}$ are the co-ordinates of the center of mass of the model adopted for the Earth's atmosphere, M_a and M_e being the mass of the atmosphere and the solid Earth and oceans respectively. The final formulae obtained in the solution referred to are summarized below.

$$h_{dp} = N_{fp} + N_{cp} , \tag{73}$$

where the Stokesian term N_{fp} is given by

$$N_{fp} = \frac{W_0 - U_0}{\gamma_p} - \bar{R}\,\frac{M\{\Delta g_c\}}{\gamma_p} + \frac{\bar{R}}{4\pi\gamma_p} \iint f(\psi)\, \Delta g_c \, d\sigma , \tag{74}$$

γ_p being the value of normal gravity at P' in Figure 1, $\bar{R}$ being the radius of the Brillouin sphere whose center is collocated with the center of mass of the solid Earth and oceans G' and contains the solid Earth and oceans. The gravity anomaly Δg_c is defined by

$$\Delta g_c = \Delta g_1 + \Delta g_2 , \tag{75}$$

where

$$\Delta g_1 = \Delta g + \frac{\partial V_a}{\partial h} + 2 \frac{V_a}{R_m}, \tag{76}$$

V_a being the potential of the atmosphere, and

$$\Delta g_2 = \frac{2V_d'}{R_m} c_\phi - \frac{1}{2} g \zeta^2 + dR \frac{\partial \Delta g}{\partial h} + o\{e^3 \Delta g\} \tag{77}$$

if

$$\frac{1}{2}(dR)^2 \frac{\partial^2 \Delta g}{\partial h^2} = o\{e^3 \Delta g\},$$

where

$$dR = \bar{R} - a(1 - f \sin^2 \phi_c) - h + o\{f\, dR\}, \tag{78}$$

$$\frac{\partial \Delta g}{\partial h} = -\gamma \left[\sum_{\gamma=1}^{2} \frac{\partial \xi_a}{\partial x_a} - \frac{\xi_1 \tan \phi_c}{R_m} - 2 \frac{N_f}{R_m^2} + o\left\{ f \frac{\partial \Delta g}{\partial h} \right\} \right], \tag{79}$$

and

$$c_\phi = f + m - 3f \sin^2 \phi_c + o\{f^2\}. \tag{79}$$

The angle ψ is computed in calculations to the order of $e^3 h_d$ from the geocentric latitude ϕ_c and longitude λ as

$$\psi = \cos^{-1} [\sin \phi_c \sin \phi_{cp} + \cos \phi_c \cos \phi_{cp} \cos d\lambda], \tag{80}$$

where

$$d\lambda = \lambda - \lambda_p, \tag{81}$$

the value without subscript referring to $d\sigma$, and those with subscript p to the point of computation P.

The indirect effect N_{cp} is given by

$$N_{cp} = \frac{V_{ap}}{\gamma_p} + \frac{1}{2\pi\gamma_p} \iint \frac{R^2}{r} \left[\frac{\partial V'_d}{\partial x_\alpha} \tan\beta_\alpha + V'_d \left(\frac{x_\alpha \tan\beta_\alpha}{r^2} + \frac{1}{2R}\left[3\left(c_\Delta + 3\frac{dR}{R}\right) - \Phi \right] \right) \right.$$

$$\left. - dR \frac{\partial \Delta g}{\partial h} + \Delta g' \left(c_\Delta + \frac{3}{2}\frac{dR}{R} \right) + o\{e^3 \Delta g\} \right] d\sigma \tag{82}$$

if

$$\frac{1}{2}(dR)^2 \frac{\partial^2 \Delta g}{\partial h^2} = o\{e^3 \Delta g\},$$

r being the distance between $d\sigma$ and P, R being the geocentric distance, given by

$$R = a(1 - f\sin^2\phi_c) + h + o\{f^2 R\}, \tag{83}$$

and h the ellipsoidal elevation. The other expressions which need definition are

$$\frac{\partial V'_d}{\partial x_\alpha} \tan\beta_\alpha = -\gamma\xi_\alpha \tan\beta_\alpha + N_f \frac{\partial \gamma}{\partial x_1} \tan\beta_1 + o\{f^2 \Delta g\}, \tag{84}$$

the two dimensional Cartesian co-ordinate system x_α having the same significance as in Figure 2,

$$\frac{x_\alpha}{r^2} \tan\beta_\alpha = \frac{R}{r^2}(1 + c_x)\sin\psi \frac{dh}{dr}, \tag{85}$$

where dh/dr is defined by Equation 67,

$$c_x = \frac{\cos\left(\frac{1}{2}\psi - \theta\right)}{\cos\left(\frac{1}{2}\psi + \theta + \delta\right)} - 1, \tag{86}$$

$$\theta = -\tan^{-1}\left[\frac{2\sin\delta\sin\frac{1}{2}\psi - \frac{\Delta R}{R_m}\cos\left(\frac{1}{2}\psi + \delta\right)}{2\cos\delta\sin\frac{1}{2}\psi + \frac{\Delta R}{R_m}\sin\left(\frac{1}{2}\psi + \delta\right)}\right] + o\{f^2\tan\theta\}$$

$$= \frac{1}{2}\frac{\Delta R}{R_m}\cot\frac{1}{2}\psi - \delta + o\{f^2\} \qquad \text{if} \qquad \psi > 5^\circ, \tag{87}$$

$$\Delta R = R_p - R, \tag{88}$$

and

$$\delta = f\sin 2\phi_c\cos\alpha_\sigma + o\{f^2\}. \tag{89}$$

The term Φ is given by

$$\Phi = \frac{2R}{r^2}\left[R - R_p\cos(\psi + \delta)\right] - 1, \tag{90}$$

and

$$c_\Delta = \frac{1 + 2\frac{dR}{R}}{(1 + c_{\bar{r}})^{1/2}} - 1, \tag{91}$$

where

$$c_{\bar{r}} = \left(\frac{\Delta R}{\bar{r}}\right)^2 - \frac{dR + dR_p}{R_m} + o\{f^2\}, \tag{92}$$

and

$$\bar{r} = 2\bar{R} \sin \frac{1}{2}\psi \tag{93}$$

Notes:

