NGR 05-002-256
OAP - 325

Gravitational-Wave Observations as a Tool for Testing Relativistic Gravity*

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ABSTRACT

Gravitational-wave observations can be powerful tools in the testing of relativistic theories of gravity — perhaps the only tools for distinguishing between certain extant theories in the [illegible] futu[illegible] field[illegible] gravi[illegible] six possible modes of polarization, which can be completely resolved by feasible experiments. We set forth a theoretical framework for classification of waves and theories, based on the Lorentz transformation properties of the six modes. We also show in detail how the six modes may be experimentally identified and to what extent such information limits the 'correct' theory of gravity.

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*Supported in part by the National Aeronautics and Space Administration [NGR-05-002-256] and the National Science Foundation [GP-36687X, GP-28027].

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ONE OF THE ORANGE AID PREPRINT SERIES
IN NUCLEAR, ATOMIC & RELATIVISTIC ASTROPHYSICS

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ABSTRACT

Gravitational-wave observations can be powerful tools in the testing of relativistic theories of gravity — perhaps the only tools for distinguishing between certain extant theories in the foreseeable future. In this paper we examine gravitational radiation in the far field using a formalism that encompasses all "metric theories of gravity." There are six possible modes of polarization, which can be completely resolved by feasible experiments. We set forth a theoretical framework for classification of waves and theories, based on the Lorentz transformation properties of the six modes. We also show in detail how the six modes may be experimentally identified and to what extent such information limits the "correct" theory of gravity.

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I. INTRODUCTION

Within the past few years, as experimental tests of gravity have been analyzed and refined, and as gravitation theories have been systematically compared,[1] most extant theories have been ruled out. Indeed, analysis of data from existing "solar system" experiments promises to distinguish more and more clearly between the theories that today remain viable. [For example, within the next two years, a search for the Nordtvedt effect[2] in lunar laser ranging data[3] should either rule out general relativity (GRT),[4] or place a limit of $\omega > 30$ on the Dicke coupling constant of Dicke-Brans-Jordan theory.[5]] An elegant theoretical formalism, the "Parametrized Post-Newtonian" (PPN) framework,[6] exists for analysis of metric theories[7] in the limit of weak gravitation and slow motion. All gravitation experiments that have played key roles in ruling out theories, except the Eötvös-Dicke experiment,[8] fall within the PPN framework. The Eötvös-Dicke experiment itself probably forces the "correct" theory of gravity to be a metric theory[7,9] and, in fact, there are no known complete[7] nonmetric theories which do not violate the Eötvös-Dicke experiment.

In the last year or so, it has become evident that the PPN framework has fundamental limitations. New metric theories of gravity,[10-13] with widely varying structures, have been invented which are virtually indistinguishable from one another and from GRT in the post-Newtonian limit. Existing and proposed solar-system experiments cannot hope to distinguish between such theories in the foreseeable future. There is, however, a strong element of hope: that new theories[10-13] and GRT differ markedly in the observable properties of their gravitational waves. With this motivation, we have embarked upon a program to develop a theoretical foundation for the analysis of gravitational waves in arbitrary metric theories of gravity — a foundation which is theory-independent and analogous to the PPN framework. (Gravitational-wave phenomena fall outside of the PPN framework.) We feel that experiments to detect gravitational waves from astronomical sources can prove to be a powerful experimental tool, in the foreseeable future, for ruling out gravitation theories.

The idea of building a theory-independent framework for analyzing gravitational-wave experiments was first conceived of in mid-1972 by Robert V. Wagoner.[14] At about the same time, and independently, our group was analyzing the gravitational-wave properties of a particular metric theory — one that two of us had recently invented.[13] When our analysis was near completion (several months after we learned of Wagoner's ideas), we suddenly realized that our theory exhibits the most general type of gravitational wave admitted by any metric theory — and that, therefore, with a mere change of viewpoint, our analysis would become the general framework that Wagoner had proposed constructing. Upon contacting Wagoner we discovered that he and Clifford M. Will had already proceeded far toward the construction of this same framework. We therefore published a brief account of the framework jointly with them in Physical Review Letters.[15] This paper presents a more detailed account of our "Caltech" version of the framework.

In a future paper we hope to treat the generation of waves by particular sources in arbitrary theories and thereby "move in from the far field."

Our fundamental results are that the most general null or nearly null wave has six independent polarization modes, which can be classified according to their behavior under Lorentz transformations. Various theories admit some subset (perhaps all) of the six possible modes. If the wave direction is known, the modes can be resolved uniquely by feasible experiments; if the direction of the wave is not known, partial but not complete resolution can be obtained. In either case detection information limits the correct theory of gravity.

Section II summarizes the properties of the general waves while Sec. III gives the details of derivations. Section IV discusses application to particular theories and their classification within the formalism; Sec. V gives a complete prescription of how to analyze and classify waves that are observed by means of gravitational-wave detectors. (For a review of the prospects of gravitational-wave astronomy, we refer the reader to Ref. 16.)

II. PROPERTIES AND CLASSIFICATION OF WEAK, PLANE, NULL WAVES: A SUMMARY OF RESULTS

A. Definition of Gravitational Waves in Metric Theories

In any metric theory of gravity,[7] just as in GRT, the response of matter to gravity is determined solely by a universal, covariant coupling to the physical metric $\underset{\sim}{g}$ (Einstein's Equivalence Principle[7]). The equation of motion of matter is given by[17]

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{T} = 0,$$

where $\underset{\sim}{\nabla}$ is the covariant derivative associated with $\underset{\sim}{g}$, and $\underset{\sim}{T}$ is the matter stress-energy tensor. This equation ensures that test particles and photons travel along time-like and null geodesics of $\underset{\sim}{g}$, respectively. Metric theories differ only in the manner that matter acts back to generate $\underset{\sim}{g}$ — i.e., only in their gravitational field equations. Some theories postulate auxiliary gravitational fields, such as the scalar field ϕ in Dicke-Brans-

Jordan theory,[5] which enter into the field equations but do not act on matter directly.

It is the universality of the coupling to the metric that permits a theory-independent discussion of the propagation and detection of gravitational waves for metric theories. On the other hand, the emission of gravitational waves involves the detailed structure of field equations, and is therefore theory-dependent. Emission will not be treated in this paper.

Consider an experiment employing matter of negligible self-gravity in a local region to measure the static or wavelike gravitational fields from faraway sources. One cannot define the absolute acceleration due to gravity at a point in the region (Einstein's Equivalence Principle[7]); only the relative, tidal acceleration between two points has observable significance. The Riemann tensor $\underset{\sim}{\mathrm{Riem}}$, formed from $\underset{\sim}{g}$, determines these relative accelerations, and is the sole locally observable imprint of gravity.

Consider a freely falling observer at any fiducial point P in the region. Let him set up an approximately Lorentz, normal coordinate system

$$\{x^{\mu}\} = \{t, x^{i}\} ,$$

with P as origin. For a particle with spatial coordinates x^i at rest or with nonrelativistic velocity in the region, the acceleration relative to P is (for sufficiently small $|x^j|$)

$$a_i^{\mathrm{GRAV}} = - R_{i0j0}\, x^j , \tag{1}$$

where R_{i0j0} are the so-called "electric" components of the $\underset{\sim}{\mathrm{Riem}}$ due to waves or other external gravitational influences.

A gravitational wave in a metric theory involves the metric field $\underset{\sim}{g}$ and any auxiliary gravitational fields that might exist. But the resultant $\underset{\sim}{\text{Riem}}$ is the only measurable field. So for this paper we define a "gravitational wave" in terms of its $\underset{\sim}{\text{Riem}}$: A "weak, plane, null wave" in a metric theory is a weak, propagating, vacuum gravitational field characterized, in some nearly Lorentz coordinate system, by a linearized $\underset{\sim}{\text{Riem}}$ with components that depend only upon a null "retarded time," $u \equiv t - z/c$:

$$R_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau}(u).$$

$\underset{\sim}{\nabla} u$, which is proportional to the wave vector, is null with respect to the physical metric $\underset{\sim}{g}$: $\underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} u = 0$. In "$u = t - z/c$," c is the speed of light, and the coordinates are oriented such that the wave travels in the $+z$ direction.

Two restrictions appear in this definition: (i) Waves must travel at exactly the local speed of light, (ii) waves must be exactly plane. These restrictions turn out to be good approximations in feasible experiments for all viable metric theories of gravity; see Sec's. III and IV for a discussion of these points.

The fundamental properties of these waves follow immediately from the algebraic and differential identities that $\underset{\sim}{\text{Riem}}$ obeys. There are six algebraically independent components of $\underset{\sim}{\text{Riem}}$ in vacuum, (Sec. III proves this assertion and succeeding ones), which correspond to six modes of polarization. In a given, nearly Lorentz coordinate frame of the above type, group these six components into amplitudes of definite helicity s (where $s = 0, \pm 1, \pm 2$) under rotations about the z-axis. There arise two real amplitudes

$$\Psi_2(u),\ (s = 0)\ ;\ \Phi_{22}(u),\ (s = 0)\ ,$$

and two complex amplitudes

$$\Psi_3(u),\ (s = \pm 1)\ ;\ \Psi_4(u)\ ,\ (s = \pm 2).$$

Here and throughout this paper one complex amplitude is equivalent to two real amplitudes. We will always describe a gravitational wave by its six amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ in the six polarization modes of a given coordinate frame.

