INVESTIGATIONS OF EARTH DYNAMICS FROM SATELLITE OBSERVATIONS

Final Report

Grant NGR 09-015-103

Principal Investigator

Dr. Edward M. Gaposchkin

March 1973

Prepared for
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ABSTRACT

The consequences of the earth's elasticity are examined for close-earth satellites. The ideas of polar motion and earth tides are developed in a form applicable to satellite studies, since the polar motion, the body tide, and the ocean tide are all suitable for study by use of satellites. Analysis of available polar-motion data is performed.

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Final Report

1. INTRODUCTION

The principal aim of this grant, initiated in 1969, was to explore the use of precision satellite-tracking data for studying the rotating elastic earth, i.e., polar motion, earth tides, and related phenomena. The data available were camera observations taken for geodetic purposes, with a limited amount of precision laser tracking data. These tracking data had been used in the determination of the 1969 Smithsonian Standard Earth (II) (Gaposchkin and Lambeck, 1971).

This study had the following aspects: 1) Theoretical investigation of polar-motion dynamics; 2) assembly of some mathematical tools; 3) study of the consequences of polar motion and tidal deformation on satellite observations and orbits; 4) incorporation of these effects into the Smithsonian Astrophysical Observatory (SAO) geodesy programs where appropriate; 5) analysis of polar-motion measurements made by classical astronomical methods; 6) assembly of the available tracking data for polar-motion determination; and 7) determination of polar motion based on satellite observations.

The principal results are as follows:

1) Improved values of polar motion and tidal parameters can be obtained from analysis of satellite-tracking data. Polar motion is most easily determined from kinematic effects, and body tides are most easily determined from analysis of satellite perturbations.

2) Analysis of existing tracking data gives reasonable results for tidal parameters, but the determination of polar motion was not possible with the data and orbits available at that time.

3) Polar motion and the motion of the principal axis of the earth can eventually be studied through satellite determination of $\bar{C}_{21}$ and $\bar{S}_{21}$. The orbital effects are approximately 1 m.

4) The body tide is most easily studied through the gravitational perturbation of satellite orbits. An important contribution to the tidal perturbation of satellite orbits comes from the ocean tide. There is so little agreement as to the size of the second-degree global ocean tide that it must be considered unknown at this time. Consequently, certain components of the ocean tides may have to be determined from satellite observations.

5) The ocean-tide effect has been ignored in previous work on the determination of the body tide. Its size could explain the variety of numerical values for the Love number k_2.

2. POLAR MOTION AND EARTH TIDES

Polar motion and earth tides depend on the elasticity of the earth – more precisely, on the rigidity $\tilde{\mu}$. The theory for polar motion, i.e., free nutation, for a rigid body was worked out by Euler. Using the precession to obtain $(C-A)/C$, Euler predicted a period of free nutation of 307 days. Attempts to observe this free nutation were unsuccessful until 1890, when Küstner demonstrated the existence of periodic variations of latitude. Shortly thereafter, Chandler showed the period for free nutation to be approximately 430 days rather than 307. Newcomb then demonstrated that the elasticity of the earth would lengthen the period of rigid-body free nutation and estimated the rigidity to be approximately that of steel or glass. Poincaré found the effects of the liquid core would shorten the free period, and Lord Kelvin demonstrated that the oceans would lengthen the period. The principal reference on polar motion and its geophysical consequences is Munk and MacDonald (1960).

The theory of forced nutation, assuming a rigid earth, was completed by Oppolzer in 1882. Schweydar concluded that the elasticity would not have any other effect on the nutation. The theory and discussion by Woolard (1963) of the forced nutation is most complete. An excellent reference on this subject is Federov (1963).

In 1876, Kelvin discussed the effects of earth deformation on the ocean and body tides. George Darwin and others attempted to measure the body tides. Darwin, in 1883, conceived of using the measured ocean tide itself, i.e., the difference between the absolute ocean tide and the absolute body tide, to establish the amount of body tide. Using long-period ocean tides (mostly monthly and semimonthly lunar tides), he found the amplitude to be two-thirds of the rigid earth tide and computed the rigidity of the earth of be the same as that of steel. A fine reference for body tides is Melchior (1966).

The interaction of body tides, ocean tides, and polar motion has never been completely understood; in some ways, our knowledge of the elastic and dissipative processes in the earth is not much advanced since the time of Kelvin and Darwin. Dissipation presents a particularly formidable problem. The polar motion would decay to zero without continuous excitation, yet no energy source seems capable of providing sufficient energy (see Section 9). Similarly, body-tide observations with tidal gravimeters, though extensive, are not sufficient to indicate how much tidal lag there is in the solid earth. The time scale of the earth–moon system, given the current rate of lunar deceleration, is too small by a factor of 2 or 3. Dissipation from the solid earth would be continuous over geologic time, where the contribution from shallow seas would depend on their distribution.

A general theory for ocean tides has never been formulated. The ocean tide is enormously complicated owing to irregular boundaries, loading, dissipation, and the necessary inclusion of inertial effects. Several recent papers are quite inconsistent in terms of the assumptions made and the results given. Compare, for example, Hendershott (1972) and Pekeris and Accad (1969); Hendershott and Munk (1970) give a general criticism of the subject. On the other hand, satellite orbits are sensitive to the ocean tide (see Section 8), and these tidal terms must be included to compute orbits of the highest precision. It is therefore imperative that the questions about ocean tides be resolved, perhaps by use of orbit analysis.

We will see that satellite measurements allow some unification of the effects of the earth's elasticity. The determination of polar motion will be accomplished from kinematic effects, and the solid-earth and ocean tides will be determined from gravitational effects.

3. DYNAMICAL PROPERTIES OF POLAR MOTION

The mathematical formulation of polar motion and body tides is most easily developed from the Liouville equation (Munk and MacDonald, 1960; hereafter referred to as MM, p. 10). Following MM, we adopt a set of geographic axes because they have a precise realization in terms of the position of observing stations. As long as the deformations and motions are small, we can treat the Liouville equation in the form:

$$
\begin{aligned}
\dot{\ell}_1 + \sigma_r \ell_2 &= \sigma_r \phi_2 , \\
\dot{\ell}_2 - \sigma_r \ell_1 &= -\sigma_r \phi_1 , \\
\dot{\ell}_3 &= \dot{\phi}_3 ,
\end{aligned}
\tag{3.1}
$$

where

$$
\sigma_r = \frac{C - A}{A} \omega_e , \quad \dot{(\ )} = \frac{d}{dt} , \tag{3.2}
$$

and

$$
\begin{aligned}
\phi_1 &= \frac{1}{(C - A)\,\omega_e^2} \left(\omega_e^2 c_{13} + \omega_e \dot{c}_{23} + \omega_e h_1 + \dot{h}_2 - L_2 \right) , \\
\phi_2 &= \frac{1}{(C - A)\,\omega_e^2} \left(\omega_e^2 c_{23} - \omega_e \dot{c}_{13} + \omega_e h_3 - \dot{h}_1 + L_1 \right) , \\
\phi_3 &= \frac{1}{\omega_e^2 C} \left(-\omega_e^2 c_{33} - \omega_e h_3 + \omega_e \int_0^t L_3 \, dt \right) .
\end{aligned}
\tag{3.3}
$$

The definitions are given in Chapter 6 of MM.

We have

$$\hat{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \quad , \text{ the direction of the spin axis,}$$

or approximately the direction of the angular momentum of the earth; A, B, and C are the principal moments of inertia; c_{ij} are the small contributions to the inertia tensor due to deformation; ω_e is the rotation rate of the earth; and h_i and L_i are the components of momentum and torque.

In this study, we are not interested in the momentum and torque contributions, or the variation in the rotation rate. By introducing the complex variables

$$\begin{aligned} \tilde{\ell} &= \ell_1 + i\ell_2 \quad , \\ \tilde{\phi} &= \phi_1 + i\phi_2 \quad , \end{aligned} \qquad (3.4)$$

where

$$i = \sqrt{-1} \quad ,$$

the equations reduce to

$$\begin{aligned} i\dot{\ell} + \sigma_r \ell &= \sigma_r \phi \quad , \\ \sigma_r \phi &= \frac{1}{A} \left[\omega_e c_{13} + \dot{c}_{23} + i(\omega_e c_{23} - \dot{c}_{13}) \right] \quad . \end{aligned} \qquad (3.5)$$

Assuming a rigid body, we have $c_{12} = c_{23} = \dot{c}_{23} = \dot{c}_{13} = 0$ and

$$\dot{\tilde{\ell}} - i\sigma_r \tilde{\ell} = 0 \quad ,$$

which has the solution

$$\tilde{\ell} = c_1 e^{-i\sigma_r t} + c_2 \quad ;$$

c_1, c_2 are constants. The free period

$$\tau = A/(C - A)/\omega_e \quad .$$

If $\omega_e \approx 1$ revolution per day and $(C - A)/A$ is obtained from the precessional constant $H = (C - A)/C = 0.0032729$, i.e., $(C - A)/A = (1 - H)(C - A)/C = 0.003262$, we have a free period of 306.5 days, the result Euler obtained.