(1) The adoption of an iterative procedure to solve Equations 73 to 93 cannot be avoided. See note (1) to section 3.2 for background. Mather (ibid, p. 49 et seq.) has suggested an iterative procedure which requires three iterations of the expression for N_{cp} to achieve an accuracy of 5 parts in 10^4 (i.e., o $\{e^3 h_d\}$), while the Stokesian contribution need be evaluated only once, but in two stages.

(2) The first two terms in Equation 74 are of zero degree and are meaningful only when the surface gravity field is sufficiently well defined to give rise to an adequate definition of the global mean value of surface gravity anomalies. Present day solutions, which are heavily dependent on satellite determined low degree harmonics of the Earth's gravitational field, tend to ignore the effect of these terms. For a further discussion of zero degree effects which would be of consequence in solutions based on adequate distributions of surface gravity alone, see section 5.

(3) It could be construed that the conditions attached to Equations 77 and 82 are a limitation on the development outlined above. The relevant terms which are omitted have a net differential effect of

$$\frac{1}{4\pi\gamma} \iint \left[\frac{1}{2}\bar{R}\, f(\psi) - \frac{R^2}{r}\right] (dR)^2 \frac{\partial^2 \Delta g}{\partial h^2}\, d\sigma$$

on h_d which would be negligible if the quantity $\partial^2 \Delta g / \partial h^2$ had random error characteristics over areas larger than 10^4 km^2 with magnitudes of order 10^{-8} mgal m^{-2}, which is only one of magnitude smaller than that of $\partial^2 \gamma / \partial h^2$.

(4) The components of the deflections of the vertical ξ_α are computed on the principles outlined in Equations 16 to 24. A series of expressions which include effects with magnitudes of the order of the flattening are given by Mather (1971c, p. 86 et seq.). Such expressions should be adequate for evaluations of h_d to o $\{e^3 h_d\}$, but fail if accuracies of this type are required from the deflections themselves. Extension would principally require the use of an ellipsoidal co-ordinate system and a more careful evaluation of

some of the higher derivatives of characteristics of the gravitational field which have been assessed as having insignificant effects on the height anomaly.

3.5 CONCLUSION

The formulation of a solution for the boundary value problem at the physical surface of the Earth, in contrast to Stokes' problem where there is no topography exterior to the geoid, calls for the evaluation of "topographical" terms which arise as a consequence of

(a) departures of the Earth's surface from a level surface; and

(b) elevation of the point of computation above or below the surrounding topography.

The first effect has contributions with long wavelength which, on present assessment of geoid determinations, should not have effects in excess of o {50 cm} unless rugged topography occurs in the vicinity of the point of computation. The second effect is a purely local one as it is scaled by the factor r^{-3}, as seen in Equations 42, 57 and 69.

The limitation in theory, of the surface layer method is the heavy reliance it places on the convergence of the series in G_i, defined at 43, in a mathematical sense. It may appear to be paradoxical in practical terms, that the slope of the topography of distant areas, which at least to a first order of approximation, are in isostatic compensation, can affect computations of the height anomaly. These terms occur because the gravity anomaly used in computations and defined by Equations 10 and 11, reflect the mass distribution of the Earth as it exists. If, on the other hand, a suggestion similar to that made by de Graaff Hunter (1958) calling for the smoothing of the Earth such that slopes in excess of 5° did not exist, were adopted, there would be little to choose between the methods outlined in this section for the solution of the boundary value problem. In such a case, all the iterative methods would require only 3 iterations to achieve 5 cm accuracy in h_d.

The conclusion that a model had to be adopted for the topography was also reached by Moritz (1972, p. 49) after a detailed study of the convergence of Molodenskii series. This approach has recently become anathema to physical geodesists (e.g., Molodenskii et al 1962, p. 118) as it involves making assumptions about the density of material comprising the upper layers of the Earth's crust. In contrast, the quantities Δg, h, $\tan \beta$ and $\partial^i \Delta g / \partial h^i$ must be considered those which can be observed. In this sense, the solution described in section 3.4 exhibits favorable convergence characteristics, as the series involved are not "open-ended", but controlled in magnitude, being terms in a rapidly convergent power series in the parameter $f (= o\{10^{-3}\})$. In practical terms, however, the higher differential coefficients $\partial^i \Delta g / \partial h^i$ are unlikely to be determined and the adequacy of Equation 69 will depend largely on the magnitude and wavelength of the series $(i!)^{-1} h^i (\partial^i \Delta g / \partial h^i)$. Cumulative magnitudes of $o\{\pm 0.5 \text{ mgal}\}$ with wavelengths of 100 km or less can be considered to be acceptable for solutions to order $e^3 h_d$, as discussed in section 4.

Reverting to the question of smoothening the topography in order that grades do not exceed 10^{-1}, the problems which have to be resolved if such a procedure were adopted as everyday practice in physical geodesy are the following.

(a) The principles underlying the transfer of mass and their associated consequences should be clearly defined.