These amplitudes are related to the "electric" components of $\underset{\sim}{\mathrm{Riem}}$, which govern relative accelerations through Eq. (1), by

$$\Psi_2(u) = -\frac{1}{6} R_{z0z0}(u)\ , \tag{2a}$$

$$\Psi_3(u) = -\frac{1}{2} R_{x0z0} + \frac{i}{2} R_{y0z0}\ , \tag{2b}$$

$$\Psi_4(u) = -R_{x0x0} + R_{y0y0} + 2i\, R_{x0y0}\ , \tag{2c}$$

$$\Phi_{22}(u) = -R_{x0x0} - R_{y0y0}\ . \tag{2d}$$

Figure 1 shows the displacement that each polarization mode induces on a sphere of test particles; Ψ_4 and Φ_{22} are purely transverse, Ψ_2 is purely longitudinal, and Ψ_3 is mixed. If an experimenter knows the wave direction, he can uniquely determine $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ by measuring the driving forces in his detector (see Sec. V for further details), and he can reconstruct $\underset{\sim}{\mathrm{Riem}}$. Therefore, currently feasible detectors can obtain all the measurable information in the most general wave permitted by any metric theory.

B. Lorentz-Invariant E(2) Classification of Plane Waves

In any metric theory, the local nongravitational laws of physics are those of special relativity. So it is fruitful to sort waves into Lorentz-invariant classes, depending on the behavior of the amplitudes under Lorentz transformations. Observers in different Lorentz frames (e.g., in relative motion) can then agree on the classification of any wave.

Rather than use the entire Lorentz group relating observers in all frames, we choose a restricted set of "standard observers" such that (i) each observer sees the wave travelling in his +z direction, and (ii) each observer sees the same Doppler shift, e.g., each measures the same frequency for a monochromatic wave. These standard observers are related by the subgroup of Lorentz transformations that leaves the vector $\underset{\sim}{\nabla} u$ invariant ["little group, E(2)"]. The parts of the Lorentz group left out of the little group are (a) [due to requirement (i)] pure rotations of $\underset{\sim}{\nabla} u$ which merely change the direction of wave propagation, and (b) [due to requirement (ii)] pure boosts along $\underset{\sim}{\nabla} u$ which merely change the observed frequency and scale each amplitude up or down independently. Without requirement (ii), different observers would see the wave travelling along the +z direction, but generally at different Doppler shifts. The subgroup relating the standard observers would be bigger (4 dimensional), but the invariant classes would be the same.

The six amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ of a wave are generally observer-dependent; their transformation law is given in Sec. III. However, there are certain "invariant" statements about them that are true for all standard observers if they are true for any one. These statements characterize <u>invariant "E(2) classes" of waves</u>: (Notation is explained in Sec. III.)

Class II_6. $\Psi_2 \not\equiv 0$. All standard observers measure the same nonzero amplitude in the Ψ_2 mode. (But the presence or absence of all other modes is observer-dependent.)

Class III_5. $\Psi_2 \equiv 0 \not\equiv \Psi_3$. All standard observers measure the absence of Ψ_2 and the presence of Ψ_3. (But the presence or absence of Ψ_4 and Φ_{22} is observer-dependent.)

Class N_3. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \not\equiv \Phi_{22}$. Presence or absence of all modes is independent of observer.

Class N_2. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \equiv \Phi_{22}$. Independent of observer.

Class O_1. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \not\equiv \Phi_{22}$. Independent of observer.

Class O_0. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \equiv \Phi_{22}$. Independent of observer. All standard observers measure no wave.

Class II_6 is the most general. As one demands that successive amplitudes vanish identically, one descends to less and less general classes. Figure 2 exhibits these relations of generality among the classes. In this paper, "more (or less) general" for classes always refers to Fig. 2. (For example: O_1 is less general than N_3, III_5, and II_6, but neither more nor less general than N_2.) The E(2) class of a particular metric theory is defined as the class of its most general wave (see Sec. IV for illustrations).

The fundamental theoretical implication of our paper is that the class of the "correct" theory of gravity is at least as general as the class of any observed wave.

Once theorists are confident of a particular classical theory of gravity, they will wish to quantize it. Then it should be possible to associate the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ with massless quanta of definite and Lorentz-invariant helicity. Section III demonstrates that the helicity content of class II_6 is not Lorentz-invariant, nor is that of III_5.

Furthermore, an associated pathology arises for these classes: The amplitudes form a nonunitary representation of the inhomogeneous Lorentz group, contradicting the tenets of relativistic quantum mechanics.[18] Attempts to quantize theories of Class II_6 or III_5 will therefore face grave difficulties.

These difficulties do not arise for theories of class N_3 or less general: There Ψ_4 and Φ_{22} act like massless quantum fields with $s = \pm 2$ and 0.

III. DERIVATIONS

This section may be skipped without essential loss of continuity.

A. Tetrad Components of $\underset{\sim}{\mathrm{Riem}}$ for Waves

A quasiorthonormal, null tetrad basis[19] is especially suitable for discussing null waves. At any point P, the null tetrad $(\underset{\sim}{k}, \underset{\sim}{\ell}, \underset{\sim}{m}, \underset{\sim}{\overline{m}})$ is related to the cartesian tetrad introduced in Sec. II by

$$\underset{\sim}{k} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{t}} + \underset{\sim}{e}_{\hat{z}}) \quad , \tag{3a}$$

$$\underset{\sim}{\ell} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{t}} - \underset{\sim}{e}_{\hat{z}}) \quad , \tag{3b}$$

$$\underset{\sim}{m} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{x}} + i\underset{\sim}{e}_{\hat{y}}) \quad , \tag{3c}$$

$$\underset{\sim}{\overline{m}} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{x}} - i\underset{\sim}{e}_{\hat{y}}) \quad . \tag{3d}$$

Throughout this section we follow Sec. II in orienting the axes such that the wave travels in the +z direction; $u \equiv t - z/c$. Equivalently, we choose $\underset{\sim}{k}$, one of the tetrad legs, proportional to the vector $\underset{\sim}{\nabla} u$. It is easily verified from Eqs. (3) that the tetrad vectors obey the relations:

$$-\underset{\sim}{k} \cdot \underset{\sim}{\ell} = \underset{\sim}{m} \cdot \underset{\sim}{\overline{m}} = 1 \quad , \tag{4}$$

while all other dot products vanish.

We adopt the following notation for null-tetrad components of tensors $\underset{\sim}{X}$:

$$X_{abc\ldots} \equiv X_{\mu\nu\sigma\ldots}\, a^{\mu} b^{\nu} c^{\sigma} \ldots \quad , \tag{5}$$

where $(a,b,c\ldots)$ range over $(k,\ell,m,\bar{m})$.

Central to our later discussions will be the transformation properties of the components of $\underset{\sim}{\mathrm{Riem}}$ under the action of some subgroup of the Poincaré group. In view of this, we first split $\underset{\sim}{\mathrm{Riem}}$ into irreducible parts: the Weyl tensor, the traceless Ricci tensor and the Ricci scalar. We follow Newman and Penrose[19] in naming their tetrad components Ψ, Φ, and Λ respectively.

In general, the ten Ψ's, nine Φ's, and Λ are all algebraically independent. When we restrict ourselves to nearly plane waves, however, we find that the differential and algebraic properties of $\underset{\sim}{\mathrm{Riem}}$ reduce the number of independent components to six by the following arguments:

Consider a weak, plane, null wave. It is characterized by the fact that the components of its $\underset{\sim}{\mathrm{Riem}}$ are functions of the retarded time u only. Of their derivatives, only those with respect to the retarded time u will be nonvanishing:

$$R_{abcd,p} = 0 \ , \tag{6}$$

where (a,b,c,d) range over $(k,\ell,m,\bar{m})$ while $(p,q,r,\ldots)$ range over $(k,m,\bar{m})$ only.

The covariant differential Bianchi identities and the symmetry properties of $R_{\mu\nu\sigma\tau}$ are necessary and sufficient to guarantee that the linearized $\underset{\sim}{\mathrm{Riem}}$ is derivable from a metric perturbation,[20]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ . \tag{7}$$

Using Eq. (6) we see that these identities imply the relations

$$R_{ab[pq,\ell]} \equiv 0 = \frac{1}{3} R_{abpq,\ell} \quad , \tag{8}$$

where ℓ is a fixed index. Equation (8) implies that

$$R_{abpq} = 0 = R_{pqab} \; , \tag{9}$$

except for a trivial, nonwavelike constant. Consequently, all nonvanishing components of $\underset{\sim}{Riem}$ must have the form $R_{p\ell q\ell}$. Taking into account the symmetries of $\underset{\sim}{Riem}$, we thus see that there are only six independent, nonvanishing components. Corresponding simplifications are induced among the Newman-Penrose quantities. For a plane wave, they are[19]:

i) Weyl Tensor

$$\Psi_0 = \Psi_1 = 0 \; , \tag{10a}$$

$$\Psi_2 = -\frac{1}{6} R_{\ell k \ell k} \; , \tag{10b}$$

$$\Psi_3 = -\frac{1}{2} R_{\ell k \ell \bar{m}} \; , \tag{10c}$$

$$\Psi_4 = -R_{\ell \bar{m} \ell \bar{m}} \quad , \tag{10d}$$

ii) Traceless Ricci Tensor

$$\Phi_{00} = \Phi_{01} = \Phi_{10} = \Phi_{02} = \Phi_{20} = 0 \; , \tag{11a}$$

$$\Phi_{22} = -R_{\ell m \ell \bar{m}} \; , \tag{11b}$$

$$\Phi_{11} = \frac{3}{2} \Psi_2 \; , \tag{11c}$$

$$\Phi_{12} = \bar{\Phi}_{21} = \bar{\Psi}_3 \; , \tag{11d}$$

iii) Ricci Scalar

$$\Lambda = -\frac{1}{2} \Psi_2 \; . \tag{12}$$

As indicated in Sec. II, we shall choose the set $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ (Ψ_3 and Ψ_4 complex) to describe, in a given null frame, the six independent components of a wave in the generic metric theory. Equations (10) and (11) give the members of this set in terms of the null-tetrad components of the Riemann tensor. Equations (2) give the members of the set in terms of the directly observable "electric" components of the Riemann tensor.