The deformation of the earth is a complicated problem in elasticity. We consider forces as derived from disturbing potentials. A. E. H. Love introduced some dimensionless parameters that summarize some of the earth's elastic properties. Their detailed theoretical computation has been carried out by Takeuchi (1950), Longman (1963), and Farrell (1972). These numbers* are defined in the following way. The displacement of the ground due to a potential of degree n is

$$\frac{h_n}{g} U_n^{(\text{surface})} \quad , \tag{3.6}$$

and the additional gravitational potential due to the motion of mass is

$$k_n U_n \left(\frac{a_e}{r}\right)^{2n+1} \quad . \tag{3.7}$$

If the force loads the earth, e.g., the ocean, then h_n' and k_n' are used. The response of the earth to deformation certainly changes with position. Kaula (1969) considers the Love numbers expressed in spherical harmonics.

---

*There is no direct way to measure k and h by terrestrial means. Only the linear combinations $1 + k - h$ from the deflection of the vertical and $1 + h - (3/2)k$ from gravity. However, k_2 is determined from the free nutation. The results are not altogether consistent, and interpretation is complicated by the frequency dependence of the Love numbers owing to resonance effects with the core. We can say that $k_2 \approx 0.3$ and $h_2 \approx 0.61$ and that $k_2 \leq 0.504\, h_2$ (Melchior, 1966).

The displacements due to rotation about ℓ_1, ℓ_2, ℓ_3 are the same as those due to the potential of second degree:

$$\frac{1}{2}\,\omega_e^2(x^2+y^2) - \omega_e^2\, z(\ell_1 x + \ell_2 y) \quad ,$$

x, y, and z being the coordinates of the displaced point. The first term adds to the oblateness and does not concern us. The second term gives rise to the additional external gravitational potential

$$-k_2\,\omega_e^2\, z(\ell_1 x + \ell_2 y)\left(\frac{a_e}{r}\right)^5 \quad . \tag{3.8}$$

MacCullagh's formula (Jeffreys and Jeffreys, 1956) gives the gravitational potential due to a deformed earth:

$$V = G\left[\frac{M_\oplus}{r} + \frac{(A+B+C)\,r^2 - 3\,(A x^2 + B y^2 + C z^2 + 2\,c_{23}\,yz + 2\,c_{13}\,xz + 2\,c_{12}\,xy)}{2\,r^5}\right] . \tag{3.9}$$

Comparing terms, we have

$$\begin{aligned} c_{23} &= k_2\,\frac{\omega_e^2\,a_e^5}{3G}\,\ell_2 = \sigma\ell_2 \quad , \\ c_{13} &= k_2\,\frac{\omega_e^2\,a_e^5}{3G}\,\ell_1 = \sigma\ell_1 \quad , \\ \sigma &= k_2\,\frac{\omega_e^2\,a_e^5}{3G} \quad . \end{aligned} \tag{3.10}$$

Inserting these into Eq. (3.5), we have

$$\begin{aligned} \sigma_r\tilde{\phi} &= \frac{\sigma}{A}\,[\omega_e\ell_1 + \dot{\ell}_2 + i\,(\omega_e\ell_2 - \dot{\ell}_1)] \\ &= \frac{\sigma}{A}\,(\omega_e\tilde{\ell} - i\dot{\tilde{\ell}}) \quad , \end{aligned} \tag{3.11}$$

giving

$$i(A + \sigma)\dot{\tilde{\ell}} + \omega_e (C - A - \sigma)\tilde{\ell} = 0 \quad . \tag{3.12}$$

This differential equation has the solution

$$\tilde{\ell} = c_1 e^{-it/\tau} + c_2 \quad , \tag{3.13}$$

with the free period

$$\tau = \frac{A + \sigma}{\omega_e (C - A - \sigma)} \quad . \tag{3.14}$$

If we now take the free period as given by latitude observations, then the Love number is determined to be $k_2 = 0.29$. Further consideration of these equations could shed more light on the dynamics of polar motion.

Before going on to the main body of the report, let us raise several points that will be useful later. Kelvin has shown (MM, p. 29) that for an incompressible homogeneous sphere of rigidity $\tilde{\mu}$,

$$k_n = \frac{2}{3} \frac{1}{(n - 1)\{1 + \mu[2(2n^2 + 4n + 3)/19n]\}} \quad , \tag{3.15}$$

where the dimensionless rigidity μ is related to the rigidity by

$$\mu = \frac{19}{2} \frac{\tilde{\mu}}{\rho g a_e} \approx 2.3 \quad . \tag{3.16}$$

It is from this formula, by use of the Love number obtained from the Chandler motion, that the rigidity of the earth was initially determined.

We can introduce dissipation into these equations by use of a complex Love number (MM, p. 153):

$$\tilde{k}_2 = k_2 \left(1 - i\,\frac{\mu}{2Q}\right) \quad . \tag{3.17}$$

Using $\tilde{k}_2$ in the previous equations includes the effect of dissipation.

It is now instructive to consider a simple solution of these modified Liouville equations. If we let

$$\begin{aligned} \tilde{\sigma} &= \tilde{k}_2 \, \frac{\omega_e^2 \, a_e^5}{3G} \quad , \\ K &= k_2 \, \frac{\omega_e^2 \, a_e^5}{3G} = \mathcal{R}e\,\tilde{\sigma} \quad , \\ \Gamma &= \frac{\mu}{2Q} \quad , \\ A' &= A + K \quad , \end{aligned} \tag{3.18}$$

where

$\mathcal{R}e(\ )$ denotes the real part of $(\ )$,

then Eq. (3.12) takes the form

$$(A' - i\Gamma K)\,\dot{\tilde{\ell}} - i(C - A' + i\Gamma K)\,\tilde{\ell} = f(t) \quad . \tag{3.19}$$

The homogeneous solution is, of course,

$$\tilde{\ell} = (\text{const})\, e^{i \sigma_c t} + \text{another constant} \quad ,$$

where

$$\sigma_c = \frac{C - A' + i\Gamma K}{A' - i\Gamma K} \quad , \qquad \sigma_n = \mathcal{R}e\,\sigma_c = \frac{A'(C - A') - (K\Gamma)^2}{(A' - iK\Gamma)(A' + iK\Gamma)} \quad . \tag{3.20}$$

Now, for a forcing function

$$f(t) = \alpha e^{i\sigma_d t} ,$$

we have the particular solution

$$\tilde{\ell} = \frac{i \alpha e^{i\sigma_d t}}{(A' - iK\Gamma)[\sigma_d - \sigma_n + i(K\Gamma/A')]} . \tag{3.21}$$

We can write the real part of the frequency as

$$\sigma_n = \frac{[(C - A')/A'] - (\Gamma K/A')^2}{[1 + (\Gamma K/A')^2]} . \tag{3.22}$$

For $k_2 = 0.3$, $\omega_e = 0.7 \times 10^{-5}$ sec^{-1}, $a_e = 6.3 \times 10^8$ cm, $G = 6.67 \times 10^{-8}$, $\mu = 2.3$, $Q = 30$ to 500, $A \approx B \approx C \approx 8 \times 10^{44}$, and $(C - A')/A' \approx 1/430 \approx 2.3 \times 10^{-3}$, we have $\Gamma K/A' \approx 10^{-5}$. Therefore, the dissipation has no effect on the free period, and further, the amplitude of the forced motion is controlled by the $K\Gamma/A'$ term.

The natural decay time is also determined by

$$\frac{\Gamma K}{A'} \approx \frac{k_2 \omega_e^2 a_e^5 \mu}{6GQ(C/Ma_e^2)\, Ma_e^2} = \frac{k_2 \omega_e^2 a_e \mu}{6GQ(C/Ma_e^2)}$$
$$\approx \frac{1.2 \times 10^{-3}}{Q} . \tag{3.23}$$

The decay time is

$$T = \frac{Q}{1.2} \times 10^3 \text{ days} = \frac{Q}{0.329} \text{ years} . \tag{3.24}$$

Finally, we wish to make a correspondence between the contribution of the deformation to the external gravity field expressed by Eq. (3.9) and the form of the potential used to develop orbital perturbations. It is customary to write the external potential as

$$V = \frac{GM}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{a_e}{r}\right)^{\ell}\overline{P}_{\ell m}(\sin\phi)\,(\overline{C}_{\ell m}\cos m\lambda + \overline{S}_{\ell m}\sin m\lambda)\right] , \tag{3.25}$$

where the $\overline{P}_{\ell m}(\sin\phi)$ are fully normalized associated Legendre polynomials, i.e.,

$$\int_{\text{Sphere}} \overline{P}^2_{\ell m}(\sin\phi)\begin{bmatrix}\sin\\ \cos\end{bmatrix}^2 m\lambda\,\cos\phi\,d\phi\,d\lambda = 4\pi . \tag{3.26}$$

If we let $x = r\cos\phi\cos\lambda$, $y = r\cos\phi\sin\lambda$, and $z = \sin\phi$ in Eq. (3.9) and compare them with the expressions in terms of Legendre polynomials, we find, for example,

$$\overline{P}_{21}(\sin\phi) = 3\sqrt{\frac{2.5}{3!}}\;\sin\phi\cos\phi ,$$

$$yz = r^2\cos\phi\sin\phi\sin\lambda = \frac{r^2}{\sqrt{15}}\,\overline{P}_{21}(\sin\phi)\,\sin\lambda .$$

Therefore, we have

$$\begin{aligned}\overline{C}_{21} &= \frac{-k_2\,\omega_e^2\,a_e^3\,\ell_1}{\sqrt{15}\,GM} ,\\ \overline{S}_{21} &= \frac{-k_2\,\omega_e^2\,a_e^3\,\ell_2}{\sqrt{15}\,GM} .\end{aligned} \tag{3.27}$$

4. POLAR MOTION AND EARTH SATELLITES (KINEMATICS)

We now turn to the observable effects of polar motion on satellite orbits and satellite observations. We will consider first the kinematic effect on observations of satellite positions. Then we will explore the orbital perturbations arising from variations in the mass distribution, i.e., gravitational perturbations.