(b) The question of assigning a density for each element of transferred will have to be dealt with, as the resulting corrections to observed gravity, will depend on the model adopted for these masses.

If such procedures were deemed to be necessary, it would be mandatory to adopt a model for the Earth with surface slopes less than 5°. The transfer of matter to achieve this goal will change both observed gravity as well as the location of the center of mass of the physical system. The exact numerical value of the corrections made will depend on the principles adopted for the mass transfer. Physical geodesists advocating this type of approach will have to face up to the philosophical problem of which elements of topography to flatten out or fill. It is most important that a single model be adopted in order that the geodetic community is not subject to a confusing variety of results which are not in agreement, not because of significant factors, but merely as a consequence of the adopted smoothening procedure. The limited surface gravity data available at the present time continues to keep the above problem in the area of academic interest alone. It is one in which continued discussion is to be encouraged.

The solution of Molodenskii's problem by means of analytical continuation (Moritz 1969; Marych 1969) has not been dealt with as it has been shown to be equivalent to the solution using the surface layer approach (Moritz 1971). Another method which may prove to have some benefits is the use of numerical integration techniques, on which published material is hard to come by.

4. PRACTICAL CONSIDERATIONS

4.1 INTRODUCTION

Practical considerations fall into two distinct categories. The first concerns the optimum sampling of data in order that the necessary precision can be achieved in numerical computations. The second is the extraction of the most probable results from whatever (inadequate) data is available. The second falls beyond the scope of this review and is covered elsewhere in this Symposium. One exception is the use of Molodenskii and Cook truncation functions to obtain the maximum information from satellite determined gravity anomalies and local gravity fields.

The problem can be summarized as follows. Over 90% of the power in h_d comes from Stokes' integral. Many regions exist where dense local gravity fields exist, but where beyond some limiting angular distance ψ_0, the available gravity data from the analysis of the orbital perturbations of near Earth satellites on combination with whatever surface gravity exists, can be represented as a set of surface harmonics, of the type given in Equations 63 and 64. The Free Air Geoid at Equation 74 can be written as (Molodenskii et al 1962, p. 147)

$$N_{fp} = \frac{W_0 - U_0}{\gamma} - R\,\frac{M\{\Delta g\}}{\gamma} + \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi} f(\psi)\,\Delta g\,\sin\psi\,d\psi\,d\alpha \tag{94}$$

$$+ \frac{R}{2\gamma}\sum_{n=0}^{\infty} Q_n\,\Delta g_n,\quad n \neq 1,$$

where the gravity anomaly Δg to be used in Stokes' integral is expressed by the set of surface harmonics

$$\Delta g = \sum_{n=0}^{\infty} \Delta g_n,\quad n \neq 1. \tag{95}$$

Q_n is Molodenskii's truncation function given by

$$Q_n = \int_{\psi_0}^{\pi} f(\psi)\,P_{n0}(\cos\psi)\,\sin\psi\,d\psi, \tag{96}$$

$P_{n0}(\cos\psi)$ being the Legendre zonal harmonic. Values of Q_n for various values of ψ_0 are given by Molodenskii and his co-workers (ibid, p. 150) to n = 8, de Witte (1967) to n = 25, and Hagiwara (1972) to n = 18. The computational efficiency of this method over the use of surface quadratures techniques for distant zone effects, in the present era where distant zone fields are heavily dependent for quality on satellite data, is a factor of 70 (Ojengbede 1973, p. 32). In practice, only a limited number (at present, up to degree and order 20) of such harmonics are available and a rounding off error will occur in the computations, due to the existence of a residual in the power spectrum of gravity anomalies, on adopting the surface harmonic representation. Molodenskii uses an elegant technique to show that the use of harmonics to n = 8 with surface gravity representations up to $\psi_0 = 23°$ results in errors less than 2 m, while extension of surface gravity coverage to $\psi_0 = 35°$ reduces the truncation error to less than 50 cm (Molodenskii et al 1962, p. 164).

Similar considerations apply to the computation of the Vening Meinesz contribution to the deflections of the vertical using Cook's truncation function (Cook 1950, p. 377), the equation equivalent to 94 in this case being (de Witte 1967, p. 455)

$$\xi_\alpha = \frac{1}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\frac{\partial\,[f(\psi)]}{\partial\psi}\,\Delta g\,\cos A_\alpha\,\sin\psi\,d\psi\,d\alpha + \frac{1}{2}\sum_{n=2}^{\infty}(n-1)\,C_{\alpha n1}\,q_n \tag{97}$$

where

$$C_{\alpha n1} = \frac{1}{4\pi(n-1)\gamma}\iint \Delta g_n\,p_{n1}(\cos\psi)\,\cos A_\alpha\,d\sigma, \tag{98}$$

A_α being defined by Equation 19, while Cook's truncation function q_n is given by

$$q_n = \int_{-1}^{\cos 1/2\,\psi_0}\frac{\partial}{\partial\psi}\,[f(\psi)]\,p_{n1}(\cos\psi)\,d(\cos\psi). \tag{99}$$

The relationship between the functions Q_n and q_n has been established by Hagiwara (1972, p. 461) who gives a proof of the equivalence of the developments by Molodenskii and Cook. On using the same values of n and ψ_0 described in the previous paragraph, Molodenskii shows that the truncation errors in ξ_α are less than 1.1 arc sec and 0.2 arc sec respectively in the two cases given.

It can be concluded with confidence that the use of truncation functions is capable of giving a resolution equivalent to the best astro-geodetic results, as is borne out by determinations in Australia (Mather, Barlow and Fryer 1971, p. 19) and the tests carried out by Ojengbede (1973, p. 41).

