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       TOPOLOGIES ON DIRECTED GRAPHS

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ABSTRACT

Given a directed graph, a natural topology is defined
and relationships between standard topological properties
and graph theoretical concepts are studied. In particular,
the properties of connectivity and separatedness are in-
vestigated. A metric is introduced which is shown to be
related to separatedness. The topological notions of con-
tinuity and homeomorphism are related to the graph notions
of homomorphism and homeomorphism. A class of maps is
studied which preserve both graph and topological properties.
Applications involving strong maps and contractions are also
presented.

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CHAPTER I

THE E-TOPOLOGIES

1. The left and right E-topologies

In this section the basic concepts needed in this paper are introduced. Throughout we will use the concept of graph to mean an ordered pair of sets (P,E) where P is a non-void point set and E is a subset of $P \times P$, the cartesian product of P. Elements of the relation E are called edges. Others prefer to call this concept a directed graph [Ore (1962)] or a digraph [Harary (1969)].

We will consider a graph $G = (P,E)$ to be infinite if the cardinalities of both P and E are infinite; otherwise the graph is said to be finite. If P is finite then E must be finite; if P is infinite and E is finite then only a finite number of points belong to edges and for all purposes only the graph on that finite subset of points need be considered. Points that belong to no edges are called isolated points.

An edge $(p,p) \in E$ is called a loop. A subgraph H of a graph $G = (P,E)$ is an ordered pair (P_H, E_H) where (i) $P_H \subseteq P$ and (ii) $(p,q) \in E_H$ if and only if $p,q \in P_H$ with $(p,q) \in E$. Unfortunately, this concept has been defined differently by some authors [Berge (1962)]. Given $P' \subseteq P$, a unique subgraph H of G is specified by condition (ii) in the definition of subgraph. We denote this subgraph by $[P']$. A partial subgraph of G is any ordered pair (Q,F) such that $Q \subseteq P$, $F \subseteq E \cap (Q \times Q)$.

The next definitions are not found in the literature but are basic to this paper.

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<u>Definition.</u> Let $G = (P,E)$ be a graph. The <u>left E-topology</u>, $\tau_L(G)$, for G is the family of all subsets A of P such that if $p \in A$ and $(q,p) \in E$ then $q \in A$. The <u>right E-topology</u>, $\tau_R(G)$, for G is the family of all subsets A of P such that if $p \in A$ and $(q,p) \in E$ then $q \in A$. When the underlying graph is evident we denote the topologies by τ_L and τ_R .

We first prove that τ_L and τ_R are in fact topologies. [See Kelley (1955) for the definition of topology and related concepts.]
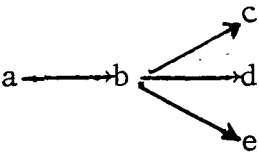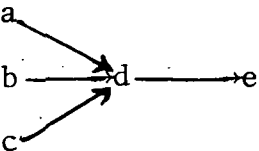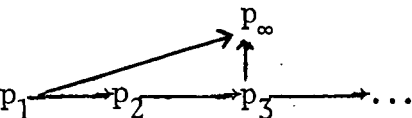
<u>Proposition 1.1.</u> The left and right E-topologies are topologies.

<u>Proof:</u> The null set $\emptyset$ and P itself are clearly members of τ_L . To show this family is closed under intersection, let $\{A_\alpha\}$ be a family of sets in τ_L and $p \in \cap A_\alpha$ with an edge $(q,p) \in E$. Since p is in each A_α and $(q,p) \in E$ we have by definition $q \in A_\alpha$ for each α; therefore $q \in \cap A_\alpha$ and this makes $\cap A_\alpha \in \tau_L$. Similarly, let $A = \cup A_\alpha$. If $p \in A$ and $(q,p) \in E$, then $p \in A_\beta$ for some β . Since A_β is in τ_L and $(p,q) \in E$ we have $q \in A_\beta \subseteq A$. Thus $A \in \tau_L$. A similar proof shows τ_R is a topology. //

If should be evident that the complements of sets in τ_L are sets in τ_R, and conversely. Since the elements of τ_L and τ_R are closed under both arbitrary union and arbitrary intersection, one could call them either open sets or closed sets, but we will avoid the use of this terminology entirely. Since the left E-topology will be the principal vehicle of our discussion, we shall assume it is the topology under consideration unless otherwise indicated.

In Figure 1.1, three graphs are pictorially represented and the elements of their left E-topologies are listed.

<u>Definition.</u> Let $G = (P,E)$ be a graph and $Q \subseteq P$; then we define

Graph	Elements of τ_L

1)

$\emptyset$, {a}, {a,b}, {a,b,c},

{a,b,d}, {a,b,e}, {a,b,c,d},

{a,b,c,e}, {a,b,d,e}, and P.

2)

$\emptyset$, {a}, {b}, {c},

{a,b,c,d}, and P.

3)

$\emptyset$, P, {p_1}, {p_1,p_2},

{p_1,p_2,p_3},...

$G = (P,E)$ where

$P = \{p_\infty, p_1, p_2 \ldots\}$

$E = \{(p_k, p_{k+1}) \mid k = 1, 2, \ldots\} \cup \{(p_k, p_\infty) \mid k = 1, 2, \ldots\}$

Figure 1.1.

$$L(Q) = \cap \{A \in \tau_L \mid Q \subseteq A\}$$

$$R(Q) = \cap \{B \in \tau_R \mid Q \subseteq B\}.$$

Since τ_L and τ_R are closed under arbitrary intersection $L(Q) \in \tau_L$ and $R(Q) \in \tau_R$. If $Q = \emptyset$ then $L(Q) = R(Q) = \emptyset$. If $Q \neq \emptyset$, $Q \subseteq L(Q)$ and $Q \subseteq R(Q)$; so these sets exist and are non-void. If $Q \in \tau_L$ then $L(Q) = Q$; if $Q \in \tau_R$ then $R(Q) = Q$.

We also observe that the topological closure of Q with respect to τ_L is $R(Q)$ and with respect to τ_R is $L(Q)$. In particular, the Kuratowski closure axioms [Kelley (1955)] hold: $L(A \cup B) = L(A) \cup L(B)$ and $L(L(A)) = L(A)$, for all $A, B \subseteq P$, and similarly for R .

An alternate way of developing τ_L and τ_R would be to follow the approach of Ore (1962). The operators $\hat{R}$, $\hat{R}^*$, $\check{R}$ and $\check{R}^*$ are defined and shown to be closure operators (in fact, they define a Galois connection). It is easily shown that for any $Q \subseteq P$, $\check{R}(Q) \in \tau_R$ and $\check{R}^*(Q) \in \tau_L$ and conversely, if $U \in \tau_R$ and $V \in \tau_L$, then there are sets P_1 and P_2 for which $\check{R}(P_1) = U$ and $\check{R}^*(P_2) = V$; hence we may regard τ_L and τ_R are the natural topologies defined by the above closure operators.

2. <u>Paths</u>

Fundamental to much of graph theory is the notion of a path between two points. The relationship defined by this concept is based on the idea of "reachability" or "accessibility" through a succession of edge relationships.

<u>Definition</u>. Let $G = (P,E)$ and $a,b \in P$. A <u>path</u> from a to b in G , denoted by $\rho_G(a,b)$, is a non-void finite sequence $\langle p_k \rangle$, $k = 0, \ldots, n$, such that

i) $p_0 = a$, $p_n = b$

ii) $(p_{k-1}, p_k) \in E$ for $1 \leq k \leq n$

iii) $p_{k-1} \neq p_k$ for $1 \leq k \leq n$.

The <u>length</u> $|\rho_G(a,b)|$ of a path $\rho_G(a,b) = \langle p_0, p_1, \ldots, p_n \rangle$ is defined to be n. If no confusion as to which G is meant will result, we shall denote the path by $\rho(a,b)$. One can verify directly that for $a, b, c \in P$, $\rho(a,b)$ and $\rho(b,c)$ imply $\rho(a,c)$, that is, the path relationship is transitive. Also, $\rho(a,a)$ is always true since $\rho(a,b) = \langle a \rangle$ is a permissible path; thus the path relation is reflexive. Throughout the rest of this paper we may use $\rho(a,b)$ to denote the existence of some path between a and b, or to denote some particular path.

The following theorem relates the topologies τ_L and τ_R to the path relation. More important, it describes the left and right sets of a point in terms of the familar path relationship.

Theorem 1.2.

(1) $x \in L(a)$ if and only if there is a path $\rho(x,a)$

(2) $y \in R(a)$ if and only if there is a path $\rho(a,y)$.

Proof: Suppose there is a path $\rho(x,a)$ of length k. If $k = 0$ we have $x = a \in L(a)$. If $k > 0$, let x' be the second point on $\rho(x,a)$, so that there is a path from x' to a of length $k-1$, and we may assume by induction hypothesis that x' is in $L(a)$. Since $L(a)$ is in τ_L, and $(x,x') \in E$, we thus have $x \in L(a)$. Conversely, let Q be the set of points from which there is a path to a. We have just shown that $Q \subseteq L(a)$. On the other hand, if $z \in Q$ and $(y,z) \in E$ with $y \neq z$, then this edge together with a path from z to a constitute a path from y to a, so that $y \in Q$. Thus $Q \in \tau_L$, and clearly $a \in Q$; hence $L(a) = \cap \{A \in \tau_L \mid a \in A\} \subseteq Q$. A similar argument holds

for (2).//

Corollary 1.3. The following statements are equivalent:

   (i)   There is a path ρ from a to b

  (ii) $L(a) \subseteq L(b)$

 (iii) $R(a) \supseteq R(b)$

Proof: If (i) holds and $x \in L(a)$, there is a path from x to a , and this can be concatenated with ρ to obtain a path from x to b , proving $x \in L(b)$. Conversely, if (ii) holds, in particular $a \in L(a) \subseteq L(b)$, so that there is a path from a to b . //

Corollary 1.4. $L(\cup A_\alpha) = \cup(L(A_\alpha))$ and $R(\cup A_\alpha) = \cup(R(A_\alpha))$ for any set of indices α .

Proof: There is a path from x to a point of $\cup A_\alpha$ if and only if there is a path from x to a point of some A_α .//

Corollary 1.5. $L(A) = \cup_{p \in A} L(p)$ and $R(A) = \cup_{p \in A} R(p)$.

Corollary 1.6. $L(A) = A \cup \{L(x) \mid (x,y) \in E$ for some $y \in A\}$; $R(A) = A \cup \{R(x) \mid (y,x) \in E$ for some $y \in A\}$.

Proof: If there is a path ρ of length ≥ 1 from z to a point y of A , then there is a path from z to the next-to-last point x of ρ .//

Corollary 1.7. If there is a path ρ from a to b , and $b \in A \in \tau_L$, then $a \in A$ (and similarly for τ_R) .

Proof: $a \in L(b) \subseteq A$ .//

Corollary 1.8. $x \in R(L(y))$ implies $y \in R(L(x))$.

Proof: $x \in R(L(y))$ implies $\rho(z,x)$ for some $z \in L(y)$, so

that $z \in L(x)$. Also, $z \in L(y)$ implies $\rho(z,y)$, so $y \in R(z) \subseteq$ R(L(x)) .//

Definition. A graph G is said to be <u>acyclic</u> if for all points a, b of G, $L(a) = L(b)$ implies $a = b$ (or equivalently $R(a) = R(b)$ implies $a = b$) .

Definition. A path $\rho(a,a)$ of length ≥ 1 is called a <u>cycle</u>. Thus a cycle is the familar "closed path", although we observe that the condition of its length prohibits our considering single points (paths of length zero) as cycles, and our definition of path excludes loops. Consequently, an acyclic graph may have loops, and further the length of a cycle must in fact, be ≥ 2.

Proposition 1.9. A graph G is acyclic if and only if it contains no cycles.

Proof: Suppose there exist distinct points a,b such that $L(a) = L(b)$. We have shown (Theorem 1.2) that $a \in L(b)$ implies a path $\rho(a,b)$ of length greater than zero and $b \in L(a)$ implies a path $\rho(b,a)$ of length greater than zero. Thus by combining these paths we have a path $\rho(a,a)$ of length greater than one. Therefore, we have shown that a non-acyclic graph must have a cycle. Conversely, if there is a cycle $\rho(a,a) = \langle p_k \rangle$ in G, then there must be $x \in p(a,a)$ such that $x \neq a$. We have $x \in L(a)$ and $L(x) \subseteq L(a)$ since there is a path from x to a . Also, $L(a) \subseteq L(x)$ because there is a path from a to x . Hence $L(a) = L(x)$, but $a \neq x$, so G is not acyclic.//

Bhargava and Ahlborn (1968) define a topology τ on a graph as follows: for $G = (P,E)$, A is an element of τ if for every pair of points p,q where $p \notin A$ and $q \in A$ we have $(p,q) \notin E$. This τ

and our left topology τ_L are equivalent, since

$b \in A$ and $(a,b) \in E$ imply $a \in A$ for all a,b is equivalent to

$b \in A$ and $a \notin A$ imply $(a,b) \notin E$ for all a,b .

Most other topologies on an ordered set are defined for lattices [Frink (1942)] and are distinct from the τ_L and τ_R topologies.

The following proposition is essentially found in Bhargava and Ahlborn (1968) as Theorem 1.4.

Proposition 1.10. Let τ_L and τ_R be the E-topologies on a graph $G = (P,E)$; then

(1) τ_L (or τ_R) is a T_0-space if and only if G is acyclic.

(2) τ_L (or τ_R) is a T_1-space if and only if $E = \emptyset$.

Proof: (1) If G is acyclic let $a,b \in P$ with $a \neq b$; then $L(a) \neq L(b)$ and either $L(a) \cap \{b\} = \emptyset$ or $L(b) \cap \{a\} = \emptyset$, since $x \in L(y)$ implies $L(x) \subseteq L(y)$. Conversely, if $L(a) = L(b)$, any set Q in τ_L must contain $L(a)$; hence $a,b \in Q$, and for τ_L to be T_0 we have $a \neq b$.

(2) If there are $a \neq b$ in P with $(a,b) \in E$ then $b \in R(a)$ so $\{a\} \notin \tau_R$ and therefore τ_L is not a T_1-space. Conversely, if τ_L is not a T_1-space then there is $a \in P$ such that there exists $b \in R(a)$ with $b \neq a$. By Corollary 1.6, $R(a) = [\cup R(p_\alpha)] \cup \{a\}$ where $(a,p_\alpha) \in E$ .//

3. Subspaces and convexity

Let H be any subgraph $H \subseteq G$. We may consider the left and right topologies on H , denoted by $\tau_L(H)$ and $\tau_R(H)$, without regard to the graph G or its topology. It readily follows that for any set $A \subseteq P_H$, the left set of A in H, $L_H(A)$, is $\{p \in P_H \mid \rho_H(p,a)$ for

some a $\in$ A} where the path $\rho_H(p,a)$ is completely contained in P_H .

A similar statement can be made for the right set of A, $R_H(A)$. It is

important to note that, in general, $L_H(A) \neq L(A) \cap P_H$.

The following definition is the usual topology for a subset of a

topological space [Kelley (1955)].

Definition. Let G = (P,E) and H = (P_H,E_H) a subgraph of G .

We define the left relative topology of H with respect to G , denoted

$\tau_L|_H$, by {L $\cap$ P_H | L $\in$ $\tau_L(G)$} . Similarly we have the right relative

topology of H , defined by $\tau_R|_H$ = {R $\cap$ P_H | R $\in$ $\tau_R(G)$} .

The example in Figure 1.2 shows that we can have $\tau_L(H) \neq \tau_L|_H$.
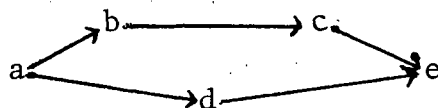
P_H = {a,c,d,e};   {c} $\in$ $\tau_L(H)$ but {c} $\notin$ $\tau_L|_H$

Figure 1.2.

However, we always have

Proposition 1.11. $\tau_L|_H \subseteq \tau_L(H)$.

Proof: Let b $\in$ L $\in$ $\tau_L|_H$ and (a,b) $\in$ E_H . Now L = L' $\cap$ P_H

for some L' $\in$ $\tau_L(G)$. Hence b $\in$ L' , and since (a,b) $\in$ $E_H \subseteq E_G$, we

have a $\in$ L' . But a $\in$ P_H, so that a $\in$ L, proving that L $\in$ $\tau_L(H)$ .//

It is now interesting to ask when the two topologies defined on a

subgraph are, in fact, the same. A sufficient condition will be given

below. A necessary condition is not known at the present time. We must

first introduce the useful notion of convexity; for a more extensive

treatment see Pfaltz (1968, 1971).

Definition. Let $G = (P,E)$. A subset Q of P is _convex_ in G if given $a,c \in Q$ such that there exist $\rho(a,b)$ and $\rho(b,c)$ then $b \in Q$. A subgraph is convex if its point set is convex in G.

The following proposition shows that the class of convex sets forms a sort of semi-subbasis of both left and right E-topologies. Indeed, it would be reasonable to define a new topology using the convex sets as a subbasis. However, in an acyclic graph single points are convex, so that this topology would be discrete.

Proposition 1.12. Let $G = (P,E)$ and $Q \subseteq P$; then Q is convex in G if and only if there are $L \in \tau_L$ and $R \in \tau_R$ such that $Q = L \cap R$.

Proof: Note first that $L(Q) \in \tau_L$, $R(Q) \in \tau_R$, and $Q \subseteq L(Q) \cap R(Q)$. If $p \in L(Q) \cap R(Q)$, there exist paths $\rho(p,q_1)$ and $\rho(q_2,p)$ for some $q_1, q_2 \in Q$. This shows $p \in Q$ since A is convex. Conversely, if $Q = L \cap R$ for $L \in \tau_L$ and $R \in \tau_R$, let $a,c \in Q$ and $\rho(a,b)$, $\rho(b,c)$; we shall show that this implies $b \in Q$. Now $\rho(a,b)$ implies $b \in R(a) \subseteq R$. Likewise, (b,c) implies $b \in L(c) \subseteq L$. Thus, $b \in R \cap L = Q$ and Q is convex.//

Corollary 1.13. Any $A \in \tau_L$ (or $\in \tau_R$) is convex.

Proof: $A = A \cap P$.//

Proposition 1.14. Let H be a subgraph of G. If H is convex in G then $\tau_L|_H = \tau_L(H)$.