Consider a reference system established with respect to some arbitrary point on the earth's surface. This point is chosen to be near the spin axis of the earth, and we will choose for this discussion the Conventional International Origin (CIO). Aside from tides and crustal motions, which we ignore here, the observing stations are rigidly fixed with respect to this point. Each station has its geographic latitude ϕ. The realization of the position of several stations in this reference system is the subject of geodetic research. Satellite-tracking data have been very important in achieving a consistent set of coordinates (Gaposchkin and Lambeck, 1971). The actual values of the station coordinates establish the origin in an analogous way to the mean observatory of the International Latitude Service.

Consider one such station, point A in Figure 1. The terrestrial system has its pole (the CIO, for example) in the direction of $\hat{P}$, which is at an angle x with the spin axis $\hat{\omega}$. The spin axis is approximately the same direction as the angular momentum $\overline{L}$ of the earth. It is the angular momentum of the earth that is constant, as we are assuming no torques. In the classical Poinsot construction, the spin axis is $f \cdot x$ from the direction of $\overline{L}$, where f is the dynamic flattening. With $1/f = 298$, and $x \approx 0.5 \times 10^{-6}$ radians, the difference is 1.6×10^{-9} radians or about 1 cm at the pole. We assume the spin axis is fixed in space.

Consider a satellite with inclination I. The inclination is defined with respect to the inertial reference system defined by $\overline{L}$. If we assume that the satellite's orbit is well known and that it can be observed at any point in the orbit, then we make an observation from A to S and 12 hours later from A′ to S′. In the first case, the angle AOS is $\phi + I - x$, and in the second, $\phi - I - x$. Using the adopted values of ϕ, we determine x.

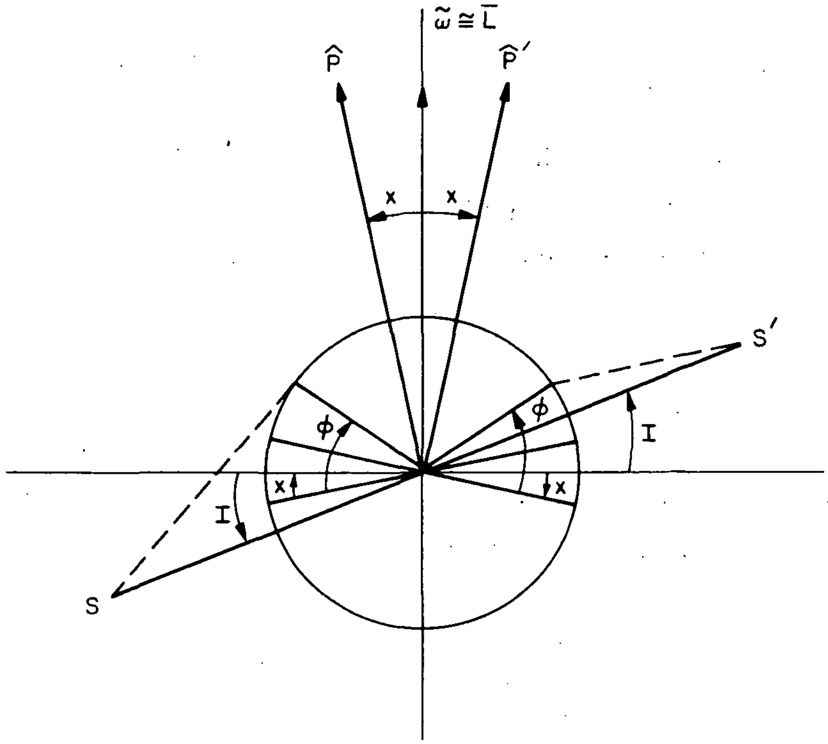

Figure 1. Kinematic determination of pole position.

This is the projection of the pole position on the meridian of A. A station 90° east or west would determine the other component. We note that ϕ - x is an apparent change in latitude that leads to the occasional use of latitude variation as a synonym for polar motion.

A polar satellite would be used in the same way. Instead, we would determine $v + \omega - x + I$ and $v + \omega - x - I$. We see that a satellite at any inclination can be used, the accuracy depending on that of the combination of along-track $v + \omega$ and cross-track I components. The inclination of a satellite is inherently more accurately known, so the optimum procedure is to make observations at the point highest in the orbit when this point is in the meridian of the station.

This analysis has been simplified by considering only observations in the meridian. Generally, a station will have observations through 10 to 20° longitude, and a single station will provide some information about both components. This will be more apparent when we consider determination of pole position as part of the process of orbit determination.

Finally, since the polar motion gives rise to diurnal variations in the apparent latitude $\phi - x$, the principal corruption in pole position determination comes from diurnal errors. There are important diurnal perturbations in the satellite position from the first-degree tesseral harmonics. Therefore, these $C_{\ell,1}$, $S_{\ell,1}$, $\ell = 3, 4, \ldots,$ geopotential coefficients must be especially well determined.

Ultimately, when orbital uncertainties are reduced, the possibility of diurnal effects in the observations will become important. Environmental conditions at a station have very strong diurnal variations.

To determine the pole position in the context of orbit determination, we consider the expression relating the observation and the computed satellite position

$$\bar{\rho} = \bar{r} - \bar{R} \quad . \tag{4.1}$$

Here, $\bar{r}$ is the position of the satellite and $\bar{R}$ is the position of the observing station. The observation equation for direction or range observations is derived from a projection of this expression. For example, the range is determined from

$$\rho = \hat{\rho} \cdot \bar{\rho} = \hat{\rho} \cdot (\bar{r} - \bar{R}) \quad . \tag{4.2}$$

The determination of any parameter p entering $\bar{r}$ or $\bar{R}$ is obtained from $\partial\bar{\rho}/\partial p$. The coordinates of the pole x, y enter $\bar{R}$ as

$$\bar{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & y \\ x & -y & 1 \end{bmatrix} \bar{X} \quad , \tag{4.3}$$

where θ is the sidereal angle, the $\bar{X}$ are the rectangular coordinates of the observing station in an earth-fixed system, and the x, y are the coordinates of the pole given in the convention of the Bureau International de l'Heure (BIH). The coordinate x is positive along the Greenwich meridian and positive y is to the west, i.e., in the opposite sense to the y coordinate of the station. The partial derivatives easily follow:

$$\frac{\partial \bar{\rho}}{\partial x} = \begin{bmatrix} z \cos \theta \\ z \sin \theta \\ -x \end{bmatrix} , \quad \frac{\partial \bar{\rho}}{\partial y} = \begin{bmatrix} z \sin \theta \\ -z \cos \theta \\ y \end{bmatrix} . \tag{4.4}$$

It remains to consider if these observation equations are in themselves singular and, further, if the resulting set is singular when they are augmented with equations for orbital elements. Both questions can be answered negatively from the following considerations.

The normal equation for the system itself for one station is

$$\begin{bmatrix} \sum_i^n (z^2 \cos^2 \theta_i + z^2 \sin^2 \theta_i + x^2) & \sum_i^n (z^2 \cos \theta_i \sin \theta_i - z^2 \cos \theta_i \sin \theta_i - xy) \\ \sum_i^n (z^2 \cos \theta_i \sin \theta_i - z^2 \cos \theta_i \sin \theta_i - xy) & \sum_i^n (z^2 \cos^2 \theta_i + z^2 \sin^2 \theta_i + y^2) \end{bmatrix}$$

$$= n \begin{bmatrix} z^2 + x^2 & -xy \\ -xy & z^2 + y^2 \end{bmatrix} , \tag{4.5}$$

with determinate

$$n R^2 z^2 ,$$

which is zero only if the station is on the equator. This does not mean that we cannot determine polar motion with an equatorial station, only that we cannot separate the x and y components. The component along the meridian can be determined.

We observe that the correlation between x and y is proportional to xy and can be reduced to zero by placing the station with either x or y = 0. This is intuitively reasonable, as the component along the meridian of the station is most easily determined. By choosing x or y = 0, one decreases the correlation and improves the determination of the component along the meridian at the expense of the accuracy of the orthogonal component. The element of the normal system corresponding to the orthogonal component is minimum and results in a maximum variance in the inverse.