4.2 SAMPLING THE GRAVITY FIELD AT THE SURFACE OF THE EARTH

The overwhelming majority of surface gravity data available at the present time has been surveyed for geophysical purposes motivated by regional considerations. Such information has to be carefully screened before being put to geodetic use. There are problems that arise in the establishment of the value of observed gravity itself. Until recently, most gravity determinations of quality were made by differential means using gravimeters. It is now possible to carry out an absolute determination of g with a resolution of ±50 μgal using a transportable apparatus (Morelli et al 1971, p. 17) while resolution at the ±3 μgal level has been reported by the apparatus at Sevres, France (Sakuma 1971).

It is all important in the first instance that all values of observed gravity are correctly referred to the unified gravity standardization network defined by the "International Gravity Standardization Network 1971" (IGSN71) or an equivalent global control network in order that datum discrepancies may be minimized if not eliminated.

The solution of the boundary value problem requires an evaluation of the gravity anomaly. This requires a knowledge of

(a) the geodetic latitude ϕ_g of the gravity station to 0.04 arcsec (±3 cm) for an accuracy of 1 μgal; and

(b) the geopotential difference ΔW with respect to the geoid to ±0.003 kgal m for a resolution of 1 μgal in the gravity anomaly Δg, in addition to the requirement stated earlier for values of observed gravity. This also calls for the definition of a datum for the geopotential differences on a global basis and to some desirable degree of resolution.

The status at the present time is as follows. While IGSN71 is available, it is most unlikely that any of the large gravity data banks are reliably connected to this network in toto at the present time. Most values of normal gravity are computed from regional geodetic co-ordinates of gravity stations which are unlikely to differ from geocentric values by more than 10 arcsec. Thus all values of normal gravity computed in a continental area covered by one of the regional datums (usually up to 5% of the Earth's total surface area) are subject to systematic errors not exceeding ±1/4 mgal.

The lack of a global datum for geopotential cannot cause errors much in excess of 1/2 mgal if the datum for elevations were based on at least one year's tide gauge readings, and as there is no evidence available at present which indicates that the magnitude of stationary sea surface topography is much in excess of ±2 m. While these magnitudes appear to be small, their effect on the evaluation of Stokes' integral is significant, being systematic in character.

Present day geoid computations from surface gravity data are therefore limited in effectiveness as a consequence of irregularly distributed data which could be subject to systematic errors due to the effect of inadequately defined datums on the data set used in the computations. The existence of such effects cannot be tolerated when the data is required for the determination of the geoid with the highest possible precision in studies of sea surface topography, whose magnitude is unlikely to exceed 2-3 m. The term sea surface topography refers to departures of the ocean surface from an equipotential surface of the Earth's gravitational field and is partially due to salinity, meteorological and tidal effects. The magnitude of the residual departures on allowing for these factors, and termed stationary effects, can only be estimated from manifestations along coastlines which have been obtained by comparing the results of geodetic levelling with tide gauge readings. Departures which cannot as yet be explained, have been reported in Australia (Hamon and Greig 1973) and the United States (Sturges 1972) with slopes approaching or in excess of 0.1 arcsec. On balancing existing satellite altimeter technology against the oceanographic requirements, it would appear that a ±10 cm resolution in the determination of the geoid is a desirable goal for this purpose (Williamstown Report 1969, 3-2).

The criteria governing the factors which constitute a "desirable" representation of the gravity field for the solution of the geodetic boundary value problem, is dependent on the requirements for the solution of Stokes' integral which, as discussed earlier, provides over 90% of the power in the representation. This would apply to any of the techniques of solution described in section 3. The following is a summary of a recent look at this problem (Mather 1973, p. 53 et seq.).

A suitable form of Stokes' integral for quadratures evaluation is

$$N_f^{(cm)} = K \sum_i n_i^2 \sum_j \mu_{ij}\, f(\psi_{ij})\, \Delta g_{ij}^{(mgal)}, \tag{100}$$

where Δg_{ij} is the value of the gravity anomaly representing a $n_i^0 \times n_i^0$ square,

$$K \doteq 1.58 \times 10^{-2}, \tag{101}$$

and $\mu_{ij} = \cos \phi_{cij}$ or $\sin \psi_{ij}$ depending on whether a latitude-longitude or azimuth-distance system of co-ordinates is used. Equation 100 would be adequate if the subdivision of the basic $n_i^0 \times n_i^0$ square into $N \left(= \left(\frac{n_i}{m}\right)^2 \right)$ $m^0 \times m^0$ squares ($m < n_i$), where the k-th such square will be represented by the gravity anomaly Δg_k at an angular distance ψ_k from the point of computation P such that

$$\Delta g_k = \overline{\Delta g} + c_{gk} \quad ; \quad F(\psi_k) = F(\overline{\psi}) + c_{\psi k}, \tag{102}$$

$\overline{\Delta g}$ and $F(\overline{\psi})$ being given by

$$\overline{\Delta g} = \frac{1}{N} \sum_{k=1}^{N} \Delta g_k \quad ; \quad F(\overline{\psi}) = \frac{1}{N} \sum_{k=1}^{N} F(\psi_k), \tag{103}$$

and the use of these smaller subdivisions in the quadratures evaluation in lieu of the $n_i^0 \times n_i^0$ and the appropriate area mean, did not reduce the quadrature error to below the desired order of accuracy ($o\{\epsilon\}$). This would happen if

$$\sum_{k=1}^{N} c_{gk}\, c_{\psi k} = o\{\epsilon\}, \tag{104}$$

implying no correlation whatever between variations in $f(\psi)$ and Δg over the $n_i^0 \times n_i^0$ area. While the function $F(\psi)$, given by

$$F(\psi) = f(\psi) \sin \psi, \tag{105}$$

has predictable variations, Δg defies accurate prediction except over very short distances and under carefully controlled conditions. As gravity has to be sampled at discrete points, the quadratures approach makes a representation procedure mandatory. Consequently, some finite element of surface area has to be represented by a single observation. It is useful to bear in mind that

(a) the global gravity standardization network available at present has a station accuracy of ±0.2 mgal (Morelli et al 1971, p.p. 6);

(b) errors in gravimeter ties seldom exceed ±0.2 mgal if performed with adequate instruments and any sort of minimal care; and

(c) geopotential errors of o {3 kgal m} give rise to a o {1 mgal} error in the gravity anomaly.