In those cases where one calculates the Riemann tensor from a metric perturbation $h_{\mu\nu}$,[21] Eq. (7), the relation between $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ and derivatives of h_{ab} may be found in Appendix 1.

B. Behavior of Tetrad Components under Lorentz Transformation

Consider two standard observers O and O', with tetrads ($\underset{\sim}{k}$, $\underset{\sim}{\ell}$, $\underset{\sim}{m}$, $\underset{\sim}{\overline{m}}$,) and ($\underset{\sim}{k}'$, $\underset{\sim}{\ell}'$, $\underset{\sim}{m}'$, $\overline{m}'$); then $\underset{\sim}{k} = \underset{\sim}{k}' \propto \underset{\sim}{\nabla} u$. Suppose O has measured the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ of a wave; how do we predict the amplitudes $\{\Psi_2', \Psi_3', \Psi_4', \Phi_{22}'\}$ measured by O'?

In group-theoretic language, we are asking the transformation properties of the amplitudes under the "little group" of Lorentz transformations that leaves the wave vector fixed. The various group representations formed by the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ provide us with a means for classifying waves.

The most general proper Lorentz transformation relating the tetrads that keeps $\underset{\sim}{k}$ fixed is[22]

$$\underset{\sim}{k}' = \underset{\sim}{k} \,, \tag{13a}$$

$$\underset{\sim}{m}' = \exp(i\varphi)(\underset{\sim}{m} + \alpha \underset{\sim}{k}) \,, \tag{13b}$$

$$\underset{\sim}{\overline{m}}' = \exp(-i\varphi)(\underset{\sim}{m} + \overline{\alpha} \underset{\sim}{k}) \,, \tag{13c}$$

$$\underset{\sim}{\ell}' = \underset{\sim}{\ell} + \overline{\alpha}\underset{\sim}{m} + \alpha\underset{\sim}{\overline{m}} + \alpha\overline{\alpha}\underset{\sim}{k} \,, \tag{13d}$$

where α is an arbitrary complex number that produces "null rotations,"[23] (particular combinations of boosts and rotations), while φ, which ranges from 0 to 2π, is an arbitrary real phase that produces a rotation about $\underset{\sim}{e}_{\hat{z}}$. The transformations described in Eqs. (13) form a subgroup of the Lorentz group which is globally isomorphic to the abstract Lie group E(2), the group of proper rigid motions in the Euclidean 2-plane. In the latter group, φ represents the rotations in the plane and α, the translations. We denote a particular element of E(2) in Eqs. (13) by (φ,α). The law of composition is $(\varphi',\alpha')(\varphi,\alpha) = (\varphi' + \varphi,\ \alpha' + \exp(i\varphi')\alpha)$.

The transformation induced on the amplitudes of a wave by (φ,α) is

$$\Psi_2{}' = \Psi_2 \,, \tag{14a}$$

$$\Psi_3{}' = e^{-i\varphi}\,(\Psi_3 + 3\bar{\alpha}\Psi_2) \,, \tag{14b}$$

$$\Psi_4{}' = e^{-2i\varphi}\,(\Psi_4 + 4\bar{\alpha}\Psi_3 + 6\bar{\alpha}^2\Psi_2) \,, \tag{14c}$$

$$\Phi_{22}' = \Phi_{22} + 2\alpha\Psi_3 + 2\bar{\alpha}\bar{\Psi}_3 + 6\alpha\bar{\alpha}\Psi_2 \,. \tag{14d}$$

Now consider a set of observers related to one another by z-axis rotations $(\varphi,0)$. A quantity M that transforms under these rotations as $M' = \exp(is\varphi)\,M$ is said to have helicity s as seen by these observers. We see from Eqs. (14) that the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ are helicity eigenstates. Furthermore, their helicity values can be read off easily from Eqs. (14), (setting $\alpha = 0 = \bar{\alpha}$):

$$\Psi_2 : s = 0 \tag{15a}$$

$$\Psi_3 : s = -1 \ ,\quad \bar{\Psi}_3 : s = +1 \,, \tag{15b}$$

$$\Psi_4 : s = -2 \ ,\quad \bar{\Psi}_4 : s = +2 \,, \tag{15c}$$

$$\Phi_{22} : s = 0 \quad . \tag{15d}$$

C. E(2) Classification of Waves

It is evident from Eqs. (14) that the various amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ cannot be specified in an observer-independent manner. [Example: O may measure a wave to have as its only nonvanishing amplitude Ψ_2(helicity 0), while O', in relative motion with respect to O, may conclude that the wave has, in addition, Ψ_3 and Ψ_4 components (helicities 0, 1, and 2).] We classify waves in an E(2)-invariant manner by uncovering all representations of E(2) embodied in Eqs. (14). Each such representation, in which certain of the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ vanish identically, is a distinct, invariant class. The name of each class is composed of the Petrov type of its nonvanishing Weyl tensor[24] (except that we do not distinguish between II and D) and the maximum number of nonvanishing amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ as seen by any observer (dimension of representation). Both the Petrov type and the dimension of representation are independent of observer.

The various classes were delineated in Sec. II, they are:

<u>Class II_6.</u> $\Psi_2 \not\equiv 0$.

<u>Class III_5.</u> $\Psi_2 \equiv 0 \not\equiv \Psi_3$.

These two classes form reducible, indecomposable representations of E(2). (See Appendix 2 for a brief resumé of the relevant group — theoretic concepts.) The maximal invariant proper subspace is the 3-dimensional one spanned by Ψ_4 and Φ_{22}. The helicity content of classes II_6 and III_5 is observer-dependent.

<u>Class N_3.</u> $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \not\equiv \Phi_{22}$.

<u>Class N_2.</u> $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \equiv \Phi_{22}$.

<u>Class O_1.</u> $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \not\equiv \Phi_{22}$.

Classes N_3, N_2, and O_1 form decomposable representations of E(2) which decompose into 1-dimensional invariant subspaces spanned by Ψ_4 and Φ_{22} respectively. Each of these invariant subspaces forms a unitary, massless-particle representation of definite, Lorentz invariant, helicity (spin). They are well studied as they occur in relativistic quantum field theory.[25]

<u>Class O_0.</u> $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \equiv \Phi_{22}$.

Class O_0 forms the trivial representation.

The foregoing classification scheme is patterned closely after Wigner's classic analysis[26] of wave functions of relativistic quantum particles as members of unitary, irreducible representations of the Poincaré group.[27] Wigner showed that each such wave function may be taken to have a definite 4-momentum $\underset{\sim}{q}$, and to transform as a member of some unitary, irreducible representation of the little group that leaves $\underset{\sim}{q}$ invariant. One determines the "spin" of the particle from the eigenvalues of the helicity operator and its square; the spin of the particle is completely determined once the representation formed by its associated wave functions under the little group is known.

For our gravitational waves, $\underset{\sim}{\nabla} u$ is null and nonvanishing, and the little group is E(2). Unfortunately, Wigner's analysis does not apply since we are not restricted to unitary representations of E(2). In fact, as we have seen, the representations generated by $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ are in general nonunitary and indecomposable. The amplitudes in classes II_6 and III_5 cannot be identified with massless particle fields. Consequently, it is impossible to give a spin decomposition for these waves.

A representation which is reducible and indecomposable can never be unitary. This applies to the little group E(2), and hence also to the

Poincaré group. In relativistic quantum theory, all invariance groups must be realized by unitary representations.[18] We therefore obtain the following result: If a theory is of class II_6 or III_5, it is impossible to quantize it in a way that is Poincaré invariant with respect to the local Lorentz metric.

D. Spherical Waves

Thus far, we have based our discussions on the properties of plane waves. The most physically satisfactory definition of a radiation field is one that carries energy off to infinity from a bounded source. For metric theories of gravity, this corresponds to that part of the Riemann tensor that falls off as 1/(distance) asymptotically. Far away from radiating sources, one may locally approximate these approximately spherical waves as plane waves. The following argument shows in a theory-independent manner that the plane wave approximation will not affect the classification scheme.

Adopt a (u,r,Θ,φ) coordinate system in the wave zone, which is assumed to be almost Minkowskian. The line element is given by

$$ds^2 = -du^2 + 2dudr + r^2\,(d\Theta^2 + \sin^2\Theta\, d\varphi^2). \tag{16}$$

Place the origin of the coordinate system somewhere inside the source. Single out the $1/r$ part of the outgoing spherical waves:

$$R_{abcd} = \frac{1}{r}\, S_{abcd}\,(u,\Theta,\varphi) + O\left(\frac{1}{r^2}\right). \tag{17}$$

In the wave zone, observer O $[r = r_0,\ \Theta = \varphi = 0]$ carries with himself a Cartesian tetrad $(\underset{\sim}{e}_{\hat{t}},\ \underset{\sim}{e}_{\hat{x}},\ \underset{\sim}{e}_{\hat{y}},\ \underset{\sim}{e}_{\hat{z}})$ oriented such that $\underset{\sim}{e}_{\hat{z}}$ is along the incident direction of the wave. The two coordinate systems are related by

$$u = t - z \,, \tag{18a}$$

$$r = z + r_0, \tag{18b}$$

$$\Theta = \frac{x}{r_0} + O\left(\frac{1}{r_0^{\,2}}\right) \,, \tag{18c}$$

$$\varphi = \frac{y}{r_0} + O\left(\frac{1}{r_0^{\,2}}\right) . \tag{18d}$$

Thus O would measure

$$R_{abcd} = \frac{1}{r_0}\, S_{abcd}\left(u, \frac{x}{r_0}, \frac{y}{r_0}\right) + O\left(\frac{1}{r_0^{\,2}}\right) . \tag{19}$$

The differential Bianchi identities then imply

$$0 \equiv R_{ab[pq;c]} = O(1/r_0^{\,2}), \quad \text{if } c \neq \ell \,, \tag{20a}$$

$$0 \equiv R_{ab[pq;\ell]} = \frac{1}{3}\,\frac{1}{r_0}\, S_{abpq,\ell} + O(1/r_0^{\,2}) \,, \tag{20b}$$

where semicolon and comma denote covariant and partial differentiation respectively. It follows immediately from Eqs. (20) that the classification scheme based on the $1/r$ part of the Riemann tensor is identical to that based on the plane waves.

