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URS/MATRIX COMPANY MAN SYSTEMS DIVISION 4702 GOVERNORS DRIVE HUNTSVILLE, ALABAMA 35805 (205) 837-4750

URS MATRIX

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DESIGN OF A TERMINAL POINTER HAND CONTROLLER FOR TELEOPERATOR APPLICATIONS

FINAL REPORT

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PREPARED FOR:

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FOREWORD

This report was prepared by the Man Systems Division of the URS/Matrix Company under Contract NAS8-27013, "Design of a Terminal Pointer Hand Controller for Teleoperator Applications", for the National Aeronautics and Space Administration, George C. Marshall Space Flight Center. The NASA technical direction was provided by Mr. Starke Cline (COR), S&E-ASTR-ME. This final report is the summary of the technical effort extending from April 25, 1972, to June 25, 1973.

ACKNOWLEDGMENTS

The authors wish to express their appreciation to Mr. Starke Cline (COR) and Mr. Wilbur Thornton, NASA MSFC, S&E-ASTR, for their technical support and direction during the contract period.

Supporting URS/Matrix Company contributors were:

- C. D. Pegden, Co-developer of Terminal Pointer Concept.
- G. L. Philpot, Developer of Hand Controller Anthropometrics and Models.

TABLE OF CONTENTS

SECTION		PAGE
	FOREWORD	i
	ACKNOWLEDGMENTS	ii
	LIST OF FIGURES	iv
	LIST OF TABLES	vi
1.0	INTRODUCTION	1-1
	1.1 Background	1-1
	1.2 Concept Description	1-1
	1.3 Potential Applications	1-6
2.0	CONCEPT DRAWINGS AND MODEL DEVELOPMENT	2-1
3.0	DEVELOPMENT OF PROTOTYPE	3-1
4.0	CONTROL LAW DEVELOPMENT	4-1
5.0	CONCLUSIONS AND RECOMMENDATIONS	5-1

APPENDICES

A	TERMINAL POINTER HAND CONTROLLER PROTOTYPE PARTS LIST	A-1
B	QUATERNIONS FOR CONTROL OF SPACE VEHICLES	B-1

LIST OF FIGURES

FIGURE		PAGE
1-1	Terminal Pointer Concept	1-4
1-2	Terminal Pointer Hand Controller Concept as Delineated in Proposal	1-5
1-3	Potential Applications of Terminal Pointer Hand Controller	1-8
1-4	Dual Station Terminal Pointer Controller Concept	1-8
2-1	Diagram of Developmental Subtasks	2-1
2-2	Terminal Pointer Hand Controller Concept as Delineated in Proposal	2-2
2-3	Terminal Pointer Hand Controller - Preliminary Configuration Layout	2-4
2-4	Terminal Pointer Hand Controller - Gimbal and Ring Details	2-5
2-5	Terminal Pointer Hand Controller - Handle Details	2-6
2-6	Model of Terminal Pointer Hand Controller (Full Scale)	2-7
3-1	Terminal Pointer Hand Controller - Modified Configuration Layout	3-2
3-2	Improved Model of Terminal Pointer Hand Controller (Full Scale)	3-3
3-3	Terminal Pointer Hand Controller Prototype Assembly Drawing	3-4
3-4	Terminal Pointer Hand Controller in Various Control Positions	3-6
3-5	Operator/Terminal Pointer Hand Controller at Console	3-7

LIST OF FIGURES (CONTINUED)

FIGURE		PAGE
3-6	Close-Up of Terminal Pointer Hand Controller	3-7
3-7	Rancho Anthropomorphic Manipulator (RAM) Arm	3-8
4-1	Desired Terminal Attitude and Translational Response	4-2
4-2	Motor Rotation and Mechanical Layout of the Rancho Anthropomorphic Manipulator	4-6
4-3	Hand Controller Angles	4-19
4-4	Terminal Pointer Hand Controller Total Systems Diagram . . .	4-23

LIST OF TABLES

TABLE		PAGE
1-1	Summary of Manipulator Controls	1-2
1-2	Controller Comparison	1-3
1-3	Terminal Pointer Controller Advantages	1-7

SECTION 1.0

INTRODUCTION

The man/machine interface remains the most difficult problem in teleoperator design. Development of an effective manipulator control concept will be of major importance to a comprehensive teleoperator technology development program. To date, a number of manipulator control concepts have been proposed for application to on-orbit teleoperators. Each control concept has its own unique advantages and disadvantages. A summary of the current manipulator control concepts is given in Tables 1-1 and 1-2.

1.1 BACKGROUND

As may be seen in the table summaries, none of the current manipulator concepts is totally adequate for use with an on-orbit teleoperator. To improve this situation, the URS/Matrix Company has developed (in-house) a manipulator control concept. This novel concept, the Terminal Pointer Manipulator Controller, was first conceived in March 1971 and reported in the Free-Flying Teleoperator Proposal (MBA/Matrix) to NASA MSFC in August 1971. This initial concept was configured with the pitch, yaw, and roll axes located forward of the handle. The concept was improved in December 1971 by placing the pitch, yaw, and roll axes of the handle in a position coincident with the operator's wrist.

1.2 CONCEPT DESCRIPTION

The basic concept of the terminal pointer hand controller is depicted in Figure 1-1. The controller consists of a three degrees of freedom hand controller

TABLE 1-1: Summary of Manipulator Controls

SWITCH CONTROL	Although time to perform certain complex tasks is quite long, switch control provides the simplest and most direct method of control. One switch is used to activate each actuator. The switch actuation causes the appropriate actuator to move at a fixed rate. A significant improvement on this technique can be attained by replacing the switch with potentiometers and generating proportional rates. This technique is generally unsuitable for force feedback or tactile feedback; however, it is quite feasible to limit the maximum force capability to avoid breaking delicate objects if the actuators are back driveable.
JOYSTICK CONTROL	Joysticks have the same basic advantages and disadvantages as switch control, but provide a capability of coordinating combinations of switches simultaneously. This type of controlling may be either isometric or non-isometric. The non-isometric control is like the classical aircraft control stick, wherein movement of the joystick in the appropriate direction results in the desired response. Both proportional and single rate non-isometric joysticks have been successfully used. Isometric joysticks are a relatively recent development made possible by the development of miniature force transducers. The isometric control remains fixed and provides output proportional to the force exerted in the desired direction. This technique has been shown to be less fatiguing to the operator than the non-isometric joystick.
RESOLVED MOTION CONTROL	A variation of the joystick controller is one wherein the operator considers only the end effector of the manipulator and "flies" that point to the desired position. In this method, the joystick motions correspond only to directions of the tip, and some intermediate system (either a complex linkage or a computer) resolves the desired actuator motions as a function of the manipulator position. This technique is currently being evaluated by the Massachusetts Institute of Technology. The major drawback of this method is that it is difficult to make precise motions in just one axis without causing small motions in the other axes.
TERMINAL FOLLOWING MASTER	Perhaps the most popular control method for currently used manipulators is that of the terminal following master. In this method a master model, usually geometrically similar to the slave, is used to control the manipulator position. Each joint of the slave is servo-controlled to the corresponding position of master. Thus, the operator controls the position of the slave directly rather than by guiding it at particular rates for a period of time and then stopping in the desired position. An advantage of this method is that transducer measurement errors and electronic biases result only in minor errors of the final position, which are rarely even detectable by the operator, whereas a rate controlled manipulator, such as described above, would continuously drift in the direction of the bias.
EXOSKELETAL MASTER	This controller is similar to the terminal following master slave, except that it is strapped to the operator's arm and hence should conform to a human arm in size, shape and range of motion. This type of controller is the most amenable to the implementation of force and tactile feedback. With these advanced control concepts, the exoskeletal master controlled manipulator becomes an extension of the operator's own arm. This type of system provides the most versatile type of man-in-the-loop control and yields the shortest times for task completion. The disadvantages of this type controller are that it encumbers the operator with a rather bulky array of transducers, actuators and rigid structure and that the operator requires a volume of space large enough that he can freely move his arms in any direction.

TABLE 1-2: Controller Comparison

TECHNIQUE	PROPOSED SPACE APPL.	SUITABILITY FOR FORCE FEEDBACK	SUITABILITY FOR TACTILE FEEDBACK	ADVANTAGES	DISADVANTAGES
Switches (rate control)	Secondary control of extremely simple task	No	No	Requires little space; simple control circuitry	Large task times for simple task; dexterous manipulations not feasible
Joystick/hand controller (rate control)	Large boom, dexterous manipulator			Allows control of several actuators at once; requires little control space	Precise motions difficult
Isotonic		Yes (wrist only)	Yes		
Isometric		No	Yes		
Terminal following master (position control)	Large boom, dexterous manipulator	Yes	Yes	Allows position control of multiple joints simultaneously	Requires large control space; requires computer interface
Exoskeletal master	Anthropomorphic manipulator	Yes	Yes	Allows more precise position control of multiple joints	Requires large control space; confines operator; limited to anthropomorphic manipulators
Terminal pointer master	Large boom, dexterous manipulator	Yes (wrist only)	Yes	Allows precise velocity control of multiple joints; requires little control space	Requires computer interface

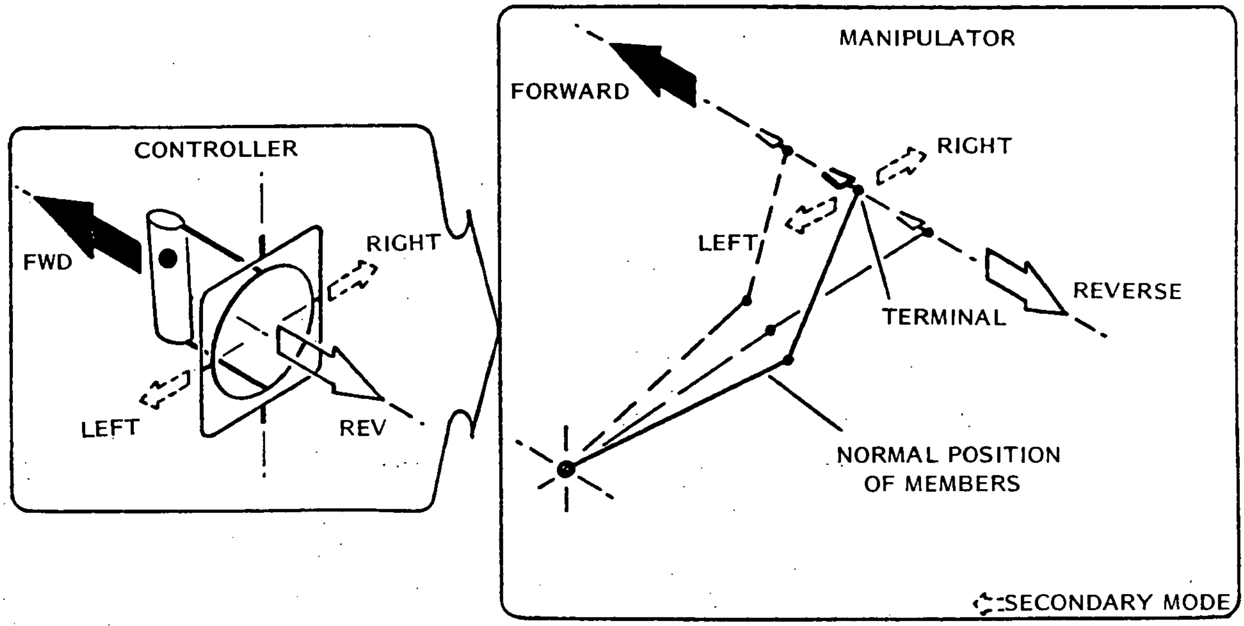

Figure 1-1: Terminal Pointer Concept

(pitch, yaw, roll) with a forward/reverse and a lateral proportional rate thumb control and a dead man trigger switch as shown in Figure 1-2 (depicts original concept). The three degrees of freedom of the handle allow the operator to point the slave end effector in any desired direction, that is, there is a one-to-one correspondence between the angular orientation of the hand controller and the angular orientation of the slave end effector in reference to the operator. Actuation of the proportional rate thumb control results in movement of the end effector in, or normal to, the direction in which it is pointed. As shown in Figure 1-1, the movement of the end effector results from actuation of the proportional rate thumb control in the forward and reverse, and right and left directions.

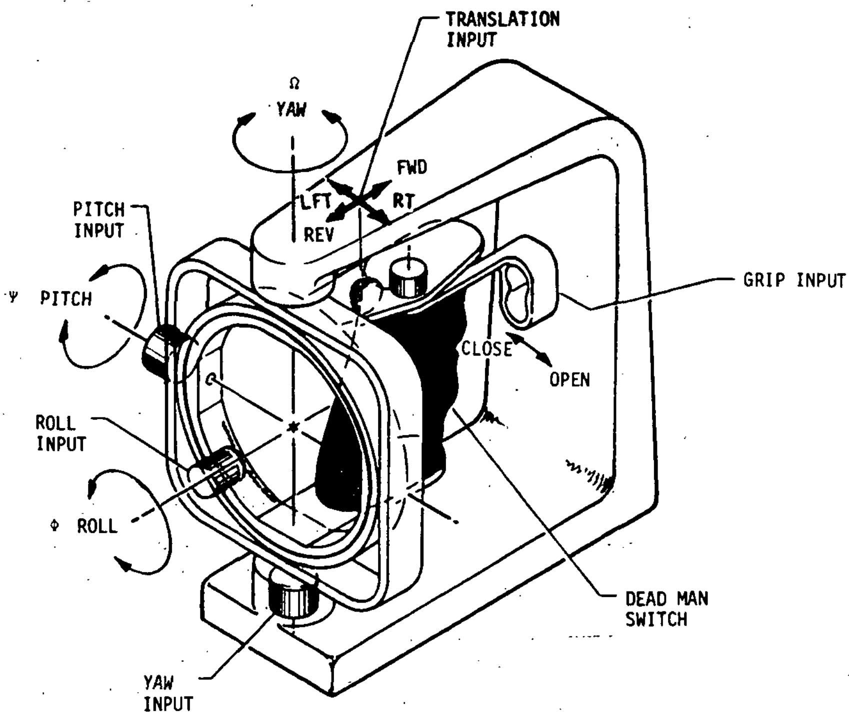

Figure 1-2: Terminal Pointer Hand Controller Concept as Delineated in Proposal

An intermediate computer subsystem transforms the input command motions from the hand controller into individual joint command motions. The computer system accomplishes this transformation utilizing a mathematical model of the slave arm.

For example, to complete the task of picking up a peg and placing it in a hole, the operator would point the hand controller such that the end effector of the slave arm pointed in the direction of the peg (note that the operator is provided with a visual indication of the direction of motion before the motion is initiated). He would then activate the proportional rate thumb control in the forward direction. This action would order the computer subsystem to command the necessary joint rates which would create a motion of the end effector in the direction that the end effector is pointed and at a velocity proportional to the displacement of the proportional rate thumb control. The operator can continuously adjust the direction of the motion and rate of motion as the end effector approaches the peg. The grip control (position) on the hand controller would be used to grasp the peg.

Next, the operator would activate the proportional rate thumb control in the reverse direction, causing the end effector to back away with the peg. Using the hand controller, the operator would point the peg towards the hole, and, while continually adjusting the end effector rate and direction of motion, he would place the peg in the hole.

The Terminal Pointer concept has several unique advantages over existing manipulator control concepts. These advantages are summarized in Table 1-3.

1.3 POTENTIAL APPLICATIONS

The Terminal Pointer concept has many applications in the controlling of space teleoperators. The applications could be to both the attached and free flying classes of teleoperators, as is shown in Figure 1-3.

A concept for a dual station Terminal Pointer Hand Controller for use in controlling the attached booms on the Space Shuttle, is shown in Figure 1-4.

The following sections of this report will discuss the four major tasks performed during the course of this contract. There were:

- Concept Drawings and Model Development (Section 2.0)
- Development of Prototype (Section 3.0)
- Development of Control Law (Section 4.0)
- Conclusions and Recommendations (Section 5.0)

TABLE 1-3: Terminal Pointer Controller Advantages

PRECISE CONTROL	The Terminal Pointer concept provides complete separation of the pointing and velocity control operations. This eliminates the difficulty (associated with resolve rate controllers) of causing small motions in the axis perpendicular to the desired direction of motion. The end effector orientation provides a predictive display to the operator of the precise direction of motion before the motion is initiated.
NATURAL MOVEMENT	All three axes of the hand controller are coincident with the three axes of the operator's wrist. Spatial correspondence exists between the operator's wrist orientation and the orientation of the end effector.
SMALL CONTROL SPACE	The Terminal Pointer has the advantage, inherent in all hand controllers, of requiring an extremely small control space.
SINGLE-HANDED CONTROL	All control functions are located on a single hand controller. The dead man switch allows the operator to completely release his hand from the controller (causing the hand controller to lock in all motions) without the risk of inadvertently actuating the manipulator. This allows a single operator to control two separate manipulators and to perform additional control functions as required.
APPLICATIONS	The Terminal Pointer concept has application to a Shuttle-Attached Boom, a dexterous manipulator arm, and to the FFTO.

