STOCHASTIC ESTIMATES OF GRADIENT FROM LASER MEASUREMENTS FOR AN AUTONOMOUS MARTIAN ROVING VEHICLE

by

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LIST OF SYMBOLS

R = Range measurement (meters)
β = Elevation angle of laser beam (degrees)
θ = Azimuth angle of laser beam (degrees)
ϕ = Roll angle of the vehicle (degrees)
ξ = Pitch angle of the vehicle (degrees)
$C(\phi)$ = Roll transformation matrix
$B(\xi)$ = Pitch transformation matrix
h = Vertical axis of unrotated coordinate system
a = Axis of unrotated coordinate system
b = Axis of unrotated coordinate system
h'' = Vertical axis of vehicle coordinate system
a'' = Cross-Path axis of vehicle coordinate system
b'' = In-Path axis of vehicle coordinate system
x_1 = Cross-path slope of terrain (degrees)
x_2 = In-path slope of terrain (degrees)
Sg = Gradient of terrain (degrees)
$\hat{x}$ = Least square estimate of x
δ = Perturbation
M = Covariance matrix of the variables
E = Expected value
σ = Standard deviation
σ^2 = Variance
v = Noise due to measurement of h and A
J = Cost function
$\tilde{x}$ = Minimum variance estimate of x
I = Identity matrix

ABSTRACT

The general problem presented in this paper is one of estimating the state vector x from the state equation $h=Ax$ where h, A, and x are all stochastic. Specifically, the problem is for an autonomous Martian Roving Vehicle to utilize laser measurements in estimating the gradient of the terrain. Error exists due to two factors - surface roughness and instrumental measurements. The errors in slope depend on the standard deviations of these noise factors. Numerically, the error in gradient is expressed as a function of instrumental inaccuracies. Certain guidlines for the accuracy of permissable gradient must be set. It is found that present technology can meet these guidlines.

PART 1

INTRODUCTION

A comprehensive navigation system will be needed for a proposed Mars rover to safely traverse the surface of Mars with reasonable speed over a long distance. Because the roundtrip communication time to earth requires more than 40 minutes, the vehicle's terrain modeling and path selection systems must be autonomous. The system is designed to collect terrain data within a 3 to 30 meter range. The range finder, which locates a point on the terrain, is a laser/dectector which gives a range measurement R,[1] azimuth angle θ, and elevation angle β. Two points along the path of the vehicle determine an in-path slope, while those across the path compute a cross path slope. However, inaccuracy in measurement can introduce very large errors in the computed slopes and heights, which are the main factors in path selections. There are some threshold values for these factors above which a change of path is required.[2]

PART 2

METHOD OF APPROACH

From the measurement data it is desirable to obtain the maximum slopes and the elevations of the terrain in front of the vehicle.

A. Transformation of Coordinate Systems

The quantities R, θ, and β are measured with respect to the coordinate system h", a", and b", fixed to the vehicle. (see Fig. 1) With laser height at 3 meters we have:

$$h'' = 3 - R\sin\beta \qquad (1a)$$

$$a'' = R\cos\beta\sin\theta \qquad (1b)$$

$$b'' = R\cos\beta\cos\theta \qquad (1c)$$

The body-bound axis rolls with the angle ϕ and pitches through an angle ξ about a reference frame h, a, and b formed by the local vertical and an axis in a plane containing the heading and the local vertical. The coordinate transformation[3] is:

$$\begin{bmatrix} h \\ a \\ b \end{bmatrix} = C(\phi)\ B(\xi) \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix} \qquad (2a)$$

where

$$C(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2b)$$

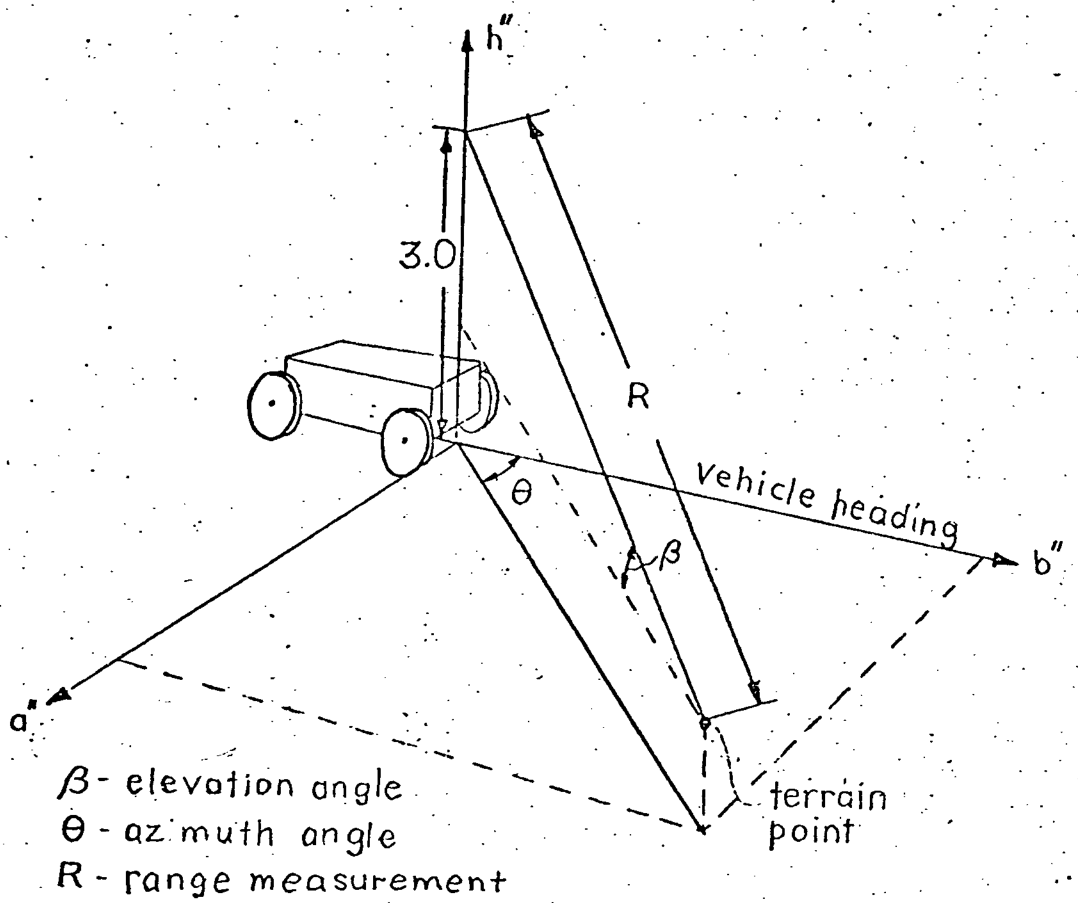

Fig. 1 Vehicle Coordinate System

$$B(\xi) = \begin{bmatrix} \cos\xi & 0 & \sin\xi \\ 0 & 1 & 0 \\ -\sin\xi & 0 & \cos\xi \end{bmatrix} \tag{2c}$$

B. Determination of Slopes and Gradient

A number of measurement points within a certain small area of surface, say 0.5m by 0.5m, can be used to determine a plane in space, which may be written as

$$h = ax_1 + bx_2 + x_3 \tag{3}$$

where x_1 and x_2 are two constant parameters to be determined. Taking the differential of equation (3) we get

$$dh = \frac{\partial h}{\partial a}da + \frac{\partial h}{\partial b}db = x_1 da + x_2 db$$

where,

$$x_1 = \frac{\partial h}{\partial a} = \text{cross-path slope}$$

$$x_2 = \frac{\partial h}{\partial b} = \text{in-path slope}$$ [4,5]

The corresponding gradient of the plane, which is defined as the gradient of the terrain in that small region on the planet's surface, is

$$Sg = (x_1{}^2 + x_2{}^2)^{1/2} \tag{4}$$

If this slope is less than a predetermined criterion, it is considered to be safe for the vehicle to travel ahead.

In order to locate the plane by a number of measurement points, one may rewrite Eq. (3) as

$$h_i = a_i x_1 + b_i x_2 + x_3 \tag{5}$$

where h_i, a_i, and b_i are found from R_i, θ_i, β_i in equations (1) and (2) for each i^{th} point. Theoretically, three points determine a plane ($i = 1,2,3$). For greater accuracy however, more than three measurements ($i = 1,2,3,..,n$) (probably $n = 4$ or 6) are needed to determine the slopes. A complete picture of the terrain in front of the vehicle can be constructed by modeling numerous adjacent planes, each covering a small area of surface.

PART 3

ANALYTIC SOLUTIONS FOR PARAMETER ESTIMATION

In this section we list the solutions of the least square estimate, the covariance matrices, and the minimum variance estimate when $n>3$.

