A PROVISIONAL ANALYSIS OF TWO-DIMENSIONAL TURBULENT MIXING WITH VARIABLE DENSITY

By Paul A. Libby
University of California at San Diego

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SUMMARY

A predictive method for the titled flows based on the Prandtl energy method is developed and assessed by comparing predicted results with experimental results. For constant-density flows, both gross properties such as spreading rate and maximum turbulent kinetic energy and detailed properties such as mean shear stress distributions are shown to be well predicted. For variable-density flows, considerable attention is devoted to the inclusion in the analysis of the added effect of pressure fluctuations and of the variation in the several extant empirical parameters on the turbulent kinetic energy. It is found that a variation with Mach number of the characteristic Reynolds number for turbulent transport is needed to account for the observed decrease in spreading rate. The predictions which result from these considerations are compared with the limited experimental data presently available for the two crucial cases: compressible adiabatic mixing and low-speed isothermal mixing of two dissimilar gases.

INTRODUCTION

There is considerable activity presently underway by several groups on the description of turbulent shear flows by methods which contain more of the physics of turbulence than do the well-known methods based on mixing length and the usual eddy-viscosity models. Bradshaw et al., Rodi and Spalding, Hanjalić and Launder, Donaldson, and Wilcox and Alber (refs. 1 to 5) provide recent entry points in this literature. Most of this activity has concerned turbulent flows with uniform properties, presumably because these new methods should first be assessed for this simpler case and because the amount of data for the case of variable density is limited. Concern here is with the simplest turbulent shear flow, the two-dimensional mixing of two different streams, but under circumstances involving variable density, either because of the high speed of one stream or because of different compositions of the two streams.

The two-dimensional mixing layer has the virtue that its description is given in terms of a similarity variable; thus, the numerical analysis associated with its study is modest. It has the further advantage that there are no solid walls present so that most effects of molecular transport are negligible and the modeling required to effect closure in the describing equations is simplified. A disadvantage is that its study in the laboratory is relatively difficult when compared with other free mixing flows, for example, wakes and jets; therefore, experimental data suitable for purposes of comparison are limited, especially for the case of variable density.

Previous theoretical work on two-dimensional, turbulent mixing for constant fluid properties is extensive; the well-known references 6 and 7 may be examined. For the case of variable density due to either compressibility effects of high speed or to heterogeneous composition, the literature involving a serious effort to compare prediction and experimental data is sparse; most entries set up a model for the case of variable properties, validate it against the data for the constant-density case, and then use it to predict the effects of variable density. Typical thereof is reference 8.

This situation is perhaps due to the disarray in the experimental data presently existent in two crucial cases of two-dimensional mixing with variable density. The data have been recently reviewed by Birch and Eggers (paper no. 2 of this conference). With respect to the case of one high-speed stream mixing with a quiescent gas of the same composition under conditions such that the stagnation temperature is constant everywhere, that is, the case of so-called "compressible adiabatic flow," several sets of data indicate no effect of high speed (that is, of high Mach number) on the spreading angle. Other data indicate a significant increase in spreading parameter, that is, a decrease in spreading angle and mixing rate, as the Mach number of the high-speed stream increases.

The other crucial case of two-dimensional mixing with variable density involves the low-speed isothermal mixing of two gases of different molecular weights. Again, Birch et al. have pointed out the inconsistency of several sets of data related to the spreading parameter, in these cases to be considered a function of the velocity and density ratios of the two participating streams.

Given this situation, any theoretical analysis such as the present one must be treated as a provisional one until at least the experimental evidence related to these two crucial cases cited is considered to be well-established. In the present work, the Brown and Roshko data (ref. 9) for heterogeneous, low-speed mixing and the data shown by Brown and Roshko and by Birch and Eggers indicating significant effect of Mach number on the spreading parameter for compressible adiabatic flows have been accepted as correct. Accordingly, comparisons have been made of the present theoretical predictions against these data and the appropriate empirical constants have been adjusted to bring prediction and these experimental data into agreement insofar as possible. If future developments do not support these experimental data as correct, the present work should still provide a methodological framework of some value, but, of course, the adjustment of parameters would require some alteration.

A cursory examination of the new methods of analysis of turbulent shear flows indicates a wide variety of approaches, each of which can be extended to the case of variable density. One of the simpler ones, that is sometimes termed the "Prandtl energy method," is followed here. It is based on the idea of an eddy viscosity proportional to a length associated with the scale of the mixing layer and to the square root of the turbulent kinetic energy. It can be seen that the extension of this idea to flows with variable density is ambiguous and must be guided by comparison with data. Comparison is made by relying heavily on the data associated with the gross behavior of the mixing layer, as reviewed and codified recently by Brown and Roshko (ref. 9) and Birch and Eggers (paper no. 2).

The appropriate conservation equations are first developed and the terms as required for closure are modeled. One representation is then selected for the eddy viscosity for constant-density flows and the predictions based thereon are compared with various experimental results. The problem of incorporating the effects of variable density in such a fashion as to achieve agreement with experimental data relative to the two crucial cases cited is then discussed. Finally, a comparison with experimental data is made.

ANALYSIS

The idealized flow shown schematically in figure 1 is considered. Two streams with different composition, velocity, and energy but with the same uniform pressure undergo turbulent mixing at the end of a splitter plate which is the origin of a x_1,x_2 coordinate system. The symbols used in the analysis are defined in appendix A.

In developing the equations for the description of this flow, several assumptions are employed. The effect of molecular transport is neglected except for certain dissipation terms. Constant pressure in the two external flows, flow similarity, and no chemical reaction are assumed. Furthermore, work is done in terms of mass-averaged quantities after Favre (ref. 10). For clarity a tilde will be used for a mass-averaged temporal mean; a double prime, for the fluctuating part left over; and a conventional bar, for a regular temporal mean; thus, $u_i(x_1,x_2,x_3,t) = \frac{\overline{\rho u_i}}{\bar{\rho}} + u_i'' = \tilde{u}_i + u_i''$ with customary notation. Note that $\overline{u_i''} \neq 0$ whereas $\overline{\rho u_i''} = 0$.

Basic Equations

In terms of Cartesian tensor notation and in accord with these assumptions, it is readily established that the describing equations for the mean flow are

$$\left.\begin{aligned}
&\frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k\right) = 0 \\
&\frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k\tilde{u}_i + \overline{\rho u_k''u_i''}\right) = -\frac{\partial\bar{p}}{\partial x_i} \qquad (i = 1, 2, \ldots, N) \\
&\frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k\tilde{h}_s + \overline{\rho u_k''h_s''}\right) = 0 \\
&\frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k\tilde{Y}_i + \overline{\rho u_k''Y_i''}\right) = 0 \qquad (i = 1, 2, \ldots, N)
\end{aligned}\right\} \tag{1}$$

where Y_i is the mass fraction of species i and where h_s is the stagnation enthalpy. In more detail

$$h_s \equiv \sum_{i=1}^{N} Y_i h_i + \frac{1}{2} u_k u_k = h + \frac{1}{2} u_k u_k \tag{2}$$

where h_i is the static enthalpy of species i. From equation (2),

$$\bar{\rho}\tilde{h}_s = \bar{\rho}\tilde{h} + \frac{1}{2}\left(\bar{\rho}\tilde{u}_k\tilde{u}_k + \overline{\rho u_k''u_k''}\right) = \bar{\rho}\tilde{h} + \frac{1}{2}\bar{\rho}\left(\tilde{u}_k\tilde{u}_k + q^2\right)$$

where the mass-averaged turbulent kinetic energy is defined as

$$q^2 \equiv \frac{\overline{\rho u_k''u_k''}}{\bar{\rho}} \tag{3}$$

Thus h_s contains all forms of energy of present interest.

For this particular closure scheme based on an eddy viscosity as discussed, a conservation equation for q^2 is needed; it is found to be

$$\frac{1}{2}\frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k q^2 + \overline{\rho u_k''u_i''u_i''}\right) + \overline{\rho u_i''u_k''}\frac{\partial\tilde{u}_i}{\partial x_k} \approx \overline{-u_k''\frac{\partial p}{\partial x_k}} - \overline{\tau_{ik}\frac{\partial u_i''}{\partial x_k}} \tag{4}$$

where τ_{ik} is the viscous stress tensor.

