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# APPLICATION OF MATCHED ASYMPTOTIC EXPANSIONS TO LUNAR AND INTERPLANETARY TRAJECTORIES

## Volume 2 - Derivations of Second-Order Asymptotic Boundary Value Solutions

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16. Abstract

Previously published asymptotic solutions for lunar and interplanetary trajectories have been modified and combined to formulate a general analytical solution to the problem of N-bodies. The earlier first-order solutions, derived by the method of matched asymptotic expansions, have been extended to second order for the purpose of obtaining increased accuracy. The complete derivation of the second-order solution, including the application of a rigorous matching principle, is given. It is shown that the outer and inner expansions can be matched in a region of order $\mu^{\alpha}$, where $2/5 < \alpha < 1/2$ and $\mu$ (the moon/Earth or planet/sun mass ratio) is much less than one. The second-order asymptotic solution has been used as a basis for formulating a number of analytical two-point boundary value solutions. These include Earth-to-moon, one- and two-impulse moon-to-Earth, and interplanetary solutions. Each is presented as an explicit analytical solution which does not require iterative steps to satisfy the boundary conditions. The complete derivation of each solution is shown, as well as instructions for numerical evaluation.

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## ADDENDUM

The results of Volume 2 were discussed recently with Professor John Breakwell of Stanford University. His comments regarding the matching (Section A14) pointed up a deficiency in the analysis. In order for the results of Section A17 to be valid to order $\mu^2$, the guage function discussed in Section A14 should be $\mu^{2+\alpha}$ rather than $\mu^2$. This stronger guage function would require the inclusion of higher order singular terms in both the outer and inner expansions in order to satisfy the limit defined by Equation (A14-1). The inclusion of such singular terms is somewhat laborious and has no effect on the results obtained from the matching and the subsequent initial and boundary value solutions derived from these results. The matching of such terms will be discussed in a forthcoming paper by Breakwell and Perko.

The limits on the overlap domain depend on which singular terms are included in the matching. The overlap domain $2/5 < \alpha < 1/2$ found in Section A14 is valid but not unique as other combinations, such as $1/2 < \alpha < 3/5$, are possible. An overlap domain which includes $\alpha = 1/2$ is also possible but requires the inclusion of more singular terms than does either of the other overlap domains mentioned. The actual choice of overlap domain also has no effect on the results of the matching.

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## INTRODUCTION

This report contains the analytical derivations of the second order asymptotic boundary value solutions for lunar and interplanetary trajectories which have been formulated under Contract No. NAS9-10526 for the NASA Manned Spacecraft Center. It is a supplementary document to the final report of the contract study.[1] Whereas the final report presents only the results of the study effort, this document contains the step by step derivations of both the initial value solution to the problem of N-bodies and the boundary value solutions which were designated in the contract work statement.

The analysis is divided into three sections. Section A contains the derivation of the second order asymptotic solution starting from the differential equations of motion for the N-body problem. It includes derivations of both the outer and inner expansions, their behavior in the overlap domain, a detailed discussion of the matching, and the solution to the initial value problem.

Section B contains the derivations of a number of boundary value problems for both lunar and interplanetary applications. These derivations are based on the results of Section A.

Finally, Section C contains discussions of how the two main types of perturbation terms in the asymptotic solution are evaluated numerically. This includes explicit formulas for evaluating the well known linear state transition matrix.

The derivations are given primarily by showing the mathematical steps involved. A minimum amount of discussion is presented in this document. For expanded discussions (and a corresponding minimum number of mathematical expressions), as well as numerical results showing the accuracy and computation speed of the asymptotic solutions, the reader is referred to Reference 1.

The notation used in this report is a combination of that of Lancaster[2] and Carlson.[3] In general, each parameter is defined as it is introduced but some which have only mathematical meaning and serve an intermediary role are defined only by an equation. Scalers are written as x or X and vectors as $\underline{x}$ or $\underline{X}$. A matrix $G(\underline{x})$ and a tensor $\underline{H}(\underline{x})$ are also used. In addition, a bar over a parameter indicates that it applies specifically to an inner solution. Finally, the order of a particular term in an expansion is given by the exponent of the parameter $\mu$ which precedes the term, i.e., $\mu^n$ is order $n$ or $O(n)$.

## Section A

## SECOND ORDER ASYMPTOTIC SOLUTION TO THE PROBLEM OF N-BODIES

### A1 N-BODY EQUATIONS OF MOTION

The problem of N-bodies will be defined as follows: The motion of a small body of negligible mass is to be determined subject to the gravitational forces of a primary body of mass $m_o$ and N-2 secondary bodies of mass $m_i$, $i = 1, 2, \ldots, N-2$, where $m_i << m_o$ for each i and the motion of the N-2 secondary bodies relative to the primary body is assumed to be a known function of time. Denoting the position of the small body relative to the primary body by $\underline{r}^*$ and the positions of the secondary bodies by $\underline{p}_i^*$ the differential equation for the small body motion is

$$\frac{d^2\underline{r}^*}{dt^{*2}} = -Gm_0\frac{\underline{r}^*}{r^{*3}} - G\sum_{i=1}^{N-2} m_i\left[\frac{\underline{r}^* - \underline{p}_i^*}{\left|\underline{r}^* - p_i^*\right|^3} + \frac{\underline{p}_i^*}{p_i^{*3}}\right] \tag{A1-1}$$

where G is the gravitational constant.

Now define dimensionless variables

$$r = r^*/L^* \tag{A1-2}$$

$$t = t^*/T^* \tag{A1-3}$$

where

$$L^* = a_j^* = \text{semi-major axis of the motion of the } j^{th} \text{ body} \tag{A1-4}$$

$$2\pi T^* = P_j^* = \text{period of the } j^{th} \text{ body motion} \tag{A1-5}$$

Also define the dimensionless mass ratios

$$\mu_i = m_i/m_o \tag{A1-6}$$

Then (A1-1) becomes

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\underline{r}}{r^3} - \sum_{i=1}^{N-2} \mu_i \left[\frac{\underline{r} - \underline{p}_i}{|\underline{r} - \underline{p}_i|^3} + \frac{\underline{p}_i}{p_i^3}\right] \tag{A1-7}$$

(A1-7) is the dimensionless differential equation of the small body where the unit of length is now L* and the unit of time T*. Defining

$$\underline{f}(\underline{x}) = -\underline{x}/x^3 \tag{A1-8}$$

$$\ddot{\underline{x}} = \frac{d^2\underline{x}}{dt^2} \tag{A1-9}$$

reduces (A1-7) to

$$\ddot{\underline{r}} = \underline{f}(\underline{r}) + \underline{F}(\underline{r}, \underline{p}_i) \tag{A1-10}$$

where

$$\underline{F}(\underline{r}, \underline{p}_i) = \sum_{i=1}^{N-2} \mu_i \left[\underline{f}(\underline{r} - \underline{p}_i) + \underline{f}(\underline{p}_i)\right] \tag{A1-11}$$

In (A1-10) $\ddot{\underline{r}}$ and $\underline{f}(\underline{r})$ are both of order unity (zero order) while $\underline{F}$ is order $\mu_i$ (first order), except in the exceptional case where $\underline{r} - \underline{p}_i = 0\,(\mu_i)$ when $\underline{F}$ is order $\mu_i^{-1}$. However, in the analysis of transfer trajectories the exceptional case is the case of interest since transfer trajectories have at least one close approach to a secondary body. The change in the order of $\underline{F}$ indicates that (A1-10) represents a type of singular perturbation problem[4]. This assumption will be verified by developing a solution to (A1-10) by the method of matched asymptotic expansions.

In addition to (A1-10) the equations of motion of the secondary bodies will be of interest. For the $k^{th}$ body they are

$$\ddot{\underline{p}}_k = (1 + \mu_k)\, \underline{f}(\underline{p}_k) + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} \mu_i \left[ \underline{f}(\underline{p}_k - \underline{p}_i) + \underline{f}(\underline{p}_i) \right] \qquad \text{(A1-12)}$$

A1.1 Interplanetary Equations of Motion

For interplanetary trajectories the primary body is the sun and the secondary bodies are whatever planets one wishes to include in the model. The $j^{th}$ planet used to determine the length and time scales, $L^*$ and $T^*$, will normally be the target planet of the trajectory. All of the $\mu_i$'s will be small, the largest being about $10^{-3}$ for the planet Jupiter. In this case it is obvious that $\underline{F}$ is small in (A1-10) except when the trajectory makes a close approach to one of the planets.

In this case the coordinate system is centered at the sun and it is the motion of the sun about the solar system center of mass which produces the $\underline{f}(\underline{p}_i)$ term in (A1-11).

A1.2 Cislunar Equations of Motion

In cislunar space the primary body is the earth and the secondary bodies are the moon and the sun. The $j^{th}$ body will be the moon so that $L^*$ is the semi-major axis of the moon's motion about the earth and $2\pi T^*$ is the moon's period. The coordinate system is centered at the earth and the $\underline{f}(\underline{p}_i)$ term in (A1-11) is due to the motion of the earth about the center of mass.

(A1-10) becomes

$$\ddot{\underline{r}} = \underline{f}(\underline{r}) + \mu_M \left[ \underline{f}(\underline{r} - \underline{p}_M) + \underline{f}(\underline{p}_M) \right] + \mu_S \left[ \underline{f}(\underline{r} - \underline{p}_S) + \underline{f}(\underline{p}_S) \right] \qquad \text{(A1-13)}$$

Since

$$\mu_S = m_S / m_E$$

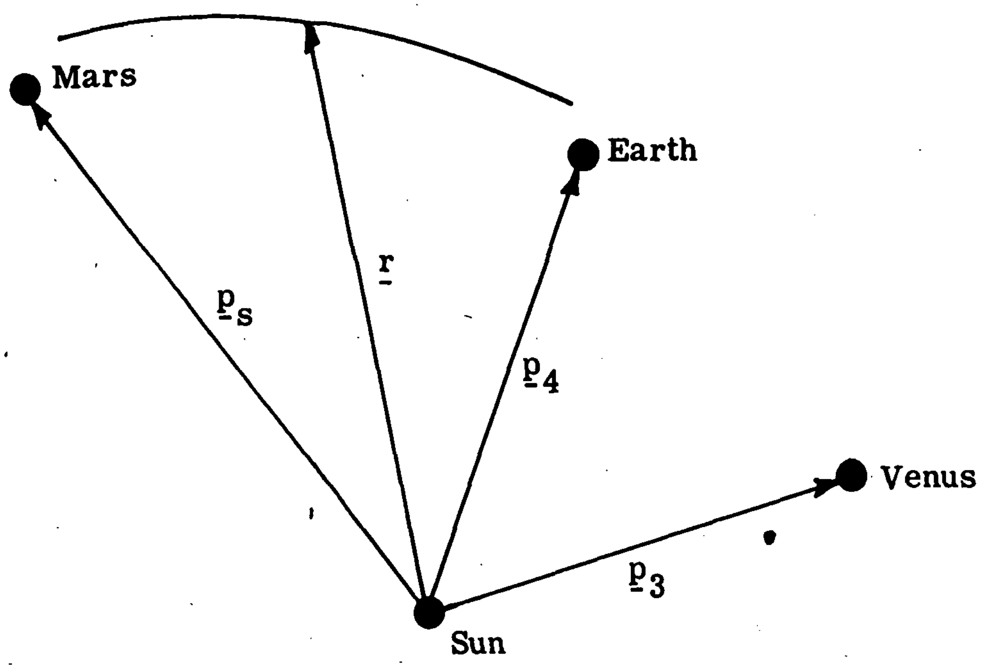


Figure A1. Coordinate System for Interplanetary Trajectories

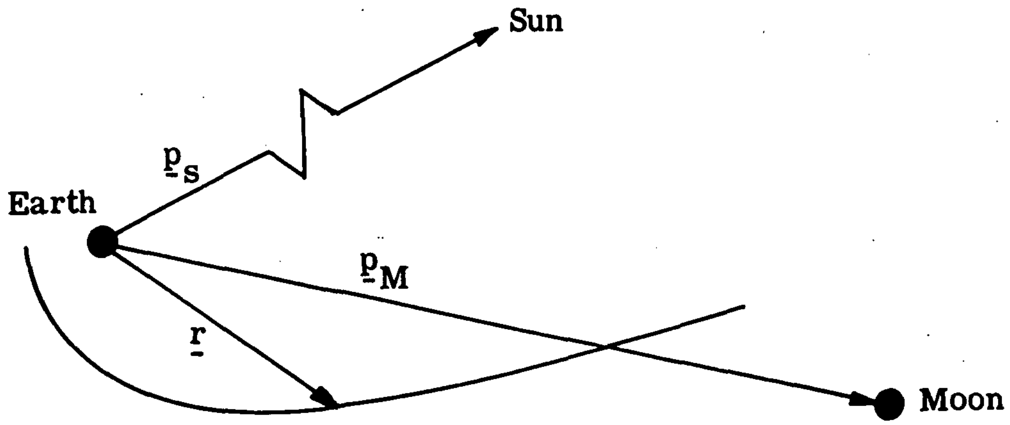


Figure A2. Coordinate System for Earth-to-Moon Trajectories it is obviously not small. Let

$$\mu = \mu_M \tag{A1-14}$$

$$M_s = \mu_s / \mu \tag{A1-15}$$

and (A1-12) becomes

$$\ddot{\underline{r}} = \underline{f}(\underline{r}) + \mu \left\{ \left[ \underline{f}(\underline{r} - \underline{p}_M) + \underline{f}(\underline{p}_M) \right] + M_s \left[ \underline{f}(\underline{r} - \underline{p}_s) + \underline{f}(\underline{p}_s) \right] \right\} \tag{A1-16}$$

It is necessary to show that the term proportional to $M_s$ is actually order $\mu$, i.e., to show that the sun's effect is the same magnitude as the moon's. Let

$$\underline{p}_s = a_s \underline{\rho}_s \tag{A1-17}$$

where $a_s$ is the mean distance of the sun from the earth (in dimensionless units). Then

$$\underline{r} - \underline{p}_s = -a_s (\underline{\rho}_s - \underline{r}/a_s) \tag{A1-18}$$

and, since $r << a_s$

$$\underline{f}(\underline{r} - \underline{p}_s) = -\underline{f}(\underline{\rho}_s - \underline{r}/a_s)/a_s^2$$

$$= -\frac{1}{a_s^2} \underline{f}(\underline{\rho}_s) + \frac{1}{a_s^3} G(\underline{\rho}_s)\, \underline{r} + O(\frac{1}{a_s^4}) \tag{A1-19}$$

where $G(\underline{\rho}_s)$ is the gravity gradient matrix defined in Section A2. Also

$$\underline{f}(\underline{p}_s) = \underline{f}(\underline{\rho}_s)/a_s^2 \tag{A1-20}$$

giving

$$f(\underline{r} - \underline{p}_s) + f(\underline{p}_s) = \frac{1}{a_s^3} G(\underline{\rho}_s)\, \underline{r} + O(\frac{1}{a_s^4}) \tag{A1-21}$$

or

$$M_s \left[ \underline{f}(\underline{r} - \underline{p}_s) + \underline{f}(\underline{p}_s) \right] = O\left( \frac{M_s \; G(\underline{\rho}_s) \; \underline{r}}{a_s^3} \right) \tag{A1-22}$$

In (A1-22)

$$M_s = \mu_s/\mu = (m_s/m_E)/(m_M/m_E) = m_s/m_M = 2.7 \times 10^7 \tag{A1-23}$$

$$G(\underline{\rho}_s) = O(1) \quad (\text{since } \underline{\rho}_s = O(1)) \tag{A1-24}$$

$$\underline{r} = O(1) \tag{A1-25}$$

$$a_s^3 = a_s^{*3}/L^{*3} = (1 \text{ au}/0.0026 \text{ au})^3 = 5.7 \times 10^7 \tag{A1-26}$$

Therefore

$$\begin{aligned} M_s \left[ \underline{f}(\underline{r} - \underline{p}_s) + \underline{f}(\underline{p}_s) \right] &= (M_s/a_s^3) \times O(1) = 0.47 \times O(1) \\ &= O(1) \end{aligned} \tag{A1-27}$$

and

$$\mu M_s \left[ \underline{f}(\underline{r} - \underline{p}_s) + \underline{f}(\underline{p}_s) \right] = O(\mu) \tag{A1-28}$$

In later analysis (A1-17) will not be introduced and it will be implied that (A1-27 and 28) hold for cislunar space. Thus the sun and moon are expected to have comparable effects on cislunar trajectories.