A. Least Square Estimate of Slopes

If $a_i=\bar{a}_i$ and $b_i=\bar{b}_i$ are assumed to be true in Eq. (5), a least square error estimate can be performed which minimizes

$$\sum_{i=1}^{n} (\hat{h}_i-\bar{h}_i)^2 \tag{6}$$

where $\bar{h}_i$ is the actual measured height and $\hat{h}_i$ is the corresponding height in the modeled plane.

In matrix-vector notation, Eq. (5) is

$$\bar{h} = \bar{A}x \tag{7}$$

where

$$\bar{h} = (\bar{h}_1,\bar{h}_2, \ldots,\bar{h}_n)^T$$

$$x = (x_1,x_2,x_3)^T$$

and

$$\bar{A} = \begin{bmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ \vdots & \vdots & \vdots \\ a_n & b_n & 1 \end{bmatrix} \qquad n = 4 \text{ or } 6$$

The least square estimate of the parameter x becomes[6]

$$\hat{x} = (\bar{A}^T\bar{A})^{-1}\bar{A}^T\bar{h} \tag{8}$$

B. Perturbation of the Variables

The least square estimate in the previous section assumes the matrix A to be completely deterministic[7].

In reality, however, there is error involved in the determination of a, b, and h, due to the errors in our measured values of ϕ, ξ, R, β, and θ.

If the symbol δ denotes a perturbation, then δh, δa, and δb in terms of $\delta\phi$, $\delta\xi$, δR, $\delta\beta$, $\delta\theta$, are

$$\begin{bmatrix}\delta h\\ \delta a\\ \delta b\end{bmatrix} = D(h'', a'', b'', \phi, \xi)\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix} + C(\phi)B(\xi)G(R,\theta,\beta)\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix} \quad (9a)$$

where

$$D(h'', a'', b'', \phi, \xi) \doteq \begin{bmatrix}(-h''\sin\phi\cos\xi - a''\cos\phi - b''\sin\phi\sin\xi) & (-h''\cos\phi\sin\xi + b''\cos\phi\cos\xi)\\ (h''\cos\phi\cos\xi - a''\sin\phi + b''\cos\phi\sin\xi) & (-h''\sin\phi\sin\xi + b''\sin\phi\cos\xi)\\ 0 & (-h''\cos\xi - b''\sin\xi)\end{bmatrix} \quad (9b)$$

and

$$G(R,\theta,\beta) = \begin{bmatrix}(-\sin\beta) & (-R\cos\beta) & 0\\ (\cos\beta\sin\theta) & (-R\sin\beta\sin\theta) & (R\cos\beta\cos\theta)\\ (\cos\beta\cos\theta) & (-R\sin\beta\cos\theta) & (-R\cos\beta\sin\theta)\end{bmatrix} \quad (9c)$$

The derivation of these matrices are given in Appendix (A).

C. Covariance Matrix of the Variables

We can define the covariance matrix of the variables as[8]

$$M = E\left\{\begin{bmatrix}\delta h\\ \delta a\\ \delta b\end{bmatrix}\begin{bmatrix}\delta h & \delta a & \delta b\end{bmatrix}\right\} \quad (10)$$

where E denotes expected value.

If $\delta\phi$, $\delta\xi$, δR, $\delta\beta$, and $\delta\theta$ are not correlated,[8] then from Appendix B we have

$$M = D\begin{bmatrix} E(\delta\phi)^2 & 0 \\ 0 & E(\delta\xi)^2 \end{bmatrix} D^T + CBG\begin{bmatrix} E(\delta R)^2 & 0 & 0 \\ 0 & E(\delta\beta)^2 & 0 \\ 0 & 0 & E(\delta\theta)^2 \end{bmatrix} G^T B^T C^T \quad (11)$$

From Eqs. (10) and (11) the standard deviations of h, a, and b, can be computed in terms of those of ϕ, ξ, R, β, and θ for each point. These are known quantities that depend upon the accuracy of the measuring devices.

D. Covariance Matrix of the Slopes

Eq. (5) can be written as

$$h = Ax \quad (12a)$$

If we set $h = \bar{h} + \delta h$, $A = \bar{A} + \delta A$, and $x = \bar{x} + \delta x$, (12b)

then with the aid of Eq. (7) we get

$$\delta h = \bar{A}\delta x + \delta A\bar{x} \quad (12c)$$

The estimate is[7] $\delta\hat{x} = F(\delta h - \delta A\bar{x})$ (13)

where $F = (\bar{A}^T\bar{A})^{-1}\bar{A}^T$ (14)

The covariance matrix of the slopes is determined in Appendix C as

$$E\left[\delta\hat{x}\delta\hat{x}^T\right] = F\left\{E\left[\delta h\delta h^T\right] - E\left[\delta A\bar{x}\delta h^T\right] - E\left[\delta h(\delta A\bar{x})^T\right] + E\left[\delta A\bar{x}(\delta A\bar{x})^T\right]\right\}F^T \quad (15)$$

where

$$\delta\hat{x} = \begin{bmatrix} \delta\hat{x}_1 \\ \delta\hat{x}_2 \\ \delta\hat{x}_3 \end{bmatrix} \quad \delta h = \begin{bmatrix} \delta h_1 \\ \vdots \\ \delta h_n \end{bmatrix} \quad A\bar{x} = \begin{bmatrix} (\delta a_1\bar{x}_1 + \delta b_1\bar{x}_2) \\ \vdots \\ (\delta a_n x_1 + \delta b_n x_2) \end{bmatrix} \quad (16)$$

Since A is in terms of a and b, then δh and δA can be expressed as functions of $\delta\phi$, $\delta\xi$, δR, $\delta\beta$, and $\delta\theta$ as given in Eq. (9a). Eq. (15) can be evaluated as shown in Appendix D.

E. Variance of the Gradient

If the symbol σ_{Sg} denotes the standard deviation of Sg, then from Eq. (4) we have

$$dSg = (x_1^2+x_2^2)^{-1/2}x_1dx_1 + (x_1^2+x_2^2)^{-1/2}x_2dx_2$$

The variance of the gradient[9] is

$$\sigma_{Sg}^2 = \frac{\bar{x}_1^2}{\bar{x}_1^2+\bar{x}_2^2}\sigma_{x_1}^2 + \frac{\bar{x}_2^2}{\bar{x}_1^2+\bar{x}_2^2}\sigma_{x_2}^2 + 2\frac{\bar{x}_1\bar{x}_2}{\bar{x}_1^2+\bar{x}_2^2}\sigma_{x_1x_2}^2 \tag{17}$$

where $\sigma_{x_1}^2 = E(\delta\hat{x}_1)^2$, $\sigma_{x_2}^2 = E(\delta\hat{x}_2)^2$, $\sigma_{x_1x_2}^2 = E(\delta\hat{x}_1\delta\hat{x}_2)$

The covariances $\sigma_{x_1}^2$, $\sigma_{x_2}^2$, and $\sigma_{x_1x_2}^2$ can be found from Eq. (15).

The value of σ_{Sg} gives a rough estimate of the accuracy in our estimation of the gradient. If the estimated Sg is 20° with $\sigma_{Sg}=2°$, then we can be 68% sure that the actual slope is between 18° and 22° and 95% sure[8] that it is between 16° and 24°. This is important when the estimation is close to the maximum permissable value.

For the case where $x_1 = x_2$ one obtains from

Eq. (4) $$Sg = \sqrt{2}\, x_1$$

Thus $$dSg = \sqrt{2}\, dx_1 \quad \text{and} \quad \sigma_{Sg}^2 = 2\sigma_{x_1}^2 \tag{17a}$$

F. Minimum Variance Estimate[8]

A generalized form of Eq. (7) is

$$v = h - A\bar{x} \tag{18}$$

where v is the noise due to measurement of h and A in Eq. (5).

If the expected values of Eq. (18) are

$$E(h) = \bar{h}\ , \quad E(A) = \bar{A} \tag{19}$$

then
$$E(v) = \bar{v} = \bar{h} - \bar{A}\bar{x} = 0 \tag{20}$$

by virtue of Eq. (12c). Subtracting Eq. (20) from Eq. (18) with the aid of the definitions of Eq. (12b), one obtains

$$v \cong \delta h - \delta A\bar{x} \tag{21}$$

Thus the covariance of v becomes

$$R = E\left[vv^T\right] = E\left[(\delta h - \delta A\bar{x})(\delta h - \delta A\bar{x})^T\right]$$
$$= E\left[\delta h \delta h^T\right] - E\left[\delta A\bar{x}\delta h^T\right] - E\left[\delta h(\delta A\bar{x})^T\right] + E\left[\delta A\bar{x}(\delta A\bar{x})^T\right] \tag{22}$$

which is the same as those terms inside the bracket in Eq. (15). These terms are evaluated in Appendix D.

