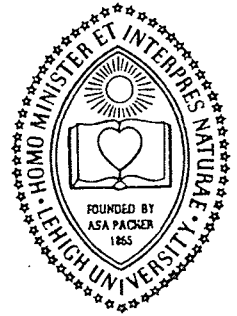


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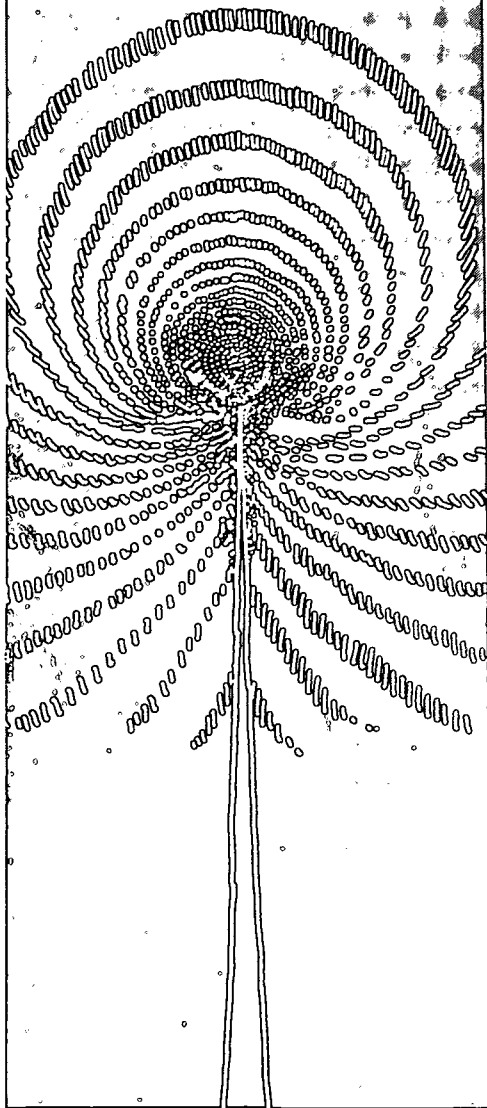
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CONTACT PROBLEM FOR AN ELASTIC REINFORCEMENT
BONDED TO AN ELASTIC PLATE*

by

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ABSTRACT

The contact problem for a thin elastic reinforcement bonded to an elastic plate is considered. The stiffening layer is treated as an elastic membrane and the base plate is assumed to be an elastic continuum. The bonding between the two materials is assumed to be either one of direct adhesion or through a thin adhesive layer which is treated as a shear spring. The solution for the simple case in which both the stiffener and the base plate are treated as membranes is also given. The contact stress is obtained for a series of numerical examples. In the direct adhesion case the contact stress becomes infinite at the stiffener ends with a typical square root singularity for the continuum model and behaving as a delta function for the membrane model. In the case of bonding through an adhesive layer the contact stress becomes finite and continuous along the entire contact area.

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1. INTRODUCTION

The purpose of this paper is to investigate the distribution of the adhesive shear stress in elastic plates reinforced by an orthotropic elastic layer. The generalized plane stress analysis of lap joints shows that if the stress variation in thickness direction is not taken into account and if the Poisson's ratios of the two bonded plates are equal, then the load transfer from one plate to the other takes place along the edges of the bonded region only [1,2]. This means that the contact shear between the layers is zero everywhere except along the line boundary of the contact area where its magnitude is infinite. Even when the two Poisson's ratios are different, the magnitude of the contact shear in the contact area away from immediate neighborhood of the boundary is found to be extremely small [2]. This result is rather disturbing because of the fact that in reinforcing the plates with unidirectionally strong layers or straps the load transfer is expected to take place through the adhesive bond and the extremely large adhesive shear along the edge of the bonded region means that the debonding would take place under relatively low values of the external loads.

In practice the thickness of the reinforcing layer is generally very small compared to the thickness of the base plate and the in-plane dimensions of the layered medium. Hence in analyzing the problem the reinforcing layer may be treated as a "membrane". On the other hand for the base plate, because of its relatively large thickness, a similar assumption neglecting

the thickness variation of stresses may not be justified and the base plate may have to be treated as an elastic continuum. Also since the nature of the bonding between the layer and the base plate is expected to be an important factor influencing the behavior of the contact stresses, it would be useful to solve the problem under different contact conditions.

In this paper we will consider the problem of an elastic plate reinforced by an orthotropic layer under the assumptions that, (a) the stress state in the layered medium is independent of the z coordinate* (see the insert in Figure 1), (b) the bending stiffness of the layer 2 is negligible, (c) the base plate 1 is an elastic continuum, (d) the only external load acting on the medium is the uniaxial tension $\sigma_{1x}^{\infty} = \sigma_0$ away from the reinforcement region, and (e) the bonding between the two materials may be accomplished through either a direct adhesion (insert in Figure 1) or an adhesive layer of finite thickness h_3 (insert in Figure 3). The solution for the special case in which the base plate is also treated as a membrane will also be given and results will be compared with that of the continuum solution. The solution for the limiting case of this problem for $h_1 = \infty$ and $h_3 = 0$ is given in [3-8].

* This means that the dimension of the composite medium in z -direction is either very small or very large compared to the x -dimension $2a$ of the stiffener. In the former case $(\sigma_z)_{ave.} = 0$. In the latter $\epsilon_{1z} = \epsilon_{2z} = \epsilon_0$, $\epsilon_0 \approx -\nu_1 \sigma_0 / E_1$ if the plate is under uniaxial tension $\sigma_{1x} = \sigma_0$ away from the stiffener and $\epsilon_0 \approx 0$ if the plate is pulled through fixed grips sufficiently close to the stiffener.