Proof: By Proposition 1.11 we need only show that $\tau_L(H) \subseteq \tau_L|_H$. Let $A \in \tau_L(H)$, so that $A \subseteq P_H$. We must show $A = A' \cap P_H$ for some $A' \in \tau_L(G)$. It is claimed that $A = L_G(A) \cap P_H$. Clearly $A \subseteq L_G(A) \cap P_H$. If $p \in L_G(A) \cap P_H$ then by Theorem 1.2 applied to G there is a $q \in A$ such that $\rho(p,q)$ is a path in G. Since $p,q \in P_H$ and P_H is convex,

$\rho(p,q) \subseteq P_H$. Consequently, $\rho(p,q)$ is a path in H and by Theorem 1.2 applied to H we have $p \in L_H(q) \subseteq L_H(A)$. But $A \in \tau_L(H)$ so $L_H(A) = A$ and we obtain $p \in A$ .//

The converse of Proposition 1.14 is not true as is shown by the example in Figure 1.3, where H is not convex but we still have $\tau_L |_H = \tau_L(H)$.
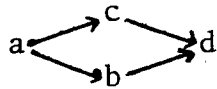
$$P_H = \{a,b,d\}$$

$$\tau_L(H) = \{\{a\}, \{a,b\}, \{a,b,d\}, \emptyset\} = \tau_L |_H$$

Figure 1.3

CHAPTER II

CONNECTIVITY

1. Topological Separability and Connectivity

In graph theory as well as in topology, the study of connectivity occupies the interest of many investigators. Furthermore, many practical problems can be reduced to questions of connectivity. It is only natural that we consider this area and relate the various topological and graph theoretical concepts.

The following definition is actually the usual topological definition of separability. Therefore, the concept of connectivity, defined below, is the same as the topological concept found in Kelley (1955).

Definition. X and Y are separated in a graph G if $L(X) \cap Y = \emptyset = X \cap L(Y)$. We show the following equivalent definition of separation in terms of the right topology.

Proposition 2.1. Two sets, X and Y , are separated in G if and only if $R(X) \cap Y = \emptyset = X \cap R(Y)$.

Proof: We assume X and Y are separated in G . Suppose $p \in X \cap R(Y)$; then $p \in R(Y)$ implies that there exists $y \in Y$ such that $\rho(y,p)$, which in turn implies $y \in L(p) \subseteq L(X)$. Thus $y \in L(X) \cap Y$, contradicting $L(X) \cap Y = \emptyset$. Similarly, $p \in R(X) \cap Y$ would contradict $X \cap L(Y) = \emptyset$. This shows $R(X) \cap Y = \emptyset = X \cap R(Y)$. The converse follows by an analogous argument with L and R interchanged.//

Definition. Let $G = (P,E)$. G is said to be separable if there exist non-empty sets $X,Y \subseteq P$ such that $X \cup Y = P$ and X and Y are separated in G . If G is not separable then G is called connected.

A topological space S is connected if and only if the only subsets of S which are both open and closed are S and $\emptyset$ [Kelley (1955)]. We recall that if the elements of τ_L are called open, then the closed sets -- their complements -- are just the elements of τ_R . Thus we have

Proposition 2.2.   G = (P,E) is connected if and only if $\tau_L(G) \cap \tau_R(G) = \{P,\emptyset\}$.

In agreement with the usual definition of connectivity for subsets of a topological space, we shall call the subgraph H connected if H is connected with respect to the relativized topology $\tau_L|_H$.

Proposition 2.3. If H is connected with respect to its own topology $\tau_L(H)$, it is connected as a subset of G .

Proof: $\{P_H,\emptyset\} \subseteq \tau_L|_H \cap \tau_R|_H \subseteq \tau_L(H) \cap \tau_R(H) = \{P_H,\emptyset\}$, using Proposition 1.11.//

By Proposition 1.14, if H is convex, the converse of Proposition 2.3 also holds.

We recall the definition of connectivity found in Tutte (1966). A graph G is connected if it has no proper non-null detached partial subgraph. A detached partial subgraph is a partial subgraph without points that belong to any edge not in the subgraph. Note that a detached partial subgraph must be a subgraph. The following proposition establishes the equivalence of Tutte's connectivity and the topologically induced connectivity for graphs.

Proposition 2.4. G is connected if and only if it is Tutte connected.

Proof: If G = (P,E) is not connected then there exist A , B

separated with $A \cup B = P$ and $A \neq \emptyset \neq B$. We claim A is detached. If not, there is $(p,q) \notin [A]$ or $(q,p) \notin [A]$. In either case $q \notin A$ and therefore $q \in B$. But, assuming $(p,q) \in E$ we have $q \in R(A) \cap B$ which contradicts separatedness. Since A is a proper detached subgraph of G , G is not Tutte connected.

Conversely, if G is not Tutte connected then there is $A = (Q,F)$ such that $\emptyset \neq Q \subsetneq P$ with A detached. Let $B = P \sim Q$. If $q \in R(A) \cap B$ then there is $\rho(p,q)$ with $p \in Q$. Clearly there must be $(\overline{p},\overline{q})$ on $\rho(p,q)$ such that $\overline{p} \in Q$, $\overline{q} \in B$ and hence A is not detached. Therefore $R(Q) \cap B = \emptyset$. Similarly we find $Q \cap R(B) = \emptyset$. This shows $[Q]$, $[B]$ are separated and consequently G is not connected.//

2. Connected Components

In this section we characterize the connected components of a graph G in terms of the L and R sets.

<u>Definition.</u> Let $G = (P,E)$ and $Q \subseteq P$. We define $C^n(Q)$ by

$C^0(Q) = Q$

$C^n(Q) = L(C^{n-1}(Q)) \cup R(C^{n-1}(Q))$ for $n \geq 1$.

Note that in particular, $C^1(Q) = L(Q) \cup R(Q)$; it follows that $C^n(Q) = C^1(C^{n-1}(Q))$ for all $n \geq 1$. For finite graphs, there is an m such that $C^n(Q)$ is the connected component of Q in G for all $n \geq m$.

<u>Proposition 2.5.</u> $C^n(Q) = C^k(C^{n-k}(Q))$ for all $0 \leq k \leq n$.

<u>Proof:</u> For any n , this is trivial for $k = 0$ or n ; in particular, it is true for $n = 0$ and $n = 1$. Suppose the assertion true for all $m < n$. Then for any $0 < k < n$ we have

$C^n(Q) = C^1(C^{n-1}(Q))$

$$= C^1(C^{k-1}(C^{n-k}(Q))) \quad \text{by the case} \quad m = n-1$$

$$= C^k(C^{n-k}(Q)) \quad \text{by the case} \quad m = k \ .//$$

<u>Definition</u>. We say that $(x,y) \in \omega$ if $y \in C^\infty(x) = \bigcup_{n=0}^\infty C^n(x)$. Clearly $\rho(x,y)$ or $\rho(y,x)$ implies $(x,y) \in \omega$, since $y \in L(x)$ or $R(x)$, respectively, hence in $C^1(x)$.

<u>Proposition 2.6</u>. ω is an equivalence relation.

<u>Proof</u>: Since $x \in C^0(x)$ we have $(x,x) \in \omega$. We next show that $y \in C^k(x)$ implies $x \in C^k(y)$ for all k (so that in particular, $(x,y) \in \omega$ implies $(y,x) \in \omega$) . This is clear for $k = 0$; suppose it true for $k-1$. Then $y \in C^k(x)$ implies $y \in L(C^{k-1}(x))$ or $y \in R(C^{k-1}(x))$, say the former. Hence there is a path $\rho(y,t)$ for some $t \in C^{k-1}(x)$. By induction hypothesis, we have $x \in C^{k-1}(t)$; and $\rho(y,t)$ implies $t \in R(y) \subseteq C^1(y)$. Thus $x \in C^{k-1}(C^1(y))$ by Proposition 2.5. Finally, if $(x,y) \in \omega$, say $y \in C^r(x)$ and $z \in C^s(y)$, we have $z \in C^s(C^r(x)) = C^{r+s}(x)$ by Proposition 2.5, so that $(x,z) \in \omega.//$

<u>Corollary 2.7</u>. If $a \in C^n(b)$ then $C^k(a) \cap C^{n-k}(b) \neq \emptyset$ for all $0 \leq k \leq n$.

<u>Proof</u>: $C^n(b) = C^k(C^{n-k}(b))$ contains a ; hence there exists $c \in C^{n-k}(b)$ such that $a \in C^k(c)$, which implies $c \in C^k(a) \ .//$

<u>Proposition 2.8</u>. $C^\infty(Q) = \bigcup_{n=0}^\infty C^n(Q)$ is in both τ_L and τ_R .

<u>Proof</u>: Let $\overline{Q} = \bigcup_{n=0}^\infty C^n(Q)$. If $x \in L(\overline{Q})$ we have $x \in L(y)$ for some $y \in \overline{Q}$, say $y \in C^k(Q)$; hence $x \in L(y) \subseteq C^1(C^k(Q)) = C^{k+1}(Q) \subseteq \overline{Q}$. The proof for R is similar.//

<u>Proposition 2.9</u>. If A is in $\tau_L \cap \tau_R$, then $C^k(A) = A$ for all k .

Proof: This is trivial for $k = 0$; suppose it true for $k-1$.
Then $C^k(A) = L(C^{k-1}(A)) \cup R(C^{k-1}(A)) = L(A) \cup R(A) = A$ .//

Proposition 2.10. $C^\infty(x)$ is the connected component containing x .

Proof: We know from Proposition 2.8 that $C^\infty(x) \in \tau_L(G) \cap \tau_R(G)$.
Thus any subset of $C^\infty(x)$ which is in $\tau_L(C^\infty(x))$ or $\tau_R(C^\infty(x))$ will
be in $\tau_L(G)$ or $\tau_R(G)$ respectively [see Kelley (1955)]. Now suppose
$C^\infty(x)$ not connected; then by Proposition 2.2, there exists $A \neq \emptyset$ and $\neq$
$C^\infty(x)$ such that $A \in \tau_L(C^\infty(x)) \cap \tau_R(C^\infty(x))$, whence as just observed,
$A \in \tau_L(G) \cap \tau_R(G)$. From Proposition 2.9 we have $C^k(A) = A$ for all
k , therefore $C^\infty(A) = A$, so that $C^\infty(x) \subseteq C^\infty(A) = A$, contradiction.

Suppose there exists C connected such that $C^\infty(x) \subsetneq C$, that is,
$C^\infty(x)$ is not maximal. $C^\infty(x) \in \tau_L(G) \cap \tau_R(G)$ and $C^\infty(x) \subseteq C$, it follows
that $C^\infty(x) \in \tau_L(C) \cap \tau_R(C)$. By Proposition 2.2, this shows C is not
connected.//

3. Other Connectivity Concepts

Definition. Let $G = (P,E)$ and $Q \subseteq P$. We say that

a) Q is weakly connected in G if $(x,y) \in \omega$ for any $x,y \in Q$
b) Q is self-connected if it is weakly connected in $[Q]$ ($[Q]$ is the
   subgraph generated by Q)
c) Q is strongly connected in G if $\rho(x,y)$ and $\rho(y,x)$ for any
   $x,y \in Q$.

Figure 2.1 shows a set Q which is weakly connected in G but not self-
connected and not connected or strongly connected in G. Note that Q
is also convex in G .

$P = \{a,b,c,d,e\}$

$Q = \{a,b,c,d\}$

Figure 2.1.

The above definitions are essentially (see Proposition 2.19) those commonly found in the literature [Tutte (1966) and Harary, et al (1965)]. In accordance with our general policy of identifying subgraphs with their point sets, unless the distinction is essential for clarity, we will say that $H = [Q]$ is connected, weakly connected, etc., if the point set Q has the appropriate property.

Proposition 2.11. $(x,y) \in \omega$ if and only if there exists a finite self-connected partial subgraph W that contains x and y.

Proof: If $y \in C^0(x)$ we can take $W = [x]$; suppose the assertion true when $y \in C^{k-1}(x)$, and let $y \in C^k(x)$. Then $y \in L(C^{k-1}(x))$ or $y \in R(C^{k-1}(x))$, say the former, so that there is a path $\rho(y,t)$ for some $t \in C^{k-1}(x)$. By induction hypothesis, there is a W' containing t and x . We can then take the partial subgraph on the points and edges of ρ together with W' as the desired W .

To prove that W is self-connected, let u,v be any points of W . If both are in W' , then $(u,v) \in \omega$ for $W' \subseteq W$ by induction hypothesis, and if both are in ρ , then $\rho(u,v)$ or $\rho(v,u)$ in $\rho \subseteq W$, implying $(u,v) \in \omega$ for W . Finally, if u is in ρ and v in W' , we have $\rho(u,t)$; hence $(u,t) \in \omega$ for ρ , while $(t,v) \in \omega$ for W' ; thus

$(u,v) \in \omega$ for W by transitivity of ω .$//$

The following propositions establish relations among the connectivity concepts.

Proposition 2.12. If Q is connected in G then Q is weakly connected in G .

Proof: Suppose that there were $p,q \in Q$ for which $w(p,q)$ did not hold. Let $A = \bigcup_{n=0}^{\infty} C^n(p)$. By Proposition 2.8, $A \in \tau_L(G) \cap \tau_R(G)$; hence $A \cap Q \in \tau_L(G)|_Q \cap \tau_R(G)|_Q$. But $A \cap Q \neq \emptyset$, since it contains p ; and $A \cap Q \neq Q$, since it does not contain q . Hence by Proposition 2.2 applied to the relative topologies on Q , Q is not connected.$//$

Proposition 2.13. If Q is self-connected then Q is connected in G .

Proof: By Proposition 2.3, it suffices to show that $[Q]$ is connected. Suppose we had $A \in \tau_L(Q) \cap \tau_R(Q)$, $A \neq \emptyset, Q$. Let $p,q \in Q$, $p \in A$, $q \notin A$. Since Q is self-connected we have $q \in C_Q^k(p)$ for some k . But then $q \in C_Q^k(A) = A$ by Proposition 2.9, contradiction.$//$ The converse of Proposition 2.13 is not true in general (see Figure 2.1). However, we have

Corollary 2.14. If Q is convex in G , the following statements are equivalent:

1) Q is self-connected

2) $[Q]$ is connected

3) Q is connected in G

Proof: (2) implies (1) by Proposition 2.12 applied to $[Q]$; (1) implies (3) is Proposition 2.13; (3) implies (2) by the remark following

Proposition 2.3.//

In particular, any graph G is self-connected if and only if it is
connected.

Proposition 2.15. Q is strongly connected in G if and only if
$L_G(x) = L_G(y)$ for all x,y in Q .

Proof: By Corollary 1.3, $\rho(x,y)$ if and only if $L(x) \subseteq L(y)$, and
$\rho(y,x)$ if and only if $L(y) \subseteq L(x)$ .//

Proposition 2.16. Q is strongly connected in G if and only if
$^{\tau}L|_{[Q]} = \,^{\tau}R|_{[Q]} = \{\emptyset,Q\}$.

Proof: Let $A \in \tau_L|_{[Q]}$, $A \neq \emptyset$. Then $A = A' \cap Q$ where
$A' \in \tau_L(G)$. Let $q \in Q$, $p \in A$. If Q is strongly connected, we
have $\rho(q,p)$, which with $A' \in \tau_L(G)$ implies $q \in A'$. Thus
$q \in A' \cap Q = A$, so that A = Q . Conversely, let $p,q \in Q$ with no
path from p to q in G ; then $p \notin L(q)$, so that $L(q) \cap Q$ is
neither $\emptyset$ (it contains q) nor Q (it does not contain p). Thus
$^{\tau}L|_{[Q]} \neq \{\emptyset,Q\}$. The proofs for R are analogous.//

Having discussed the connectivity of point sets in a graph, we now
turn our attention to edges which reduce the connectivity of a point
set.

Definition. Consider a graph G = (P,E) . An edge $(p,q) \in E$ is
said to be a disconnecting edge if {p,q} is not weakly connected in
the partial subgraph (P,E~(p,q)) . A disconnecting edge is also called
an isthmus [Tutte (1966)].

Definition. A tree is a connected graph in which every edge is a

disconnecting edge. A _forest_ is a graph whose connected components
are trees.

This definition of tree is applicable to both finite and infinite
graphs, and reduces to one of the standard definitions in the finite
case. However, in the finite case, there are several well known
equivalent definitions. One such states that a connected graph is
a tree if the number of points is equal to the number of edges plus one.

The following theorem is of interest in that is allows the use
of the finite conditions even for infinite graphs. In particular, if
a graph is not a tree then we know there is some finite subgraph which
will provide the counterexample.

Theorem 2.17. A connected graph is a tree if and only if every
finite self-connected subgraph is a tree.

Proof: If G is a tree and H a finite self-connected subgraph,
H must be a tree since a non-disconnecting edge in H would also
be a non-disconnecting edge in G. Conversely, suppose G is not
a tree. Then there is a non-disconnecting edge (a,b), that is, $w(a,b)$
in the partial subgraph $G' = (P, E \sim (a,b))$. By Proposition 2.11, there
exists a finite, self-connected subgraph $W = (P_W, E_W)$ of G' that
contains a and b. If we define $W^* = [P_W]$ we have W^* finite, self-
connected and (a,b) an edge in it; but (a,b) is not a disconnecting
edge, hence W^* is not a tree.//

4. Walks

In most developments of graph theory, the concept of walk has
been awkwardly defined. The problem lies in the fact that certain sequences
of edges should not be considered walks. For example, repeating one
or more edges in not desired unless the edge is arrived at by a

"different" route. Simply stated, walks should not retrace themselves. Another objection has been the often sudden introduction of edge sequences. In this section, we define walk indirectly using the concept of weak connectedness.

Let us denote by $\Omega(x,y)$ the family of all self-connected partial subgraphs containing x and y . Proposition 2.11 assures that there are finite members of $\Omega(x,y)$ whenever $\Omega(x,y) \neq \emptyset$. We can partially order $\Omega(x,y)$ by edge inclusion, that is $W_1(P_1,E_1) \leq W_2(P_2,E_2)$ if $P_1 \subseteq P_2$ and $E_1 \subseteq E_2$. Then there must exist minimal elements of $\Omega(x,y)$. We now have

Definition. A <u>walk</u> between x and y , denoted $w(x,y)$, is any minimal element of $\Omega(x,y)$. As in the case of paths our notation $w(x,y)$ denotes both the existence of a minimal self-connected partial subgraph and a specific such partial subgraph.

It is easy to establish the following lemma.

Lemma 2.18. When $x \neq y$, any $w(x,y)$ must be acyclic.

