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Introduction

The motion of a charged particle in electromagnetic fields of various geometric configurations and arising from a variety of sources is of intrinsic interest in electromagnetic theory. The results of the analysis of the motions generated by the field-charge interaction are in many cases also pertinent to the discussion of applied problems in plasma physics and astrophysics. The particular configuration consisting of a plane wave propagating in the presence of a static uniform magnetic field whose direction is parallel to the wave normal is examined in this report. The analysis presented here is developed along lines different from those pursued in a previous study of the same configuration by Roberts and Buchshaum [1964]. It should be remarked that the problem under consideration is treated within the context of classical electromagnetic theory. The corresponding quantum mechanical problem - but neglecting the quantization of the field - has been discussed by Redmond [1965] who solved the appropriate Klein-Gordon and Dirac equations. In neither of the references cited above is the radiative reaction force taken into account. While from a physical point of view a knowledge of the influence of the radiative reaction on the motion of the charge would be very useful, the analytical difficulties associated with the Lorentz-Dirac equation are frequently sufficiently formidable to preclude the obtaining of the desired information. In fact it was hoped originally, on the basis of some success with a simpler field configuration, Mitchell et al. [1971], to include this radiative reaction in the following analysis, but to date mathematical difficulties have proved insurmountable in the search for a purely <u>analytical</u> solution. A numerical solution - at least to the approximate Lorentz-Dirac equation - could, of course, be obtained in a relatively straightforward way.

As a final comment prior to developing the analysis, it may be pointed out that an approach to this problem through the Hamilton-Jacobi formulation, although not adopted here, is an alternative possibility. Such an approach would naturally have much in common with the quantum mechanical formulation. The basic non-linearity of the Hamilton-Jacobi equation, especially in its relativistic form introduces major difficulties, however; see e.g., Corben and Stehle [1965], Landau and Lifshitz [1959].

Motion in Moving Frame of Reference

The direction of propagation of the wave, and consequently the uniform magnetic field direction, is taken to be along the positive z-axis of the usual cartesian xyz triad. The field can then, without significant restriction, be represented as the superposition of a circularly polarized monochromatic wave and a uniform field, $\vec{H}_o$, in the form

$$\left.\begin{aligned} E_x &= E \sin(\omega t - \alpha z) \quad ; \quad H_x = H \cos(\omega t - \alpha z) \\ E_y &= -E \cos(\omega t - \alpha z) \quad ; \quad H_y = H \sin(\omega t - \alpha z) \\ E_z &= 0 \qquad\qquad\qquad\quad ; \quad H_z = H_o \end{aligned}\right\} \quad \ldots (1)$$

all quantities being measured relative to a reference frame in which the electrical conductivity is zero and the constitutive and field equations are the usual

$$\left.\begin{aligned} \vec{D} &= \varepsilon\vec{E} \quad ; \quad \nabla\cdot\vec{H} = 0 \quad ; \quad \nabla\times\vec{E} + \mu\frac{\partial\vec{H}}{\partial t} = 0 \\ \vec{B} &= \mu\vec{H} \quad ; \quad \nabla\cdot\vec{E} = 0 \quad ; \quad \nabla\times\vec{H} - \varepsilon\frac{\partial\vec{E}}{\partial t} = 0 \end{aligned}\right\} \quad \ldots (2)$$

expressed in rationalized m.k.s. units. It follows, Stratton [1941], that the phase velocity, v_1, of the wave is given by

$$v_1 = \omega/\alpha = 1/\sqrt{\mu\varepsilon} = E/\mu H \tag{3}$$

and that the index of refraction, n, of the medium in which the motion takes place satisfies the relations

$$n = c/v_1 = c/\omega = cB/E \tag{4}$$

where the speed of light in vacuo is represented by c. Subsequently, unless explicitly stated otherwise, it is assumed that $n > 1$. The exception to this inequality will be the case in which $n = 1$, i.e., the case in which the motion takes place in a vacuum.

The equation of motion of the charge, q, is

$$\frac{d\vec{p}}{dt} = q[\vec{E} + \vec{v}x\vec{B} + \vec{v}x\vec{B}_o] \tag{5}$$

in which the momentum of the charge is

$$\vec{p} = m\vec{v}/\sqrt{1 - v^2/c^2} \tag{6}$$

m, being the rest mass.