2. THE CASE OF DIRECT ADHESION

Consider an isotropic elastic plate of thickness h_1 stiffened by an orthotropic layer of thickness h_2 and width $2a$ (insert in Figure 1). Let the base plate be subjected to a uniaxial tension $\sigma_{1x}^\infty = \sigma_0$ away from the stiffening region. Assuming that the dimensions of the plate in x and z directions are sufficiently large, the strains in z -direction may be expressed as

$$\epsilon_{1z} = \epsilon_{2z} \cong - \frac{\nu_1 \sigma_0}{E_1}, \quad (1)$$

where E_1 and ν_1 are the elastic constants of the base plate. Also, for this case it may be assumed that the stresses in the composite medium are independent of the z coordinate. Defining the contact stress at $y = 0$ by

$$\sigma_{1xy}(x, 0) = -p(x) \quad (2)$$

and treating the stiffening layer as a membrane, the equilibrium of the layer and the condition of symmetry give

$$h_2 \sigma_{2x}(x) = \int_x^a p(t) dt, \quad \int_{-a}^a p(t) dt = 0, \quad p(x) = -p(-x) \quad (3.a-c)$$

Using (1) and the stress-strain relations

$$\begin{aligned} \epsilon_{2x} &= \frac{\partial u_2}{\partial x} = \frac{1}{E_{2x}} (\sigma_{2x} - \nu_{2x} \sigma_{2z}), \\ \epsilon_{2z} &= - \frac{\nu_1 \sigma_0}{E_1} = \frac{1}{E_{2z}} (\sigma_{2z} - \nu_{2z} \sigma_{2x}), \end{aligned} \quad (4.a, b)$$

from (3.a) we obtain

$$\frac{\partial u_2}{\partial x} = \frac{1 - \nu_{2x} \nu_{2z}}{h_2 E_{2x}} \int_x^a p(t) dt + \frac{\nu_1 \nu_{2z} \sigma_0}{E_1}, \quad (|x| < a) \quad (5)$$

For the base plate solving the related field equations under the boundary conditions*

$$\begin{aligned}\sigma_{1y}(x,0) &= 0, & \sigma_{1xy}(x,0) &= \begin{cases} -p(x), & |x| < a \\ 0, & |x| > a \end{cases}, \\ \sigma_{1y}(x,-h_1) &= 0, & \sigma_{1xy}(x,-h_1) &= 0\end{aligned}\quad (6)$$

and adding a homogeneous solution due to σ_0 , for the displacement derivative at $y = 0$ we find

$$\frac{\partial}{\partial x} u_1(x,0) = \frac{\sigma_0}{E_1} - \frac{1-\nu_1}{\pi \mu_1} \int_{-a}^a \left[\frac{1}{t-x} - k(x,t) \right] p(t) dt \quad (7)$$

where the bounded function $k(x,t)$ is given by

$$k(x,t) = \int_0^\infty \frac{4\alpha^2 h_1^2 + 2 - 4\alpha h_1 - 2e^{-2h_1\alpha}}{4\alpha^2 h_1^2 + 2 - e^{-2\alpha h_1} - e^{2\alpha h_1}} \sin\alpha(t-x) d\alpha \quad (8)$$

Now, if the bonding along the interface $y = 0$ is one of direct adhesion, using (5) and (7) from the continuity condition $u_2(x) = u_1(x,0)$, $(-a < x < a)$ we obtain the following singular integral equation to determine the unknown function $p(x)$:

$$\begin{aligned}\frac{1}{\pi} \int_{-a}^a \frac{p(t)}{t-x} dt - \frac{1}{\pi} \int_{-a}^a k(x,t) p(t) dt + \frac{\lambda}{a} \int_x^a p(t) dt \\ = \frac{1 - \nu_1 \nu_{2z}}{2(1 - \nu_1^2)} \sigma_0, \quad (-a < x < a),\end{aligned}\quad (9)$$

where $k(x,t)$ is given by (8) and

$$\lambda = \frac{a}{h_2} \frac{\mu_1(1 - \nu_{2x} \nu_{2z})}{E_{2x}(1 - \nu_1)} \quad (10)$$

At the end points $x = \pm a$, $p(x)$ must have integrable

*Using, for example, a technique similar to that outlined in [9].

singularities. Hence, referring to [10], it may be shown that the index of the integral equation is $\kappa = +1$ and the solution contains an arbitrary constant which is determined from the equilibrium condition (3.b).

Let us now assume that h_1/a too is sufficiently small so that the base plate can also be treated as an elastic membrane. In this case, in addition to (1), (3) and (5), from the equilibrium of the base plate and from the stress-strain relations we obtain

$$\sigma_{1x} = \sigma_0 - \frac{1}{h_1} \int_x^a p(t) dt ,$$

$$\frac{\partial u_1}{\partial x} = \frac{1-\nu_1^2}{E_1} \sigma_{1x} + \frac{\nu_1^2 \sigma_0}{E_1} , \quad (|x| < a) \quad (11.a,b)$$

Substituting from (5) and (11), the continuity condition

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} , \quad (-a < x < a), \text{ gives}$$

$$B - A \int_x^a p(t) dt = 0 ,$$

$$B = \frac{1 - \nu_1 \nu_{2z}}{E_1} \sigma_0 , \quad A = \frac{1 - \nu_1^2}{h_1 E_1} + \frac{1 - \nu_{2x} \nu_{2z}}{h_2 E_{2x}} . \quad (12.a-c)$$

From (12) and (3.b,c) it is easily seen that

$$p(x) = \frac{B}{A} [\delta(x-a) - \delta(x+a)] , \quad (13)$$

that is, $p(x)$ is zero everywhere except at $x = \pm a$ where it is infinite. For this simple model, the stresses in the stiffener and in the base plate are found to be

$$\sigma_{2x}(x) = \frac{B}{Ah_2}, \quad (|x| < a),$$

$$\sigma_{1x}(x) = \begin{cases} \sigma_0, & (|x| > a), \\ \sigma_0 - \frac{B}{Ah_1}, & (|x| < a). \end{cases} \quad (14.a,b)$$