Proof. If $w(x,y)$ contains a cycle, an edge can be deleted without destroying self-connectivity; thus $w(x,y)$ was not minimal.//

The following additional properties are easily verified:

(1) $w(\cdot,\cdot)$ is an equivalence relation, in fact, $w(x,y)$ if and only if $w(y,x)$

(2) $a, b \in w(x,y)$ then there is $w(a,b) \subseteq w(x,y)$

(3) if $w_1(x,y) \cap w_2(a,b) \neq \emptyset$ then for any $p,q \in w_1 \cup w_2$ there exists $w(p,q) \subseteq w_1 \cup w_2$.

This last statement follows from (1) and (2).

We now are in a position to show the equivalence of the walk concept defined above and the simple (undirected) path concept found

elsewhere in the literature.

Proposition 2.19. A walk between x and y, where $x \neq y$, and a simple path between x and y (as defined by Tutte (1966)) are equivalent.

Proof: From Tutte (1966) we know a simple path is equivalent to an arc with end points x and y (Propositions 4.31 and 4.35). An arc is defined by Tutte to be a tree with just two points, called end points, that belong to only one edge.

Let $W = (P,E)$ be an arc with end points x and y, and let $W' = (P',E')$ be a proper connected partial subgraph of W that still contains x and y. If $P' \subsetneq P$, there must exist $p \in P'$, $q \in P \sim P'$ with $e = (p,q)$ or $(q,p) \in E$. Thus $(P' \cup \{q\}, E' \cup \{e\})$ is still connected. If it is all of W, then q is an end point of W. Contradiction; hence it is still proper. Using this argument repeatedly, we can obtain a W' with $P' = P$, so that $E' \subsetneq E$; but since W is a tree, such a W' cannot be connected, contradiction.

Let W be a walk and let e be any edge in W. From Tutte (1966), Proposition 3.13, we know that the graph G' obtained by deleting one edge from G has the same number of components that G has if the edge is not an isthmus, or the same number plus one if it is an isthmus. Since W is a minimal connected partial subgraph, W has only one component, but $W \sim \{e\}$ must have at least two. This shows that every edge of W is an isthmus, so that W is a tree.

We need only show that there are exactly two points of W which belong to only one edge. First suppose $p \in W$, $p \neq x,y$ and p belongs to only one edge, say (p,q), in W. As we have observed, $W \sim \{(p,q)\}$ has exactly two components, and since $\{p\}$ is clearly one component, it follows that x and y are both in the other. This contradicts the

the minimality of W as a connected partial subgraph containing x

and y .

Now we show both x and y belong to only one edge. Let x

belong to the edge e , where e = (x,p) or (p,x). As shown before, for

e any edge, W∿{e} has exactly two components, and by definition of

isthmus, the end points of the edge are in different components.

Clearly y cannot be in the same component as x in V=W∿{e} ; therefore

y and p are in the same component, call it C.

Let q ≠ p be any other point such that f = (x,q) or (q,x) is an

edge. Then q must be in the same component of V that x is in, since

f was not deleted from W . We now can define a new partial subgraph

D consisting of C together with x and e . Since q is not in

D , D is properly contained in W . However, D is obviously

connected; thus we have a contradiction, showing that x belongs

to only one edge. A similar proof holds for y ://

Another characterization of trees in terms of the walk concept is

possible [compare Harary (1969), Theorem 4.1]. Thus, the following

Proposition further substantiates that our definition of walk is

consistent with the usual one.

Proposition 2.20. G is a tree if and only if for any two points

x, y there exists an unique walk w(x,y) .

Proof: If G is a tree by Proposition 2.12 for any two points

x, y we have $x \in \bigcup_{n=0}^{\infty} C^n(y)$ and thus $\Omega(x,y) \neq \emptyset$; that is, there

exists at least one w(x,y) .

Suppose there exist distinct walks $w_1(x,y)$ and $w_2(x,y)$. By

minimality of walks there must be an edge (a,b) in $w_1(x,y)$ but

not in $w_2(x,y)$. We know we have a walk $w_1(x,a) \subseteq w_1(x,y)$ and

$w_1(y,a) \subseteq w_1(x,y)$. If (a,b) is in both $w_1(x,a)$ <u>and</u> $w_1(y,a)$ then we have $w(x,b)$ and $w(y,b)$ be deleting (a,b). Moreover, $W = w(x,b) \cup w(y,b) \subsetneq w_1(x,y)$ is a self connected partial subgraph, so $w_1(x,y)$ was not minimal, contradiction. Consequently the edge (a,b) is <u>not</u> in both $w_1(x,a)$ and $w_1(y,a)$, say (a,b) not in $w_1(x,a)$. Similarly we can show (a,b) is not in both $w_1(x,b)$ and $w_1(y,b)$ where both these walks are contained in $w_1(x,y)$. If (a,b) is not in $w_1(x,b)$, we have $U = w_1(x,a) \cup w_1(x,b) \in \Omega(a,b)$ and (a,b) not in U . This shows (a,b) is <u>not</u> a disconnecting edge which contradicts the assumption that G is a tree. If (a,b) is not in $w_1(y,b)$, we have $V = w_2(x,y) \cup w_1(x,a) \cup w_1(y,b)$ in $\Omega(a,b)$ and (a,b) not in V . Thus, in this case too (a,b) is not a disconnecting edge.

Conversely, if there is an unique walk between any two points of G , for (x,y) an edge, the partial subgraph $(\{x,y\}, \{(x,y)\})$ must be the only walk between x and y . This shows (x,y) is a disconnecting edge, which proves G is a tree.//

CHAPTER III

REVERSAL ORDER, NORMAL

GRAPHS, AND BASIC GRAPHS

1. Reversal order

In this section we define a metric -- essentially, the minimum number of reversals in a walk between two points, plus one -- which can be used to develop a more quantitative concept of separation of a graph by deleting edges.

Definition. Let a,b be points of the graph G. We say that the reversal order of a and b, denoted by $r(a,b)$, is equal to $n \geq 1$ if $a \notin C_G^{n-1}(b)$ but $a \in C_G^n(b)$. We also say that $r(a,b) = 0$ if $a = b$ (i.e., $a \in C_G^0(b)$), and that $r(a,b) = \infty$ if $a \notin C_G^\infty(b)$. Intuitively, $r(a,b)$ is 1 greater than the number of path direction reversals required to walk from a to b. The following proposition establishes some elementary properties of reversal order -- in particular, that it is a metric.

Proposition 3.1.

1) $r(a,b) \leq n$ if and only if $a \in C_G^n(b)$

2) $r(a,b) = 1$ if and only if $a \neq b$ and $\rho(a,b)$ or $\rho(b,a)$

3) $r(b,a) = r(a,b)$ for all a,b

4) $r(a,c) \leq r(a,b) + r(b,c)$ for all a,b,c

5) Let $r(a,b) = n$ and $0 \leq k \leq n$; then there exists c such that $r(a,c) = k$ and $r(c,b) = n-k$.

Proof:

(1) is clear since $C^0 \subseteq C^1 \subseteq \ldots \subseteq C^n$.

(2) is immediate from the definitions of C^0 and C^1 and from Theorem 1.2.

(3) follows from the proof of Proposition 2.6.

(4) follows from Proposition 2.5.

To prove (5), let $c \in C^k(a) \cap C^{n-k}(b)$ as guaranteed by Proposition 2.7. If $c \in C^{k-1}(a)$ we would have $r(a,b) \le r(a,c) + r(b,c) \le (k-1) + (n-k) < n$, contradiction; hence $r(a,c) = k$, and similarly $r(c,b) = n-k$ .//

Using the concept of reversal order, we can define a generalized notion of "separating edge" as follows:

Definition. The edge (a,b), $a \ne b$, of the graph $G = (P,E)$ will be called <u>separating edge of order</u> k if $r(a,b) > k$ in the partial graph $G' = (P, E-(a,b))$.

The relationship between separating edges and disconnecting edges is given by

Proposition 3.2. A disconnecting edge is a separating edge of order k for all k, and conversely.

Proof: If we had $r(a,b) \le k$ we would have $b \in C^k_{G'}(a)$, contradicting the fact that $\{a,b\}$ is not weakly connected in G' .//

We characterize separating edges of orders 1 and 2 in the next two propositions:

Proposition 3.3. (a,b) is a separating edge of order 1 if and only if $\{a\}$ and $\{b\}$ are separated in G' .

Proof: If $\{a\}$ and $\{b\}$ are separated we have $L_{G'}(a) \cap \{b\} = \{a\} \cap L_{G'}(b) = \emptyset$, i.e., $a \notin L_{G'}(b)$ and $b \notin L_{G'}(a)$, i.e., there is no path from a to b or from b to a in G', so that $r(a,b) > 1$ in G' . Conversely, if they are not separated we have either $a \in L_{G'}(b)$ or $b \in L_{G'}(a)$, i.e., $\rho(a,b)$ or $\rho(b,a)$ in G', so that $r(a,b) = 1$

in G' .//

We shall omit the subscript G' from now on unless confusion would result.

Proposition 3.4. The following statements are all equivalent:

1) (a,b) is a separating edge of order 2

2) {a} and both L(b) and R(b) are separated in G'

3) {b} and both L(a) and R(a) are separated in G'

4) L(a) $\cap$ L(b) = R(a) $\cap$ R(b) = $\emptyset$ in G' .

Proof: (2) means L(a) $\cap$ L(b) = {a} $\cap$ L(L(b)) = R(a) $\cap$ R(b) = {a} $\cap$ R(R(b)) = $\emptyset$ (see Proposition 2.1). Since L(L(b)) = L(b) , the condition {a}$\cap$ L(L(b)) = $\emptyset$ is implied by L(a) $\cap$ L(b) = $\emptyset$, and similarly the fourth condition is implied by the third; hence (2) is equivalent to (4), and analogously for (3).

If L(a) $\cap$ L(b) or R(a) $\cap$ R(b) were $\neq \emptyset$, say the former, we would have ρ(c,a) and ρ(c,b) for some c , so that $b \in C^2(a)$, contradicting (1). Conversely, if r(a,b) $\leq$ 2 we have $b \in C^2(a)$, i.e., b is in L(L(a)) = L(a), R(R(a)) = R(a), L(R(a)), or R(L(a)) . In the first case L(a) $\cap$ R(b) $\neq \emptyset$ (it contains b); in the second case, R(a) $\cap$ R(b) $\neq \emptyset$; in the third case, ρ(b,c) for some $c \in R(a)$, so that R(b) $\cap$ R(a) $\neq \emptyset$; and similarly the last case implies L(b) $\cap$ L(a) $\neq \emptyset$ -- a contradiction to (4) in all cases.//

We now introduce two special types of separating edge which turn out to have interesting properties.

Definition. (a,b) will be called a normal edge if L(a) and R(b) are separated in G' ; an antinormal edge, if R(a) and L(b) are separated in G' .

Proposition 3.5. A normal or antinormal edge is a separating edge

of order 2.

Proof: L(a) and R(b) separated means L(a) ∩ L(R(b)) = L(L(a)) ∩ R(b) = L(a) ∩ R(R(b)) = R(L(a)) ∩ R(b) = ∅ , where the second and third conditions are redundant; the first and fourth conditions evidently imply L(a) ∩ L(b) = ∅ and R(a) ∩ R(b) = ∅ , respectively. The proof for antinormal is analogous.//

Proposition 3.6. An edge is a separating edge of order 3 if and only if it is both normal and antinormal.

Proof: Analogous to that of Proposition 3.4.//

Proposition 3.7. The following statements are equivalent:

1) (a,b) is normal

2) a ∉ R(L(R(b))) in G'

3) b ∉ L(R(L(a))) in G'

Proof: If a ∈ R(L(R(b))) there exists c ∈ L(a) ∩ L(R(b)) , so that (a,b) is not normal by the proof of Proposition 3.5., and conversely, proving (2). The proof of (3) is similar.//
An analogous result, with L and R interchanged, is true for antinormal edges.

In Section 3 we shall characterize graphs all of whose edges are separating edges of order 1 (they turn out to be just the graphs that are basic and acyclic). To conclude the present section, we prove

Proposition 3.8. The following statements about the graph G are equivalent:

1) Every edge of G is normal

2) Every edge of G is a separating edge of order 2

3) Every path in G is convex.

Proof: (1) implies (2) by Proposition 3.5. To see (2) implies (3),

suppose ρ is not convex; then there exist x,y on ρ such that $\rho'(x,y)$ where $\rho' \not\subseteq \rho$. Let a,b be consecutive points of ρ' such that $a \notin \rho$, $b \in \rho$. If x precedes y on ρ we have $\rho(x,b) \subseteq \rho$ and $\rho(x,a) \subseteq \rho'$, so that $x \in L(a) \cap L(b)$ in G' . If y precedes x on ρ we have $\rho(b,y) \subseteq \rho'$, $\rho(y,x) \subseteq \rho$ and $\rho(x,a) \subseteq \rho'$, so that $b \in L(a)$ in G' . In either case this proves (a,b) is not separating of order 2 by Proposition 3.4.

Finally, to see (3) implies (1), if (a,b) not normal then there exists $x \in L(a) \cap L(R(b))$ in G', say $\rho(x,y)$ where $y \in R(b)$ and $\rho(x,y)$ does not have a,b as consecutive points. We also have $\tilde{\rho}(x,y)$ through (a,b) . It is clear that at least one of ρ and $\tilde{\rho}$ has at least three points; thus the other path is not convex.//

2. Normal graphs

Definition. G will be called normal if every normal edge is a disconnecting edge. G is completely normal if every partial subgraph is normal. It is clear that completely normal implies normal.

An "antinormal" graph can be defined analogously with antinormal replacing normal; however, this concept will not be needed. An example of a non-normal graph is shown in Figure 3.1, as well as an example of normal but not completely normal graph. Note that any non-normal graph must contain a normal edge, since otherwise the graph would be vacuously normal.

normal but
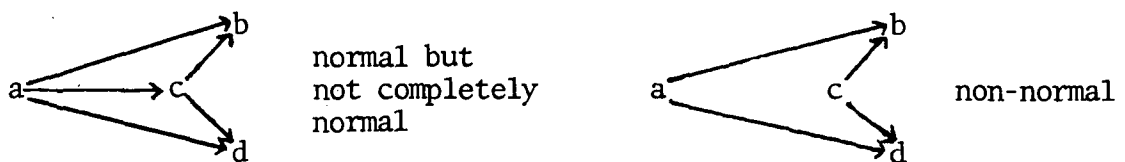not completely
normal

non-normal

Figure 3.1

Any tree is a normal (and "antinormal") graph, since every edge is a disconnecting edge. Because every connected partial subgraph of a tree is a tree, any tree is also completely normal.

As an immediate consequence of Proposition 3.8 we have

Proposition 3.9. A connected graph G is a tree if and only if G is normal and all paths in G are convex.

Proof: If G is normal and all paths are convex, every edge of G must be disconnecting edge, and conversely.//

Definition. The point s is called a left terminal point of G if L(s) = {s} and R(s) = G . Similarly, t is called a right terminal point if L(t) = G and R(t) = {t} . Clearly G can have at most one left and one right terminal point. If it has both, we call it a two-terminal graph. Such a graph is evidently connected.

Proposition 3.10. Every two-terminal graph is normal.

Proof: If the edge (a,b) is normal then for any path $\rho(x,y)$ such that $x \in L(a)$, $y \in R(b)$, a,b must occur consecutively on $\rho(x,y)$. Since $s \in L(a)$, $t \in R(b)$, any path $\rho(s,t)$ must have a,b as consecutive points. Since every point p of G belongs to some path $\rho(s,t)$, it follows that $p \in L(a)$ or $p \in R(b)$, that is, $P = L(a) \cup R(b)$.

Now $L(a) \in \tau_L(G')$, since $\rho(x,a)$ in G evidently implies $\rho(x,a)$ in G'. Also $R_{G'}(L(a)) \cap R(b) = \emptyset$ by normality; since $L(a) \cup R(b) = P$, it follows that $R_{G'}(L(a)) = L(a)$, that is, $L(a) \in \tau_R(G')$. Since $b \notin L(a)$ by normality, we have $\emptyset \subsetneq L(a) \subsetneq P$, so that L(a) is a proper open and closed set in G' , proving that G' is not connected, that is, the edge (a,b) is a disconnecting edge.//

Definition. Let G = (P,E) be acyclic and let $A \subseteq P$. We say

that m is a <u>lower bound</u> of A if $m \in \bigcap_{a \in A} L(a)$, that is, if $A \subseteq R(m)$.
We say that m is a <u>maximal lower bound</u> (or an <u>inf</u>) of A if m is a
lower bound of A and there are no lower bounds of A in $R(m) \sim \{m\}$.
In a similar manner we can define <u>upper bounds</u> and <u>minimal upper bounds</u>
(<u>sups</u>) of A .

<u>Proposition 3.11</u>. Let G be a finite, acyclic, and completely
normal graph. If points a,b of G have a lower bound, then they have
a <u>unique</u> inf.

<u>Proof</u>: Since G is finite and acyclic, there readily exists at
least one inf. If $a \in L(b)$ or $b \in L(a)$, it is evident that a or
b is the unique inf. Otherwise, we have $a \notin C^1(b)$. Let m_1 , m_2 be
infs of $\{a,b\}$ and $m_1 \neq m_2$. Since $m_j \in L(a) \cap L(b)$ for $j = 1,2$, we have
the paths $\rho_1(m_1,a)$, $\rho_2(m_1,b)$, $\rho_3(m_2,a)$, $\rho_4(m_2,b)$. Let $\rho_1 = \langle m_1,a_1,\ldots,a_n \rangle$,
and let $\rho_2 = \langle m_1,b_1,\ldots,b_m \rangle$ where $a_n = a$ and $b_m = b$. We show that
the edge (m_1,a_1) is normal in the partial subgraph H defined by the
union of ρ_1,ρ_2,ρ_3 , and ρ_4 . Let H' denote H with the edge (m_1,a_1)
deleted. We have $L_H(m_1) = L_{H'}(m_1) = \{m_1\}$ and $R_{H'}(L_H(m_1)) = R_{H'}(m_1)$
is contained in the set of points of ρ_2 and ρ_4 , since if we had
$\rho(m_1,x)$ in H' for some point x of ρ_1 or ρ_3 , the second point on
ρ would have to be b_1 , so that $b_1 \in L(a) \cap L(b)$ would contra-
dict the maximality of m_1 . Similarly, $R_H(a_1)$ is contained in the
set of points of ρ_1 and ρ_3 . Hence $R_H(a_1) \cap R_{H'}(L_H(m_1)) = \emptyset$, so
that (m_1,a_1) is normal in H; however, it clearly is not a discon-
necting edge. This shows that partial subgraph H is not normal, and
therefore G is not completely normal, contradiction.//

The converse of this Proposition is not true; there exists a finite,
acyclic graph which is not completely normal but for which every pair

of points with a lower bound has a unique inf (Figure 3.2).
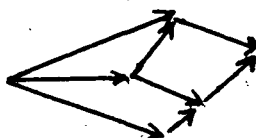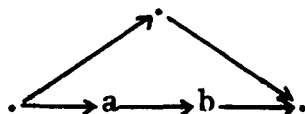

Figure 3.2.