Closure Assumptions

The closure of these equations as applicable to a thin shear layer wherein the boundary-layer approximations apply is next considered. First, in all the applications considered herein, it appears adequate to assume a single eddy transport coefficient essentially based on the mean velocity field. The generalization to a separate coefficient for each species, for the energy, and so forth is straightforward if an appropriate, separate length scale for each coefficient is introduced. Additional comments on this will be made. Thus for the present a single eddy transport coefficient is introduced and

$$
\left.
\begin{aligned}
\frac{\partial}{\partial x_k}\left(\overline{\rho u_k''u_1''}\right) &\approx \frac{\partial}{\partial x_2}\left(\overline{\rho u_2''u_1''}\right) \approx -\frac{\partial}{\partial x_2}\left(\bar{\rho}\epsilon\,\frac{\partial\tilde{u}_1}{\partial x_2}\right) \\
\frac{\partial}{\partial x_k}\left(\overline{\rho u_k''h_s''}\right) &\approx \frac{\partial}{\partial x_2}\left(\overline{\rho u_2''h_s''}\right) \approx -\frac{\partial}{\partial x_2}\left(\bar{\rho}\epsilon\,\frac{\partial\tilde{h}_s}{\partial x_2}\right) \\
\frac{\partial}{\partial x_k}\left(\overline{\rho u_k''Y_i''}\right) &\approx \frac{\partial}{\partial x_2}\left(\overline{\rho u_2''Y_i''}\right) \approx -\frac{\partial}{\partial x_2}\left(\bar{\rho}\epsilon\,\frac{\partial\tilde{Y}_i}{\partial x_2}\right)
\end{aligned}
\right\} \tag{5}
$$

The remaining terms to be modeled arise in the equation for the turbulent kinetic energy. Dependence rests heavily on previous work related to the new phenomenology of turbulent shear flows; previous ideas are adapted to the a priori and formal introduction of an eddy viscosity. Thus for the dissipation term,

$$
\overline{\tau_{ik}\,\frac{\partial u_i''}{\partial x_k}} \approx \beta_1\left(\frac{q}{\epsilon}\right)\bar{\rho}q^3 \tag{6}
$$

where β_1 is a constant for flows with constant density but is considered provisionally a function of an appropriate density ratio for flows with variable density. Equation (6) may be compared with the usual form $\overline{\tau_{ik}\left(\partial u_i''/\partial x_k\right)} \propto \left(q^3/L\right)$ where L is a suitable length scale. Here dimensional arguments are used to let $L \propto \epsilon/q$ but a density ratio $\bar{\rho}/\rho_1$ raised to some power could be introduced without compromising dimensionality.

It is customary in previous work devoted principally to constant-density flows to group the triple correlation terms and the pressure-velocity correlation together. What remains is a pressure-strain correlation which for constant-density flows is zero. Here more care must be taken; in appendix B the maintenance of the customary grouping and the addition of several new terms to account for the pressure-strain correlation prevailing in variable-density flows are heuristically justified. There results (see eq. (B9))

$$
\begin{aligned}
\frac{\partial}{\partial x_k}\left(\overline{\frac{1}{2}\,\rho u_k''u_i''u_i''}\right) + \overline{u_i''\,\frac{\partial p}{\partial x_i}} &\approx -\beta_2\,\frac{\partial}{\partial x_2}\left(\bar{\rho}\epsilon\,\frac{\partial q^2}{\partial x_2}\right) \\
&\quad + (1-\varphi)\alpha\,\frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k q^2\right) - \alpha q^2\bar{\rho}^2\tilde{u}_k\,\frac{\partial}{\partial x_k}\left(\frac{1}{\bar{\rho}}\right)
\end{aligned} \tag{7}
$$

where β_2 and α are at most functions of a density ratio, otherwise constants, and φ is a thermodynamic parameter. (See appendix B.) This procedure should be compared with the usual form for constant density, that is, $\frac{\partial}{\partial x_2}\left(qL\,\frac{\partial}{\partial x_2}\right)$. Here again L in terms of ϵ has been eliminated and the terms appropriate for variable density have been added.

The mean pressure $\bar{p}$ is now considered; the x_2-momentum equation with the pressure in the external flows set to zero yields $\bar{p} = \overline{-\rho u_2''u_2''}$. Thus in the x_1-momentum equation,

$$\frac{\partial}{\partial x_1}\left(\bar{p} + \overline{\rho u_1''u_2''}\right) = \frac{\partial}{\partial x_1}\left(\overline{\rho u_1''u_1''} - \overline{\rho u_2''u_2''}\right)$$

Although it is recognized that unless the turbulent kinetic energy is equally distributed among the three velocity components, this term is nonzero; it does not appear to contribute significantly and will be dropped.

Similarity Form

The describing equations are now transformed to similarity form and the nondimensional variables are introduced. It is considered to be convenient to include a density distortion of the x_2-coordinate since the resulting equations are formally simpler and since the inverse transformation back to the physical variables is readily performed once a similarity solution is obtained. Thus, let

$$\eta \equiv \frac{1}{x_1}\int_0^{x_2} \frac{\bar{\rho}}{\rho_1}\, dx_2' \tag{8}$$

and introduce

$$\left.\begin{aligned} f' &\equiv \frac{\tilde{u}_1}{U} \\ H &\equiv \frac{\tilde{h}_s}{h_{s1}} \\ Q^2 &\equiv \frac{q^2}{U^2} \\ \epsilon_0 &\equiv \left(\frac{\bar{\rho}}{\rho_1}\right)^2 \frac{\epsilon}{Ux_1} \end{aligned}\right\} \tag{9}$$

where a prime denotes differentiation with respect to η. Thus,

$$\left.\begin{aligned} &\left(\epsilon_0 f''\right)' + ff'' = 0 \\ &\left(\epsilon_0 H'\right)' + fH' = 0 \\ &\beta_2\left[\epsilon_0\left(Q^2\right)'\right]' + \left[\frac{1}{2} + (1 - \varphi)\alpha\right]f\left(Q^2\right)' - \beta_1\left(\frac{\bar{\rho}}{\rho_1}\right)^2 \frac{Q^4}{\epsilon_0} + \epsilon_0 f''^2 - \alpha f Q^2 \frac{\left(\rho_1/\bar{\rho}\right)'}{\left(\rho_1/\bar{\rho}\right)} = 0 \end{aligned}\right\} \tag{10}$$

The first two of these equations imply a Crocco relation

$$H = 1 + \frac{H_2 - 1}{\gamma}(1 - f') \tag{11}$$

where $H_2 \equiv \left(h_{s2}/h_{s1}\right)$. Thus, equations (10) may be considered two equations for the two unknowns $f(\eta)$ and $Q(\eta)$, provided the density ratio $\left(\bar{\rho}/\rho_1\right)$ and the eddy-viscosity parameter ϵ_0 are appropriately given.

The boundary conditions are

$$\left.\begin{aligned} & f'(\infty) = 1, \quad f'(-\infty) = 1 - \gamma, \quad f(0) = 0 \\ & Q(\pm\infty) = 0 \end{aligned}\right\} \tag{12}$$

Several remarks as to the boundary conditions on $f(\eta)$ are perhaps indicated.

First, it is known from the work of Ting (ref. 11) that in applications of analyses of laminar or turbulent two-dimensional mixing of the sort considered here, the actual location of the dividing streamline $f = 0$ depends on details of the external flow, the presence of walls, and so forth. Thus, the coordinates x_1 and x_2 must be considered to be boundary-layer coordinates along the actual dividing streamline whose location in physical coordinates may be found for a particular flow situation by application of reference 11.

Second, in the presentation of experimental data relative to two-dimensional mixing it is customary to select as the origin of x_2 the line along which f' has a particular value, for example, $f'(0) = 1 - \frac{1}{2}\gamma$. The difference between these two means for selecting an origin is simply a translation in η so that comparison between the solutions and such data is readily possible.

Form for ϵ_0

Since the eddy transport parameter ϵ_0 clearly plays a central role in these equations, it is appropriate to make some remarks about it. In terms of an academic investigation, devoid of connection with the available data, a wide variety of relations for ϵ_0 can be assumed. Indeed, even if experimental results are conscientiously considered, there remain several such relations and in the course of this study several have been examined for flows with constant and variable density. Only those which involve coupling between the mean velocity and the turbulent kinetic energy, that is, those consistent with the spirit of the Prandtl energy method are reported.

Comparisons of prediction and experiment lean heavily on the gross property spreading rate, as specified by Brown and Roshko (ref. 9). Define a parameter σ such that

$$\sigma = \frac{1.32}{\left[\left(x_2/x_1\right)_1 + \left(-x_2/x_1\right)_2\right]}$$

where $\left(x_2/x_1\right)_i$, $i = 1, 2$ are defined by the velocity ratios

$$
\left.\begin{aligned}
\frac{\tilde{u}_1 - (1 - \gamma)U}{\gamma U} &= (0.9)^{1/2} && (i = 1) \\
\frac{\tilde{u}_1 - (1 - \gamma)U}{\gamma U} &= (0.1)^{1/2} && (i = 2)
\end{aligned}\right\} \tag{13}
$$

Note that in defining σ in terms of the mass-averaged velocity $\tilde{u}_1$ rather than in terms of the usual $\bar{u}_1$ which is implicitly assumed to be the mean velocity given by experimental data, the density-velocity correlations near the outer edges of the mixing layer are neglected since $\tilde{u}_1 = \bar{u}_1 + \left(\overline{\rho' u_1'}\right)\Big/\bar{\rho}$.

Note also that there are other possible definitions of quantities defining the spreading rate but equation (13) is convenient and according to Brown and Roshko (ref. 9) yields values of σ "very close to those obtained from the more elaborate . . ." definitions. Accordingly, in considering some experimental data not treated by Brown and Roshko, the values of σ given by the experimentalist have been taken and have been assumed to be equal to that which would be given by equations (13) if the detailed velocity profiles had been available. For the case $\gamma = 1$, the most often quoted value $\sigma \equiv \sigma_0 = 11.3$ is due to Liepmann and Laufer (ref. 12). However, Birch and Eggers (paper no. 2) have pointed out that many experimentalists assume that this value is the value they would measure if they, in fact, did so, and has provided a more rational means for estimating the value of σ_0 peculiar to their setup.