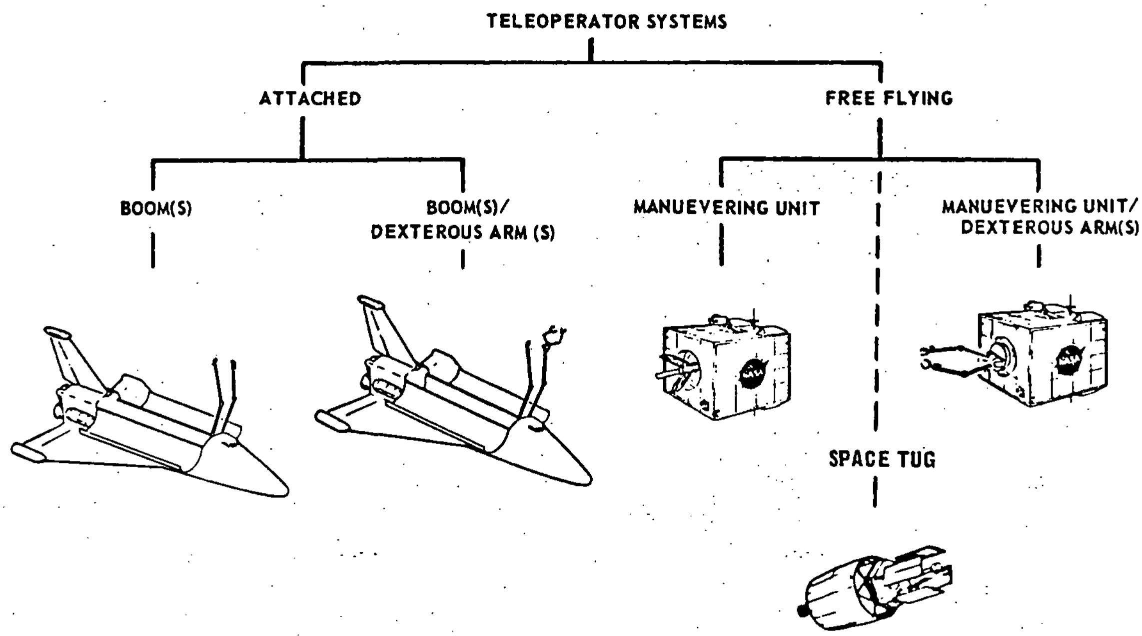

Figure 1-3: Potential Applications of Terminal Pointer Hand Controller

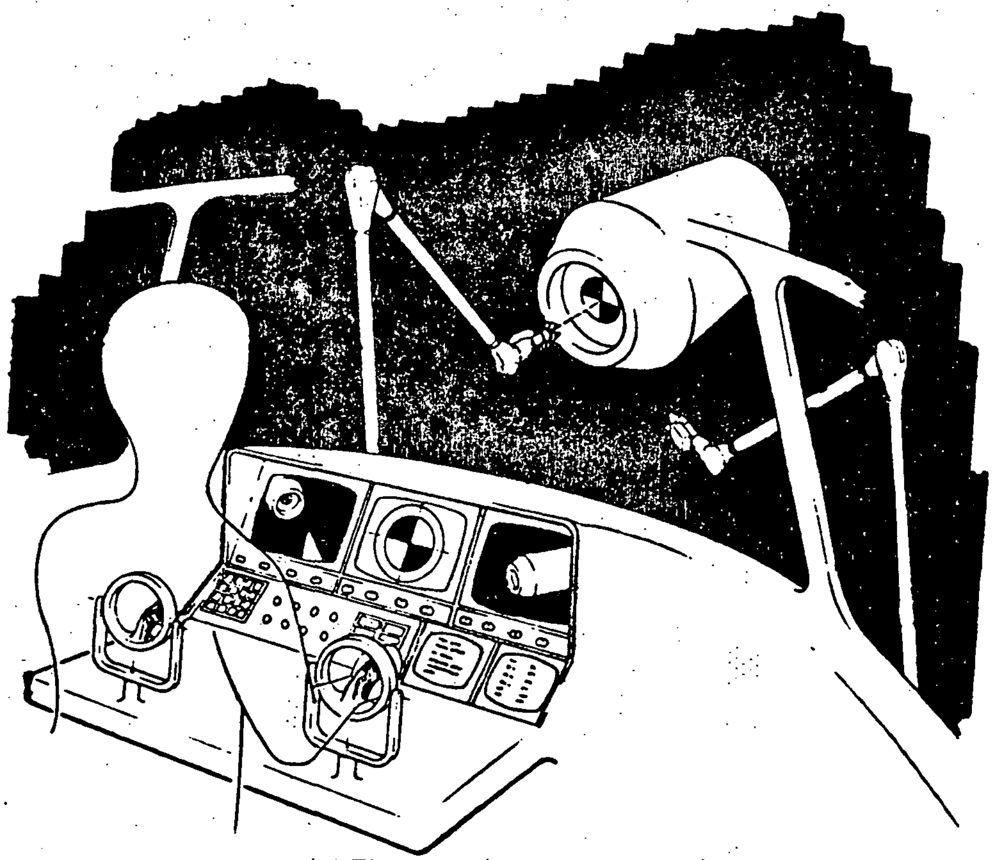

Figure 1-4: Dual Station Terminal Pointer Controller Concept

SECTION 2.0

CONCEPT DRAWINGS AND MODEL DEVELOPMENT

This section describes the design metamorphosis of a hand controller intended to achieve the highest possible compatibility with the hand of the human operator. The development of the final form of the Terminal Pointer Hand Controller and the generation of a layout drawing and model depicting the concept comprised the principal task in this effort, and its successful completion was accomplished by means of the subtasks shown below in Figure 2-1.

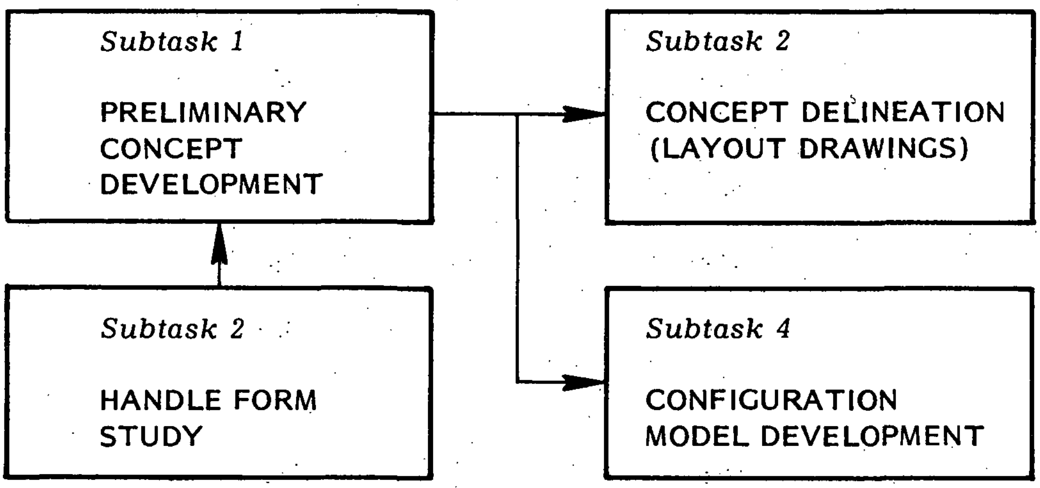

Figure 2-1: Diagram of Developmental Subtasks

Subtask 1 - Preliminary Concept Development

In this subtask, URS/Matrix developed a preliminary concept configuration, based on its initial proposed sketches (see Figure 2-2) and on the results of the handle form study in Subtask 2. This effort finalized areas such as the maximum controller envelope, the position of the control inputs, the relationships of the inputs and handle to gimbals, etc.

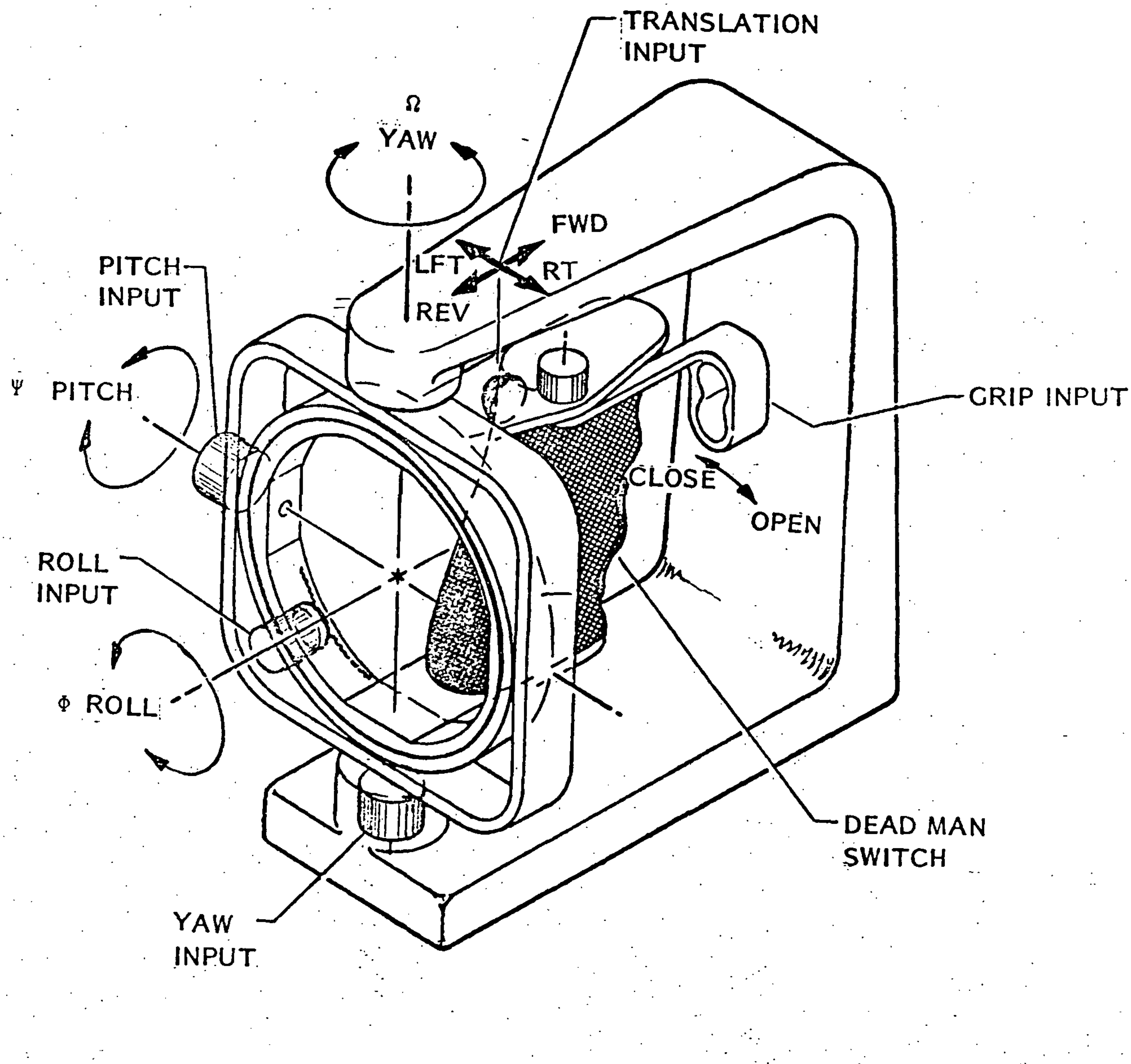

Figure 2-2: Terminal Pointer Hand Controller Concept as Delineated in Proposal

Subtask 2 - Handle Form Study

During this subtask, URS/Matrix studied various handle configurations to determine which form would provide the following:

- A handle that conformed anthropomorphically to the operator's hand.
- Use compatibility for a wide range of the user population.
- Gimbal and roll inputs to take advantage of natural hand and wrist movements.
- Optimum location for the thumb actuated translation control input.

Provision of each item listed above was attained by using many foam only and foam and clay models of candidate configurations. The form finally selected is incorporated in the design shown in Figure 2-3.

Subtask 3 - Concept Delineation

Using the outputs of Subtasks 1 and 2, URS/Matrix generated a set of full-scale layouts to be used by MSFC-ASTR in its preparation of manufacturing and assembly drawings of the hand controller. These drawings are shown in Figures 2-3, 2-4, and 2-5. The layout includes all relevant information concerning materials, dimensions, tolerances, and special components. The find numbers are referenced and identified in Appendix A.

Subtask 4 - Configuration Model Development

After completing Subtasks 1, 2, and 3, URS/Matrix fabricated a full-scale model of the Terminal Pointer Hand Controller (see Figure 2-6), according to the configuration delineated in Subtask 3, and used the model to verify the operator/controller interface. The model was further used by the government contracter during design, drafting and fabrication.