## A2 EXPANSION OF $\underline{f}(\underline{x})$

Let

$$\underline{f}(\underline{x}) = f_i(\underline{x}) \, \underline{e}_i \tag{A2-1}$$

where $f_i$ are components of $\underline{f}$ and $\underline{e}_i$ are orthogonal base vectors. Also let

$$\underline{x} = \underline{x}_o + \delta\underline{x} \tag{A2-2}$$

where

$$\delta\underline{x} = \epsilon\underline{x}_1 + \epsilon^2\underline{x}_2 + O(\epsilon^3) \tag{A2-3}$$

$$= (\epsilon x_{1i} + \epsilon^2 x_{2i} + \ldots)\,\underline{e}_i \tag{A2-4}$$

Substituting (A2-2) into (A2-1) and expanding in a Taylor series gives

$$f_i(\underline{x}) = f_i(\underline{x}_o) + \frac{\partial f_i(\underline{x}_o)}{\partial x_j}\,\delta x_j + \frac{1}{2}\,\frac{\partial^2 f_i(\underline{x}_o)}{\partial x_j \partial x_k}\,\delta x_j\,\delta x_k + O(\delta x^3)$$

$$= f_i(\underline{x}_o) + \frac{\partial f_i(\underline{x}_o)}{\partial x_j}\left[\epsilon x_{1j} + \epsilon^2 x_{2j} + O(\epsilon^3)\right]$$

$$+ \frac{1}{2}\,\frac{\partial^2 f_i(\underline{x}_o)}{\partial x_j\,\partial x_k}\,\epsilon^2\,x_{1j}\,x_{1k} + O(\epsilon^3) \tag{A2-5}$$

Let

$$G_{ij}(\underline{x}) = \frac{\partial f_i(\underline{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\frac{x_i}{x^3}\right) = -\,x_i\,\frac{\partial}{\partial x_j}\left(\frac{1}{x^3}\right) - \frac{1}{x^3}\,\frac{\partial x_i}{\partial x_j} \tag{A2-6}$$

But

$$x^2 = x_m\,x_m$$

$$2x\,\frac{\partial x}{\partial x_j} = 2x_m\,\frac{\partial x_m}{\partial x_j}$$

$$\frac{\partial x}{\partial x_j} = \frac{x_m}{x}\frac{\partial x_m}{\partial x_j} \qquad \text{(A2-7)}$$

and

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \text{kronecker delta} \qquad \text{(A2-8)}$$

Therefore

$$\frac{\partial x}{\partial x_j} = \frac{x_j}{x} \qquad \text{(A2-9)}$$

Using (A2-7) and (A2-9) in (A2-6) gives

$$G_{ij}(\underline{x}) = \frac{3x_i x_j}{x^5} - \frac{\delta_{ij}}{x^3} \qquad \text{(A2-10)}$$

Now let

$$\underline{H}_{ijk}(\underline{x}) = \frac{\partial^2 f_i(\underline{x})}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_k} G_{ij}(\underline{x}) = \frac{\partial}{\partial x_k}\left(3\frac{x_i x_j}{x^5} - \frac{\delta_{ij}}{x^3}\right)$$

$$= -\frac{15x_i x_j x_k}{x^7} + \frac{3}{x^5}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}) \qquad \text{(A2-11)}$$

Summarizing, if

$$\underline{f}(\underline{x}) = -\underline{x}/x^3 \qquad \text{(A2-12)}$$

and

$$\underline{x} = \underline{x}_o + \varepsilon\,\underline{x}_1 + \varepsilon^2\,\underline{x}_2 + O(\varepsilon^3) \qquad \text{(A2-13)}$$

then

$$\underline{f}(\underline{x}) = \underline{f}(\underline{x}_o) + \varepsilon G(\underline{x}_o)\,\underline{x}_1 + \varepsilon^2\left[G(\underline{x}_o)\,\underline{x}_2 + \frac{1}{2}\,\underline{H}\,(\underline{x}_o)\,\underline{x}_1^2\right] + O(\varepsilon^3) \qquad \text{(A2-14)}$$

with

$$G_{ij}(\underline{x}) = \frac{3x_i x_j}{x^5} - \frac{\delta_{ij}}{x^3} \qquad \text{(A2-15)}$$

$$\underline{H}_{ijk}(\underline{x}) = -\frac{15x_i x_j x_k}{x^7} + \frac{3}{x^5}\,(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}) \qquad \text{A2-16)}$$

The matrix $G(\underline{x})$ is the gravity gradient matrix and is also defined by

$$G(\underline{x}) = \frac{d\underline{f}(\underline{x})}{d\underline{x}} \qquad \text{(A2-17)}$$

An expansion of $G(\underline{x})$ is similar to (A2-5):

$$G_{ij}(\underline{x}) = G_{ij}(\underline{x}_o) + \frac{\partial G_{ij}}{\partial x_k}(\underline{x}_o)\,\delta x_k + \frac{\partial^2 G_{ij}(\underline{x}_o)}{\partial x_k \partial x_\ell}\,\delta x_k \delta x_\ell + O(\delta x^3)$$

$$= G_{ij}(\underline{x}_o) + \frac{\partial G_{ij}(\underline{x}_o)}{\partial x_k}\left[\varepsilon x_{1j} + \varepsilon^2 x_{2j} + O(\varepsilon^3)\right]$$

$$+ \frac{1}{2}\,\frac{\partial^2 G_{ij}(\underline{x}_o)}{\partial x_k \partial x_\ell}\,\varepsilon^2 x_{1j} x_{1\ell} + O(\varepsilon^3) \qquad \text{(A2-18)}$$

Let

$$\underset{\sim}{T}_{ijk\ell} = \frac{\partial^2 G_{ij}(\underline{x})}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_\ell}\,\underline{H}_{ijk}(\underline{x}) = \frac{105\,x_i x_j x_k x_\ell}{x^9} - \frac{15}{x^7}\Big[x_i(x_j\delta_{k\ell} - x_\ell\delta_{jk})$$

$$+ x_j(x_k\delta_{i\ell} - x_\ell\delta_{ik}) + x_k(x_i\delta_{j\ell} - x_\ell\delta_{ij})\Big]$$

$$- \frac{3}{x^5}\,(\delta_{i\ell}\delta_{jk} + \delta_{j\ell}\delta_{ik} + \delta_{k\ell}\delta_{ij}) \qquad \text{(A2-19)}$$

Then

$$\underline{G}(\underline{x}) = G(\underline{x}_o) + \epsilon \underline{H}(\underline{x}_o)\underline{x}_1 + \epsilon^2\left[\underline{H}(\underline{x}_o)\underline{x}_2 + \underset{\sim}{T}(\underline{x}_o)\underline{x}_1^2/2\right] + O(\epsilon^3) \qquad \text{(A2-20)}$$

A3 LINEAR DIFFERENTIAL EQUATIONS; STATE TRANSITION MATRIX

In order to develop an analytical solution to (A1-10) it is necessary to solve a system of first order differential equations of the form

$$\dot{\underline{r}}_n = \underline{v}_n \qquad \text{(A3-1)}$$

$$\dot{\underline{v}}_n = G(t)\underline{r}_n + \underline{F}_n(t) \qquad \text{(A3-2)}$$

where $G(t)$ is the gravity gradient matrix and the $\underline{F}_n(t)$ are given functions of time. This system can be written in the condensed form

$$\dot{\underline{x}}_n = K(t)\underline{x}_n + \underline{u}_n(t) \qquad \text{(A3-3)}$$

where

$$\underline{x}_n = \begin{pmatrix} \underline{r}_n \\ \underline{v}_n \end{pmatrix} \qquad \text{(A3-4)}$$

$$\underline{u}_n = \begin{pmatrix} \underline{O} \\ \underline{F}_n \end{pmatrix} \qquad \text{(A3-5)}$$

$$K = \begin{pmatrix} O & I \\ G & O \end{pmatrix} \qquad \text{(A3-6)}$$

The solution of (A3-3) is well documented[3, 5, 6] and can be written in the form

$$\underline{x}_n(t) = \Phi(t, t_o)\, \underline{x}_n(t_o) + \int_{t_o}^{t} \Phi(t, \tau)\, \underline{u}_n(\tau)d\tau \qquad \text{(A3-7)}$$

where the matrix $\Phi$ satisfies the differential equation

$$\frac{d\Phi}{dt}(t,\tau) = K(t)\Phi(t,\tau), \qquad \Phi(t,t) = I \tag{A3-8}$$

or

$$\frac{d\Phi}{d\tau}(t,\tau) = -\Phi(t,\tau)K(\tau), \qquad \Phi(\tau,\tau) = I \tag{A3-9}$$

It is easily verified that (A3-7) is a solution of (A3-3) by differentiation:

$$\dot{\underline{x}}_n(t) = \dot{\Phi}(t,t_o)\underline{x}_n(t_o) + \int_{t_o}^{t} \dot{\Phi}(t,\tau)\underline{u}_n(\tau)d\tau + \Phi(t,t)\underline{u}_n(t) \tag{A3-10}$$

The last two terms in (A3-10) come from differentiating the integral in (A3-7). Replacing $\dot{\Phi}(t, t_o)$ and $\Phi(t, t)$ by (A3-8) gives

$$\dot{\underline{x}}_n(t) = K(t)\Phi(t,\tau)\underline{x}_n(t_o) + \int_{t_o}^{t} K(t)\Phi(t,\tau)\underline{u}_n(\tau)d\tau + \underline{u}_n(t)$$

$$= K(t)\left[\Phi(t,\tau)\underline{x}_n(t_o) + \int_{t_o}^{t} \Phi(t,\tau)\underline{u}_n(\tau)d\tau\right] + \underline{u}_n(t) \tag{A3-11}$$

Replacing the bracketed term in (A3-11) by (A3-7) gives

$$\dot{\underline{x}}_n(t) = K(t)\underline{x}_n(t) + \underline{u}_n(t) \tag{A3-12}$$

and (A3-12) is identical to (A3-3). QED.

The matrix $\Phi$ is the well known state transition matrix[3, 5], also known as the matrizant[6] and the fundamental matrix[7]. It is a 6 x 6 matrix which can be partitioned (following Carlson's notation) into four 3 x 3 partial derivative matrices

$$\Phi(t,\tau) = \begin{pmatrix} \frac{\partial \underline{r}(t)}{\partial \underline{r}(\tau)} & \frac{\partial \underline{r}(t)}{\partial \underline{v}(\tau)} \\ \frac{\partial \underline{v}(t)}{\partial \underline{r}(\tau)} & \frac{\partial \underline{v}(t)}{\partial \underline{v}(\tau)} \end{pmatrix} \tag{A3-13}$$

$$\equiv \begin{pmatrix} A(t,\tau) & B(t,\tau) \\ C(t,\tau) & D(t,\tau) \end{pmatrix} \tag{A3-14}$$

The four partial derivative matrices have derivatives given by[3]

$$\frac{\partial}{\partial t} A(t,\tau) = C(t,\tau) \tag{A3-15}$$

$$\frac{\partial}{\partial t} B(t,\tau) = D(t,\tau) \tag{A3-16}$$

$$\frac{\partial}{\partial t} C(t,\tau) = G(t)\, A(t,\tau) \tag{A3-17}$$

$$\frac{\partial}{\partial t} D(t,\tau) = G(t)\, B(t,\tau) \tag{A3-18}$$

$$\frac{\partial}{\partial \tau} A(t,\tau) = -B(t,\tau)\, G(\tau) \tag{A3-19}$$

$$\frac{\partial}{\partial \tau} B(t,\tau) = -A(t,\tau) \tag{A3-20}$$

$$\frac{\partial}{\partial \tau} C(t,\tau) = -D(t,\tau)\, G(\tau) \tag{A3-21}$$

$$\frac{\partial}{\partial \tau} D(t,\tau) = -C(t,\tau) \tag{A3-22}$$

Also[3]

$$A(t,t) = A(\tau,\tau) = I \tag{A3-23}$$

$$B(t,t) = B(\tau,\tau) = O \tag{A3-24}$$

$$C(t,t) = C(\tau,\tau) = O \tag{A3-25}$$

$$D(t,t) = D(\tau,\tau) = I \tag{A3-26}$$

From (A3-15) through (A3-26) various Taylor series expansions can be derived. Some examples are[3]

$$A(t,\tau) = I + G(t)(t-\tau)^2/2! + O(t-\tau)^3 \tag{A3-27}$$

$$B(t,\tau) = I(t-\tau) + G(t)(t-\tau)^3/3! + O(t-\tau)^4 \tag{A3-28}$$

$$C(t,\tau) = G(t)(t-\tau) + O(t-\tau)^2 \tag{A3-29}$$

$$D(t,\tau) = I + G(t)(t-\tau)^2/2! + O(t-\tau)^3 \tag{A3-30}$$

Some additional forms of the partial derivative matrices are given in Section C .

Two properties of the state transition matrix which are used in later sections are

$$\Phi(t_2, t_1) = \Phi(t_1, t_2)^{-1} \tag{A3-31}$$

$$\Phi(t_2, t_1) = \Phi(t_2, t_o)\,\Phi(t_o, t_1) \tag{A3-32}$$

These properties are especially useful in formulating the solutions of Section B.

## A4 OUTER LIMIT

The outer limit of (A1-10) is the limit where $\underline{r} - \underline{p}_i = 0(1)$ for all $i = 1, 2, \ldots, N-2$. In other words, the outer limit specifically excludes the exceptional case where $\underline{r} - \underline{p}_i = 0(\mu_i)$. Therefore the function $\underline{F}$ in

(A1-10) is always order $\mu_i$ in the outer limit. In the outer domain, i.e., the domain defined by the outer limit, the solution of (A1-10) is assumed to be of the form

$$\underline{r} = \underline{r}_o + \mu \underline{r}_1 + \mu^2 \underline{r}_2 + \mu^3 \underline{r}_3 + \dots \quad \text{(A4-1)}$$

where

$$\mu \in (\mu_1, \mu_2, \dots, \mu_{N-2}) \quad \text{(A4-2)}$$

(A4-1) is a representation of the solution $\underline{r}$ in the form of an asymptotic expansion in powers of $\mu$. It is an exact solution of (A1-10) when $\mu = \mu_1 = \mu_2 = \dots = \mu_{N-2} = 0$ and approximates the exact solution as long as $\mu$ is small.

## A5 OUTER DIFFERENTIAL EQUATIONS

Substituting (A4-1) into (A1-10) gives

$$\ddot{\underline{r}} = \ddot{\underline{r}}_o + \mu \ddot{\underline{r}}_1 + \mu^2 \ddot{\underline{r}}_2 + O(\mu^3) \quad \text{(A5-1)}$$

$$\underline{f}(\underline{r}) = \underline{f}(\underline{r}_o) + \mu G(\underline{r}_o)\underline{r}_1 + \mu^2 \left[G(\underline{r}_o)\underline{r}_2 + \frac{1}{2} \underline{H}(\underline{r}_o)\underline{r}_1^2\right] + O(\mu^3) \quad \text{(A5-2)}$$

$$\underline{F}(\underline{r}, \underline{p}_i) = \sum_{i=1}^{N-2} \mu M_i \Big[\underline{f}(\underline{r}_o - \underline{p}_i) + \mu G(\underline{r}_o - \underline{p}_i)\, \underline{r}_1$$

$$+ O(\mu^2) + \underline{f}(\underline{p}_i)\Big] \quad \text{(A5-3)}$$

where

$$M_i = \mu_i/\mu \quad \text{(A5-4)}$$

Equating power of $\mu$ in (A5-1) through A5-3) gives

$$\ddot{\underline{r}}_o = \underline{f}(\underline{r}_o) \quad \text{(A5-5)}$$

$$\ddot{\underline{r}}_i = G(\underline{r}_o)\underline{r}_1 + \underline{F}_1(\underline{r}_o, \underline{p}_i) \quad \text{(A5-6)}$$

$$\ddot{\underline{r}}_2 = G(\underline{r}_o)\underline{r}_2 + \underline{F}_2(\underline{r}_o, \underline{r}_1, \underline{p}_i) \tag{A5-7}$$

or, in general,

$$\ddot{\underline{r}}_n = G(\underline{r}_o)\underline{r}_n + \underline{F}_n(\underline{r}_o, \ldots, \underline{r}_{n-1}, \underline{p}_i) \tag{A5-8}$$

The functions $\underline{F}_1$ and $\underline{F}_2$ are given by

$$\underline{F}_1(\underline{r}_o, \underline{p}_i) = \sum_{i=1}^{N-2} M_i \left[\underline{f}(\underline{r}_o - \underline{p}_i) + \underline{f}(\underline{p}_i)\right] \tag{A5-9}$$

$$\underline{F}_2(\underline{r}_o, \underline{r}_1, \underline{p}_i) = \frac{1}{2} \underline{H}(\underline{r}_o)\, \underline{r}_1^2 + \sum_{i=1}^{N-2} M_i\, G(\underline{r}_o - \underline{p}_i)\, \underline{r}_1 \tag{A5-10}$$

The general term $\underline{F}_n$ would involve tensors up to order $(n + 1)$. Because of this behavior it appears somewhat impractical to go beyond $n = 2$ which already includes the third order tensor $\underline{H}(\underline{r}_o)$.