IV. APPLICATIONS TO PARTICULAR THEORIES

A. Two-Metric Theories

In all of the preceding discussion we have assumed that the components of the Riemann tensor are functions of the retarded time associated with the "physical metric" $g_{\alpha\beta}$, i.e.,

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(u) \ , \tag{21a}$$

where

$$u_{,\alpha} u_{,\beta} g^{\alpha\beta} = 0 \quad . \tag{21b}$$

This is indeed the proper approach, since the physical metric is associated with the physical local Lorentz frames, which are in turn the basis for our classification scheme. In some theories of gravity,[10,13] however, gravitational waves travel along null geodesics of a "flat space," global, background metric $\underset{\sim}{\eta}$, while electromagnetic waves (and neutrinos) travel along null geodesics of the physical metric $\underset{\sim}{g}$. Equations (21) are then not rigorously satisfied. On the other hand, if $\underset{\sim}{g}$ differs from $\underset{\sim}{\eta}$ locally by only a small amount in the above-mentioned theories, Eqs. (21) are approximately correct and all of the formalism developed in Secs. II and III is applicable to a high degree of accuracy. In all such "two-metric" theories that we have studied, present experimental limits on "preferred-frame effects"[1,11] require, in the mean rest frame of the solar system,

$$\frac{|g_{\alpha\beta} - \eta_{\alpha\beta}|}{|\eta_{\alpha\beta}|} < 10^{-2} \ , \tag{22}$$

where $|\eta_{\alpha\beta}|$ refers to the magnitude of a typical element of $\eta_{\alpha\beta}$, etc. In fact, if the difference between $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$ is due entirely to solar system or galactic matter, then the 10^{-2} in Eq. (22) becomes 10^{-7}. Equation (22) is equivalent to the relation, again as measured in the mean rest frame of the solar system,

$$\frac{|c_g - c_{em}|}{c} < 10^{-2} \ , \tag{23}$$

where c_g and c_{em} are the speeds of gravitational and electromagnetic waves respectively. Thus, for all Lorentz observers who move at low speeds ($v \ll c$) with respect to the mean rest frame of the solar system, two-metric theories that are viable [in the sense of no preferred frame effects and so compliance with Eq. (22)] may be included in the formalism of Secs. II and III.

A further important point is that Eq. (23), a distinctive feature of two-metric theories, suggests that a search for time delays between simultaneously emitted gravitational and electromagnetic bursts could prove a valuable experimental tool. An experimental limit of $\lesssim 10^{-8}$ for $|c_g - c_{em}|/c$ would disprove most "two-metric" theories and would stringently constrain future theory-building. If current experimental efforts continue unabated, by 1980 one may detect gravitational-wave bursts from supernovae in the Virgo cluster (~ 3 supernovae per year). Then a limit of

$$|c_g - c_{em}|/c \lesssim 10^{-9} \times (\text{time-lag precision})/(1 \text{ week})$$

will be possible.

B. Degrees of Freedom Versus Polarization Modes

We have enumerated the various independent gravitational wave modes in the general metric theory. This does not mean, however, that for a given theory the maximum number of nonvanishing modes for any observer is equal to the number of dynamical degrees of freedom[28] in the gravitational field. For a given theory, there may be fewer or more degrees of freedom than the number of modes; if fewer, amplitudes in the various modes are linearly dependent in a manner dictated by the detailed structure of the theory (see discussion following Stratified Theories below).

C. Classification of Particular Theories

Table I gives the E(2) classification (see Secs. II and III) of some metric theories in the literature (some of which have already been ruled out, e.g., the conformally flat and stratified theories[29]). The classification procedure involves examining the far-field, linearized, vacuum field equations of a theory and is illustrated below by several examples. In the examples, the relevant approximated vacuum equations of a theory will be quoted whenever necessary.

1. General Relativity[4]

$$R_{\alpha\beta} = 0 \tag{24a}$$

From Eqs. (10), (11), and (A1.3) one can deduce that

$$R_{\ell k \ell k} = R_{\ell m \ell \bar{m}} = R_{\ell k \ell m} = R_{\ell k \ell \bar{m}} = 0, \tag{24b}$$

or

$$\Psi_2 = \Psi_3 = \Phi_{22} = 0. \tag{24c}$$

Since there are no further constraints, $\Psi_4 \neq 0$ and the E(2) classification is N_2.

2. Dicke-Brans-Jordan Theory[5]

$$\Box\varphi = 0, \tag{25a}$$

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}R = \omega\varphi^{-2} \left(\varphi_{,\alpha}\varphi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta}\, \varphi_{,\gamma}\varphi^{,\gamma}\right) + \varphi^{-1}\, \varphi_{,\alpha,\beta}\, , \tag{25b}$$

$$R = \omega\varphi^{-2}\, \varphi_{,\gamma}\varphi^{,\gamma}\, . \tag{25c}$$

The monochromatic plane wave solution to Eq. (25a) is[30]

$$\varphi = \varphi_0 + \varphi_1\, e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} \tag{25d}$$

where φ_0 and φ_1 are constants and the wave vector $\underset{\sim}{q}$ is null. The quantity φ_0 is the cosmological boundary value of the scalar field, and φ_1 is a small amplitude of a wave (work only to first order in φ_1). Then from Eq. (25c),

$$R = 0, \tag{25e}$$

and Eq. (25b) yields

$$R_{\alpha\beta} = - \varphi_0^{-1} \varphi_1 e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} q_\alpha q_\beta \ . \tag{25f}$$

Thus $R_{\ell\ell}$ is the only nonvanishing tetrad component of the Ricci tensor and one can conclude that

$$R_{\ell k \ell k} = R_{\ell k \ell m} = R_{\ell k \ell \bar{m}} = 0 \neq R_{\ell m \ell \bar{m}} \ , \tag{25g}$$

or

$$\Psi_2 = \Psi_3 = 0 \ , \quad \Phi_{22} \text{ and } \Psi_4 \neq 0 \ . \tag{25h}$$

Therefore for the Dick-Brans-Jordan theory, <u>the E(2) classification is N_3</u>.

3. Will-Nordtvedt Theory[11]

$$\Box K_\alpha = 0 \tag{26a}$$

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = K_{\alpha,\gamma} K_\beta{}^{,\gamma} + K_{\gamma,\alpha} K^\gamma{}_{,\beta} - \tfrac{1}{2} g_{\alpha\beta} K^\gamma{}_{,\delta} K_\gamma{}^{,\delta}$$
$$+ \tfrac{1}{2}\left[K^\gamma (K_{\alpha,\beta} + K_{\beta,\alpha}) - K_\alpha (K^\gamma{}_{,\beta} + K_\beta{}^{,\gamma}) - K_\beta (K^\gamma{}_{,\alpha} + K_\alpha{}^{,\gamma})\right]_{,\gamma} \ . \tag{26b}$$

The plane wave solution to Eq. (26a) is

$$K_\alpha = A_\alpha e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} + B_\alpha \ , \tag{26c}$$

where A_α and B_α are constant vectors and the wave vector q is null. Again, assume A_α is small and work only to linear order in that quantity. The vector B_α is of cosmological origin. Taking the trace of Eq. (26b) and using Eqs. (26c), (A1.2b), and (A1.4), we obtain

$$R = 0 = \Psi_2 \quad . \tag{26d}$$

Equation (26b) then reads

$$R_{\alpha\beta} = e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} \left[(\underset{\sim}{q} \cdot \underset{\sim}{A})\, q_{(\alpha}B_{\beta)} - (\underset{\sim}{B} \cdot \underset{\sim}{q})\, A_{(\alpha}q_{\beta)} \right] \quad . \tag{26e}$$

Equation (26e) indicates the relations

$$R_{\ell m} \neq 0 \ , \quad R_{\ell \bar{m}} \neq 0 \ , \quad R_{\ell\ell} \neq 0 \quad , \tag{26f}$$

or, from Eqs. (A1.3),

$$\Psi_3 \neq 0 \ , \quad \Phi_{22} \neq 0 \quad . \tag{26g}$$

Using Eqs. (26g), Eq. (26d), and the fact that there are no other constraints or the Riemann tensor ($\Psi_4 \neq 0$), one concludes that for the Will-Nordtvedt theory, <u>the E(2) classification is III_5.</u>

4. Stratified Theories[29]

$$\Box\, \varphi = 0 \quad , \tag{27a}$$

$$g = e^{2h(\varphi)}\, \underset{\sim}{\eta} + \left(e^{2f(\varphi)} - e^{2h(\varphi)}\right) \underset{\approx}{dt} \otimes \underset{\approx}{dt} \quad , \tag{27b}$$

or

$$g_{\alpha\beta} = e^{2h}\, \eta_{\alpha\beta} + (e^{2f} - e^{2h})\, \delta^0{}_\alpha\, \delta^0{}_\beta \quad , \tag{27c}$$

in a particular coordinate system, where f and h are given, unequal functions of the scalar field φ and $\underset{\approx}{dt}$ is a time-like one-form. The wave solution to Eq. (27a) is

$$\varphi = \varphi_0 + \varphi_1 e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} , \tag{27d}$$

as in Eq. (25d) and one can compute the Riemann tensor from $g_{\alpha\beta}$ using Eqs. (A1.1), (27c), and (27d). Contraction with $g_{\alpha\beta}$ then gives the linearized Ricci tensor:

$$R_{\beta\delta} = \varphi_1 e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} \left[(f' + g') q_\beta q_\delta - 2(f' - g') q^0 \delta^0{}_{(\delta} q_{\beta)} \right] , \tag{27e}$$

where $f' \equiv df/d\varphi$, etc. From Eq. (27e) one finds

$$R = - 2\varphi_1 (f' - g') e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} (q^0)^2 \neq 0 . \tag{27f}$$

From Eq. (27f), one concludes $\Psi_2 \neq 0$ [cf. Eq. (A1.4)], and consequently, for stratified theories, <u>the E(2) classification is II_6</u>.