In the remainder of this section, we investigate another class of graphs, TTSPN's, which like trees, are completely normal (but not "antinormal"; see Figure 3.3). We give a new charaterization of these graphs below, and in Chapter IV we show that they are invariant under certain kinds of mappings.

Definition. A graph $G = (P,E)$ is called a two-terminal series-parallel network (TTSPN) if

a) $P = \{u,v\}$, $E = \{(u,v)\}$

or b) $P = P_1 \cup P_2$, $E = E_1 \cup E_2$, where $G_1 = (P_1,E_1)$ and $G_2 = (P_2,E_2)$ are TTSPN's, $E_1 \cap E_2 = \emptyset$, and

(b_S) $P_1 \cap P_2 = \{z\}$, where $\rho_{G_1}(x,z)$ for all $x \in P_1$ and $\rho_{G_2}(z,y)$ for all $y \in P_2$

or (b_p) $P_1 \cap P_2 = \{s,t\}$, where $\rho_{G_1}(s,x)$ and $\rho_{G_1}(x,t)$ for all $x \in P_1$, and $\rho_{G_2}(s,y)$ and $\rho_{G_2}(y,t)$ for all $y \in P_2$.

(a,b) is antinormal but not

disconnecting

Figure 3.3.

In case (b_S), G is called the serial composition of G_1 and G_2; in case (b_p), it is called their parallel composition.

Aside from their importance in electrical network applications,

where they originated, TTSPN's are of interest in the theory of graphs for

two reasons. First, with the exception of trees and acyclic graphs, there

are few other classes of directed graphs that admit a reasonable topological

characterization. Second, since TTSPN's are "generated" from a single edge,

they provide a means of obtaining a richer class of graphs from a given

class by a composition operation, namely replacing edges with TTSPN's.

For example, Husimi trees [Ore (1962)] can be generated from trees by

replacing edges with TTSPN's. Many of the results can be extended to

larger classes of graphs which are formed by composition of TTSPN's.

For any TTSPN, G, the following observations can be readily proved by

induction on the number of edges of G:

1) There is at least one edge in G

2) G is connected

3) G is acyclic

4) G is a two-terminal graph, hence is normal

5) Every two-terminal connected partial subgraph of G is a TTSPN;

   hence G is completely normal.

The example in Figure 3.4 shows that (1) through (5) do <u>not</u>

characterize the class of two terminal series parallel networks.
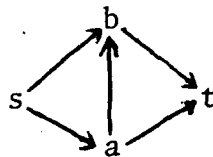

Figure 3.4.

A graph theoretical characterization of a TTSPN is important since

the inductive definition does not immediately lend itself to proving

topological results. In particular, such a characterization is needed

later to prove that the class of TTSPN's is closed under certain types of mappings.

The concept of series-parallel graphs arose in electrical network theory. In this context the prototype of a non-series-parallel network is the familiar Wheatstone Bridge (Figure 3.4 with the edges having resistors on them). In this case the cross-connection between a and b destroys the series-parallel property. The following definition seems to be a natural graph-theoretical generalization of this concept.

Definition. Two points x and y of a graph G with $y \in R(x)$ are said to be cross-connected if (1) there exist two paths $\rho_1(x,y)$ and $\rho_2(x,y)$ and (2) there is a path $\rho_3(a,b)$, where $a \neq b$, such that $\rho_1 \cap \rho_3 = \{a\}$ and $\rho_2 \cap \rho_3 = \{b\}$. A graph without any cross-connections will be called cross-connection-free (CCF). It is trivial to see that $a \neq x,y$, $b \neq x,y$, and $a, b \in L(y) \cap R(x)$. We also note that any partial subgraph of a CCF graph must be CCF.

In Figure 3.5 a cyclic two-terminal graph with a cross-connection between s and t is illustrated.

$\rho_1 = <s,c,a,t>$

$\rho_2 = <s,b,c,t>$
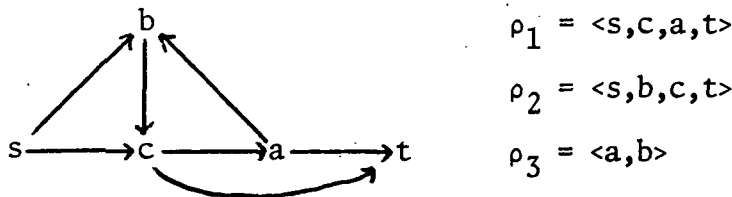
$\rho_3 = <a,b>$

Figure 3.5.

Theorem 3.12. A finite, two-terminal acyclic graph is a TTSPN if and only if it is CCF.

Proof: Since a one-edge graph is CCF, to prove that any TTSPN is CCF, it suffices to show that serial or parallel composition preserves the CCF property. Let ρ_1, ρ_2, ρ_3 be a cross-connection in the

composition of G_1 and G_2 .

In the serial case, if a and b are both in G_1 , then ρ_3 must lie in G_1 . If we restrict ρ_1 and ρ_2 to G_1 , replacing y by the composition point z if y is in G_2 , this yields a cross-connection in G_1 , contradiction. If a is in G_1 and b in G_2 , then $x \in L(a) \subseteq G_1$ and $y \in R(b) \subseteq G_2$, so that ρ_1, ρ_2, ρ_3 all pass through z , and so they cannot constitute a cross-connection.

In the parallel case, ρ_1, ρ_2, and ρ_3 must all lie in (say) G_1 , since no path can pass through either of the composition points s,t ; hence we immediately have a cross-connection in G_1. (The possibility that ρ_1 lies in G_1, ρ_2 lies in G_2 and x,y = s,t is ruled out since ρ_3 must then contain s or t and so is not a cross-connection.)

Conversely, suppose G = (P,E) is two-terminal, acyclic, and CCF, and let s,t be its terminal points. If G has only one edge, it is trivially a TTSPN. Suppose the desired result true for all G's having fewer edges than the given one. If G can be shown to be the serial or parallel composition of two of its partial subgraphs G_1, G_2, where G_1, G_2 are two-terminal graphs, then G_1, G_2 are TTSPN by induction hypothesis (since the acyclic and CCF properties pass to partial subgraphs), so that G is also TTSPN. In particular, if (s,t) $\in$ E , evidently G is the parallel composition of its two-terminal partial subgraphs ({s,t}, {(s,t)}) and (P,E~{(s,t)}), and we are done; we may thus assume (s,t) $\notin$ E.

Let $F(s) = \{x \in P \mid L(x) = \{s,x\}\}$, and $E(t) = \{x \in P \mid (x,t) \in E\}$ Define $Q(x) = \{s\} \cup R(L(x) \sim \{s\})$. Suppose that Q(x) = P for all x $\in$ E(t) . We claim that for any y $\neq$ s in F(s) we have $R(y) \supseteq E(t)$. Indeed, if not, let z $\in$ E(t), z $\notin$ R(y) , so that y $\notin$ L(z) . Now Q(z) = P , so we must have $y \in R(L(z)~\{s\})$, i.e., $w \in L(y)$ for some

$w \in L(z) \sim \{s\}$. But $L(y) = \{s,y\}$, and $w \neq s$; hence $w = y$ and we have $y \in L(z)$, contradiction. We have thus shown that one of the following must be true:

1) $Q(z) \neq P$ for some $z \in E(t)$

or 2) $R(y) \supseteq E(t)$ for some $y \neq s$ in $F(s)$, so that $E(t)$ has a lower bound $\neq s$.

We shall show that in case (1), G has a parallel decomposition, and in case (2), a serial decomposition.

In case (1), there can be no edge (a,b) in G with $a \in Q(z)$, $b \notin Q(z)$ unless $a = s$, since $Q(z) \sim \{s\}$ is an R-set. We show that there can also be no edge (a,b) with $a \notin Q(z)$, $b \in Q(z)$ unless $b = t$. Since $b \in Q(z)$, and clearly $b \neq s$, there exists a lower bound $\neq s$ for b and z . Since $a \notin L(z)$, we have $b \notin L(z)$, so that b is not an inf of b and z; let $q \neq s,b$ be such an inf. Clearly $q \in Q(z)$. Consider three paths

$$\rho_1(s,t) = \rho(s,q) + \rho(q,z) + (z,t)$$
$$\rho_2(s,t) = \rho(s,a) + (a,b) + \rho(b,t)$$
$$\rho_3(q,b)$$

For any $r \in \rho_3$ except q , we cannot have $r \in \rho(s,q)$ or $r \in \rho(q,z)$, since r would then be a lower bound for b and z , contradicting the maximality of q . Also $r \neq t$ since $b \neq t$; hence $\rho_3 \cap \rho_1 = \{q\}$. Moreover, for any $r \in \rho_3$ except b we cannot have $r \in \rho(s,a)$ since this would imply $a \in R(q) \subseteq Q(z)$, and we cannot have $r \in \rho(b,t)$ since G is acyclic; hence $\rho_3 \cap \rho_2 = \{b\}$, so that ρ_1, ρ_2, ρ_3 constitute a cross-connection, contradiction.

Let $G_1 \equiv (P_1,E_1) = [Q(z)]$, $G_2 \equiv (P_2,E_2) = [P \sim Q(z) \cup \{s,t\}]$. Clearly G_1 and G_2 have s and t as terminals. Since $(s,t) \notin E$ we have $E_1 \cap E_2 = \emptyset$, $P_1 \cap P_2 = \{s,t\}$. By the preceding paragraph, $E_1 \cup E_2 = E$, and clearly $P_1 \cup P_2 = P$; thus G is the parallel composition of G_1 and G_2.

In case (2) since $E(t)$ has a lower bound $\neq s$, we have $p = \inf E(t) \neq s$. We shall show that $a \notin R(p)$, $b \in R(p)$ and $(a,b) \in E$ implies $b = p$. Clearly $b \neq t$, for otherwise $a \in E(t)$ so that $a \in R(p)$ (since p is a lower bound), contradiction. If $E(t) \subseteq R(b)$, then b is also a lower bound, and since $b \in R(p)$ where p is a maximal lower bound, we have $b = p$, and we are done. Otherwise, we have $z \notin R(b)$ for some $z \in E(t)$. Since b,z are in $R(p)$, they have p as a lower bound, hence they have a maximal lower bound q in $R(p)$, where $q \neq b$. Consider three paths

$$\rho_1(s,t) = \rho(s,a) + (a,b) + \rho(b,t)$$

$$\rho_2(s,t) = \rho(s,p) + \rho(p,q) + \rho(q,z) + (z,t)$$

$$\rho_3(q,b)$$

For any $r \in \rho_3$ except b we have $r \in R(q) \subseteq R(p)$. Hence $r \notin \rho(s,a)$, since we would then have $a \in R(r) \subseteq R(p)$, and $r \notin \rho(b,t)$, since $b \in R(r)$ and G is acyclic. Thus $r \notin \rho_1$, so that $\rho_3 \cap \rho_1 = \{b\}$, and in particular $q \notin \rho_1$, so that $\rho_2 \neq \rho_1$. Moreover, if any $r \in \rho_3$ except q were on ρ_2, then r would be a lower bound of b and z, contradicting the maximality of q (we cannot have $r = t$ since G is acyclic). Thus $\rho_3 \cap \rho_1 = \{q\}$, and ρ_1, ρ_2, ρ_3 constitute a cross-connection, contradiction.

Since G is acyclic, $L(p) \cap R(p) = \{p\}$. Also, if $w \notin R(p)$ there is a path ρ from w to $t \in R(p)$, so that a point not in $R(p)$ and a point in $R(p)$ must occur consecutively on ρ; but by the preceding paragraph, the second of these points can only be p, so that there is a path from w to p. Thus $w \notin R(p)$ implies $w \in L(p)$. Let $G_1 = [L(p)]$, $G_2 = [R(p)]$. Since G_1 and G_2 are subgraphs of G, they are acyclic and CCF; and as we have just seen, G_1 has the two terminals s and p, and G_2 has the two terminals p and t. Thus

by induction hypothesis, G_1 and G_2 are TTSPN's. Moreover, there are no edges from points of G_1 to points of G_2 , so that G is the serial composition of G_1 and G_2 , proving that G is a TTSPN.//

As a useful application of Theorem 3.12 we have

Proposition 3.13. A finite, two-terminal acyclic graph is a TTSPN if and only if any walk between its terminals is a path.

Proof: "Only if" is clear if G has only one edge. If G is the parallel composition of G_1 and G_2 , then readily any walk between the terminals s and t must be entirely contained in either G_1 or G_2 , so that the induction hypothesis applies immediately. In the serial composition case, any walk between s and t must pass through the common point z of G_1 and G_2 , and by minimality it can only pass this point once. Hence we can break it up into two walks $w(s,z)$ and $w(z,t)$. By induction hypothesis, each of these is a path, and readily this implies that the original walk is also a path.

Conversely, it suffices to show, by the proof of Theorem 3.12, that there is no cross-connection between any two paths from s to t . Suppose ρ_1, ρ_2, ρ_3 were such a cross-connection, say $\rho_1 = \langle x_1, \ldots, x_m \rangle$, $\rho_2 = \langle y_1, \ldots, y_n \rangle$, and $\rho_3 = \langle z_1, \ldots, z_r \rangle$ where $\rho_3 \cap \rho_1 = \{x_i\} = \{z_1\}$, $\rho_3 \cap \rho_1 = \{y_j\} = \{z_r\}$. If $\{x_{i+1}, \ldots, x_m\} \cap \{y_1, \ldots, y_{j-1}\} = \emptyset$, then $y_1, \ldots, y_{j-1}, y_j = z_r, \ldots, z_1 = x_i, x_{i+1}, \ldots, x_m$ is a walk, since its points are all distinct and is not a path, contradiction. Otherwise, let y_h be the last of $y_1, \ldots, y_{j-1}$ that is equal to any of $x_{i+1}, \ldots, x_m$, say to x_k . We cannot have $y_u = x_v$ for any $h \le u \le j-1$ and $1 \le v \le i-1$, since $x_v, \ldots, x_k = y_h, \ldots, y_u = x_v$ would then be a cycle. Hence $x_1, \ldots, x_{i-1}, z_1, \ldots, z_r, y_{j-1}, \ldots, y_h = x_k, \ldots, x_m$ is a walk, since its points are all distinct, and is not a path, contradiction.//

Note that in any two-terminal graph, no walk between the terminals can have reversal order exactly 2.

3. Basic graphs

When considering a graph as representing a transitive relation, some edges may be superfluous. Given a graph we can safely delete these edges without altering the relation. We call a graph without these "extra" edges a basic graph.

Definition. A graph G is basic if it has no proper partial subgraph H on the same point set such that $\rho_G(x,y)$ implies $\rho_H(x,y)$ for all x,y .

It is easily seen that G is basic if and only if for any edge (a,b) , the points a and b occur consecutively on any path from a to b. Indeed, if this condition is violated by some path $\rho(a,b)$, the edge (a,b) can be deleted from G without changing the path structure, since one can use ρ to get from a to b without using (a,b) . Conversely, if G is not basic, let H be a proper partial subgraph with the same path structure as G . Since H is proper, some edge (u,v) of G is not an edge of H . But (u,v) constitutes a path from u to v in G ; hence there is a path ρ' from u to v in H , which cannot have u and v as consecutive points since (u,v) is not an edge of H . Since H is a partial subgraph, ρ' is also a path in G.

If G is acyclic, a and b cannot occur at all on a path from a to b except as the endpoints; hence they cannot occur consecutively if the path has length ≥ 2. We thus have

Proposition 3.14. An acyclic graph $G = (P,E)$ is basic if and only if $|\rho(x,y)| \geq 2$ implies $(x,y) \notin E$ for all $x,y \in P$.

40

Corollary 3.15. An acyclic graph is basic if and only if $(x,y) \in E$ implies $R(x) \cap L(y) = \{x,y\}$.

Proof: If $a \in R(x) \cap L(y)$ with $x \neq a \neq y$, then $\rho(x,a)$ and $\rho(a,y)$, so that $\rho(x,y)$ with $|\rho| \geq 2$, and conversely.//

Proposition 3.16. G is basic and acyclic if and only if every edge is a separating edge of order 1.

Proof: Suppose G has an edge which is not a separating edge of order 1, in other words, an edge (x,y) such that $r(x,y) \leq 1$ in $G' = G^\sim\{(x,y)\}$. This implies $x \in L(y) \cup R(y)$. If $x \in L(y)$ then G is non-basic; if $x \in R(y)$ then G is non-acyclic.

Conversely, if there is a cycle $\rho(x,x)$ then any edge (a,b) on it would have $r_{G'}(a,b) = 1$; contradiction. If G is non-basic and acyclic there is a path $\rho(x,y)$ with $|\rho| \geq 2$ and an edge (x,y), so that the edge (x,y) is not a separating edge of order 1.//

Definition. By a basis graph G^b of G [Ore (1962)] we mean any basic partial subgraph of G that has the same point set as G. An arbitrary graph G need not have a basis graph, and if a basis graph exists, it need not be unique; see Figures 3.6 and 3.7.

It can be easily shown that for finite graphs G, a basis graph always exists. This is done by methodically deleting edges for which there exists a path of length ≥ 2 between the endpoints. We will show that for finite acyclic graphs, the basis graph is unique [Ore (1962)]. Additional conditions which are sufficient to show the existence of a basis graph are given in Chapter 8 of Ore (1962).
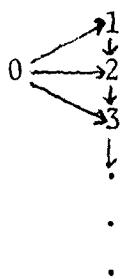
Proposition 3.17. A finite acyclic graph has a unique basis graph.

Proof: Let $H_1 = (P,E_1)$ and $H_2 = (P,E_2)$ be two basis graphs of $G = (P,E)$. We need only show $E_1 = E_2$. Let $e = (x,y) \in E_1 \subseteq E$.