The cost functional J can be expressed as

$$J = v^T R^{-1} v = (h-A\bar{x})^T R^{-1}(h-A\bar{x})$$
$$= h^T R^{-1} h - \bar{x}^T A^T R^{-1} h - h^T R^{-1} A\bar{x} + \bar{x}^T A^T R^{-1} A\bar{x} \tag{23}$$

The minimum variance estimate of $\bar{x}$ can be obtained by taking the minimum of J with respect to x.

$$\mathrm{Min}_{\bar{x}}\ J = \frac{\partial J}{\partial \bar{x}} = -2A^T R^{-1} h + 2A^T R^{-1} A\bar{x} = 0 \tag{24}$$

Thus the minimum variance estimate x becomes

$$\tilde{x} = (A^T R^{-1} A)^{-1} A^T R^{-1} h \tag{25}$$

where R can be obtained from Eq. (22).

The minimum variance estimate given by Eq. (25) is a weighted least square estimate. It will provide a better estimate of the gradient then the standard least square estimate of Eq. (8) because it gives correct weight to the individual measurement points.[9] The minimum variance estimate utilizes the value $\hat{x}$ given by the least square estimate to evaluate the covariance matrix R given by Eq. (22).

However, the numerical results in the example computed from Eq. (25) is very close to the results given by Eq. (8) because the weighting factor in the covariance matrix R is nearly proportional to an identity matrix. That is to say, the weighting factors in the diagonal terms of R are equal. This is equivalent to an unweighted least square estimate. The details of the derivation are shown in Appendix E where R=kI.

G. Determination of Gradient

Once the values for the cross-path and in-path slopes are determined from the minimum variance estimate, the gradient can be determined from Eq. (4). Knowing the value of the gradient and its variance, the rover can make decisions concerning safe and unsafe terrain.

PART 4

NUMERICAL RESULTS

It is assumed that the rover will use a split beam where $\Delta\beta$ is the difference in the elevation angle between the two beams and $\Delta\theta$ is the difference in azimuth angle between anY two laser pulses. This is shown in Fig. 2.

The values of β and θ for each data point, along with the magnitude of the cross-path and in-path slopes of the terrain, determine the data point spacing[5] and the distance of the data points from the vehicle. After a number of data points are measured, they are transformed to the non-rotating reference system by Eqs. (1a to 2c). Then a least square estimate of the slopes is performed which is given by Eq. (8). This least square estimate depends directly upon the values of h,a, and b for each measurement point. Utilizing the value of the vector $\hat{x}$ found in Eq. (8), the values of ϕ, ξ, R, β, and θ for each data point, and the standard deviations of these quantities, we can ultimately find the magnitude of the gradient and its variance thru the proceedures outlined in sections 3b-3f.

The variance of the gradient depends in part upon the magnitude of the cross-path and in-path slopes of the terrain and the accuracy of the measuring devices. It also depends upon ϕ, ξ, R, β, and θ, or alternately,

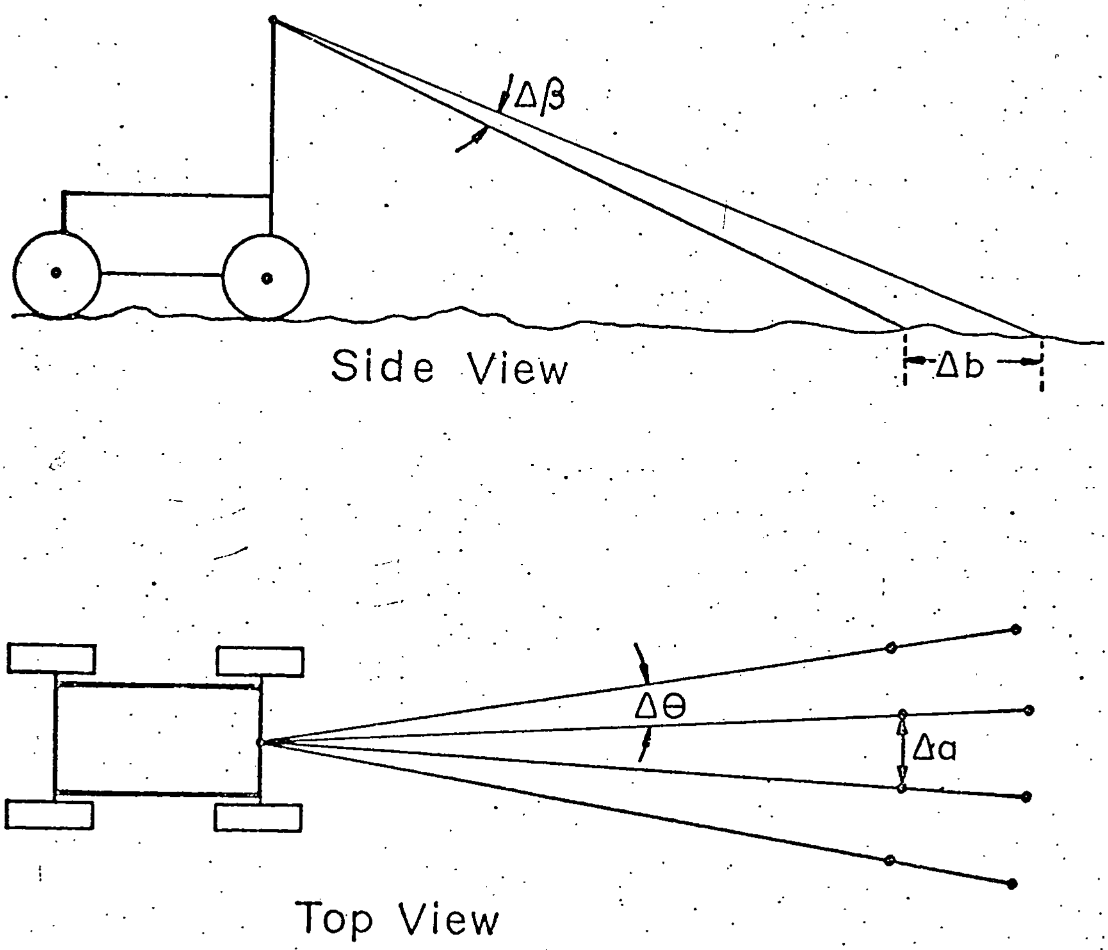

Fig. 2 Split Beam Laser

upon the roll and pitch angles, the data point spacing, and the distance of the data points from the vehicle. How much each of these factors affects the standard deviation of the gradient is to be found.

The standard deviation in gradient for any combination of these can be determined by utilizing Eqs. (9b),(9c),(11),(14),(15), and (17). The data point spacing and the distance of the data points from the vehicle can be varied by changing the values of β and θ for each data point. The vector x completely determines the gradient of the plane we are 'measuring' and along with specified values of ϕ, ξ, β, and θ we can geometrically find the values for R, h, a, b, h", a", and b" which are utilized in above equations.

For simplicity, we first set $\phi = \xi = 0°$ for all the measurement points and assume the reference gradient of the measured plane to be 0° wherever it may be located. By applying Eqs. (9b), (9c), (11), (14), (15), and (17a) and setting $\sigma_\beta = \sigma_\theta = .1°$, and $\sigma_\phi = \sigma_\xi = 1°$ in Eq. 11 we find that for any reasonable data point spacing at 20 - 30 meters from the vehicle, σ_{sg} was on the order of 30° - 60°. On the other hand, if $\sigma_\phi = \sigma_\xi = 0°$ in Eq. (11), σ_{sg} decreased to only 2°-3° at that distance.

Since it can be expected that σ_ϕ and σ_ξ are really in the neighborhood of 1° (due to the constant rock and roll of the vehicle as it traverses the surface), the rover will have to be provided with a 'rapid scan' laser. If the scan rate is on the order of milliseconds, which is perfectly feasible with electronic scanning,[1] then each set of 4 adjacent data points is measured practically instantaneously, since the rover motion is on the order of seconds. Each of the 4 data points will have the same value for ϕ and ξ and therefore all 4 points will retain the same relative position to each other when they are transformed from the vehicle coordinate system to the fixed system. Consequently, the rover can model the planes in the vehicle coordinate system and then transform the planes to the non-rotating system, instead of first transforming each point to the non-rotating frame. In the analysis, this corresponds to setting $\phi = \xi = 0^\circ$ in Eqs. (9b), (9c), and (11)′ and setting h"=h, a"=a, and b"=b in these equations. Since the maximum positive or negative slope that the vehicle can navigate is $\pm 25^\circ$, relative slopes might be as high as $\pm 50^\circ$ and still be navigable. This is shown in Fig. 3. Therefore, in my analysis, I consider a range of slope changes from $+50^\circ$ to -50° where possible.