Here we have a perfect example of a discrepancy between the number of dynamical degrees of freedom and the number of nonzero modes in the E(2) classification. Stratified theories clearly have only one dynamical degree of freedom, arising from the scalar field φ — yet some Lorentz observers see all six gravitational wave modes. The reason for this apparent paradox is that the "prior geometric"[7] one-form $\underset{\sim\sim}{dt}$ introduces another vector into the problem in addition to the wave vector $\underset{\sim}{q}$ — a vector which transforms in a complicated way under the Lorentz transformations which leave $\underset{\sim}{q}$ fixed. The Ricci tensor does not "point" only along the $\underset{\sim}{q}$ direction [cf., Eq. (27e)] and any pure mode feeds all the other modes under Lorentz transformations.

V. EXPERIMENTAL DETECTION AND CLASSIFICATION OF WAVES

A. The Ideal Detection Experiment

An experimenter attempting any foreseeable experiment to detect gravitational waves[16] faces two fundamental limitations which hinder the E(2) classification of detected waves: (i) He can measure only the six "electric" components R_{i0j0} of $\underset{\sim}{Riem}$, not all twenty.[31] (ii) He may not know that wave direction *a priori*; he may be hoping to infer it from his data, as does Weber.[32] We will find that the consequences of these limitations are that the experimenter can generally classify a wave unambiguously only if he knows the direction *a priori*, and that he can *never* determine the direction using a single detector. Other limitations (antenna pattern, noise, time-resolution, bandwidth, need for coincidence detection) complicate the task further, but to treat the heart of the classification problem, we will ignore them.

Consider an ideal detection experiment: The experimenter uses the coordinate system of Sec. II. He measures the relative accelerations of test masses and obtains via Eq. (1) the six components R_{i0j0} of $\underset{\sim}{Riem}$, with perfect accuracy and infinite time-resolution. He expresses his data as a 3×3, symmetric, "driving-force matrix" $\underset{\sim}{S}(t)$, with components

$$S_{ij}(t) \equiv R_{i0j0}(u) \quad ;$$

here t is his proper time, and he takes his spatial origin at his detector, so $t = u$.

The experimenter knows, by time-coherence of the signal or by some other means, that the wave originates in a single, localized source. He denotes the wave direction (which he may or may not know *a priori*) by a spatial unit vector $\vec{k}$. (In previous sections we have taken $\vec{k} = \vec{e}_z$; here it is arbitrary.)

Let us rename, for this section only, the amplitudes of a wave with direction $\vec{k}$, measured at the detector:

$$p_1(\vec{k},t) \equiv \Psi_2(u) \quad , \tag{28a}$$

$$p_2(\vec{k},t) \equiv \mathrm{Re}\ \Psi_3(u) \quad , \tag{28b}$$

$$p_3(\vec{k},t) \equiv \mathrm{Im}\ \Psi_3(u) \quad , \tag{28c}$$

$$p_4(\vec{k},t) \equiv \mathrm{Re}\ \Psi_4(u) \quad , \tag{28d}$$

$$p_5(\vec{k},t) \equiv \mathrm{Im}\ \Psi_4(u) \quad , \tag{28e}$$

$$p_6(\vec{k},t) \equiv \Phi_{22}(u) \quad . \tag{28f}$$

Let the index A = 1, 2, ... 6 run over these six modes. The amplitudes $p_A(\vec{k},t)$ are real.

For the case $\vec{k} = \vec{e}_{\hat{z}}$, Eqs. (2) imply

$$\underset{\sim}{S} = \begin{pmatrix} -\frac{1}{2}(p_4 + p_6) & \frac{1}{2}p_5 & -2p_2 \\ \frac{1}{2}p_5 & \frac{1}{2}(p_4 - p_6) & 2p_3 \\ -2p_2 & 2p_3 & -6p_1 \end{pmatrix} ,$$

or

$$\underset{\sim}{S}(t) = \Sigma_A\ p_A(\vec{e}_{\hat{z}},t)\ \underset{\sim}{E}_A(\vec{e}_{\hat{z}}) \quad , \tag{29}$$

where "basis polarization matrices" $\underset{\sim}{E}_A(\vec{e}_{\hat{z}})$ belonging to wave direction $\vec{k} = \vec{e}_{\hat{z}}$ are defined by

$$\underset{\sim}{E}_1(\vec{e}_{\hat{z}}) = -6\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underset{\sim}{E}_2(\vec{e}_{\hat{z}}) = -2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ,$$
$$\underset{\sim}{E}_3(\vec{e}_{\hat{z}}) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \underset{\sim}{E}_4(\vec{e}_{\hat{z}}) = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$
$$\underset{\sim}{E}_5(\vec{e}_{\hat{z}}) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{\sim}{E}_6(\vec{e}_{\hat{z}}) = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{30}$$

Equation (29) represents $\underset{\sim}{S}(t)$ as a superposition of modes with $\vec{k} = \vec{e}_{\hat{z}}$.

For any other $\vec{k}$, just rotate these matrices: Let $\underset{\sim}{R}$ be a 3×3 rotation matrix[33] that takes $\vec{e}_{\hat{z}}$ into $\vec{k}$:

$$\vec{k} = \underset{\sim}{R} e_{\hat{z}} \quad .$$

Define unit polarization matrices $\underset{\sim}{E}_A(\vec{k})$ for wave direction $\vec{k}$ by

$$\underset{\sim}{E}_A(\vec{k}) = \underset{\sim}{R}\, \underset{\sim}{E}_A(\vec{e}_{\hat{z}})\, \underset{\sim}{R}^T \quad .$$

Then for any $\underset{\sim}{S}(t)$ and any $\vec{k}$, there is the unique representation

$$\underset{\sim}{S}(t) = \Sigma_A\, p_A(\vec{k}, t)\, \underset{\sim}{E}_A(\vec{k}) \quad ; \tag{31}$$

the amplitudes $p_A(\vec{k}, t)$ may be extracted from $\underset{\sim}{S}(t)$ by

$$p_A(\vec{k}, t) = C_A \;\mathrm{Trace}\Big(\underset{\sim}{E}_A(\vec{k})\; \underset{\sim}{S}(t)\Big) \quad , \tag{32}$$

where C_A are normalization constants:

$$C_A = \left(\frac{1}{36}\, , \;\; \frac{1}{8}\, , \;\; \frac{1}{8}\, , \;\; 2,\; 2,\; 2\right) \quad .$$

Equation (32) follows from Eq. (31) and an orthogonality property of the $\underset{\sim}{E}_A(\vec{k})$:

$$C_A \;\mathrm{Trace}\Big(\underset{\sim}{E}_A(\vec{k})\; \underset{\sim}{E}_B(\vec{k})\Big) = \delta_{AB} \quad .$$

Equations (31) and (32) embody an important principle: Any measured $\underset{\sim}{S}(t)$ can be represented uniquely as a superposition of the six modes belonging to any arbitrary wave direction $\vec{k}$. Equation (32) specifies the amplitude in each mode of this wave. This wave is generally of class II_6, but it can be less general for certain $\underset{\sim}{S}(t)$ and certain $\vec{k}$.

The classification procedure now splits into two cases: $\vec{k}$ known and $\vec{k}$ unknown.

B. The Case of Known Direction

The experimenter knows $\vec{k}$ <u>a priori</u> if the source of a gravitational wave that he detects can be identified with an object observed by means of electromagnetic radiation (light, radio, X-ray). There are also purely gravitational methods for determining $\vec{k}$. For example, if several detectors a distance $\gtrsim D$ apart, each with time-resolution $\ll D/c$, detect a sharp wave burst with pulse-width $\ll D/c$, then experimenters can determine $\vec{k}$ from the relative time-of-arrival at each detector. For $D \sim$ radius of Earth, $D/c \sim 13$ msec.

Knowing $\vec{k}$, the experimenter extracts from $\underset{\sim}{S}(t)$ the amplitudes $p_A(\vec{k}, t)$ by Eq. (32). Knowing the amplitudes, he classifies the wave unambiguously, using the prescription given in Sec. II. The theoretical implications of his results are discussed in subsection E below.

C. The Case of Unknown Direction

If the experimenter does not know $\vec{k}$ <u>a priori</u>, he cannot hope to determine it from $\underset{\sim}{S}(t)$ without further assumptions; he can fit $\underset{\sim}{S}(t)$ equally well for any $\vec{k}$ in the sky by using Eqs. (31) and (32). Neither can he extract the p_A unambiguously. However, knowledge of $\underset{\sim}{S}(t)$ always provides information which limits the E(2) class of the wave and also the class of the correct theory of gravity (see E below).