A precision of ±1 mgal in the gravity anomaly is relatively easy to obtain in areas where the regional geodetic level network is reasonably dense. The gravity anomaly also undergoes changes with position at different points in the basic square it is expected to represent. This penchant is characterized by a quantity introduced by de Graaff Hunter (1935), called the error of representation E $\{\Delta g_{mn}\}$ for an $n^0 \times m^0$ square, which in the case of a fully represented square, is given by

$$(E\{\Delta g\}_{nm})^2 = \sum_{i=1}^{N} \frac{(\Delta g_i - \overline{\Delta g})^2}{N}. \tag{106}$$

A reliable value for E $\{\Delta g_{nm}\}$ is obtained from N evenly spaced values of Δg_i covering the $n^0 \times m^0$ square, $\overline{\Delta g}$ being the mean value of the gravity anomaly, given by

$$\overline{\Delta g} = \frac{1}{N} \sum_{i=1}^{N} \Delta g_i . \tag{107}$$

Several estimates of this statistical characteristic of the gravity anomaly field at the surface of the Earth are available in the literature (e.g., ibid; Hirvonen 1956; Molodenskii et all 1962, p. 172; Mather 1967, p. 131). Samples which are available at the present time from different parts of the globe, reflect the flatter continental areas. E $\{\Delta g\}_n$ in such areas is a function of square size and, in general terms, can be expressed by the relations

$$E\{\Delta g\}_n = \begin{cases} \pm C_1 \sqrt{n} & \tfrac{1}{4}^\circ < n < 5^\circ \\ \pm C_2\, n & n < \tfrac{1}{4}^\circ \end{cases} \tag{108}$$

for an $n^0 \times n^0$ square, where n is in degrees and E $\{\Delta g\}_n$ in mgal, when $C_1 \doteqdot 12$ and $C_2 \doteqdot 3 \times 10$. It can also be shown that E $\{\Delta g\}_n$ is a function of unsigned groundslope $|\beta|$, with magnitudes which can be as much as 5 times as great in very rugged mountainous areas especially when n is small. As such variations are not a function of elevation but of ground slope, it is estimated that about 2-5% of the Earth's surface will require values of C_1 and C_2 which are significantly greater than these given above, for an adequate representation of variations in the gravity anomaly.

The number of terms involved in the quadratures evaluation is a function of the accuracy desired in the computation. If the requirements of sea surface topography determinations (1 part in 10^4) were to be met, it would be necessary, for estimation purposes, to restrict square sizes to those over which the contributions of the terms containing the second differential coefficient of $F(\psi)$ were held to $o\{e^3 h_d\}$. The required number of summations is $o\{10^6\}$. The study of the propagation of systematic and random error characteristics through Equation 100, under these conditions, shows that an adequate representation of the surface gravity field which would enable the achievement of an accuracy of ±10 cm in the final result would be one which had an $E\{\Delta g\}$ value of ±3 mgal, if the data were not subject to systematic error in excess of ±50 μgal. Such a representation is afforded by a 10 km grid in nonmountainous areas. While the estimation characteristics of gravitationally disturbed regions are covered by the above figures, which assume that oceanic fields will have a similar tendency to vary as continental data, regions characterized with larger ground slopes, have significantly greater values of $E\{\Delta g\}$. It would be necessary to reduce the size of the grid in such cases to retain $E\{\Delta g\}$ at ±3 mgal. The use of smoothening techniques described in section 3.5 would of course reduce these values. It follows that present day techniques for establishing surface gravity anomalies are adequate for the determination of sea surface topography. It is interesting to note that the station spacing required on the above basis, is already available over large continental areas, like the United States, Canada and Australia, at the present time.

The consequences of systematic errors in Δg which hold the same sign over considerable extents, have significant effects on computed values of h_d. A systematic error $e_{\Delta g}$ which holds its magnitude over a $n^0 \times n^0$ area but has random error characteristics over larger extents, is shown to have an effect e_{Ns} on the computed value of h_d given by (ibid, p. 65)

$$e_{Ns} = \pm o\{K'' n\, e_{\Delta g}\}, \tag{109}$$

where $K'' \doteqdot 10$, for e_{Ns} in cm, n in degrees and $e_{\Delta g}$ in mgal. If e_{Ns} were held at ±5 cm, the estimate of the magnitude of the permissible systematic error $e_{\Delta g}$, which is inversely proportional to its wavelength, varies from $o\{\pm 5 \text{ mgal}\}$ when $n = 0.1°$ to $o\{\pm 0.1 \text{ mgal}\}$ when $n = 5°$.

Likely sources of systematic error have been listed at the commencement of this subsection. The following conclusions can be drawn.