G:

G₁:

G₂:
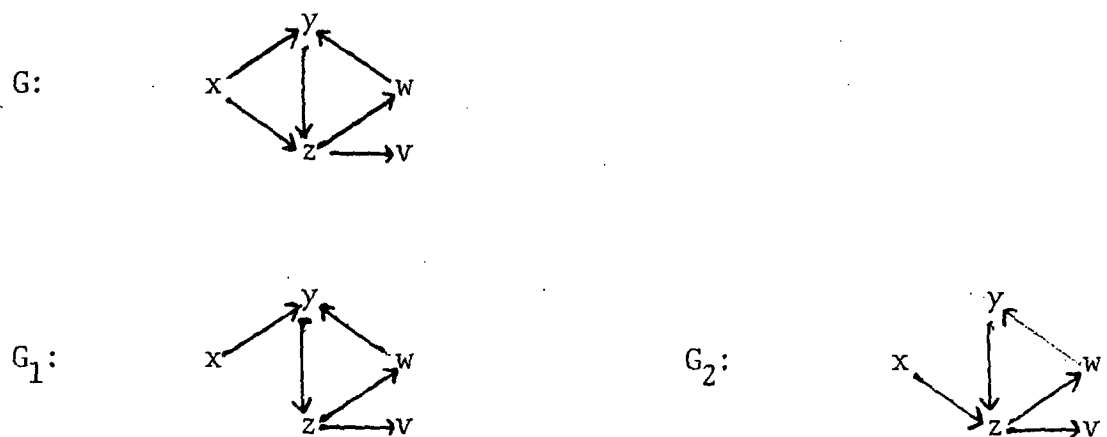
G_1 and G_2 are both basis graphs of G

Figure 3.6.

$G = (P,E)$

$P = \{0,1,2,\ldots\}$

$E = \{(0,n)\,|\,n = 1,2,3,\ldots\}$

$\cup\ \{(n,n\text{-}1)\,|\,n = 2,3,\ldots\}$

G has no basis graph

Figure 3.7.

Then e constitutes a path from x to y in G , and since H_2 is basic there exists $\rho_2(x,y)$ in H_2. If $\rho_2(x,y)$ consists of e alone, then $e \in E_2$ and we are done. Otherwise, there exists a path from x to y in H_2 , hence in G , through some point $z \neq x,y$, so that there exist paths $\rho_3(x,z)$ and $\rho_4(z,y)$ in G . Since H_1 is basic, we must thus have $\rho_5(x,z)$ and $\rho_6(z,y)$ in H_1 , which combine to give $\rho_1(x,y)$ of length ≥ 2 in H_1 , contradicting $e \in E_1$. Thus $E_1 \subseteq E_2$. The reverse inclusion is shown similarly.//

Definition. A graph $G^T = (P,E^T)$ is said to be the transitive closure of $G = (P,E)$ if $\rho_G(x,y)$ is equivalent to $(x,y) \in E^T$.

Proposition 3.18. Let $G_1 = (P,E_1)$, $G_2 = (P,E_2)$; then $\tau_L(G_1) = \tau_L(G_2)$ if and only if $G_1^T = G_2^T$.

Proof: Let $A \in \tau_L(G_1)$, $p \in A$, and $(q,p) \in E_2$. Then $(q,p) \in E_2^T = E_1^T$; hence $\rho(q,p)$ in G_1 , so that $q \in L(p) \subseteq A$ by Theorem 1.2. This shows $A \in \tau_L(G_2)$, and thus $\tau_L(G_1) \subseteq \tau_L(G_2)$; the reverse inclusion is proved similarly. Conversely, if $\tau_L(G_1) = \tau_L(G_2)$ we have, for all $x,y \in P$, $\rho(x,y)$ in G_1 iff. $x \in L_{G_1}(y)$ iff. $x \in L_{G_2}(y)$ iff. $\rho(x,y)$ in G_2 , proving $G_1^T = G_2^T$ .//

Corollary 3.19. Let $G = (P,E)$; then the largest graph on P with topology $\tau_L(G)$ is G^T .

Proposition 3.20. $G = (P,E)$ is basic if and only if no proper partial subgraph (P,E_H) of G has the same topology as G .

Proof: If (P,E_H) had the same topology, then we would have $\rho_G(x,y)$ implies $x \in L_G(y) = L_H(y)$ implies $\rho_H(x,y)$, contradiction. Conversely, if ρ_G implies ρ_H , then ρ_G iff. ρ_H , so that $L_G(y) = L_H(y)$ for all y , implying $\tau_L(G) = \tau_L(H)$ .//

Thus a basic graph $G = (P,E)$ is the same as a minimal graph on P having topology $\tau_L(G)$. In particular, in the finite acyclic case, there is exactly one such minimal graph, i.e., there is a smallest such graph.

As a consequence of the previous propositions, we see that an E-topology characterizes a family of graphs which are bounded above by their common transitive closure and bounded below by a set of basic graphs.

The basis graph of a given graph can be obtained in still another way as is described in the following proposition.

Proposition 3.21. Let $G = (P,E)$ be an acyclic graph that has a basis graph G^b , and let $G^a = (P^a, E^a)$ be defined by $P^a = \{L(x) \mid x \in P\}$ and $E^a = \{(L(x), L(y)) \mid (x,y) \in E$ and there exists no z such that $L(x) \subsetneq L(z) \subsetneq L(y)\}$. Then G^a is isomorphic to G^b .

Proof: Let $G^b = (P, E^b)$ and $f:P \to P^a$ where $f(p) = L(p)$. If $(p,q) \in E^b$ we have no $\rho(p,q)$ in G^b such that $|\rho| \geq 2$. Clearly $L(p) \subseteq L(q)$. If $L(p) \subsetneq L(r) \subsetneq L(q)$ then r is on a path from p to q which thus has length ≥ 2 , contradiction; thus $(L(p), L(q)) \in E^a$.

Conversely, if $(L(p), L(q)) \in E^a$, we have $(p,q) \in E$. Suppose there were $\rho(p,q)$ in G^b such that $|\rho| \geq 2$. Then there would exist $r \in \rho(p,q), r \neq p,q$ and therefore $L(p) \subseteq L(r) \subseteq L(q)$. Since $(L(p), L(q)) \in E^a$, we must have $L(p) = L(r)$ or $L(q) = L(r)$; both cases are impossible since G is acyclic. This shows $|\rho| < 2$ or $(p,a) \in E^b$. Moreover, f is 1:1 since G is acyclic. Hence f is an isomorphism.//

CHAPTER IV

CONTINUOUS MAPS

1. Continuous maps and homomorphisms

A key concept in the development of mathematics is that of a function, map, or transformation. In a functional approach one begins with a space S of objects, then defines transformations which may the space either into itself or into some new space S'. In general one is then interested in:

a. those transformations which are well-behaved, for example preserve some property of the original space

or b. those spaces which are well-behaved under particular classes of transformations.

There are two natural ways to establish a functional approach to graph theory (which makes the extreme paucity of any literature on functional graph theory rather surprising). The first is to construct a space in which the objects of interest are individual graphs. Such a set of graphs might be organized as a space by establishing a notion of similarity or distance between individual graphs. One would then have essentially a single space of graphs, which might be organized in different fashions depending on one's definition of the concept of similarity. This would be analogous to the space of real numbers, which can be organized according to the usual metric topology, the half open interval topology, or various other "irregular" topologies.

The second approach is to regard each individual graph as a space. In this approach the points of the graph P are the objects of the space and it is the relation E that determines their organization. It is this latter approach that we take here. It appears to be the more

basic of the two approaches; in fact, it may be essential as a prerequisite to the definition of similarity between graphs. Furthermore, the entire development of the edge topologies τ_L and τ_R, together with the exploration of their implications with respect to separability and connectivity, has been built up with this view in mind.

We therefore will consider maps that can be defined on many different graphs (spaces) although they may be of greatest interest when restricted to a specific class of graphs. In general the range graph (space) will be distinct from the domain graph. We will interchangeably use the notations $f:G \to G'$ and $f:P \to P'$ to denote such maps, since usually E' (the organization of G') is determined by f and E.

The first class of maps to consider is, of course, those which are continuous (in the usual sense) with respect to τ_L and τ_R. This will turn out to be a very large class; for example, all graph homomorphisms (under any of the half-dozen different possible definitions) are continuous. For this reason we will investigate a more restrictive class of functions, called ideal maps, in Section 4.

Relative to the topologies $\tau_L(G)$ and $\tau_L(G')$ we can define a continuous map in the usual way, namely,

Definition. $f:G \to G'$ is a <u>continuous map</u> if
$A' \in \tau_L(G')$ implies $f^{-1}(A') \in \tau_L(G)$.

For our definition of a homomorphism between graphs, we shall use a stronger version [Pfaltz (1968,1971)] then the one found in Ore (1962).

Definition. The mapping $f:G \to G'$ is called a graph homomorphism if
(1) $(a,b) \in E$ implies $(f(a), f(b)) \in E'$
  and
(2) $(a', b') \in E'$ implies there exist $a \in f^{-1}(a')$ and

$b \in f^{-1}(b')$ such that $(a,b) \in E$.

A function satisfying only condition (1) will be called a <u>weak homomorphism</u>. Both definitions are common in the literature under the name "homomorphism". For most of our results we need not distinguish between the two variants.

Our first proposition shows that a continuous function is a weaker concept than a weak homomorphism. Thus, continuous maps are a very general class of functions which relate graphs.

<u>Proposition 4.1.</u> Weak homomorphisms are continuous.

<u>Proof</u>: Let $A' \in \tau_L(G')$ and $A = f^{-1}(A')$. If $a \in A$ and $(p,a) \in E$, we have $(f(p), f(a)) \in E'$ since f is a homomorphism. Since $A' \in \tau_L(G')$ and $f(a) \in A'$, we thus have $f(p) \in A'$. This implies $p \in A = f^{-1}(A')$ and therefore $A \in \tau_L(G)$ .//

The graph in Figure 4.1 shows the converse of Proposition 4.1 is not true. For this continuous map f we have $(c,b) \in E$ but $(f(c), f(b)) = (c', b') \notin E'$ and thus f is not a weak homomorphism.

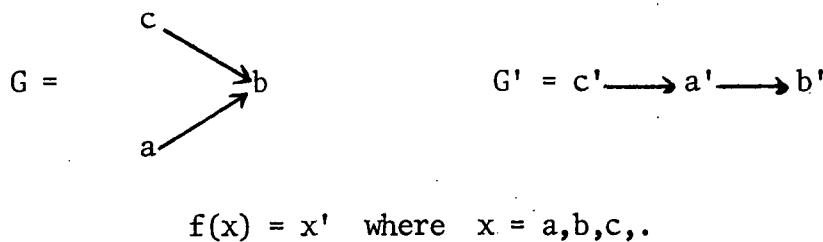$$f(x) = x' \quad \text{where} \quad x = a,b,c,.$$

Figure 4.1.

Continuous maps with respect to the topologies $\tau_R(G)$ and $\tau_R(G')$ can be defined similarly. However, the class of functions so defined is the same as for the τ_L topologies. That is, a function is continuous

with respect to $\tau_R(G)$ and $\tau_R(G')$ if and only if it is continuous with respect to $\tau_L(G)$ and $\tau_L(G')$. This is evident since the τ_L and τ_R topologies of a graph are complements of each other, and one can invoke the well-known equivalent condition for continuity that the complements of elements of $\tau_L(G')$ are mapped onto complements of elements of $\tau_L(G)$ under f^{-1} .

We now characterize continuous maps by examining the preservation of the path structure. This characterization provides a convenient method for determining which maps are continuous.

Proposition 4.2. $f: G \to G'$ is continuous if and only if $\rho_G(p,q)$ implies $\rho_{G'}(f(p), f(q))$.

Proof: Let $\rho(p,q)$ be a path in G. Since f is continuous and $L(f(q))$ is an element of $\tau_L(G')$, the set $A = f^{-1}(L(f(q)))$ is an element of $\tau_L(G)$. Having $q \in A$ and $\rho(p,q)$, we find $p \in A$ and $f(p) \in f(A) = L(f(q))$, which implies $\tilde{\rho}(f(p), f(q))$. Conversely, suppose $\rho(p,q)$ implies $\tilde{\rho}(f(p), f(q))$. Let $A' \in \tau_L(G')$ and define $A = f^{-1}(A')$. If $p \in A$ and $(q,p) \in E$, then $\rho(q,p)$ and therefore $\tilde{\rho}(f(q), f(p))$. Since $A' \in \tau_L(G')$ and $f(p) \in A'$, we must have $f(q) \in A'$. Thus $q \in f^{-1}(A') = A$ and $A \in \tau_L(G)$ .//

Corollary 4.3.   The map f is continuous if and only if $r(f(a), f(b)) \leq r(a,b)$ for all a,b.

When f in Proposition 4.2 is not onto then the path $\tilde{\rho}(f(p), f(q))$ may not be in the range of the mapping. Consider the example in Figure 4.2. The map f is continuous but $\tilde{\rho}(f(a), f(b))$ is not in the range of f .

$$G = a \longrightarrow b \qquad\qquad G' = a' \longrightarrow x' \longrightarrow b'$$

$$f(a) = a', \ f(b) = b' \qquad \tilde{\rho}(a', b') = a', x', b'$$

Figure 4.2.

Although Proposition 4.2 gives a useful test for continuity in terms of paths, the left and right sets defined in Chapter I have played the central role in our development and, in effect, have replaced the notion of path. Thus, a suitable characterization of continuity in terms of these sets would be desirable.

<u>Proposition 4.4.</u> Let f be a map from G to G'. The following are equivalent:

(1) f is continuous

(2) $f(L(a)) \subseteq L(f(a))$ for all a

(3) $f(R(a)) \subseteq R(f(a))$ for all a

<u>Proof</u>: If $b' \in f(L(a))$, then there exists $a_1 \in L(a)$ such that $f(a_1) = b'$. We have $\rho(a_1, a)$, and therefore $\tilde{\rho}(f(a_1), f(a))$ in G' since f is continuous. Hence $b' = f(a_1) \in L(f(a))$, proving (2). Conversely, if (2) holds, let $A' \in \tau_L(G')$ and $A = f^{-1}(A')$. If $(a,b) \in E$ with $b \in A$ then $a \in L(b)$ and consequently $f(a) \in f(L(b)) \subseteq L(f(b))$. Since $A' \in \tau_L(G')$, we thus have $f(a) \in L(f(b)) \subseteq L(f(A)) = L(a') = A'$. Therefore $a \in f^{-1}(A') = A$ and $A \in \tau_L(G)$. The proofs for (3) are analogous.//

<u>Corollary 4.5.</u> If f is continuous, we have $f(C^n(A)) \subseteq C^n(f(A))$ for all A and all n.

2. Convex and open maps

A mapping on a topological space is said to be open if the map takes open sets in the domain onto open sets in the range. For our situation two topologies are defined on the domain and range graphs, namely the left and right topologies. In addition, by Proposition 1.12, convex subgraphs are closely related to left and right open sets; thus we define convex maps here also.

Definition. A map $f: G \to G'$ is

    (1) left open if $A \in \tau_L(G)$ implies $f(A) \in \tau_L(G')$

    (2) right open if $B \in \tau_R(G)$ implies $f(B) \in \tau_R(G')$

    (3) convex if C convex in G implies $f(C)$ is convex in G'.

It is necessary to distinguish between left open and right open maps since they do not imply one another (see Figure 4.3). This is not the case for continuity; "left" continuity implies "right" continuity and conversely. When a map is both left and right open, we will call the map open.

The examples in Figure 4.3 show the independence of these concepts.

Continuous maps were characterized in Proposition 4.4; the next proposition gives a dual characterization of left and right open maps.

Proposition 4.6. The map f is

    (1) left open if and only if

$$L(f(a)) \subseteq f(L(a)) \text{ for all } a$$

    (2) right open if and only if

$$R(f(a)) \subseteq f(R(a)) \text{ for all } a.$$

Proof: If f is left open, $f(L(a)) \in \tau_L(G')$ since $L(a) \in \tau_L(G)$; but $f(a) \in f(L(a))$, so we have $L(f(a)) \subseteq f(L(a))$. Conversely, if A is
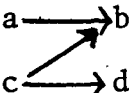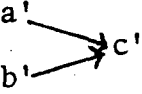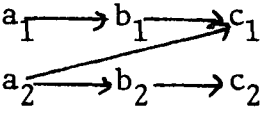
Example	G	G'	Remarks
1	a ——→ b / c ——→ d	a' ——→ b' / c' ⨯ d'	continuous, convex; not left or right open.
2	a ↘ / c ↗ b	a' ——→ b' = c'	continuous, right open not left open.
3	a ↗ b / ↘ c	a' = b' ——→ c'	continuous, left open; not right open
4	a ——→ b ——→ c	a' ↘ / b' ↗ c'	convex, open; not continuous
5	$\overset{o}{a}$ $\overset{o}{b}$	a' ——→ c' ——→ b'	continuous; not convex, not left or right open.
6	a_1 ——→ b_1 ——→ c_1 / a_2 ——→ b_2 ——→ c_2	a' ——→ b' ——→ c'	open, continuous, not convex.

Figure 4.3.

left open, we have $A = U_{a \in A} L(a)$. Hence $f(A) = U_{a \in A} f(L(a)) \supseteq$

$U_{a \in A} L(f(a)) = U_{b \in f(A)} L(b) = L(f(A))$; and $f(A) \supseteq L(f(A))$ implies $f(A)$

left open.//

As an immediate consequence of this proposition we have

Corollary 4.7. If f is open, $C^n(f(A)) \subseteq f(C^n(A))$ for all A and

all $n \geq 0$.

If f is left open, in particular f(P) is left open, so that

P' ~ f(P) is left closed, and similarly on the right. It follows that

if f is open, f(P) is both open and closed, hence is a union of con-

nected components of G'. We thus have

Proposition 4.8. If G' is connected, an open map is onto.

If f is one-to-one onto and left open, its inverse function is

continuous, and thus preserves paths, and similarly for f one-to-one

onto and right open. Conversely, if f is one-to-one onto, and its

inverse preserves paths, then f is both left and right open, i.e., is

open. The following two propositions generalize these remarks to the

case where f is not necessarily one-to-one.

Proposition 4.9. Let f map G onto G'; then the following

statements are equivalent:

1) f is left open

2) For all a',b' $\in$ G' such that $\rho(a',b')$, and all b $\in$ $f^{-1}(b')$,

there exists a $\in$ $f^{-1}(a')$ such that $\rho(a,b)$.