The analysis of the motion can be simplified considerably by carrying out a Lorentz Transformation to a frame moving in the positive z-direction with velocity v_1. Relative to this frame the particle is subjected to the influence of a magnetic field alone, the electric field of the wave being entirely eliminated. In fact, after some reduction one finds the magnetic field in the moving frame to be

$$\vec{B}' = \vec{i}\,\frac{\mu H}{\gamma_1}\cos\frac{\omega}{v_1\gamma_1}z' - \vec{j}\,\frac{\mu H}{\gamma_1}\sin\frac{\omega}{v_1\gamma_1}z' + \vec{k}\mu H_o \tag{7}$$

and the equation of motion reduced to

$$\frac{d\vec{p}'}{dt'} = q\,\vec{v}'\,x\,\vec{B}' \tag{8}$$

In equations (7) and (8) the superscript prime has been introduced to denote quantities measured relative to the moving reference frame, the unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are along the x', y', z' axes, respectively, and the constant factor, γ_1, represents $(1 - v_1^2/c^2)^{-1/2}$. Equation (8) in cartesian component form is

$$\frac{dp'_x}{dt'} = \sigma_2 \frac{dy'}{dt'} + \sigma_1 \frac{dz'}{dt'} \sin \sigma_3 z' \tag{9}$$

$$\frac{dp'_y}{dt'} = \sigma_1 \frac{dz'}{dt'} \cos \sigma_3 z' - \sigma_2 \frac{dx'}{dt'} \tag{10}$$

and

$$\frac{dp'_z}{dt'} = - \sigma_1 \frac{dx'}{dt'} \sin \sigma_3 z' - \sigma_1 \frac{dy'}{dt'} \cos \sigma_3 z' \tag{11}$$

the three constants

$$\sigma_1 = q\mu H/\gamma_1 \quad , \quad \sigma_2 = q\mu H_o \quad , \quad \sigma_3 = \omega/\gamma_1 v_1 \tag{12}$$

being introduced for conciseness.

The first two component equations can be integrated immediately to give

$$p'_x = \sigma_2 y' - \sigma_4 \cos \sigma_3 z' + C_1 \tag{13}$$

and

$$p'_y = - \sigma_2 x' + \sigma_4 \sin \sigma_3 z' + C_2 \tag{14}$$

respectively, where C_1 and C_2 are constants of integration and the constant $\sigma_4 = \sigma_1/\sigma_3$.

The energy of the charge is

$$E' = mc^2/\sqrt{1 - (\frac{v'}{c})^2} \tag{15}$$

and so

$$\frac{dE'}{dt'} = v' \cdot \frac{dp'}{dt'} \tag{16}$$

From equations (9), (10), (11), and (16) it follows that

$$\frac{dE'}{dt'} = 0 \tag{17}$$

and therefore the energy E' is a constant. This, of course, is to be expected because the motion is taking place in a pure magnetic field. Accordingly, equations (13) and (14) can be written

$$m\gamma' \frac{dx'}{dt'} = \sigma_2 y' - \frac{\sigma_1}{\sigma_3} \cos \sigma_3 z' + C_1 \tag{18}$$

and

$$m\gamma' \frac{dy'}{dt'} = -\sigma_2 x' + \frac{\sigma_1}{\sigma_3} \sin \sigma_3 z' + C_2 \tag{19}$$

in which the constant $\gamma' = (1 - (\frac{v'}{c})^2)^{-\frac{1}{2}}$. Although the third component of the equation of motion cannot be integrated as directly as the first two, the explicit spatial dependence of p_z' can be found as follows. On eliminating the terms dependent on z' from the right hand side of equation (11) one finds after some development an integrable equation which leads to

$$p_z' = -\frac{1}{2} \sigma_2\sigma_3(x'^2 + y'^2) + \sigma_3(C_2x' - C_1y') + C_3 \tag{20}$$

and consequently

$$m\gamma' \frac{dz'}{dt'} = -\frac{1}{2} \sigma_2\sigma_3(x'^2 + y'^2) + \sigma_3(C_2x' - C_1y') + C_3 \tag{21}$$

in which C_3 is a constant of integration.