3. ADHESION THROUGH AN ELASTIC SHEAR LAYER

If the bonding between the base plate and the stiffener is accomplished by using an adhesive material with elastic properties different than that of the plate and the stiffener, the solution given in the previous section will not be valid. In this section the effect of the adhesive layer will be taken into account by assuming that due to its relatively very small thickness the adhesive may be treated as a "shear spring". That is, referring to the insert in Figure 3 if $u_2(x)$ and $u_1(x,y)$ are the x-components of the displacement vectors in the stiffener and in the base plate, respectively, and if h_3 is the thickness and μ_3 is the shear modulus of the adhesive, the continuity condition along the interface may be expressed by

$$u_1(x,0) - u_2(x) = \frac{h_3}{\mu_3} p(x) \quad (15)$$

where $p(x)$ is the contact shear defined by (2), and the stiffener is again treated as a membrane. Noting that $u_2(0) = 0$, from (5) we obtain

$$\begin{aligned} u_2(x) &= \frac{\nu_1 \nu_{2z} \sigma_0}{E_1} x + \frac{1 - \nu_{2x} \nu_{2z}}{h_2 E_{2x}} \int_0^x ds \int_s^a p(t) dt \\ &= \frac{\nu_1 \nu_{2z}}{E_1} \sigma_0 x + \frac{1 - \nu_{2x} \nu_{2z}}{h_2 E_{2x}} p x - \frac{1 - \nu_{2x} \nu_{2z}}{h_2 E_{2x}} \int_0^x (x-t) p(t) dt, \end{aligned} \quad (16)$$

where P is a constant defined by

$$P = \int_0^a p(t) dt . \quad (17)$$

Similarly, expressing (7) in the range $(0, a)$ and integrating, we find

$$u_1(x, 0) = \frac{\sigma_0}{E_1} x - \frac{1-\nu_1}{\pi\mu_1} \int_0^a [\log|t+x| - \log|t-x| - k_1(x, t)] p(t) dt , \quad (0 < x < a) , \quad (18)$$

where

$$k_1(x, t) = \int_0^\infty \frac{4\alpha^2 h_1^2 + 2 - 4\alpha h_1 - 2e^{-2\alpha h_1}}{\alpha(4\alpha^2 h_1^2 + 2 - e^{-2\alpha h_1} - e^{2\alpha h_1})} [\cos\alpha(t-x) - \cos\alpha(t+x)] d\alpha . \quad (19)$$

Substituting now from (16) and (18) into (15) we obtain the following integral equation to determine the unknown function $p(x)$:

$$\begin{aligned} \frac{1}{\pi} \int_0^a [\log|\frac{t+x}{t-x}| - k_1(x, t)] p(t) dt - \frac{\lambda}{a} \int_0^x (x-t) p(t) dt + \frac{\mu_1 h_3}{\mu_3(1-\nu_1)} p(x) \\ = \left[\frac{1-\nu_1\nu_2z}{2(1-\nu_1^2)} \sigma_0 - \frac{P\lambda}{a} \right] x , \quad (0 < x < a) \end{aligned} \quad (20)$$

where the constants λ and P , and the kernel $k_1(x, t)$ are given by (10), (17), and (19), respectively.

If h_1/a is sufficiently small for this problem too a very simple solution may be obtained by treating the base plate as well as the stiffening layer as a membrane. In this case the equations (3), (5) and (11) remain valid. Thus, differentiating

the continuity condition (15) and substituting from (5) and (11), we obtain the following simple integrodifferential equation to determine the unknown function $p(x)$:

$$B - A \int_x^a p(t) dt = \frac{h_3}{\mu_3} \frac{d}{dx} p(x), \quad (0 < x < a) \quad (21)$$

where B and A are given by (12.b,c). Solving (21) under the condition that $p(0) = 0$, we obtain

$$p(x) = \frac{\beta}{\alpha} \frac{\text{sh} \alpha x}{\text{ch} \alpha a}, \quad \alpha = \left(\frac{A \mu_3}{h_3} \right)^{\frac{1}{2}}, \quad \beta = \frac{B \mu_3}{h_3}. \quad (22.a-c)$$

It may be seen that (22) reduces to (13) as $h_3 \rightarrow 0$. To do this note that for small values of h_3 we have

$$\frac{\beta}{\alpha} = O(h_3^{-\frac{1}{2}}), \quad \alpha = O(h_3^{-\frac{1}{2}}), \quad \frac{\text{sh} \alpha x}{\text{ch} \alpha a} = O(e^{-\alpha(a-x)}). \quad (23)$$

Thus, since $\lim_{z \rightarrow \infty} z^{-\frac{1}{2}} e^{-\epsilon z} \rightarrow 0$ as $z \rightarrow \infty$ for any $\epsilon > 0$, from (22) we obtain

$$\lim_{h_3 \rightarrow 0} p(x) = \begin{cases} 0, & (0 \leq x < a), \\ \infty, & (x = a), \end{cases}$$

$$\lim_{h_3 \rightarrow 0} \int_0^a p(x) dx = \frac{B}{A}, \quad (24.a,b)$$

which is the result found in the previous section.