The differential equations (A5-5), (A5-6) and (A5-7) are the zeroth, first and second order <u>outer</u> differential equations. The equation for $\underline{r}_o$ is simply the two-body differential equation while those for $\underline{r}_1$ and $\underline{r}_2$ are of the linear type discussed in Section A3.

## A6 OUTER SOLUTION

The zeroth order outer differential equation can be written

$$\frac{d^2 \underline{r}_o}{dt^2} = - \frac{\underline{r}_o}{r_o^3} \tag{A6-1}$$

The solution to this differential equation is well known and many forms exist. The motion is elliptical with respect to the primary body and a useful form is

$$\underline{r}_o(t) = f_o(t)\, \underline{r}_o(t_o) + g_o(t)\, \underline{v}_o(t_o) \tag{A6-2}$$

where $\underline{r}_o$ is the two-body position and $\underline{v}_o$ the velocity

$$\underline{v}_o = \frac{d\underline{r}_o}{dt} \tag{A6-3}$$

The functions $f_o$ and $g_o$ are infinite series in time but have closed form expressions as functions of eccentric anomaly E where

$$n_o(t - t_{po}) = E - e_o \sin E \tag{A6-4}$$

In (A6-4) $n_o$ is the two-body mean motion, $e_o$ the eccentricity and $t_{po}$ the time of pericenter passage. The functions $f_o$ and $g_o$ are given by

$$f_o(t) = 1 - a_o \left[1 - \cos \Delta E\,(t, t_o)\right] \Big/ r_o(t_o) \tag{A6-5}$$

$$g_o(t) = \left[r_o(t_o) \sin \Delta E\,(t, t_o)\right] \Big/ (n_o a_o)$$

$$+ e_o \sin E\,(t_o) \left[1 - \cos \Delta E\,(t, t_o)\right] \Big/ n_o \tag{A6-6}$$

where

$$\Delta E(t, t_o) = E(t) - E(t_o) \tag{A6-7}$$

The higher order solutions are obtained by first writing (A5-8) as

$$\dot{\underline{r}}_n = \underline{v}_n \tag{A6-8}$$

$$\dot{\underline{v}}_n = G(\underline{r}_o)\, \underline{r}_n + \underline{F}_n\, (\underline{r}_o, \ldots, \underline{r}_{n-1}, \underline{p}_i) \tag{A6-9}$$

Since (A6-8) and (A6-9) are similar to (A3-1) and (A3-2) the solution follows from (A3-7)

$$\begin{pmatrix} \underline{r}_n(t) \\ \\ \underline{v}_n(t) \end{pmatrix} = \Phi(t, t_o) \begin{pmatrix} \underline{r}_n(t_o) \\ \\ \underline{v}_n(t_o) \end{pmatrix} + \int_{t_o}^{t} \Phi(t, t_o) \begin{pmatrix} 0 \\ \\ \underline{F}_n(\tau) \end{pmatrix} d\tau \tag{A6-10}$$

Using (A3-14) gives

$$\underline{r}_1(t) = A(t, t_o)\, \underline{r}_1(t_o) + B(t, t_o)\, \underline{v}_1(t_o) + \int_{t_o}^{t} B(t,\tau)\, \underline{F}_1(\tau) d\tau \tag{A6-11}$$

$$\underline{r}_2(t) = A(t, t_o)\, \underline{r}_2(t_o) + B(t, t_o)\, \underline{v}_2(t_o) + \int_{t_o}^{t} B(t,\tau)\, \underline{F}_2(\tau) d\tau \tag{A6-12}$$

The outer solutions are therefore given by (A6-2), (A6-11) and (A6-12). They are functions of the time t, the initial time $t_o$, and the initial position and velocity

$$\underline{r}(t_o) = \underline{r}_o(t_o) + \mu \underline{r}_1(t_o) + \mu^2 \underline{r}_2(t_o) \tag{A6-13}$$

$$\underline{v}(t_o) = \underline{v}_o(t_o) + \mu \underline{v}_1(t_o) + \mu^2 \underline{v}_2(t_o) \tag{A6-14}$$

It will be shown that using these solutions the outer expansion (A4-1) contains a non-uniformity when $\underline{r} - \underline{p}_i = 0(\mu_i)$. That is, the individual terms in the expansion do not remain small compared to the preceding terms. In this case it is necessary to investigate another limit of (A1-10).

## A7 INNER LIMIT

When the trajectory representing a solution of (A1-10) passes close to one of the secondary bodies the outer solution is no longer valid due to a non-uniformity in the outer expansion (A4-1). In order to study another limit the origin is transferred to the kth secondary body and the length scaled such that the new position vector is order unity. Such a transformation is given by

$$\underline{r}_\alpha = (\underline{r} - \underline{p}_k)/\mu_k^\alpha \tag{A7-1}$$

It is obvious that when $\underline{r} - \underline{p}_k = 0(\mu_k^\alpha)$ then $\underline{r}_\alpha = 0\ (1)$. It is also necessary to make a similar transformation of the independent variable, i.e.,

$$t_\beta = (t - t_{pk})/\mu_k^\beta \tag{A7-2}$$

where $t_{pk}$ is some fixed time, such as time of closest approach, associated with motion close to $\underline{p}_k$. When $t - t_{pk} = 0(\mu_k^\beta)$ then $t_\beta = 0(1)$.

The next step in determining the inner limit is to transform (A1-10) into the new variables. From (A7-1)

$$\underline{r} = \underline{p}_k + \mu_k^\alpha \underline{r}_\alpha \tag{A7-3}$$

and differentiating gives

$$\dot{\underline{r}} = \dot{\underline{p}}_k + \mu_k^{\alpha-\beta}\frac{d\underline{r}_\alpha}{dt_\beta} \tag{A7-4}$$

$$\ddot{\underline{r}} = \ddot{\underline{p}}_k + \mu_k^{\alpha-2\beta}\frac{d^2\underline{r}_\alpha}{dt_\beta^2} \tag{A7-5}$$

Also

$$\underline{f}(\underline{r}) = \underline{f}(\underline{p}_k + \mu_k^\alpha \underline{r}_\alpha)$$

and, using (A2-14)

$$\underline{f}(\underline{r}) = \underline{f}(\underline{p}_k) + \mu_k^\alpha G(\underline{p}_k)\underline{r}_\alpha + \mu_k^{2\alpha} \underline{H}(\underline{p}_k)\underline{r}_\alpha^2/2 + O(\mu_k^{3\alpha}) \tag{A7-6}$$

Likewise

$$\begin{aligned}\underline{F}(\underline{r}, \underline{p}_i) &= \sum_{i=1}^{N-2} \mu_i\left[\underline{f}(\underline{p}_k + \mu_k^\alpha \underline{r}_\alpha - \underline{p}_i) + \underline{f}(\underline{p}_i)\right] \\ &= \mu_k\left[\underline{f}(\mu_k^\alpha \underline{r}_\alpha) + f(\underline{p}_k)\right] + \sum_{\substack{i=1\\ i\neq k}}^{N-2} \mu_i\left[\underline{f}(\underline{p}_k - \underline{p}_i) + \underline{f}(\underline{p}_i)\right. \\ &\quad \left. + \mu_k^\alpha G(\underline{p}_k - \underline{p}_i)\underline{r}_\alpha + O(\mu_k^{2\alpha})\right]\end{aligned} \tag{A7-7}$$

where

$$\underline{f}(\mu_k\, \underline{r}_\alpha) = \underline{f}(\underline{r}_\alpha)/\mu_k^2 \tag{A7-8}$$

Substituting (A1-12) into (A7-5) and (A7-8) into (A7-7) and then substituting (A7-5), (A7-6) and (A7-7) into (A1-10) gives

$$\underline{f}(\underline{p}_k) + \mu_k \underline{f}(\underline{p}_k) + \sum_{\substack{i=1\\ i\neq k}}^{N-2} \mu_i \left[ \underline{f}(\underline{p}_k - \underline{p}_i) + \underline{f}(\underline{p}_i) \right]$$

$$+ \mu_k^{\alpha-2\beta} \frac{d^2 \underline{r}_\alpha}{\alpha\, dt_\beta^2} = \underline{f}(\underline{p}_k) + \mu_k^\alpha\, G(\underline{p}_k)\underline{r}_\alpha + \frac{1}{2} \mu_k^{2\alpha}\, \underline{H}(\underline{p}_k)\underline{r}_\alpha^2$$

$$+ \mu_k^{1-2\alpha}\, \underline{f}(\underline{r}_k) + \mu_k \underline{f}(\underline{p}_k) + O(\mu_k^{3\alpha})$$

$$+ \sum_{\substack{i=1\\ i\neq k}}^{N-2} \mu_i \left[ \underline{f}(\underline{p}_k - \underline{p}_i)\, \underline{f}(\underline{p}_i) \right.$$

$$\left. + \mu_k^\alpha\, G(\underline{p}_k - \underline{p}_i)\, \underline{r}_\alpha + O(\mu_k^{2\alpha}) \right] \tag{A7-9}$$

or, cancelling similar terms gives

$$\frac{d^2 \underline{r}_\alpha}{dt_\beta^2} = \mu_k^{1+2\beta-3\alpha}\, \underline{f}(\underline{r}_\alpha) + \mu_k^{2\beta}\, G(\underline{p}_k)\underline{r}_\alpha + \frac{1}{2} \mu_k^{\alpha+2\beta}\, \underline{H}(\underline{p}_k)\underline{r}_\alpha^2$$

$$+ \mu_k^{2\beta} \sum_{\substack{i=1\\ i\neq k}}^{N-2} \mu_i\, G(\underline{p}_k - \underline{p}_i)\underline{r}_\alpha + O(\mu_k^{2\alpha+2\beta},\ \mu_k^{\alpha+2\beta}\, \mu_i) \tag{A7-10}$$

(A7-10) still includes functions of t through $\underline{p}_k$ and $\underline{p}_i$. From (A7-2)

$$t = t_{pk} + \mu_k^\beta t_\beta \tag{A7-11}$$

Let

$$t_{pk} = t_k + \mu_k^\beta \tau_k \tag{A7-12}$$

giving

$$t = t_k + \mu_k^\beta (t_\beta + \tau_k) \tag{A7-13}$$

The significance of $t_k$ and $\tau_k$ will be demonstrated later in the matching. Using (A7-13)

$$G\left(\underline{p}_k(t)\right) = G\left(\underline{p}_k(t_k)\right) + \mu_k^\beta \frac{d}{dt} G\left(\underline{p}_k(t_k)\right) (t_\beta + \tau_k) + O\left(\mu_k^{2\beta}\right) \tag{A7-14}$$

The derivative dG/dt is found as follows:
From (A2-15)

$$\frac{d}{dt} G_{ij}(\underline{x}) = \frac{d}{dx_k}\left(\frac{3x_i x_j}{x^5} - \frac{\delta_{ij}}{x^3}\right)\frac{dx_k}{dt} = \left[-\frac{15x_i x_j}{x^6}\frac{\partial x}{\partial x_k} + \frac{3x_i}{x^5}\delta_{jk}\right.$$

$$\left. + \frac{3x_j}{x^5}\delta_{ik} + \frac{3\delta_{ij}}{x^4}\frac{\partial x}{\partial x_k}\right]\dot{x}_k \tag{A7-15}$$

Using (A2-9), (A7-15) becomes

$$\frac{d}{dt} G_{ij}(\underline{x}) = \left[-\frac{15x_i x_j x_k}{x^7} + \frac{3}{x^5}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})\right]\dot{x}_k$$

$$= \underline{H}_{ijk}(\underline{x})\underline{\dot{x}}_k \tag{A7-16}$$

Then

$$G\left(\underline{p}_k(t)\right) = G_k + \mu_k^{\beta}\, \underline{H}_k\, \dot{\underline{p}}_k\,(t_k)\,(t_\beta + \tau_k) + O\left(\mu_k^{2\beta}\right) \tag{A7-17}$$

where

$$G_k = G\left(\underline{p}_k(t_k)\right) \tag{A7-18}$$

$$\underline{H}_k = \underline{H}\left(\underline{p}_k(t_k)\right) \tag{A7-19}$$

Putting

$$G\left(\underline{p}_k(t_k) - \underline{p}_i(t_k)\right) = G_k^i \tag{A7-20}$$

(A7-10) can be written

$$\begin{aligned} \frac{d^2\underline{r}}{dt^2} = {} & \mu_k^{1+2\beta-3\alpha}\, \underline{f}(\underline{r}_\alpha) + \mu_k^{2\beta}\, G_k\, \underline{r}_\alpha + \mu_k^{3\beta}\, \underline{H}_k\, \dot{\underline{p}}_k(t_k)\, \underline{r}_\alpha\,(t_\beta + \tau_k) \\ & + \frac{1}{2}\mu_k^{\alpha+2\beta}\, \underline{H}_k\, \underline{r}_\alpha^2 + \mu_k^{2\beta} \sum_{\substack{i=1 \\ i\neq k}}^{N-2} \mu_i\, G_k^i\, \underline{r}_\alpha \\ & + O\left(\mu_k^{2\alpha+2\beta},\ \mu_k^{\alpha+2\beta}\, \mu_i\right) \end{aligned} \tag{A7-21}$$

The inner limit is partially defined by balancing the inertial term with the gravitational term of the $k^{th}$ body, i.e., matching $(d^2\underline{r}_\alpha/dt^2)$ with $\underline{f}(\underline{r}_\alpha)$. The two terms will be balanced if both are order unity. This is obtained if

$$1 + 2\beta - 3\alpha = O \tag{A7-22}$$

Also, from (A7-4)

$$\frac{d\underline{r}_\alpha}{dt_\beta} = \frac{\dot{\underline{r}} - \dot{\underline{p}}_k}{\mu_k^{\alpha-\beta}} \tag{A7-23}$$

In (A7-1) it is assumed that the numerator, $\underline{r} - \underline{p}_k$, is small; this gives a close approach to the kth body. In (A7-22) it is not necessary for $\dot{\underline{r}} - \dot{\underline{p}}_k$, the relative velocity, to be small. In fact, this velocity difference will depend on the energy of the outer solution and the only conclusion that can be made at this point is that it is order unity. Thus it is most reasonable to make the denominator in (A7-23) the same order, i.e., let

$$\alpha - \beta = O \tag{A7-24}$$

The inner limit will be defined by (A7-22) and (A7-23), i.e.,

$$\alpha = \beta = 1 \tag{A7-25A}$$

As a result (A7-21) becomes

$$\frac{d^2\underline{r}_\alpha}{dt^2_\beta} = \underline{f}(\underline{r}_\alpha) + \mu_k^2 G_k \underline{r}_\alpha + \mu_k^3 \left[ \underline{H}_k \dot{\underline{p}}_k(t_k)\, \underline{r}_\alpha (t_\beta + \tau_k) + \frac{1}{2} \underline{H}_k \underline{r}_\alpha^2 + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i G_k^i\, \underline{r}_\alpha \right] + O(\mu_k^4) \tag{A7-25B}$$

In the terminology of singular perturbation theory, (A7-24) is called a distinguished limit.

The inner variables will be defined by

$$\underline{R}_k = (\underline{r} - \underline{p}_k)/\mu_k \tag{A7-26}$$

$$S_k = (t - t_{pk})/\mu_k \tag{A7-27}$$

and the associated differential equation is

$$\frac{d^2 \underline{R}_k}{dS_k^2} = \underline{f}(\underline{R}_k) + \underline{P}(\underline{R}_k, \underline{p}_k, \underline{p}_i) \tag{A7-28}$$

where

$$\underline{P}(\underline{R}_k, \underline{p}_k, \underline{p}_i) = \mu_k^2 G_k \underline{R}_k + \mu_k^3 \left[ \underline{H}_k \underline{p}_k(t_k) \underline{R}_k(S_k + \tau_k) + \frac{1}{2} \underline{H}_k \underline{R}_k^2 + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i G_k^i \underline{R}_k \right] + O(\mu_k^4) \tag{A7-29}$$

The term of order $\mu_k^3$ in $\underline{P}$ is included here although it is not actually needed in a general second order theory. It is important in problems where only an inner solution is used. It may also be important in problems where the model includes satellites of secondary bodies, i.e., including moons in the analysis of interplanetary trajectories. Suppose the $m^{th}$ body is close to the $k^{th}$ body such that

$$\underline{p}_k - \underline{p}_m = \varepsilon \underline{\rho}_m \tag{A7-30}$$

Then

$$\begin{aligned} G_k^m &= G(\underline{p}_k - \underline{p}_m) \\ &= G(\varepsilon \underline{\rho}_m) \\ &= G(\underline{\rho}_m)/\varepsilon^3 \end{aligned} \tag{A7-31}$$

Then the term

$$\mu_k^3 M_m G_k^m = (\mu_k^2 \mu_m / \varepsilon^3) G(\underline{\rho}_m)$$

For the moon, earth, sun case

$$\mu_m = m_m/m_o = 3.7 \times 10^{-8}$$

$$\varepsilon = 2.6 \times 10^{-3} \text{ au}$$

so that

$$(\mu_m/\varepsilon^3) = 2.1 = O(1)$$

and

$$\mu_k^3 M_m G_k^m = O(\mu_k^2) \qquad \text{(A7-32)}$$

(A7-32) shows that the effect of a nearby moon $\left(\text{i.e., } \mu_k^3 M_m G_k^m \underline{R}_k\right)$ on the inner solution of an interplanetary trajectory may be the same order of magnitude as the effect of the Sun (i.e., $\mu_k^2 G_k \underline{R}_k$).