Also, due to rapid scan, the error introduced by a standard deviation in σ_ϕ and σ_ξ will be consistently between 1° and 2° because it only involves the error in transforming the already modeled plane from the vehicle system of reference to the fixed system. Consequently,

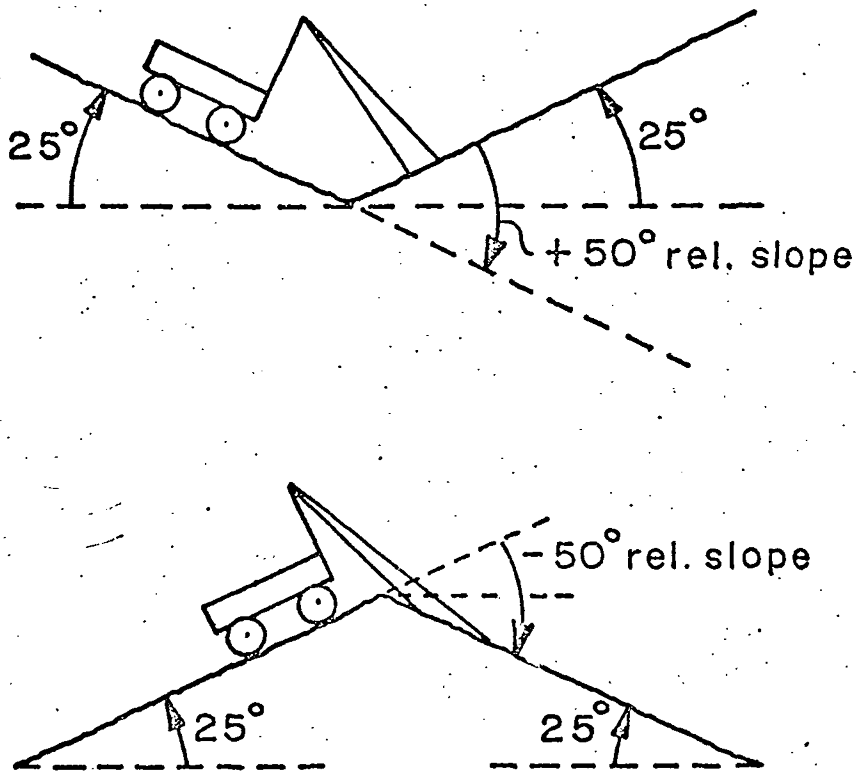

Fig. 3 Relative In-Path Slopes

the effect of the standard deviations in ϕ and ξ is not considered in the analysis and σ_ϕ and σ_ξ are set to 0° in Eq.(11). It must also be realized that any relative slopes between $+50^\circ$ and -50° can represent critical values in the fixed system and therefore all relative slopes are important.

If the vector $\overline{x}=0$ in Eqs.(15) and (17a), $\sigma_\beta = \sigma_\theta = 1'$ and $\sigma_R = 5\text{cm}$ in Eq.(11), and $\Delta\beta$ and $\Delta\theta$ (and consequently the data point spacing) are varied for each set of 4 points, then the graph shown in Fig.4 results. This corresponds to collecting data from a flat terrain. Each solid line represents constant values for $\Delta\beta$ and $\Delta\theta$. The quantities Δb and Δa are the data point spacings along the in-path and cross-path directions respectively. By looking at any solid line it can be seen that for any constant values of $\Delta\beta$ and $\Delta\theta$, the quantities Δb and Δa decrease very rapidly as we scan closer to the vehicle and consequently σ_{S_g} rises very rapidly.

In choosing an 'optimum' spacing it must be kept in mind that by increasing the spacing Δb and Δa, the standard deviation σ_{S_g} decreases. This also renders the data less meaningful, as more terrain has been overlooked. At the 30 meter range, a general picture of the terrain with a spacing of 2-3 meters is sufficient. At close range, (4-7 meters) the points should be at least as close as 0.66 meters because this is the width of the widest

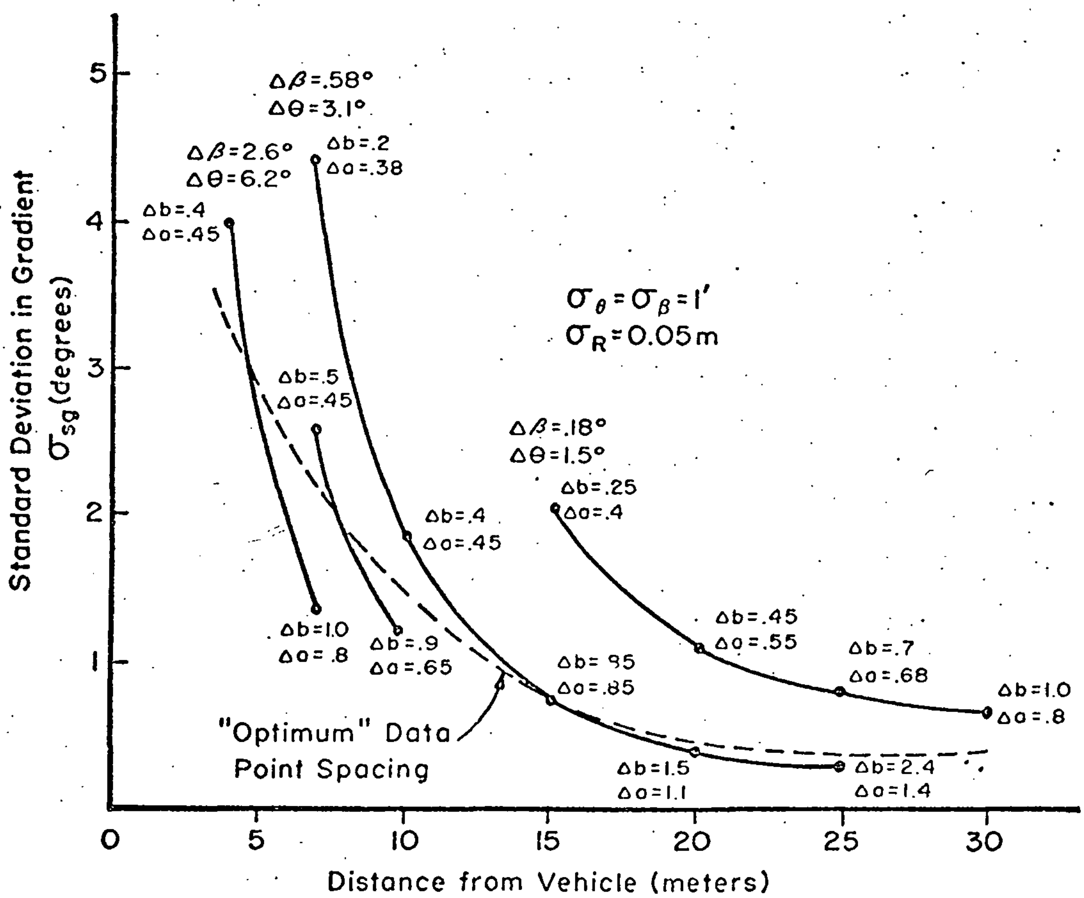

Fig. 4 Standard Deviation in Gradient vs. Distance from Vehicle on Flat Surface.

navigable crevice.[2] In Fig.4, the dotted line represents σ_{sg} vs. distance from the vehicle for such a scheme, where the data point spacing varies from about .5 meters at a distance of 4 meters from the vehicle, to a spacing of about 3 meters at a distance of 30 meters from the vehicle.

Each of the plots in Figs.5 and 6 utilize the same optimum scheme as the dotted line in Fig.4. Once again the vector $\bar{x}=0$ (corresponding to flat terrain). In Fig.5, σ_R is kept at 5cm inEq.(11) and σ_β and σ_θ are set to 0°, $.017^\circ$(1'), and $.1^\circ$(6'). It is seen that σ_{sg} is markedly reduced by decreasing σ_β and σ_θ from 6' to 1', but any further increases in accuracy will result in bulkier and heavier equipment which is not very beneficial. By comparing the plots for $\sigma_\beta = \sigma_\theta = 1'$ and $\sigma_\beta = \sigma_\theta = 0^\circ$, it is seen that σ_β and σ_θ have very little effect on σ_{sg} when they have a value of 1'. In Fig.6, σ_R is varied from 1cm to 10cm in Eq.(11) while σ_β and σ_θ are kept constant at 1'(.017). The quantity σ_{sg} increases rapidly for each value of σ_R at close range in Fig.6.