Since the data are taken at arbitrary times, the geometry of the observations controls the part of the normal system relating the orbital elements. In general, we can expect (and in fact find) no singularity in determining pole position with orbital elements.

5. POLAR MOTION AND EARTH SATELLITES (DYNAMICS)

In the theory of free nutation, we found that the elastic deformation that lengthens the period of free nutation also gives rise to tesseral harmonics $\bar{C}_{21}$ and $\bar{S}_{21}$, which depend on the pole positions ℓ_1 and ℓ_2. Can we hope to determine these coefficients by satellite perturbation analysis and recover the pole position?

The orbital system defined by the direction of $\bar{L}$ is the system in which we study orbital perturbations. The gravity field is expressed in the system of principal axis. We want to explore the consequences of orbit determination in which the geopotential is moving. The dominant part of the anomalous gravity field is due to $J_2 = -\sqrt{5} \times \bar{C}_{20}$. We assume the system of principal axis has coordinates ξ, η with respect to $\bar{L}$. In the principal-axis system centered on ξ, η, the gravity field can be approximated by

$$V = \frac{GM}{r}\left[1 + \bar{C}_{20}\left(\frac{a_e}{r}\right)^2 \bar{P}_{20}(\sin\phi')\right] \quad . \tag{5.1}$$

We could use the general transformation of spherical harmonics to find the expression for V in the system oriented along $\bar{L}$. However, if we assume ξ, η to be small, a simple expansion suffices and we have

$$\bar{P}_{20}(\sin\phi') = \sqrt{5}\left(\frac{3}{2}\sin^2\phi' - \frac{1}{2}\right) = \sqrt{5}\left(\frac{3}{2}z'^2 - \frac{1}{2}\right) \quad .$$

In the orbital system,

$$z' = z - \xi x - \eta y$$

and

$$\overline{P}_{20}(\sin\phi') = \sqrt{5}\left(\frac{3}{2}z^2 - \frac{1}{2} - 3\,\xi xz - 3\eta yz + \cdots\right)$$

$$= \overline{P}_{20}(\sin\phi) - \xi\sqrt{3}\,\overline{P}_{21}(\sin\phi)\cos\lambda - \eta\sqrt{3}\,\overline{P}_{21}(\sin\phi)\sin\lambda \quad , \tag{5.2}$$

so we have

$$V = \frac{GM}{r}\left\{1 + \overline{C}_{20}\left(\frac{a_e}{r}\right)^2\left[\overline{P}_{20}(\sin\phi) - \sqrt{3}\,\overline{P}_{21}(\sin\phi)(\xi\cos\lambda + \eta\sin\lambda)\right]\right\} \quad . \tag{5.3}$$

Therefore, the motion of the principal axis appears as tesseral harmonics of degree 2 and order 1:

$$\overline{C}_{21} = -\overline{C}_{20}\sqrt{3}\,\xi \quad ,$$
$$\overline{S}_{21} = -\overline{C}_{20}\sqrt{3}\,\eta \quad . \tag{5.4}$$

The gravitational potential sensed by the satellite in orbit referred to $\overline{L}$ contains the terms

$$\overline{C}_{21} = -\overline{C}_{20}\sqrt{3}\,\xi - \frac{k_2\,\omega_e^2\,a_e^3\,\ell_1}{\sqrt{15}\,GM} \quad ,$$
$$\overline{S}_{21} = -\overline{C}_{20}\sqrt{3}\,\eta - \frac{k_2\,\omega_e^2\,a_e^3\,\ell_2}{\sqrt{15}\,GM} \quad . \tag{5.5}$$

Using quantities known with sufficient accuracy, we have

$$\overline{C}_{21} = 0.838\times 10^{-3}\,\xi - k_2\,0.893\times 10^{-3}\,\ell_1 \quad ,$$
$$\overline{S}_{21} = 0.838\times 10^{-3}\,\eta - k_2\,0.893\times 10^{-3}\,\ell_2 \quad . \tag{5.6}$$

If we can choose the origin of the geodetic system to be the same as the principal axis, which is <u>not</u> the case for the CIO, then $\xi = \ell_1$, $\eta = \ell_2$, and the values become

$$\bar{C}_{21} = (0.838 - k_2\, 0.893)\, \ell_1 \times 10^{-3} \quad ,$$
$$\bar{S}_{21} = (0.838 - k_2\, 0.893)\, \ell_2 \times 10^{-3} \quad , \tag{5.7}$$

with the elasticity reducing the effect by about one-third. The minimum effect would be zero for $k_2 = 0.936$, i.e., if the Love number was the fluid Love number (MM, p. 26).

If we assume $\ell_1 = \xi = 0.5 \times 10^{-5}$ and $k_2 = 0.30$, we have

$$\bar{C}_{21} = -3 \times 10^{-9} \quad .$$

Until now, this accuracy has been achieved only for the zonal harmonics of the geopotential (Kozai, 1969). Zonal harmonics cause long-period and secular perturbations that can be averaged for months or years. For close-earth satellites, the tesseral harmonics $\bar{C}_{21}$ and $\bar{S}_{21}$ give rise to diurnal perturbations. These perturbations will have the same frequency spectrum as do all the first-order tesseral harmonics. Synchronous satellites are also sensitive to the first-order coefficients. These synchronous satellites will be in the resonant state and the orbital effects considerably amplified. The use of synchronous satellites to study the polar motion has been previously discussed by Gaposchkin (1968).

We consider the use of laser retroreflector satellites as the optimum method for determining the geopotential today. Table 1 contains the amplitude of the perturbation for the seven existing laser reflector satellites and the nominal orbit of Geos C. The perturbation is estimated by using the approach given by Gaposchkin (1970).

The principal perturbation due to $\bar{C}_{21}$ and $\bar{S}_{21}$ has a diurnal variation, which is the same frequency as the kinematic effects (see Section 4). The 1-m perturbation comes from a $0.5 \times 10^{-5} \times 6 \times 10^{6}$ m = 30-m polar motion. We can imagine a process where we determine ℓ_1 and ℓ_2 from the kinematic effects and then use them in

Eqs. (5.7) with a nominal value for k_2. We can then redetermine ℓ_1 and ℓ_2, using these values of $\overline{C}_{21}$ and $\overline{S}_{21}$. Since the error in computing the perturbation is 1/30 the error in the assumed position of the pole, the process will converge.

Table 1. Perturbations due to $\overline{C}_{21}$ and $\overline{S}_{21}$ with $\ell_1 = 0.5 \times 10^{-5}$ and k = 0.30.

Satellite		Inclination	a (Mm)	Eccentricity	Amplitude (m)
6406401	BE-B	79°.69	7.360	0.013	0.70
6503201	BE-C	41.17	7.502	0.025	1.41
6508901	Geos 1	59.36	8.074	0.071	1.07
6701101	D1C	39.96	7.319	0.051	1.51
6701401	D1D	39.43	7.603	0.084	1.40
6800201	Geos 2	105.80	7.708	0.032	0.88
7010901	Peole	15.0	6.999	0.017	1.20
	Geos C	67	7.28	0.006	1.12

The analysis of pole position, by using values of $\overline{C}_{21}$ and $\overline{S}_{21}$ determined by satellite perturbation analysis, can be approached as follows. We imagine employing a satellite with minimum sensitivity to the gravitational perturbations, i.e., BE-B or a high, heavy satellite such as Lageos, to determine the pole position. Then, by using the gravitational perturbations of a sensitive satellite, such as BE-C, D1C, or D1D, we can determine $\overline{C}_{21}$ and $\overline{S}_{21}$ as functions of time. We can then: 1) Assume a value of k_2 and determine ξ, η from Eq. (5.5) or (5.6). 2) Assume a value of ξ, η, say $\ell_1 = \xi$, $\ell_2 = \eta$, and determine k_2. The temporal variation of k_2 could be investigated, (Gaposchkin, 1968, 1972). 3) Assume a particular form for the variation of ξ, η, e.g., $\xi = \xi_0 \cos \omega t$, $\eta = \eta_0 \sin \omega t$, and determine ξ_0, η_0 and k_2. This would require a long series of data, several oscillations, i.e., several years.