In the inner domain, i.e., the domain defined by the inner limit, the solution of (A7-28) is assumed to be of the form

$$\underline{R}_k = \underline{R}_{ko} + \mu_k^2 \underline{R}_{k2} + \mu_k^3 \underline{R}_{k3} + \mu_k^4 \underline{R}_{k4} + \ldots \qquad \text{(A7-33)}$$

(A7-33) is also an asymptotic representation and reduces to the exact solution as $\mu_k \to 0$.

It should be noted that another distinguished limit exists besides (A7-24). If the velocity difference in (A7-22) is assumed to be smaller than order unity then the other limit is

$$\alpha = 1/3, \quad \beta = 0 \qquad \text{(A7-34)}$$

in which case (A7-10) becomes

$$\frac{d^2 \underline{r}_\alpha}{dt_\beta^2} = \underline{f}(\underline{r} \ ) + G(\underline{p}_k)\, \underline{r}_\alpha + O(\mu^{1/3}) \qquad \text{(A7-35)}$$

(A7-35) is a form of Hill's equation and it's solution is beyond the scope of this study.

## A8 INNER DIFFERENTIAL EQUATIONS

Substituting (A7-33) into (A7-28) gives

$$\frac{d^2\underline{R}_k}{dS_k^2} = \frac{d^2\underline{R}_{ko}}{dS_k^2} + \mu_k^2 \frac{d^2\underline{R}_{k2}}{dS_k^2} + \mu_k^3 \frac{d^2\underline{R}_{k3}}{dS_k^3} + O(\mu_k^4) \qquad \text{(A8-1)}$$

$$\underline{f}(\underline{R}_k) = \underline{f}(\underline{R}_{ko}) + \mu_k^2\, G(\underline{R}_{ko})\, \underline{R}_{k2}$$

$$+ \mu_k^3\, G(\underline{R}_{ko})\, \underline{R}_{k3} + O(\mu_k^4) \qquad \text{(A8-2)}$$

$$\underline{P}(\underline{R}_k, \underline{p}_k, \underline{p}_i) = \mu_k^2\, G_k\, \underline{R}_{ko} + \mu_k^3 \left[ \underline{H}_k \dot{\underline{p}}_k(t_k) \underline{R}_{ko}\, (S_k + \tau_k) \right.$$

$$\left. + \frac{1}{2}\, \underline{H}_k\, \underline{R}_{ko}^2 + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i\, G_k^i\, \underline{R}_{ko} \right] + O(\mu_k^4) \qquad \text{(A8-3)}$$

Equating powers of $\mu_k$ in (A8-1) through (A8-3) gives

$$\frac{d^2\underline{R}_{ko}}{dS_k^2} = \underline{f}(\underline{R}_{ko}) \qquad \text{(A8-4)}$$

$$\frac{d^2\underline{R}_{k2}}{dS_k^2} = G(\underline{R}_{ko})\, \underline{R}_{k2} + \underline{P}_2\, (\underline{R}_{ko}) \qquad \text{(A8-5)}$$

$$\frac{d^2\underline{R}_{k3}}{dS_k^2} = G(\underline{R}_{ko})\, \underline{R}_{k3} + \underline{P}_3\, (\underline{R}_{ko}) \qquad \text{(A8-6)}$$

where

$$\underline{P}_2(\underline{R}_{ko}) = G_k \underline{R}_{ko} \tag{A8-7}$$

$$\underline{P}_3(\underline{R}_{ko}) = \frac{1}{2} \underline{H}_k \underline{R}_{ko}^2 + \underline{H}_k \dot{\underline{p}}_k(t_k) \underline{R}_{ko} (S_k + \tau_k)$$

$$+ \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i \, G_k^i \, \underline{R}_{ko} \tag{A8-8}$$

The differential equations (A8-4) and (A8-6) are the zeroth, second and third order <u>inner</u> differential equations. The equation for $\underline{R}_{ko}$ is simply the two-body differential equation while those for $\underline{R}_{k2}$ and $\underline{R}_{k3}$ are of the linear type discussed in Section A3.

## A9 INNER SOLUTION

The solution to the zeroth order differential equation is hyperbolic motion with respect to the $k^{th}$ secondary body and can be represented by

$$\underline{R}_{ko} (S_k) = \bar{f}_o(S_k) \, \underline{R}_{ko} (S_{ko}) + \bar{g}_o (S_k) \, \underline{V}_{ko} (S_{ko}) \tag{A9-1}$$

The functions $\bar{f}_o$ and $\bar{g}_o$ have closed form expressions as functions of the eccentric anomaly $F_k$ where

$$\bar{n}_k S_k = \bar{e}_k \sinh F_k - F_k \tag{A9-2}$$

In (A9-2) $\bar{n}_k$ is the two-body mean motion and $\bar{e}_k$ is the eccentricity. The functions $\bar{f}_o$ and $\bar{g}_o$ are given by

$$\bar{f}_o = 1 - \bar{a}_k \left[ 1 - \cosh \Delta F_k (S_k, S_{ko}) \right] \Big/ R_{ko}(S_{ko}) \tag{A9-3}$$

$$\bar{g}_o = S_k - S_{ko} - \left[ \sinh \Delta F_k (S_k, S_{ko}) - \Delta F_k (S_k, S_{ko}) \right] \Big/ \bar{n}_k \tag{A9-4}$$

where

$$\Delta F_k(S_k, S_{ko}) = F_k(S_k) - F_k(S_{ko}) \tag{A9-5}$$

It was shown by Carlson[3] that it is desirable to choose the initial conditions for the higher order solutions to be zero at pericenter of $\underline{R}_{ko}$, i.e., the higher order solutions vanish at $S_k = 0$. A slightly more general approach is to put

$$\underline{R}_{k2}(S_{ko}) = \underline{V}_{k2}(S_{ko}) = \underline{R}_{k3}(S_{ko}) = \underline{V}_{k3}(S_{ko}) = 0 \tag{A9-6}$$

so that the higher order solutions vanish at $S_k = S_{ko}$. Since (A8-5) and (A8-6) are identical in form to (A5-6) and (A5-7) the solutions will be similar to (A6-11) and (A6-12)

$$\underline{R}_{k2}(S_k) = \int_{S_{ko}}^{S_k} \overline{B}(S_k, \sigma)\, \underline{P}_2(\sigma)\, d\sigma \tag{A9-7}$$

$$\underline{R}_{k3}(S_k) = \int_{S_{ko}}^{S_k} \overline{B}(S_k, \sigma)\, \underline{P}_3(\sigma)\, d\sigma \tag{A9-8}$$

where $\overline{B}$ is a partial derivative matrix evaluated along the hyperbola $\underline{R}_{ko}$.

The inner solutions are given by (A9-1), (A9-7) and (A9-8). It will be shown that using these solutions the inner expansion (A7-33) contains a non-uniformity as $S_k \to \infty$.

## A10 OVERLAP DOMAIN

The outer expansion has already been given as

$$\underline{r}(t) = \underline{r}_o(t) + \mu \underline{r}_1(t) + \mu^2 \underline{r}_2(t) + O(\mu^3) \tag{A10-1}$$

Rewriting (A7-26) using (A7-33) gives the outer form of the inner expansion as

$$\underline{r}(t) = \underline{p}_k(t) + \mu_k \underline{R}_{ko}(S_k) + \mu_k^3 \underline{R}_{k2}(S_k)$$

$$+ \mu_k^4 \underline{R}_{k3}(S_k) + O(\mu_k^5) \qquad \text{(A 10-2)}$$

These two solutions are valid in the outer and inner domains, repectively. It is assumed that both solutions are valid in an intermediate domain called the overlap domain which connects the outer and the inner domains around the $k^{th}$ secondary body. It will be shown later in the matching that one of the conditions for the two solutions overlapping is that for some $t = t_k$

$$\underline{r}_o(t_k) = \underline{p}_k(t_k) \qquad \text{(A 10-3)}$$

where $t_k$ was first introduced in (A7-12). The time $t_k$ is the nominal time at which a close approach to the $k^{th}$ body occurs. The actual closest approach time is given by (A7-12).

In (A6-13) $\underline{r}_o(t_o)$ is the initial position for the two body ellipse $\underline{r}_o(t)$. If $t_k$ is specified and an ephemeris is used to determine $\underline{p}_k(t_k)$ then (A10-3) gives the final position $\underline{r}_o(t_k)$ for the two-body ellipse. The initial and final positions and the time interval $(t_k - t_o)$ constitute the standard Lambert problem. Solution of the Lambert problem completely determines $\underline{r}_o(t)$.

The overlap domain will be defined as the domain where $t \rightarrow t_k$ in the outer solution and where $|S_k| \rightarrow \infty$ in the inner solution. Thus the behavior of the outer solution must be determined in a near neighborhood of the $k^{th}$ secondary body and the behavior of the inner solution must be determined in a region sufficiently removed from the $k^{th}$ body.

## A11 BEHAVIOR OF THE OUTER SOLUTION IN THE OVERLAP DOMAIN

### A11.1 Zeroth Order

The overlap domain is defined as the domain where $t \to t_k$. It is therefore necessary to expand the outer solution in terms of $t - t_k$. The zeroth order solution is easily expanded since a two-body ellipse is defined by functions which can be expanded in Taylor series' about $t = t_k$. The result is

$$\underline{r}_o(t) = \underline{a}_{ok} + \underline{b}_{ok}(t - t_k) + \underline{c}_{ok}(t - t_k)^2 + \underline{d}_{ok}(t - t_k)^3$$

$$+ \underline{e}_{ok}(t - t_k)^4 + O\left((t - t_k)^5\right) \qquad \text{(A11-1)}$$

where

$$\underline{a}_{ok} = \underline{r}_o(t_k) \qquad \text{(A11-2)}$$

$$\underline{b}_{ok} = \dot{\underline{r}}_o(t_k) = \underline{v}_o(t_k) \qquad \text{(A11-3)}$$

$$\underline{c}_{ok} = \ddot{\underline{r}}_o(t_k)/2 = \underline{f}\left(\underline{r}_o(t_k)\right)/2 \qquad \text{(A11-4)}$$

$$\underline{d}_{ok} = \dddot{\underline{r}}_o(t_k)/6 = \dot{\underline{f}}\left(\underline{r}_o(t_k)\right)/6$$

$$= \frac{d\underline{f}}{d\underline{r}_o}\left(\underline{r}_o(t_k)\right)\dot{\underline{r}}_o(t_k)/6$$

$$= G\left(\underline{r}_o(t_k)\right)\underline{v}_o(t_k)/6 \qquad \text{(A11-5)}$$

$$\underline{e}_{ok} = \ddddot{\underline{r}}_o(t_k)/24$$

$$= \frac{d}{dt}\left[G\left(\underline{r}_o(t_k)\right)\underline{v}_o(t_k)\right]\Big/24$$

$$= \left[\underline{H}\left(\underline{r}_o(t_k)\right)\underline{v}_o^2(t_k) + G\left(\underline{r}_o(t_k)\right)\underline{f}\left(\underline{r}_o(t_k)\right)\right]\Big/24 \qquad \text{(A11-6)}$$

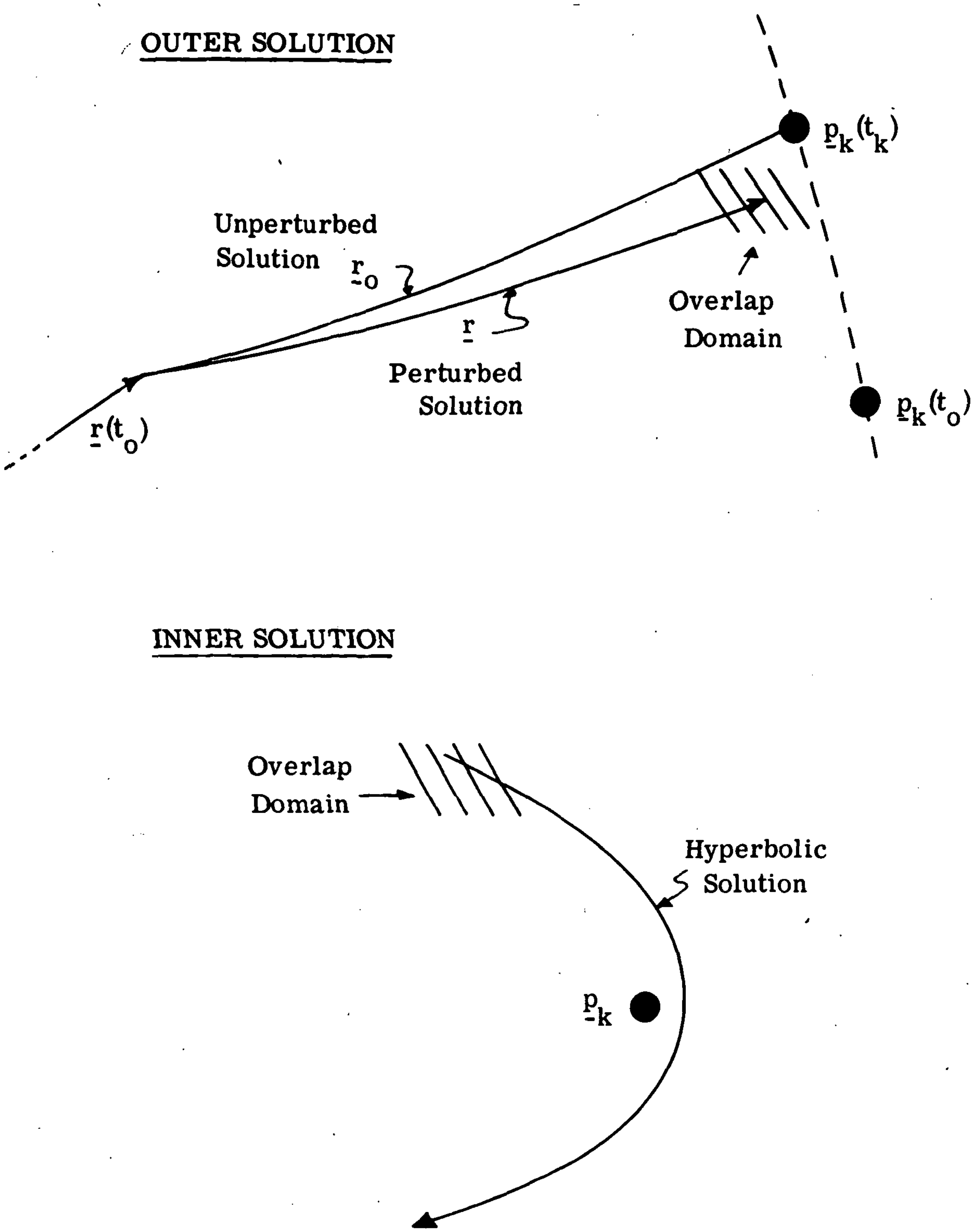


Figure A3. Outer Solution, Inner Solution and Overlap Domain

The expressions for $\underline{c}_{ok}$, $\underline{d}_{ok}$ and $\underline{e}_{ok}$ follow from using (A5-5), (A2-17) and (A7-16).