He limits the possible class of the wave in the following way: For each arbitrary $\vec{k}$ in the sky, he computes the $p_A(\vec{k}, t)$ via Eq. (32) and determines the E(2) class associated with that $\vec{k}$. By letting $\vec{k}$ range all over the sky, he obtains the set of possible E(2) classes for that wave.

For a given $\underset{\sim}{S}(t)$, the following recipe yields a complete analysis of the possible E(2) classes of the wave. One distinguishes several cases according to the form of $\underset{\sim}{S}(t)$. Figure 3 diagrams this recipe as a flow-chart.

Case 1. Driving forces remain in a fixed line. There is a fixed coordinate system in which

$$\underset{\sim}{S}(t) = \begin{pmatrix} \lambda(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{33}$$

Pattern of forces is as in Fig. 1(d); but propagation direction need not be as in Fig. 1(d). Conclusion: Wave is II_6 or N_3.

Case 2. Driving forces remain in a fixed plane: There is a fixed coordinate system in which

$$\underset{\sim}{S}(t) = \begin{pmatrix} \lambda(t) & \mu(t) & 0 \\ \mu(t) & \nu(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{34}$$

but none in which Eq. (33) holds. Wave may always be II_6. In addition, two separate determinations must be made: (a) Can the wave be O_1, N_2, or N_3? (b) Can the wave be III_5?

Test 2.a; for O_1, N_2, or N_3.

Subcase 2.a.i. Driving forces are "pure monopole":

$$\lambda(t) \equiv \nu(t) , \quad \mu(t) \equiv 0 . \tag{35}$$

Pattern of forces is as in Fig. 1(c); but wave need not be pure Φ_{22}. Conclusion: Wave may be O_1. (Furthermore, wave cannot be III_5; test (b) is always failed.)

Subcase 2.a.ii. Driving forces are "pure quadrupole":

$$\lambda(t) \equiv - \nu(t) . \tag{36}$$

Pattern of forces is as in Fig. 1(a) (and the principal axes may rotate with time in the transverse plane); but propagation direction need not be as in Fig. 1(a). Conclusion: Wave may be N_2.

Subcase 2.a.iii. Driving forces are neither "pure monopole" nor "pure quadrupole": Neither Eq. (35) nor Eq. (36) holds. Conclusion: Wave may be N_3.

Test 2.b; for III_5. Wave may be III_5 if and only if there exists a fixed unit vector $\vec{k}$ not normal the plane of the forces [i.e.,

$$\vec{k} \neq \vec{e}_{\hat{z}} \quad ,$$

in the coordinates of Eq. (34)] such that

$$\vec{k} \cdot \underset{\sim}{S}(t) \cdot \vec{k} \equiv 0 \quad . \tag{37}$$

The complete set of possibilities for Case 2 is II_6 plus the outcomes of Test 2.a and Test 2.b.

Case 3. Driving forces do not remain in any fixed plane: Equation (34) does not hold in any fixed coordinate system. Wave may always be II_6. It may be III_5 if and only if there exists a fixed unit vector $\vec{k}$ such that

$$\vec{k} \cdot \underset{\sim}{S}(t) \cdot \vec{k} \equiv 0 \quad . \tag{38}$$

Note that when the driving forces do not occur in one plane and Eq. (38) is violated, the wave must be II_6.

D. Guessing $\vec{k}$

We have emphasized that $\vec{k}$ can never be extracted from $\underset{\sim}{S}(t)$. However, the fact that a certain $\underset{\sim}{S}(t)$ can be fitted by a wave of a certain class less general than II_6 must weigh as strong circumstantial evidence that the wave is actually of that class. If one is willing to assume that the simplest allowed classification is correct, then $\vec{k}$ is generally fixed uniquely (up to an inevitable antipodal ambiguity, $\vec{k} \to -\vec{k}$).

Referring to the recipe above, the information that one can guess in this way is as follows.

Case 1. If the wave is N_3, $\vec{k}$ lies anywhere in the plane spanned by $\vec{e}_{\hat{y}}$ and $\vec{e}_{\hat{z}}$ in the coordinates of Eq. (33).

Case 2. If the wave is O_1, N_2, or N_3, $\vec{k}$ is normal to the plane of the forces:

$$\vec{k} = \pm \vec{e}_{\hat{z}} \; ,$$

in the coordinates of Eq. (34). If the wave is III_5, $\vec{k}$ is as in Eq. (37).

Case 3. If the wave is III_5, $\vec{k}$ is as in Eq. (38).

One can never limit the direction of a II_6 wave in this way.

E. Theorerical Implications of Experimental Results

The E(2) class of the correct theory of gravity is at least as general as that of any observed wave: This is always the fundamental implication of any observation. We must always qualify, "at least as general," because in any particular theory a particular source may couple poorly or not at all to some of the admissable modes, and therefore it may radiate only special classes of waves. But the observation of a wave of a certain class always rules out all theories of less general classes.

If the wave direction is unknown, an observed wave cannot be classified unambiguously (except for some waves of class II_6). However, there is always a least general possible class for each such wave, which limits the correct theory.

There are still sharper implications for particular theories. In the case of a well-understood source (e.g., binary star system), each particular theory should make a precise prediction about the mixture of modes radiated, leading to a crucial test. We shall discuss this point in a future paper. In the case of a theory for which the number of degrees of freedom is less than the dimension of the E(2) class (see Sec. IV.B), the various admissable modes should appear only in definite mixtures by <u>any</u> source, again leading to a crucial test. Finally, the difference in propagation speed for light and for gravitational waves leads to a crucial test for many theories (see Sec. IV.A).

ACKNOWLEDGMENTS

We are grateful to K. S. Thorne, R. V. Wagoner, C. M. Will, and W. H. Press for conversations and helpful comments, and to K. S. Thorne for a careful reading of the manuscript.

APPENDIX 1. USEFUL FORMULAE FOR PLANE WAVES

General linearized Riemann tensor in terms of flat space perturbation $h_{\mu\nu}$:

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) \quad . \tag{A1.1}$$

Tetrad components of Riemann tensor in terms of h_{ab}:

$$\Psi_2 \equiv -\frac{1}{6} R_{\ell k \ell k} = \frac{1}{12}\ddot{h}_{kk} \quad , \tag{A1.2a}$$

$$\Psi_3 \equiv -\frac{1}{2} R_{\ell k \ell \bar{m}} = \frac{1}{4}\ddot{h}_{k\bar{m}} \quad , \tag{A1.2b}$$

$$\Psi_4 \equiv -R_{\ell \bar{m} \ell \bar{m}} = \frac{1}{2}\ddot{h}_{\bar{m}\bar{m}} \quad , \tag{A1.2c}$$

$$\Phi_{22} \equiv -R_{\ell m \ell \bar{m}} = \frac{1}{2}\ddot{h}_{m\bar{m}} \quad , \tag{A1.2d}$$

(where $\ddot{h} \equiv d^2h/du^2$).

Tetrad components of Ricci tensor:

$$R_{\ell k} = R_{\ell k \ell k} \quad , \tag{A1.3a}$$

$$R_{\ell \ell} = 2\, R_{\ell m \ell \bar{m}} \quad , \tag{A1.3b}$$

$$R_{\ell m} = R_{\ell k \ell m} \quad , \tag{A1.3c}$$

$$R_{\ell \bar{m}} = R_{\ell k \ell \bar{m}} \quad . \tag{A1.3d}$$

Ricci scalar:

$$R = -2\, R_{\ell k} = -2\, R_{\ell k \ell k} \quad . \tag{A1.4}$$

APPENDIX 2. INDECOMPOSABLE GROUP REPRESENTATIONS

Let G be a group and $\mathcal{D}$ a linear representation of G on a linear space V. $\mathcal{D}$ is reducible, if it has an invariant proper subspace, $V_1 \subset V$. $\mathcal{D}$ is decomposable, if V is the direct sum of invariant proper subspaces. A decomposable representation is always reducible but not vice versa; $\mathcal{D}$ is indecomposable, if it is reducible but not decomposable. $\mathcal{D}$ is decomposable, if, and only if, there is a basis of V for which each $g \in G$ is represented by a block-triangular matrix

$$\begin{pmatrix} g_1 & 0 \\ g_3 & g_2 \end{pmatrix},$$

with not all g_3 vanishing.

Indecomposable representations never occur for a finite group G, for finite-dimensional representations of a semi-simple Lie group G, or for unitary representations of any Lie group G. Because of these facts, physicists are not well acquainted with indecomposable representations. For a physicist, indecomposable representations have two unpleasant attributes: (i) They are always nonunitary. (ii) There is no analog of Schur's lemma: An invariant operator is not generally constant on an indecomposable representation; e.g., "spin" is undefined.

See Ref. 27 or Ref. 34 for a discussion of these concepts.

For waves of E(2) class II_6 or III_5, we deal with 6- or 5-dimensional indecomposable representations of E(2). The only finite-dimensional decomposable representations of E(2) decompose to the familiar 1-dimensional unitary representations that describe a massless quantum particle of integral or half-integral helicity[25-27]; these representations arise for E(2) classes N_3, N_2, and O_1.

TABLE I. E(2) classification of various metric theories of gravity. See Sec. IV.

Theory	E(2) class	Degrees of freedom[28]	$c_g = c_{em}$?	Currently viable?	Equal to GRT in PPN limit[a]?
GRT[4]	N_2	2	yes	yes	yes
Dicke-Brans-Jordan[5,b]	N_3	3	yes	yes	no
Conformally flat theories[29]	O_1	1	yes	no	no
Stratified theories[29]	II_6	1	c	no	no
Will-Nordtvedt[11]	III_5	5	yes	yes	yes
Lightman-Lee[13]	II_6	6	no	yes	yes
Ni[10]	II_6	1	no	yes	yes
Hellings-Nordtvedt[12]	N_3	?	yes	yes	yes

[a] If a theory can be made to coincide with GRT in the PPN limit[6] by a particular choice of arbitrary constants and/or possible cosmological boundary values, we put a "yes" in this column.