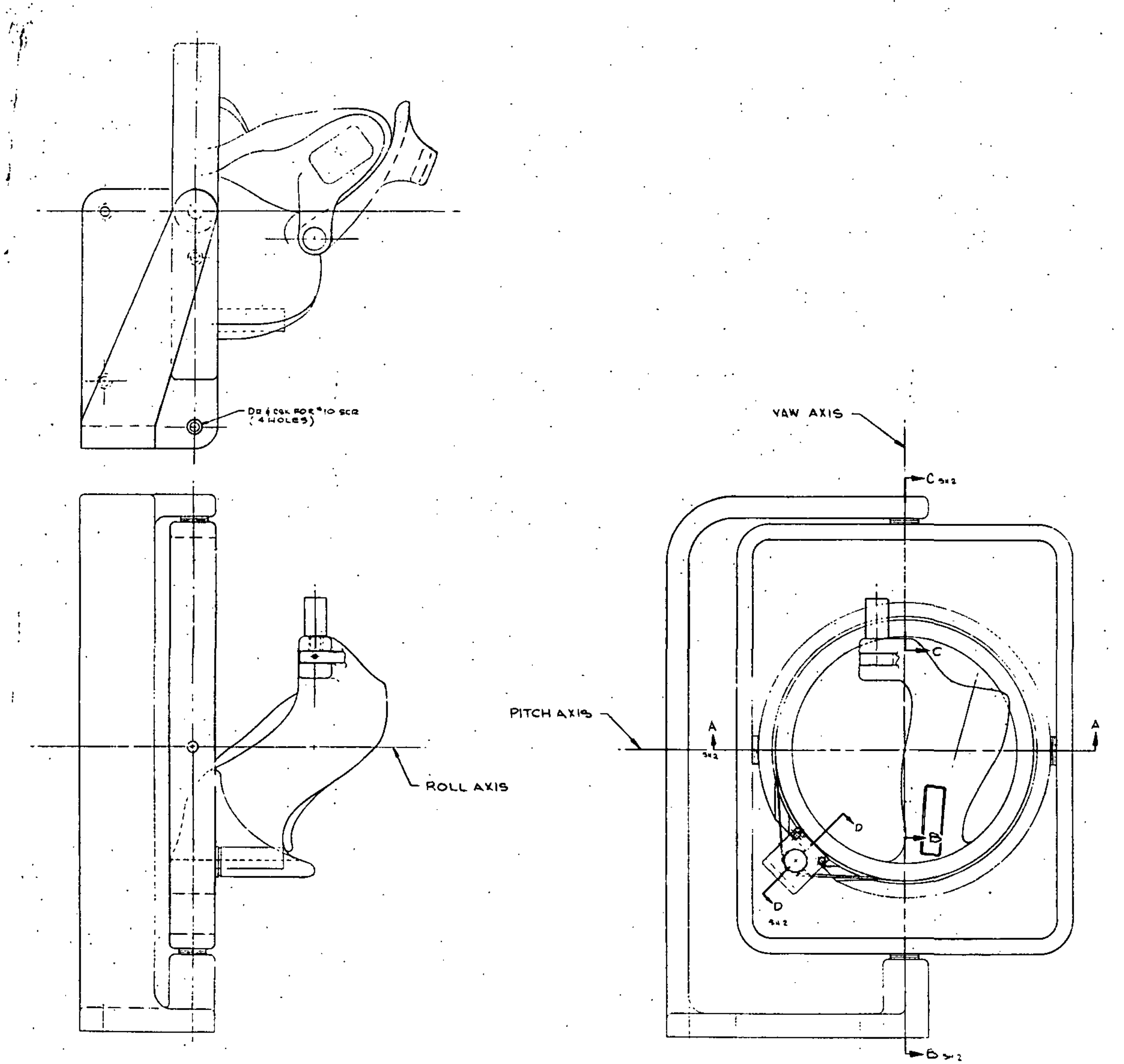

Figure 2-3: Terminal Pointer Hand Controller - Preliminary Configuration Layout

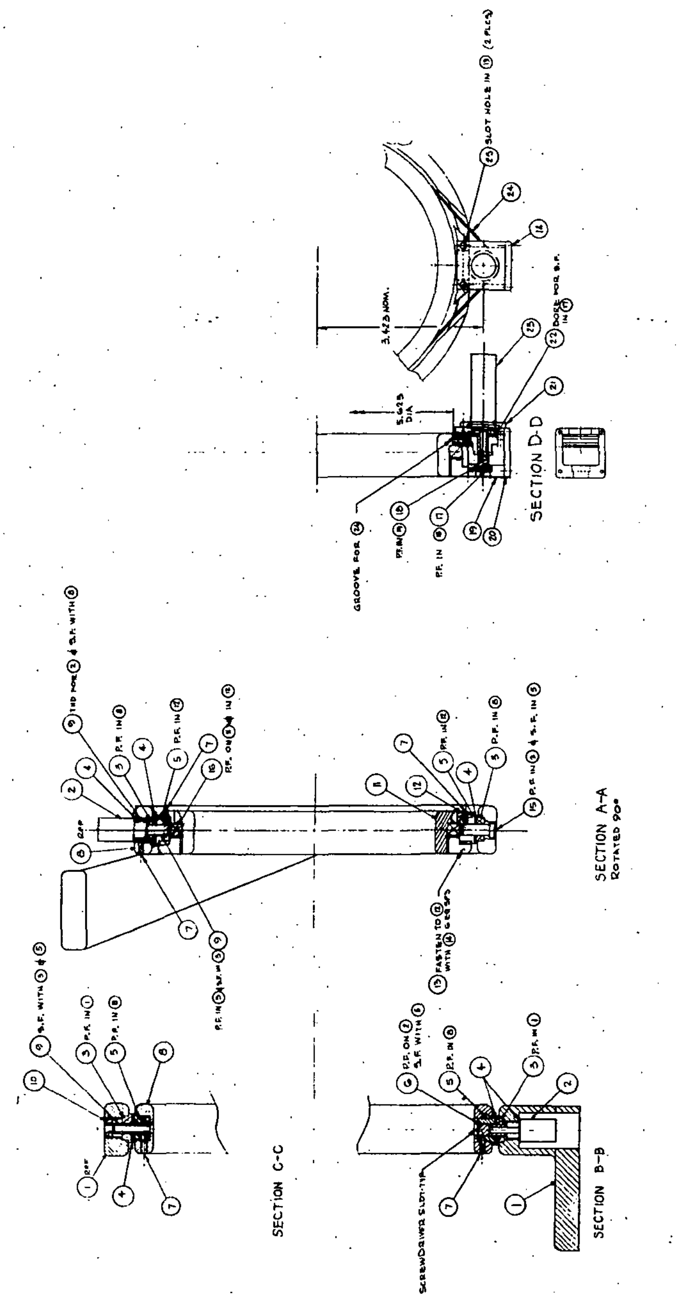

Figure 2-4: Terminal Pointer Hand Controller - Gimbal and Ring Details

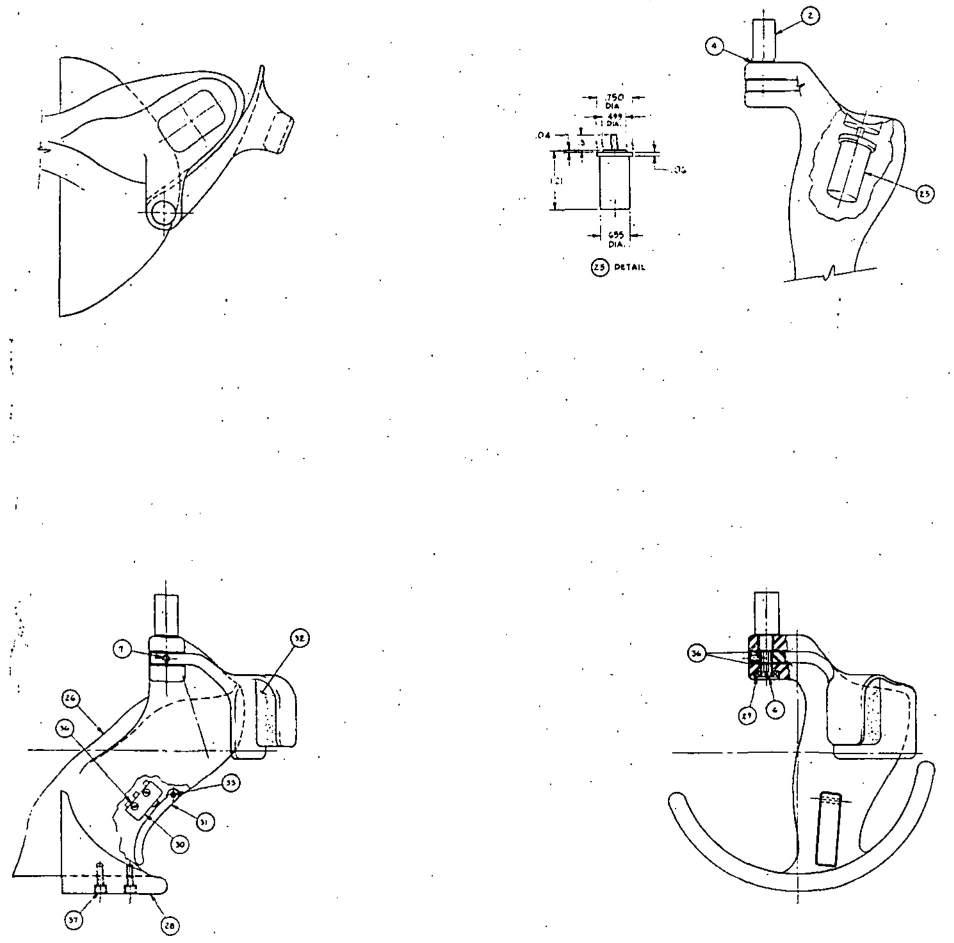

Figure 2-5: Terminal Pointer Hand Controller - Handle Details

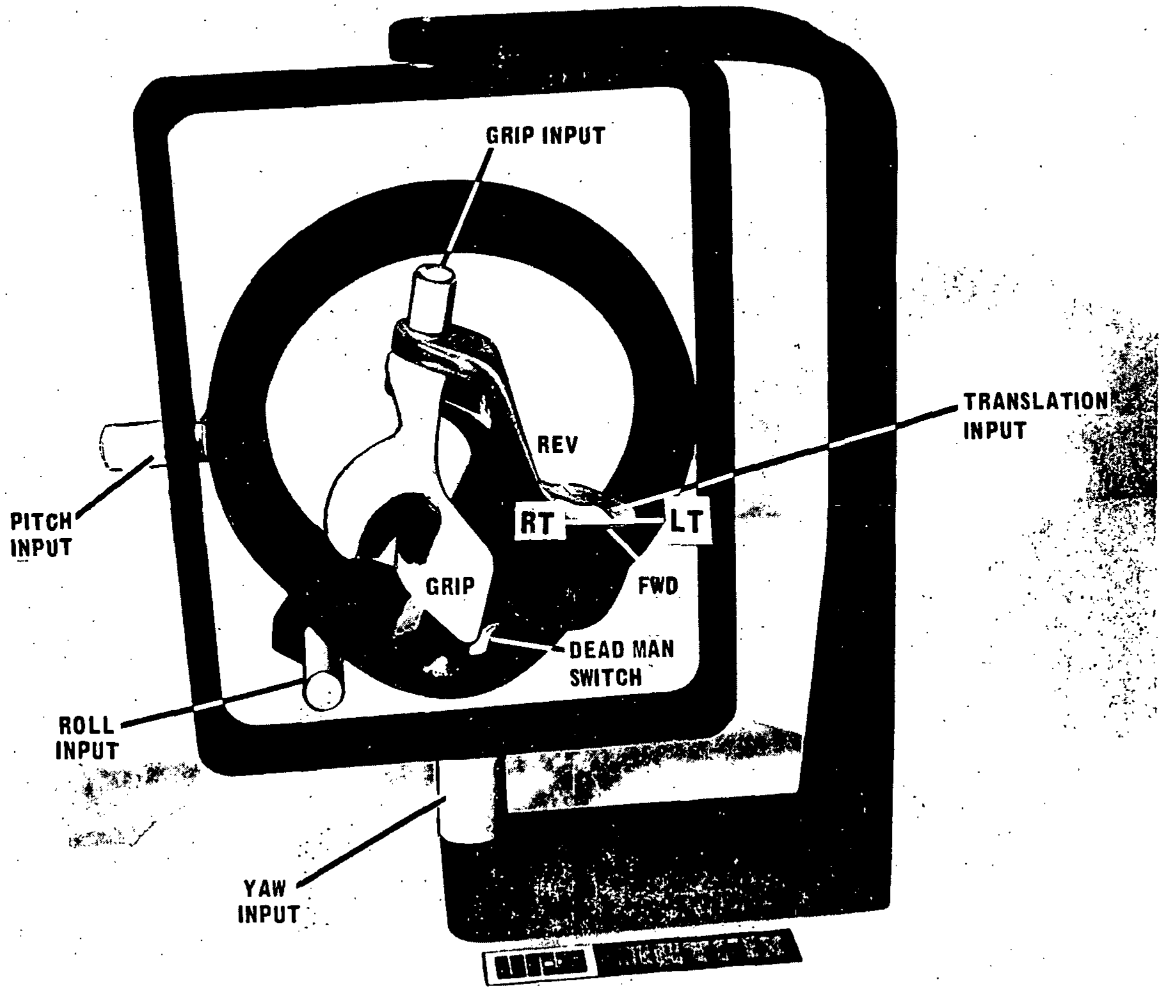

Figure 2-6: Model of Terminal Pointer Hand Controller (Full Scale)

The layout drawings and model configurations generated in this effort were delivered to MSFC.

SECTION 3.0

DEVELOPMENT OF PROTOTYPE

The URS/Matrix Company assisted the MSFC-ASTR Laboratory and Support Contractor (Sperry Rand Corp.) by providing liaison and coordination support from the design/drafting phase, thru the manufacturing/assembly phases of the prototype development.

URS/Matrix also furnished NASA/Sperry with an epoxy model of the handle (as shown in Figure 2-5), on a loan basis. This was done in order to provide the prototype fabricator a dimensionally stable form from which to mold the prototype handle. Molding directly from the original model eliminated the costly activity of measuring the model, recording the dimensions on the detailed drawings, and translation of the drawings into a machined component.

During the early phases of the detail design effort, URS/Matrix determined that controller gimbal frames could be modified to provide a configuration that was not only more simple in design, but also enabled the operator to have a greater amount of flexibility during its operation. The drawing configuration is illustrated in Figure 3-1, and the full scale model is shown in Figure 3-2.

The Terminal Pointer Hand Controller prototype assembly drawing is depicted in Figure 3-3. This drawing, along with the detail drawings, was produced by Sperry Rand.

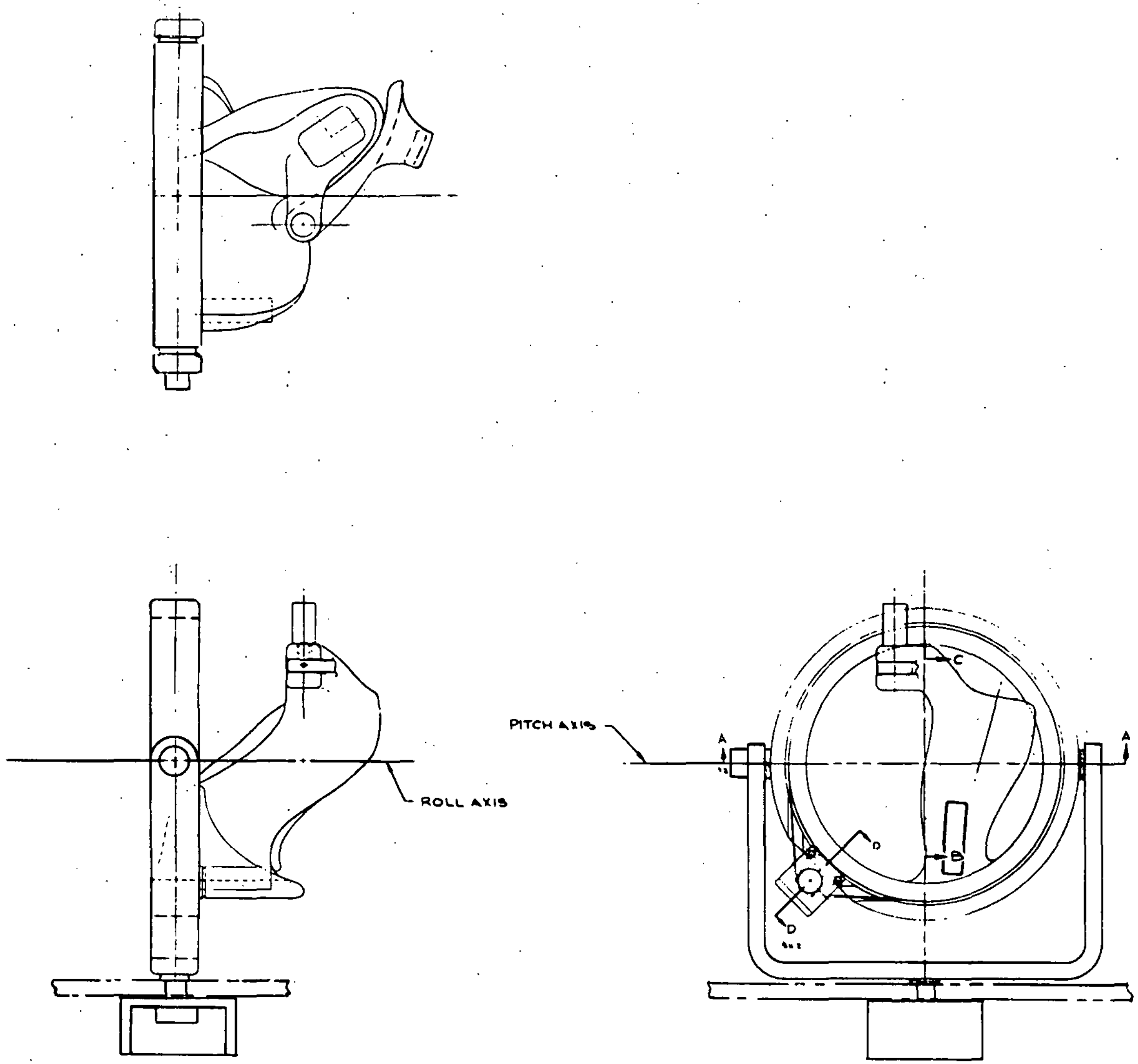

Figure 3-1: Terminal Pointer Hand Controller - Modified Configuration Layout

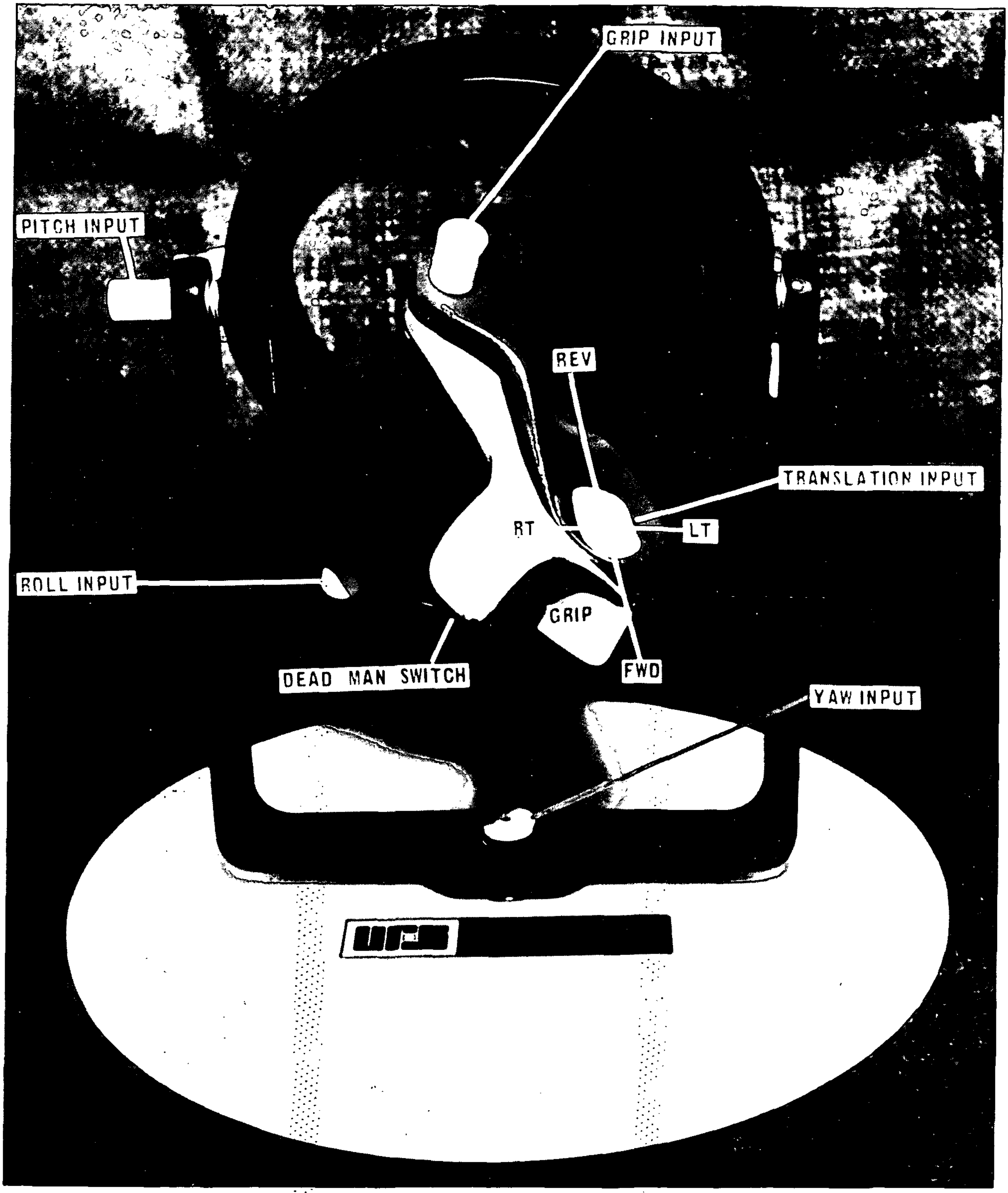

Figure 3-2: Terminal Pointer Hand Controller Full Scale Model (Mod. 2)