Since the motion of the $k^{th}$ body is assumed to be a known function of time it can also be expanded in a Taylor series:

$$\underline{p}_k(t) = \underline{p}_{ko} + \underline{p}_{k1}(t - t_k) + \underline{p}_{k2}(t - t_k)^2 + \underline{p}_{k3}(t - t_k)^3$$

$$+ \underline{p}_{k4}(t - t_k)^4 + O\left((t - t_k)^5\right) \tag{A11-7}$$

where

$$\underline{p}_{ko} = \underline{p}_k(t_k) \tag{A11-8}$$

$$\underline{p}_{k1} = \dot{\underline{p}}_k(t_k) \tag{A11-9}$$

$$\underline{p}_{k2} = \ddot{\underline{p}}_k(t_k)/2$$

$$= \left[\underline{f}\left(\underline{p}_k(t_k)\right) + \mu\underline{p}_k^{*}\right]/2 \tag{A11-10}$$

$$\underline{p}_{k3} = \dddot{\underline{p}}_k(t_k)/6$$

$$= \left[\frac{d}{dt}\underline{f}\left(\underline{p}_k(t_k)\right) + \mu\underline{p}_k^{**}\right]/6$$

$$= \left[G\left(\underline{p}_k(t_k)\right)\dot{\underline{p}}_k(t_k) + \mu\underline{p}_k^{**}\right]/6 \tag{A11-11}$$

$$\underline{p}_{k4} = \ddddot{\underline{p}}_k(t_k)/24$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\underline{f}\left(\underline{p}_k(t_k)\right)\right]/24 + O(\mu)$$

$$= \left[\underline{H}\left(\underline{p}_k(t_k)\right)\dot{\underline{p}}_k^2(t_k) + G\left(\underline{p}_k(t_k)\right)\underline{f}\left(\underline{p}_k(t_k)\right)\right]/24$$

$$+ O(\mu) \tag{A11-12}$$

and

$$\underline{p}_k^* = \left[\ddot{\underline{p}}_k(t) - \underline{f}\left(\underline{p}_k(t)\right)\right]_{t=t_k} \Big/ \mu$$

$$= M_k \underline{f}\left(\underline{p}_k(t_k)\right) + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i \left[\underline{f}\left(\underline{p}_k(t_k) - \underline{p}_i(t_k)\right)\right.$$

$$\left. + \underline{f}\left(\underline{p}_i(t_k)\right)\right] \qquad \text{(A11-13)}$$

$$\underline{p}_k^{**} = \frac{d}{dt}\left[\ddot{\underline{p}}_k(t) - \underline{f}\left(\underline{p}_k(t_k)\right)\right]_{t=t_k} \Big/ \mu$$

$$= M_k G_k \dot{\underline{p}}_k(t_k) + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i \left[G_k^i \left(\dot{\underline{p}}_k(t_k) - \dot{\underline{p}}_i(t_k)\right)\right.$$

$$\left. + G\left(\underline{p}_i(t_k)\right) \dot{\underline{p}}_i(t_k)\right] \qquad \text{(A11-14)}$$

The expressions for $\underline{p}_{k2}$, $\underline{p}_{k3}$, $\underline{p}_{k4}$, $\underline{p}_k^*$ and $\underline{p}_k^{**}$ follow from using (A1-12), (A2-17), (A5-5), (A7-16), (A7-18) and (A7-20).

Now define $\underline{r}_{ko}$ by

$$\underline{r}_{ko} \equiv \underline{r}_o - \underline{p}_k$$

$$= (\underline{a}_{ok} - \underline{p}_{ko}) + (\underline{b}_{ok} - \underline{p}_{k1})(t - t_k)$$

$$+ (\underline{c}_{ok} - \underline{p}_{k2})(t - t_k)^2 + (\underline{d}_{ok} - \underline{p}_{k3})(t - t_k)^3$$

$$+ (\underline{e}_{ok} - \underline{p}_{k4})(t - t_k)^4 + O\left((t - t_k)^5\right) \qquad \text{(A11-15)}$$

where

$$\underline{a}_{ok} - \underline{p}_{ko} = \underline{r}_o(t_k) - \underline{p}_k(t_k) \tag{A11-16}$$

$$\underline{b}_{ok} - \underline{p}_{k1} = \underline{v}_o(t_k) - \dot{\underline{p}}_k(t_k) \tag{A11-17}$$

$$\underline{c}_{ok} - \underline{p}_{k2} = \left[\underline{f}\left(\underline{r}_o(t_k)\right) - \underline{f}\left(\underline{p}_k(t_k)\right) - \mu \underline{p}_k^*\right] \Big/ 2 \tag{A11-18}$$

$$\underline{d}_{ok} - \underline{p}_{k3} = \Big[G\left(\underline{r}_o(t_k)\right) \underline{v}_o(t_k) - G\left(\underline{p}_k(t_k)\right) \dot{\underline{p}}_k(t_k)$$

$$- \mu \underline{p}_k^{**}\Big] \Big/ 6 \tag{A11-19}$$

$$\underline{e}_{ok} - \underline{p}_{k4} = \Big[\underline{H}\left(\underline{r}_o(t_k)\right) \underline{v}_o^2(t_k) - H\left(\underline{p}_k(t_k)\right) \dot{\underline{p}}_k^2(t_k)$$

$$+ G\left(\underline{r}_o(t_k)\right) \underline{f}\left(\underline{r}_o(t_k)\right) - G\left(\underline{p}_k(t_k)\right) \underline{f}\left(\underline{p}_k(t_k)\right) + O(\mu)\Big] \Big/ 24 \tag{A11-20}$$

From (A10-3) and (A11-16)

$$\underline{a}_{ok} - \underline{p}_{ko} = O \tag{A11-21}$$

Let

$$\underline{b}_{ok} - \underline{p}_{k1} = \underline{v}_o(t_k) - \dot{\underline{p}}_k(t_k) \equiv \underline{V}_k \tag{A11-22}$$

Again using (A10-3) and (A11-22)

$$\underline{c}_{ok} - \underline{p}_{k2} = -\mu \underline{p}_k^* / 2 \tag{A11-23}$$

$$\underline{d}_{ok} - \underline{p}_{k3} = \left[G\left(\underline{p}_k(t_k)\right) \underline{V}_k - \mu \underline{p}_k^{**}\right] \Big/ 6 \tag{A11-24}$$

$$\underline{e}_{ok} - \underline{p}_{k4} = \left[\underline{H}\left(\underline{p}_k(t_k)\right)\left(\underline{v}_o^2(t_k) - \dot{\underline{p}}_k^2(t_k)\right)\right.$$

$$\left. + O(\mu)\right] \Big/ 24 \qquad \text{(A11-25)}$$

Introducing (A7-18) and (A7-19) gives

$$\underline{r}_{ko} \doteq \underline{V}_k(t - t_k) - \mu \overset{*}{\underline{p}}_k (t - t_k)^2 \Big/ 2 + \left[G_k \underline{V}_k\right.$$

$$\left. -\mu \overset{**}{\underline{p}}_k\right] (t - t_k)^3 \Big/ 6 + \left[\underline{H}_k\left(\underline{v}_o^2(t_k) - \dot{\underline{p}}_k^2(t_k)\right)\right] (t - t_k)^4 \Big/ 24$$

$$+ O\left((t - t_k)^5\right) + O\left(\mu(t - t_k)^4\right) \qquad \text{(A11-26)}$$

(A11-26) represents the behavior, in the overlap domain, of the zeroth order relative position between the trajectory and the $k^{th}$ body. It is used to determine the behavior of the higher order terms and in the matching.

## A11.2 First Order

Let

$$\underline{r}_{ko} = \underline{X}_o + \underline{X}_1 + \underline{X}_2 + \underline{X}_3$$

where

$$\underline{X}_o = \underline{V}_k(t - t_k) \qquad \text{(A11-28)}$$

$$\underline{X}_1 = -\mu \overset{*}{\underline{p}}_k (t - t_k) \big/ 2 \qquad \text{(A11-29)}$$

$$\underline{X}_2 = \left[G_k \underline{V}_k - \mu \overset{**}{\underline{p}}_k\right] (t - t_k)^3 \Big/ 6 \qquad \text{(A11-30)}$$

$$\underline{X}_3 = O\left((t - t_k)^4\right) \qquad \text{(A11-31)}$$

Then, using (A2-14)

$$\underline{f}(\underline{r}_{ko}) = \underline{f}(\underline{X}_o) + G(\underline{X}_o)\left[\underline{X}_1 + \underline{X}_2 + O(\underline{X}_3)\right]$$

$$+ \underline{H}(\underline{X}_o)\left[\underline{X}_1^2 + O(\underline{X}_1\ \underline{X}_2)\right]\Big/2 + O(\underline{X}_1^3) \qquad \text{(A11-33)}$$

In order to use the solution being developed for both approach to and departure from the $k^{th}$ body, let

$$Q_k = \text{sgn}\,(t - t_k) = \frac{(t - t_k)}{|t - t_k|} \qquad \text{(A11-34)}$$

Then

$$\underline{f}(\underline{X}_o) = \underline{f}\Big(\underline{V}_k(t - t_k)\Big)$$

$$= - \underline{V}_k(t - t_k)\Big/\left|\underline{V}_k(t - t_k)\right|^3$$

$$= Q_k\,\underline{f}(\underline{V}_k)\Big/(t - t_k)^2 \qquad \text{(A11-35)}$$

By a similar analysis using (A2-15) and (A2-16)

$$G(\underline{X}_o) = Q_k\,G(\underline{V}_k)\Big/(t - t_k)^3 \qquad \text{(A11-36)}$$

$$\underline{H}(\underline{X}_o) = Q_k\,\underline{H}(\underline{V}_k)\Big/(t - t_k)^4 \qquad \text{(A11-37)}$$

Now (A11-33) can be written

$$\underline{f}(\underline{r}_{ko}) = \frac{Q_k}{(t - t_k)^2}\left[\underline{f}(\underline{V}_k) - \frac{\mu}{2}\,G(\underline{V}_k)\,\underline{p}_k^{*}\,(t - t_k)\right.$$

$$+ \frac{1}{6}\,G(\underline{V}_k)\,(G_k\,\underline{V}_k - \mu\underline{p}_k^{**})\,(t - t_k)^2$$

$$\left. + O\Big((t - t_k)^3\Big)\right] \qquad \text{(A11-38)}$$

The first order force function is given by (A5-9). It can be rewritten in terms of a singular part, $\underline{F}_{1s}$, and a non-singular part, $\underline{F}_{1n}$, i.e.,

$$\underline{F}_1(\underline{r}_o, \underline{p}_i) = \underline{F}_{1s}(\underline{r}_o, \underline{p}_k) + \underline{F}_{1n}(\underline{r}_o, \underline{p}_i) \quad \text{(A11-39)}$$

where

$$\underline{F}_{1s}(\underline{r}_o, \underline{p}_k) = M_k \, \underline{f}(\underline{r}_{ko}) \quad \text{(A11-40)}$$

$$\underline{F}_{1n}(\underline{r}_o, \underline{p}_i) = M_k \, \underline{f}(\underline{p}_k) + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i \left[ \underline{f}(\underline{r}_o - \underline{p}_i) + \underline{f}(\underline{p}_i) \right] \quad \text{(A11-41)}$$

Using (A11-38) and (A11-40) the singular force function is

$$\underline{F}_{1s} = \frac{Q_k M_k}{(t - t_k)^2} \left[ \underline{f}(\underline{V}_k) - \frac{u}{2} G(\underline{V}_k) \, \underline{p}_k^{*} (t - t_k) \right.$$

$$\left. + \frac{1}{6} G(V_k) \, (G_k \underline{V}_k - \mu \underline{p}_k^{**}) \, (t - t_k)^2 + O\left((t - t_k)^3\right) \right] \quad \text{(A11-42)}$$

From (A3-28)

$$B(t, \tau) = I(t - \tau) + G(t) \, (t - \tau)^3/6 + O\left((t - \tau)^4\right) \quad \text{(A11-43)}$$

where

$$G(t) = G\left(\underline{r}_o(t)\right) \quad \text{(A11-44)}$$

According to (A6-11) the first order solution is a function of $B(t, \tau) \, \underline{F}_1(\tau)$.

From (A11-42) and (A11-43)

$$B(t, \tau) \underline{F}_{1s}(\tau) = \underline{\beta}_{1k} \frac{(t - \tau)}{(\tau - t_k)^2} + \underline{\beta}_{2k} \frac{(t - \tau)}{(\tau - t_k)} + \underline{\beta}_{3k} (t - \tau)$$

$$+ \underline{\beta}_{4k} \frac{(t - \tau)^3}{(\tau - t_k)^2} + O\left[(t - \tau)(\tau - t_k), \right.$$

$$\left. \frac{(t - \tau)^4}{(\tau - t_k)^2}, \mu \frac{(t - \tau)^3}{(\tau - t_k)}\right] \qquad \text{(A11-45)}$$

where

$$\underline{\beta}_{1k} = Q_k M_k \underline{f}(\underline{V}_k) \qquad \text{(A11-46)}$$

$$\underline{\beta}_{2k} = \mu Q_k M_k G(\underline{V}_k) \underline{p}_k^{*}/2 \qquad \text{(A11-47)}$$

$$\underline{\beta}_{3k} = Q_k M_k G(\underline{V}_k)(G_k \underline{V}_k - \mu \underline{p}_k^{**})/6 \qquad \text{(A11-48)}$$

$$\underline{\beta}_{4k} = \underline{\beta}_{4k}(t)$$

$$= Q_k M_k G(t) \underline{f}(\underline{V}_k)/6 \qquad \text{(A11-49)}$$

(A11-45) represents the singular behavior of the integrand in (A6-11). Although only the first term of (A11-45) is actually singular as $t \to t_k$ and $\tau \to t_k$ the other terms play an important role in evaluating the second order solution. It is not necessary to develop a corresponding expansion of the non-singular part of the integrand in (A6-11), i.e., $B(t, \tau) \underline{F}_{1n}(\tau)$, as this contribution can be evaluated directly as will be shown.

Now let

$$\underline{\Phi}^{s}_{1k}(t,\tau) = B(t,\tau)\,\underline{F}_{1s}(\tau) - \underline{\beta}_{1k}\frac{(t-\tau)}{(\tau-t_k)^2} - \underline{\beta}_{2k}\frac{(t-\tau)}{(\tau-t_k)}$$

$$- \underline{\beta}_{3k}(t-\tau) - \underline{\beta}_{4k}(t)\frac{(t-\tau)^3}{(\tau-t_k)^2} \qquad \text{(A11-50)}$$

From (A11-45)

$$\underline{\Phi}^{s}_{1k}(t,\tau) = O\left[(t-\tau)(\tau-t_k),\ \frac{(t-\tau)^4}{(\tau-t_k)^2},\ \mu\frac{(t-\tau)^3}{(\tau-t_k)}\right] \qquad \text{(A11-51)}$$

The singular part of (A6-11) is

$$\int_{t_o}^{t} B(t,\tau)\,\underline{F}_{1s}(\tau)\,d\tau = \int_{t_o}^{t}\left[\underline{\beta}_{1k}\frac{(t-\tau)}{(\tau-t_k)^2} + \underline{\beta}_{2k}\frac{(t-\tau)}{(\tau-t_k)} + \underline{\beta}_{3k}(t-\tau)\right.$$

$$\left. + \beta_{4k}(t)\frac{(t-\tau)^3}{(\tau-t_k)^2} + \underline{\Phi}^{s}_{1k}(t,\tau)\right]d\tau \qquad \text{(A11-52)}$$

At this point (A11-52) is exact since the last term on the right integrates the error in the expansion (A11-45). The first four terms on the right integrate to

$$\int_{t_o}^{t}\frac{(t-\tau)}{(\tau-t_k)^2}\,d\tau = -\log Q_k(t-t_k) + \log Q_k(t_o-t_k)$$

$$-1 + (t-t_k)/(t_o-t_k) \qquad \text{(A11-53)}$$

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)} d\tau = (t_o - t_k) + (t - t_k) \log Q_k(t - t_k)$$

$$- \left[ 1 + \log Q_k(t_o - t_k) \right] (t_o - t_k) \tag{A11-54}$$

$$\int_{t_o}^{t} (t - \tau) \, d\tau = (t_o - t_k)^2 / 2 - (t_o - t_k)(t - t_k)$$

$$+ (t - t_k)^2 / 2 \tag{A11-55}$$

$$\int_{t_o}^{t} \frac{(t - \tau)^3}{(\tau - t_k)^2} d\tau = (t_o - t_k)^2 / 2 - 3(t_o - t_k)(t - t_k)$$

$$- 3(t - t_k)^2 \log Q_k(t - t_k)$$

$$+ 3 \left[ 1/2 + \log Q_k(t_o - t_k) \right] (t - t_k)^2$$

$$+ (t - t_k)^3 / (t_o - t_k) \tag{A11-56}$$

Note that (A11-53) contains a logarithmic singularity as $t \rightarrow t_k$. This is the singular behavior of (A6-11). On the other hand, the first two terms in (A11-50) cancel out the singular behavior of $B(t, \tau) \, \underline{F}_{1s} (\tau)$ and $\underline{\Phi}^s_{1k}$ is finite as t  $t_k$. Therefore the last term in (A11-52) can be expanded about $t = t_k$.