In the special case of constant density,

$$
\epsilon_0 = \left(\frac{\epsilon}{Ux_1}\right) = \left(\frac{\epsilon}{q\Lambda}\right)\left(\frac{\Lambda}{x_1}\right)Q \tag{14}
$$

where Λ is a scale length on the order of the mixing-layer thickness. Now in the spirit of the Prandtl energy method it is assumed that at least for constant density

$$
\frac{q\Lambda}{\epsilon} \equiv R_q = \text{Constant} \tag{15}
$$

that is, there is a characteristic Reynolds number whose exact value depends on how Λ is defined. In cases requiring several eddy transport coefficients, it would be hoped that R_q would be constant for each value, provided consistent definitions of Λ are applied to each property being mixed.

For the single eddy transport coefficient based on the velocity and for the two-dimensional mixing considered here, it is convenient to choose for Λ a thickness defined by

$$
\Lambda = \left[U - U(1 - \gamma)\right]\left[\frac{\partial \tilde{u}_1}{\partial x_2}(x_1,0)\right]^{-1} \tag{16}
$$

This procedure is reminiscent of one means for defining the thickness of a shock wave and is readily generalized to other properties being mixed. Thus, the form of ϵ_0 examined for flows with constant density becomes

$$\epsilon_0 = \frac{\gamma}{R_q f''(0)} Q \tag{17}$$

The predictions of the analysis using equation (17) are compared with a variety of data for mixing having constant density and then ϵ_0 is reconsidered for the case of variable density.

In connection with the comparison of experiment and prediction, it is convenient to identify the similarity variable $\xi = \int_0^{\eta} \left(\frac{\rho_1}{\bar{\rho}}\right) d\eta' = \xi(\eta) = \frac{x_2}{x_1}$.

CONSTANT-DENSITY FLOWS

For constant-density flows with equation (17) taken to relate ϵ_0 to the dependent variables, a single parameter γ specifies the flow situation but R_q, β_1, and β_2 must be selected. The parameter $\varphi = 1$, and α is immaterial for these flows. For β_1 the following consideration is made: if in the last of equations (10), specialized to $\left(\bar{\rho}/\rho_1\right) \equiv 1$, it is assumed that production and dissipation of turbulent kinetic energy are roughly balanced, that is, that $\left(\beta_1 Q^4/\epsilon_0\right) \approx \epsilon_0 f''^2$, it is found that $\epsilon_0 f'' = \left(\beta_1\right)^{1/2} Q^2$. This result can be identified with the relation between the mean shear stress and the turbulent kinetic energy widely used in the new methods of analysis of turbulent shear flows. (See, for example, ref. 1.) The generally accepted constant in this relation suggests $\beta_1 = 0.024$, the value used.

Furthermore, for β_2, which appears not to be critical, it is found on the basis of numerical experimentation that an appropriate value is 0.5. Finally, again on the basis of numerical experimentation, $R_q = 22$ is taken.

Each of these three parameters has not been systematically varied to establish a set which is "optimum" in some sense but rather it has been determined on the basis of the comparisons of experiment and prediction that the cited set is adequate for many purposes. Also note that equations (10) have been solved by finite-difference methods and by iteration to handle their nonlinearity, the problem of the three-point boundary conditions, and the implicit appearance in the differential equations of $f''(0)$, a quantity obtained from the solution. No features of the numerical analysis appear to warrant special mention; in fact, the computer program was designed primarily to provide means for altering readily the several parameters and the various flow situations to be studied and not to provide efficient computation of any one case.

Now the predictions can be compared with experimental data. Solutions have been computed for a variety of values of γ and several parameters are shown in table I which are obtained from these solutions and which may have residual value. Note that $\sigma_0 = 11.4$ is predicted.

In accord with previous remarks, figure 2 shows the prediction for σ_0/σ as well as the analytic, empirical approximation due to Sabin (ref. 13), $\sigma_0/\sigma = \gamma/(2 - \gamma)$. This approximation was shown by Birch et al. to represent well a wide variety of data if σ_0 is appropriately estimated for those cases in which it is not directly measured. It is seen that the predictions for σ_0/σ are in good agreement with this empirical approximation and thus with experiment.

Next, consider a second gross property. Yule (ref. 14) has recently developed an empirical equation for the maximum turbulent kinetic energy as a function of γ. In terms of the variables used herein and with $\sigma_0 = 11.4$ for consistency, his equation becomes

$$Q^2(0) = 0.054(2 - \gamma)\gamma^2 \tag{18}$$

In figure 3 the predictions of this paper are compared with those of equation (18). Again, very reasonable agreement is obtained.

Because of the apparent high quality and completeness of the measured details of Spencer and Jones (ref. 15), they have been used to make comparisons with the predictions of this analysis. In figure 4 for $\gamma = 0.7$, the measured and predicted mean shear stresses in the form $-\epsilon_0 f'' = \left(\overline{u_1' u_2'}/U^2\right)$ are compared; in figure 5 a comparison is made for the turbulent kinetic energy in the form $Q^2 = \left(\overline{u_i' u_i'}/U^2\right)$. In both cases the agreement is good. It, of course, follows from the agreement in figure 4 that the measured and predicted mean velocity profiles $\left(\bar{u}_1/U\right)$ will also agree well. There seems little point in actually showing this comparison.

It is also perhaps of interest to compare the distribution of eddy viscosity inferred from the measurements of mean shear and mean velocity with that computed. Although the experimental results may be considered subject to possible error, the results of carrying out the comparison are shown in figure 6 for $x_1 = 23.5$ inches (59.69 cm) (distance from the virtual origin), and for $\gamma = 0.7$. The agreement must be considered highly satisfactory.

Thus on the basis of these comparisons, it is concluded that the analysis for two-dimensional mixing flows with constant density gives satisfactory agreement over the full span of velocity ratios with gross properties such as spreading rate and with some of the available detailed properties. Further refinements of the constants β_1, β_2, and R_q could presumably be made but to do so is not the main purpose of the present work.

In the case of variable-density flows the constants β_1, β_2, and R_q must be supplemented with the parameter α and the thermodynamic parameter φ. It is perhaps appropriate to make some general remarks about the approach used to incorporate the variable-density effects into the analysis.

Crucial Cases and Alterations of Analysis

As mentioned in "Introduction" there appear to be two crucial cases of variable-density, two-dimensional turbulent mixing. In the compressible adiabatic mixing, $\gamma = 1$ and to a good approximation

$$\frac{\rho_1}{\bar{\rho}} = 1 + \frac{\tilde{m}}{1 - \tilde{m}}\left(1 - f'^2\right) \tag{19}$$

where $\tilde{m} \equiv \left(U^2/2h_{s1}\right) = \frac{1}{2}(\gamma - 1)M_1{}^2\left[1 + \frac{1}{2}(\gamma - 1)M_1{}^2\right]^{-1}$ for the case of a calorically perfect gas, and where M_1 is the Mach number in the high speed stream.* Equation (19) gives the density ratio $\rho_1/\rho_2 = (1 - \tilde{m})^{-1}$.

The second crucial case is the low-speed isothermal mixing of two gases of different molecular weights, W_1 and W_2. In this case,

$$\frac{\rho_1}{\bar{\rho}} = 1 + \frac{w - 1}{\gamma}(1 - f') \tag{20}$$

where $w \equiv W_1/W_2$. In this case the density ratio for $\gamma = 1$ is $\rho_1/\rho_2 = w$.

The experimental data which apply to the first case, compressible adiabatic mixing, and which appears to be most reliable (see paper no. 2 by Birch and Eggers) indicate that a significant reduction in σ occurs as M_1 increases; in particular, $\sigma \approx 38$ for $\tilde{m} = 0.833$ $\left(M_1 = 5\right)$. On the contrary, the most reliable data for the second case seem to be that of Brown and Roshko (ref. 9) which indicate no significant change in σ with w over the range $0.143 \lessgtr w \lessgtr 7$. The essential data for $\gamma = 1$ are shown in figure 7 in terms of σ_0/σ plotted against ρ_1/ρ_2. On the basis of a comparison of equations (19) and (20), one would not expect such a disparate difference in mixing behavior.

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*In this relation for $\tilde{m}$, γ is the ratio of specific heats; there should be no confusion with our other use of γ. In deriving equation (19), the contribution of turbulent kinetic energy to the stagnation enthalpy is essentially canceled by the variation of static pressure due to $\overline{\rho u_2'' u_2''}$. It is perhaps appropriate to observe here that $\tilde{m}$ is the proper similarity parameter for compressible adiabatic mixing of calorically perfect gases so that high-speed mixing of gases other than air, for example, high-speed helium with quiescent helium, should be correlated in terms of $\tilde{m}$.

In the interest of exposition it is indicated now that with the parameters β_1, β_2, and R_q equal to the values used for constant-density flows and with $\alpha = 0.25$ and φ appropriately selected, the analysis results in predictions in reasonable agreement with those of Brown and Roshko (ref. 9). However, it underpredicts very significantly the effect of high speed on the spreading parameter. Therefore, accepting the experimentally observed difference in the two crucial cases to be correct, that is, anticipating that it will be substantiated by further experimental results, the implications thereof are now discussed, that is, determination of the effect of Mach number through the parameter $\tilde{m}$. This apparent difference, accepted herein, in behavior of the two crucial cases rules out several possible means for incorporating the effects of variable density. Several means have been studied but have been abandoned, at least for the time being.