4. SOLUTION OF THE INTEGRAL EQUATIONS AND NUMERICAL RESULTS

Referring to [10], the solution of the singular integral equation (9) is of the form

$$p(x) = f(x)(a^2 - x^2)^{-\frac{1}{2}}, \quad (25)$$

where $f(x)$ is a bounded odd function. The integral equation is solved by using the technique described in [11]. The major

numerical problem here is the evaluation of the kernel $k(x,t)$ given by (8) for which a modification of Filon's integration formula given in [12] is used. In this problem the physical parameters which may be varied are the thickness ratio h_1/a for the base plate and the dimensionless constant λ defined by (10). In the numerical examples given in this section the following material constants are used:

Base plate: Aluminum ($E_1 = 10^7$ psi, $\nu_1 = 0.3$),

Stiffener: Boron-Epoxy Composite ($E_{2x} = 32.4 \times 10^7$ psi,
 $\nu_{2x} = 0.23$, $E_{2z} = 3.5 \times 10^6$ psi, $\nu_{2z} = 0.025$,
 $\mu_2 = 1.23 \times 10^6$ psi),

Adhesive: Epoxy ($\mu_3 = 1.65 \times 10^5$ psi, $\nu_3 = 0.35$).

For a fixed value of $h_1/a = 0.25$ the contact stress $p(x)$ obtained from (9) for various values of λ is shown in Table 1. Similar results for a fixed value of $\lambda = 0.5$ and various values of h_1/a are shown in Figure 1. At the end points $x = \pm a$ the contact stress has a conventional square root singularity the strength of which may be characterized by a constant defined by (see (25))

$$K = \lim_{x \rightarrow a} \sqrt{2(a-x)} p(x) = f(a)/\sqrt{a} . \quad (26)$$

Figure 2 shows the variation of the constant K with h_1/a and λ . The values of K corresponding to the results given in Table 1 are shown in Table 2 where

$$K_0 = \frac{1 - \nu_1 \nu_{2z}}{2(1 - \nu_1^2)} \sigma_0 \sqrt{a}$$

Table 1. Distribution of contact stress for a stiffener directly bonded to an elastic plate (Materials: 1 Aluminum, 2 Boron-Epoxy composite, $h_1/a = 0.25$)

t_i/a	$\frac{2(1-\nu_1^2)}{1-\nu_1\nu_{2z}} \frac{p(t_i)}{\sigma_0}$		
	$\lambda = 5$	$\lambda = 2$	$\lambda = 1$
T= 9.9862953E-01	4.1003254E+00	4.7358320E+00	5.0147577E+00
T= 9.8768834E-01	1.1602731E+00	1.4171615E+00	1.5312509E+00
T= 9.6592583E-01	5.3547896E-01	7.0067146E-01	7.7660112E-01
T= 9.3358043E-01	2.6658982E-01	3.7500242E-01	4.2715300E-01
T= 8.9100652E-01	1.2799551E-01	1.9403152E-01	2.2748018E-01
T= 8.3867057E-01	5.4958676E-02	9.0503116E-02	1.0958186E-01
T= 7.7714596E-01	1.9528145E-02	3.5737633E-02	4.5058720E-02
T= 7.0710678E-01	5.1368259E-03	1.1123089E-02	1.4889820E-02
T= 6.2932039E-01	7.3009177E-04	2.4095920E-03	3.6196458E-03
T= 5.4463904E-01	-9.8345780E-05	2.0800832E-04	4.9638807E-04
T= 4.5399050E-01	-9.4163598E-05	-8.6114432E-05	-4.5557425E-05
T= 3.5836795E-01	-2.9878907E-05	-4.7722433E-05	-5.0250121E-05
T= 2.5881905E-01	-5.4920568E-06	-1.3046450E-05	-1.6648501E-05
T= 1.5643447E-01	-4.0977364E-07	-2.2574134E-06	-3.5199361E-06
T= 5.2335956E-02	6.0685120E-08	-2.0693287E-07	-4.5071192E-07
t_i/a	$\lambda = 0.5$	$\lambda = 1/3$	$\lambda = 0.2$
T= 9.9862953E-01	5.1721770E+00	5.2277471E+00	5.2734079E+00
T= 9.8768834E-01	1.5959247E+00	1.6187991E+00	1.6376110E+00
T= 9.6592583E-01	8.2024154E-01	8.3577334E-01	8.4858310E-01
T= 9.3358043E-01	4.5769981E-01	4.6866637E-01	4.7774724E-01
T= 8.9100652E-01	2.4751027E-01	2.5477516E-01	2.6081943E-01
T= 8.3867057E-01	1.2129973E-01	1.2560020E-01	1.2919782E-01
T= 7.7714596E-01	5.0959695E-02	5.3156164E-02	5.5005804E-02
T= 7.0710678E-01	1.7369433E-02	1.8309233E-02	1.9107330E-02
T= 6.2932039E-01	4.4618109E-03	4.7891450E-03	5.0703810E-03
T= 5.4463904E-01	7.1678844E-04	8.0595704E-04	8.8397127E-04
T= 4.5399050E-01	-6.3512786E-06	1.0913307E-05	2.6573531E-05
T= 3.5836795E-01	-4.8336798E-05	-4.6879506E-05	-4.5328367E-05
T= 2.5881905E-01	-1.8338700E-05	-1.8822525E-05	-1.9162521E-05
T= 1.5643447E-01	-4.3008397E-06	-4.5782013E-06	-4.8039158E-06
T= 5.2335956E-02	-6.2258680E-07	-6.8811183E-07	-7.4347031E-07

Table 2. Strength of stress singularity K
corresponding to Table 1 ($\frac{h_1}{a} = 0.25$)

λ	5	2	1	0.5	1/3	0.2
$\frac{K}{K_0}$	0.2187	0.2511	0.2653	0.2733	0.2762	0.2785

For the stiffened plate problem in which a separate adhesive layer is used to join the stiffener to the base plate the numerical results obtained from the solution of (20) are shown in Figures 3 and 4. Here the kernel of the integral equation is square integrable. Hence the equation is that of a Fredholm integral equation of the second kind. Consequently, its solution $p(x)$ will be bounded and continuous in the closed interval $0 \leq x \leq a$. Because of the logarithmic term in the kernel, from the integral equation (20), it can be shown that at the end point $x = a$ dp/dx will be infinite. Figure 3 shows the effect of h_1/a on $p(x)$ for a constant $\lambda = 5$ (which, for the materials under consideration, corresponds to $h_2/a = 0.0336$) and $h_3/a = 0.004$. For $h_1/a = 0.25$ the effect of λ is seen in Figure 4. The figures indicate that there is a severe stress concentration at the end point $x = a$ and for relatively small h_1/a , because of the "bending" of the base plate, there is a sign reversal in the shear stress $p(x)$ acting through the adhesive. The same stress reversal effect (in a smaller degree) is also observed in the direct adhesion (i.e., $h_3 = 0$) case shown in Table 1.