The form of the expansion is

$$\int_{t_o}^{t} \underline{\Phi}_{1k}^{s}(t,\tau)d\tau = \left[\int_{t_o}^{t} \underline{\Phi}_{1k}^{s}(t,\tau)d\tau\right]_{t=t_k}$$

$$+ \left[\frac{d}{dt}\int_{t_o}^{t} \underline{\Phi}_{1k}^{s}(t,\tau)d\tau\right]_{t=t_k} (t - t_k)$$

$$+ \left[\frac{d^2}{dt^2}\int_{t_o}^{t} \Phi_{1k}^{s}(t,\tau)d\tau\right]_{t=t_k} (t - t_k)^2/2$$

$$+ O\left((t - t_k)^3\right) \qquad \text{(A11-57)}$$

The individual integrals in (A11-57) can be evaluated as follows:

$$\left[\int_{t_o}^{t} \underline{\Phi}_{1k}^{s}(t,\tau)d\tau\right]_{t=t_k} = \int_{t_o}^{t_k} \underline{\Phi}_{1k}^{s}(t_k, \tau)d\tau$$

$$= \int_{t_o}^{t_k} \left[B(t_k, \tau)\, \underline{F}_{1s}(\tau) + \underline{\beta}_{1k} \frac{1}{(\tau - t_k)}\right] d\tau$$

$$+ \underline{\beta}_{2k}\int_{t_o}^{t_k} d\tau + \underline{\beta}_{3k} + \int_{t_o}^{t_k} (\tau - t_k)d\tau$$

$$+ \beta_{4k}(t_k) \int_{t_o}^{t_k} (\tau - t_k) d\tau$$

$$= \int_{t_o}^{t_k} \underline{\Phi}^s_{10k}(t_k, \tau) d\tau - \underline{\beta}_{2k}(t_o - t_k)$$

$$- \left[\underline{\beta}_{3k} + \underline{\beta}_{4k}(t_k)\right] (t_o - t_k)^2 / 2 \qquad \text{(A11-58)}$$

where

$$\underline{\Phi}^s_{10k}(t_k, \tau) = B(t_k, \tau) \underline{F}_{1s}(\tau) + \underline{\beta}_{1k} / (\tau - t_k) \qquad \text{(A11-59)}$$

$$\left[\frac{d}{dt} \int_{t_o}^{t} \underline{\Phi}^s_{1k}(t, \tau) d\tau\right]_{t=t_k} = \left[\int_{t_o}^{t} \frac{d}{dt} \underline{\Phi}^s_{1k}(t, \tau) d\tau + \underline{\Phi}^s_{1k}(t, t)\right]_{t=t_k}$$

$$= \int_{t_o}^{t} \frac{d}{dt} \underline{\Phi}^s_{1k}(t, \tau) d\tau \qquad \text{(A11-60)}$$

since, by (A11-51)

$$\underline{\Phi}^s_{1k}(t, t) = O \qquad \text{(A11-61)}$$

Using (A3-16) and (A11-50)

$$\frac{d\underline{\Phi}^s_{1k}}{dt} = D(t, \tau) \underline{F}_{1s}(\tau) - \frac{\underline{\beta}_{1k}}{(\tau - t_k)^2} - \frac{\underline{\beta}_{2k}}{(\tau - t_k)} - \underline{\beta}_{3k}$$

$$-3\underline{\beta}_{4k}(t) \frac{(t - \tau)^2}{(\tau - t_k)^2} - \frac{d\beta_{4k}(t)}{dt} \frac{(t - \tau)^3}{(\tau - t_k)^2} \qquad \text{(A11-62)}$$

$$\int_{t_o}^{t_k} \frac{d}{dt}\underline{\Phi}^s_{1k}(t_k, \tau)d\tau = \int_{t_o}^{t_k} D(t_k, \tau)\, \underline{F}_{1s}(\tau) - \left[\frac{\underline{\beta}_{1k}}{(\tau - t_k)^2} - \frac{\underline{\beta}_{2k}}{(\tau - t_k)^2}\right] d\tau$$

$$- \int_{t_o}^{t_k} \left[\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k) - \frac{d}{dt}\underline{\beta}_{4k}(t_k)(\tau - t_k)\right] d\tau$$

$$= \int_{t_o}^{t_k} \underline{\Phi}^s_{11k}(t_k, \tau)d\tau + \left[\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k)\right](t_o - t_k)$$

$$- \frac{d}{dt}\underline{\beta}_{4k}(t_k)(t_o - t_k)^2/2 \qquad \text{(A11-63)}$$

where

$$\underline{\Phi}^s_{11k}(t_k, \tau) = D(t_k, \tau)\, \underline{F}_{1s}(\tau) - \underline{\beta}_{1k}/(\tau - t_k)^2 - \underline{\beta}_{2k}/(\tau - t_k) \qquad \text{(A11-64)}$$

Finally

$$\left[\frac{d^2}{dt^2}\int_{t_o}^{t} \underline{\Phi}^s_{1k}(t, \tau)d\tau\right]_{t=t_k} = \left[\int_{t_o}^{t} \frac{d^2}{dt^2}\underline{\Phi}^s_{1k}(t, \tau)d\tau + \frac{d}{dt}\underline{\Phi}^s_{1k}(t, t)\right]_{t=t_k}$$

$$= \int_{t_o}^{t_k} \frac{d^2}{dt^2}\underline{\Phi}^s_{1k}(t_k, \tau)d\tau + O(\mu) \qquad \text{(A11-65)}$$

since, by (A11-51)

$$\frac{d}{dt}\underline{\Phi}^s_{1k}(t,\tau) = O\left[(\tau - t_k),\ \frac{(t-\tau)^3}{(\tau - t_k)^2},\ \mu\,\frac{(t-\tau)^2}{(\tau - t_k)}\right]$$

$$\frac{d}{dt}\underline{\Phi}^s_{1k}(t,\tau) = O\left[(t - t_k),\ (t - t_k),\ \mu\,(t - t_k)\right]$$

$$\frac{d}{dt}\underline{\Phi}^s_{1k}(t_k, t_k) = o\,(t - t_k) \tag{A11-66}$$

Using (A3-18) and (A11-62)

$$\frac{d^2\underline{\Phi}^s_{1k}}{dt^2} = G(t)\,B(t,\ \tau)\,\underline{F}_{1s}(\tau) - 6\underline{\beta}_{4k}\,\frac{(t-\tau)}{(\tau - t_k)^2}$$

$$-6\,\frac{d\underline{\beta}_{4k}}{dt}\,\frac{(t-\tau)^2}{(\tau - t_k)^2} - \frac{d^2\underline{\beta}_{4k}}{dt^2}\,\frac{(t-\tau)^3}{(\tau - t_k)^2} \tag{A11-67}$$

$$\int_{t_o}^{t_k} \frac{d^2}{dt^2}\underline{\Phi}^s_{1k}(t_k,\tau)\,d\tau = \int_{t_o}^{t_k}\left[G(t_k)\,B\,(t_k,\ \tau)\,\underline{F}_{1s}(\tau) + 6\,\frac{\underline{\beta}_{4k}(t_k)}{(\tau - t_k)}\right]d\tau$$

$$-\int_{t_o}^{t_k}\left[6\,\frac{d}{dt}\,\underline{\beta}_{4k}\,(t_k) - \frac{d^2}{dt^2}\,\underline{\beta}_{4k}(t_k)\,(\tau - t_k)\right]d\tau$$

$$= \int_{t_o}^{t_k} \underline{\Phi}^s_{12k}\,(t_k,\ \tau)\,d\ + 6\,\frac{d}{dt}\,\underline{\beta}_{4k}(t_k)\,(t_o - t_k)$$

$$-\frac{d^2}{dt^2}\,\underline{\beta}_{4k}(t_k)\,(t_o - t_k)^2/2 \tag{A11-68}$$

where

$$\underline{\Phi}^{s}_{12k}(t_k, \tau) = G(t_k)\, B(t_k, \tau)\, \underline{F}_{1s}(\tau) + 6\underline{\beta}_{4k}(t_k)/(\tau - t_k) \tag{A11-69}$$

Using (A11-49) and then (A11-44) and (A11-25)

$$\underline{\Phi}^{s}_{12k}(t_k, \tau) = G_k\, \underline{\Phi}^{s}_{10k}(t_k, \tau) \tag{A11-70}$$

Substituting the results of (A11-58) through (A11-70) into (A11-57) gives

$$\begin{aligned}
\int_{t_o}^{t} \underline{\Phi}^{s}_{1k}(t, \tau)\, d\tau = & \int_{t_o}^{t_k} \underline{\Phi}^{s}_{10k}(t_k, \tau)\, d\tau - \underline{\beta}_{2k}(t_o - t_k) - \Big[\underline{\beta}_{3k} \\
& + \beta_{4k}(t_k)\Big](t_o - t_k)^2/2 + \left\{\int_{t_o}^{t_k} \underline{\Phi}^{s}_{11k}(t_k, \tau)\, d\tau \right. \\
& + \Big[\underline{\beta}_{3k} + 3\beta_{4k}(t_k)\Big](t_o - t_k) \\
& \left. - \frac{d}{dt}\underline{\beta}_{4k}(t_k)(t_o - t_k)^2/2 \right\}(t - t_k) \\
& + \left\{ G_k \int_{t_o}^{t_k} \underline{\Phi}^{s}_{10k}(t_k, \tau)\, d\tau + 6\frac{d}{dt}\underline{\beta}_{4k}(t_k)(t_o - t_k) \right. \\
& \left. - \frac{d^2}{dt^2}\underline{\beta}_{4k}(t_k)(t_o - t_k)^2/2 \right\}(t - t_k)^2/2 \\
& + O\Big((t - t_k)^3\Big)
\end{aligned} \tag{A11-71}$$

The derivatives of $\underline{\beta}_{4k}(t)$ are simply coefficients in the expansion

$$\underline{\beta}_{4k}(t) = \underline{\beta}_{4k}(t_k) + \frac{d}{dt}\underline{\beta}_{4k}(t_k)(t - t_k) + \frac{d^2}{dt^2}\underline{\beta}_{4k}(t_k)(t - t_k)^2/2$$

$$+ O\left((t - t_k)^3\right) \tag{A11-72}$$

Substituting (A11-53), (A11-54), (A11-55), (A11-56), (A11-71) and (A11-72) into (A11-52) and collecting terms gives

$$\int_{t_o}^{t} B(t, \tau)\underline{F}_{1s}(\tau)\, d\tau = -\underline{\beta}_{1k}\log Q_k(t - t_k) + \underline{\beta}_{1k}\left[\log Q_k(t_o - t_k) - 1\right]$$

$$+ \int_{t_o}^{t_k} \underline{\Phi}^{s}_{10k}(t_k, \tau)\, d\tau + \underline{\beta}_{2k}(t - t_k)\log Q_k(t - t_k)$$

$$+ \left\{\underline{\beta}_{1k}/(t_o - t_k) - \underline{\beta}_{2k}\left[\log Q_k(t_o - t_k) + 1\right]\right.$$

$$\left. + \int_{t_o}^{t_k} \underline{\Phi}^{s}_{11k}(t_k, \tau)\, d\tau\right\}(t - t_k)$$

$$- 3\,\underline{\beta}_{4k}(t_k)(t - t_k)^2 \log Q_k(t - t_k)$$

$$+ \left\{\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k) + 6\underline{\beta}_{4k}(t_k)\log Q_k(t_o - t_k)\right.$$

$$\left. + G_k \int_{t_o}^{t_k} \underline{\Phi}^{s}_{10k}(t_k, \tau)\, d\tau\right\}(t - t_k)^2/2$$

$$+ O\left((t - t_k)^3\right) \tag{A11-73}$$

(A11-73) is an expansion of the singular part of (A6-11). Expansions of the non-singular terms in (A6-11) can be obtained using some of the expressions found in Section A3 and Taylor series formulas. The results are

$$A(t, t_o) = A(t_k, t_o) + C(t_k, t_o)(t - t_k) + G_k A(t_k, t_o)(t - t_k)^2/2 + O\left((t - t_k)^3\right) \tag{A11-74}$$

$$B(t, t_o) = B(t_k, t_o) + D(t_k, t_o)(t - t_k) + G_k B(t_k, t_o)(t - t_k)^2/2 + O\left((t - t_k)^3\right) \tag{A11-75}$$

$$\begin{aligned}\int_{t_o}^{t} B(t, \tau)\, \underline{F}_{1n}(\tau)d\tau = & \int_{t_o}^{t} B(t_k, \tau)\, \underline{F}_{1n}(\tau)d\tau \\ & + \left[\int_{t_o}^{t_k} D(t_k, \tau)\, \underline{F}_{1n}(\tau)d\tau\right](t - t_k) \\ & + \left[G_k \int_{t_o}^{t_k} B(t_k, \tau)\, \underline{F}_{1n}(\tau)d\tau \right. \\ & \left. + \underline{F}_{1n}(t_k)\right](t - t_k)^2/2 + O\left((t - t_k)^3\right)\end{aligned} \tag{A11-76}$$

where

$$\underline{F}_{1n}(t_k) = \underline{p}_k^* \tag{A11-77}$$

Substituting (A11-73) through (A11-76) into (A6-11) gives

$$\underline{r}_1(t) = \underline{a}_{1k} \log Q_k (t - t_k) + \underline{b}_{1k} + \mu \underline{c}_{1k} (t - t_k) \log Q_k (t - t_k)$$

$$+ \underline{d}_{1k} (t - t_k) + \mu \underline{e}_{1k} (t - t_k) + \underline{f}_{1k} (t - t_k)^2 \log Q_k (t - t_k)$$

$$+ \underline{g}_{1k} (t - t_k)^2 + O\left((t - t_k)^3\right) \qquad \text{(A11-78)}$$

where

$$\underline{a}_{1k} = - \underline{\beta}_{1k} \qquad \text{(A11-79)}$$

$$\underline{b}_{1k} = \underline{\beta}_{1k} \left[\log Q_k (t_o - t_k) - 1\right] + A (t_k, t_o) \underline{r}_1 (t_o)$$

$$+ B (t_k, t_o) \underline{v}_1 (t_o) + \underline{K}_{10k} (t_k, t_o) \qquad \text{(A11-80)}$$

$$\mu \underline{c}_{1k} = \underline{\beta}_{2k} \qquad \text{(A11-81)}$$

$$\underline{d}_{1k} = \underline{\beta}_{1k}/(t_o - t_k) + C (t_k, t_o) \underline{r}_1 (t_o) + D (t_k, t_o) \underline{v}_1 (t_o)$$

$$+ \underline{K}_{11k} (t_k, t_o) \qquad \text{(A11-82)}$$

$$\mu \underline{e}_{1k} = - \underline{\beta}_{2k} \left[\log Q_k (t_o - t_k) + 1\right] \qquad \text{(A11-83)}$$

$$\underline{f}_{1k} = -3 \underline{\beta}_{4k} (t_k) \qquad \text{(A11-84)}$$

$$\underline{g}_{1k} = \underline{\beta}_{3k}/2 + 3 \underline{\beta}_{4k} (t_k) \left[\log Q_k (t_o - t_k) + 1/2\right] + \underline{p}_k^*/2$$

$$+ G_k \left[A (t_k, t_o) \underline{r}_1 (t_o) + B (t_k, t_o) \underline{v}_1 (t_o) + \underline{K}_{10k} (t_k, t_o)\right]/2 \qquad \text{(A11-85)}$$

and

$$\underline{K}_{10k}(t_k, t_o) = \int_{t_o}^{t_k} \left[\underline{\Phi}^s_{10k}(t_k, \tau) + B(t_k, \tau)\,\underline{F}_{1n}(\tau)\right] d\tau \tag{A11-86}$$

$$\underline{K}_{11k}(t_k, t_o) = \int_{t_o}^{t_k} \left[\underline{\Phi}^s_{11k}(t_k, \tau) + D(t_k, \tau)\,\underline{F}_{1n}(\tau)\right] d\tau \tag{A11-87}$$

(A11-78) gives the behavior of $\underline{r}_1(t)$ in the overlap domain. The logarithmic singularity as $t \to t_k$ produces the non-uniformity in the first order outer solution mentioned at the end of Section A6. It will be shown that a similar singularity occurs in the inner solution. The integrals given by (A11-86) and (A11-87) cannot be evaluated analytically, i.e., expressed in closed form. They must be evaluated numerically and this problem is discussed in Section C.