(1) IGSN71 would be an adequate gravity standardization net for sea surface topography studies only if the station density were 1 per 5×10^4 km^2 and the errors of adjacent stations were not correlated at the 200 μgal level. Neither of these conditions is likely to be satisfied. An adequate net would be afforded by stations at which absolute determinations had been carried out to ± 50 μgal resolution, and a representation of 1 station per 10^6 km^2. In the interim, it would be advisable that all gravity data should be subject to randomzation procedures at the level of the precision of the gravity standardization network, prior to use in solutions of the boundary value problem.

(2) Gravity anomaly information on each geodetic datum should be corrected for changes in normal gravity due to the datum not being geocentric (ibid, p. 16).

(3) The term "geoid" which is assumed to be synonymous with both the global datum for elevations as well as the "undisturbed" free level of the sea, should be defined on the basis of models which afford resolution with an accuracy of ± 10 cm.

A possible problem of some significance in the determination of sea surface topography and other high precision determinations of h_d, is the existence of the sea surface topography itself with not insignificant amplitudes (e.g., 3-4 m) and substantial wavelengths. The evidence for the existence of such phenomena is widespread but based on purely coastal phenomena, as obtained from levelling-tide gauge comparisons. Extended studies of the sea surface using short pulse high resolution satellite altimeters should go a long way toward clarifying whether stationary sea surface topography is merely a coastal phenomenon, and if not, the dominant wavelengths with which it is prone to occur. The existence of stationary sea surface topography with 4000 km wavelengths and 2 m amplitudes would cause errors of $o\,\{\pm 1\ \mathrm{m}\}$ in h_d. While this estimate is based on the largest estimate of the phenomenon presently available, the existence of such an effect will require an iteration of the determinations of h_d. The rapidity with which these iterations converge are more a function of the wavelength of the stationary sea surface topography than of its amplitude.

5. GRAVITY AND EARTH SPACE

5.1 GRAVITY AND SCALE

A problem which requires careful scrutiny is the possibility or otherwise of defining a scale for Earth space from gravity determinations at the surface of the Earth. It must be clearly emphasized that the ensuing development excludes

6

consideration of satellite data which constitutes the basis of low degree representations of the Earth's gravity field at the present time. The problem could be stated as follows. Given an adequate distribution of determinations of surface gravity, how are effects of zero degree h_{d0} in the global distribution of height anomalies to be interpreted. This effect can be written as

$$h_{d0} = \frac{W_0 - U_0}{\gamma} - R\,\frac{M\{\Delta g_c\}}{\gamma} + N_{c0} + o\{f\,h_{d0}\}. \tag{110}$$

on considering Equations 73 and 74, N_{c0} being the contribution of zero degree by the indirect effect N_c. W_0 is not known and it is common practice to assume the first term to be zero. The second and third terms will have finite magnitudes which could be made equal to zero by changing the value of GM, and hence $M\{\Delta g_c\}$. This would, of course change the value of U_0, implying an equivalent change in the estimate of W_0 if h_{d0} is to remain zero. h_{d0} could only be forced to take zero value if this is justified by some external condition.

A more realistic procedure is to establish the numerical value of h_{d0} by analyzing the differences

$$v_i = d_{di} + h_n - h_{0i}, \tag{111}$$

where h_{0i} is the ellipsoidal elevation based on the same reference ellipsoid as used in the gravimetric determination, but from independent observations, e.g., geocentric satellite solutions. The adoption of a value for GM used in defining the system of reference will in turn define a value for W_0, on extracting a zero degree residual from 111 and using it in Equation 110.

Any "improvement" in the value of GM obtained from gravimetric determinations is based on the assumption that the potential of the geoid W_0 is equal to that on the ellipsoid of reference U_0. The geoid is a physical reality, being a manifestation of the mass distribution which gives the observed phenomena at the surface of the Earth, while U_0 is defined by the values chosen for each of the parameters a, GM, ω and f defining the system of reference. It has been deduced that the term $(W_0 - U_0)/\gamma$ is approximately 3 m if the ellipsoid were one of best fit to the geoid and GM were the best estimate available for the Earth (Mather 1971c, p. 98), provided the free air anomaly had no zero degree harmonic.

Thus any deductions which can be drawn about scale from gravimetric determinations alone are subject to ambiguity, if restricted to a single epoch. It would be prudent to refrain from distorting the physical characteristic of the Earth represented by the value of G M, as determined with reliability, by independent means. The effect of zero degree deduced from comparisons described by Equation 111, can preferably be used in Equation 110 to deduce the true value of W_0. The effect of zero degree is therefore fully accounted for in Equation 74, and values of h_d will no longer give rise to effects of zero degree in Equation 111.

A second effect of importance is the term of zero degree obtained on studying changes in observed gravity determined by the use of absolute techniques to resolutions approaching ±1 μgal, as determined on specially designed observing platforms, well distributed around the Earth, between successive epochs in time. Such changes can be interpreted as either reflecting an expansion of the Earth, as measured within the framework of the velocity of light, or else a change in the value of G M. For a discussion see (Mather 1972, p. 15).

5.2 GRAVITY AND GEODETIC REFERENCE SYSTEMS

The preceding development has assumed that the Earth has a fixed mass distribution subject to some periodic changes due to effects like tides. Such a description would only be adequate if observations referred to a limited period of time, such as a few decades. There is considerable evidence which appears to point to the large scale redistribution of at least the masses constituting the Earth's crust over very long periods of time, with the attendant possibility of mass variations at greater depth depending on the nature of the mechanism which could give rise to such crustal motions.