6. BODY TIDES AND EARTH SATELLITES (KINEMATICS)

The application of an external potential of degree n, $U_n(r, \phi, \lambda)$, causes the surface of the earth to move vertically by an amount

$$\Delta r_n = \frac{h_n}{g} U_n(r, \phi, \lambda)$$

(see Section 3). Now, U_2 can be written (MM, p. 68) as

$$U_2 = g \frac{3}{2} \frac{M'}{M} \frac{a_e^4}{r'^3} b f(\phi) \cos \beta \quad .$$

The largest term is M_2 with b = 0.908, and we have

$$\Delta r_{M_2} = h_2 (53.7 \text{ cm}) \; 0.908 \frac{\cos^2 \phi}{2} \cos 2 (\lambda_\odot - \lambda' - 79\overset{\circ}{.}8 + \lambda) \quad .$$

By using the rough value of $h_2 = 0.61$,

$$\Delta r_{M_2} = 14.9 \text{ (cm)} \cos^2 \phi \cos 2 (\lambda_\odot - \lambda' - 79\overset{\circ}{.}8 + \lambda) \quad .$$

We see that the kinematic semidiurnal raising and lowering of the observing station is 14.9 cm, which is maximum on the equator. Takeuchi (1950) has computed the complete deformation by use of a model earth and obtains 22 cm for the total change. We conclude that only when observations and data-analysis capability reach centimeter accuracy will this effect be observable. The current value for h_2 has sufficient accuracy to be used as a preliminary number in the reduction of observations.

The previous calculation was based on uniform properties of the earth. There are local variations; for example, tectonic conditions can control the deformation due to atmospheric pressure. With certain pressure distributions, the vertical movement amounts to 15 cm in northern Europe (Tomaschek).

7. BODY TIDES AND EARTH SATELLITES (DYNAMICS)

We have defined the additional potential due to the deformation from a potential U_n as

$$\Delta V_n = k_n U_n \left(\frac{a_e}{r}\right)^{2n+1} . \tag{7.1}$$

The total potential acting on the satellite is then

$$\left[1 + k_n \left(\frac{a_e}{r}\right)^{2n+1}\right] U_n . \tag{7.2}$$

The potential that acts on the satellite owing to the moon (or sun) can be written

$$V = GM' \left(\frac{1}{\Delta} - \frac{\overline{r} \cdot \overline{r}'}{|r'|^3}\right) , \tag{7.3}$$

where $\overline{r}$ and $\overline{r}'$ are the position of the disturbed body (the satellite) and the disturbing body (the moon or sun), respectively. We can write

$$\frac{1}{\Delta} = \mathcal{R}e \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{1}{2\ell + 1} \frac{r^{\ell}}{r'^{\ell+1}} \overline{P}_{\ell m}(\sin \phi) \overline{P}_{\ell m}(\sin \phi') e^{im(\lambda-\lambda')} , \tag{7.4}$$

where r, ϕ, λ are the spherical coordinates of the satellite and the primed symbols are those of the moon (or sun). To calculate orbital perturbations, we will use the gradient of Eq. (7.4) with respect to the satellite position, and we can drop the $\ell = 0$ term. The $\ell = 1$ term just cancels the $(\overline{r} \cdot \overline{r}')/|r'|^3$, so we have for the third-body potential

$$V' = GM' \mathcal{R}_e \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{1}{2\ell + 1} \frac{r^{\ell}}{r'^{\ell+1}} \overline{P}_{\ell m}(\sin \phi) \overline{P}_{\ell m}(\sin \phi') e^{im(\lambda-\lambda')} \quad . \tag{7.5}$$

Therefore, the potential acting on the satellite, including the tide, is

$$V' = GM' \mathcal{R}_e \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{1}{2\ell + 1} \left(\frac{r^{\ell}}{r'^{\ell+1}} + \frac{k_{\ell}\, a_e^{2\ell+1}}{r'^{\ell+1} r^{\ell+1}} \right) \overline{P}_{\ell m}(\sin \phi) \overline{P}_{\ell m}(\sin \phi') e^{im(\lambda-\lambda')} \quad . \tag{7.6}$$

To include the effects of the tidal phase lag, one introduces a fictitious moon lagging the real moon by Δt. In this case, we cannot write the disturbing potential in such a compact form. We proceed, assuming $\Delta t = 0$, i.e., no phase lag, the revision of the theory being relatively straightforward if lag is desired, either by use of Δt or a complex k_n.

To develop perturbations in satellite position, we have to express the potential in orbital elements. The first step is through the relation (see Appendix 1)

$$\overline{P}_{\ell m}(\sin \phi) e^{im\lambda} = \sum_{p=0}^{\ell} (i)^{\ell-m} D_{\ell mp}(i) e^{i[(\ell-2p)(v+\omega)+m\Omega]} \quad , \tag{7.7}$$

which gives

$$V' = GM' \mathcal{R}_e \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{p'=0}^{\ell} \frac{(1)^{\ell+m}}{2\ell + 1} \left(\frac{r^{\ell}}{r'^{\ell+1}} + \frac{k_{\ell}\, a_e^{2\ell+1}}{r'^{\ell+1} r^{\ell+1}} \right) D_{\ell mp}(I) D_{\ell(-m)p'}(I') e^{i\psi} , \tag{7.8}$$

where

$$\psi = (\ell - 2p)(v + \omega) + (\ell + 2p')(v' + \omega') + m(\Omega - \Omega') \quad ,$$

by noting

$$P_{\ell m}(\mu) = (-1)^m P_{\ell(-m)}(\mu) \quad .$$

The final step in expressing Eq. (7.6) in Kepler elements involves expressing $r^p\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\} mv$ as a series in a; Kaula (1966) gives formulas for

$$\frac{1}{r^{\ell+1}} e^{i[(\ell-2p)v+\psi]} = \frac{1}{a^{\ell+1}} \sum_{q=-\infty}^{\infty} G_{\ell pq}(e)\, e^{i[(\ell-2p+q)m+\psi]} \quad . \tag{7.9}$$

A more general form is given by Plummer (1918, p. 44):

$$r^n e^{imv} = a^n \sum_{q=-\infty}^{\infty} X_q^{n,m}(e)\, e^{iqM} \quad , \tag{7.10}$$

where $X_q^{n,m}(e)$ are known as Hansen coefficients. There is a classical development for $X_q^{n,m}(e)$ given in Plummer, where

$$X_q^{n,m} = (1 + \beta^2) \sum_p J_p(qe)\, X_{q,p}^{n,m} \quad ,$$

$$X_{q,p}^{n,m} = (-\beta)^{q-p-m} \binom{n+1-m}{q-p-m} F\,(q-p-n-1,\ -m-n-1,\ q-p-m+1,\ \beta^2) \quad ,$$

$$q - p - m > 0 \quad , \tag{7.11}$$

(Eq. cont. on next page)

$$X^{n,m}_{q,p} = (-\beta)^{-q+p+m} \binom{n+1+m}{-q+p+m} F(-q+p-n-1,\ m-n-1,\ -q+p+m+1,\ \beta^2),$$

$$q-p-m<0 ,$$

$$X^{n,m}_{q,p} = F(m-n-1,\ -m-n-1,\ 1,\ \beta^2), \qquad q-p-m=0 , \tag{7.11}$$

$$\beta = \frac{e}{1+\sqrt{1-e^2}} ,$$

in the notation for a hypergeometric series. These may suffice for theoretical studies. For practical problems, we prefer the expression as a polynomial in e. Andoyer (1903) developed a recursive method for computing this polynomial, which was rediscovered by Izsak et al. (1964). Cherniack (1972) gives the practical details for computing the $X^{n,m}_q(e)$.

By using Eq. (7.10), V′ becomes

$$V' = \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{p'=0}^{\ell} \sum_{q=-\infty}^{\infty} \sum_{q'=-\infty}^{\infty} V'_{\ell mpp'qq'} ,$$

where

$$V'_{\ell mpp'qq'} = \frac{(-1)^{\ell+m}}{2\ell+1} D_{\ell mp}(I)\, D_{\ell(-m)p}(I')$$

$$\times \left[\frac{a^{\ell}}{a'^{\ell+1}} X_q^{\ell,m}(e)\, X_{q'}^{-\ell-1,m}(e') + \frac{k_\ell\, a_e^{2\ell+1}}{a'^{\ell+1}\, a^{\ell+1}} X_q^{-\ell-1,m}(e)\, X_{q'}^{-\ell-1,m}(e') \right] e^{i\psi} ,$$

and

$$\psi = qM + q'M' + (\ell-2p)\omega + (\ell-2p')\omega' + m(\Omega-\Omega') . \tag{7.12}$$

Considering the size of the perturbations, we chose to use a linear theory developed in terms of the Lagrange Planetary Equations (LPE). These are given in Kaula (1966):

$$
\begin{aligned}
\frac{d\omega}{dt} &= \frac{-\cos I}{na^2(1-e^2)^{1/2}\sin I}\frac{\partial R}{\partial I} + \frac{(1-e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial e}\ , \\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1-e^2)^{1/2}\sin I}\frac{\partial R}{\partial I}\ , \\
\frac{dI}{dt} &= \frac{\cos I}{na^2(1-e^2)^{1/2}\sin I}\frac{\partial R}{\partial \omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin I}\frac{\partial R}{\partial \Omega}\ , \\
\frac{de}{dt} &= \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial \omega}\ , \\
\frac{dM}{dt} &= n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}\ , \\
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M}\ .
\end{aligned}
\tag{7.13}
$$

By using Eq. (7.12) for R in (7.13) and integrating, we obtain the lunar or solar perturbations. They have been developed by Gaposchkin et al. (1973) by use of computer algebra, and we will not give the full expressions here. From the argument ψ, we observe that there is a rich spectrum of perturbations. The orbital effects are large for those terms involving slow rates, i.e., $q = 0$.