### A11.3 Second Order

In deriving the first order expansion it was first necessary to derive $\underline{f}(\underline{r}_{ko})$, c.f. (A11-38). According to (A5-10) the second order force function requires an expansion of $G(\underline{r}_{ko})$. From (A2-20) and (A11-27)

$$G(\underline{r}_{ko}) = G(\underline{x}_o) + \underline{H}(\underline{x}_o)\left[\underline{x}_1 + \underline{x}_2 + O(\underline{x}_3)\right]$$
$$+ \underset{\sim}{T}(\underline{x}_o)\left[\underline{x}_1^2 + O(\underline{x}_1\,\underline{x}_2)\right]\Big/2 + O(\underline{x}_1^3) \tag{A11-88}$$

Continuing the sequence (A11-35), (A11-36) and (A11-37) gives

$$\underset{\sim}{T}(\underline{x}_o) = Q_k\,\underset{\sim}{T}(\underline{V}_k)/(t - t_k)^5 \tag{A11-89}$$

Then (A11-88) becomes

$$G(\underline{r}_{ko}) = \frac{Q_k}{(t - t_k)^3} \left[ G(\underline{V}_k) - \frac{\mu}{2} \underline{H}(\underline{V}_k)\, \underline{p}_k^* (t - t_k) \right.$$

$$+ \frac{1}{6} \underline{H}(\underline{V}_k)\,(G_k \underline{V}_k - \mu \underline{p}_k^{**})(t - t_k)^2$$

$$\left. + \frac{\mu^2}{8} \underset{\sim}{T}(\underline{V}_k)\, \underline{p}_k^{*2} (t - t_k)^2 + O\left((t - t_k)^3\right) \right] \qquad (A11\text{-}90)$$

The second order force function can be written in terms of a singular part, $\underline{F}_{2s}$, and a non-singular part, $\underline{F}_{2n}$, i.e.,

$$\underline{F}_2(\underline{r}_o, \underline{r}_1, \underline{p}_i) = \underline{F}_{2s}(\underline{r}_o, \underline{r}_1, \underline{p}_k) + \underline{F}_{2n}(\underline{r}_o, \underline{r}_1, p_i) \qquad (A11\text{-}91)$$

where

$$\underline{F}_{2s}(\underline{r}_o, \underline{r}_1, \underline{p}_k) = M_k\, G(\underline{r}_{ko})\, \underline{r}_1 \qquad (A11\text{-}92)$$

$$\underline{F}_{2n}(\underline{r}_o, \underline{r}_1, p_i) = \frac{1}{2} \underline{H}(\underline{r}_o)\, \underline{r}_1^2 + \sum_{\substack{i=1 \\ i \neq k}}^{N-2} M_i\, G(\underline{r}_o - \underline{p}_i)\, \underline{r}_1 \qquad (A11\text{-}93)$$

Since $\underline{r}_o$ is a non-zero vector, $\underline{H}(\underline{r}_o)$ is non-singular. The dyadic $\underline{r}_1^2$ contributes a factor of $(\log Q_k (t - t_k))^2$ but

$$\int (\log x)^2\, dx = x (\log x)^2 - 2x \log x + 2x \rightarrow O$$

as $x \rightarrow O$. Therefore the vector $\underline{H}(\underline{r}_o)\, \underline{r}_1^2$ is finite as $t \rightarrow t_k$ and is included in $\underline{F}_{2n}$ (also see Carlson,[3] pg. 27).

Multiplying (A11-78) by (A11-90) and substituting into (A11-92) gives

$$
\begin{aligned}
\underline{F}_{2s} = {} & G_k^* \underline{a}_{1k} \frac{\log Q_k(t - t_k)}{(t - t_k)^3} + \frac{G_k^* \underline{b}_{1k}}{(t - t_k)^3} \\
& + \mu \left(G_k^* \underline{c}_{1k} - J_k^* \underline{a}_{1k}\right) \frac{\log Q_k(t - t_k)}{(t - t_k)^2} \\
& + \left[G_k^* \underline{d}_{1k} + \mu \left(G_k^* \underline{e}_{1k} - J_k^* \underline{b}_{1k}\right)\right] \frac{1}{(t - t_k)^2} \\
& + \left[G_k^* \underline{f}_{1k} + H_k^* \underline{a}_{1k} - \mu^2 J_k^* \underline{c}_{1k}\right] \frac{\log Q_k (t - t_k)}{(t - t_k)} \\
& + \left[G_k^* \underline{g}_{1k} + H_k^* \underline{b}_{1k} - \mu J_k^* \underline{d}_{1k} - \mu^2 J_k^* \underline{e}_{1k}\right] \frac{1}{(t - t_k)} \\
& + O(\log |t - t_k|)
\end{aligned}
\tag{A11-94}
$$

where

$$G_k^* = M_k Q_k G(\underline{V}_k) \tag{A11-95}$$

$$H_k^* = \frac{M_k Q_k}{2} \left[\frac{1}{3} \underline{H}(\underline{V}_k) (G_k \underline{V}_k - \mu \underline{p}_k^{**}) + \frac{\mu^2}{4} \underset{\sim}{T}(\underline{V}_k) \underline{p}_k^{*2}\right] \tag{A11-96}$$

$$J_k^* = \frac{M_k Q_k}{2} \underline{H}(\underline{V}_k) \underline{p}_k^* \tag{A11-97}$$

According to (A6-12) the second order solution is a function of (B(t, $\tau$) $\underline{F}_2(\tau)$).

From (A11-43) and (A11-94)

$$B(t, \tau)\, \underline{F}_{2s}(\tau) = \underline{\psi}_{1k} \frac{(t - \tau)}{(\tau - t_k)^3} \log Q_k(\tau - t_k) + \underline{\psi}_{2k} \frac{(t - \tau)}{(\tau - t_k)^3}$$

$$+ \underline{\psi}_{3k} \frac{(t - \tau)}{(\tau - t_k)^2} \log Q_k(\tau - t_k) + \underline{\psi}_{4k} \frac{(t - \tau)}{(\tau - t_k)^2}$$

$$+ \underline{\psi}_{5k} \frac{(t - \tau)}{(\tau - t_k)} \log Q_k(\tau - t_k) + \underline{\psi}_{6k} \frac{(t - \tau)}{(\tau - t_k)}$$

$$+ \underline{\psi}_{7k} \frac{(t - \tau)^3}{(\tau - t_k)^3} \log Q_k(t - t_k) + \underline{\psi}_{8k} \frac{(t - \tau)^3}{(\tau - t_k)^3}$$

$$+ O\left[(t - \tau) \log |\tau - t_k|, \frac{(t - \tau)^3}{(\tau - t_k)^2} \log |\tau - t_k|, \right.$$

$$\left. \frac{(t - \tau)^4}{(\tau - t_k)^3} \log |\tau - t_k| \right] \qquad \text{(A11-98)}$$

where

$$\underline{\psi}_{1k} = G_k^* \underline{a}_{1k} \qquad \text{(A11-99)}$$

$$\underline{\psi}_{2k} = G_k^* \underline{b}_{1k} \qquad \text{(A11-100)}$$

$$\underline{\psi}_{3k} = \mu\, (G_k^* \underline{c}_{1k} - J_k^* \underline{a}_{1k}) \qquad \text{(A11-101)}$$

$$\underline{\psi}_{4k} = G_k^* \underline{d}_{1k} + \mu\, (G_k^* \underline{e}_{1k} - J_k^* \underline{b}_{1k}) \qquad \text{(A11-102)}$$

$$\underline{\psi}_{5k} = G_k^* \underline{f}_{1k} + H_k^* \underline{a}_{1k} - \mu^2\, J_k^* \underline{c}_{1k}) \qquad \text{(A11-103)}$$

$$\underline{\psi}_{6k} = G_k^* \underline{g}_{1k} + H_k^* \underline{b}_{1k} - \mu J_k^* \underline{d}_{1k} - \mu^2 J_k^* \underline{e}_{1k} \tag{A11-104}$$

$$\underline{\psi}_{7k} = \underline{\psi}_{7k}(t)$$

$$= G(t)\, G_k^* \underline{a}_{1k}/6 \tag{A11-105}$$

$$\underline{\psi}_{8k} = \underline{\psi}_{8k}(t)$$

$$= G(t)\, G_k^* \underline{b}_{1k}/6 \tag{A11-106}$$

(A11-98) represents the singular behavior of the integrand in (A6-12). It is not necessary to develop a corresponding expansion of the non-singular part of the integrand in (A6-12), i.e., $B(t, \tau)\, \underline{F}_{2n}(\tau)$, as this contribution can be evaluated directly.

Now let

$$\begin{aligned}\underline{\Phi}_{2k}^{s}(t, \tau) = {} & B(t, \tau)\, \underline{F}_{2s}(\tau) - \underline{\psi}_{1k} \frac{(t-\tau)}{(\tau - t_k)^3} \log Q_k (\tau - t_k) \\ & - \underline{\psi}_{2k} \frac{(t-\tau)}{(\tau - t_k)^3} - \underline{\psi}_{3k} \frac{(t-\tau)}{(\tau - t_k)^2} \log Q_k (\tau - t_k) \\ & - \underline{\psi}_{4k} \frac{(t-\tau)}{(\tau - t_k)^2} - \underline{\psi}_{5k} \frac{(t-\tau)}{(\tau - t_k)} \log Q_k (\tau - t_k) \\ & - \underline{\psi}_{6k} \frac{(t-\tau)}{(\tau - t_k)} - \underline{\psi}_{7k}(t) \frac{(t-\tau)^3}{(\tau - t_k)^3} \log Q_k (\tau - t_k) \\ & - \underline{\psi}_{8k}(t) \frac{(t-\tau)^3}{(\tau - t_k)^3}\end{aligned} \tag{A11-107}$$

From (A11-98)

$$\underline{\Phi}^{s}_{2k}(t,\ \tau) = O\left[(t-\tau)\log\left|\tau - t_k\right|,\ \frac{(t-\tau)^3}{(\tau - t_k)^2}\log\left|\tau - t_k\right|,\right.$$

$$\left.\frac{(t-\tau)^4}{(\tau - t_k)^3}\log\left|\tau - t_k\right|\right] \qquad \text{(A11-108)}$$

The exact expression for the singular part of (A6-12) can be written

$$\int_{t_o}^{t} B(t,\ \tau)\ \underline{F}_{2s}(\tau)d\tau = \int_{t_o}^{t}\left[\underline{\psi}_{1k}\frac{(t-\tau)}{(\tau - t_k)^3}\log Q_k(\tau - t_k) + \underline{\psi}_{2k}\frac{(t-\tau)}{(\tau - t_k)^3}\right.$$

$$+ \underline{\psi}_{3k}\frac{(t-\tau)}{(\tau - t_k)^2}\log Q_k(\tau - t_k) + \underline{\psi}_{4k}\frac{(t-\tau)}{(\tau - t_k)^2}$$

$$+ \underline{\psi}_{5k}\frac{(t-\tau)}{(\tau - t_k)}\log Q_k(\tau - t_k) + \underline{\psi}_{6k}\frac{(t-\tau)}{(\tau - t_k)}$$

$$+ \underline{\psi}_{7k}(t)\frac{(t-\tau)^3}{(\tau - t_k)^3}\log Q_k(\tau - t_k)$$

$$\left. + \underline{\psi}_{8k}(t)\frac{(t-\tau)^3}{(\tau - t_k)^3}\right]d\tau$$

$$+ \int_{t_o}^{t}\underline{\Phi}^{s}_{2k}(t,\ \tau)\ d\tau \qquad \text{(A11-109)}$$

The first eight terms on the right of (A11-109) integrate to

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)^3} \log Q_k (\tau - t_k)\, d\tau = \frac{\left[\log Q_k (t - t_k)\right]}{2(t - t_k)} + \frac{3}{4(t - t_k)}$$

$$- \frac{\left[\log Q_k (t_o - t_k) + 1\right]}{(t_o - t_k)}$$

$$+ \frac{\left[2 \log Q_k (t_o - t_k) + 1\right]}{4(t_o - t_k)^2} (t - t_k) \qquad \text{(A11-110)}$$

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)^3}\, d\tau = \frac{1}{2(t - t_k)} - \frac{1}{(t_o - t_k)} + \frac{(t - t_k)}{2(t_o - t_k)^2} \qquad \text{(A11-111)}$$

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)^2} \log Q_k (\tau - t_k)\, d\tau = - \frac{1}{2} \log^2 Q_k (t - t_k) - \log Q_k (t - t_k)$$

$$+ \frac{1}{2} \log^2 Q_k (t_o - t_k)$$

$$- 1 + \frac{\left[\log Q_k (t_o - t_k) + 1\right]}{(t_o - t_k)} (t - t_k) \qquad \text{(A11-112)}$$

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)^2}\, d\tau = - \log Q_k (t - t_k) + \log Q_k (t_o - t_k)$$

$$- 1 + \frac{(t - t_k)}{(t_o - t_k)} \qquad \text{(A11-113)}$$

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)} \log Q_k (\tau - t_k)\, d\tau = (t_o - t_k) \left[\log Q_k (t_o - t_k) - 1\right]$$

$$+ \frac{1}{2} (t - t_k) \log^2 Q_k (t - t_k)$$

$$- (t - t_k) \log Q_k (t - t_k)$$

$$- \left[\frac{1}{2} \log^2 Q_k (t_o - t_k) - 1\right] (t - t_k) \tag{A11-114}$$

$$\int_{t_o}^{t} \frac{(t - \tau)}{(\tau - t_k)}\, d\tau = (t_o - t_k) + (t - t_k) \log Q_k (t - t_k)$$

$$- \left[\log Q_k (t_o - t_k) + 1\right] (t - t_k) \tag{A11-115}$$

$$\int_{t_o}^{t} \frac{(t - \tau)^3}{(\tau - t_k)^3} \log Q_k (\tau - t_k)\, d\tau = (t_o - t_k) \left[\log Q_k (t_o - t_k) - 1\right]$$

$$+ \frac{3}{2} (t - t_k) \log^2 Q_k (t - t_k)$$

$$+ \frac{3}{2} (t - t_k) \log Q_k (t - t_k)$$

$$- \left[\frac{3}{2} \log^2 Q_k (t_o - t_k) - \frac{15}{4}\right] (t - t_k)$$

$$+ O\left((t - t_k)^2\right) \tag{A11-116}$$

$$\int_{t_o}^{t} \frac{(t-\tau)^3}{(\tau - t_k)^3} d\tau = (t_o - t_k) + 3(t - t_k) \log Q_k (t - t_k)$$

$$- \left[3 \log Q_k (t_o - t_k) - 1\right](t - t_k) + O\left((t - t_k)^2\right) \tag{A11-117}$$