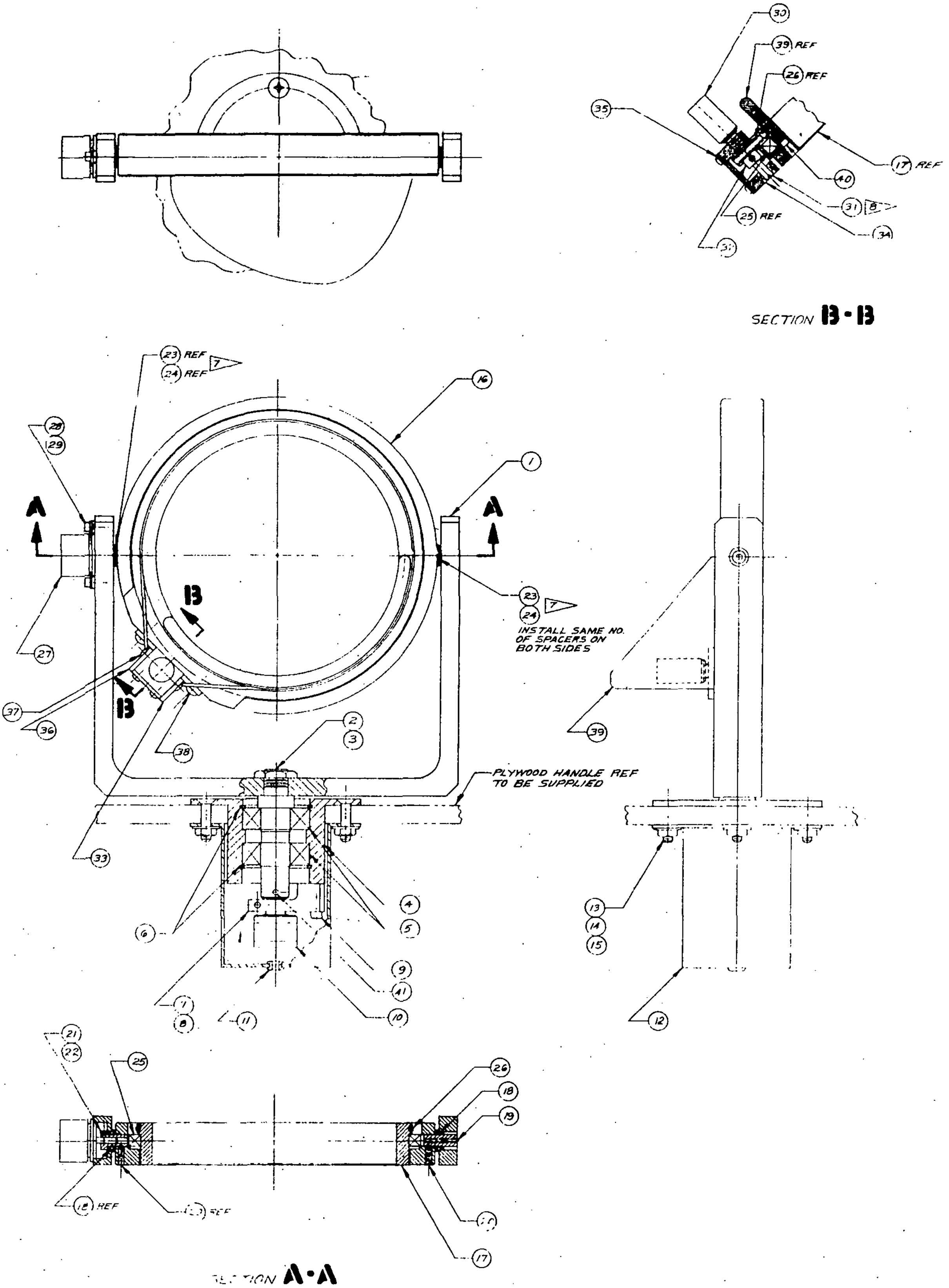

Figure 3-3: Terminal Pointer Hand Controller Prototype Assembly Drawing

Figure 3-4 shows the operator/controller interface with the controller in various input positions. Figure 3-5 shows an operator seated at the MSFC-ASTR teleoperator console. A close up view of the prototype is shown in Figure 3-6. Finally, Figure 3-7 shows the Rancho Anthropomorphic Manipulator arm in various positions.