Proof: If (1) holds, then since a' $\in$ L(b') = L(f(b)) $\subseteq$ f(L(b)),

there exists a $\in$ L(b) such that f(a) = a', proving (2). Conversely,

(2) says that for all a' $\in$ L(f(b)) there exists a $\in$ L(b) such that

f(a) = a', i.e., a' $\in$ f(L(b)), proving f left open.//

Analogously we have

Proposition 4.10. Let f map G onto G'; then the following

statements are equivalent:

1) f is right open

2) For all a',b' $\in$ G' such that $\rho(a',b')$, and all a $\in$ f^{-1}(a'),

there exists b $\in$ f^{-1}(b') such that $\rho(a,b)$.

By Proposition 4.2, continuous maps preserve the path relationship

of the domain in the range. In essence, Proposition 4.9 shows that open

maps preserve the path relationship of the range in the domain. It is

also possible to have non-onto maps which preserve this relationship.

For example, consider the graphs in Figure 4.4; none of them are left

or right open. Nevertheless, all these maps seem to be one-to-one onto

a range whose path structure is reflected in the domain. A relaxed

condition for openness which these maps satisfy is the following:

Definition. A map f:G $\to$ G' is <u>relatively</u> <u>left</u> [<u>right</u>] <u>open</u> if

A $\in$ τ_L(G) implies f(A) $\in$ $\tau_L \mid f(G)$ [f(A) $\in$ $\tau_R \mid f(G)$]. A <u>relatively</u> <u>open</u>

map is both relatively left and right open.

We emphasize that the topology used in the range is the relativized

topology of the image subgraph. Evidently every left (right) open map

is relatively left (right) open. Another concept could also be defined

by considering the $\tau_L(f(G))$ and $\tau_R(f(G))$ topologies. This definition

of openness is equivalent to relative openness when the image of G is

convex in G'. All the maps depicted in Figure 4.4 are relatively open.

Figure 4.5 illustrates a map that is not relatively left open.

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Example	G	G'	Remarks
1	a ⟶ b	a' ⟶ x' ⟶ b'	
2	a ⟶ b ⟶ c	a' ⟶ b' ⟶ c' ↗ x'	not convex
3	a ⟶ b, a ↗ c, d ⟶	a' ⟶ x' ⟶ b', a' ↗ c', d'	
4	a ⟶ b ⟶ c	a' ↘ x' ↘ c', b' ⟶ y' ↗	not continuous not convex

Figure 4.4.

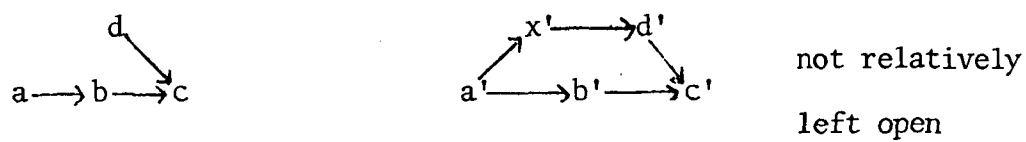a ⟶ b ⟶ c, d ↘   a' ⟶ b' ⟶ c', x' ⟶ d'   not relatively
left open

Figure 4.5.

3. Homomorphisms and isomorphisms

As a consequence of the definition of continuity, we have a well defined notion of homeomorphism [Kelley (1955)], namely, one-to-one map which is continuous and whose inverse is continuous.

Definition. A map $f: G \to G'$ is an isomorphism if f is one-to-one and $(a,b) \in E$ if and only if $(f(a), f(b)) \in E'$.

It is well known that f is an isomorphism if and only if f is a one-to-one graph homomorphism. Graphs G and G' are called isomorphic if there exists an isomorphism between them. We observe that an isomorphism is also a homomorphism, and hence is a homeomorphism.

A homeomorphism need not in general be a homomorphism: but it must be one when the graphs involved are basic and acyclic. Specifically, we have

Proposition 4.11. If the map $f: G \to G'$ is a homeomorphism onto, and G is basic and acyclic, then f is a weak homomorphism; if G' is also basic and acyclic, f is an isomorphism.

Proof: Let $(a,b) \in E$; then $\rho_1(a,b)$ in G, so that $\rho_2(f(a), f(b))$ in G_2 by the continuity of f. If we assume $|\rho_2| \geq 2$ then there is a point p' on ρ_2 with $\rho_3(f(a), p')$, $\rho_4(p', f(b))$, where $f(a) \neq p' \neq f(b)$. Since f^{-1} is continuous and one-to-one we thus have $\rho_5(a, f^{-1}(p'))$ and $\rho_6(f^{-1}(p'), b)$ with $a \neq f^{-1}(p') \neq b$. This contradicts the basicness of G, since $\rho_5(a, f^{-1}(p')) + \rho_6(f^{-1}(p'), b)$ is a path from a to b with length greater than 1. Therefore $|\rho_2| < 2$, i.e., $(f(a), f(b)) \in E'$, which shows that f is a weak homomorphism. The second part of the theorem follows analogously, interchanging the roles of G' and G .//

Corollary 4.12. Let G^b and G'^b be the basis graphs of G and G', respectively, where G and G' are acyclic; then f is an isomorphism between G^b and G'^b.

Proof: We can regard f as a one-to-one map of G^b onto G'^b. Moreover, $\rho(x,y)$ in G iff. $\rho(x,y)$ in G^b, and similarly for G' and G'^b; hence f is still bicontinuous.//

In Proposition 4.11 G' need not be basic or acyclic even though G is. It is of interest to consider conditions under which basic acyclic graphs are mapped into basic acyclic graphs. By Proposition 7.7 in Pfaltz (1968), a convex homomorphism g takes acyclic graphs into acyclic graphs. Moreover, using Pfaltz's Proposition 7.10, we can show that if g is onto, it takes acyclic basic graphs into acyclic basic graphs.

The one-to-one onto requirement for a topological homeomorphism is a severe restriction in the case of finite graphs. One really wants a notion of homeomorphism between graphs of different cardinality. In the graph literature one therefore finds a different concept of homeomorphism, which we shall now define; first we need the concept of a subdivision.

Definition. $G' = (P',E')$ is a subdivision of $G = (P,E)$ if

1) $P' = P \cup \{x\}$ where $x \notin P$, $E' = E \cup \{(a,x), (x,b)\} - \{(a,b)\}$, where $(a,b) \in E$ (in this case, we call G' an elementary subdivision),

or 2) G' is a subdivision of another subdivision of G.

Following Tutte (1966) and Harary (1969) we now have

Definition. Two graphs G and G' are graph-homomorphic if G and G' have subdivisions which are isomorphic.

If $\tilde{G}$ is an elementary subdivision of G, the identity map $\tilde{s}$ from

G into $\tilde{G}$ is one-to-one, continuous, and relatively open. Similarly, we can define a canonical map $\tilde{t}$ from $\tilde{G}$ onto G, as the identity map on the points of G and as taking x into a, and readily $\tilde{t}$ is continuous and left open. It follows that if $\tilde{G}$ is an arbitrary subdivision, the composite s of the $\tilde{s}$ maps is one-to-one, continuous, and relatively open, and the composite t of the $\tilde{t}$ maps is onto, continuous, and left open. Let f be an isomorphism of $\tilde{G}$ with a subdivision $\tilde{G}'$ of G', and let t' be the canonical map from $\tilde{G}'$ onto G' as just defined; then

$$t \circ f \circ s$$

is a continuous, relatively left open map from G into G'; but G and G' are not necessarily homeomorphic.

4. Ideal Maps

If a graph G is acyclic, so that the path relation ρ is a partial ordering, then the left or right ideal generated by a set A can be defined. Clearly the left ideal generated by a set A is simply L(A) and the right ideal is R(A) . By Propositions 4.4 and 4.6, open, continuous maps preserve left and right ideals. Thus we call an open, continuous map an _ideal map_. The remainder of this thesis will be devoted to the study and application of these ideal maps. By Propositions 4.4 and 4.6 we also have immediately

Proposition 4.13. The map $f:G \to G'$ is ideal if and only if $f(L(p)) = L(f(p))$ and $f(R(p)) = R(f(p))$ for all points p of G .

Example   G   G'

(1)   a —→ b ——→ c   a'⟶b' ——⟶c'

(2)

$$a_1 \longrightarrow b_1 \longrightarrow c_1$$

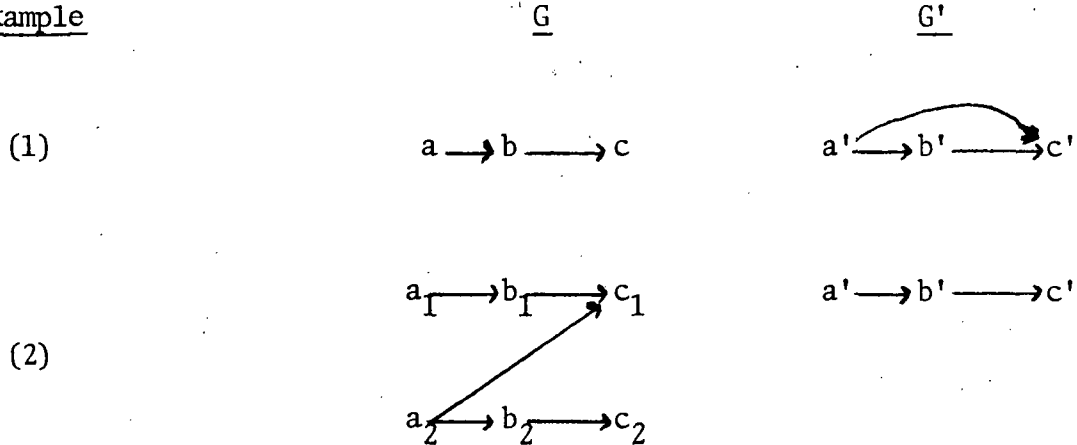$$a_2 \longrightarrow b_2 \longrightarrow c_2$$

a' ⟶ b' ——→ c'

Figure 4.4

The examples in Figure 4.4 show that an ideal map need not be a homo-morphism or even a weak homomorphism.

Corollaries 4.5 and 4.7 show that if f is ideal, then $C^n(f(A)) = f(C^n(A))$ for all A and all n. It follows that $C^\infty(f(A)) = f(C^\infty(A))$. We also have

Proposition 4.14. Let $f: G \to G'$ be ideal and onto, let a', $b' \in G'$ with $r(a',b') = n$, and let $b \in f^{-1}(b')$. Then there exists $a \in f^{-1}(a')$ with $r(a,b) = n$.

Proof: We have $a' \in C^n(f(b)) = f(C^n(b))$, so that there exists $a \in f^{-1}(a')$ with $a \in C^n(b)$, which implies $r(a,b) \leq n$. On the other hand, $r(a,b) < n$ would imply $r(a',b') < n$ by Proposition 4.3.//

Propositions 4.9-10 show that under an onto ideal map, paths have "inverses". A stronger result is

Proposition 4.15. Let $f: G \to G'$ be ideal and onto, let G' be basic and acyclic, and let ρ' be any path in G'. Then there exists

a path $\bar{\rho}$ in G such that $r \in \bar{\rho}$ implies $f(r) \in \rho'$ and $r' \in \rho'$ implies $f^{-1}(r') \cap \bar{\rho} \neq \emptyset$. In fact, we can require $\bar{\rho}$ to begin at a specified preimage of the first point of ρ', or to end at a specified preimage of its last point. .

   Proof: Let $\rho' = \langle p'_0, \ldots, p'_n \rangle$, and apply Proposition 4.9 (or 4.10) to the edges (p'_k, p'_{k+1}) , $0 \leq k < n$, to obtain paths $\rho(p_k, p_{k+1})$ in G , where $f(p_k) = p'_k$ for each k . Let $\bar{\rho} = \rho_0 + \ldots + \rho_{n-1}$; clearly any $r' \in \rho'$ has a preimage in $\bar{\rho}$. Conversely, let $r \in \bar{\rho}$, say $r \in \rho_k$; then $\rho(p_k, r)$ and $\rho(r, p_{k+1})$, and since f is continuous, this implies $\rho(f(p_k), f(r))$ and $\rho(f(r), f(p_{k+1}))$ in G' , i.e., $\rho(p'_k, f(r))$ and $\rho(f(r), p'_{k+1})$. But (p'_k, p'_{k+1}) is an edge, and G' is basic and acyclic; hence we cannot have $p'_k \neq f(r) \neq p'_{k+1}$ -- in other words, we must have $f(r) = p'_k$ or p'_{k+1}, so that $f(r) \in \rho'$ .//
Clearly if f is one-to-one and ideal, then f^{-1} is ideal, and f is a homeomorphism. Proposition 4.15 suggests that even if f is not one-to-one, the path topologies are still "similar".

   Corollary 4.16.   In Proposition 4.15, let (a', b') be any edge of G' , and let $f(\bar{b}) = b'$; then there exists an edge (a, b) of G such that $f(a) = a'$ and $\rho(b, \bar{b}) \subseteq f^{-1}(b')$.

   Proof: Let $\bar{\rho}(a, \bar{b})$ be a path in G constructed from $\rho' = \langle a', b' \rangle$ as in Proposition 4.15; then $f(x) = a'$ or b' for all $x \in \bar{\rho}$, and $f(a) = a'$. Hence somewhere in $\bar{\rho}$ there must be two consecutive points the first of which has image a' and the second b' .//
Similarly, we can fix $\bar{a} \in f^{-1}(a')$ and find an edge (a, b) with $f(b) = b'$. Note that the conclusion of Corollary 4.16 implies one of the two defining properties of a graph homomorphism, namely that for any $(a', b') \in E'$ there exists $(a, b) \in E$ such that $f(a) = a', f(b) = b'$.

Corollary 4.17. In Proposition 4.15, let $\bar{\rho} = \langle x_o, \ldots, x_m \rangle$.
Then for all $o \leq i < m$ we have either $f(x_i) = f(x_{i+1})$ or $(f(x_i),$
$f(x_{i+1}))$ an edge in G' .

Proof: Since f is continuous we have $\rho_i' (f(x_i), f(x_{i+1}))$ in G' .
By construction of $\bar{\rho}$, we know that any r' on ρ_i' has a preimage r
on $\bar{\rho}$. Since x_i and x_{i+1} are consecutive on $\bar{\rho}$, we must have
$r \in L(x_i) \cup R(x_{i+1})$. If $r \in L(x_i)$, then $f(r) = r' \in L(f(x_i))$ by
continuity. But r' is on ρ_i' , hence in $R(f(x_i))$; hence $r' = f(x_i)$
since G' is acyclic. Similarly, if $r \in R(x_{i+1})$, we must have
$r' = f(x_{i+1})$. Thus the only points on the path ρ_i' are $f(x_i)$ and
$f(x_{i+1})$ themselves, so that ρ_i' has length ≤ 1 .//

We next generalize Proposition 4.15 from paths to walks; this result
will be needed to prove the Theorems which conclude this section.

Proposition 4.18. Let $f : G \to G'$ be ideal, onto and let G' be
basic, acyclic with $x' \neq y'$ in G' , $x \in f^{-1}(x')$. If $W' = w(x',y')$
in G' then there exists a walk $W = w(x,y)$ in G for some $y \in f^{-1}(y')$
with the properties

(1) $a \neq b$, $(a,b) \in W$ implies $(f(a), f(b)) \in W'$

(2) $a' \neq b'$, $(a',b') \in W'$ implies there is $(a,b) \in W$ such that
     $f(a) = a'$, $f(b) = b'$.

(3) $r(x',y') = n$ in W' implies $r(x,y) = n$ in W .

Proof: If $r_{w'}(x',y') = 1$ then $W' = \rho(x',y')$ or $\rho(y',x')$.
In either case by Proposition 4.15 there exists $\rho(x,y)$ or $\rho(y,x)$ in
G with $r \in \rho$ implying $f(r) \in W'$ and $r' \in W'$ implying $f^{-1}(r') \cap$
$\rho \neq \emptyset$. In particular, $r(x,y) = 1$. Clearly ρ is a minimal finite,
self-connected partial subgraph containing x and y . If $a \neq b$

occur consecutively on ρ , so that (a,b) is an edge, then since G'
is basic, $(f(a), f(b))$ must be an edge in W , proving (1). Moreover,
(2) holds by the proofs of Proposition 4.15 and Corollary 4.16.

Suppose $r_{W'}(x',y') = n$ and the proposition is true for all walks
such that $r(x',y') < n$ with respect to W' . By Corollary 2.7, there
exists $z' \in C_{W'}^{n-1}(x') \cap C_{W'}^1(y')$. Thus, $r_{W'}(x',z') = n-1 < n$ and
$r_{W'}(z',y') = 1$. By our induction hypothesis there exists $\tilde{W}$ in G
which satisfies the conditions of the theorem for $W_x' = w(x',z') \subseteq W'$.
Since $z' \in C_{W'}^1(y')$ there exists $W_y' = \rho(z',y')$ or $\rho(y',z')$ in W' .
Fixing $z \in f^{-1}(z')$ in $\tilde{W}$, by Proposition 4.9 or 4.10 we have $\tilde{\rho} =$
$\rho(z,y)$ or $\rho(y,z)$ in G for some $y \in f^{-1}(y')$. We define $\bar{W} = \tilde{W} \cup \tilde{\rho}$.
Clearly $\bar{W}$ is a finite, self-connected partial subgraph containing x
and y . Therefore there exists a minimal $W \subseteq \bar{W}$ and readily W still
contains x and y . If $a \neq b$ and (a,b) in W then $(a,b) \in \bar{\rho}$ or
$(a,b) \in \tilde{W}$, so that $(f(a), f(b)) \in W'$ by induction hypothesis. If
$a' \neq b'$ and $(a',b') \in W'$ then (a',b') in W_x' or W_y' implies there
is (a,b) as in (2) in $\tilde{W}$ or $\tilde{\rho}$, hence in $\bar{W}$. Now $f(W) \subseteq f(\bar{W}) = W'$
is a connected partial subgraph containing x' and y' . If no such
(a,b) were in W , there could be no edge (a',b') in $f(W)$, which
would contradict the minimality of W' . Finally, $r(x,z) = n-1$ in $\tilde{W}$
and $r(z,y) = 1$ in $\tilde{\rho}$, so that $r(x,y) \leq n$ in W . If $r(x,y) < n$
in W then $r(f(x), f(y)) < n$ in W' by Corollary 4.3. But $r(x',y') =$
n ; therefore $r(x,y) = n$. //

We conclude this chapter by applying the above results to prove that
ideal maps onto basic graphs preserve trees, TTSPN's, normal and com-
pletely normal graphs. (It should be noted that homomorphisms do not
preserve these properties.) We first prove

Proposition 4.19. Let $f:G \to G'$ be an onto ideal map. If G is finite and acyclic then G' is acyclic.