Alternatively, the temporal and spatial and dependence of p_z' can be determined from the general energy-momentum relationship

$$E' = (p'c)^2 + m^2c^4 \tag{22}$$

The trajectory of the charge in the moving frame of reference is given in parametric form by equations (18), (19), and (21), or in simultaneous form by

$$\frac{dx'}{\sigma_2 y' - \sigma_4 \cos \sigma_3 z' + C_1} = \frac{dy'}{-\sigma_2 x' + \sigma_4 \sin \sigma_3 z' + C_2}$$

$$= \frac{dz'}{-\sigma_2\sigma_3(x'^2 + y'^2)/2 + \sigma_3(C_2 x' - C_1 y') + C_3} \tag{23}$$

Without attempting an explicit determination of the trajectory it can be asserted on the basis of equations (13), (14), (20), and (15) that the motion takes place on the surface

$$\left[\frac{\sigma_2\sigma_3}{2}(x'^2 + y'^2) - \sigma_3(C_2 x' - C_1 y') - C_3\right]^2$$

$$+ (\sigma_2 y' - \sigma_4 \cos \sigma_3 z' + C_1)^2 + (\sigma_2 x' - \sigma_4 \sin \sigma_3 z' - C_2)^2$$

$$= \text{constant} \tag{24}$$

where the constant is determined by the initial conditions.

Motion in Fixed Frame of Reference

The application of the Lorentz Transformation to the momentum four-vector

$$M' = \{m\gamma' \frac{\vec{v}'}{c}, \quad i \frac{E'}{c^2}\} \tag{25}$$

produces the equations which describe the motion in the fixed frame. After some algebraic manipulation one finds the momentum components to be

$$p_x = \sigma_2 y - \sigma_4 \cos(\omega t - \alpha z) + C_1 \tag{26}$$

$$p_y = -\sigma_2 x - \sigma_4 \sin(\omega t - \alpha z) + C_2 \tag{27}$$

and

$$p_z = -\gamma_1 \sigma_2 \sigma_3 (x^2 + y^2)/2 + \gamma_1 \sigma_3 (C_2 x - C_1 y) + \gamma_1 C_3 + \gamma_1 v_1 \frac{E'}{c^2} \tag{28}$$

The energy of the motion relative to the fixed frame can be written in the form

$$\begin{aligned} E &= \gamma_1 E' + \gamma_1 v_1 p_z' \\ &= \gamma_1 E' - \gamma_1 v_1 \sigma_2 \sigma_3 (x^2 + y^2)/2 + \gamma_1 {}_1\sigma_3 (C_2 x - C_1 y) + \gamma_1 v_1 C_3 \end{aligned} \tag{29}$$

It is worthy of note that equations (28) and (29) can be combined to give

$$E' = \gamma_1 (E - v_1 p_z) \tag{30}$$

Thus the quantity $E - v_1 p_z$ remains constant throughout the motion.

An additional constant of the motion can be obtained by combining equations (26) and (27) in the form

$$(p_x - \sigma_2 y - C_1)^2 + (p_y + \sigma_2 x - C_2)^2 = \sigma_4^2 \tag{31}$$

A more direct link between the relationship expressed in equation (30) and the equation of motion (5) may be established as follows. The rate of change of the energy E is

$$\frac{dE}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt} \tag{32}$$

which

$$= q\vec{v} \cdot \vec{E} \tag{33}$$

on using the equation of motion (5). However, for a plane wave

$$\vec{E} = v_1 \vec{B} \times \vec{K} \tag{34}$$

and again from equation (5)

$$\vec{K} \cdot \frac{d\vec{p}}{dt} = \frac{d\vec{p}_z}{dt} = q\vec{K} \cdot (\vec{v} \times \vec{B}) \tag{35}$$

Therefore,

$$\frac{dE}{dt} = v_1 \frac{dp_z}{dt} \tag{36}$$

and

$$E = E_o + v_1 (p_z - p_{z_o}) \tag{37}$$

in which the subscript zero denotes values at some particular time t_o. It is now seen that the constant E' introduced in equation (15) satisfies the equation

$$E' = \gamma_1 (E_o - v_1 p_{z_o}) \tag{38}$$

In conclusion one notices that a simultaneous set of differential equations which determine the trajectory relative to the fixed frame can be written by combining equations (26), (27), and (28). In contradistinction to equation (23), however, the formulation under discussion would explicitly involve the time.