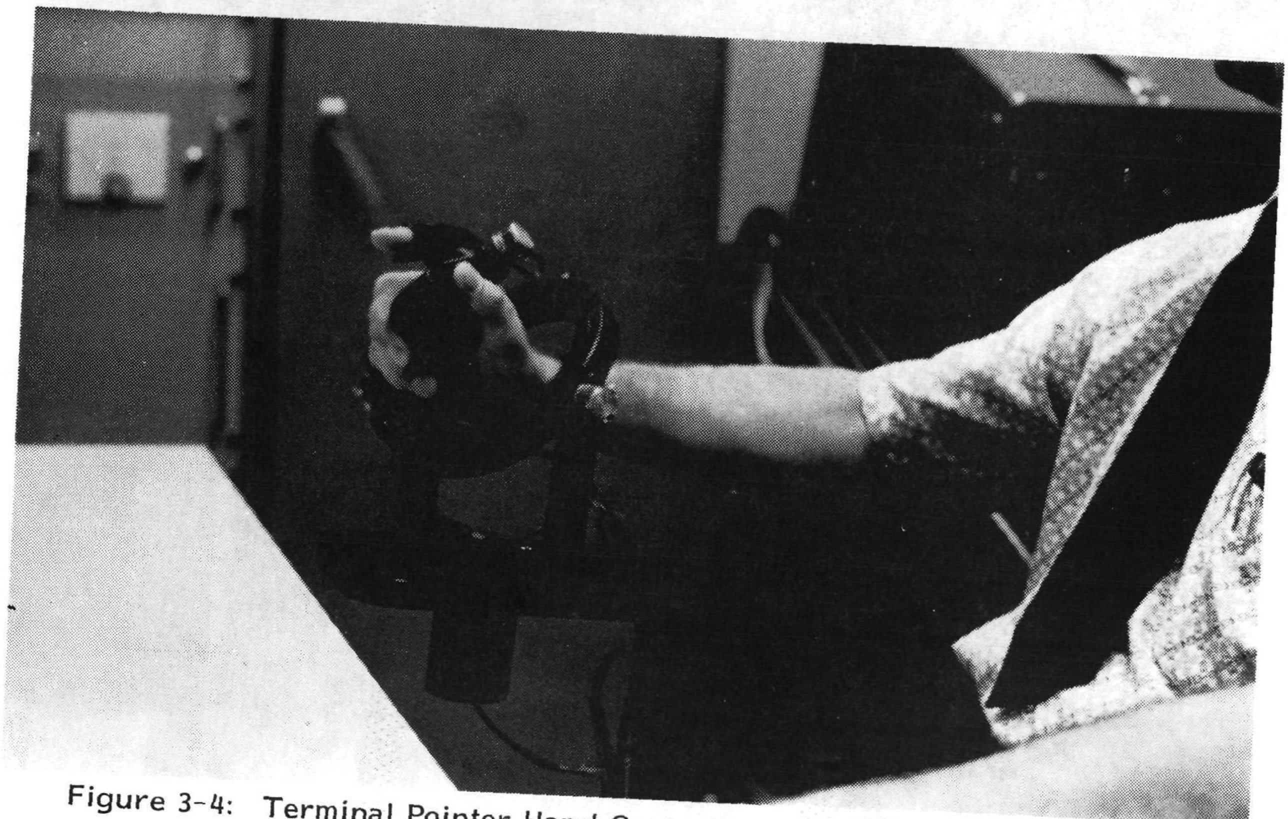

Figure 3-4: Terminal Pointer Hand Controller in Various Control Positions

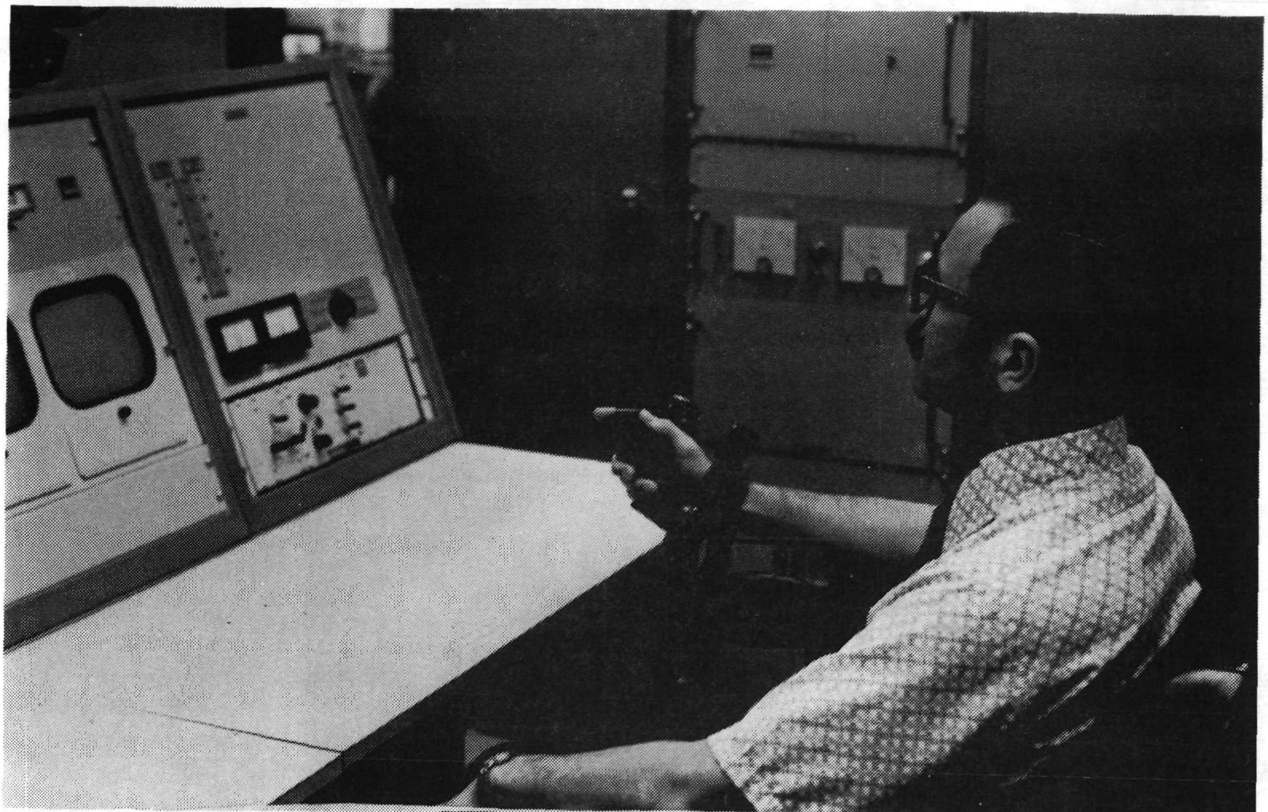

Figure 3-5: Operator/Terminal Pointer Hand Controller at Console

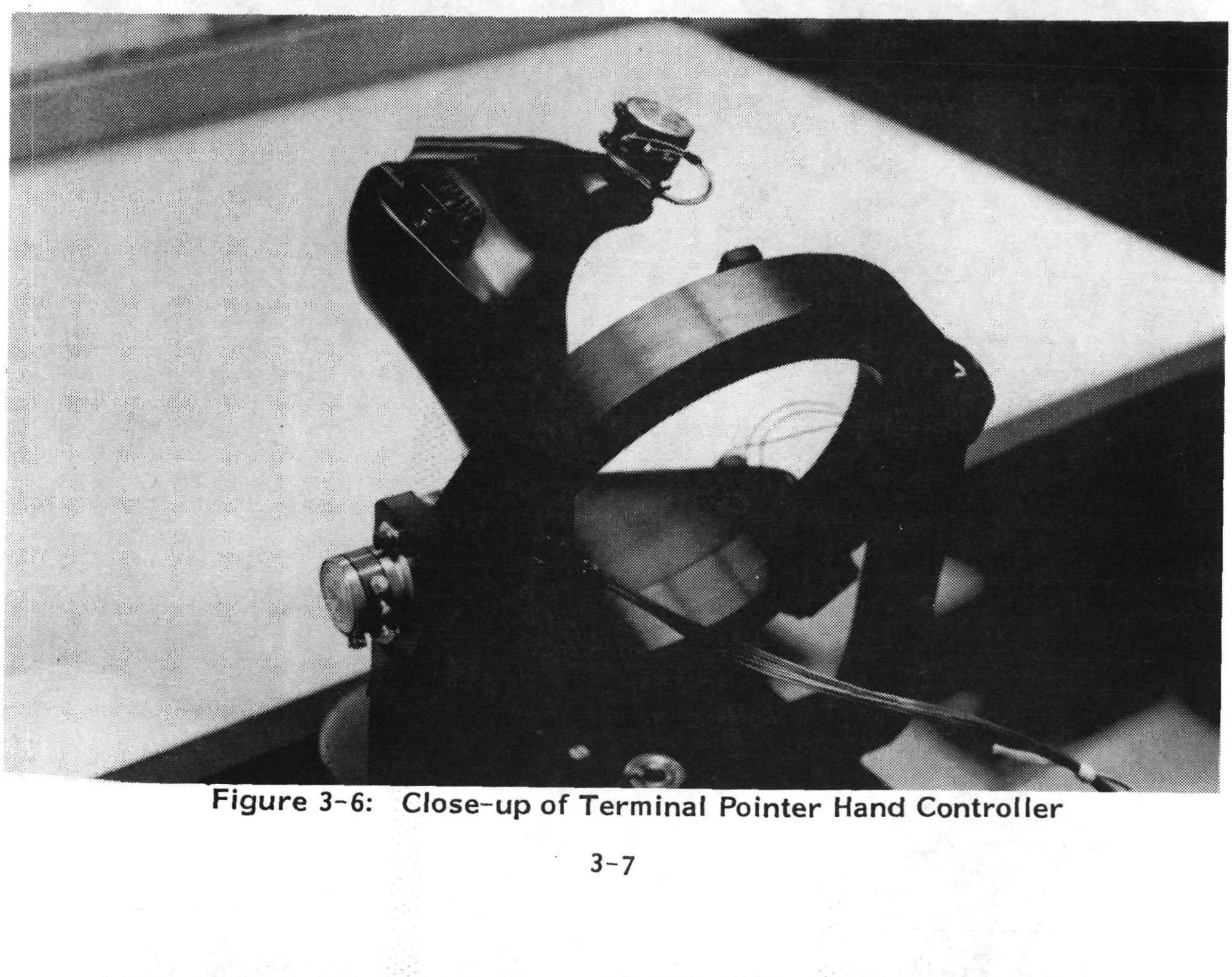

Figure 3-6: Close-up of Terminal Pointer Hand Controller

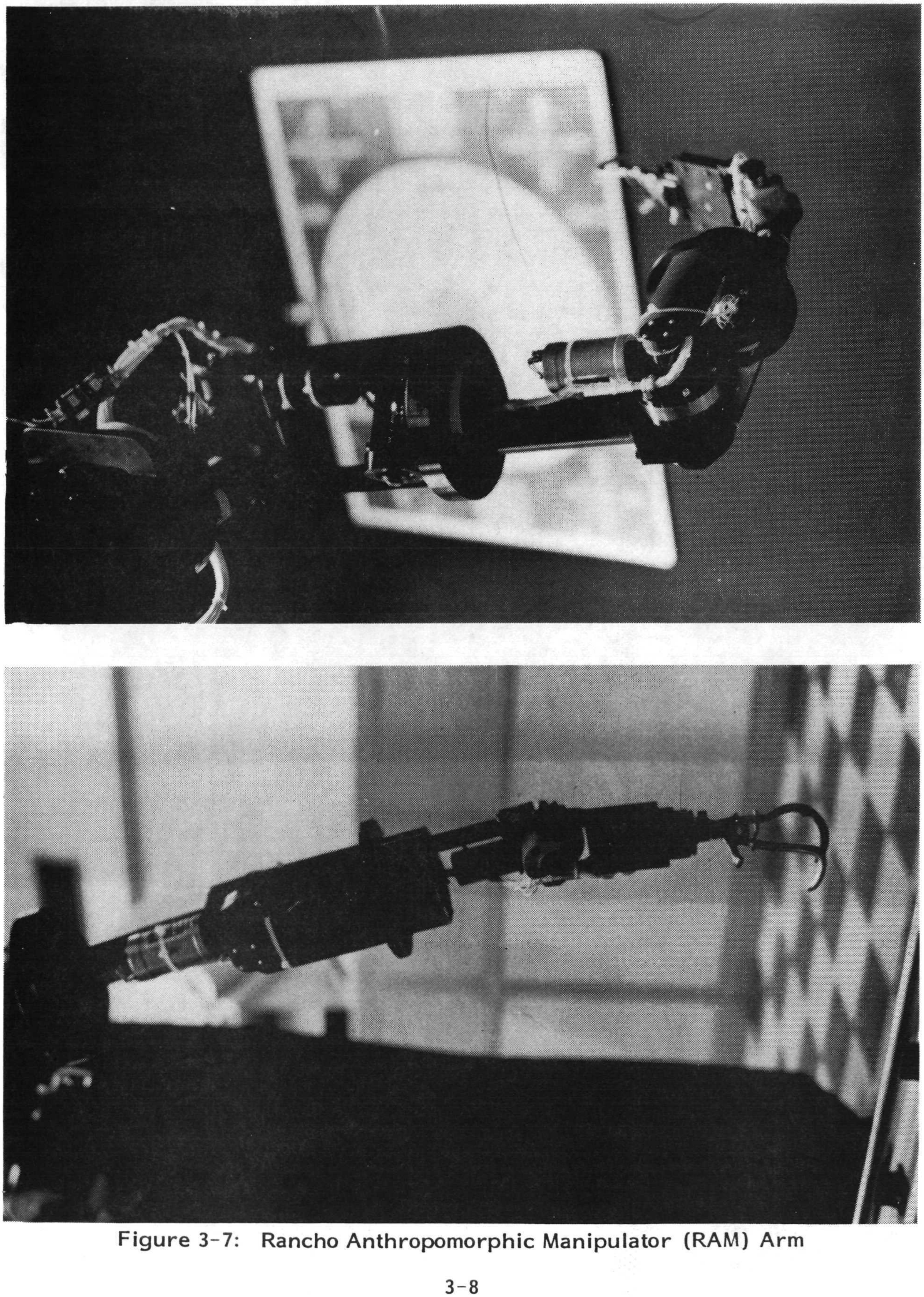

Figure 3-7: Rancho Anthropomorphic Manipulator (RAM) Arm

SECTION 4.0

CONTROL LAW DEVELOPMENT

The use of the URS/Matrix hand controller with the MSFC Rancho Anthropomorphic Manipulator (RAM) requires conversion of the hand controller input command motions to motor rate commands for the six RAM motors. This conversion is accomplished by an on-line SEL 840 digital computer. In order to perform the conversion, the computer must be programmed with a set of equations (control law), the development of which is presented below.

The general form of the control law is:

$$\begin{Bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \\ \dot{\Theta}_4 \\ \dot{\Theta}_5 \\ \dot{\Theta}_6 \end{Bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ 0 \\ \alpha \\ \beta \\ \phi \end{Bmatrix} \qquad (1)$$

where $\dot{\Theta}_1$ - $\dot{\Theta}_6$ are the six RAM motor rate commands,

K_x and K_y are the thumb switch lateral and forward/reverse rate commands,

A is the control transformation matrix, and

α, β, and ϕ are the hand controller pitch, yaw, and roll commands.

The desired response of the RAM is to have the terminal segment oriented in the same attitude as the hand controller (see Figure 4-1), with translation in the forward/reverse and lateral (y_7 and x_7) directions, and rotation about the terminal x_7, y_7 and z_7 axes corresponding to the hand controller α, ϕ, and β commands.

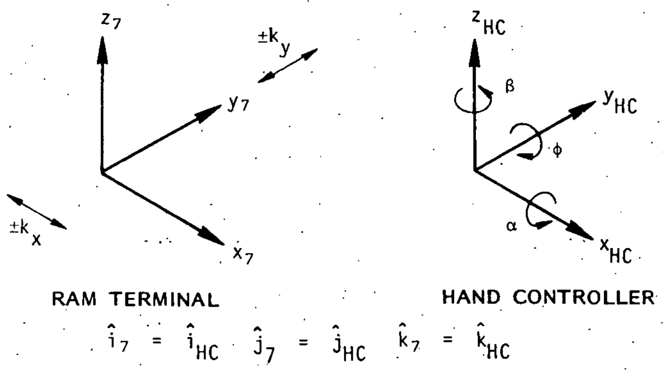

Figure 4-1: Desired Terminal Attitude and Translational Response

Defining

$$u = \frac{dx_7}{dt} = V_{x_7} ,$$

$$v = \frac{dy_7}{dt} = V_{y_7} ,$$

$$w = \frac{dz_7}{dt} = V_{z_7} ,$$

$$\omega_x = \dot{\theta}_{x_7} ,$$

$$\omega_y = \dot{\theta}_{y_7} , \text{ and}$$

$$\omega_z = \dot{\theta}_{z_7} , \text{ we can write}$$

the translation and rotation of the terminal in terms of the six motor rates as:

$$\begin{Bmatrix} u \\ v \\ w \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} J \end{bmatrix} \begin{Bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \\ \dot{\Theta}_4 \\ \dot{\Theta}_5 \\ \dot{\Theta}_6 \end{Bmatrix} \qquad (2)$$

or

$$\begin{Bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \\ \dot{\Theta}_4 \\ \dot{\Theta}_5 \\ \dot{\Theta}_6 \end{Bmatrix} = \begin{bmatrix} J^{-1} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \qquad (3)$$

where J is the Jacobian for the system of dynamic equations relating $(u, v, w, \omega_x, \omega_y, \omega_z)$ to $\dot{\Theta}_1 - \dot{\Theta}_6$:

$$J = \begin{bmatrix} \frac{\delta u}{\delta \dot{\Theta}_1} & \frac{\delta u}{\delta \dot{\Theta}_2} & \frac{\delta u}{\delta \dot{\Theta}_3} & \frac{\delta u}{\delta \dot{\Theta}_4} & \frac{\delta u}{\delta \dot{\Theta}_5} & \frac{\delta u}{\delta \dot{\Theta}_6} \\ \frac{\delta v}{\delta \dot{\Theta}_1} & \frac{\delta v}{\delta \dot{\Theta}_2} & \frac{\delta v}{\delta \dot{\Theta}_3} & \frac{\delta v}{\delta \dot{\Theta}_4} & \frac{\delta v}{\delta \dot{\Theta}_5} & \frac{\delta v}{\delta \dot{\Theta}_6} \\ \frac{\delta w}{\delta \dot{\Theta}_1} & \frac{\delta w}{\delta \dot{\Theta}_2} & \frac{\delta w}{\delta \dot{\Theta}_3} & \frac{\delta w}{\delta \dot{\Theta}_4} & \frac{\delta w}{\delta \dot{\Theta}_5} & \frac{\delta w}{\delta \dot{\Theta}_6} \\ \frac{\delta \omega_x}{\delta \dot{\Theta}_1} & \frac{\delta \omega_x}{\delta \dot{\Theta}_2} & \frac{\delta \omega_x}{\delta \dot{\Theta}_3} & \frac{\delta \omega_x}{\delta \dot{\Theta}_4} & \frac{\delta \omega_x}{\delta \dot{\Theta}_5} & \frac{\delta \omega_x}{\delta \dot{\Theta}_6} \\ \frac{\delta \omega_y}{\delta \dot{\Theta}_1} & \frac{\delta \omega_y}{\delta \dot{\Theta}_2} & \frac{\delta \omega_y}{\delta \dot{\Theta}_3} & \frac{\delta \omega_y}{\delta \dot{\Theta}_4} & \frac{\delta \omega_y}{\delta \dot{\Theta}_5} & \frac{\delta \omega_y}{\delta \dot{\Theta}_6} \\ \frac{\delta \omega_z}{\delta \dot{\Theta}_1} & \frac{\delta \omega_z}{\delta \dot{\Theta}_2} & \frac{\delta \omega_z}{\delta \dot{\Theta}_3} & \frac{\delta \omega_z}{\delta \dot{\Theta}_4} & \frac{\delta \omega_z}{\delta \dot{\Theta}_5} & \frac{\delta \omega_z}{\delta \dot{\Theta}_6} \end{bmatrix} \qquad (4)$$

The desired translation and rotation can be described in terms of the hand controller input command motions as

$$\begin{Bmatrix} u \\ v \\ w \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [B] \begin{Bmatrix} K_x \\ K_y \\ 0 \\ \alpha_{HC} - \alpha_7 \\ \phi_{HC} - \phi_7 \\ \beta_{HC} - \beta_7 \end{Bmatrix} \qquad (5)$$

Combining (3) and (5) the solution for the motor rate commands becomes

$$\begin{Bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \\ \dot{\Theta}_4 \\ \dot{\Theta}_5 \\ \dot{\Theta}_6 \end{Bmatrix} = J^{-1} \; B \begin{Bmatrix} K_x \\ K_y \\ 0 \\ \alpha' \\ \phi' \\ \beta' \end{Bmatrix} \qquad (6)$$

where $\alpha' = \alpha_{HC} - \alpha_7$ $\phi' = \phi_{HC} - \phi_7$ and $\beta' = \beta_{HC} - \beta_7$. The problem is now reduced to finding J and B.

The motor rotations and mechanical layout of the RAM are shown in simplified form in Figure 4-2. Seven coordinate systems, corresponding to the seven segments of the RAM, are used in the solution.

Derivation of the Jacobian

The J matrix is found by determining the velocity and rotation of coordinate system 7 caused by each of the six motors.

In general, the velocity of a point (in this case, the origin of coordinate system 7) due to an angular rotation is given by

$$\vec{V} = \vec{\Omega} \times \vec{r}$$

where $\vec{\Omega}$ is the rotational rate and $\vec{r}$ is the position vector from the center of rotation to the point. Thus,

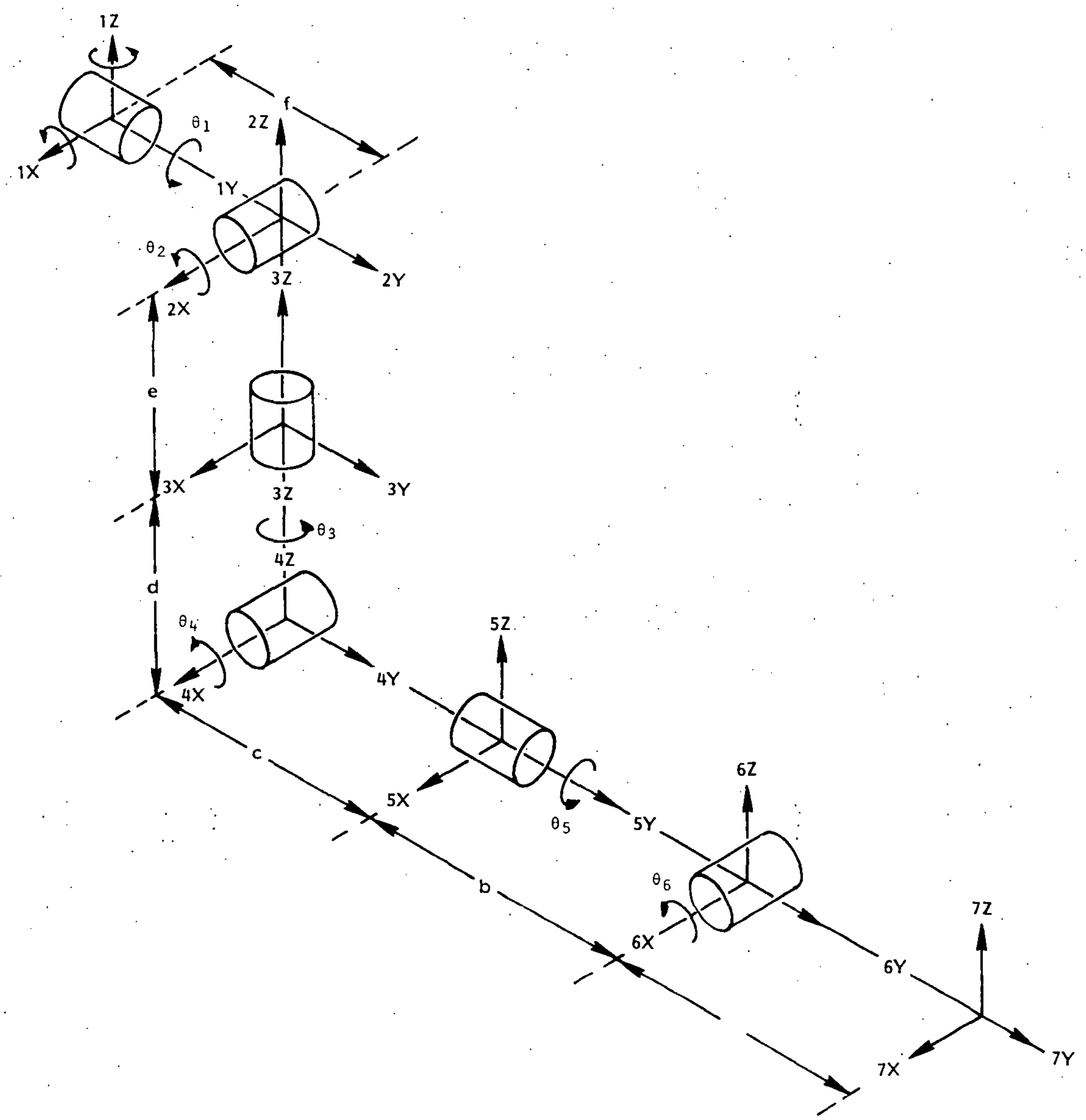

Figure 4-2: Motor Rotation and Mechanical Layout of the Rancho Anthropomorphic Manipulator

$$\vec{V}_7 = [\dot{\Theta}_6\hat{i}_6 \times a\hat{j}_7] + [\dot{\Theta}_5\hat{j}_5 \times (a\hat{j}_7 + b\hat{j}_6)] +$$
$$+ [\dot{\Theta}_4\hat{i}_4 \times (a\hat{j}_7 + b\hat{j}_6 + c\hat{j}_5)] + [\dot{\Theta}_3\hat{k}_3 \times (a\hat{j}_7 + b\hat{j}_6 + c\hat{j}_5 - d\hat{k}_4)] +$$
$$+ [\dot{\Theta}_2\hat{i}_2 \times (a\hat{j}_7 + b\hat{j}_6 + c\hat{j}_5 - d\hat{k}_4 - e\hat{k}_3)] + \qquad (7)$$
$$+ [\dot{\Theta}_1\hat{j}_1 \times (a\hat{j}_7 + b\hat{j}_6 + c\hat{j}_5 - d\hat{k}_4 - e\hat{k}_3 + f\hat{k}_2)]$$

The velocity components u, v, and w are obtained as the dot product of $\vec{V}$, with $\hat{i}_7$, $\hat{j}_7$ and $\hat{k}_7$, respectively:

$$u = \vec{V}_7 \cdot \hat{i}_7$$
$$v = \vec{V}_7 \cdot \hat{j}_7 \qquad (8)$$
$$w = \vec{V}_7 \cdot \hat{k}_7.$$

Equation (7) must be expressed in terms of a single coordinate system. Coordinate system 7 is chosen since the required velocity and rotational movements are expressed in this system. Transformation matrices used to convert unit vectors in systems 1 - 6 to system 7 are derived by inspection and presented below.