The most obvious candidate to account for the difference in the two cases is the pressure fluctuations which are expected to become more significant with Mach number. It is shown in appendix B that in the variable-density case, the pressure correlations introduce effects into the equation for the turbulent kinetic energy through the diffusion term, the convection term, and through an added term directly associated with the density gradient. However, this equation, and thus presumably the turbulent kinetic energy, is dominated by the production and dissipation terms, neither of which are directly influenced by the pressure fluctuations. Thus it is found by numerical experimentation that unreasonable values for β_2, treated as a function of $\tilde{m}$, must be assumed in order to achieve a significant alteration of σ with $\tilde{m}$. It thus appears that at least up to a Mach number of 5, pressure effects cannot account for a significant compressibility effect on two-dimensional turbulent mixing. This result is in accord with the finding of Laufer (ref. 16) for turbulent boundary layers.

If the changes in the empirical parameters with $\tilde{m}$ that might be rationalized to account for the observed effect on σ are considered, the parameter β_1 which accounts directly for dissipation and R_q appearing in ϵ_0 remain. Again by numerical experimentation, it is found that unreasonably large changes in β_1 with $\tilde{m}$ are required to bring prediction and experiment into agreement. Moreover, there appears to be no way to rationalize such changes; in this regard, note that such would imply a considerable deviation with $\tilde{m}$ of the coefficient in the frequently assumed relation between mean shear and turbulent kinetic energy. Although it is not certain that some alteration of this coefficient is not in fact indicated, the alteration associated with the required changes in β_1 is probably excessive.

Thus, alterations of R_q with Mach number, that is, of the only parameter at our disposal, are considered. As is seen in more detail subsequently, rather modest changes in R_q with $\tilde{m}$ can bring the predictions in agreement with the data for compressible adiabatic mixing. Consider here the possible physical explanation therefor, and keep in mind a distinction between density and Mach number effects. With all the other parameters in ϵ_0 fixed, the value of R_q determines the rate of entrainment of external flow into the turbulent layer. This entrainment is known to be related to the detailed mechanisms connected with the superlayer at the interface between the potential and turbulent flows. Because of detailed experiments involving conditioned sampling (see Kovasznay et al. (ref. 17) and Kaplan and Laufer (ref. 18)), a great deal is known about these mechanisms for low-speed turbulent boundary layers, for example, about the relative speed of the large turbulent eddies and the external flow. Although the corresponding data for the high-speed mixing layer do not exist and will be extremely difficult to obtain, it appears intuitively clear that when the relative speed of the large-scale turbulent eddies and the external flow approaches a significant fraction of the local speed of sound, major changes in the detailed entrainment mechanisms could occur. The observed decrease in spreading angle and mixing rate with free-stream Mach number, for example, as implied by figure 7, supports this view. These effects are concluded to be attributed to R_q in the present analysis.

Accordingly, a strategy of seeking $R_q = R_q(\tilde{m})$ is adopted to bring the analysis into agreement with the case of compressible adiabatic mixing. Beyond the case of $\gamma = 1$, that is, of mixing with a quiescent gas, it is expected that R_q will depend on γ as well as on $\tilde{m}$ when the second stream becomes supersonic. Resolution of this effect awaits further experimental data.

Isothermal Binary Mixing

Consider now the low-speed isothermal mixing of two dissimilar gases and treat $w \equiv \left(W_1/W_2\right)$ and the velocity ratio by means of γ as parameters. The assumed values of w include helium-air cases with the molecular weight of air assumed to be 28. The value of φ is taken for simplicity to be three and represents its arithmetic mean for air and helium; these results are independent of any reasonable value for φ since it enters only in the convection term, that is, in one of the small terms, in the equation for the turbulent kinetic energy. Also $\alpha = 0.25$ is taken in all cases since it too does not enter importantly. Several of the combinations of w and γ are chosen to permit comparison with the experimental results of Brown and Roshko (ref. 9) for helium and air mixing.

Table II gives the principal gross results from this series of calculations. Several points are indicated there. For a given velocity ratio, that is, γ, the effect of the molecular weight ratio w on the spreading parameter σ is small. This result is in accord with the main conclusion of Brown and Roshko (ref. 9). Moreover, simple calculations using the results in table II show that for a given w, the effect of γ on σ is closely given by the equation of Sabin (ref. 13) developed for constant-density flows. This prediction may be useful in determining σ_0 for binary mixing flows after the suggestion of Birch and Eggers (paper no. 2).

The results with respect to the spreading parameter σ for $\gamma = 1$ are shown in figure 7 in terms of σ_0/σ plotted against $\rho_1/\rho_2 = w$ in this case. The comparisons with the results deduced by Brown and Roshko (ref. 9) are seen to be good, as suggested earlier; that is, when no changes in the effective parameters β_1, β_2, and R_q are made, the spreading parameter is insensitive to density variations.

Attention is now turned to some of the detailed results of Brown and Roshko (ref. 9). In figure 8 a comparison is made of the predicted velocity and density profiles for one of their experiments corresponding to $\gamma = 0.622$, $w = 0.143$, that is, helium at a higher velocity mixing with slower air. It is seen that the velocity profile is well predicted but that the density profile reflects the single eddy transport coefficient assumed a priori in the analysis, whereas the experimental results indicate a considerably larger coefficient for the species. Contrast this result with another case shown in figure 9, $\gamma = 0.622$, $w = 7$, corresponding to faster moving air mixing with slower helium. Within the ability to read the plots in reference 9, it is not possible to distinguish the prediction from experiment so only the predicted results are shown. The reason for the difference in the predictability of the two cases is not clear.

It is of interest to consider the predicted alteration in the distribution of some mean quantities with molecular weight or density ratio. Accordingly, the velocity profiles for the constant density case ($w = 1$) and the hydrogen-air cases $w = 14$, (1/14), each with zero velocity on one side of the mixing layer, are shown in figure 10. Although the spreading parameter is about the same in all three cases, it may be seen that the variation in mean density results in altered velocity profiles.

In figures 11 and 12 are shown the predicted distributions of production of turbulent kinetic energy in terms of $\epsilon_0 f''^2(\eta)$ and of dissipation in terms of $-\beta_1\left(\bar{\rho}/\rho_1\right)^2 Q^4 \epsilon_0^{-1}$ plotted against $\xi = x_2/x_1$ for these same three cases. There is a shift in the peaks of production and dissipation toward the low-density side of the dividing streamline and, as expected, the dissipation term is less than the production term in the central portion of the mixing layer. From the calculations it is found that the diffusion term provides most of the difference in the two main terms in the equation for the turbulent kinetic energy. It would be highly desirable if some measurements of the turbulent kinetic energy could be made in binary mixing flows to permit comparisons with predictions of the sort shown in figures 11 and 12 in order to establish whether the energy balances are as predicted or whether changes in the modeling are required.

It is thus concluded that the analysis for constant-density flows without essential change (the changes due to $\overline{p\left(\partial u_k''/\partial x_k\right)}$ are considered to be "unessential" here) largely agrees with the results of Brown and Roshko (ref. 9) and shows little effect on the gross quantity, spreading parameter σ, although detailed distributions are as is to be expected altered by density variations.

Compressible Adiabatic Mixing

Attention is now turned to the case of compressible adiabatic mixing. Only $\gamma = 1$, that is, mixing with a quiescent gas, is considered and thus there is a single parameter, that associated with the Mach number in the moving stream, $\tilde{m}$. The previous values $\beta_1 = 0.024$, $\beta_2 = 0.5$, and $\alpha = 0.25$ are retained and $\varphi = 7/2$ is taken, corresponding to air. Again, the values of α and φ are unimportant. However, as discussed previously, R_q is adjusted to bring into agreement prediction and experiment with respect to the spreading parameter σ.

As a result of preliminary numerical experimentation, it is found that a simple variation of R_q with $\tilde{m}$ gives a variation of spreading parameter with $\rho_1/\rho_2 = (1 - \tilde{m})^{-1}$ in reasonable agreement with the data shown in figure 7; thus,

$$\left.\begin{aligned} R_q &= 22 + 5.2(1 - \tilde{m})^{-1} && (0 \leqq \tilde{m} \lessapprox 0.7) \\ R_q &= 39 && (0.7 \lessapprox \tilde{m} < 1) \end{aligned}\right\} \tag{21}$$

This variation of R_q is used for the calculations discussed here. The implication of this result in terms of the previous discussion of the possible physical explanation of a change in R_q with Mach number appears to be that at a Mach number of about 3 the large-scale eddies in the mixing layer readjust so that further effects of Mach number on the spreading parameter are small and so that the characteristic Reynolds number R_q may be taken as a constant at higher Mach numbers. This explanation is, of course, to be considered a conjecture based on a careful consideration of the present analysis and the experimental data presently considered definitive.

With the use of equations (21) a series of compressible adiabatic flows corresponding to a range of $\tilde{m}$ have been computed; the principal gross results are given in table III and the variation of the spreading parameter in terms of σ_0/σ with ρ_1/ρ_2 is shown in figure 7. It is seen that the simple variation of R_q with $\tilde{m}$ given by equations (21) results in increases in σ with $\tilde{m}$ in good agreement with the experimental data.

In figures 13 to 15 are shown the same distributions of mean quantities as in the case of binary mixing in order to provide some indication of the effect of high speed on turbulent mixing. From figure 13 it can be seen that a significant change in the velocity profile takes place when $\tilde{m}$ increases from 0 to 0.667 $\left(M_1\right.$ from 0 to $\left.\approx 3\right)$ but that little change seems to take place as $\tilde{m}$ increases further despite the doubling of the density ratio ρ_1/ρ_2 as $\tilde{m}$ increases from 0.667 to 0.833. This behavior is also reflected in figures 14 and 15 which show that the distributions of production and dissipation of turbulent kinetic energy for $\tilde{m} = 0.667$ and $\tilde{m} = 0.833$ are alike but are very different from the low-speed case ($\tilde{m} = 0$).