[b] Typical of scalar-tensor theories.[29]

[c] Depends on the particular theory.

REFERENCES

[1] C. M. Will, Cource 56 in Proceedings of the International School of Physics "Enrico Fermi," B. Bertotti ed. (Academic Press, in press); also distributed as Caltech preprint OAP-289 (1972).

[2] K. Nordtvedt, Jr., Phys. Rev. **180**, 1293 (1969); C. M. Will, Astrophys. J. **163**, 611 (1971).

[3] C. O. Alley, R. F. Chang, D. G. Currie, S. K. Poultney, P. L. Bender, R. H. Dicke, D. T. Wilkinson, J. E. Faller, W. M. Kaula, G. J. F. MacDonald, J. D. Mulholland, H. H. Pitkin, W. Carrion, and E. J. Wampler, Science, **167**, 458 (1970).

[4] See any standard textbook on general relativity, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman and Co., San Francisco, 1973).

[5] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[6] The PPN formalism was developed largely by Kenneth Nordtvedt, Jr. and Clifford M. Will. For a complete discussion, see Ref. 1. See also C. M. Will and K. Nordtvedt, Jr., Astrophys. J. **177**, 757 (1972).

[7] For the definitions of certain terms and concepts in this paper, see K. S. Thorne, D. L. Lee, and A. P. Lightman, Phys. Rev. in press (1973).

[8] P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (N. Y.) **26**, 422 (1964); V. B. Braginsky and V. I. Panov, Zh. Eksp. Teor. Fiz. **61**, 875 (1971). [Sov. Phys.—JETP **34**, 3, 463 (1972).]

[9] A. P. Lightman and D. L. Lee, Phys. Rev. (in press). Also distributed as Caltech preprint OAP-314.

[10] W.-T. Ni, Phys. Rev. in press (1973). Also available as Caltech preprint OAP-306.

[11] C. M. Will and K. Nordtvedt, Jr., Astrophys. J. **177**, 757 (1972).

[12] R. W. Hellings and K. Nordtvedt, Jr., submitted to Phys. Rev. We consider this theory only for parameters $\omega \neq 0$, $\eta = 0$.

[13] A. P. Lightman and D. L. Lee, submitted to Phys. Rev. (1973).

[14] R. V. Wagoner, paper presented at the Caltech Conference on Relativity, October 27, 1972.

[15] D. M. Eardley, D. L. Lee, A. P. Lightman, R. V. Wagoner, and C. M. Will, Phys. Rev. Letters **30**, 884 (1973).

[16] W. H. Press and K. S. Thorne, Ann. Rev. Astron. & Astrophys. **10**, 335 (1972).

[17] In this paper Latin indices range over 1, 2, 3, and Greek indices over 0, 1, 2, 3. We choose units such that $c = G = 1$ and we denote 4-vectors, 3-vectors, and tensors by $\underset{\sim}{v}$, $\vec{k}$, $\underset{\approx}{g}$ respectively. Other conventions are as in Ref. 5.

[18] See, e.g., S. Gasiorowicz, Elementary Particle Physics (John Wiley and Sons, Inc., New York, 1966) p. 10.

[19] For details of the null-tetrad formalism, see, e.g., E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962); see errata, J. Math. Phys. **4**, 998 (1962). Note that we use a different convention in contracting the Riemann tensor to obtain the Ricci tensor, i.e., $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$, and the opposite signature; hence the sign differences in Φ's and Λ.

[20] A. Trautman, Gravitation — An Introduction to Current Research, L. Witten ed. (John Wiley and Sons, Inc., New York, 1962), p. 184.

[21] The metric perturbation $h_{\mu\nu}$ is, of course, not invariant under infinitesimal coordinate ("gauge") transformations. For any weak, plane, null wave, there exists $h_{\mu\nu}$ with the property that it is a function of u only, but this property is not gauge invariant.

[22] F. A. E. Pirani in Lectures on General Relativity, 1964 Brandeis Summer Institute in Theoretical Physics, A. Trautman ed. (Prentice-Hall, Englewood Cliffs, New Jersey), Vol. I.

[23] R. K. Sachs in Relativity, Groups and Topology, C. DeWitt and B. DeWitt eds. (Gordon and Breach, New York, 1964).

[24] A. Z. Petrov, Sci. Nat. Kazan State University **114**, 55 (1954); see also Ref. 19.

[25] See, e.g., S. Gasiorowicz, Elementary Particle Physics (John Wiley and Sons, Inc., New York, 1966), Chap. 4.

[26] E. Wigner, Ann. Math. **40**, 39 (1939).

[27] Iu. M. Shirokov, Zh. Eksp. Teor. Fiz. **33**, 861 (1957); **33**, 1196 (1957); **33**, 1208 (1957); **34**, 717 (1958). [Sov. Phys.—JETP **6**, 664 (1958); **6**, 919 (1958); **6**, 929 (1958); **7**, 493 (1958).] Shirokov says "completely reducible" where we say "decomposable."

[28] We define the "number of degrees of freedom" as being the number of independent components of dynamical variables obeying a wave equation, after constraints and coordinate arbitrariness have been subtracted.

[29] For a list of "stratified theories" and related discussion, and for a similar discussion of "conformally flat theories," see W.-T. Ni, Astrophys. J. **176**, 769 (1972).

[30] All of the coordinate components of Riemann and Ricci tensors are real, and one should always take the real part of expressions for these quantities. The reader should not confuse this with the complex tetrad components, obtained by projecting real coordinate components onto complex basis vectors.

[31] Detectors have been proposed that measure the "magnetic" components R_{i0jk}, but none seem practical; see Ref. 16, also F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. **5**, 1629 (1964). Using such a detector in conjunction with a conventional one, an experimenter could uniquely classify any wave and determine its direction.

[32] J. Weber, Phys. Rev. Letters **22**, 1302 (1969); **24**, 276 (1969); **25**, 180 (1970).

[33] There is actually a one-parameter family of such R; the members differ only in a final rotation about $\vec{k}$. This final rotation only changes the phase of Ψ_3 and Ψ_4 and hence cannot change the ultimate classification.

[34] H. Weyl, The Theory of Groups and Quantum Mechanics (Dover, New York, 1931), Chap. III, Sec. 4. Weyl says "completely reducible" where we say "decomposable."

FIGURE CAPTIONS

Fig. 1. The six polarization modes of a weak, plane, null gravitational wave permitted in the generic metric theory of gravity. Shown is the displacement that each mode induces on a sphere of test particles. The wave is propagating in the +z direction (arrow at upper right) and has time dependence cos ωt. The solid line is a snapshot at $\omega t = 0$, the broken line one at $\omega t = \pi$. There is no displacement perpendicular to the plane of the figure.

Fig. 2. The E(2) classes of weak, plane, null waves, displayed in order of increasing generality toward the top. Descending along a line represents specializing the class by demanding that some amplitude vanish for all observers. One class is said to be more general than another if it is possible to descend from one to the other along lines.

Fig. 3. Prescription for finding possible E(2) classes for a wave of unknown direction $\vec{k}$, given the driving-force matrix $\underset{\sim}{S}(t)$. Boxes contain tests involving $\underset{\sim}{S}(t)$ and circles contain possible classes. See text of Sec. V.