${}^{j}M_{i}$ is the transformation matrix <u>from</u> system i <u>to</u> system j, thus

$$\begin{Bmatrix} \hat{i}_j \\ \hat{j}_j \\ \hat{k}_j \end{Bmatrix} = {}^{j}M_{i} \begin{Bmatrix} \hat{i}_i \\ \hat{j}_i \\ \hat{k}_i \end{Bmatrix}$$

c_i and s_i are cos Θ_i and sin Θ_i, respectively

$$C \equiv {}^7M_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & s_6 \\ 0 & -s_6 & c_6 \end{bmatrix} \tag{9}$$

$${}^6M_5 = \begin{bmatrix} c_5 & 0 & -s_5 \\ 0 & 1 & 0 \\ s_5 & 0 & c_5 \end{bmatrix}$$

$${}^5M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_4 & s_4 \\ 0 & -s_4 & c_4 \end{bmatrix}$$

$${}^4M_3 = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^3M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{bmatrix}$$

$${}^2M_1 = \begin{bmatrix} c_1 & 0 & -s_1 \\ 0 & 1 & 0 \\ s_1 & 0 & c_1 \end{bmatrix}$$

Therefore:

$$
{}^{7}M_5 = {}^{7}M_6\ {}^{6}M_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & s_6 \\ 0 & -s_6 & c_6 \end{bmatrix} \begin{bmatrix} c_5 & 0 & -s_5 \\ 0 & 1 & 0 \\ s_5 & 0 & c_5 \end{bmatrix} =
$$

$$
D \equiv {}^{7}M_5 = \begin{bmatrix} c_5 & 0 & -s_5 \\ s_5s_6 & c_6 & c_5s_6 \\ s_5c_6 & -s_6 & c_5c_6 \end{bmatrix} \tag{10}
$$

$$
{}^{7}M_4 = {}^{7}M_5\ {}^{5}M_4 = \begin{bmatrix} c_5 & 0 & -s_5 \\ s_5s_6 & c_6 & c_5s_6 \\ s_5c_6 & -s_6 & c_5c_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_4 & s_4 \\ 0 & -s_4 & c_4 \end{bmatrix} =
$$

$$
E \equiv {}^{7}M_4 = \begin{bmatrix} c_5 & s_4s_5 & -c_4s_5 \\ s_5s_6 & (c_4c_6 - s_4c_5s_6) & (s_4c_6 + c_4c_5s_6) \\ s_5c_6 & (-c_4s_6 - s_4c_5c_6) & (-s_4s_6 + c_4c_5c_6) \end{bmatrix} \tag{11}
$$

$$
{}^{7}M_3 = {}^{7}M_4\ {}^{4}M_3 = \begin{bmatrix} C_5 & S_4S_5 & -C_4S_5 \\ S_5S_6 & (C_4C_6 - S_4C_5S_6) & (S_4C_6 + C_4C_5S_6) \\ S_5C_6 & (-C_4S_6 - S_4C_5C_6) & (-S_4S_6 + C_4C_5C_6) \end{bmatrix} \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} =
$$

$$
F \equiv {}^{7}M_3 = \begin{bmatrix} C_3C_5 - S_3S_4S_5 & S_3C_5 + C_3S_4S_5 & -C_4S_5 \\ C_3S_5S_6 - S_3(C_4C_6 - S_4C_5S_6) & S_3S_5S_6 + C_3(C_4C_6 - S_4C_5S_6) & S_4C_6 + C_4C_5S_6 \\ C_3S_5C_6 + S_3(C_4S_6 + S_4C_5C_6) & S_3S_5S_6 - C_3(C_4S_6 + S_4S_5S_6) & -S_4S_6 + C_4C_5C_6 \end{bmatrix} \quad (12)
$$

$$G \equiv {}^{7}M_2 = {}^{7}M_3\,{}^{3}M_2 = \begin{bmatrix} c_3c_5-s_3s_4s_5 & c_2(s_3c_5+c_3s_4s_5)+s_2c_4s_5 & s_2(s_3c_5+c_3s_4s_5)-c_2c_4s_5 \\ c_3s_5s_6-s_3(c_4c_6-s_4c_5s_6) & \begin{matrix} c_2[s_3s_5s_6+c_3(c_4c_6+ \\ -s_4c_5s_6)]+ \\ -s_2[s_4c_6+c_4c_5s_6] \end{matrix} & \begin{matrix} s_2[s_3s_5s_6+c_3(c_4c_6+ \\ -s_4c_5s_6)]+ \\ +c_2[s_4c_6+c_4c_5s_6] \end{matrix} \\ \begin{matrix} c_3s_5c_6+s_3(c_4s_6+ \\ +s_4c_5c_6) \end{matrix} & \begin{matrix} c_2[s_3s_5c_6-c_3(c_4s_6+ \\ +s_4c_5c_6)]+ \\ +s_2[s_4s_6-c_4c_5c_6] \end{matrix} & \begin{matrix} s_2[s_3s_5c_6-c_3(c_4s_6+ \\ +s_4s_5s_6)]+ \\ -c_2[s_4s_6-c_4c_5c_6] \end{matrix} \end{bmatrix} \quad (13)$$

$$H \equiv {}^{7}M_1 = {}^{7}M_2\,{}^{2}M_1 = \begin{bmatrix} \begin{matrix} c_1(c_3c_5-s_3s_4s_5)+ \\ +s_1[s_2(s_3c_5+ \\ +c_3s_4s_5)-c_2c_4s_5] \end{matrix} & \begin{matrix} c_2(s_3c_5+c_3s_4s_5)+ \\ +s_2c_4s_5 \end{matrix} & \begin{matrix} -S_1(c_3c_5-s_3s_4s_5)+ \\ +c_1[s_2(s_3c_5+c_3s_4s_5)+ \\ -c_2c_4s_5] \end{matrix} \\ \begin{matrix} c_1[c_3s_5s_6-s_3(c_4c_6+ \\ -s_4c_5s_6)]+s_1\{s_2[s_3s_5s_6+ \\ +c_3(c_4c_6-s_4c_5s_6)]+ \\ +c_2[s_4c_6+c_4c_5s_6]\} \end{matrix} & \begin{matrix} c_2[s_3s_5s_6+c_3(c_4c_6+ \\ -s_4c_5s_6)]+ \\ -s_2[s_4c_6+c_4c_5s_6] \end{matrix} & \begin{matrix} -s_1[c_3s_5s_6-s_3(c_4c_6+ \\ -s_4c_5s_6)]+c_1\{s_2[s_3s_5s_6+ \\ +c_3(c_4c_6-s_4c_5s_6)]+ \\ +c_2[s_4c_6+c_4c_5s_6]\} \end{matrix} \\ \begin{matrix} c_1[c_3s_5c_6+s_3(c_4s_6+s_4c_5c_6)]+ \\ +s_1\{s_2[s_3s_5c_6-c_3(c_4s_6+ \\ +s_4s_5s_6)]-c_2[s_4s_6-c_4c_5c_6]\} \end{matrix} & \begin{matrix} c_2[s_3s_5c_6-c_3(c_4s_6+ \\ +s_4c_5c_6)]+ \\ +s_2[s_4s_6-c_4c_5c_6] \end{matrix} & \begin{matrix} -s_1[c_3s_5c_6+s_3(c_4s_6+ \\ +s_4c_5c_6)]\ +c_1\{s_2[s_3s_5c_6+ \\ -c_3(c_4s_6+s_4s_5s_6)]+ \\ -c_2[s_4s_6-c_4c_5c_6]\} \end{matrix} \end{bmatrix} \quad (14)$$

Using the above transformations we can rewrite the unit vectors in systems 1 - 6 as

$$\begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} = H^T \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{21} & H_{31} \\ G_{12} & G_{22} & G_{32} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \quad (15)$$

$$\begin{Bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{Bmatrix} = G^T \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ G_{12} & G_{22} & G_{32} \\ G_{13} & G_{23} & G_{33} \end{bmatrix} \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \quad (16)$$

$$\begin{Bmatrix} \hat{i}_3 \\ \hat{j}_3 \\ \hat{k}_3 \end{Bmatrix} = F^T \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \quad (17)$$

$$\begin{Bmatrix} \hat{i}_4 \\ \hat{j}_4 \\ \hat{k}_4 \end{Bmatrix} = E^T \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \begin{bmatrix} D_{11} & D_{21} & D_{31} \\ E_{12} & E_{22} & E_{32} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \quad (18)$$

$$\begin{Bmatrix} \hat{i}_5 \\ \hat{j}_5 \\ \hat{k}_5 \end{Bmatrix} = D^T \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \begin{bmatrix} D_{11} & D_{21} & D_{31} \\ 0 & c_6 & -s_6 \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \quad (19)$$

$$\begin{Bmatrix} \hat{i}_6 \\ \hat{j}_6 \\ \hat{k}_6 \end{Bmatrix} = C^T \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & -s_6 \\ 0 & s_6 & c_6 \end{bmatrix} \begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} \quad (20)$$

The velocity $\vec{V}_7$ of coordinate system 7 from equation (7) becomes then (noting first that $\hat{i}_7 = \hat{i}_6$, $\hat{j}_6 = \hat{j}_5$, $\hat{i}_5 = \hat{i}_4$, $\hat{k}_4 = \hat{k}_3$, $\hat{i}_3 = \hat{i}_2$, and $\hat{j}_2 = \hat{j}_1$:

$$
\begin{aligned}
\vec{V}_7 = {} & \mathring{\Theta}_6(a\hat{k}_7)+\mathring{\Theta}_5(as_6\hat{i}_7)-\mathring{\Theta}_4\{[D_{31}(a+(b+c)c_6)+(b+c)D_{21}s_6)]\hat{i}_7+ \\
& -D_{11}[a+(b+c)s_6]\hat{j}_7-[(b+c)D_{11}c_6]\hat{k}_7\}+\mathring{\Theta}_3\{[-E_{33}(a+(b+c)c_6)+ \\
& -E_{23}(b+c)s_6]\hat{i}_7+[(b+c)E_{13}s_6]\hat{j}_7+E_{13}[a+(b+c)c_6]\hat{k}_7\}+ \\
& +\mathring{\Theta}_2\{[-F_{31}(a+(b+c)c_6-(d+e)E_{23})+F_{21}((b+c)s_6+(d+e)E_{33})]\hat{i}_7+ \\
& +[F_{11}((b+c)s_6+(d+e)E_{33}]-(d+e)F_{31}E_{13})]\hat{j}_7+[F_{11}(a+(b+c)c_6-(d+e)E_{23})+ \\
& +(d+e)F_{21}E_{13})]\hat{k}_7\}+\mathring{\Theta}_1\{[-G_{32}(a+(b+c)c_6-(d+e)E_{23})+ \\
& -G_{22}((b+c)s_6+(d+e)E_{33})+fF_{11}]\hat{i}_7+ \\
& +[G_{12}((b+c)s_6+(d+e)E_{33})-(d+e)G_{32}E_{13})+fF_{21}]\hat{j}_7+ \\
& +[G_{12}(a+(b+c)c_6-(d+e)E_{23})-(d+e)G_{22}E_{13})+fF_{31}]\hat{k}_7\}
\end{aligned}
\tag{21}
$$

and so

$$
\begin{aligned}
u = {} & \mathring{\Theta}_5as_6-\mathring{\Theta}_4\{D_{31}[a+(b+c)c_6]+(b+c)D_{21}s_6\}+ \\
& -\mathring{\Theta}_3\{E_{33}[a+(b+c)c_6]+(b+c)E_{23}\}-\mathring{\Theta}_2\{F_{31}[a+(b+c)c_6]+ \\
& +F_{21}[(b+c)s_6+(d+e)E_{33}]-(d+e)F_{31}E_{23})\}-\mathring{\Theta}_1\{G_{32}[a+(b+c)c_6]+ \\
& +G_{22}[(b+c)s_6+(d+e)E_{33}]+(d+e)G_{32}E_{23}+fF_{11}\}
\end{aligned}
\tag{22}
$$

$$v = \overset{\circ}{\Theta}_4 D_{11}[a+(b+c)s_6]+\overset{\circ}{\Theta}_3(b+c)E_{13}s_6+\overset{\circ}{\Theta}_2\{[(b+c)s_6+(d+e)E_{33}]F_{11}+$$
$$-(d+e)F_{31}E_{13}\}+\overset{\circ}{\Theta}_1\{[(b+c)s_6+(d+e)E_{33}]G_{12}-(d+e)E_{13}G_{32}\} \tag{23}$$

$$w = \overset{\circ}{\Theta}_6 a+\overset{\circ}{\Theta}_4(b+c)D_{11}c_6+\overset{\circ}{\Theta}_3E_{13}[a+(b+c)c_6]+\overset{\circ}{\Theta}_2\{[a+(b+c)c_6]F_{11}+$$
$$-(d+e)(F_{11}E_{23}-F_{21}E_{13})\}+\overset{\circ}{\Theta}_1\{[a+(b+c)c_6]G_{12}-(d+e)(G_{12}E_{23}-G_{22}E_{13})+$$
$$+fF_{31}\} \tag{24}$$

The top three rows of the Jacobian are now easily found from equations (22) - (24). For ease in computer programming and for computational speed the following substitutions are made:

$$e_1 = (b + c) \tag{25}$$

$$e_2 = (b + c)\,c_6 = e_1 c_6 \tag{26}$$

$$e_3 = (b + c)\,s_6 = e_1 s_6 \tag{27}$$

$$e_4 = a + (b + c)\,c_6 = a + e_2 \tag{28}$$

$$e_5 = (d + e) \tag{29}$$

$$e_6 = (d + e)\,E_{13} = e_5 E_{13} \tag{30}$$

$$e_7 = (d + e)\,E_{23} = e_5 E_{23} \tag{31}$$

$$e_8 = (d + e)\,E_{33} = e_5 E_{33} \tag{32}$$

$$e_9 = (b + c)\,s_6 + (d + e)\,E_{33} = e_3 + e_8 \tag{33}$$

$$e_{10} = a + (b + c)\,c_6 - (d + e)\,E_{23} = e_4 - e_7 \tag{34}$$

Using (25) - (34) in (22) - (24) and differentiating, the first three rows of the Jacobian are obtained.

The last three rows of the Jacobian are the partial derivatives of ω_x ω_y and ω_z with respect to the individual motor rates ($\dot{\theta}_1 - \dot{\theta}_6$). The rotational rates about the ($\hat{i}_7$ $\hat{j}_7$ $\hat{k}_7$) axes (ω_x ω_y ω_z) are found as sum of the dot products of those axes with the individual rotational rates ($\dot{\theta}_1$ - $\dot{\theta}_6$). Thus

$$\omega_x = \hat{i}_7 \cdot \vec{\Omega}_T$$

$$\omega_y = \hat{j}_7 \cdot \vec{\Omega}_T$$

$$\omega_z = \hat{k}_7 \cdot \vec{\Omega}_T$$

where $\qquad \vec{\Omega}_T = (\dot{\Theta}_1\hat{j}_1 + \dot{\Theta}_2\hat{i}_2 + \dot{\Theta}_3\hat{k}_3 + \dot{\Theta}_4\hat{i}_4 + \dot{\Theta}_5\hat{j}_5 + \dot{\Theta}_6\hat{i}_6)$

Carrying out the above products yields

$$\omega_x = \dot{\Theta}_1 G_{12} + \dot{\Theta}_2 F_{11} + \dot{\Theta}_3 E_{13} + \dot{\Theta}_4 D_{11} + \dot{\Theta}_6 \quad , \tag{35}$$

$$\omega_y = \dot{\Theta}_1 G_{22} + \dot{\Theta}_2 F_{21} + \dot{\Theta}_3 E_{23} + \dot{\Theta}_4 D_{21} + \dot{\Theta}_5 c_6 \quad , \tag{36}$$

and

$$\omega_z = \dot{\Theta}_1 G_{32} + \dot{\Theta}_2 F_{31} + \dot{\Theta}_3 E_{33} + \dot{\Theta}_4 D_{31} - \dot{\Theta}_5 s_6 \quad , \tag{37}$$

The total Jacobian is now easily obtained from equations (22) - (24) and (35) - (37), using the substitutions of equations (25) - (34). Equation (4) becomes

$$J = \begin{bmatrix} G_{32}e_{10} + G_{22}e_9 + F_{11}f & (F_{31}e_{10} + F_{21}e_9) & (-E_{33}e_4) & (D_{31}e_4 + D_{21}e_3) & (as_6) & 0 \\ (G_{12}e_9 - G_{32}e_6) & (F_{11}e_9 - F_{31}e_6) & (E_{13}e_3) & D_{11}(a + e_3) & 0 & 0 \\ G_{12}e_{10} + G_{22}e_6 + F_{31}f & (F_{11}e_{10} + F_{21}e_6) & (E_{13}e_4) & D_{11}(e_2) & 0 & a \\ G_{12} & F_{11} & E_{13} & D_{11} & 0 & 1 \\ G_{22} & F_{21} & E_{23} & D_{21} & c_6 & 0 \\ G_{32} & F_{31} & E_{33} & D_{31} & -s_6 & 0 \end{bmatrix} \tag{38}$$

It is seen from equation (38) that all elements of each transformation matrix are not required. In fact, the required portions of each transformation matrix are:

$$D_{req} = \begin{bmatrix} D_{11} & & \\ D_{21} & & \\ D_{31} & & \end{bmatrix}$$

$$E_{req} = \begin{bmatrix} & & E_{13} \\ & & E_{23} \\ & & E_{33} \end{bmatrix}$$

$$F_{req} = \begin{bmatrix} F_{11} & & \\ F_{21} & & \\ F_{31} & & \end{bmatrix}$$

$$G_{req} = \begin{bmatrix} & G_{12} & \\ & G_{22} & \\ & G_{32} & \end{bmatrix}$$

A significant savings in computer time is realized by calculating only these elements. It will be shown later that the required elements from H are

$$H_{req} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & H_{13} \\ H_{21} & \ddots & \ddots & \ddots & \ddots & H_{23} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

<u>Development of the Control Relations</u>

From equation (6)

$$\begin{Bmatrix} u \\ v \\ w \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = B \begin{Bmatrix} k_x \\ k_y \\ 0 \\ \alpha' \\ \phi' \\ \beta' \end{Bmatrix} \tag{6}$$

The desired motions are:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = G_x \begin{Bmatrix} k_x \\ k_y \\ 0 \end{Bmatrix} \tag{39}$$

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = G_\Theta \begin{Bmatrix} \alpha' \\ \phi' \\ \beta' \end{Bmatrix} \tag{40}$$

where G_x is the gain for translational motion

and G_Θ is the gain for rotational motion.