It may also be noted from figures 14 and 15 that the peaks in production and dissipation shift toward the low-density side of the dividing streamline as in the case of binary mixing. Also to be noted is the close balance between production and dissipation which indicates, as suggested earlier, that the other terms in the equation for conservation of turbulent kinetic energy are dominated by these two terms.

The General Case

On the basis of the results for the two crucial cases of turbulent mixing studied in some detail, the present analysis could be used with the parameters β_1, β_2, and α fixed, with φ determined from thermodynamic considerations, and with $R_q(\tilde{m})$ as given by equations (21) to make predictions of the properties of a variety of mixing flows involving heterogeneous composition, high-speed effects, and nonadiabaticity. These have not been carried out in the present work because there appear to be no data with which to compare prediction and experiment. When such data are available, it would be of considerable interest to determine the extent of the agreement and/or disagreement.

Of considerable value in assessing the a priori assumption of a single transport coefficient and the overall accuracy of the present analysis for these fundamental flows, that is, for two-dimensional turbulent mixing, would be cases of heterogeneous mixing under nonisothermal conditions, for example, heated helium mixing with relatively cold air. The greater the statistical detail constituting the data the more valuable such experiments would be.

CONCLUDING REMARKS

A simple turbulent flow, the two-dimensional mixing layer, involving significant variations in density due either to heterogeneity in composition or to compressibility effects associated with the high speed of one flow has been analyzed in some detail. One of the simpler of the new methods of analysis of turbulent shear flows, that usually termed the "Prandtl energy method," has been used. It involves relating the eddy transport coefficient to the turbulent kinetic energy and to an appropriate length scale of the large-scale turbulent motion. Thus, one equation in the second level in the hierarchy of describing equations, that describing the conservation of turbulent kinetic energy, must be added to the usual set of equations describing the variables of the mean flow.

For two-dimensional mixing layers with constant density it is found that the analysis provides satisfactory predictions for gross properties of the flow over the entire range of velocity ratios of the two streams and for detailed properties for the one case, that is, one velocity ratio, examined.

With respect to flows involving variable density, there is discussed the existing experimental data for two crucial cases, the low-speed isothermal mixing of two dissimilar gases and the high-speed adiabatic mixing of air. It has been assumed provisionally that the results of Brown and Roshko, which show little alteration of the spreading rate with large density differences in the first case, and the results which show significant reduction in the spreading rate with Mach number in the second case are correct, and that they will be confirmed by further experiments. With this assumption fixing the strategy to be followed, it is found that an alteration with Mach number of the empirical parameters entering the analysis is required to bring prediction and experiment into agreement. The analysis suggests that neither the modeling of pressure rate of strain nor alterations in two parameters, those relating to diffusion and dissipation in the equation for the turbulent kinetic energy, can reasonably account for the observed Mach number effect. However, it is shown that a modest change in the Reynolds number characterizing the turbulence reduces the spreading rate in accord with experiment. A suggestion is made of how this required change is related to the physics of the turbulence.

It is emphasized that if further experiments do not support the assumed behavior of the two-dimensional mixing layer in the two crucial cases cited, the methodology of the present work may retain some value, but the strategy used to bring prediction and experiment into agreement would, of course, be altered. It is also emphasized that this study indicates the dangers of casually extending the new methods of analysis of turbulent shear flows to cases involving significant variations of density.

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APPENDIX A

SYMBOLS

c_p	coefficient of specific heat
c_{pi}	coefficient of specific heat of i species
f	stream function (see eq. (9))
H	stagnation enthalpy ratio (see eq. (9))
H_2	stagnation enthalpy ratio, h_{s2}/h_{s1}
h	static enthalpy
h_i	static enthalpy of i species
h_s	stagnation enthalpy (see eq. (2))
L	length scale
M_1	Mach number in stream 1
$\tilde{m}$	Mach number parameter (see eq. (19))
p	static pressure
Q	nondimensionalized turbulent kinetic energy
q	turbulent kinetic energy, $\dfrac{\overline{\rho u_i' u_i'}}{\bar{\rho}}$
R_0	universal gas constant
R_q	Reynolds number of turbulence
T	static temperature
t	time

U	x-component of velocity in stream 1
u_i	Cartesian velocity components, $i = 1, 2, 3$
W	molecular weight of mixture
W_i	molecular weight of i species
w	molecular weight ratio, W_2/W_1
x_i	Cartesian coordinates, $i = 1, 2, 3$
Y_i	mass fraction of i species
α	fraction of turbulent kinetic energy due to u_2
β_1,β_2	empirical constants
γ	parameter determining velocity of stream 2; ratio of coefficients of specific heat
ϵ	eddy viscosity
ϵ_0	nondimensionalized eddy viscosity
η	transformed similarity variable (see eq. (8))
Λ	length scale
ν	kinematic viscosity coefficient
ξ	similarity variable, x_2/x_1
ρ	density
ρ_1	density in stream 1
σ	spreading parameter

σ_0	spreading parameter when $u_i = 0$
τ_{ik}	viscous stress tensor
φ	thermodynamic parameter (see eq. (B7))

APPENDIX B

THE PRESSURE-STRAIN CORRELATION

Here the modeling of the velocity-pressure gradient correlation appearing in equation (4) is considered. The first step is to write

$$\overline{u_k'' \frac{\partial p}{\partial x_k}} = \frac{\partial}{\partial x_k}\left(\overline{pu_k''}\right) - \overline{p \frac{\partial u_k''}{\partial x_k}} \tag{B1}$$

For constant-density flows the second set of terms on the right-hand side is zero. Here the general case of a gas mixture described by a perfect gas law is considered. First

$$p = \rho \frac{R_0 T}{W} = \rho R_0 T \sum_{i=1}^{N} \frac{Y_i}{W_i} \tag{B2}$$

where R_0 is the universal gas constant and W is the mixture molecular weight. The conservation equations devoid of significant molecular effects are

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x_k}\left(\rho u_k\right) = 0 \\
&\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_k}\left(\rho u_k u_i\right) = -\frac{\partial p}{\partial x_i} \qquad (i = 1, 2, 3) \\
&\frac{\partial}{\partial t}\left(\rho h_s\right) + \frac{\partial}{\partial x_k}\left(\rho u_k h_s\right) = \frac{\partial p}{\partial t} \\
&\frac{\partial}{\partial t}\left(\rho Y_i\right) + \frac{\partial}{\partial x_k}\left(\rho u_k Y_i\right) = 0 \qquad (i = 1, 2, . . ., N)
\end{aligned}\right\} \tag{B3}$$

The pressure relative to that in the two streams external to the mixing layer is measured so that $p\left(x_1, \pm\infty, x_3, t\right) \equiv 0$. In addition, the stagnation enthalpy is defined as

$$h_s \equiv h + \frac{1}{2} u_k u_k = \sum_{i=1}^{N} Y_i h_i(T) + \frac{1}{2} u_k u_k \tag{B4}$$

From mass conservation and the equation of state, it is easy to show that

$$\overline{p \frac{\partial u_k''}{\partial x_k}} = R_0 \left\{ \overline{\frac{\rho}{W}\left(\frac{\partial T}{\partial t} + u_k \frac{\partial T}{\partial x_k}\right)} + \overline{\rho T\left[\frac{\partial}{\partial t}\left(\frac{1}{W}\right) + u_k \frac{\partial}{\partial x_k}\left(\frac{1}{W}\right)\right]} \right\} - \overline{u_k \frac{\partial p}{\partial x_k}} - \bar{p} \frac{\partial \tilde{u}_k}{\partial x_k} \tag{B5}$$

Next from equations (B3) and (B4),

$$\overline{\frac{\rho}{W}\left(\frac{\partial T}{\partial t} + u_k \frac{\partial T}{\partial x_k}\right)} = \overline{\left(\frac{u_k}{c_p W}\right)\frac{\partial p}{\partial x_k}}$$

and

$$\overline{\rho T\left[\frac{\partial}{\partial t}\left(\frac{1}{W}\right) + u_k \frac{\partial}{\partial x_k}\left(\frac{1}{W}\right)\right]} = \sum_{i=1}^{N} \overline{T \frac{\rho}{W_i}\left(\frac{\partial Y_i}{\partial t} + u_k \frac{\partial Y_i}{\partial x_k}\right)} = 0$$

where

$$c_p = \sum_{i=1}^{N} c_{pi} Y_i$$

Thus

$$\overline{p \frac{\partial u_k''}{\partial x_k}} = \overline{\left(\frac{R_0}{c_p W} - 1\right) u_k \frac{\partial p}{\partial x_k}} - \bar{p} \frac{\partial \tilde{u}_k}{\partial x_k} \tag{B6}$$