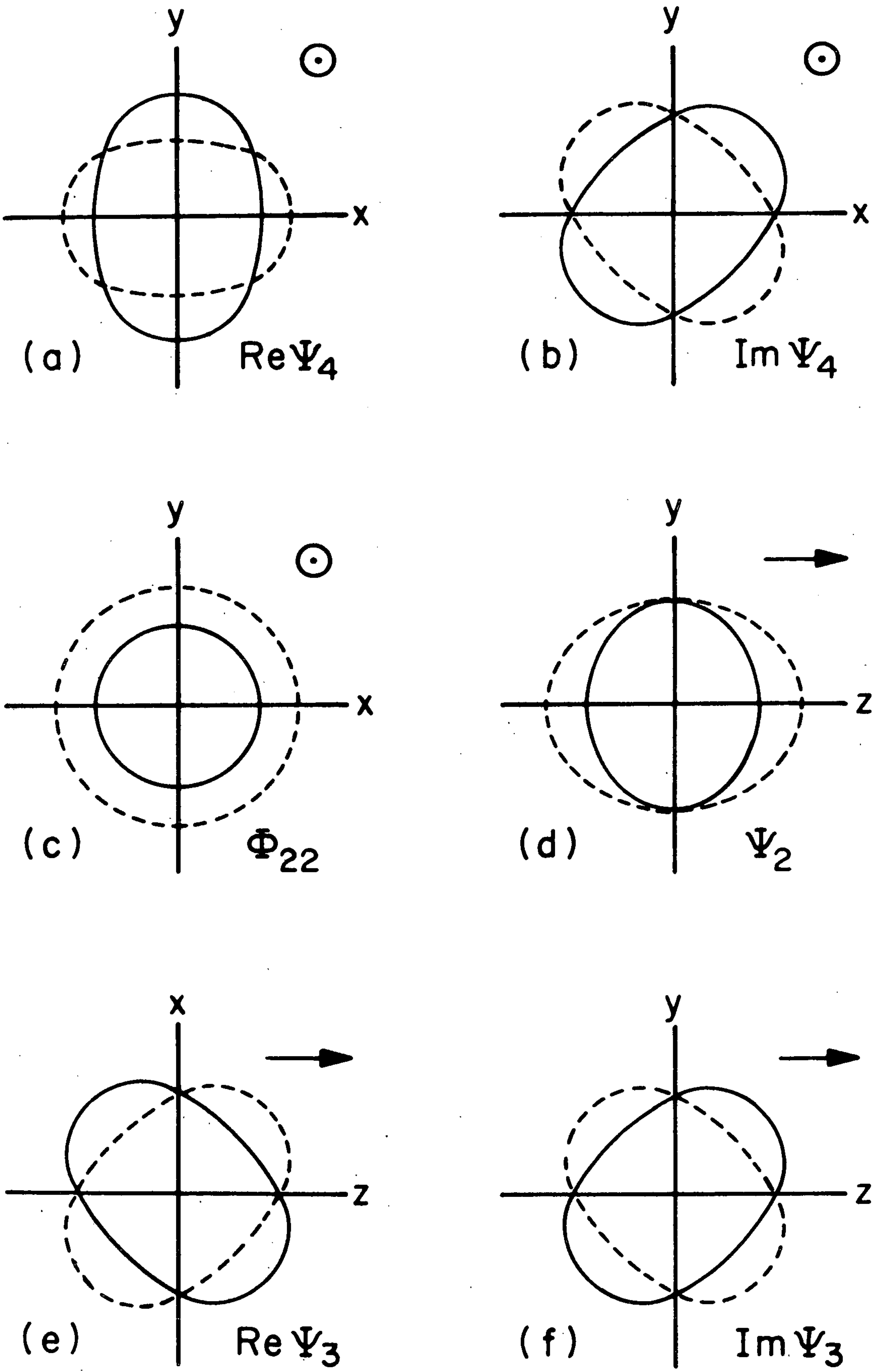

y
x
(a)
Re Ψ_4
y
x
(b)
Im Ψ_4
y
x
(c)
Φ_{22}
y
z
(d)
Ψ_2
x
z
(e)
Re Ψ_3
y
z
(f)
Im Ψ_3

Fig. 1

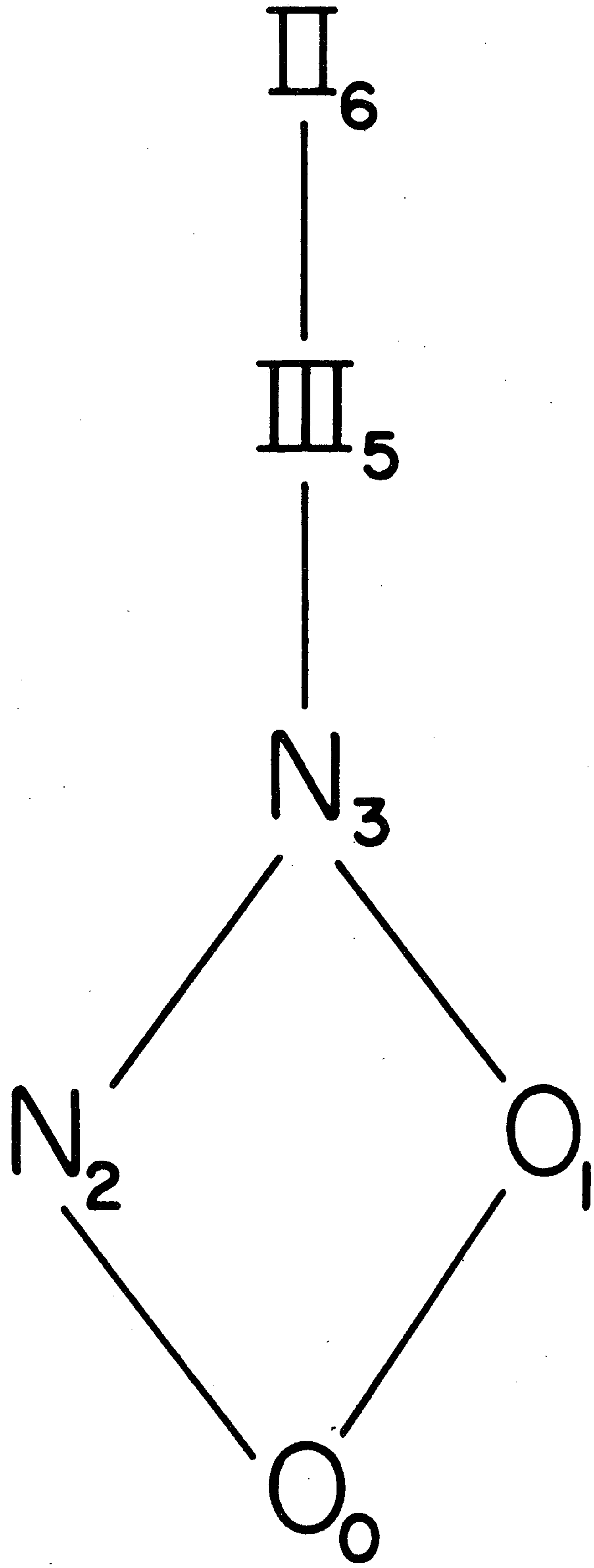
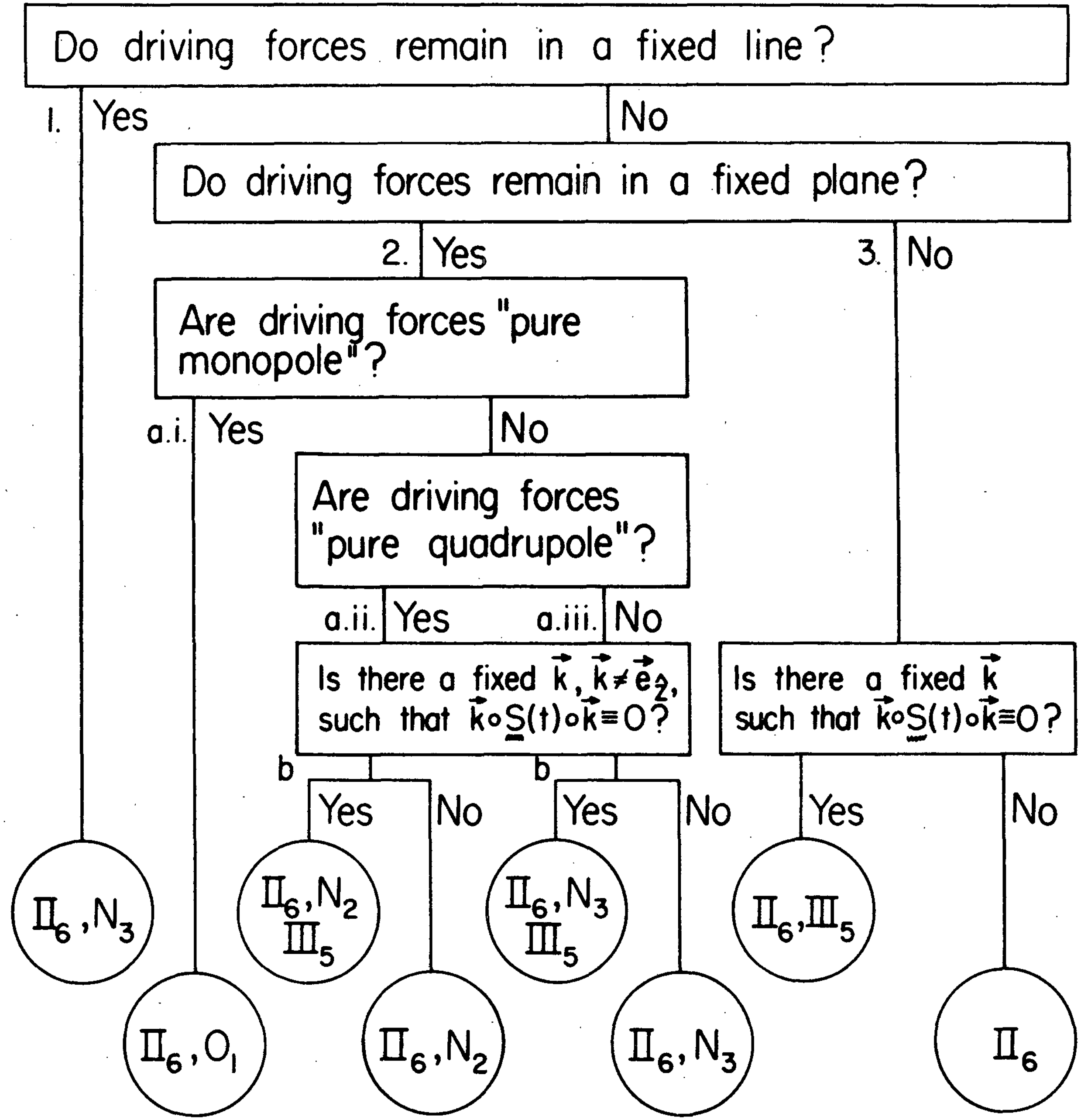


Fig. 2

41

42



Do driving forces remain in a fixed line?
1. Yes
No
Do driving forces remain in a fixed plane?
2. Yes
3. No
Are driving forces "pure monopole"?
a.i. Yes
No
Are driving forces "pure quadrupole"?
a.ii. Yes
a.iii. No
Is there a fixed $\vec{k}$, $\vec{k} \neq \vec{e}_2$, such that $\vec{k} \circ \underline{S}(t) \circ \vec{k} \equiv 0$?
Is there a fixed $\vec{k}$ such that $\vec{k} \circ \underline{S}(t) \circ \vec{k} \equiv 0$?
b
Yes
No
b
Yes
No
Yes
No
II_6, N_3
II_6, O_1
II_6, N_2 III_5
II_6, N_2
II_6, N_3 III_5
II_6, N_3
II_6, III_5
II_6

Fig. 3