Proof: If G' has a cycle $\rho(x',x')$ with $|\rho| \geq 2$ then by Proposition 4.10 there is a path $\rho(x_1,x_2)$ in G with $f(x_1) = f(x_2) = x'$ and $|\rho| \geq 2$. If $x_1 = x_2$ we have a contradiction. If $x_1 \neq x_2$ we can fix x_2 and obtain another path $\rho(x_2,x_3)$ with $f(x_3) = x'$, $x_3 \neq x_1, x_2$, and $|\rho| \geq 2$. Continuing this process we have $x_1, x_2, x_3,$ ... all distinct; but G is finite, contradiction.//

Theorem 4.20. Let $f:G \to G'$ be onto ideal and G' basic. If G is a TTSPN then G' is a TTSPN.

Proof: It is evident that the terminal points s,t of G map onto terminal points s',t' of G' and that G' cannot have additional terminal points. By Proposition 4.19, since G is finite and acyclic, we have G' acyclic. By Proposition 4.19, if there were a walk between s' and t' in G' with reversal order > 1, there would also be such a walk between s and t in G, contradicting Proposition 3.13.//

Lemma 4.21. For $G = (P,E)$, and $G' = (P',E')$ basic and acyclic, let $f:G \to G'$ be an onto ideal map and let $(a',b') \in E'$, $a' \neq b'$. Define $H' = (P',E'-\{(a',b')\})$ and $H = (P,E-\{(u,v)|$ a' and b' occur consecutively on a path from $f(u)$ to $f(v)\})$. Then $f:H \to H'$ is ideal.

Proof: Let $y' \in L_{H'}(f(x))$. We have $\rho(y',f(x))$ in H', hence in G'. So by Proposition 4.18 there is a path $\rho(y,x)$ in G for some $y \in f^{-1}(y')$. If there are consecutive points u,v on $\rho(y,x)$ such that a' and b' are consecutive on a path from $u' = f(u)$ to $v' = f(v)$, we have $u' \neq v'$ and u',v' consecutive points on $\rho(y',f(x))$ by the proof of Proposition 4.15. Therefore, $u' = a'$ and

$v' = b'$ since G' is basic. Furthermore we have u', v' consecutive on $\rho(y',f(x))$; hence $(a',b') = (u',v')$ is an edge in H', contradiction. So $\rho(y,x) \subseteq H$, that is, $y \in L_H(x)$ and it follows that $f(y) = y' \in f(L_H(x))$.

Conversely, let $y' \in f(L_H(x))$. Thus, there is $y \in f^{-1}(y') \cap L_H(x)$. This implies $\rho(y,x)$ in H. If $\rho(y,x) = \langle p_0, p_1, \ldots, p_n \rangle$, we have $\rho(y',x') = \sum_{k=0}^{n-1} \rho(p_k', p_{k+1}')$ in G' by the continuity of f on G. Suppose (a',b') were on $\rho(y',x')$; then (a',b') would be on some $\rho(p_k', p_{k+1}')$. This implies (p_k, p_{k+1}) on $\rho(y,x)$ is not in H, contradiction. This shows $\rho(y',x')$ in H' and it follows that $y' \in L_{H'}(f(x))$.

A similar proof shows $R_{H'}(f(x)) = f(R_H(x))$. //

Theorem 4.22. Let $f:G \to G'$ be an onto ideal map and G' basic. If G is a tree then G' is a tree.

Proof: G' is acyclic by Proposition 4.19. Suppose $G' = (P',E')$ is not a tree; then there exists an edge (x',y') in G' which is not disconnecting. Thus we have a walk $w(x',y')$ in $H' = (P',E' - \{(x',y')\})$. Let H be defined as in Lemma 4.21. It follows that $f:H \to H'$ is an ideal map.

By Proposition 4.18 for any $x_1 \in f^{-1}(x')$ there exists a walk $w(x_1,\bar{y}_1)$ in H with $\bar{y}_1 \in f^{-1}(y')$. It is evident that H does not contain any edge (x,y) such that $x \in f^{-1}(x')$ and $y \in f^{-1}(y')$. But by Corollary 4.16, G does contain an edge (x_2,y_2) such that $x_2 \in f^{-1}(x')$, $y_2 \in f^{-1}(y')$ and the path $\rho(y_2,\bar{y}_1)$ exists. Let F_1 be the union of walk $w(x_1,\bar{y}_1)$ and the partial subgraph consisting of the points of $\rho(y_2,\bar{y}_1)$. It is clear that F_1 is weakly connected in G even if the edge (x_2,y_2) is deleted. Since $w(x,\bar{y})$ is in H and

(x_2, y_2) is not, and since $\rho \subseteq f^{-1}(y')$ so that $x_2 \notin \rho$.

In a similar manner there exists a walk $w(x_2, \bar{y}_2)$ in H . If $w(x_2, \bar{y}_2)$ and F_1 had any point in common then x_2 and y_2 would be weakly connected in G even if the edge (x_2, y_2) is deleted; thus (x_2, y_2) would not be a disconnecting edge. In particular, $x_2 \neq x_1$. We also must have an edge (x_3, y_3) such that $x_3 \in f^{-1}(x')$, $y_3 \in f^{-1}(y')$ and $\rho(y_3, \bar{y}_2)$. Let F_2 be the union of the edge (x_2, y_2) , the walk $w(x_2, \bar{y}_2)$, the path $\rho(y_3, \bar{y}_2)$, and F_1. It is evident that F_2 is weakly connected in G even if (x_3, y_3) is deleted.

Again, we must have a walk $w(x_3, \bar{y}_3)$ in H . If $w(x_3, \bar{y}_3)$ and F_2 had any point in common then x_3, y_3 would be weakly connected in G even if (x_3, y_3) is deleted; therefore the edge (x_3, y_3) would not be a disconnecting edge. In particular, $x_3 \neq x_1, x_2$. Continuing this process we obtain walks $w(x_i, \bar{y}_i)$ which do not have points in common with F_{i-1} and, hence $x_i \neq x_1, x_2, \ldots, x_{i-1}$. This yields an infinite set of distinct points $\{x_1, x_2, \ldots\}$ - a contradiction, since G is finite.//

Theorem 4.23. Let $f: G \to G'$ be onto ideal and G' basic and acyclic. If G is finite and normal then so is G' .

Proof: If G' is not normal there is an edge (x', y') which is normal but not disconnecting. In the proof of Theorem 4.22 we have shown that if an edge in G' is not disconnecting then there exists an edge in G which is not disconnecting. Thus, we have an edge (x, y) in G where $x \in f^{-1}(x')$, $y \in f^{-1}(y')$ and (x, y) is not disconnecting. However, from Proposition 3.7 if (x, y) is not normal then $x \in R(L(R(y)))$. Since f is an ideal map, $f(x) \in f(R(L(R(y)))) = R(L(R(f(y))))$, that is, $(f(x), f(y)) = (x', y')$ is not a normal edge,

contradiction. Thus (x,y) must be a normal edge; but it is not dis-connecting, contradiction to the normality of G .//

Lemma 4.24. Let f be an onto ideal map from $G = (P,E)$ to a basic and acyclic $G' = (P',E')$ and let $x' \in P'$. Then $f_H:H \to H'$ is an ideal onto map where

$H = (P \sim f^{-1}(x')$, $E \sim \{(u,v) \mid x'$ is on a path from $f(u)$ to $f(v)\}$

and

$H' = (P' \sim \{x'\}, E' \sim \{(u',x'), (x',v') \mid u',v'$ in $P'\})$.

Proof: Let $a' \in L_{H'}(f_H(b))$ for some b in H . Then we have a path $\rho(a',f(b)) = \langle p'_0, p'_1, \ldots, p'_n \rangle$ in H' . Since f is an ideal map onto G' , by Proposition 4.15 we have for some $a \in f^{-1}(a')$ a path $\rho(a,b) = \langle q_0, \ldots, q_m \rangle$ where for all i , $f(q_i) = p'_j$ for some j . It follows that $q_i \notin f^{-1}(x')$ since $p'_j \neq x'$ for all j . By Corollary 4.17, either $f(q_i) = f(q_{i+1})$ or $(f(q_i), f(q_{i+1}))$ is an edge in G'; and since $f(q_i) \neq x' \neq f(q_{i+1})$, we know in the latter case that $(f(q_i), f(q_{i+1}))$ is an edge in H' . Because G' is basic, the only possible path of length ≥ 1 from $f(q_i)$ to $f(q_{i+1})$ is the edge $(f(q_i), f(q_{i+1}))$, so that $x' \notin \rho(f(q_i), f(q_{i+1}))$; therefore (q_i,q_{i+1}) is in H . This shows $\rho(a,b)$ is in H , that is, $a \in L_H(b)$. Therefore, $a' = f_H(a) \in f_H(L_H(b))$.

Conversely let $a' \in f_H(L_H(b))$; so there exists $a \in L_H(b) \cap f^{-1}(a')$. We let $\rho(a,b) = \langle q_0, \ldots, q_n \rangle$. Since (q_i,q_{i+1}) is an edge in H , it is not possible that $x' \in \rho(f(q_i), f(q_{i+1}))$ for any path ρ in G' . Consequently we have $\rho(f(q_i), f(q_{i+1}))$ in H' ; thus, $\rho(f(a), f(b))$ is in H' , that is, $a' \in L_{H'}(f_H(b))$.

A similar proof will show $f_H(R_H(b)) = R_{H'}(f_H(b))$ .//

Given any subgraph H_1' , of G' , by repeated application of Lemma 4.24, discarding all points of $G' \sim H_1'$, and all edges involving these points, we can find an ideal map from some partial subgraph H_1 of G onto H_1' . In particular, given any partial subgraph $H' = (P',E')$ of G' , we can find an ideal map from some H_1 onto $H_1' = [P']$. Then, by repeated application of Lemma 4.21, discarding all edges of $[P']$ that are not in H' , we can find an ideal map from some partial subgraph H of G onto H' . These remarks, together with Theorem 4.23, give us

Theorem 4.25. Let $f:G \to G'$ be onto ideal and G' basic and acyclic. If G is finite and completely normal then so is G' .

CHAPTER V

APPLICATIONS

1. Contractions and ρ-congruence

In Pfaltz (1972) the idea of mappings between graphs is used to generalize the concept of lists, list structures, and graph structures. The purpose is to obtain a more effective computer representation of acyclic graphs and search algorithm for these data structures. In order to do this, that paper introduces three concepts, two of which we reproduce here.

Definition. An equivalence relation Σ on the point set P of a graph $G = (P,E)$ is called a ρ-congruence if $\rho(p_1,q_1)$ implies $\rho(p_2,q_2)$ whenever (1) (p_1,p_2) and (q_1,q_2) are in Σ and (2) (p_1,q_1) is not in Σ .

Every function $f: G \to G'$ obviously induces an inverse image partition on its domain G and conversely every partition defines a function (namely, all points in the same equivalence class map to the same image). The essential question is what kinds of maps correspond to ρ-congruences.

Definition. A map $f: G \to G'$ is called a contraction if G' is acyclic.

A key proposition in Pfaltz' development is the proof that every "m-M contraction" (whose formal definition will not be needed) induces a ρ-congruence on G . Then this fact is used to show that the generated graph structure is a faithful representation of the original; and to develop computationally efficient algorithms to find "m-M contractions." It turns out that "m-M contractions" are simply a restricted class of ideal

maps. In fact, we will show that if the inverse image partition of

f is a ρ-congruence then f must be a convex ideal contraction.

First, we obtain a characterization of ideal maps by combining
Propositions 4.2, 4.9, and 4.10. Suppose a partition π of a graph G
is given. Readily any partition of a graph induces a map from P onto
P' which we may extend to a continuous map of G onto G' where
P' = π and E' = $\{(\pi_1, \pi_2) \mid \pi_1, \pi_2 \in P', \pi_1 \neq \pi_2$, and there exist
$p_1 \in \pi_1$, $p_2 \in \pi_2$ such that $(p_1, p_2) \in E\}$. Conversely, any map f: G → G'
induces a partition of G , namely, $\pi = \{f^{-1}(p') \mid p' \in P'\}$. Let $\pi(x)$
be the element of the partition π which contains x .

Proposition 5.1. Let G be a graph and π a partition on G .
The map f induced by the partition π is ideal if and only if

(1) if $x_0 \in L(y_0)$ and $x_0 \notin \pi(y_0)$, then for all $y \in \pi(y_0)$
there exists $x \in \pi(x_0) \cap L(y)$ and

(2) if $y_0 \in R(x_0)$ and $y_0 \notin \pi(x_0)$, then for all $x \in \pi(x_0)$
there exists $y \in \pi(y_0) \cap R(x)$.

Proof: By construction of the induced map, f is continuous.
Conditions (1) and (2) are equivalent to Propositions 4.9 and 4.10,
respectively. This shows f is continuous and open; and conver-
sely.//

Proposition 5.2. Let G = (P,E) be acyclic and Σ a finite
equivalence class. A ρ-congruence of Σ on P induces a convex
ideal contraction f: G → G' .

Proof: We define f(x) = f(y) whenever $(x,y) \in \Sigma$. Thus,

P' = $\{f(x) \mid$ for all $x \in P\}$, and

E' = $\{(f(x), f(y)) \mid f(x) \neq f(y)$ and $(x,y) \in E\}$.

In order to see that $G' = (P',E')$ is acyclic, suppose there exists $\rho(x',x')$ in G' with $|\rho| \geq 2$. It follows that there is z' on ρ such that $z' \neq x'$. This implies $\rho_1(x',z')$ and $\rho_2(z',x')$ in G' . Consequently, there exist $\rho_1(x_1,z_1)$ and $\rho_2(z_2,x_2)$ in G . In other words, $(z_1,z_2) \in \Sigma$ and $(x_1,x_2) \in \Sigma$ with $f(x_1) \neq f(z_1)$, so we have $(x_1,z_1) \notin \Sigma$. Because Σ is a ρ-congruence, $\rho(x_1,z_1)$ implies $\rho(x_2,z_2)$. However, $\rho(z_2,x_2)$ and $\rho(x_2,z_2)$ form a cycle contradicting the assumption that G is acyclic. Hence G' is acyclic and f is a contraction.

To prove f is an ideal map onto G' , we need only show conditions (1) and (2) of Propositions 5.1 are satisfied. These conditions follow immediately by observing $z \in R(x_0) \cap \pi(y)$ for some $x_0 \notin \pi(y)$ implies $\rho(x_0,z)$ which, by ρ-congruence, implies $\rho(x,z)$ for any $x \in \pi(x_0)$. Similarly we have condition (2) satisfied.

To show convexity, we let A be convex in G , $\rho(x',y')$ in G' and x', $y' \in f(A)$. If $p' \in \rho(x',y')$ it follows that $\rho(x_1,p_1)$ and $\rho(p_2,y_2)$ exist in G with x_1, $y_2 \in A$. By ρ-congruence of Σ we have $\rho(x_1,p_2)$ which gives us $p_2 \in \rho(x_1,y_2)$. Since A is convex $p_2 \in A$ and, hence $f(p_2) = p' \in f(A)$ .//

The example in Figure 5.1 shows a convex ideal contraction which does not induce a ρ-congruence.
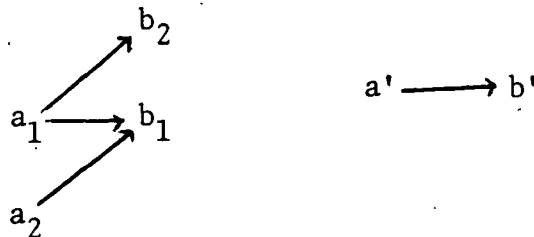

Figure 5.1

It is generally convenient to decompose a contraction f into several simpler maps f_i and then represent f as a composition of these, namely, $f = f_1 \circ f_2 \circ \cdots \circ f_n$. It appears that these simple maps are of relevance when the graphs are viewed as data or control structures, see Pfaltz (1972).

Definition. A map f_H: G → G' = (P',E') is called a simple contraction if H is a non-void subgraph of G such that (1) f(p) = p' for all p ∈ H and (2) f is a homeomorphism of G ∿ H onto [P' ∿ {p'}].

If G and G' are basic and acyclic then the homeomorphism of condition (2) must be an isomorphism (Proposition 4.11). Pfaltz, in fact, uses the more restrictive isomorphic condition and then introduces a sequence of homeomorphic contractions, called σ-contractions, to "reduce" graphs.

Proposition 5.3. Let H be a non-empty subgraph of a basic finite acyclic graph G . The map f_H: G → G' = (P',E') is a simple ideal contraction if and only if H induces a ρ-congruence on G .

Proof: Suppose H = (Q,F) induces a ρ-congruence on G , that is, Q and the singleton sets {p} for all p ∉ Q form a partition which is ρ-congruent. By Proposition 5.2, f is an ideal contraction so we need only show G ∿ H is homeomorphic to [P' ∿ {h'}] where f(Q) = {h'} . Clearly f is one-to-one and onto from G ∿ H to [P' ∿ {h'}] . If $\rho(a,b) = \langle q_0, \ldots, q_n \rangle$ in G ∿ H , we have $(q_i, q_{i+1}) \in E$ and $q_i \notin Q$ for all i . Thus, there are $\rho(f(q_i), f(q_{i+1}))$ for all q_i by the continuity of f (Proposition 4.2). If $h' \in \rho(f(q_i), f(q_{i+1}))$ for any i , since f is one-to-one outside of H , then there exists a

path $\rho(q_i, q_{i+1})$ with some $h \in Q$ on it. Contradiction; G is basic. Therefore, $(f(q_i), f(q_{i+1}))$ in $[P' \sim \{h'\}]$ and $\rho(f(a), f(b))$ in $[P' \sim \{h'\}]$. This shows f is continuous from $G \sim H$ to $[P' \sim \{h'\}]$. For the map $f^{-1}: [P' \sim \{h'\}] \rightarrow G \sim H$ we have a path $\rho(a', b')$ in $[P' \sim \{h'\}]$ implying $\rho(f^{-1}(a'), f^{-1}(b')) = \rho(a, b)$. By Proposition 4.15 we can choose $\rho(a, b)$ so that $x \in \rho(a, b)$ implies $f(x) \in \rho(a', b')$; thus $\rho(a, b)$ in $G \sim H$. This shows f^{-1} is continuous from $[P' \sim \{h'\}]$ to $G \sim H$.