Except for certain dissipation terms which would enter if molecular effects were included in equations (B3), equation (B6) is exact. Drastic simplifications are now made. It hardly seems reasonable to add to the result of the crude modeling usually associated with the first set of terms on the right-hand side of equation (B1) and complicated modeled terms arising from the second set. First note that $R_0/c_p W$ is to a good approximation constant because the model specific heat $c_{pi} W_i$ is roughly constant. Thus let

$$\overline{p \frac{\partial u_k''}{\partial x_k}} \approx \left(\overline{\frac{R_0}{c_p W}} - 1\right)\left(\tilde{u}_k \frac{\partial \bar{p}}{\partial x_k} + \overline{u_k'' \frac{\partial p}{\partial x_k}}\right) - \bar{p} \frac{\partial \tilde{u}_k}{\partial x_k}$$

Then by letting $\varphi^{-1} \equiv \left(\overline{R_0/c_p W}\right)$, it is found from equation (B1) that

$$\overline{u_k'' \frac{\partial p}{\partial x_k}} \approx \varphi \frac{\partial}{\partial x_k}\left(\overline{p u_k''}\right) - (1 - \varphi)\tilde{u}_k \frac{\partial \bar{p}}{\partial x_k} + \varphi \bar{p} \frac{\partial \tilde{u}_k}{\partial x_k} \tag{B7}$$

Note now that the x_2-momentum equation in the boundary-layer approximation and with the external pressure set equal to zero yields

$$\bar{p} = \overline{-\rho u_2'' u_2''} = -\alpha \bar{\rho} q^2 \tag{B8}$$

where α is a parameter (taken to be constant) representing the fraction of the total turbulent kinetic energy in the u_2-velocity fluctuations. Most low-speed data for turbulent shear flows indicate $\alpha \approx 1/4$ but there does not appear to be any data related thereto for variable-density flows.

With equation (B8), equation (B7) can be rewritten to yield

$$\overline{u_k'' \frac{\partial p}{\partial x_k}} \approx \varphi \frac{\partial}{\partial x_k}\left(\overline{pu_k''}\right) + (1 - \varphi)\alpha \frac{\partial}{\partial x_k}\left(\bar{\rho}\tilde{u}_k q^2\right) - \alpha q^2 \bar{\rho}^2 \tilde{u}_k \frac{\partial}{\partial x_k}\left(\frac{1}{\bar{\rho}}\right) \tag{B9}$$

Several remarks about equation (B9) are appropriate. Equation (B9) reduces to that for constant density flows if $\varphi = 1$ so the usual model in this case is recovered. For compressible adiabatic flows, $\varphi = \gamma/(\gamma - 1)$ where γ is the ratio of specific heats. For binary isothermal flows, φ is really a function of concentration but an adequate approximation would appear to be an average value across the mixing layer. Finally, note that the inclusion of density effects in the velocity-pressure gradient correlation introduces two new terms (one directly related to the mean density gradient, and one to the convection of turbulent kinetic energy) and modifies the usual combination of the diffusion of turbulent kinetic energy and the pressure-velocity correlation.

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TABLE I.- NUMERICAL RESULTS FOR CONSTANT DENSITY FLOWS

γ	f'(0)	f''(0)	$Q^2(0)$	σ_0
1	0.578	6.07	6.42×10^{-2}	11.4
0.8	0.643	7.18	4.17	16.8
0.7	0.680	7.74	3.22	20.6
0.6	0.721	8.31	2.37	25.8
0.4	0.808	9.46	1.06	43.1
0.2	0.902	10.6	0.265	97.0

TABLE II.- NUMERICAL RESULTS FOR BINARY FLOWS

γ	w	$f'(\eta = 0)$	$f''(\eta = 0)$	$Q^2(0)$	$\frac{\rho_1}{\bar{\rho}(0)}$	σ_0
1	1	0.578	6.07	6.42×10^{-2}	1	11.4
	2	0.624	8.63	6.92	1.38	11.2
	4	0.664	13.0	6.88	2.01	11.6
	7	0.694	18.7	6.82	2.83	12.1
	14	0.729	30.7	6.70	4.53	13.2
	0.5	0.544	4.59	6.94	0.772	10.9
	0.25	0.505	3.68	6.92	0.629	11.1
	0.143	0.487	3.20	6.87	0.560	11.1
	0.0714	0.450	2.79	6.88	0.489	11.8
0.8	7	0.744	22.3	4.40	2.92	17.2
0.622		0.794	25.6	2.67	2.99	24.8
0.4		0.861	29.8	1.11	3.08	42.5
0.2		0.928	33.7	0.279	3.15	91.0
0.857	0.143	0.529	3.45	5.10	0.529	15.0
0.622		0.635	3.93	2.70	0.497	25.2
0.4		0.756	4.42	1.12	0.478	50.0
0.2		0.875	4.93	0.279	0.466	105

TABLE III.- NUMERICAL RESULTS FOR COMPRESSIBLE ADIABATIC MIXING

$[\gamma = 1]$

$\tilde{m}$	$f'(\eta = 0)$	$f''(\eta = 0)$	$Q^2(0)$	$\frac{\rho_1}{\bar{\rho}(0)}$	σ_0	R_q
0	0.578	6.07	6.42×10^{-2}	1	11.4	22
0.333	0.610	14.6	4.12	1.31	19.8	30
0.5	0.626	21.6	3.52	1.61	24.4	32
0.667	0.649	38.2	2.77	2.16	31.6	38
0.75	0.664	54.3	2.47	2.68	36.4	39
0.833	0.684	77.3	2.43	3.66	38.4	39
0.875	0.700	99.3	2.41	4.57	40.0	39

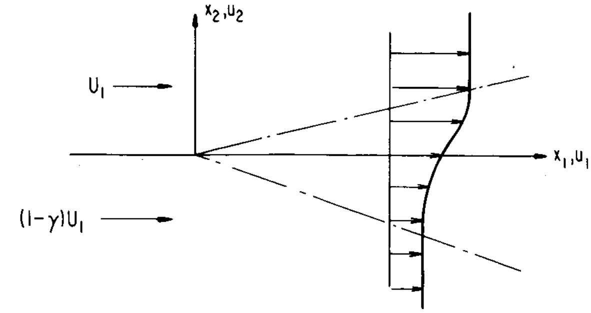

Figure 1.- Schematic representation of the flow.

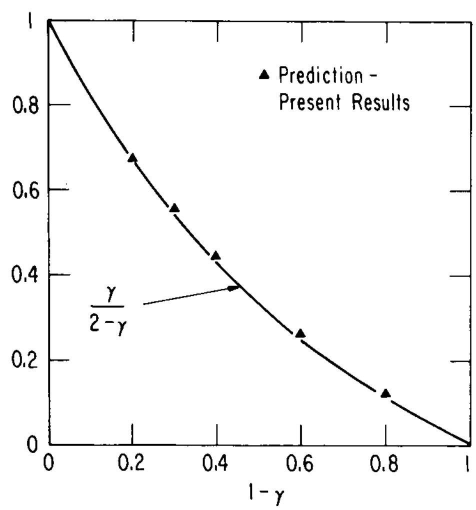

Figure 2.- Variation of spreading parameter with velocity ratio. Constant density flows.

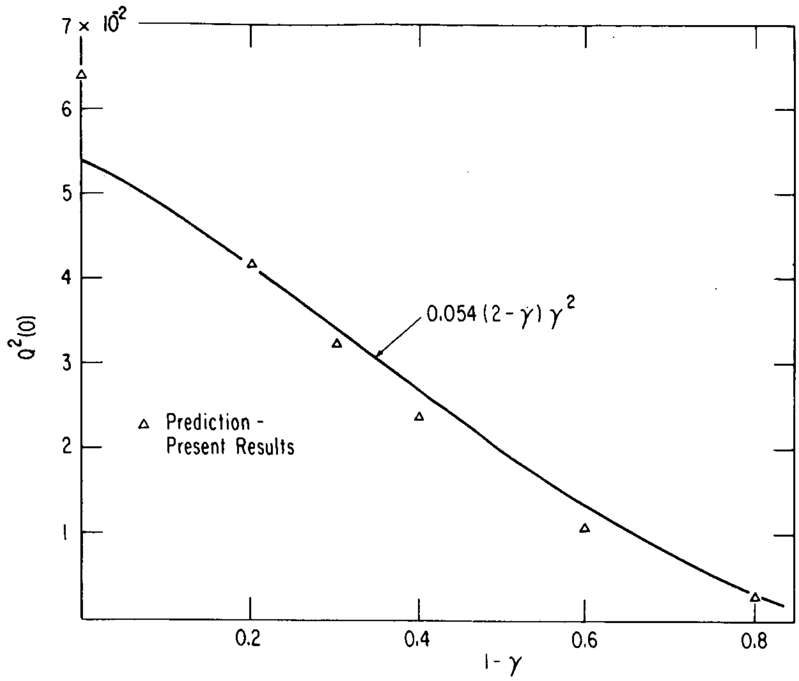

Figure 3.- Variation of turbulent kinetic energy on the dividing streamline with velocity ratio. Constant density flows.