With respect to tidal studies, Kozai (1970) has used the $\Omega - \Omega'$ terms, which have a very long period. Using these terms requires careful study of all long-period effects. The principal difficulties arise from radiation pressure and drag and are complicated by our imperfect knowledge of the satellite aspect, characteristics, and the atmospheric density. For this reason, we concentrate on the two-week lunar tides. Though the orbital effects are smaller, the other effects are manageable and the theoretical difficulties are reduced.

We estimate the size of the direct lunar perturbation as

$$\frac{\mu(n')^2}{2nn'} \approx 1.5 \times 10^{-5} \approx 120 \text{ m} \quad ,$$

where

$$\mu = \frac{M'}{M} \quad ,$$

which is quite a good estimate. The tidal effect is $k_2(a_e/a)^5$ times the direct effect and is about 15 m.

8. OCEAN TIDES AND EARTH SATELLITES (DYNAMICS)

The observations of ocean tides are extensive, but mainly along coastlines. The theoretical treatment of ocean tides remains a fundamental problem. Hendershott and Munk (1970) give an excellent review of this subject. For our purposes, we note that the bulk of the tide observations are taken on coastlines that have very important boundary effects, and that all treatments existing today are inadequate in one way or another. Although there is superficial agreement in the cotidal charts, the spherical harmonic coefficients differ by more than 100% and are often different in sign. Only a few terms in the general tidal development are important for satellite perturbation analysis, since in exactly the same way that the satellite is a frequency filter for the gravity anomalies, it selects only a few of the tidal terms.

Most of the tidal theory has been worked out for the M_2 tide, the principal semi-diurnal tide (Pekeris and Accad, 1969; Hendershott and Munk, 1970; Hendershott, 1972). It is the two-week tide, in the inertial reference system, that would arise from the moon in the earth's equator.

Current determinations of the low degree and low-order terms in the tide are quite inconsistent. An equilibrium theory, which surely is inadequate, predicts the $\overline{P}_{22}(\sin\phi)\, e^{i2\lambda}$ tide of 30 cm. Pekeris and Accad give a solution in terms of a number map. Lambeck and Cazenave (1973) have determined by harmonic analysis an amplitude of 4.4 cm and a phase of 330°. Hendershott gives an amplitude of 50 cm. We can only comment that these estimates roughly agree as to the order of magnitude.

For further analysis, we assume a tide raised by a disturbing body in a reference system centered at the earth's center and oriented toward the disturbing body. We have $1/\Delta$ from Eq. (7.4), and the tide raised is developed in spherical harmonics:

$$\zeta = \mathcal{R}_e \sum_{\ell, m} \overline{\mathcal{C}}_{\ell m}\ \overline{P}_{\ell m}(\sin\phi'')\, e^{im\lambda''} \quad , \tag{8.1}$$

where

$$\overline{\mathscr{C}}_{\ell m} = \overline{C}_{\ell m} - i\overline{S}_{\ell m} \quad . \tag{8.2}$$

If the tide is static in this system, it follows the disturbing body and $\overline{\mathscr{C}}_{\ell m}$ is constant. If there is no lag, then $\overline{S}_{\ell m} = 0$. The external potential due to this tide, including the loading effect, is

$$V' = \mathcal{R}_e \sum_{\ell m} \frac{(1 + k'_\ell)\, 4\pi G\, \rho_\omega\, a_e^{\ell+2}}{(2\ell + 1)\, r^{\ell+1}}\, \overline{\mathscr{C}}_{\ell m}\, \overline{P}_{\ell m}(\sin \phi'')\, e^{im\lambda''} \quad , \tag{8.3}$$

where ρ_ω is the density of the water, k'_ℓ is the Love number for a deformation that loads the earth, and ϕ'' and λ'' are the coordinates of a point referred to the position of the disturbing body. The quantity ϕ'' is measured from the lunar orbit and λ'' is measured from the position of the moon $v' + \omega'$ in the lunar orbit. If λ' is referred to the lunar node, then $\lambda' = \lambda'' - v' - \omega'$. We can express this potential in the inertial system by using the transformation (see Appendix 1)

$$\overline{P}_{\ell m}(\sin \phi')\, e^{i[m\lambda' - m(v'+\omega')]} = \sum_{s=-\ell}^{\ell} (i)^{S-m}\, E_{\ell ms}(-I')\, \overline{P}_{\ell s}(\sin \phi)\, e^{i[s\lambda - m(\Omega' + v' + \omega')]} \quad . \tag{8.4}$$

We can now express the potential due to ocean tides in the orbital elements of the satellite as

$$V' = \mathcal{R}_e \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \sum_{p=0}^{\ell} V'_{\ell msp} \quad ,$$

where

$$V_{\ell msp} = \Gamma_{\ell m}\, (i)^{\ell+s}\, (-1)^m\, E_{\ell ms}(-I')\, D_{\ell sp}(I)\, e^{i[(\ell-2p)(v+\omega) - m(v'+\omega') + s\Omega - m\Omega']} \quad , \tag{8.5}$$

in which

$$\Gamma_{\ell m} = \frac{4\pi G \rho_\omega (1 + k'_\ell)}{2\ell + 1} \frac{a_e^{\ell+2}}{r^{\ell+1}} \overline{\mathcal{C}}_{\ell m} \quad . \tag{8.6}$$

Similarly, if we can take the definition of the M_2 tide with $I' = 0$, we have

$$V^{m_2} = \mathcal{R} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} V^{m_2}_{\ell mp} \quad ,$$

where (8.7)

$$V^{m_2}_{\ell mp} = \Gamma_{\ell m} (i)^{\ell - m} D_{\ell mp}(I) \, e^{i[(\ell-2p)(v+\omega)+m(\Omega-v'-\omega'-\Omega')]} \quad .$$

With this expression for the potential, we can use the Hansen coefficients, Eq. (7.11), to express it in Kepler elements and then put it into the LPE, Eqs. (7.13). Integrating the resulting equations gives the perturbations due to the ocean tides:

$$\Gamma_{\ell m} = \Gamma'_{\ell m} \frac{r^{\ell+1}}{a_e^{\ell+2}} = \frac{4\pi G \rho_\omega (1 + k'_\ell)}{2\ell + 1} \overline{\mathcal{C}}_{\ell m} \quad , \tag{8.8}$$

$$\psi = qM + q'M' + (\ell - 2p)\,\omega + m\,(\Omega - \Omega' - \omega') \quad ,$$

(8.9)

$$\dot{\psi} = qn + q'n' + (\ell - 2p)\,\dot{\omega} + m\,(\dot{\Omega} - \dot{\Omega}' - \dot{\omega}') \quad ,$$

$$\delta a_{\ell mpqq'} = \mathcal{R}\, (i)^{\ell-m} \, \Gamma_{\ell m} \frac{a_e^{\ell+2}}{na^{\ell+2}} D_{\ell mp}(I)\, q\, X_q^{-\ell-1,\, \ell-2p}(e)\, X_{q'}^{o,\, m}(e') \frac{e^{i\psi}}{\dot{\psi}} \quad ,$$

(8.10)

(Eq. cont. on next page)

$$\delta e_{\ell mpqq'} = \mathcal{R}e \, (i)^{\ell-m} \, \Gamma_{\ell m} \, \frac{a_e^{\ell+2}}{na^{\ell+3} e} \, D_{\ell mp}(I) \, X_q^{-\ell-1, \ell-2p}(e) \, X_{q'}^{o, m}(e')$$

$$\times \left[(1 - e^2) q - (\ell - 2p)(1 - e^2)^{1/2}\right] \frac{e^{i\psi}}{\dot{\psi}} \quad ,$$

$$\delta \omega_{\ell mpqq'} = \mathcal{R}e \, (i)^{\ell-m-1} \, \Gamma_{\ell m} \, \frac{a_e^{\ell+2}}{na^{\ell+3}} \left\{ \frac{(1 - e^2)^{1/2}}{e} \, D_{\ell mp}(I) \left[\frac{d}{de} X_q^{-\ell-1, \ell-2p}(e)\right] \right.$$

$$\left. - \frac{\cos I \, X_q^{-\ell-1, \ell-2p}(e)}{\sin I \, (1 - e^2)^{1/2}} \left[\frac{d}{dI} D_{\ell mp}(I)\right] \right\} X_{q'}^{o, m}(e') \, \frac{e^{i\psi}}{\dot{\psi}} \quad ,$$

$$\delta I_{\ell mpqq'} = \mathcal{R}e \, (i)^{\ell-m} \, \Gamma_{\ell m} \, \frac{a_e^{\ell+2} D_{\ell mp}(I) \, X_q^{-\ell-1, \ell-2p}(e) \, X_{q'}^{o, m}(e') \left[(\ell - 2p) \cos I - m\right] e^{i\psi}}{na^{\ell+3} (1 - e^2)^{1/2} \sin I \, \dot{\psi}} \quad ,$$

$$\delta \Omega_{\ell mpqq'} = \mathcal{R}e \, (i)^{\ell-m-1} \, \Gamma_{\ell m} \, \frac{a_e^{\ell+2} X_q^{-\ell-1, \ell-2p}(e) \, X_{q'}^{o, m}(e') \left[\frac{d}{dI} D_{\ell mp}(I)\right]}{na^{\ell+3} (1 - e^2)^{1/2} \sin I \, \dot{\psi}} \, e^{i\psi} \quad ,$$

$$\delta M_{\ell mpqq'} = \mathcal{R}e \, (i)^{\ell-m-1} \, \Gamma_{\ell m} \, \frac{a_e^{\ell+2}}{na^{\ell+3}} \, D_{\ell mp}(I) \left\{ - \frac{1 - e^2}{e} \left[\frac{d}{de} X_q^{-\ell-1, \ell-2p}(e)\right] \right.$$

$$\left. + 2(\ell + 1) \, X_q^{-\ell-1, \ell-2p}(e) \right\} X_{q'}^{o, m}(e') \, \frac{e^{i\psi}}{\dot{\psi}} \quad . \qquad (8.10)$$