Fig.7 plots σ_{sg} vs. relative in-path slopes of from -30° to $+50^\circ$ at 4 meters from the vehicle. This means that the vector $\bar{x}$ is varied in Eq.(15) and (17) such that the cross path slope $\bar{x}_1$ remains 0°, the in-path slope $\bar{x}_2$ varies from -30° to $+50^\circ$, and $\bar{x}_3$ changes in such a manner that there is always at least 1 data point with a

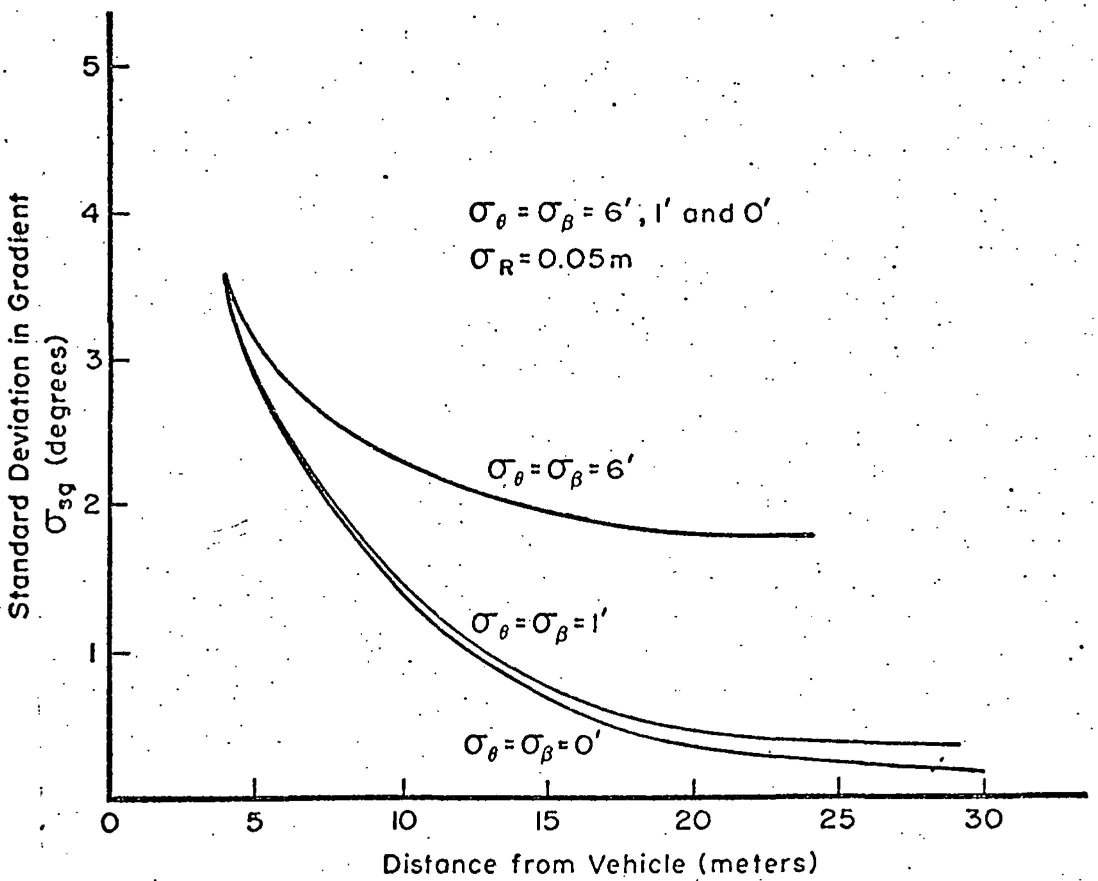

Fig. 5 Standard Deviation in Gradient vs. Distance from Vehicle for "Optimum" Data Point Spacing on Flat Surface.

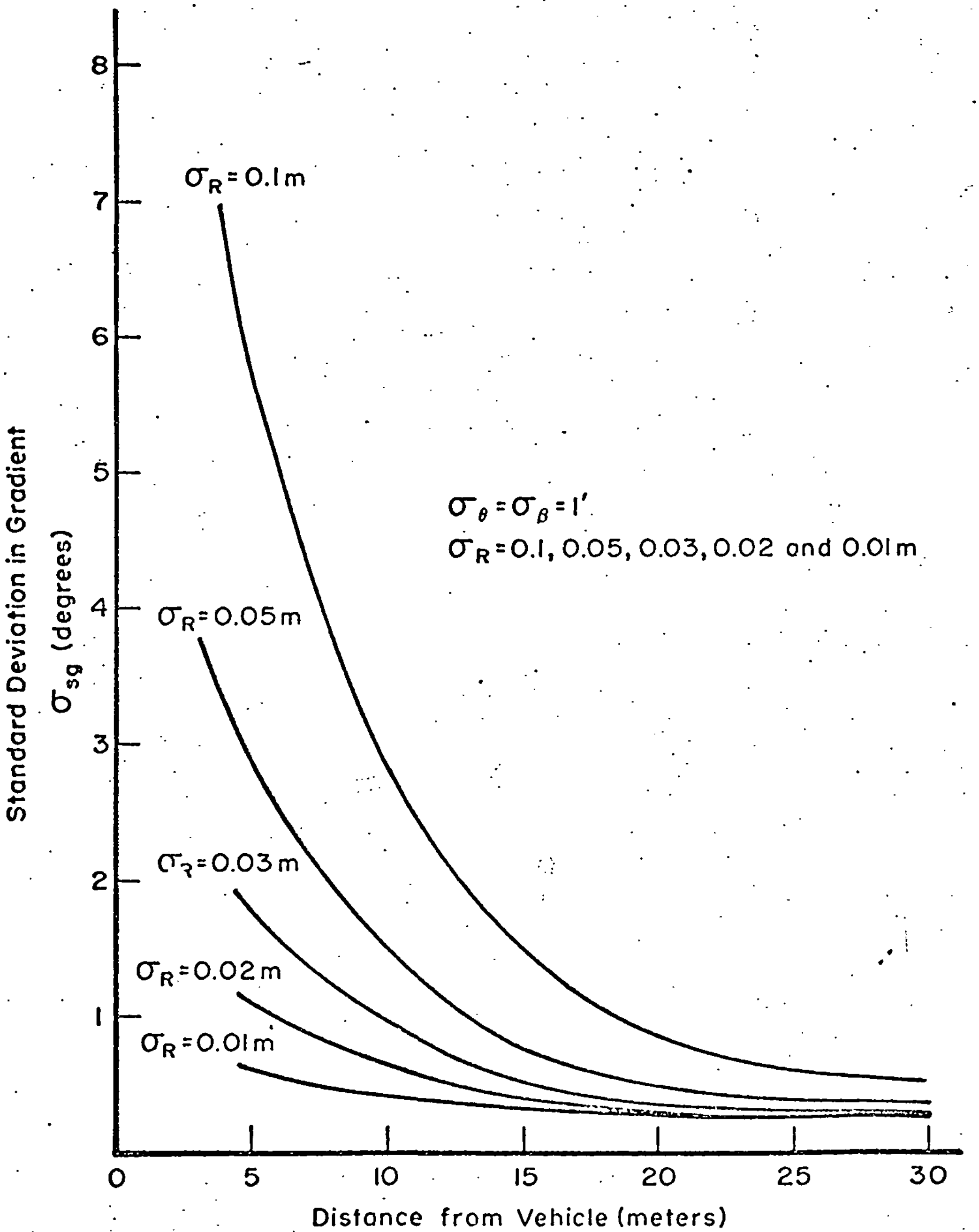

Fig. 6 Standard Deviation in Gradient vs. Distance from Vehicle for "Optimum" Data Point Spacing on Flat Surface.

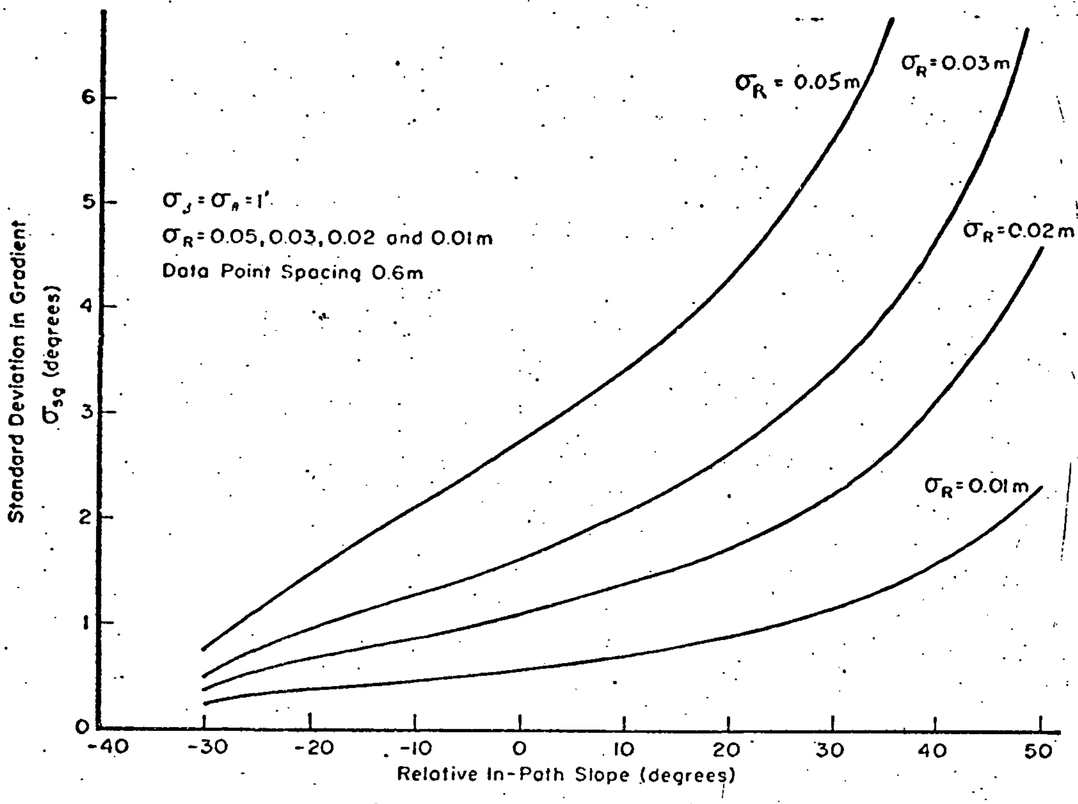

Fig. 7 Standard Deviation in Gradient vs. Relative In-Path Slope At 4m from the Vehicle.