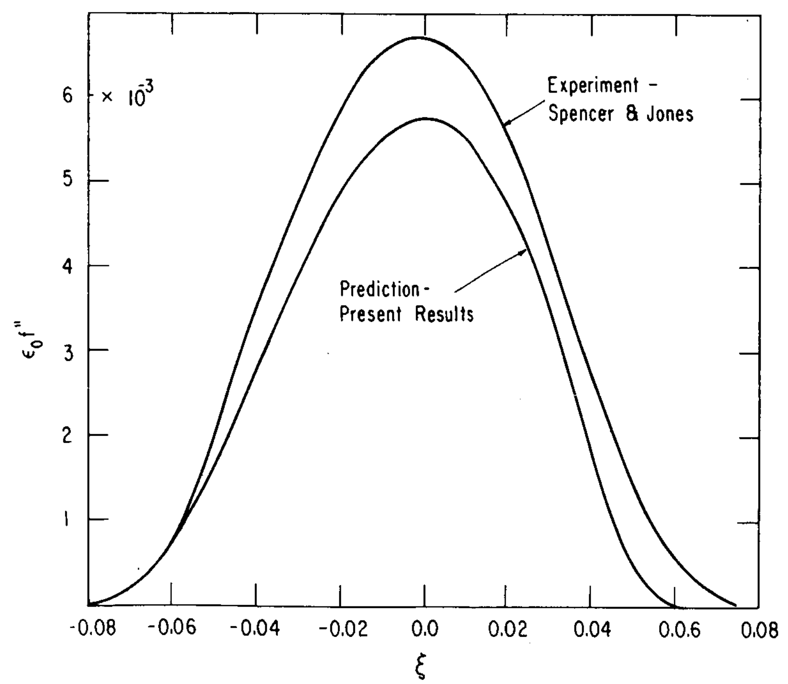

Figure 4.- Comparison of predicted and experimental mean shear stress. $\gamma = 0.7$; constant density flows.

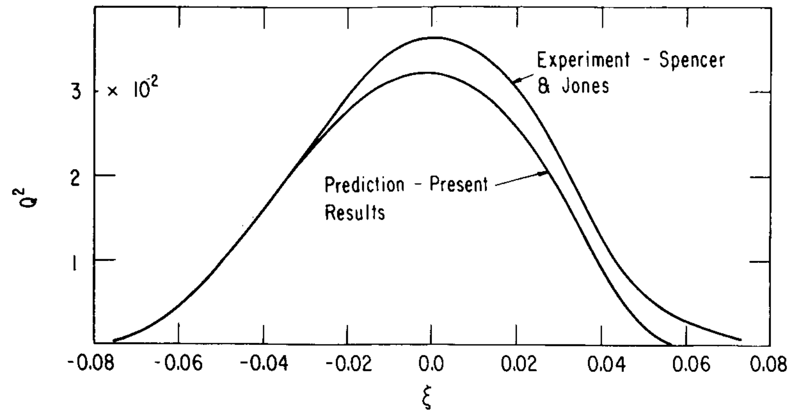

Figure 5.- Comparison of predicted and measured turbulent kinetic energy. γ = 0.7; constant density flows.

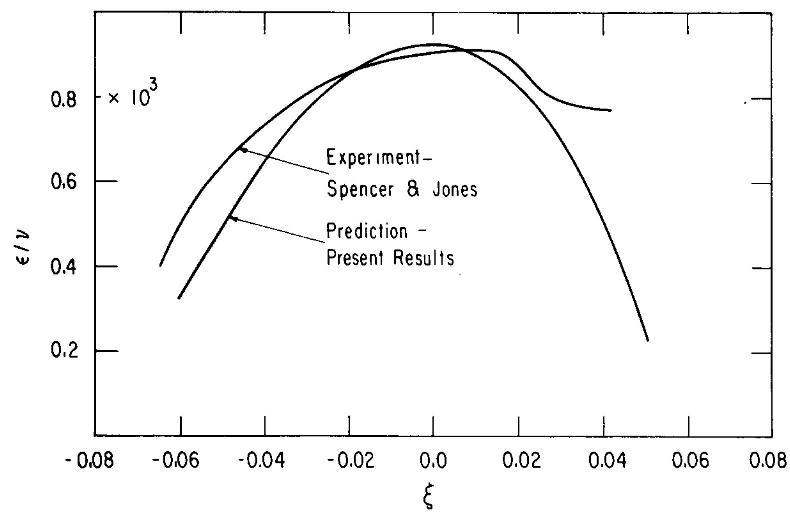

Figure 6.- Comparison of predicted and measured distributions of eddy viscosity. γ = 0.7; constant density flows; x_1 = 23.5 inches (59.69 cm).

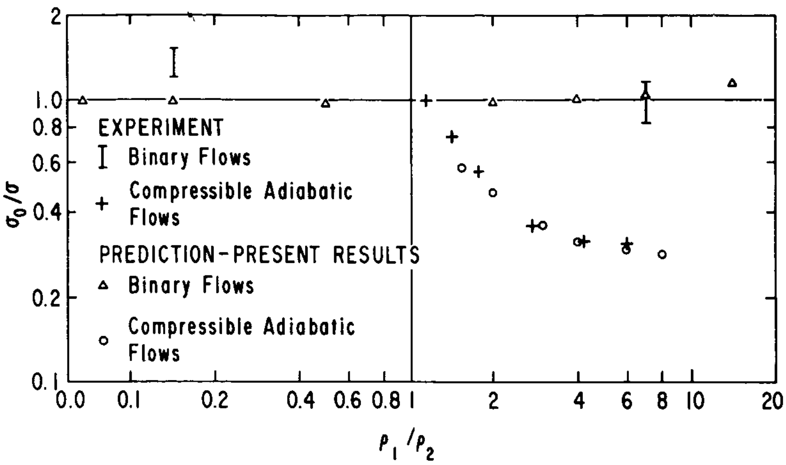

Figure 7.- Variation of spreading parameter in variable-density flows.

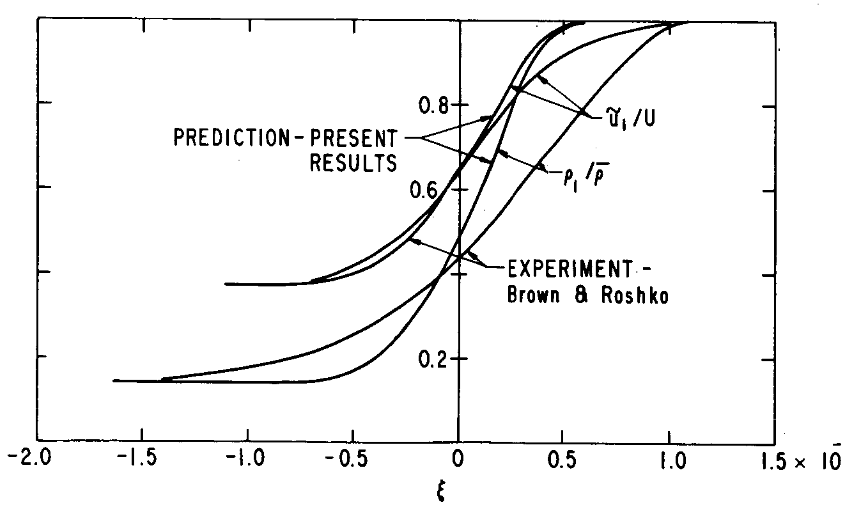

Figure 8.- Comparison of predicted and experimental mean velocity and mean density profiles. $\gamma = 0.622$; w = 0.143.

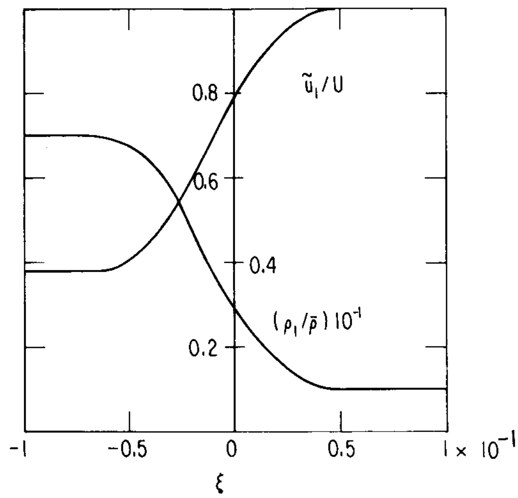

Figure 9.- Velocity and mean density profiles. $\gamma = 0.622$; w = 7.

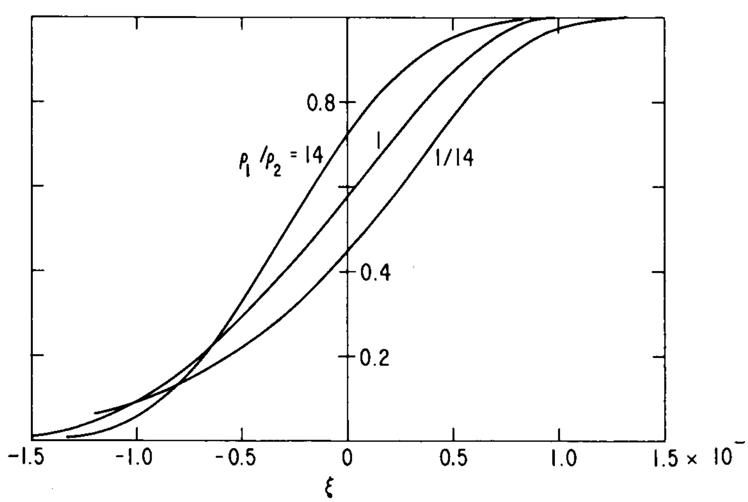

Figure 10.- Predicted velocity profiles for several density ratios. Binary mixing; $\gamma = 1$.