Figure 5 shows the comparison of the results obtained by

using three different models. Curve (a) corresponds to the direct adhesion ($h_3 = 0$) problem in which the stiffener is treated as a membrane and the base plate is an elastic continuum (equation (9)). Curve (b) contains the effect of the adhesive layer in addition to the assumptions made for (a) (equation (20)). Curve (c) is obtained from (22) where both the stiffener and the base plate are treated as membranes and the bonding is through an adhesive layer. The fourth model (i.e., two membranes bonded without any adhesive) would simply give a peak of infinite magnitude at $x = a$ and zero values elsewhere (equation (13)). The numerical solutions of the integral equations (9) and (20) indicate that as h_1/a and h_2/a decrease the convergence becomes slower and hence computations become costlier. On the other hand Figure 5 shows that even for a relatively large h_1/a the difference between the membrane solution (c) and continuum base plate solution (b) may not be considered high enough to justify the elaborate and complicated analysis required by the latter. For smaller values of h_i/a ($i=1,2$) the difference would, of course, be even smaller. Hence, it may be concluded that in the type of problems discussed in this paper the closed form solution given by (22) may give adequate results provided h_1/a and h_2/a are "sufficiently small"*. The results also show that in the

*The results given in this paper may be a useful guide in attempting to interpret the phrase "sufficiently small" quantitatively. For example, from these results it is clear that for the practical problems considered in [13], the membrane assumption would be perfectly adequate. Another model for problems of this kind would be the treatment of the elastic layers as "plates" with certain bending stiffnesses and the approximation of the adhesive by a combination of uncoupled shear and tension springs. This

presence of an adhesive layer the direct adhesion model described in section 2 is totally unsuitable for the analysis of bond rupture.

would give rise to normal as well as shear contact stresses along the contact area. For geometries such as lap joints the concentration of the normal stresses would be quite significant, playing an important role in bond rupture studies. For the symmetric geometries, however, one would expect that the dominant contact stress would be the shear stress which would be approximately the same as that obtained from the membrane theory.

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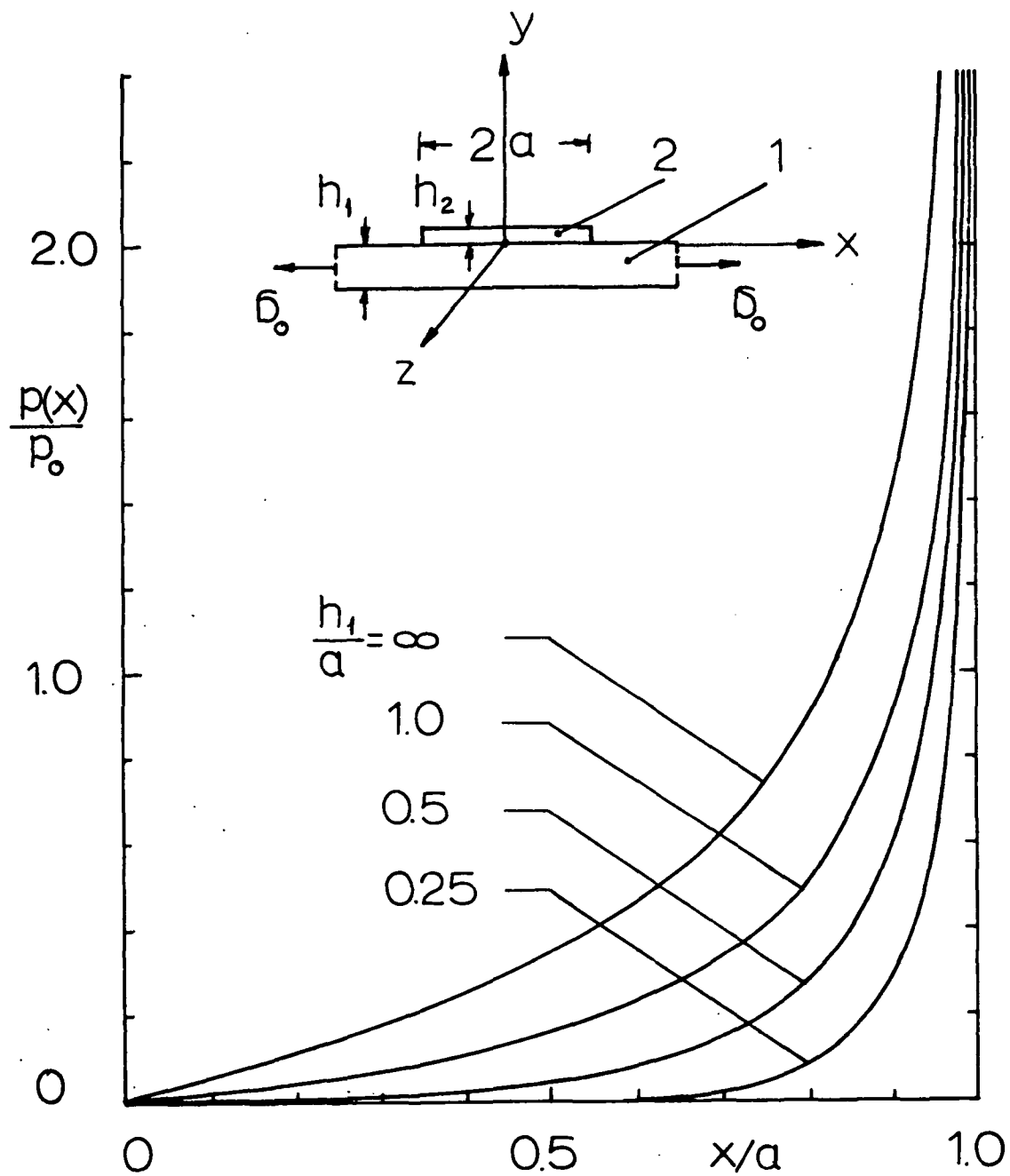