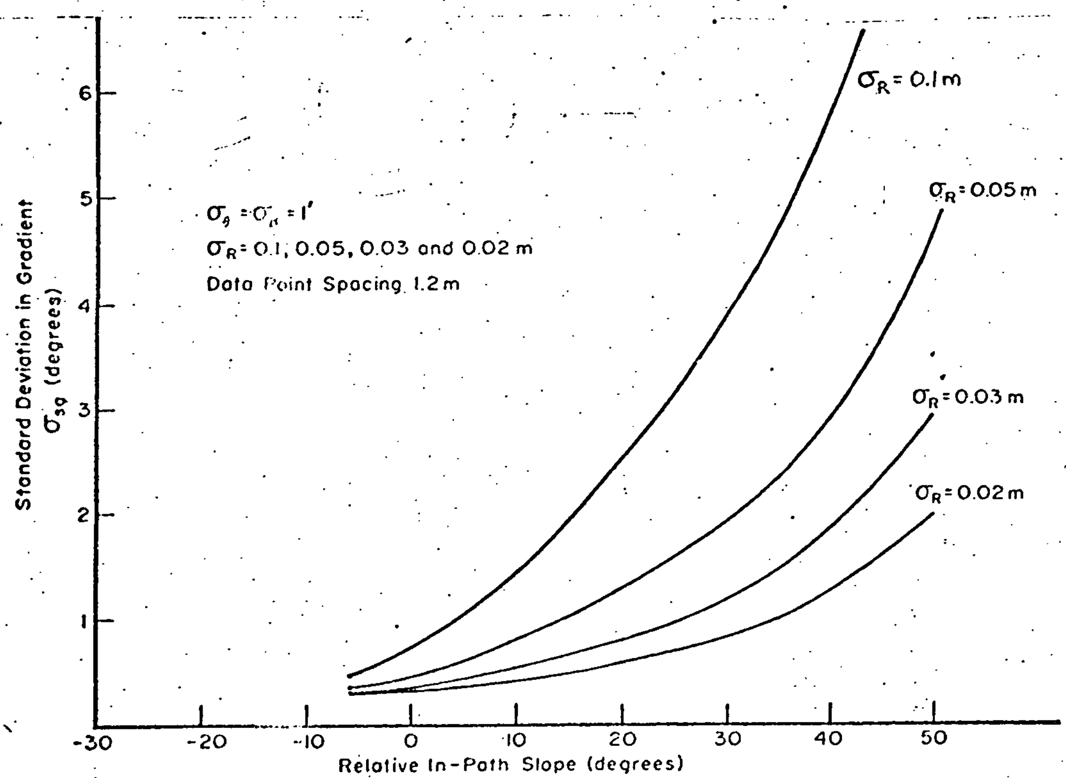

Fig. 8 Standard Deviation in Gradient vs. Relative In-Path Slope At 20m from the Vehicle.

height h of 0 meters in each group of 4 points. The graph only extends to -30° because the laser beam cannot 'see' over a negative in-path slope greater than this value at 4 meters distance. In this graph, $\sigma_\beta = \sigma_\theta = 1'$ in Eq.(11) and σ_R is varied from 1cm to 5cm. The data point spacing is .6 meters. Fig.8 is also a graph of σ_{sg} vs. relative in-path slope from -6° to $+50^\circ$ at 20 meters from the vehicle. All the other conditions are the same as in the last graph except that the data point spacing here is 1.2 meters instead of 0.6 meters. Finally, Fig.9 plots graphs of σ_{sg} vs. relative cross-path slopes of from 0° to $+50^\circ$ and for distances of 4 meters and 20 meters from the vehicle. Once again $\sigma_\beta = \sigma_\theta = 1'$ and σ_R is varied from 2 to 10 cm. The data point spacing at 20 meters from the vehicle is 1.2 meters and the spacing at 4 meters from the vehicle is 0.6 meters.

Present technology cannot improve σ_R below 1cm, and even this figure is quite low.[10] It is obvious from Fig.7 that σ_R should be as close to 1 cm as possible in order that σ_{sg} will be within an acceptable range at 4 meters distance and high values of relative in-path slopes. Cross-path slopes do not present as much of a problem and a σ_R of 2 cm is quite acceptable for the worst cases.

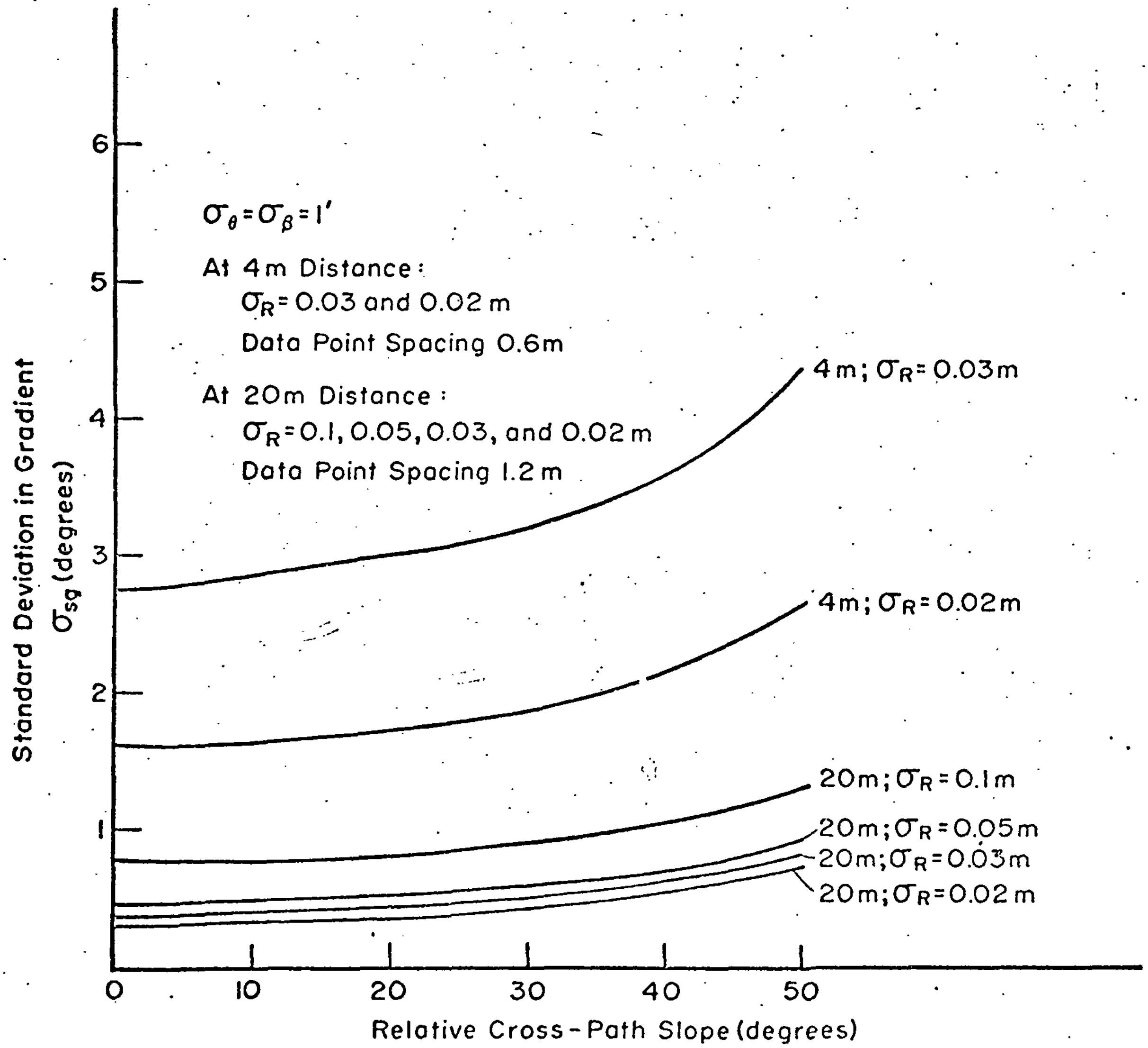

Fig. 9 Standard Deviation in Gradient vs. Relative Cross-Path Slope At 20m and 4m from the Vehicle.