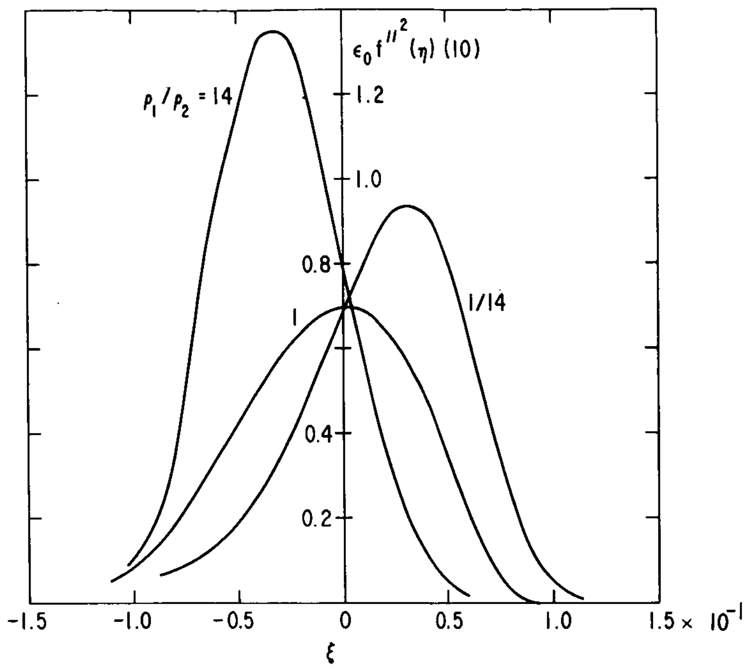

Figure 11.- Predicted distributions of production of turbulent kinetic energy for several density ratios. Binary mixing; $\gamma = 1$.

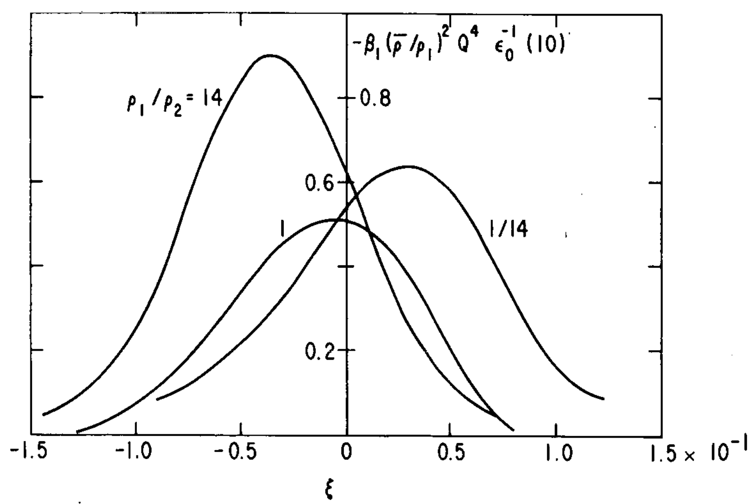

Figure 12.- Predicted distributions of dissipation of turbulent kinetic energy for several density ratios. Binary mixing; $\gamma = 1$.

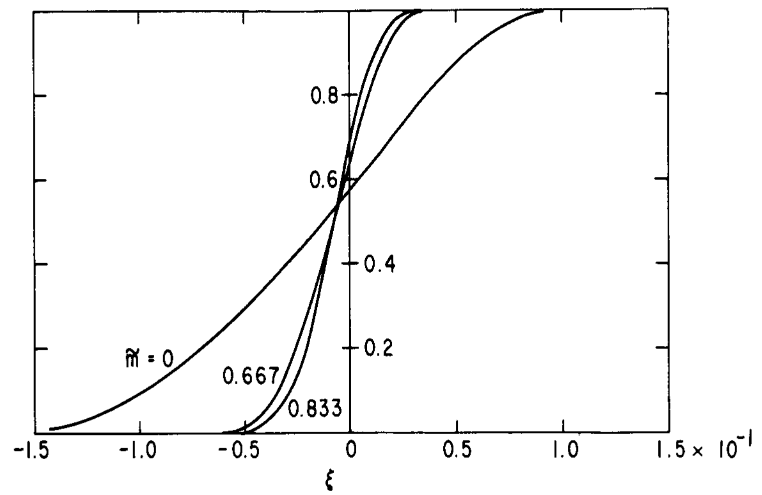

Figure 13.- Predicted velocity profiles for several values of the Mach number parameter. Compressible adiabatic mixing; $\gamma = 1$.

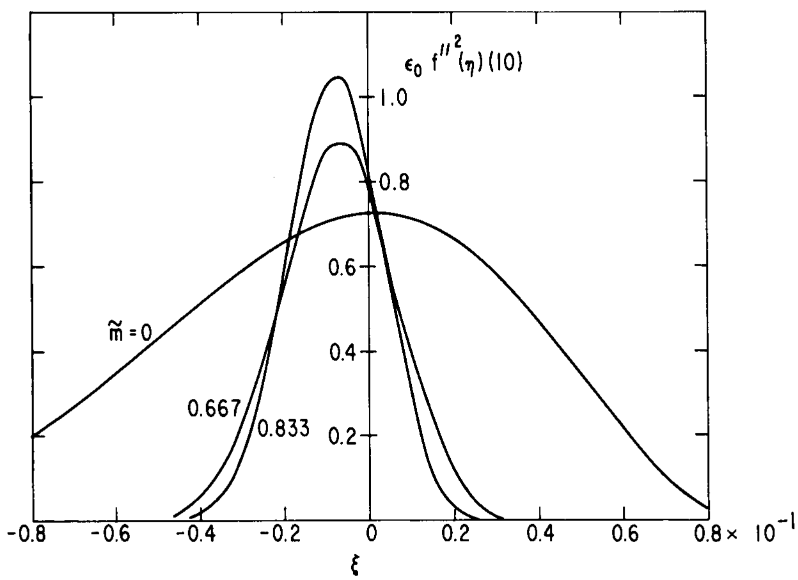

Figure 14.- Predicted distributions of production of turbulent kinetic energy for several values of the Mach number parameter. Compressible adiabatic mixing; $\gamma = 1$.

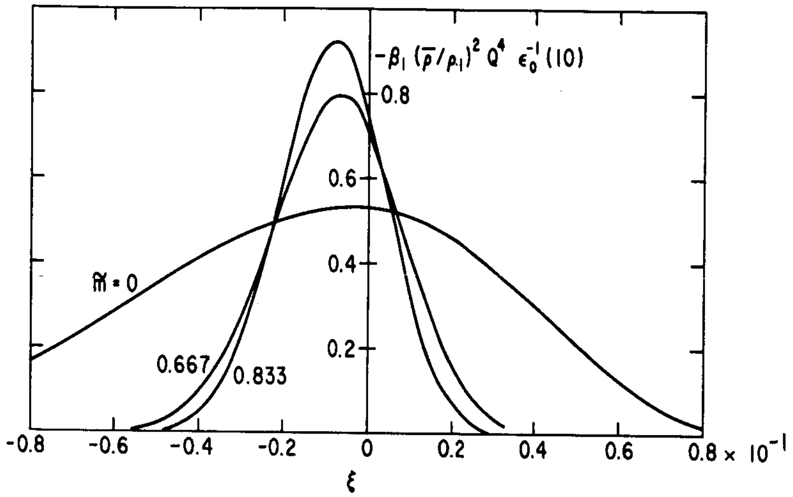

Figure 15.- Predicted distributions of dissipation of turbulent kinetic energy for several values of the Mach number parameter. Compressible adiabatic mixing; $\gamma = 1$.

DISCUSSION

J. Laufer: I think Libby showed us very well the extent of our ignorance as far as density fluctuations in the turbulent flow are concerned. At this stage I would like to just ask one question – and maybe that question should be more properly addressed to Professor Roshko. Dr. Libby pointed out the difference between the conventional average and mass average velocity. I suspect that in many cases, the experimentalist measures a mass average quantity with a pitot tube. Professor Roshko, could you tell us, is it a mass average or a mean velocity that you are measuring?

A. Roshko: Well, that is a little hard to say. The velocities are measured by getting a pitot tube reading for ρu^2, but there is a little question of what you are reading with a pitot tube because you have the turbulence part of the term. When we do have an accurate measurement of the average density, we simply divide one by the other. What that average means is hard to say. We estimate that it's perhaps within 10 percent.

J. Laufer: I would like to suggest just on the basis of the simple Bernoulli's equation that one uses in obtaining the velocity that that value is, in fact, a mass average and not a conventional average.

S. J. Kline: On the same point, we have some experiments going on in which we are trying to vary the fluctuations and see what happens in a pitot under some controlled conditions. I don't want to go into details, but perhaps we can shed some light on your question. One more comment, we are getting ready to measure time averages by using laser Doppler techniques that we are developing, and then you will have a little more information on this point.

P. A. Libby: John, that's exactly why I said, "if we take the experimenter's word for it that he is giving us the usual average," then we are, in fact, comparing two slightly different things. I understand, as we have talked before, about the fact that the experimenter may be wrong, but I do think that we have to take his word for it.

J. Ito: We have been working with heterogenous shear flows for applications to the gas-gas auxiliary propulsion on the space shuttle system. We have obtained quite a bit of data in the past 6 to 8 months with cold flow and chemical reactions. One of the things that we have determined empirically was that we can best correlate mixing rates with pressure-gradient-type terms when you look at heterogenous flows of variable densities. In our case, we used gaseous hydrogen and gaseous nitrogen to simulate our oxygen, and we found that our best correlating parameters were really $\frac{1}{2}\rho u^2$ of the two differing jets, and the ratios of them.