Figure 1. Distribution of contact stress for the direct adhesion case. Materials: 1 Aluminum, 2 Boron-Epoxy Composite;

$$\lambda = 0.5, \quad p_0 = \frac{1 - \nu_1 \nu_2}{2(1 - \nu_1^2)} \sigma_0.$$

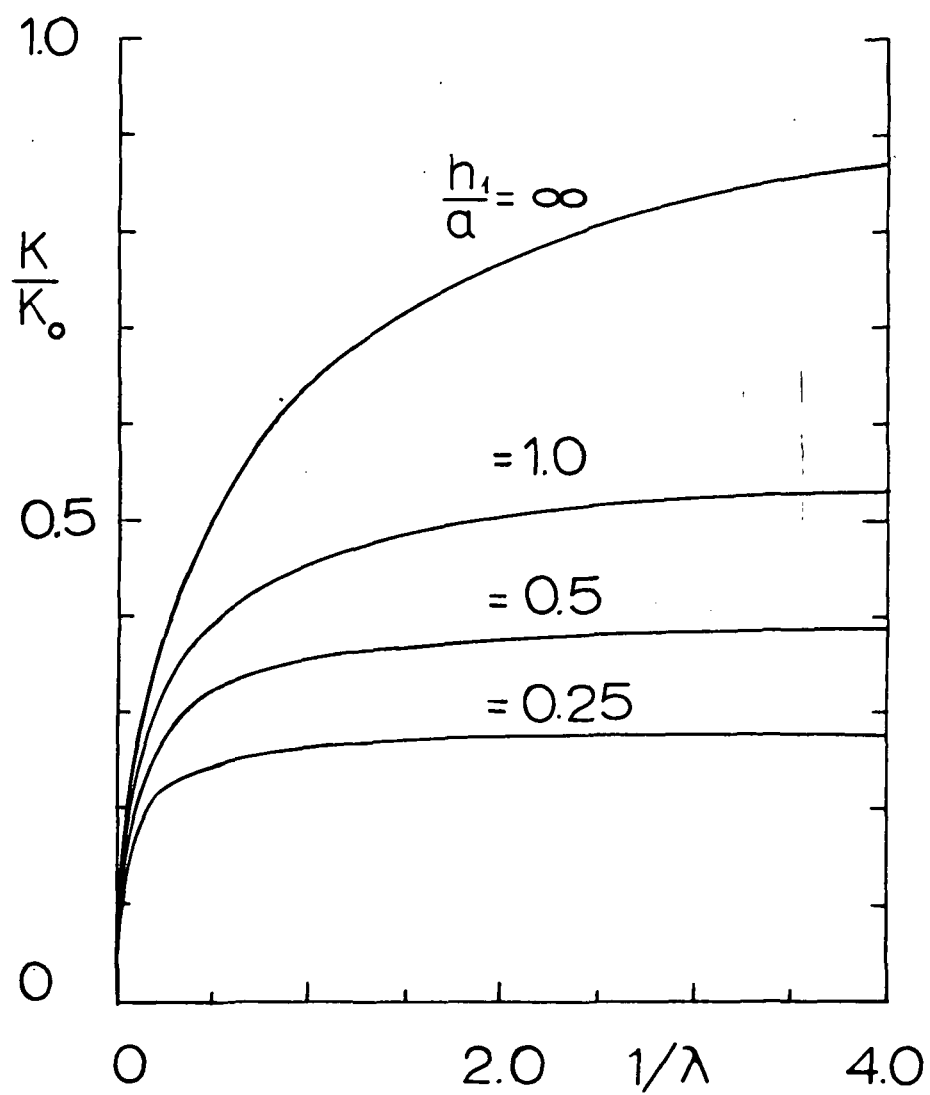


Figure 2. Strength of stress singularity for the direct adhesion case, $K_0 = \frac{1 - \nu_1 \nu_2 z}{2(1 - \nu_1^2)} \sigma_0 \sqrt{a}$.