A possible consequence of such mass redistributions could be a motion of the Earth's center of mass (geocenter) with respect to the Earth's crust. The analysis of high precision determinations of absolute gravity at a well distributed net of observing platforms as described in the previous subsection, could provide a means of recovering the motion of the geocenter between epochs on analyzing the first degree harmonic of changes in absolute g (ibid 1972, p. 15). It should be pointed out that a problem in filtering out short period effects due to meteorological causes has to be overcome before results of reliability are likely to be obtained. Fortunately, an estimate of the same effect can be obtained on studying changes in geocentric position of a global network of laser tracking stations, using dynamic techniques, to provide a verification of the effectiveness of the determination.

5.3 THE ROLE OF GRAVIMETRIC METHODS IN EARTH AND OCEAN PHYSICS

Until recently, it was generally held that gravimetric methods if used with adequate data, provided the only non-controversial technique for computing ellipsoidal elevations with the same resolution as that available from geodetic levelling, thus completing the definition of geocentric position of points on the Earth's surface in three dimensions. Position determination at the present time has not provided resolutions which can confidently be claimed to be better than 1 part in 10^6.

It is now clear that the most precise determination of geocentric position is required primarily for studies in Earth and ocean physics, rather than for any direct engineering or technological purpose. It would not be exaggeration to state that resolution to 1 part in 10^8 would be the aim of geodetic techniques being currently developed for such schemes. While there is no clear indication that surface methods, subject to restrictions imposed by atmospheric uncertainties, can be improved to meet these goals, extra-terrestrial techniques, like laser ranging to near Earth satellites and VLBI, promise that such goals may well be attained in the foreseeable future. There also is no reason to doubt at this stage, that transportable versions of these systems could not achieve this same degree of resolution.

It would therefore appear that, with the passage of time, there would be less use of geodetic levelling and the systems of reference implicit in its concept, for use in Earth physics. The exception is of course the study of the instantaneous geocentric position of the ocean surface and the interpretation of these results for the study of ocean circulation. The determination of the geoid with the highest possible precision are a necessary prerequisite for such studies. Gravity information will still have to be assembled and anomalies computed on the basis of elevations referred to an equipotential surface, the most convenient being the geoid.

Three matters of significance which should be closely studied before undertaking the task of assembling an adequate gravity anomaly field for computation of geoid heights to ±10 cm, are the following.

(a) The definition of a physical model to serve as a datum for elevations with an accuracy which is not more than a factor of 3 less the highest precision sought in the geoid solution.

(b) Techniques to be used for minimizing the effect of gravity base station errors on geoid computations.

(c) The question of whether it is necessary to adopt a model for the "surface of measurement" and, if so, the nature of an acceptable model and the procedure to be adopted in converting measurements on the Earth's surface to equivalent quantities on the model.

6. ACKNOWLEDGMENTS

This paper was written while the author was the holder of a National Academy of Sciences Resident Research Associateship at National Aeronautics and Space Administration's Goddard Space Flight Center, Greenbelt, Maryland, while on leave of absence from the University of New South Wales, Sydney, Australia.

7. REFERENCES

Arnold, K. 1959. Zur Bestimmung der Geoidundulationen aus Freiluftanomalien. Report 12, Geodatischen Instituts in Potsdam.

Bjerhammar, A. 1964. A New Theory of Geodetic Gravity, Trans. R. Inst. Techn. Nr 243, 75 pp, Stockholm.

Bjerhammar, A. 1968. On Gravity. Royal Inst. of Techn., Division of Geodesy, Stockholm, 129 pp.

Bjerhammar, A. 1969. On the Boundary Value Problem of Physical Geodesy, Tellus, 21, 451-516.

Brovar, V. V. 1964. On the Solution of Molodensky's Boundary Value Problem, Bull. geodes., 72, 167-173.

Bursa, M. 1972. Variations of the Earth's Gravity Field due to the Free Nutation, Studia geophys. et geod., 16, 122-125.

Cook, A. H. 1950. The Calculation of Deflexions of the Vertical from Gravity Anomalies, Proc. R. Soc. A, 204, 374-395.

Cook, A. H. 1965. A New Absolute Determination of g, Nature, 208, 5007, 279.

de Graaff-Hunter, J. 1935. On the Figure of the Earth from Gravity Observations etc., Phil. Trans. R. Soc. 234, 377-431.

de Graaff-Hunter, J. 1958. Reduction of Observed Gravity, Report of Study Group 8, Bull. geodes., 50, 1-16.

de Witte, L. 1967. Truncation Errors in the Stokes and Vening Meinesz Formulas for Different Order Spherical Harmonic Gravity Terms, Geophys. J. R. astr. Soc., 12, 449-464.

Esposito, P. B. and Wong, S. K. 1972. Geocentric Gravitational Constant Determined from Mariner 9 Radio Tracking Data, Microfilm Proceedings, International Symposium on Earth Gravity Models and Related Problems, St. Louis, (Abstr) EOS Trans. Am. Geophys. U., 53, 891.

Faller, J. E. 1965. An Absolute Interferometric Determination of the Acceleration of Gravity, Bull. geodes., 77, 203-4.

Hamon, B. V. and Greig, M. A. 1973. Mean Sea Level in Relation to Geodetic Land Levelling Around Australia (Submitted to J. geophys. Res.)

Heiskanen, W. A. and Mortiz, H. 1967. Physical Geodesy, Freeman, San Francisco.

Hendershott, M. C. 1972. The Effect of Solid Earth Deformation on Global Ocean Tides, Geophys. J. R. astr. Soc., 29, 389-402.

Hirvonen, R. A. 1956. On the Precision of the Gravimetric Determination of the Geoid, Trans. Am. geophys. U., 37, 1-8.