The angles α', ϕ', and β' are the differences between the hand controller pitch, roll, and yaw angles and those of the terminal (coordinate system 7). The hand controller pitch, roll, and yaw angles (α_{HC}, ϕ_{HC}, β_{HC}) are obtained directly from the hand controller resolvers. The position of the hand controller can be found from the reference (zero) position by the following three rotations (see Figure 4-3):

(1) rotate about z_{ref} the angle β to obtain $(\hat{i}'_{HC}, \hat{j}'_{HC}, \hat{k}'_{HC})$, then

(2) rotate about $\hat{i}'_{HC}$ the angle α to obtain $(\hat{i}''_{HC}, \hat{j}''_{HC}, \hat{k}''_{HC})$, and then

(3) rotate about $\hat{j}''_{HC}$ the angle ϕ to obtain $(\hat{i}_{HC}, \hat{j}_{HC}, \hat{k}_{HC})$.

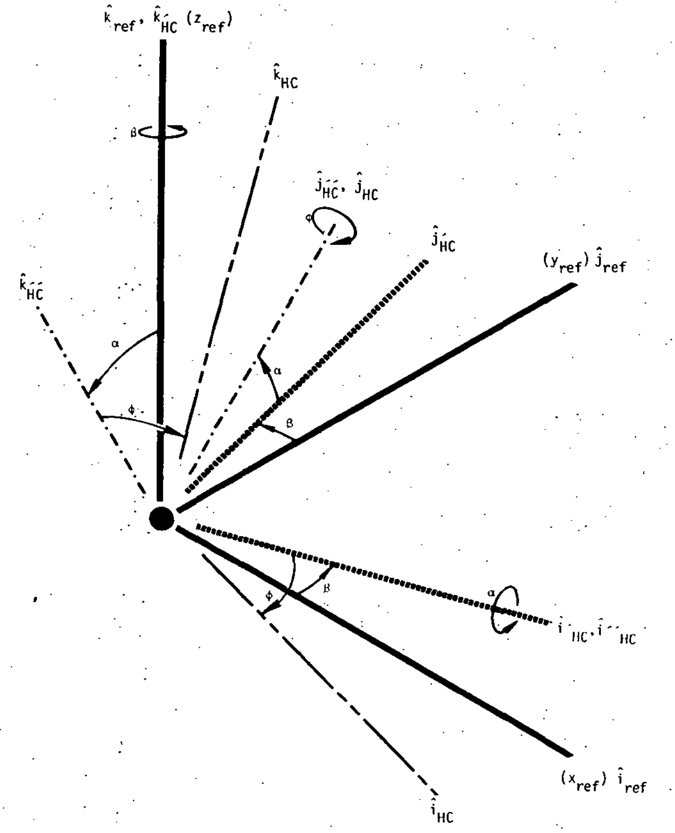

Figure 4-3: Hand Controller Angles

The reference frame (X_{ref}, Y_{ref}, Z_{ref}) for the hand controller corresponds to coordinate system 1 (X_1, Y_1, Z_1) for the arm. When the hand controller axes are parallel to the reference axes, the terminal axes will be commanded to be parallel to coordinate system 1. The terminal (coordinate system 7) orientation relative to coordinate system 1 is described by the three angles α_7, β_7, and ϕ_7 in the same sequence of rotations described for the hand controller. Thus

$$\begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} = \begin{bmatrix} \cos\phi_7 & 0 & -\sin\phi_7 \\ 0 & 1 & 0 \\ \sin\phi_7 & 0 & \cos\phi_7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_7 & \sin\alpha_7 \\ 0 & -\sin\alpha_7 & \cos\alpha_7 \end{bmatrix} \begin{bmatrix} \cos\beta_7 & \sin\beta_7 & 0 \\ -\sin\beta_7 & \cos\beta_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} = [P] \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} \quad (41)$$

Carrying out the multiplication and dropping the subscript 7's we obtain

$$\begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} = \begin{bmatrix} \cos\beta\cos\phi+\sin\alpha\sin\beta\sin\phi & \sin\beta\cos\phi+\sin\alpha\cos\beta\sin\phi & -\cos\alpha\sin\phi \\ \cos\alpha\sin\beta & \cos\alpha\cos\beta & \sin\alpha \\ \cos\beta\sin\phi-\sin\alpha\sin\beta\cos\phi & \sin\beta\sin\phi-\sin\alpha\cos\beta\cos\phi & \cos\alpha\cos\phi \end{bmatrix} \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} \quad (42)$$

$$\begin{Bmatrix} \hat{i}_7 \\ \hat{j}_7 \\ \hat{k}_7 \end{Bmatrix} = {}^7M_1 \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} = H \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} \tag{43}$$

therefore

$$[P] = [H]. \tag{44}$$

From equation (44) we find

$$\alpha_7 = \sin^{-1}(H_{23}) \tag{45}$$

$$\phi_7 = \sin^{-1}\left(\frac{H_{13}}{-\cos\alpha}\right) \tag{46}$$

$$\beta_7 = \sin^{-1}\left(\frac{H_{21}}{-\cos\alpha}\right) \tag{47}$$

The above forms are chosen as the most direct and the least likely to fail due to angles of 0° or 90°. Note that if $\alpha = 90°$, equations (46) and (47) are undefined, however the likelihood of ever having $\alpha = 90°$ is remote. Infact, the human operator would find it very difficult to move the hand controller to $\alpha_{HC} = 90°$, so that this restriction is not considered to be significant.

Combining equations (6), (38) and (45) - (47), the control law becomes

$$
\begin{Bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \\ \dot{\Theta}_4 \\ \dot{\Theta}_5 \\ \dot{\Theta}_6 \end{Bmatrix} = J^{-1} \quad G \quad G \quad 0 \quad G_\theta \quad G_\theta \quad G_\theta \begin{Bmatrix} K_x \\ K_y \\ 0 \\ \alpha_{HC} - \sin^{-1}(H_{23}) \\ \phi_{HC} + \sin^{-1}\left(\frac{H_{13}}{\cos\alpha_7}\right) \\ \beta_{HC} + \sin^{-1}\left(\frac{H_{21}}{\cos\alpha_7}\right) \end{Bmatrix} \tag{48}
$$

The operation of the hand controller, computer subsystem, and the RAM is as follows (See Figure 4-4):

(1) the computer reads the current joint angles from the A/D converter and calculates the elements of the transformation matrices, and the inverse of the J matrix,

(2) the computer reads the hand controller resolver and thumb switch signals from the A/D converter,

(3) the computer evaluates the differences between the hand controller and terminal pitch, roll, and yaw angles,

(4) based on the differences obtained in (3) and the thumb switch signals obtained in (2) and using the J^{-1} matrix generated in (1), the motor rate commands are calculated and sent to the D/A converter,

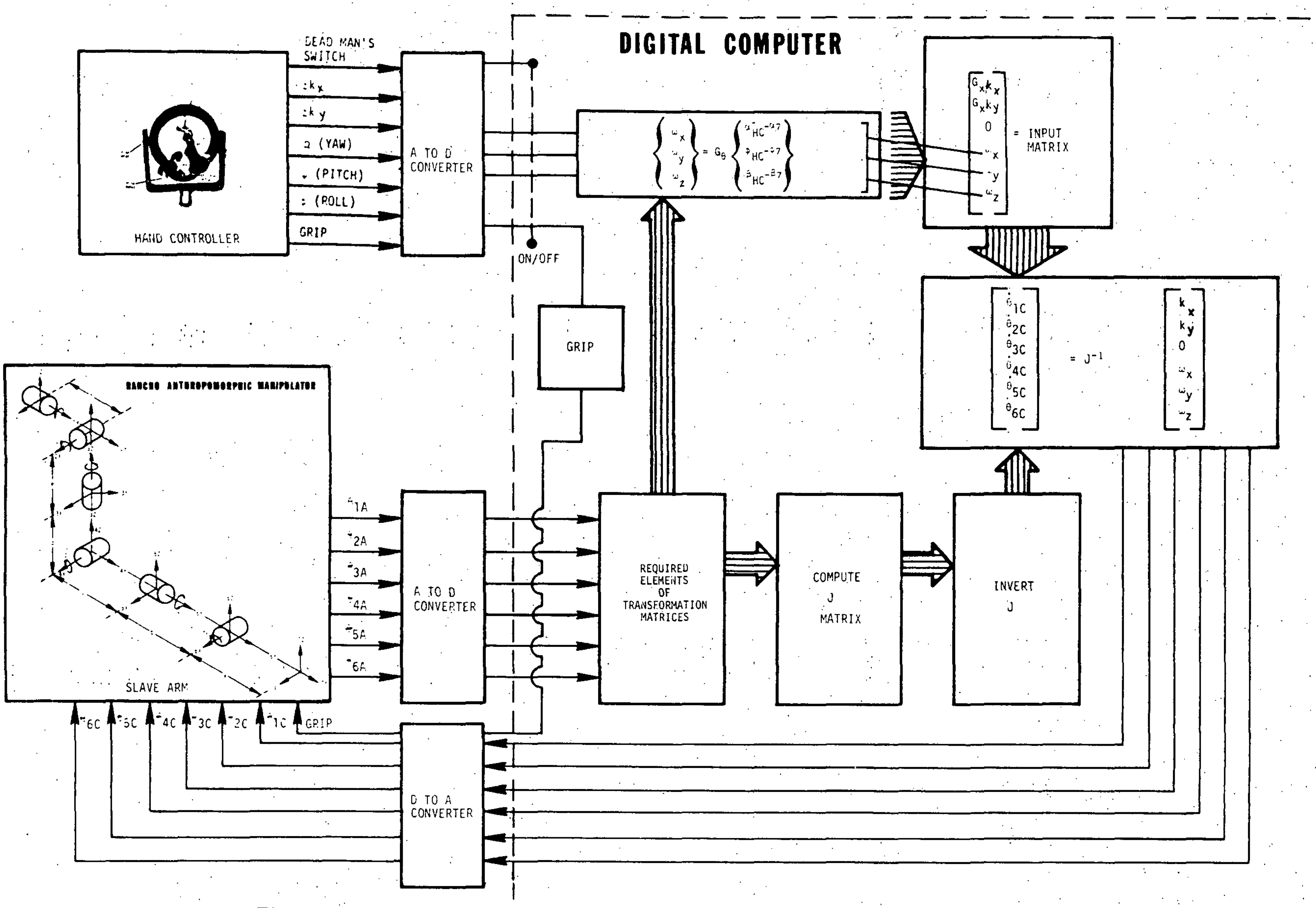

Figure 4-4: Terminal Pointer Hand Controller Total Systems Diagram

(5) the D/A converter generates motor rate commands which cause the arm to move as required.

(6) the grip motor is linked directly to the grip resolver on the hand controller--no calculations are required.

NOTE: The control law incorporates the following:

(1) instantaneous motor start/stop is assumed (linearized equations),

(2) no rate feedback is used,

(3) the pitch angle of the hand controller is assumed to be $-90^\circ < \alpha_{HC} < +90^\circ$.

SECTION 5.0

CONCLUSIONS AND RECOMMENDATIONS

The URS/Matrix hand controller and the necessary mathematical control law have been developed for use with the MSFC Rancho Anthropomorphic Manipulator. The hand controller allows control of the RAM with a single hand, and the mathematical control law causes the attitude of the RAM and effector to correspond directly to the attitude of the hand controller. This correspondence greatly simplifies control of the RAM by a human operator.

Two important points must be made regarding the operation of the system. First, a digital computer has been chosen to convert the hand controller input motions to individual motor rate commands. The digital computer must perform operations sequentially, i.e., only one operation can be carried out at any given instant in time. The time required to complete a single cycle through the system is therefore the sum of the times required for each operation. The operations required for the RAM system are:

(1) sampling and storing the six joint angles (one-by-one),

(2) calculating the trigonometric functions for the joint angles,

(3) calculating the required elements of the transformation matrices,

(4) calculating the J matrix,

(5) inverting the J matrix,

(6) sampling and storing the hand controller input angles and thumb switch commands,

(7) calculating the differences between the hand controller and terminal pointer angles,

(8) calculating the motor rate commands, and

(9) sending the motor rate commands to the motors (one-by-one).

Step (1) requires A/D conversion of the six joint angles, step (6) requires A/D conversion of the five hand controller input motions, and step (9) requires D/A conversion of the six motor rate commands. As all of these operations must be performed one at a time, the response of the RAM will be less than instantaneous. In fact, the computation and conversion rates could be so slow as to significantly limit the usefulness of the system. The response of the system, and therefore its usefulness, might be improved in several ways:

(1) analog computation, which allows simultaneous operations,

(2) quaternion representation of the dynamic equations (see Appendix B) which reduces the number of operations,

(3) polar coordinate representation of the dynamic equations, which reduces the number of terms, or

(4) series expansion of the sine and cosine terms with truncation of higher order terms, which reduces the number of terms.

Each of these areas offer possible significant savings of computer time and should be investigated. Time and funds provided for the current effort did not allow such an investigation.

The second point to be made is that, due to the construction of the RAM, attitude commands can be generated by the hand controller that are impossible for the RAM to execute (due to mechanical limits). Significantly, a twisting motion about the $\hat{i}_1$ axis is not possible. Also, rotation about the $\hat{k}_7$ axis is limited.

The first problem could be resolved by reversing segments 5 and 6 so that the final motion is a roll motion, or both problems could be resolved by adding a roll motor after segment 6.

APPENDIX A

TERMINAL POINTER HAND CONTROLLER

PROTOTYPE PARTS LIST

ITEM NO.	NO. REQ'D.	NOMENCLATURE:	PART NO./MAT'L.:	VENDOR:
1	1	Support, Gimbal	Alum	-
2	3	Potentiometer, Linear, Bushing Mount	#50 Linear	Computer Instrument Corp., Hempstead, N.Y.
3	4	Precision Ball Bearing, Flanged, 1/4 I.D., 3/8 O.D. x 1/8 W.	E2-11	PIC Design Corp., Benrus Corp.
4	7	Washer	1/32 THK, Teflon	-
5	4	Bushing	1/2 DRA STK, CRES	-
6	2	Sleeve	1/4 STK, CRES	-
7	7	Set Screw, Int. Hex.	#4-40, CRES	-
8	1	Gimbal Ring	Alum	-
9	1	Sleeve	1/4 STK, CRES	-
10	1	Sleeve	1/2 STK, CRES	-
11	1	Inner Ring, Roll	Alum	-
12	1	Outer Ring, Roll	Alum	-
13	1	Ring	Alum	-
14	10	Screw Ft. Hd.	#2-56, CRES	-
15	1	Shaft	3/8 STK, CRES	-

(Item numbers are referenced from Figures 2-4 and 2-5, Section 2.0)

PROTOTYPE PARTS LIST (CONTINUED)

ITEM NO.	NO. REQ'D	NOMENCLATURE:	PART NO./MAT'L.:	VENDOR:
16	1	Ball Bearing, Type X, 5 1/2 I.D., 6 O.D. x 1/4 W.	KA055XPO	Keene Corp., Kaydon Bearing Div., Dixie Bearing Co., Decatur, Ala.
17	1	Sleeve	3/16 STK, CRES	-
18	1	Precision Ball Bearing, 3/16 I.D., 3/8 O.D. x 1/8 W.	E1-5	PIC
19	1	Body	Alum	-
20	1	Cover	Alum	-
21	2	Retainer Clip	Alum	PIC
22	1	Gear Pulley, "No-Slip"	FC4-18	PIC
23	1	Potentiometer - 10 Turn, Std. Mount	#5010 Linear	See Item 2
24	1	Positive Drive Belt, "No-Slip", 18.557 Dia.	FA-189	PIC
25	1	Switch, Isometric Two Axis	#469	Measurement Sys.
26	1	Handle	Plastic, Cast	-
27	1	Grip Lever	Alum	-
28	1	Handle Mount	Alum	-

PROTOTYPE PARTS LIST (CONTINUED)

ITEM NO.	NO. REQ'D	NOMENCLATURE:	PART NO./MAT'L.:	VENDOR:
29	1	Precision Bearing, 1/4 I.D., 3/8 O.D. x 1/4 W.	E1-5	PIC
30	1	Microswitch	-	-
31	1	Lever	Alum	-
32	1	Pad	Rubber, Foam	-
33	1	Pin	1/16 STK, CRES	-
34	1	Thumb Pad	Plastic	-
35	2	Washer	1/64 STK, Teflon	-
36	2	Screw, Rd. Hd.	#2-56, CRES	-
37	2	Screw, Ft. Hd.	#8-32, CRES	-

APPENDIX B

QUATERNIONS FOR CONTROL OF SPACE VEHICLES

by

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ABSTRACT

The historical method of representing successive rotations of a coordinate frame by quaternion multiplication requires that each quaternion be referenced to a common coordinate frame. This paper presents a method whereby quaternion representation of successive rotations utilizes quaternions referenced in the same manner as direction cosine matrices. A product quaternion is formed which locates the final position of the coordinate frame with respect to its original position. If the quaternion defining the instantaneous relative position of two coordinate frames is known, the necessary control information to drive them toward coincidence can be determined directly. Incremental updating of the quaternion is accomplished by means of quaternion multiplication involving the original quaternion and the incremental quaternions. Each incremental quaternion defines the incremental movement of its associated coordinate frame. The product quaternion is the new quaternion relating the two coordinate frames after their incremental movements. A complete development of the foregoing updating procedure is shown. This is followed by a simplified quaternion update procedure currently being implemented for strapdown calculations on the Skylab vehicle.