We have given the formulas for the M_2 tide caused by a fictitious moon in an equatorial orbit. These formulas are easily extended to a tide oriented in the plane of the lunar orbit. We would use the potential given by Eq. (8.5). We acquire another summation index, and the formal expressions for the perturbations would be

$$\delta \mathcal{E}_{\ell mpqq's} = \delta \mathcal{E}_{\ell mpqq'} \, E_{\ell ms}(-I') \quad , \tag{8.11}$$

with

$$\psi = qM + q'M' + (\ell - 2p)\,\omega + s\Omega - m(\omega' + \Omega')$$

and

$$\dot{\psi} = qn + q'n' + (\ell - 2p)\,\dot{\omega} + s\dot{\Omega} - m\,(\dot{\omega}' + \dot{\Omega}') \quad ,$$

with $\mathcal{E}$ a generic element.

It has been convenient to develop the theory for orbital perturbations in complex notation. It is easy to utilize these formulas as given by automatic use of complex variables in modern Fortran compilers. To work in terms of real variables, one can easily separate the real and the complex parts of $\bar{\mathcal{C}}_{\ell m}$, $(i)^{\ell - m}$, k_ℓ, $e^{i\psi}$; the other variables that appear are real.

It is instructive to determine the ocean-tide equivalent of the body tide; however, we can do this only approximately. We make the correspondence by comparing the potentials in Eqs. (7.8) and (8.7) for a particular ℓmp combination. We have

$$V^{\text{body}}_{\ell mp} = \frac{GM'\,(-1)^{\ell+m}}{2\ell + 1} \, \frac{k_\ell \, a_e^{\ell+1}}{r'^{\ell+1}\, r^{\ell+1}} \, D_{\ell mp}(I) \sum_{p'=0}^{\ell} D_{\ell(-m)p'}(I')\, e^{i\phi} \quad ,$$

where (8.12)

$$\phi = (\ell - 2p)(v + \omega) + (\ell - 2p')(v' + \omega') + m\,(\Omega - \Omega') \quad ,$$

and

$$V^{M_2}_{\ell mp} = \frac{4\pi G \rho_\omega (1 + k'_\ell)}{2\ell + 1} \frac{a_e^{\ell+2}}{r^{\ell+1}} \overline{\mathcal{C}}_{\ell m} (i)^{\ell - m} D_{\ell mp}(I) \, e^{i\psi} \quad ,$$

where (8.13)

$$\psi = (\ell - 2p)(v + \omega) - m(v' + \omega' + \Omega' - \Omega) \quad .$$

We note that the lunar inclination is $I' = 23° \pm 5°$, and that $D_{2(-2)0} = 0.925$, $D_{2(-2)1} = 0.160$, and $D_{2(-2)2} = 0.0036$. So for the principal semidiurnal term, we can take $\ell = 2$, $m = 2$, $\ell - 2p = 2$, $p = 0$, and $p' = 0$, which gives

$$\frac{k_2}{1 + k'_2} = \frac{4\pi G \rho_\omega \overline{\mathcal{C}}_{22}}{\mu n'^2 a_e D_{2(-2)0}(I')} \tag{8.14}$$

or

$$\overline{\mathcal{C}}_{22} = \frac{k_2}{1 + k'_2} \frac{\mu n'^2 a_e D_{2(-2)0}(I')}{4\pi G \rho_\omega} \quad , \tag{8.15}$$

where k_2 would have a complex value as defined by Eq. (3.17). Using typical values, we have

$$k_2 = 0.0114 \, \overline{\mathcal{C}}_{\ell m} / D_{2(-2)0}(I') \quad . \tag{8.16}$$

From Lambeck and Cazenave (1973), the Pekeris and Accad (1969) solution with dissipation gives

$$\overline{\mathcal{C}}_{22} = 4.4 \, e^{-i\, 330\, \pi/180} = -2.19 - i\, 3.81 \text{ (cm)} \quad .$$

We then have $k_2^{ocean} = -0.026 - 0.047\, i$. If we then add this to the body tide, we obtain the effective Love number that a satellite would sense. Choosing $k_2^{body} = 0.30$ with no dissipation, we have

$$k_2^{\text{effective}} = k_2^{\text{body}} + k_2^{\text{ocean}}$$

$$= 0.274 - 0.047\, i \quad .$$

We would conclude that the Love number was 0.276 with a phase lag of $9^\circ\!.73$, or 38.9 min.

Conversely, we could determine the ocean tide if we were to adopt a value of the effective Love number and the body Love number.

We have analyzed the $\overline{P}_{22}$ component with a 14-day period. The ocean tide has a much richer spectrum than does the body tide, with important contributions from $\overline{P}_{42}$ and $\overline{P}_{62}$. They will give rise to similar perturbations to the $\overline{P}_{22}$, in both amplitude and frequency. Therefore, the oceanic contribution to the satellite perturbations must include these terms, and we cannot easily determine just one $\overline{\mathcal{C}}_{\ell m}$.

Analysis of tides by Anderle (1971) and Smith, Kolenkiewicz, and Dunn (1972) used these two-week terms. Newton (1968) and Kozai (1970) used the terms in $\Omega - \Omega'$. The perturbations due to those terms from the ocean tide are much smaller, owing to the expansion in Hansen coefficients, and should not influence the determination. Therefore, we can roughly reconcile the determinations of Anderle, Kozai, Smith, Kolenkiewicz, and Dunn. Newton's result could be explained by the fact that he used $k_2 = k_2'$, whereas $k_2 \cong -k_2'$, which emphasizes the importance of atmospheric tides in the treatment of long-period perturbations.

9. ANALYSIS OF EXISTING POLAR-MOTION DATA

The availability of information on pole position from 1846 through 1970 invited a rediscussion of the data and properties of the polar motion. The results of the analysis have been reported by Gaposchkin (1968, 1972).

Since completing the analysis, I obtained from Madame Rykhlova the computed pole positions published in graphic form, which we read to obtain tabulated values. Reprocessing these improved data gave better results in the sense that the high-frequency-noise part of the spectrum was very much reduced in amplitude. The properties of the motion and the amplitudes remained within the expected uncertainties.

Since the analysis was completed, the complicated spectrum in the Chandler Band has been studied in other ways. The polar motion causes an oceanic tide called the pole tide. This pole tide has been studied with a frequency spectrum analysis, and the two main frequencies have been found. Clearly, the pole tide, being excited by the polar motion, would exhibit the same spectrum. However, this corollary evidence rules out the possibility that the complicated frequency spectrum is in some way due to some property of the reduction methods. One such study has been carried out by Smets (1970).

10. ANALYSIS OF SATELLITE-TRACKING DATA

The 1969 Smithsonian Standard Earth (II) was determined with a combination of camera observations, simultaneous triangulation, and dynamical data, from surface-gravity data compiled in 1964. In addition, the Deep Space Network of the Jet Propulsion Laboratory provided observation equations based on some deep-space-probe data. A limited amount of laser data were also used.

The camera data had an assumed accuracy of 3 to 4 arcsec. At 2×10^6 m, this corresponds to a 30-m observation. The station coordinates were determined to between 5 and 10 m. This figure was given at the time and has been verified subsequently by comparison with the Smithsonian Standard Earth (III). The orbit computation seemed to be ~10 m for the Geos 1 type satellites, though this proved to be optimistic. When sufficient laser data for good orbital coverage became available from the ISAGEX program, the orbital fits were 20 to 30 m. One significant contribution to this uncertainty is the tide perturbations. Though the tides had been studied by Kozai with long-period terms, no short-period, 14-day, tidal terms were used in the analytical theory. Only with the complete treatment of lunar and solar perturbations by use of computer algebra have we satisfactorily accounted for the body tides. In this report, we give the necessary formulas to include the effects due to the body and ocean tides.