Hotine, M. 1969. Mathematical Geodesy, ESSA Monograph 2, 416 pp.

Hagiwara, Y. 1972. Truncation Error Formulas for the Geoidal Height and the Deflection of the Vertical, Bull. geodes., 106, 453-466.

IAG 1970. Geodetic Reference System 1967. Spec. Publ., International Association of Geodesy, Paris.

Koch, K.-R. 1965. Der Einfluss der Topographie auf die Forme von Stokes, Z. f. Vermess Wes., 90, 356-361.

Krarup, T. 1969. A Contribution to the Mathematical Foundation of Physical Geodesy, Rep 44, Danish Geodetic Institute, Copenhagen.

Krarup, T. 1972. Letters on Molodenskiy's Problem, Unpublished circulation to Members of Study Group 4:31, International Association of Geodesy.

Lerch, F. J., Wagner, C. A., Smith, D. E., Sandson, M. L., Brownd, J. E. and Richardson, J. A. 1972. Gravitational Field Models for the Earth (GEM 1 & 2) (GEM 3 & 4), GSFC X-553-72-146.

Marych, M. J. 1969. On the Second Approximation of Molodenskii for the Perturbing Potential (in Russian), Geodezija, Kartografija i Aerofotos'jemka, 10, 17-27, L'vov.

Mather, R. S. 1967. The Extension of the Gravity Field in South Australia, Oster. Z. F. Vermess Wes, 25, 126-138.

Mather, R. S. 1971a. The Analysis of the Earth's Gravity Field, Monograph 2, Sch. of Surveying, Univ. of NSW, Kensington NSW. Australia, 172 pp.

Mather, R. S. 1971b. The Geocentric Orientation Vector for the Australian Geodetic Datum, Geophys. J. R. astr. Soc., 22, 55-81.

Mather, R. S. 1971c. A World Geodetic System from Gravimetry, Geophys. J. R. astr. Soc., 23, 75-100.

Mather, R. S. 1972. Four Dimensional Studies in Earth Space, GSFC X-553-72-230, 25 pp., (Abstr). EOS Trans. Am. geophys. U., 53, 896, Bull. geodes. (in press)

Mather, R. S. 1973. A Solution of the Geodetic Boundary Value Problem to Order e^3, GSFC-S-592-73-11, 126 pp., (Abstr) EOS Trans. Am. geophys. U., 54, 229.

Mather, R. S., Barlow, B. C., and Fryer, J. G. 1971. A Study of the Earth's Gravitational Field in the Australian Region. XV General Assembly, IAG, Moscow, 1971. (in UNISURV Rep. 22, 1-41, Univ. of New South Wales, Kensington NSW Australia)

Melchior, P. 1966. The Earth Tides. Pergamon, New York.

Molodenskii, M. S., Eremeev, V. F. and Yurkina, M. I. 1962. Methods for Study of the External Gravitational Field and Figure of the Earth. Israel Program for Scientific Translations, Jerusalem.

Morelli, C., Gantar, C., Honkasalo, T., McConnell, R. K., Szabo, B., Tanner, J. G., Uotila, U. A., and Whalen, C. T. 1971. The International Gravity Standardization Net 1971. XV General Assembly, IAG, Moscow, 1971

Moritz, H. 1965. The Boundary Value Problem of Physical Geodesy, Suomal. Tiedeakat. Toim A III, 83.

Moritz, H. 1966. Linear Solutions of the Geodetic Boundary Value Problem, Report 79, Dept. of Geodetic Science, Ohio State Univ., Columbus, Ohio, 135 pp.

Moritz, H. 1969. Monlinear Solutions of the Geodetic Boundary Value Problem, Report 126, Op. cit. supra, 56 pp.

Moritz, H. 1970. Molodensky's Series and Analytical Continuation, Report 145, Op. cit. supra, 35 pp.

Moritz, H. 1972. Convergence of Molodensky's Series, Report 183, Op. cit. supra, 50 pp.

Ojengbede, O. A. 1973. The Use of Truncation Functions in the Gravimetric Solution of the Geoid. Dissertation, Sch. of Surveying, Univ. of NSW, Kensington NSW Australia, 109 pp.

Pellinen, L. P. 1972. On the Identity of Various Solutions of the Molodensky's Problem with help of a Small Parameter κ. Microfilm Proceedings, International Symposium on Earth Gravity Models and Related Problems, St. Louis, (Abstr) EOS Trans Am. geophys. U., 53, 894.

Sakuma, A. 1971. Une Tendance de la Variation de la Pesanteur Observee au B.I.P.M., Sevres France., XV General Assembly, IAG, Moscow 1971.

Stokes, G. G. 1849. On the Variation of Gravity at the Surface of the Earth, Trans. Camb. Phil. Soc., Math. Phys. 2, 131-171.

Sturges, W. 1972. Comments on Ocean Circulation with Regard to Satellite Altimetry (in Sea Surface Topography from Space (J. R. Apel (ed.)), Vol. 2, 24 (1-17) NOAA Tech. Rep. ERL 228-AOML 7-2).

Vening Meinesz, F. A. 1928. A Formula for Expressing the Deflection of the Plumb Line in the Gravity Anomalies and some Formulae for the Gravity Field and the Gravity-Potential outside the Geoid, Koninkl. Akad. van Wetenschap. (Science Sec.) 31, 315-331, Amsterdam.

Williamstown Report 1969. Solid Earth and Ocean Physics, MIT Measuring Systems Lab., Cambridge Mass.

Yeremeev, V. F. 1969. On the Numerical Solution of the Molodensky's Integral Equation for Layer Density, TsNIIGAiK, 176, 16-23.