PART 5

CONCLUSION

This paper suggests a solution to the general problem of estimating the state vector x from the equation h=Ax where the measurements h and A are all stochastic. The proceedures outlined here are applied to a specific case in estimating the gradient of the terrain by laser measurements from an autonomous Martian Roving Vehicle. Due to terrain irregularity and instrumental uncertainties, the four measurement points do not fall in the same plane. First, a least square estimate is performed which assumes the matrix A to be deterministic. A minimum variance estimate, which takes into account the error in the matrix A as well as h, gives approximately the same numerical results because the weighting factors are nearly equal to one another. The error involved in these algorithms can also be estimated and a complete error analysis has been presented.

The variance of the gradient should be as low as possible but an upper bound of 2 degrees has been set. Figs. 4-9 show that the standard deviation σ_{S_g} of the gradient is dependent upon the distance of the modeled plane away from the vehicle, the data point spacing, the gradient of the terrain, and the values for σ_R, σ_β, and σ_θ. By using a rapid scan laser, we do not have to take into consideration the error due to the motion of the vehicle. If σ_β and σ_θ are reduced below 1 arc minute, bulkier and heavier equipment would result. In fact, with $\sigma_\beta = \sigma_\theta$ of 1 arc minute very little error is introduced into the calculations. Most of the errors come from σ_R. In order to keep the upper bound of σ_{Sg} within 2 degrees, the quantity σ_R must be as low as 1 cm.

PART 6

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APPENDIX A

DERIVATION OF THE PERTURBATIONS

To prove the result in Eq. (9), the coordinates in Eq. (2) are first perturbed as

$$\begin{bmatrix}\delta h\\ \delta a\\ \delta b\end{bmatrix} = \delta C(\phi)B(\xi)\begin{bmatrix}h''\\ a''\\ b''\end{bmatrix} + C(\phi)\,B(\xi)\begin{bmatrix}h''\\ a''\\ b''\end{bmatrix} + C(\phi)B(\xi)\begin{bmatrix}\delta h''\\ \delta a''\\ \delta b''\end{bmatrix} \tag{A-1}$$

where

$$\delta C=\delta C_{ij}=\begin{bmatrix}(-\sin\phi\delta\phi) & (-\cos\phi\delta\phi) & 0\\ (\cos\phi\delta\phi) & (-\sin\phi\delta\phi) & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}-\sin\phi & -\cos\phi & 0\\ \cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{bmatrix}\delta\phi=C^1\delta\phi \tag{A-2}$$

and

$$\delta B=\delta b_{ij}=\begin{bmatrix}(-\sin\xi\delta\xi) & 0 & \cos\xi\delta\xi\\ 0 & 0 & 0\\ (-\cos\xi\delta\xi) & 0 & -\sin\xi\delta\xi\end{bmatrix} = \begin{bmatrix}(-\sin\xi) & 0 & (\cos\xi)\\ 0 & 0 & 0\\ (-\cos\xi) & 0 & (-\sin\xi)\end{bmatrix}\delta\xi=B^1\delta\xi \tag{A-3}$$

perturbing Eq. (1a) - (1c)

$$\begin{bmatrix}\delta h''\\ \delta a''\\ \delta b'\end{bmatrix} = G(R,\theta,\beta)\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}$$

$$= \begin{bmatrix}(-\sin\beta) & (-R\cos\beta) & 0\\ (\cos\beta\sin\theta) & (-R\sin\beta\sin\theta) & (R\cos\beta\cos\theta)\\ (\cos\beta\cos\theta) & (-R\sin\beta\cos\theta) & (-R\cos\beta\sin\theta)\end{bmatrix}\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix} \tag{A-4}$$

By substituting (A-4), (A-3) and (A-2) into (A-1) one obtains

$$\begin{bmatrix}\delta h\\ \delta a\\ \delta b\end{bmatrix} = \left\{C^1(\phi)B(\xi)\begin{bmatrix}h''\\ a''\\ b''\end{bmatrix} \vdots C(\phi)B^1(\xi)\begin{bmatrix}h''\\ a''\\ b''\end{bmatrix}\right\}\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix} + C(\phi)B(\xi)G(R,\beta,\theta)\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix} \tag{A-5}$$

Compare the above Eq. with Eq. (9a), to see that

$$D(h'',a'',b''\phi,\xi) = \left\{ C^1(\phi)B(\xi) \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix} \vdots C(\phi)B^1(\xi) \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix} \right\} \tag{A-6}$$

The first term on the right is determined from (A-2) and (2c)

$$C^1B \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix} = \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\xi & 0 & \sin\xi \\ 0 & 1 & 0 \\ -\sin\xi & 0 & \cos\xi \end{bmatrix} \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h''\cos\xi+b''\sin\xi \\ a'' \\ -h''\sin\xi+b''\cos\xi \end{bmatrix}$$

$$= \begin{bmatrix} -h''\sin\phi\cos\xi - b''\sin\phi\sin\xi - a''\cos\phi \\ +h''\cos\phi\cos\xi + b''\cos\phi\sin\xi - a''\sin\phi \\ 0 \end{bmatrix} \tag{A-7}$$

The second term of (A-6) is determined from (2b) and(A-3)

$$C\,B^1 \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\xi & 0 & \cos\xi \\ 0 & 0 & 0 \\ -\cos\xi & 0 & -\sin\xi \end{bmatrix} \begin{bmatrix} h'' \\ a'' \\ b'' \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -h''\sin\xi + b''\cos\xi \\ 0 \\ -h''\cos\xi - b''\sin\xi \end{bmatrix}$$

$$= \begin{bmatrix} -h''\cos\phi\sin\xi + b''\cos\phi\cos\xi \\ -h''\sin\phi\sin\xi + b''\sin\phi\cos\xi \\ -h''\cos\xi - b''\sin\xi \end{bmatrix} \tag{A-8}$$

By substituting Eqs. (A-7) and (A-8) into (A-6) one obtains the results in Eq. (9b).

APPENDIX B

DERIVATION OF THE COVARIANCE MATRICES OF THE VARIABLES

To prove the result in Eq. (11) we multiply Eq.(9) by its transpose.

$$\begin{bmatrix}\delta h\\ \delta a\\ \delta b\end{bmatrix}[\delta h \;\; \delta a \;\; \delta b] = \left\{D\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix} + CBG\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}\right\}\left\{D\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix} + CBG\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}\right\}^T$$

$$= D\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix}[\delta\phi\,\delta\xi]D^T + CBG\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}[\delta R \;\; \delta\beta \;\; \delta\theta]G^TB^TC^T$$

$$+CBG\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}[\delta\phi\,\delta\xi]D^T + D\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix}[\delta R \;\; \delta\beta \;\; \delta\theta]G^TB^TC^T \qquad \text{(B-1)}$$

Taking the expected value of (B-1) one obtains

$$M = E\left\{\begin{bmatrix}\delta h\\ \delta a\\ \delta b\end{bmatrix}[\delta h \;\; \delta a \;\; \delta b]\right\}$$

$$= D\,E\left\{\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix}[\delta\phi \;\; \delta\xi]\right\}D^T + CBG\,E\left\{\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}[\delta R \;\; \delta\beta \;\; \delta\theta]\right\}G^TB^TC^T$$

$$+ CBG\,E\left\{\begin{bmatrix}\delta R\\ \delta\beta\\ \delta\theta\end{bmatrix}[\delta\phi \;\; \delta\xi]\right\}D^T + D\,E\left\{\begin{bmatrix}\delta\phi\\ \delta\xi\end{bmatrix}[\delta R \;\; \delta\beta \;\; \delta\theta]\right\}G^TB^TC^T \qquad \text{(B-2)}$$

Since $\delta\phi$, $\delta\xi$, δR, $\delta\beta$, $\delta\theta$ are not correlated, (B-2) becomes Eq. (11) in the text.