The attempt to determine polar motion in 1969–70 by use primarily of photographic data was unsuccessful. The investigations were successful, however, in exposing several aspects of the analytical process that required refinement. With the advances that have been made in orbit computations through a revised gravity field, improved station coordinates, and a complete tidal treatment, and with the improvements that have been obtained in the accuracy, reliability, and coverage in significant bodies of laser-ranging data, we can expect that future efforts will produce useful determinations of polar motion.

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APPENDIX 1

ROTATION OF SPHERICAL HARMONICS

Legendre functions arise naturally in the solution of Laplace's equation in spherical coordinates. They are also used as a complete set of orthogonal base functions for mapping arbitrary functions in spherical coordinates, particularly in potential theory. As a set of base functions, we naturally need their expression, transformed from one coordinate system to another. We consider here the rotation of a coordinate system. This transformation has been used in quantum mechanics, and proofs of the result described here can be found in that literature. For our purposes, we follow Jeffreys (1965).

We first consider conventional Legendre polynomials, which can be defined with

$$P_{\ell m}(z) = \frac{1}{2^{\ell}\,\ell!}\,(1 - z^2)^{m/2}\,\frac{d^{\ell+m}}{dz^{\ell+m}}\,(z^2 - 1)^{\ell} \quad ,$$

$$z = \sin\phi \quad .$$

For computational purposes, we can use

$$P_{\ell m}(z) = \frac{(1 - z^2)^{m/2}}{2^{\ell}} \sum_{k=0}^{\{(\ell-m)/2\}} \frac{(-1)^k\,(2\ell - 2k)!}{k!\,(\ell - k)!\,(\ell - m - 2k)!}\, z^{\ell-m-2k} \quad ,$$

where we use $\{x\}$ to be the greatest integer in x. These polynomials are orthogonal such that

$$\int_{\text{sphere}} P_{\ell m}(z)\, P_{\ell' m'}(z) \begin{bmatrix}\sin\\ \cos\end{bmatrix} m\lambda \begin{bmatrix}\sin\\ \cos\end{bmatrix} m'\lambda \cos\phi\, d\phi\, d\lambda$$

$$= \frac{2\pi(\ell + m)!}{\epsilon_m(2\ell + 1)(\ell - m)!} \quad , \qquad \ell = \ell', \; m = \ell' \quad ,$$

$$= 0 \quad , \qquad \ell \neq \ell', \; m \neq m', \text{ or both}$$

where

$$\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases} \quad .$$

Each $P_{\ell m}(z)$ can have a scale factor, called the normalization. We can choose this scale factor such that

$$\int_{\text{sphere}} [\overline{P}_{\ell m}(z)]^2 \begin{bmatrix}\sin\\ \cos\end{bmatrix}^2 m\lambda \cos\phi\, d\phi\, d\lambda = 4\pi \quad ,$$

i.e.,

$$\overline{P}_{\ell m}(z) = \sqrt{\frac{\epsilon_m(2\ell + 1)(\ell - m)!}{(\ell + m)!}}\; P_{\ell, m}(z) \quad .$$

We note that Jeffreys uses

$$p_\ell^m(z) = \frac{(\ell - m)!}{\ell!}\; P_\ell^m(z) \quad .$$

We want the expression for $\overline{P}_{\ell m}(z)\, e^{im\lambda}$ in another coordinate system defined by the conventional Euler angles (see Fig. A1). We can write

$$\overline{P}_{\ell m}(\sin\phi)\, e^{im\lambda} = \sum_{s=-\ell}^{\ell} (i)^{s-m}\, E_{\ell ms}(I)\, \overline{P}_{\ell, s}(\sin\phi')\, e^{i[s(\lambda'+\omega')+m\Omega]} \quad ,$$

with

$$E_{\ell ms}(I) = N_{\ell ms} \sum_{r=\max\left\{\begin{smallmatrix}0\\-(m+s)\end{smallmatrix}\right.}^{\min\left\{\begin{smallmatrix}\ell-s\\\ell-m\end{smallmatrix}\right.} (-1)^{\ell-m-r} \binom{\ell+m}{m+s+r} \binom{\ell-m}{r} \gamma^{2r+m+s}\, \sigma^{2(\ell-r)-m-s} \quad ,$$

where

$$\gamma = \cos\frac{I}{2} \quad , \qquad \sigma = \sin\frac{I}{2} \quad ,$$

and

$$N^2_{\ell ms} = \frac{(\ell - s)!\,(\ell + s)!\,\epsilon_m}{(\ell - m)!\,(\ell + m)!\,\epsilon_s} \quad .$$

We can write this in a more compact form, if $\phi' \equiv 0$, as

$$\overline{P}_{\ell m}(\sin\phi)\, e^{im\lambda} = \sum_{p=0}^{\ell} (i)^{\ell-m}\, D_{\ell mp}(I)\, e^{\,i[m\Omega+(\ell-2p)(\lambda'+\omega')]} \quad ,$$

where

$$-\ell \geq m \geq \ell$$

$$D_{\ell mp}(I) = \frac{1}{N_{\ell m}}\, \frac{(\ell + m)!}{2^{\ell}\,\ell!} \sum_{r=\max\left\{\begin{smallmatrix}0\\2p-\ell-m\end{smallmatrix}\right.}^{\min\left\{\begin{smallmatrix}\ell-m\\2p\end{smallmatrix}\right.} (-1)^{\ell-m-r} \binom{\ell}{p} \binom{2p}{r} \binom{2\ell-2p}{\ell-m-r}$$

$$\times\ \gamma^{\ell+m+2r-2p}\, \sigma^{\ell-m-2r+2p} \quad ,$$

in which

$$\gamma = \cos \frac{I}{2} \quad , \qquad \sigma = \sin \frac{I}{2} \quad ,$$

and

$$N_{\ell m}^2 = \frac{(\ell + m)!}{\epsilon_m (2\ell + 1)(\ell - m)!} \quad .$$

We note that

$$\overline{P}_{\ell m}(z) = (-1)^m \, \overline{P}_{\ell (-m)}(z) \quad .$$

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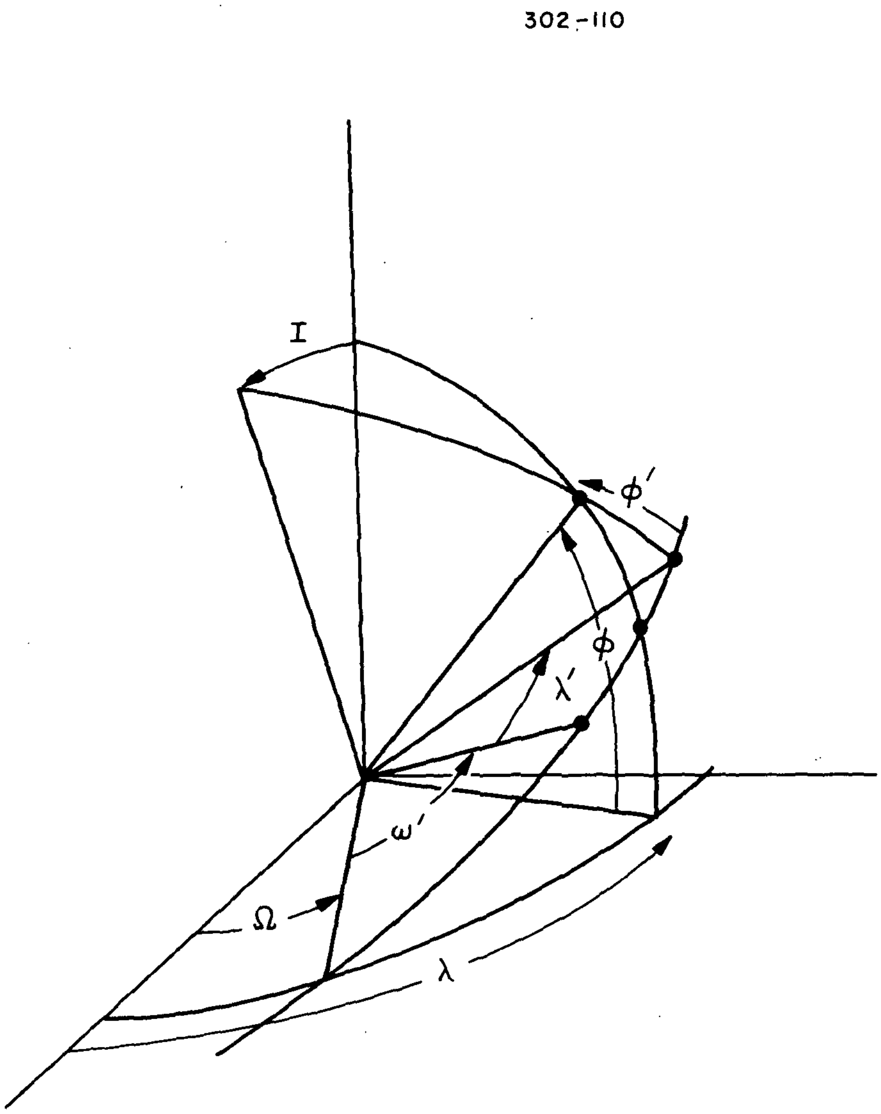

Figure A1. Geometry of coordinate transformation.