Motion in a Vacuum: The Case $n = 1$

The analysis contained in the preceding pages has been developed subject to the restriction that the index of refraction, n, of the medium in which the motion takes place is greater than unity. However, the case in which $n = 1$ is also of considerable interest. It cannot be treated by the Lorentz Transformation already used because the phase velocity of the wave is $v_1 = c$ when $n = 1$, and hence $\gamma_1 \to \infty$ rendering the transformation singular. One observes, nevertheless that the expressions (26), (27), and (28) for the components of the linear momentum are independent* of the factor γ_1. It can be shown, in fact, by direct substitution in the equation of motion (5) that the three relations (26), (27), and (28) are valid for any value of n. Accordingly, the constants of the motion already established exist independently of any restriction on the value of the index of refraction. For the case $n = 1$, the energy relationship (37) takes the form

$$E = E_o + c(p_z - p_{z_o}) \tag{39}$$

On using the general energy-momentum relationship, one finds that

$$\begin{aligned} p_z = \frac{1}{2}\,(p_{z_o} - E_o/c)^{-1}\,\{&(p_{z_o} - E_o/c)^2 - m^2c^2 \\ &- [\sigma_2 y - \sigma_4 \cos\,(\omega t - \alpha z) + C_1]^2 \\ &- [\sigma_2 x + \sigma_4 \sin\,(\omega t - \alpha z) - C_2]^2\} \end{aligned} \tag{40}$$

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* Equation (28) exhibits a superficial dependence on γ_1. It can be removed upon recourse to equation (12).

Equation (40) is equivalent to equation (28) which, although derived for $n > 1$, is now seen to be valid for arbitrary n.

In addition equation (40) when used in conjunction with equation (39) provides the explicit dependence of the energy E on the position of the charge and on time. This functional dependence, as in the case $n > 1$, can be expressed in the form $E(x, y, \omega t - \alpha z)$ without difficulty.

Motion in Momentum Space

Considerable insight into the particle's motion can be obtained by examining the dynamics in a "momentum-space" in which the momentum components are taken to be the rectangular cartesian coordinates of a point. In particular, the difference in the behavior of the charge in the two cases $n > 1$ and $n = 1$ is quickly made apparent by the formulation in momentum-space. For this reason, and also for mathematical convenience, the two cases referred to are once again treated separately.

1. Index of Refraction $n > 1$

On squaring equation (37), which actually expresses the constancy throughout the motion of a certain function of the momentum components, it is seen that the charge is confined in momentum-space to the surface defined by

$$p_x^2 + p_y^2 + \frac{1}{\gamma_1^2} p_z^2 - 2\frac{v_1}{c^2}(E_o - v_1 p_{z_o})p_z + m^2c^2 - \frac{1}{c^2}(E_o - v_1 p_{z_o})^2 = 0 \tag{41}$$

This equation represents a prolate ellipsoid whose axis of symmetry coincides with p_z-axis. The semi-major axis, a, of this spheroid is

$$a = (p_{z_{max}} - p_{z_{min}})/2 = \gamma_1[(E_o - v_1 p_{z_o})^2(\frac{1}{c})^2 - m^2c^2]^{1/2}$$

where the maximum and minimum values of p_z are, respectively

$$p_{z_{max}} = \gamma_1\{\frac{E'}{c}\cdot\frac{v_1}{c} + [(\frac{E'}{c})^2 - m^2c^2]^{1/2}\} \tag{43}$$

$$p_{z_{min}} = \gamma_1 \{ \frac{E'}{c} \cdot \frac{v_1}{c} - [(\frac{E'}{c})^2 - m^2c^2]^{1/2} \} \qquad (44)$$

Owing to the closed nature of the surface of the ellipsoid it is immediately clear that the energy of the charge remains finite and bounded at all times. In fact, it is possible, although the results will not be reproduced here, to calculate explicit upper and lower bounds on the energy.

2. *Motion in a Vacuum; Index of Refraction $n = 1$.*

In this case the appropriate point to start the discussion form is equation (39) which leads, on squaring it, to

$$p_x^2 + p_y^2 - 2(\frac{E_o}{c} - p_{z_o})p_z + m^2c^2 - (\frac{E_o}{c} - p_{z_o})^2 = 0 \qquad (45)$$

The significant difference between equations (41) and (45) is the absence of a quadratic term in p_z in the latter. This results in an opening up of the ellipsoid into a paraboloid, its surface being a surface of revolution about the p_z-axis. Consequently, it is no longer possible to establish an upper bound on the particle's energy which may therefore increase without limit as the motion develops. Some aspects of these results are discussed in a different context by Roberts and Buchsbaum [1964].

As a final comment, it is remarked that if the index of refraction n is less than unity, the corresponding surface in momentum-space is determined by an equation formally similar to equation (41). However, the condition $n < 1$ implies that the coefficient $\gamma_1^2 < 0$ and hence the surface is a hyperboloid of revolution. As in the case $n = 1$, it is not possible to fix an upper bound on the energy.

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