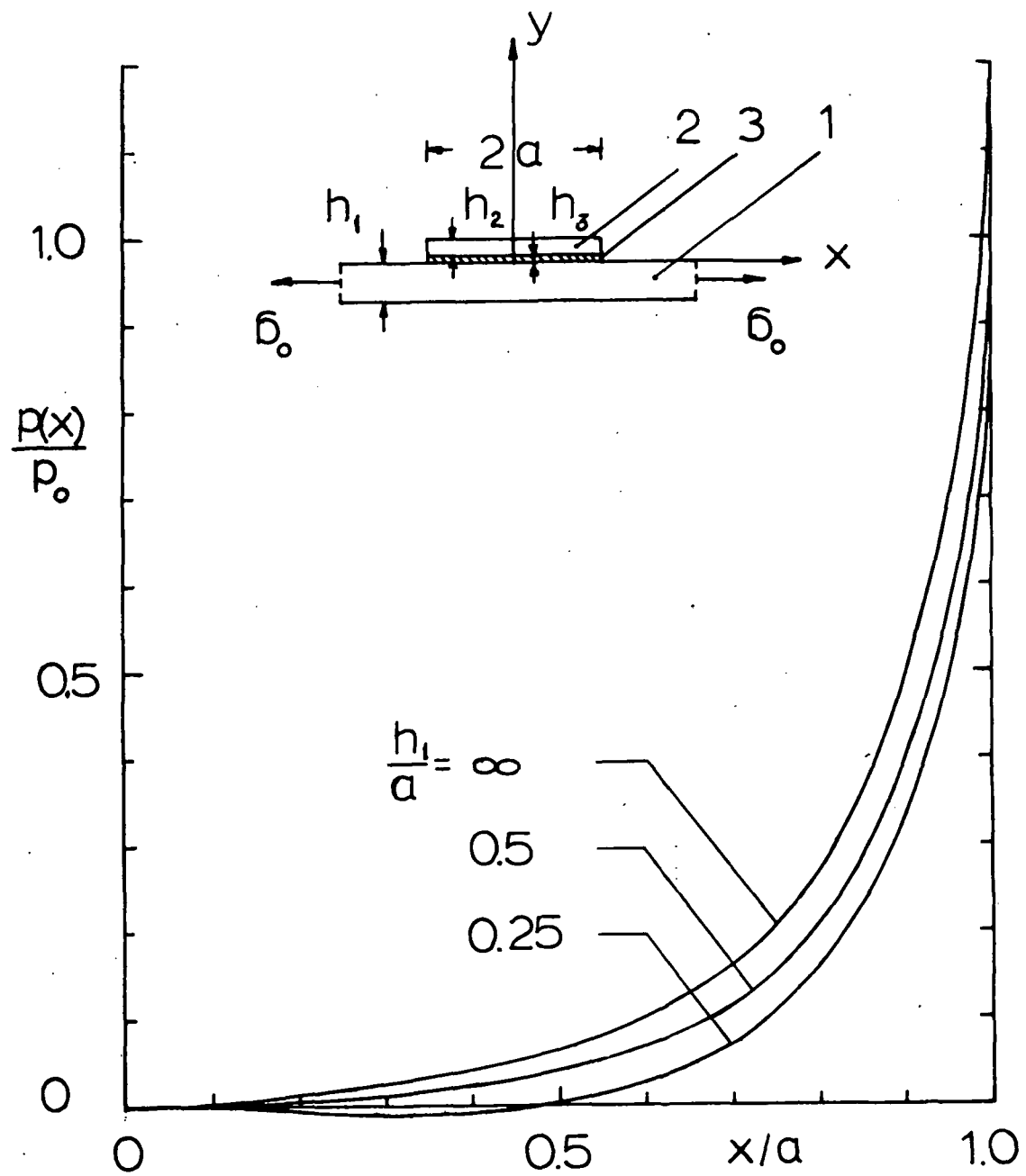


Figure 3. Distribution of contact stress for the case of bonding through an adhesive layer. Materials: 1 Aluminum, 2 Boron-Epoxy Composite, 3 Epoxy; $\lambda = 5$, $h_3/a = 0.004$, $p_0 = \sigma_0(1 - \nu_1\nu_{2z})/(2(1 - \nu_1^2))$.