INTRODUCTION

Quaternions were invented by Sir William R. Hamilton in 1843 and resulted from his attempts to form a three dimensional vector algebra in which vector multiplication and division could be performed. For approximately a century, quaternions were little used, being supplanted by the vector analysis of Professor Willard Gibbs. Recently, however, the use of quaternions has been shown to offer significant computational advantages when applied to the digital solution of time varying coordinate frame transformations. In addition, if the quaternion defining the instantaneous relative position of two coordinate frames is known, the necessary information to optimally drive them toward coincidence can be directly determined.

This paper presents a review of some of the properties of quaternions, a development of the strapdown equations for Skylab and a brief comparison of three methods used to formulate the direction cosine matrix relating two coordinate frames.

THE UNIT QUATERNION

A quaternion may be written as:

$$q = Q_4 + iQ_1 + jQ_2 + kQ_3$$

where Q_4 is a scalar and Q_1, Q_2 and Q_3 are scalar components along the orthogonal dextral vector triad i, j, k. The properties assigned i, j, and k are such that:

$$ij = k; \quad jk = i; \quad ki = j; \quad ijk = -1$$

$$ji = -k; \quad kj = -i; \quad ik = -j; \quad jik = 1$$

If the condition is imposed that:

$$Q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 = 1$$

the following relation can be established:

$$q = Q_4 + iQ_1 + jQ_2 + kQ_3 = \cos\frac{\theta}{2} + e\sin\frac{\theta}{2}$$

where

$$\cos\frac{\theta}{2} = Q_4$$

$$e = (iQ_1 + jQ_2 + kQ_3)/(Q_1^2 + Q_2^2 + Q_3^2)^{1/2}$$

$$e = i\cos\alpha + j\cos\beta + k\cos\gamma$$

$$\sin\frac{\theta}{2} = (Q_1^2 + Q_2^2 + Q_3^2)^{1/2}$$

the unit quaternion has the property:

$$qq^{-1} = (Q_4 + iQ_1 + jQ_2 + kQ_3)(Q_4 - iQ_1 - jQ_2 - kQ_3) = 1$$

$$= (\cos\frac{\theta}{2} + e\sin\frac{\theta}{2})(\cos\frac{\theta}{2} - e\sin\frac{\theta}{2}) = 1$$

thus the complex conjugate of the unit quaternion is its inverse.

GEOMETRICAL INTERPRETATION OF THE UNIT QUATERNION

If the unit quaternion is written in its trigonometric form:

$$q = \cos\frac{\theta}{2} + e\,\sin\frac{\theta}{2}$$

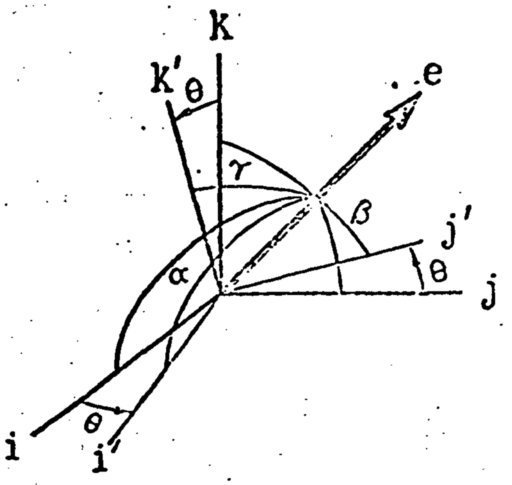

Figure 1.

The unit vector e can be regarded as the axis about which the i, j, k coordinate frame is rotated through the angle θ to the new position i', j', k'. The direction cosines locating e with respect to either i, j, k or i', j', k' are the same with respect to either coordinate frame. The vector e is known as the Euler axis or the eigenaxis. A unit quaternion can thus be considered to represent a rotation of θ degrees about an eigenaxis e of a coordinate frame to another position.

COORDINATE TRANSFORMATIONS USING QUATERNIONS

It has been shown by Hamilton that a vector transformation is achieved by the following quaternion multiplication:

$$\overline{V}' = q\overline{V}q^{-1}$$

where V' is the transformed vector V. By treating each of the orthogonal axes i, j and k as unit vectors and performing the indicated quaternion multiplications:

$$\begin{aligned} i' &= q\,i\,q^{-1} \\ j' &= q\,j\,q^{-1} \\ k' &= q\,k\,q^{-1} \end{aligned} \qquad \text{Equation 1}$$

If the indicated operations are performed in terms of $Q_4 + iQ_1 + jQ_2 + kQ_3$ and its inverse $Q_4 - iQ_1 - jQ_2 - kQ_3$; and the results are arrayed in matrix format there is obtained:

$$\begin{array}{c} i' \\ j' \\ k' \end{array} = \begin{bmatrix} Q_4^2 + Q_1^2 - Q_2^2 - Q_3^2 & 2(Q_1 Q_2 + Q_3 Q_4) & 2(Q_1 Q_3 - Q_2 Q_4) \\ 2(Q_1 Q_2 - Q_3 Q_4) & Q_4^2 - Q_1^2 + Q_2^2 - Q_3^2 & 2(Q_2 Q_3 + Q_1 Q_4) \\ 2(Q_1 Q_3 + Q_2 Q_4) & 2(Q_2 Q_3 - Q_1 Q_4) & Q_4^2 - Q_1^2 - Q_2^2 + Q_3^2 \end{bmatrix} \begin{array}{c} i \\ j \\ k \end{array}$$

or in concise notation:

$$R' = [Q]R \qquad \text{Equation 2}$$

This is the direction cosine matrix relating the i, j, k and the i', j', k' coordinate frames. It is expressed in terms of the four parameters (Q_4, Q_1, Q_2, Q_3) which comprise the quaternion originally used to locate the relative positions of the two coordinate frames. Conversely, if the transformation matrix is known, regardless of how it was obtained, the four parameters (Q_4, Q_1, Q_2, Q_3) may be determined by a method presented in Reference 1 as follows; let the transformation matrix be represented by:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

Then:

$$Q_4^2 + Q_1^2 - Q_2^2 - Q_3^2 = Q_{11}$$

$$Q_4^2 - Q_1^2 + Q_2^2 - Q_3^2 = Q_{22}$$

$$Q_4^2 - Q_1^2 - Q_2^2 + Q_3^2 = Q_{33}$$

$$Q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 = 1$$

By selectively combining these four equations:

$$Q_4 = \pm \quad .5\sqrt{1 + Q_{11} + Q_{22} + Q_{33}}$$

$$Q_1 = \pm \quad \sqrt{Q_4^2 - .5(Q_{22} + Q_{33})}$$

$$Q_2 = \pm \quad \sqrt{Q_4^2 - .5(Q_{11} + Q_{33})}$$

$$Q_3 = \pm \quad \sqrt{Q_4^2 - .5(Q_{11} + Q_{22})}$$

The proper signs of the four parameters may be determined as follows:

a. Choose Q_4 positive

b. The sign of $(Q_{23} - Q_{32})$ is then the sign of Q_1

c. The sign of $(Q_{21} - Q_{13})$ is then the sign of Q_2

d. The sign of $(Q_{12} - Q_{21})$ is then the sign of Q_3

QUATERNIONS AND TRANSFORMATION MATRICES FOR SUCCESSIVE ROTATIONS

In the previous section it has been shown how the transformation matrix relating two coordinate frames can be generated by using a quaternion and its inverse. It will now be shown how two or more quaternions can be combined and the results used to generate a transformation matrix relating the resultant coordinate frame to the original coordinate frame. Euler has shown that the result of two eigenaxis rotations represented by the equaternions q_1 and q_2, where q_1 is the first rotation and q_2 is the second rotation, can be represented by:

$$q = q_2 \, q_1$$

The resultant quaternion q is dependent on the order of rotation. Most important q_1 and q_2 must both be referenced to the same coordinate frame. Consider the quaternion:

$$q_1 = \cos \frac{90}{2} + \hat{i} \sin \frac{90}{2}$$

This represents a rotation of 90° about the eigenaxis, i, in the positive sense. See Figure 2.

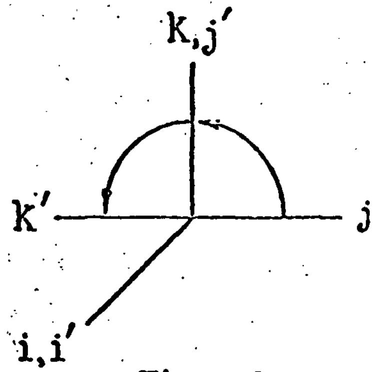

Figure 2.

If now a second rotation about the original k axis of 90° is taken, the quaternion representation is:

$$q_2 = \cos \frac{90}{2} + \hat{k} \sin \frac{90}{2}$$

Referring to Figure 3, it is seen that after the two eigenaxis rotations:

$$i'' = j$$

$$j'' = k$$

$$k'' = i$$

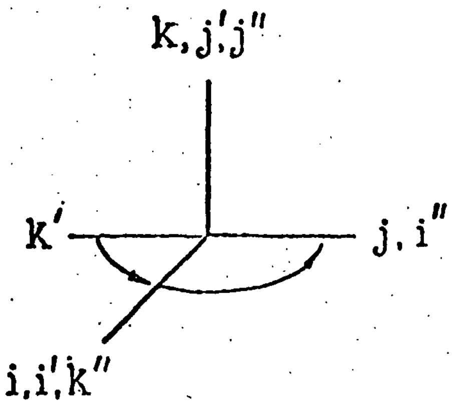

Figure 3.

Also, if the resultant quaternion q is sought:

$$q = q_2 q_1 = (\cos \frac{90}{2} + k \sin \frac{90}{2}) \; (\cos \frac{90}{2} + i \sin \frac{90}{2}) \qquad \text{Equation 3}$$

$$q = (\frac{1}{\sqrt{2}} + \frac{k}{\sqrt{2}}) \; (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$$

$$q = \frac{1}{2} + \frac{i + j + k}{\sqrt{3}} \; \frac{\sqrt{3}}{2} = \cos \frac{\varphi}{2} + e \sin \frac{\varphi}{2}$$

$$\cos \frac{\varphi}{2} = \frac{1}{2} \; ; \; \sin \frac{\varphi}{2} = \frac{\sqrt{3}}{2}$$

$$\varphi = 120^\circ$$

$$\hat{e} = (\hat{i} + \hat{j} + \hat{k}) \, / \, \sqrt{3}$$

The resultant rotation is 120° about an eigenaxis e which has equal direction cosines with respect to i, j and k.

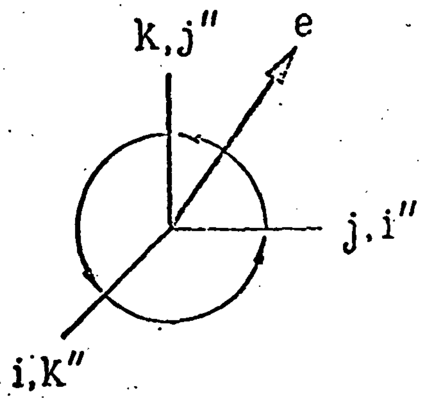

Figure 4.

The problem with the foregoing procedure is that q_2 is defined in terms of the original coordinate system (i, j, k). It is desirable that it be defined with respect to the immediately prior rotated or moving coordinate frame (i', j', k'). If this is done:

$$q_2' = \cos \frac{90}{2} + j' \sin \frac{90}{2}$$

Where q_2' indicates a quaternion rotation with respect to the previously rotated (i', j', k') frame. Using this convention:

$$q = q_2' q_1 = (\cos \frac{90}{2} + j' \sin \frac{90}{2}) \; (\cos \frac{90}{2} + i \sin \frac{90}{2}) \qquad \text{Equation 4}$$

This quaternion multiplication cannot be performed directly because the eigenaxes, j' and i, are referenced to different coordinate frames. This difficulty is resolved in the following manner; let:

$$q_2 = \cos \frac{\theta}{2} + e \sin \frac{\theta}{2}$$

where e is any eigenaxis referenced to the original coordinate frame. Transform q_2 as follows:

$$q_1 \, q_2 \, q_1^{-1} = q_1 \, q_1^{-1} \cos \frac{\theta}{2} + q_1 \, e \, q_1^{-1} \sin \frac{\theta}{2}$$

$$= \cos \frac{\theta}{2} + e' \sin \frac{\theta}{2} = q_2'$$

From which:

$$q_2' = q_1 \, q_2 \, q_1^{-1}$$

This is the quaternion transformation previously given in equations 1 and 2. The eigenaxis e referenced to the original coordinate frame has been rotated into e' in the rotated coordinate frame by means of the quaternion q_1 and its inverse. By this procedure, the eigenaxis e', about which the rotated coordinate frame is again rotated, can be expressed in terms of the original coordinate frame (i, j, k). Then:

$$q = q_2' \, q_1 = (q_1 \, q_2 \, q_1^{-1}) \, q_1 = (q_1 \, q_2) \, (q_1^{-1} \, q_1) = q_1 \, q_2$$

Using this result, equation 4 becomes:

$$q = q_1 \, q_2 = (\cos \frac{90}{2} + i \sin \frac{90}{2}) \; (\cos \frac{90}{2} + j \sin \frac{90}{2}) \qquad \text{Equation 5}$$

$$q = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2} = \frac{1}{2} + \frac{i + j + k}{\sqrt{3}} \, \frac{\sqrt{3}}{2}$$

The results obtained using equation 3 or equation 5 are the same. In the one case (equation 3) the second rotation is easily referenced to the original coordinate system due to the simplicity of the first rotation (90° about i). In equation 5, the second rotation is referenced to the rotated or moving coordinate frame. This is an important advantage when a previous rotation or rotations are not easily visualized. By an extension of the above process, it can be shown that a series of quaternion rotations each taken with respect to the immediately prior rotated or moving coordinate frame can be given by:

$$q = q_1 \, q_2 \, q_3 \cdots q_N$$

Here again $q_1 \, q_2 \, q_3 \cdots q_N$ refers to quaternion multiplication. Note that if a series of rotations are defined by a chain matrix:

$$R''' = [Q_3''] \, [Q_2'] \, [Q_1] R = q_3'' q_2' q_1 \, R \, q_1^{-1} \, q_2'^{-1} \, q_3''^{-1}$$

$$= q_1 \, q_2 \, q_3 \, R \, q_3^{-1} \, q_2^{-1} \, q_1^{-1} \, ,$$

The quaterion order of multiplication is reversed from the matrix order of multiplication. This property will be used in developing the Strapdown Equations.

For example; with respect to Figure 5.

i, j and k	are axes of the original frame
i', j' and k'.	are axes of the first transformation (90° about i)
i'', j'' and k''	are axes of the second transformation (90° about j')
i''', j''' and k'''	are axes of the third transformation (90° about k'')

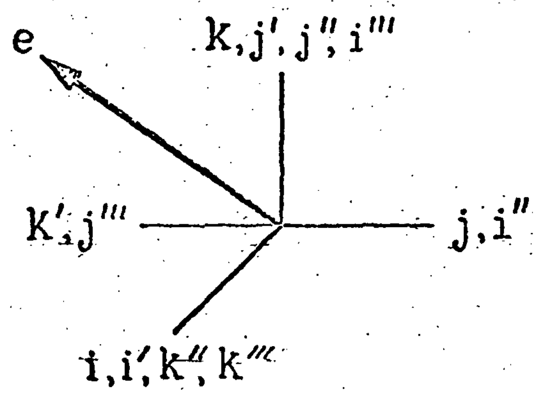

Figure 5.

Then:

$$q = (\cos\frac{90}{2} + i\sin\frac{90}{2})\ (\cos\frac{90}{2} + j\sin\frac{90}{2})\ (\cos\frac{90}{2} + k\sin\frac{90}{2}) = q_1\, q_2\, q_3$$

$$q = \frac{i}{\sqrt{2}} + \frac{k}{\sqrt{2}} = \cos\frac{\varphi}{2} + e\sin\frac{\varphi}{2}$$

$$\cos\frac{\varphi}{2} = 0\ ;\ \frac{\varphi}{2} = 90^\circ\ ;\ \varphi = 180^\circ$$

$$e\sin\frac{\varphi}{2} = \frac{i+k}{\sqrt{2}}\ ;\ e = \frac{i+k}{\sqrt{2}}\ ;\ \sin\frac{\varphi}{2} = 1$$

Here q represents a rotation of 180° about the eigenaxis $e = (i + k)/\sqrt{2}$. The three quaternion rotations used to effect the coordinate transformation:

$$i''' = k$$

$$j''' = -j$$

$$k''' = i$$

are found to be represented by the single quaternion rotation q. By inspection of Figure 5, it can be seen that single rotation of 180° about the eigenaxis $(i + k)/\sqrt{2}$ results in the same coordinate transformation as that obtained by three successive rotations of 90° each respectively about i, j' and k''

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