Conversely, suppose $f_H: G \rightarrow G'$ is a simple ideal contraction. Let $(x_1, x_2) \in \Sigma$, that is $x_1, x_2 \in Q$, and let $y \notin Q$ such that $\rho(x_1, y)$ in G . By continuity we have $\rho(x', y')$ in G' . Fixing $x_2 \in f^{-1}(x')$ and applying Proposition 4.10 we obtain $\rho(x_2, \overline{y})$ for some $\overline{y} \in f^{-1}(y')$. But f is one-to-one outside of H so $\overline{y} = y$, and $\rho(x_2, y)$ exists. Similarly, if $z \notin Q$ and $\rho(z, x_1)$ in G we get by Proposition 4.9 a path $\rho(x, x_2)$. //

2. Strong Maps

Pfaltz (1968) develops the idea of a convex subgraph lattice S_G and shows that lattice theoretic properties of S_G reflect the essential graph theoretic properties of G itself. In this paper we have shown the relationships between topological properties and the graph theoretic properties. Using results developed in this paper, we will now extend some of the results in Pfaltz (1968).

Pfaltz maintains that if f is a homorphism of G to G' which induces a lower semi-homeomorphism $\sigma: S_G \rightarrow S_{G'}$ then f "strongly" preserves the graph theoretic properties of G ; he calls these maps "strong homomorphisms". Since they do preserve the internal subgraph structure in a "natural" way, "strong" maps appear to be a valuable

concept in the theory of directed graphs. Unfortunately, he provides only one condition which is sufficient for establishing that a particular kind of map (convex homomorphism) is "strong". A more general sufficient condition for the existence of a "strong" map can be established for a wider class of maps (ideal maps) by applying results developed for trees and TTSPN's. In particular we show that if G is either a forest or TTSPN, then any ideal map defined on G is strong. First we must prove a proposition which is interesting in its own right. It establishes the existence of a partial subgraph, I , which is a one-to-one "homeomorph" of the range of an ideal map. (The notation I is to suggest the notion of identity.)

Proposition 5.4. Let G = (P,E) be a finite, basic and acyclic graph, and let f: G → T' be any ideal map with T' any forest. Then there exists I ⊆ P such that (1) f is one-to-one from I onto the points of T' and (2) for p,q ∈ I, f(p) = p', and f(q) = q' we have $\rho_G(p,q)$ if and only if $\rho_{T'}(p',q')$.

Proof: We note that constructing I "simply" consists of choosing a single representative p from each inverse image set $f^{-1}(p')$ where p' in T' . Since for any choice of I we have $\rho_G(p,q)$ implies $\rho_{T'}(p',q')$ because f is continuous, the problem lies in showing $\rho_{T'}(p',q')$ implies $\rho_G(p,q)$. Our proof is by induction on the edge set E' of T' .

For |E'| = 0, 1 the proposition is clearly evident. Now let |E'| = n+1 . In any finite forest it is easily seen that there exists at least one point z' which belongs to only one edge. For concreteness we may assume that this sole edge is (y',z') . Let the partial subgraph

T* be T' with (y',z') deleted. Clearly we still have T* basic
and acyclic. Let G* be the graph formed by deleting edges of G
as in Lemma 4.21. Now by applying Lemma 4.21 we have a map f*: G* → T*
which is ideal. Since the edge set of T* is strictly less than E',
our induction assumption shows the existence of a set I* in G* with
the desired properties. In particular there is $y \in I^* \cap f^{-1}(y')$. Since
f is ideal there exists $z \in f^{-1}(z')$ such that $\rho(y,z)$ by Proposition
4.15. Define $I = I^* \cup \{z\}$. We need only show that for $\rho(x',z')$
there is a path $\rho(x,z)$ in G . Because $\rho(x',z')$ must end with the
consecutive points y', z', we have $\rho(x,y)$ in G and thus combining
this path $\rho(y,z)$ we obtain $\rho(x,z)$ .//

Theorem 5.5. If T is a finite tree and f is any ideal map
defined on T with its image basic then f is a strong map of T
onto its range f(T).

Proof: By Theorem 4.22 we know T' = f(T) must be a tree.
We need only show that f induces a lower semi-homeomorphism (LSH)
$\sigma: S_T \to S_{T'}$. We follow the proof and notation of Theorem 7.11 in
Pfaltz (1968). From Proposition 5.4 there exists I in T such
that I is one-to-one with T' and $\rho(p^*,q^*)$ if and only if
$\rho(p',q')$ for all $p^*, q^* \in I$ and $f(p^*) = p'$, $f(q^*) = q'$. Define
σ as follows:

   i) $\sigma(\emptyset) = \emptyset$

   ii) $\sigma(p^*) = p'$ for $p^* \in I$

     $\sigma(p) = \emptyset$ for $p \notin I$

   iii) $\sigma(H) = \sup(\sigma(A_H))$ for all convex subgraphs, i.e. $H \in S_T$.

By definition the map σ is order preserving with respect to

subgraph containment. Also, for $H' \in S_T$, we know $H = f^{-1}(H')$ is convex since the inverse image of convex sets are convex under all continuous maps. Thus, $H \in S_T$. It is evident that σ is onto $S_{T'}$.

The only remaining problem is to show that σ takes full sets of atoms onto full sets of atoms. Following Pfaltz (1968) let A be full and $q' = ch(\sigma(A))$. Thus, either $q' = \sigma(q)$ for some $q \in A$, in which case we are done or there exists r^*, s^* in A with $\sigma(r^*) = r' \neq \emptyset$ and $\sigma(s^*) = s' \neq \emptyset$ with $q' \in \rho(r', s')$. From Proposition 5.4 there exist paths $\rho_1(r^*, q^*)$ and $\rho_2(r^*, s^*)$ in T. Since A is full $q^* \in A$ and $\sigma(q^*) = q' \in \sigma(A)_{\tau}$. This shows that $\sigma(A)$ is full.

The remainder of the proof is exactly the same as the proof of Theorem 7.11 in Pfaltz (1968).//

We note that in view of Proposition 5.4 we could have stated a slightly stronger version of the preceding Theorem, namely "if f is an ideal map and its range is a forest then f is strong."

Our next goal is to show a similar result for TTSPN's, that is, ideal maps defined on TTSPN's are strong. First, the next two lemmas must be proved.

Lemma 5.6. Let $G = (P,E)$ be the serial composition of two TTSPN's $H_1 = (P_1, E_1)$ and $H_2 = (P_2, E_2)$, with terminal points s, z and z, t respectively. Then $f: G \to G'$ is an ideal map where

$G' = (P', E')$, $P' = P_1 \cup \{t\}$,

$E' = E_1 \cup \{(z,t)\}$, and

$f(p) = p$ if $p \in P_1$

$f(p) = t$ if $p \in P_2 \sim \{z\}$.

Proof: We show the conditions of Proposition 5.1 are satisfied

for the partition $\pi = \{f^{-1}(p') \mid p' \in P'\}$. Suppose $x_0 \in L(y_0)$

and $x_0 \notin \pi(y_0)$. Case (a) if $y_0 \notin \pi(t)$ then $\pi(y_0) = \{y_0\}$ so that

for all $y \in \pi(y_0)$ we have $\pi(x_0) \cap L(y) = \pi(x_0) \cap L(y_0) \neq \emptyset$.

Case (b) if $y_0 \in \pi(t)$ then $x_0 \notin \pi(t)$ and $\pi(x_0) = \{x_0\}$. Since

$x_0 \in L(z)$ and $z \in L(y)$ for all $y \in P_2$ we have $\pi(x_0) \cap L(y) \neq \emptyset$

for all $y \in \pi(t) \subseteq P_2$. In a like manner condition (2) of Proposition

5.1 can be proved.//

Lemma 5.7. Let $f: G \to G'$ be an onto ideal map, G be a TTSPN,

and G' be the serial composition of $H_1' = [\Theta_1']$ and $H_2' = [\Theta_2']$ and

z' the common point of H_1' and H_2'. Then for any $z \in f^{-1}(z')$ the

map $f: [R(z)] \to H_2'$ is ideal.

Proof: Since f is ideal $f(R(z)) = R(f(z)) = R(z') = \Theta_2'$.

Define the partition $\pi_H = \{f^{-1}(p') \cap R(z) \mid p' \in \Theta_2'\}$ for $H = [R(z)]$.

We shall show this partition satisfies the conditions of Proposition 5.1

and, therefore, $f: H \to H_2'$ is an ideal map. In H let $x_0 \in L(y_0)$

and $f(x_0) \neq f(y_0)$ and $y \in f^{-1}(y_0)$. If for some $x \in \pi(x_0)$ such

that $x \in L(y)$ in H then condition (1) of Proposition 5.1 is satisfied.

Suppose $\pi(x_0) \cap L(y) = \emptyset$, then in particular $x_0 \notin L(y)$. Since

f is an ideal map on G, there exists $x \in L(y)$ in G such that

$f(x) = f(x_0)$; so $x \notin R(z)$. We have $y \in R(x) \cap R(z)$, thus, let

u be a minimal upper bound of x and z. Now $z \in L(u) \cap L(x_0)$,

so let v be a maximal lower bound of u and x_0. It follows that

we can define

$\rho_1(s,t) = \rho(s,v) + \rho(v,x_0) + \rho(x_0,t)$

$\rho_2(s,t) = \rho(s,x) + \rho(x,u) + \rho(u,t)$ and

$\rho_3(v,u)$ where s and t are the terminal points of G.

We have $v \neq u$ for otherwise the paths $\rho(x,u)$ and $\rho(v,x_0)$ imply $u = v \in \pi(x_0)$ (G' acyclic) and, thus, $\pi(x_0) \cap L(y) \neq \emptyset$.

Let $r \in \rho_3$ and $r \neq v$. If $r \in \rho(s,v)$ then G has a cycle; if $r \in \rho(v,x_0)$ then v is not a maximal lower bound of u and x_0 ; if $r \in \rho(x_0,t)$ then $x_0 \in L(u) \subseteq L(y)$ - in all cases we have a contradiction. Now let $r \in \rho_3$ and $r \neq u$. If $r \in \rho(s,x)$ then $x \in R(z)$; if $r \in \rho(x,u)$ then u is not a minimal upper bound of x and z ; if $r \in \rho(u,t)$ then G has a cycle; a contradiction in all cases. Therefore we have shown $\rho_1 \cap \rho_3 = \{v\}$ and $\rho_2 \cap \rho_3 = \{u\}$, and, thus, ρ_1, ρ_2, ρ_3 constitute a cross-connection. By Theorem 3.12, G is CCF, hence we have a contradiction. This proves $\pi(x_0) \cap L(y) \neq \emptyset$.

Condition (2) of Proposition 5.1 is immediately satisfied since if $y_0 \in R(x_0)$ and $f(y_0) \neq f(x_0)$ then for $x \in \pi(x_0)$ there exists $y \in \pi(y_0) \cap R(x)$ in G , and $y \in R(x) \subseteq R(z)$ so $y \in \pi(y_0) \cap R(x)$ in H .//

<u>Proposition 5.8.</u> If f: $G \to G'$ is ideal and G and G' are TTSPN then there exists $I \subseteq P$ such that (1) f is an one-to-one map from I onto G' and (2) for $p,q \in I$, $f(p) = p'$, $f(q) = q'$ we have $\rho_G(p,q)$ if and only if $\rho_{G'}(p',q')$. Moreover, we can pick the terminal points of G to be in I .

<u>Proof:</u> When $|E'| = 1,2$ the graph G' is a tree and, thus we can apply Proposition 5.4. For $|E'| \geq 3$, let G' be the composition of two TTSPN's, H_1' and H_2' . Let s' and t' be the terminal points of G' . In the parallel case, if the edge $(s',t') \in E'$, then clearly the identity map from G' onto $G'' = (P', E' \sim \{(s',t')\})$

is an ideal map with G'' having fewer edges than G' ; so we have a map from G onto G'' that is ideal (composition of ideal maps is ideal). By our induction hypothesis there exists $I \subseteq P$ with the desired properties. It is evident that this set I satisfies the desired properties for $f: G \to G'$.

By the remarks following Lemma 4.24 we can obtain partial subgraphs H_1 and H_2 of G so that $f: H_1 \to H_1'$ and $f: H_2 \to H_2'$ are ideal maps. If we can show H_1 and H_2 are TTSPN then we can apply the induction hypothesis for f restricted to H_1 and H_2 ; thus we have I_1 of H_1 and I_2 of H_2 . Let $I = I_1 \cup I_2$. Since s and t are terminal points for both H and H , we have $s, t \in I_1 \cap I_2$. Moreover, $f(I_1 \cap I_2) = \{s', t'\}$ because the image set must be in H_1' and H_2' . This shows f is an one-to-one map from I onto G' . Now suppose $p, q \in I$, $f(p) = p'$ and $f(q) = q'$. By continuity of f , $\rho(p,q)$ implies $\rho(p',q')$. Conversely if $\rho(p',q')$ in G' we have two cases. (i) If p' and q' are in H_1' (or H_2') then $p, q \in I_1$ (or I_2) and (p,q) exists. (ii) If p' is in H_1' and q' is in H_2' , it follows that either $p' = s'$ (in which case p' is in H_2') or $q' = t'$ (in which case q' is in H_1'); therefore only the first case (i) is possible.

Now we show H_1 and H_2 are TTSPN. Clearly s and t are in H_1 since $f(\{s,t\}) = \{s',t'\}$ (which is in H_1'). If s and t are the terminal points of H_1 then H_1 will be a TTSPN by remark (5) following the definition of TTSPN. Obviously, $L(s) = \{s\}$ and $R(t) = \{t\}$ in H_1 . For any point x in H_1 , we have $f(x) = x'$ in H_1' so that there exist paths $\rho(s',x)$ and $\rho(x',t')$ in H' . Since f is an ideal map from H_1 onto H_1' , by Proposition 4.15 we have paths

$\rho(s_1,x)$ and $\rho(x,t_1)$ in H_1 where $f(s_1) = s'$ and $f(t_1) = t'$.
By the constructing of H_1 the paths $\rho(s,s_1)$ and $\rho(t_1,t)$ must be
in H_1 ; thus we have the paths $\rho(s,t)$ and $\rho(x,t)$ in H_1 . This
proves s and t are left and right terminals respectively for H_1 .
Similarly we prove s and t are terminal points for H_2' and, therefore,
H_2' is a TTSPN.

In the serial case let z' be the common point of H_1' and H_2' .
Since $|E'| \geq 3$ then either H_1' or H_2' must have at least two edges,
say H_2' . We may also assume s' is in H_1' and t' is in H_2' . (A
similar argument holds if s' is in H_2 and t' is in H_2' .) By
Lemma 5.6 we have an ideal onto map $g: G' \to G''$ where the points of
H' except for z' map onto t' and $g(x') = x'$ for all other points
of G' . Clearly G'' has fewer edges than G' . Since the composition
of ideal maps is ideal, it follows that $f \circ g: G \to G''$ is an ideal map.
By the induction hypothesis, there exists I_1 in G where $f \circ g$ is
one-to-one from I_1 onto G'' .

In particular there exists $z \in I_1$ such that $f(z) = z'$.
Let $H = [R(z)]$; since f is ideal we have $f(R(z)) = R(z')$ so that
$f(H)$ maps onto H_2' . By Lemma 5.7 the map $f: H \to H_2'$ is ideal.
Therefore, by the induction hypothesis there exists I_2 in H such
that f is one-to-one from I_2 onto H_2' . In particular $z, t \in I_2$.
Define $I = I_1 \cup I_2$. Clearly $I_1 \cap I_2 = \{z,t\}$ and f is one-to-one
from I onto G' . By the continuity of $f: G \to G'$ we have for
$p,q \in I$ the path $\rho(p,q)$ implying $\rho(p',q')$. Conversely let
$\rho(p',q')$ in G' ; we have two cases. If p',q' are both in H_1'
(or H_2') then we have $\rho(p,q)$ in G for $p,q \in I_1$ (or I_2). If
p' in H_1' and q' in H_2' we must have $\rho(p',z')$ in H_1' and

$\rho(z',q')$ in $H_2^!$. It follows that $\rho(p,z)$ and $\rho(z,q)$ are in G for $p,q,z \in I$ .//

Theorem 5.9. If G is a TTSPN and f is an ideal map defined on G , then f is a strong map of G onto its range.

Proof: By Theorem 4.20 the range of f is a TTSPN, call it G' . Proposition 5.8 enables us to pick a point set I of G such that $\rho_G(p^*,q^*)$ if and only if $\rho_{G'}(p',q')$ for $p^*,q^* \in I$. The proof is now the same as that of Theorem 5.5.//

In earlier chapters we have shown that ideal maps, when defined on suitable domains, have many desirable properties. In particular ideal maps preserve 1) trees, 2) TTSPN's, and 3) completely normal graphs. In this section we have shown that in the first two cases ideal maps are strong. It seems natural to conjecture that ideal maps with completely normal domains are also strong; especially since we have shown that trees and TTSPN's are completely normal. This conjecture does appear to be true.

REFERENCES

Berge, C. (1962). The Theory of Graphs and its Applications.
   Wiley, N. Y.

Bhargava, T. N., and Ahlborn, T. J. (1968). On topological
   spaces associated with digraphs. Acta Mathematica
   Academiae Scientiarum Hungaricae 19, 47-52.

Frink, O. (1942). Topology in lattices. Trans. Am. Math.
   Soc. 51, 569-582.

Harary, F. (1969). Graph Theory. Addison-Wesley, Reading,
   Mass.

Harary, F., Norman, R. Z., and Cartwright, D. (1965).
   Structural Models: An Introduction to the Theory
   of Directed Graphs. Wiley, N. Y.

Kelley, J. L. (1955). General Topology. Van Nostrand, N. Y.

Ore, O. (1962). Theory of Graphs. Am. Math. Soc. Colloq.
   Pubs. 38, Providence, R. I.

Pfaltz, J. L. (1968). Convexity in Graphs. TR-68-74,
   Computer Science Center, University of Maryland,
   College Park, Maryland.

Pfaltz, J. L. (1971). Convexity in directed graphs.
   J. Combinatorial Theory 10, 143-162.

Pfaltz, J. L. (1972). Graph Structures. J. Assoc. Comp.
   Mach. 19, 411-422.

Tutte, W. T. (1966). Connectivity in Graphs. Univ. of
   Toronto Press, Toronto, Canada.