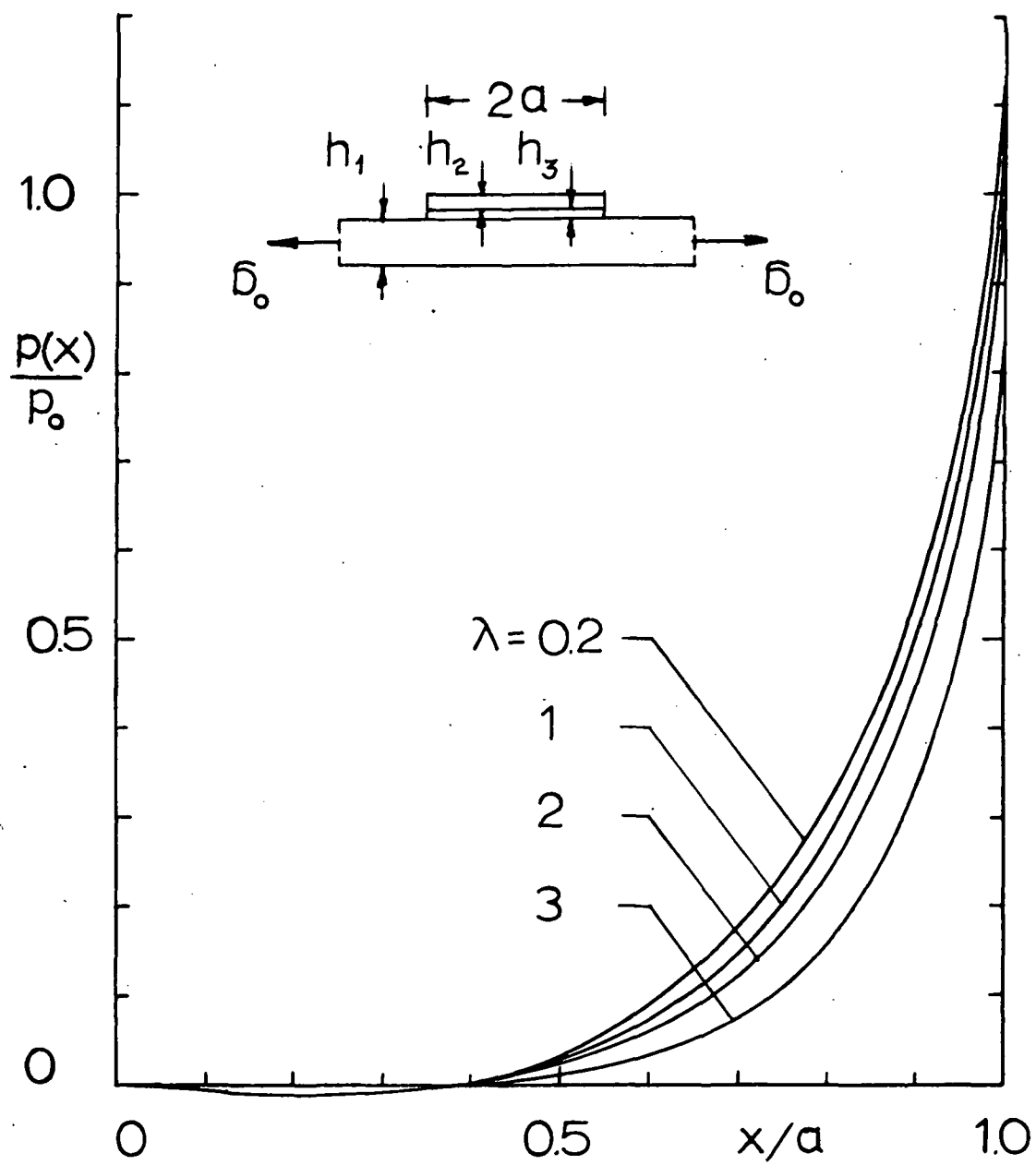


Figure 4. Same as Figure 3 for constant $h_1/a = 0.25$ and $h_0/a = 0.004$ and varying λ ; $p_0 = \sigma_0(1 - \nu_1\nu_{2z})/(2(1 - \nu_1^2))$.

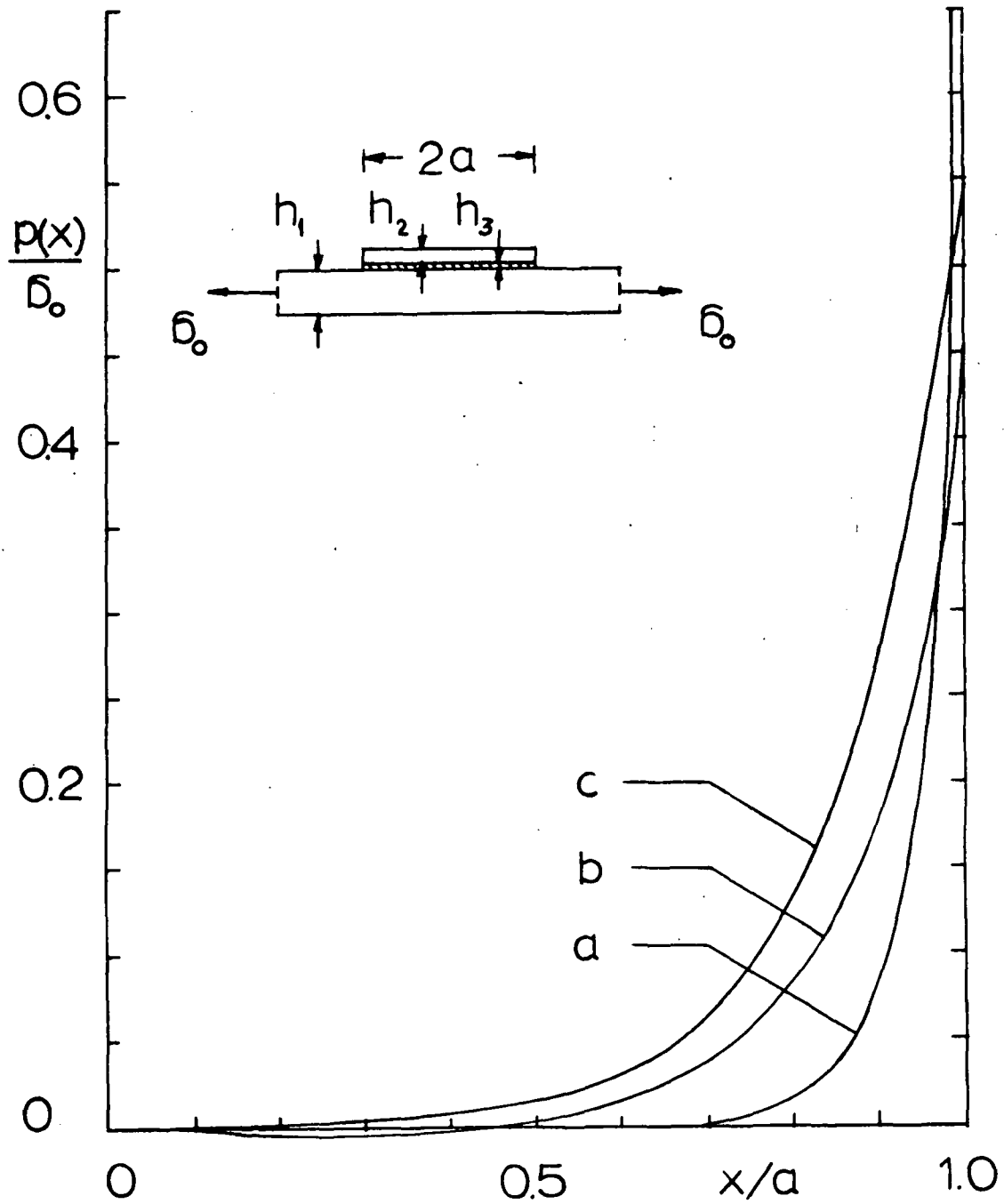


Figure 5. Contact stress distribution calculated from various models: (a) Stiffener: membrane, Base Plate: elastic continuum, Contact: direct adhesion; (b) Stiffener: membrane, Base Plate: elastic continuum, Contact: bonding through an adhesive layer; (c) Stiffener and Base Plate: membrane, Contact: bonding through an adhesive layer. $\lambda = 5$, $a/h_1 = 4$, $h_0/a = 0.004$.