APPENDIX C

DERIVATION OF SLOPE COVARIANCE (I)

Here we will prove Eq. (14) and Eq. (15). Substituting $h=\bar{h}+\delta h$, $A=\bar{A}+\delta A$, and $x=\bar{x}+\delta x$ into Eq.(12) we have

$$\bar{h}+\delta h=(\bar{A}+\delta A)(\bar{x}+\delta x)=\overline{Ax} + \delta A\bar{x} + \bar{A}\delta x + \delta A\delta x \tag{C-1}$$

Since $\bar{h}=\overline{Ax}$, and neglecting the second order term $\delta A\delta x$, Eq. (C-1) becomes

$$\bar{A}\delta x = \delta h-\delta A\bar{x}$$

by premultiplying the above equation by $\bar{A}^T$

$$\bar{A}^T\bar{A}\delta x = \bar{A}^T(\delta h-\delta A\bar{x})$$

Thus the estimate of δx becomes

$$\delta\hat{x} = (\bar{A}^T\bar{A})^{-1}\bar{A}^T(\delta h-\delta A\bar{x}) = F(\delta h-\delta A\bar{x}) \tag{C-2}$$

which is equivalent to Eqs.(13) and (14). Multiply Eq. (C-2) by its transpose to obtain

$$\begin{aligned}\delta\hat{x}\delta\hat{x}^T &= F(\delta h-\delta A\bar{x})(\delta h-\delta A\bar{x})^T F^T \\ &= F\left[\delta h\delta h^T-\delta A\bar{x}\delta h^T-\delta h(\delta A\bar{x})^T+\delta A\bar{x}(\delta A\bar{x})^T\right]F^T\end{aligned} \tag{C-3}$$

The expected value of Eq. (C-3) becomes Eq. (15) in the text.

APPENDIX D

DERIVATION OF SLOPE COVARIANCES (2)

Replacing, term by term, Eq. (16) into Eq. (15) we get

$$E[\delta\hat{x}\delta\hat{x}^T] = E\left\{\begin{bmatrix}\delta\hat{x}_1\\ \delta\hat{x}_2\\ \delta\hat{x}_3\end{bmatrix}[\delta\hat{x}_1 \delta\hat{x}_2 \ \delta\hat{x}_3]\right\} \tag{D-1}$$

$$E\,[\delta h\delta h^T] = E\left\{\begin{bmatrix}\delta h\\ \vdots\\ \delta h_n\end{bmatrix}[\delta h \ \ldots \delta h_n]\right\} = \begin{bmatrix} E(\delta h_1)^2 & E(\delta h_1\delta h_2) & \ldots & E(\delta h_1\delta h_n)\\ E(\delta h_2\delta h_1) & E(\delta h_2)^2 & & E(\delta h_2\delta h_n)\\ \vdots & \vdots & & \vdots\\ E(\delta h_n\delta h_1) & E(\delta h_n\delta h_2) & \ldots & E(\delta h_n\delta h_n)\end{bmatrix} \tag{D-2}$$

The different measurement points are uncorrelated. Therefore, the off diagonal terms are equal to zero. Eq. (D-2) reduces to

$$E[\delta h\delta h^T] = \begin{bmatrix} E(\delta h_1)^2 & & 0\\ & \ddots & \\ 0 & & E(\delta h_n)^2\end{bmatrix} \tag{D-3}$$

Similarly:

$$E[\delta A\bar{x}\delta h^T] = E\left\{\begin{bmatrix}\delta a_1\bar{x}_1 + \delta b_1\bar{x}_2\\ \vdots\\ \delta a_n\bar{x}_1 + \delta b_n\bar{x}_2\end{bmatrix}[\delta h \ \ldots \delta h_n]\right\}$$

$$= \left\{\begin{matrix}[E(\delta a_1\delta h_1)\bar{x}_1 + E(\delta b_1\delta h_1)\bar{x}_2 & & 0\\ & \ddots & \\ 0 & & [E(\delta a_n\delta h_n)\bar{x}_1 + E(\delta b_n\delta h_n)\bar{x}_2]\end{matrix}\right\} \tag{D-4}$$

$$E[\delta h(\delta A\bar{x})^T] = E\left\{\begin{bmatrix}\delta h \\ \vdots \\ \delta h_n\end{bmatrix} [(\delta a_1\bar{x}_1 + \delta b_1\bar{x}_2)\ldots(\delta a_n\bar{x}_1 + \delta b_n\bar{x}_2]\right\}$$

$$= \left\{\begin{matrix}[E(\delta h_1\delta a_1)\bar{x} + E\delta h_1\delta b_1)\bar{x}_2] & & 0 \\ & \ddots & \\ 0 & & [E(\delta h_n\delta a_n)\bar{x}_1 + E(\delta h_n\delta b_n)\ \bar{x}_2]\end{matrix}\right\} \tag{D-5}$$

$$E[\delta A\bar{x}(\delta A\bar{x})^T] = E\left\{\begin{bmatrix}\delta a_1\bar{x}_1 + \delta b_1\bar{x}_2 \\ \vdots \\ \delta a_n\bar{x}_1 + \delta b_n\bar{x}_2\end{bmatrix}[(\delta a_1\bar{x}_1 + \delta b_1\bar{x}_2 \ \ldots(\delta a_n\bar{x}_1 + \delta b_n\bar{x}_2)]\right\}$$

$$\left\{\begin{matrix}[E(\delta a_1)^2\bar{x}_1^2 + 2E(\delta a_1\delta b_1)\bar{x}_1\bar{x}_2 + E(\delta b_1)^2\bar{x}_2^2] & & 0 \\ & \ddots & \\ 0 & & [E(\delta a_n)^2\bar{x}_1^2 + 2E(\delta a_n\delta b_n)\bar{x}_1\bar{x}_2 + E(\delta b_n)^2\bar{x}_2^2]\end{matrix}\right\} \tag{D-6}$$

Where the expected value terms in Eqs. (D-2) - (D-6) can be evaluated from Eq. (11).

APPENDIX E

THE NOISE COVARIANCE MATRIX

From the quantities of δh and $\delta A\bar{x}$ given in Eq.(16) one obtains for Eq. (21)

$$v = \begin{bmatrix} \delta h_1 - \delta a_1 \bar{x}_1 - \delta b_1 \bar{x}_2 \\ \vdots \\ \delta h_4 - \delta a_4 \bar{x}_1 - \delta b_1 \bar{x}_2 \end{bmatrix} \qquad \text{(E-1)}$$

Since the measurement points are close together, the expected values of the errors are nearly the same, i.e.,

$$\begin{aligned} E(\delta h_1) &\cong E(\delta h_2) \cong E(\delta h_3) \cong E(\delta h_4) \\ E(\delta a_1) &\cong E(\delta a_2) \cong E(\delta a_3) \cong E(\delta a_4) \\ E(\delta b_1) &\cong E(\delta b_2) \cong E(\delta b_3) \cong E(\delta b_4) \end{aligned} \qquad \text{(E-2)}$$

similarly

$$\begin{aligned} E(\delta h_1)^2 &\cong E(\delta h_2)^2 \cong E(\delta h_3)^2 \cong E(\delta h_4)^2 \\ E(\delta a_1)^2 &\cong E(\delta a_2)^2 \cong E(\delta a_3)^2 \cong E(\delta a_4)^2 \\ E(\delta b_1)^2 &\cong E(\delta b_2)^2 \cong E(\delta b_3)^2 \cong E(\delta b_4)^2 \end{aligned} \qquad \text{(E-3)}$$

From Eq. (D-3) we have

$$E\left[\delta h \delta h^T\right] \cong E(\delta h_1)^2 I \qquad \text{(E-4)}$$

Similarly from Eqs. (D-4) and (D-5) one obtains.

$$\begin{aligned} E\left[\delta A\bar{x}\delta h^T\right] &\cong E\left[\delta h(\delta A\bar{x})^T\right] \\ &\cong \left[E(\delta a_1 \delta h_1)\bar{x}_1 + E(\delta b_1 \delta h_1)\bar{x}_2\right] I \end{aligned} \qquad \text{(E-5)}$$

Moreover, Eq. (D-6) results

$$E\left[\delta A\bar{x}(\delta A\bar{x})^T\right] = \left[E(\delta a_1)^2 \bar{x}_1^2 + 2E(\delta a_1 \delta b_1)\bar{x}_1\bar{x}_2 + E(\delta b_1)^2 \bar{x}_2^2\right] I \qquad \text{(E-6)}$$

Substituting Eqs. (E-4), (E-5) and (E-6) into Eq. (22) gives

$$R = kI \qquad \text{(E-7)}$$

where $k = E(\delta h_1)^2 - 2E(\delta a_1 \delta h_1)\bar{x}_1 - 2E(\delta b_1 \delta h_1)\bar{x}_2$

$$+E(\delta a_1)^2\bar{x}_1^2 + 2E(\delta a_1 \delta b_1)\bar{x}_1\bar{x}_2 + E(\delta b_1)^2\bar{x}_2^2 \qquad \text{(E-8)}$$

which is the predicted result in section 3f.