Note that (A11-110), (A11-111), (A11-112) and (A11-113) contain singularities as $t \quad t_k$. This is the singular behavior of (A6-12). On the other hand, the first five terms on the right side of (A11-107) cancel out the singular behavior of $B(t, \tau)\, \underline{F}_{2s}(\tau)$ and $\underline{\Phi}_{2k}^{s}$ is finite as $t \to t_k$. Therefore the last term in (A11-109) can be expanded about $t = t_k$. The form of the expansion is

$$\int_{t_o}^{t} \underline{\Phi}_{2k}^{s}(t, \tau) d\tau = \left[\int_{t_o}^{t} \underline{\Phi}_{2k}^{s}(t, \tau) d\tau\right]_{t=t_k}$$

$$+ \left[\frac{d}{dt} \int_{t_o}^{t} \underline{\Phi}_{2k}^{s}(t, \tau) d\tau\right]_{t=t_k} (t - t_k)$$

$$+ O\left((t - t_k)^2\right) \tag{A11-118}$$

The individual integrals in (A11-118) can be evaluated as follows:

$$\left[\int_{t_o}^{t} \underline{\Phi}_{2k}^{s}(t, \tau) d\tau\right]_{t=t_k} = \int_{t_o}^{t_k} \underline{\Phi}_{2k}^{s}(t_k, \tau) d\tau$$

$$= \int_{t_o}^{t_k} \left[B(t_k, \tau)\, \underline{F}_{2s}(\tau) + \underline{\psi}_{1k} \frac{\log Q_k (\tau - t_k)}{(\tau - t_k)^2}\right.$$

$$+ \underline{\psi}_{2k} \frac{1}{(\tau - t_k)^2} + \underline{\psi}_{3k} \frac{\log Q_k (\tau - t_k)}{(\tau - t_k)}$$

$$\left. + \underline{\psi}_{4k} \frac{1}{(\tau - t_k)} \right] d\tau$$

$$+ \left[ \underline{\psi}_{5k} + \underline{\psi}_{7k} (t_k) \right] \int_{t_o}^{t_k} \log Q_k (\tau - t_k) \, d\tau$$

$$+ \left[ \underline{\psi}_{6k} + \underline{\psi}_{8k} (t_k) \right] \int_{t_o}^{t_k} d\tau$$

$$= \int_{t_o}^{t_k} \underline{\Phi}^s_{20k} (t_k, \tau) \, d\tau - \left[ \underline{\psi}_{5k} + \underline{\psi}_{7k} (t_k) \right] (t_o - t_k) \cdot$$

$$\left[ \log Q_k (t_o - t_k) - 1 \right] - \left[ \underline{\psi}_{6k} + \underline{\psi}_{8k} (t_k) \right] (t_o - t_k) \tag{A11-119}$$

where

$$\underline{\Phi}^s_{20k} (t_k, \tau) = B(t_k, \tau) \, \underline{F}_{2s}(\tau) + \underline{\psi}_{1k} \frac{\log Q_k (\tau - t_k)}{(\tau - t_k)^2}$$

$$+ \underline{\psi}_{2k} \frac{1}{(\tau - t_k)^2} + \underline{\psi}_{3k} \frac{\log Q_k (\tau - t_k)}{(\tau - t_k)} + \underline{\psi}_{4k} \frac{1}{(\tau - t_k)} \tag{A11-120}$$

$$\left[\frac{d}{dt}\int_{t_o}^{t} \underline{\Phi}_{2k}^{s}(t,\ \tau)\, d\tau\right]_{t=t_k} = \left[\int_{t_o}^{t} \frac{d}{dt}\underline{\Phi}_{2k}^{s}(t,\ \tau)\, d\tau + \underline{\Phi}_{2k}^{s}(t,\ t)\right]_{t=t_k}$$

$$= \int_{t_o}^{t_k} \frac{d}{dt}\underline{\Phi}_{2k}^{s}(t,\ \tau)\, d\tau \qquad \text{(A11-121)}$$

since, by (A11-108)

$$\underline{\Phi}_{2k}^{s}(t,\ t) = O \qquad \text{(A11-122)}$$

Using (A3-16) and (A11-107)

$$\frac{d}{dt}\underline{\Phi}_{2k}^{s}(t,\tau) = D(t,\tau)\,\underline{F}_{2s}(\tau) - \underline{\psi}_{1k}\frac{\log Q_k(\tau - t_k)}{(\tau - t_k)^3}$$

$$- \underline{\psi}_{2k}\frac{1}{(\tau - t_k)^3} - \underline{\psi}_{3k}\frac{\log Q_k(\tau - t_k)}{(\tau - t_k)^2}$$

$$- \underline{\psi}_{4k}\frac{1}{(\tau - t_k)^2} - \underline{\psi}_{5k}\frac{\log Q_k(\tau - t_k)}{(\tau - t_k)}$$

$$- \underline{\psi}_{6k}\frac{1}{(\tau - t_k)} - 3\underline{\psi}_{7k}(t)\frac{(t - \tau)^2}{(\tau - t_k)^3}\log Q_k(\tau - t_k)$$

$$- 3\underline{\psi}_{8k}(t)\frac{(t - \tau)^2}{(\tau - t_k)^3} - \frac{d}{dt}\underline{\psi}_{7k}(t)\frac{(t - \tau)^3}{(\tau - t_k)^3}\log Q_k(\tau - t_k)$$

$$- \frac{d}{dt}\underline{\psi}_{8k}(t)\frac{(t - \tau)^3}{(\tau - t_k)^3} \qquad \text{(A11-123)}$$

$$\int_{t_o}^{t_k} \frac{d}{dt} \underline{\Phi}_{2k}^{s}(t_k, \tau)\, d\tau = \int_{t_o}^{t_k} \underline{\Phi}_{21k}^{s}(t_k, \tau)\, d\tau - \frac{d}{dt}\underline{\psi}_{7k}(t_k)(t_o - t_k) \cdot$$

$$\left[\log Q_k (t_o - t_k) - 1\right] - \frac{d}{dt}\underline{\psi}_{8k}(t_k)(t_o - t_k)$$

where (A11-124)

$$\underline{\Phi}_{21k}^{s}(t_k, \tau) = \frac{d}{dt}\underline{\Phi}_{2k}^{s}(t_k, \tau) - \frac{d}{dt}\underline{\psi}_{7k}(t_k) \log Q_k (\tau - t_k)$$

$$- \frac{d}{dt}\underline{\psi}_{8k}(t_k)$$

(A11-125)

Substituting (A11-119) and (A11-124) into (A11-118) gives

$$\int_{t_o}^{t} \underline{\Phi}_{2k}^{s}(t, \tau)\, d\tau = \int_{t_o}^{t_k} \underline{\Phi}_{20k}^{s}(t_k, \tau)\, d\tau - \left[\underline{\psi}_{5k} + \underline{\psi}_{7k}(t_k)\right](t_o - t_k) \cdot$$

$$\left[\log Q_k (t_o - t_k) - 1\right] - \left[\underline{\psi}_{6k} + \underline{\psi}_{8k}(t_k)\right](t_o - t_k)$$

$$+ \left\{ \int_{t_o}^{t_k} \underline{\Phi}_{21k}^{s}(t_k, \tau)\, d\tau - \frac{d}{dt}\underline{\psi}_{7k}(t_k)(t_o - t_k) \cdot \right.$$

$$\left. \left[\log Q_k (t_o - t_k) - 1\right] - \frac{d}{dt}\underline{\psi}_{8k}(t_k)(t_o - t_k) \right\} (t - t_k)$$

$$+ \left((t - t_k)^2\right)$$

(A11-126)

The derivatives of $\underline{\psi}_{7k}(t)$ and $\underline{\psi}_{8k}(t)$ are coefficients in the expansions

$$\underline{\psi}_{7k}(t) = \underline{\psi}_{7k}(t_k) + \frac{d}{dt}\underline{\psi}_{7k}(t_k)(t - t_k) + O\left((t - t_k)^2\right) \tag{A11-127}$$

$$\underline{\psi}_{8k}(t) = \underline{\psi}_{8k}(t_k) + \frac{d}{dt}\underline{\psi}_{8k}(t_k)(t - t_k) + O\left((t - t_k)^2\right) \tag{A11-128}$$

Substituting (A11-110) through (A11-117) and (A11-126) through (A11-128) into (A11-109) and collecting terms gives

$$\begin{aligned}
\int_{t_o}^{t} B(t, \tau)\, \underline{F}_{2s}(t)\, d\tau = {} & \frac{\underline{\psi}_{1k}}{2} \frac{\log Q_k (t - t_k)}{(t - t_k)} + \frac{1}{2}\left(\frac{3\underline{\psi}_{1k}}{2} + \underline{\psi}_{2k}\right)\frac{1}{(t - t_k)} \\
& - \frac{\underline{\psi}_{3k}}{2} \log^2 Q_k (t - t_k) - (\underline{\psi}_{3k} + \underline{\psi}_{4k}) \log Q_k (t - t_k) \\
& - \underline{\psi}_{1k} \frac{\left[\log Q_k (t_o - t_k) + 1\right]}{(t_o - t_k)} - \frac{\underline{\psi}_{2k}}{(t_o - t_k)} \\
& + \underline{\psi}_{3k}\left[\frac{1}{2}\log^2 Q_k (t_o - t_k) - 1\right] \\
& + \underline{\psi}_{4k}\left[\log Q_k (t_o - t_k) - 1\right] + \int_{t_o}^{t_k} \underline{\Phi}^{s}_{20k}(t_k, \tau)\, d\tau \\
& + \frac{1}{2}\left[\underline{\psi}_{5k} + 3\underline{\psi}_{7k}(t_k)\right](t - t_k) \log^2 Q_k (t - t_k) \\
& - \left[\underline{\psi}_{5k} - \underline{\psi}_{6k} - \frac{3}{2}\underline{\psi}_{7k}(t_k) - 3\underline{\psi}_{8k}(t_k)\right] \cdot \\
& (t - t_k) \log Q_k (t - t_k)
\end{aligned}$$

$$+\left\{ \underline{\psi}_{1k} \frac{\left[2 \log Q_k (t_o - t_k) + 1\right]}{4(t_o - t_k)^2} + \frac{\underline{\psi}_{2k}}{2(t_o - t_k)^2}\right.$$

$$+ \underline{\psi}_{3k} \frac{\left[\log Q_k (t_o - t_k) + 1\right]}{(t_o - t_k)} + \frac{\underline{\psi}_{4k}}{(t_o - t_k)}$$

$$- \underline{\psi}_{5k} \left[\frac{1}{2} \log^2 Q_k (t_o - t_k) - 1\right]$$

$$- \underline{\psi}_{6k} \left[\log Q_k (t_o - t_k) + 1\right]$$

$$- \underline{\psi}_{7k} (t_k) \left[\frac{3}{2} \log^2 Q_k (t_o - t_k) - \frac{15}{4}\right]$$

$$- \underline{\psi}_{8k} (t_k) \left[3 \log Q_k (t_o - t_k) - 1\right]$$

$$\left. + \int_{t_o}^{t_k} \underline{\Phi}^s_{21k} (t_k, \tau) \, d\tau \right\} (t - t_k)$$

$$+ O\left((t - t_k)^2\right) \qquad \text{(A11-129}$$

Apart from being the longest expression derived thus far, (A11-129) represents the singular part of (A6-12). Expansions of the non-singular terms in (A6-12) can be obtained using some of the expressions found in Section A3 and Taylor Series formulas. The results are given by (A11-74) and (A11-75) as well as

$$\int_{t_o}^{t} B(t, \tau) \, \underline{F}_{2n} (\tau) \, d\tau = \int_{t_o}^{t_k} B (t_k, \tau) \, \underline{F}_{2n} (\tau) \, d\tau$$

$$+ \left[\int_{t_o}^{t_k} D(t_k, \tau) \, \underline{F}_{2n}(\tau) \, d\tau\right] (t - t_k) + O\left((t - t_k)^2\right) \qquad \text{(A11-130)}$$

which is just (A11-76) with $\underline{F}_{1n}$ replaced by $\underline{F}_{2n}$.

Substituting (A11-74), (A11-75), (A11-129) and (A11-130) into (A6-12) gives

$$\underline{r}_2(t) = \underline{a}_{2k}(t - t_k)^{-1} \log Q_k (t - t_k) + \underline{b}_{2k}(t - t_k)^{-1}$$

$$+ \underline{c}_{2k} \log Q_k (t - t_k) + \underline{d}_{2k}$$

$$+ \underline{e}_{2k}(t - t_k) \log^2 Q_k (t - t_k)$$

$$+ \underline{f}_{2k}(t - t_k) \log Q_k (t - t_k) + \underline{g}_{2k}(t - t_k)$$

$$+ O\left[(t - t_k)^2, \ \mu \log^2 \left|t - t_k\right|\right] \qquad \text{(A11-131)}$$

where

$$\underline{a}_{2k} = \underline{\psi}_{1k}/2 \qquad \text{(A11-132)}$$

$$\underline{b}_{2k} = (3\underline{\psi}_{1k} + 2\underline{\psi}_{2k})/4 \qquad \text{(A11-133)}$$

$$\underline{c}_{2k} = -\underline{\psi}_{4k} \qquad \text{(A11-134)}$$

$$\underline{d}_{2k} = -\underline{\psi}_{1k}\left[\log Q_k (t_o - t_k) + 1\right](t_o - t_k)^{-1} + \underline{\psi}_{2k}(t_o - t_k)^{-1}$$

$$+ \underline{\psi}_{4k}\left[\log Q_k (t_o - t_k) - 1\right] + \underline{K}_{20k}(t_k, t_o)$$

$$+ A(t_k, t_o)\,\underline{r}_2(t_o) + B(t_k, t_o)\,\underline{v}_2(t_o) \qquad \text{(A11-135)}$$

$$\underline{e}_{2k} = \left[\underline{\psi}_{5k} + 3\underline{\psi}_{7k}(t_k)\right]/2 \qquad \text{(A11-136)}$$

$$\underline{f}_{2k} = -\left[2\underline{\psi}_{5k} - 2\underline{\psi}_{6k} - 3\underline{\psi}_{7k} - 6\underline{\psi}_{8k}(t_k)\right]/2 \tag{A11-137}$$

$$\begin{aligned}
\underline{g}_{2k} = {} & \underline{\psi}_{1k}\left[2 \log Q_k (t_o - t_k) + 1\right](t_o - t_k)^{-2} /4 \\
& + \underline{\psi}_{2k}(t_o - t_k)^{-2} /2 + \underline{\psi}_{4k}'(t_o - t_k)^{-1} \\
& - \underline{\psi}_{5k}\left[\log^2 Q_k (t_o - t_k) - 2\right]/2 \\
& - \underline{\psi}_{6k}\left[\log Q_k (t_o - t_k) + 1\right] \\
& - 3\underline{\psi}_{7k}(t_k)\left[2 \log^2 Q_k (t_o - t_k) - 5\right]/4 \\
& - \underline{\psi}_{8k}(t_k)\left[3 \log Q_k (t_o - t_k) - 1\right] + \underline{K}_{21k}(t_k, t_o) \\
& + C(t_k, t_o)\,\underline{r}_2(t_o) + D(t_k, t_o)\,\underline{v}_2(t_o)
\end{aligned} \tag{A11-138}$$

and

$$\underline{K}_{20k}(t_r, t_o) = \int_{t_o}^{t_k}\left[\underline{\Phi}^s_{20k}(t_k, \tau) + B(t_k, \tau)\,\underline{F}_{2n}(\tau)\right] d\tau \tag{A11-139}$$

$$\underline{K}_{21k}(t_k, t_o) = \int_{t_o}^{t_k}\left[\underline{\Phi}^s_{21k}(t_k, \tau) + D(t_k, \tau)\,\underline{F}_{2n}(\tau)\right] d\tau \tag{A11-140}$$

(A11-131) gives the behavior of $\underline{r}_2(t)$ in the overlap domain. Like the first order solution it contains singularities as $t \to t_k$ which must be cancelled by similar singularities in the inner solution. Also, just as in the first order expansion, the integrals (A11-139) and (A11-140) must be evaluated numerically; a problem which is discussed in Section C.

### A11.4 Third Order

The third order solution is not actually used but knowing its behavior in the overlap domain is important in the matching. From (A5-8)

$$\ddot{\underline{r}}_3 = G(\underline{r}_o)\, \underline{r}_3 + \underline{F}_3\, (\underline{r}_o, \underline{r}_1, \underline{r}_2, \underline{p}_i) \tag{A11-141}$$

with a solution similar to (A6-11) and (A6-12)

$$\underline{r}_3\,(t) = A(t, t_o)\, \underline{r}_3\,(t_o) + B\,(t, t_o)\, \underline{v}_3\,(t_o) + \int_{t_o}^{t} B(t, \tau)\, \underline{F}_3\,(\tau)\, d\tau \tag{A11-142}$$

Carlson[3] has shown that in the overlap domain

$$\underline{r}_3\,(t) = O\left((t - t_k)^{-2} \log^2 |t - t_k|\right) \tag{A11-143}$$

## A12 BEHAVIOR OF THE INNER SOLUTION IN THE OVERLAP DOMAIN

### A12.1 Zeroth Order

According to (A9-1), (A9-3) and (A9-4) the zeroth order inner solution is a function of the eccentric anomaly $F_k$. The behavior of the outer solution in the overlap domain has been developed as a function of time. Therefore it is necessary to find the behavior of the inner solution as a function of time, i.e., find

$$F_k = F_k(S_k) \tag{A12-1}$$

Such a result is found by inverting (A9-2) but, in general, this is not possible due to the transcendental nature of the equation. However, the overlap domain for the inner solution corresponds to the region far out on the asymptotes of the zeroth order hyperbola. This region is characterized by large values of the time $S_k$ and this makes it possible to invert (A9-2).

In order to be completely general it will be assumed that on the approach asymptote of the inner hyperbola $S_k$ and $F_k$ are negative and on the departure asymptote they are positive. This will require two solutions of (A9-2).

Dropping the subscript k for the time being, (A9-2) can be written

$$\bar{n}S = (\bar{e}e^{F} - \bar{e}e^{-F} - 2F)/2 \tag{A12-2}$$

For $F \to \infty$ (A12-2) is rewritten as

$$e^{F} = (2\bar{n}\,S + 2F + \bar{e}e^{-F})/\bar{e} \tag{A12-3}$$

Since F and $e^{-F}$ are much smaller than $e^{F}$ (A12-3) can be approximated by

$$e^{F} = 2\bar{n}\,S/\bar{e} + O(F) \tag{A12-4}$$

and then

$$F = \log\left(\frac{2\bar{n}\,S}{\bar{e}}\right) + O\left(\frac{\log S}{S}\right) \tag{A12-5}$$

$$e^{-F} = O\,(\frac{1}{S}) \tag{A12-6}$$

Substituting (A12-5) and (A12-6) into (A12-3) gives

$$e^{F} = \frac{2\bar{n}S}{\bar{e}} + \frac{2}{\bar{e}} \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + 0\left(\frac{\log S}{S}\right) \qquad \text{(A12-7)}$$

(A12-7) is simply a more accurate form of (A12-4). From (A12-7)

$$F = \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + \frac{1}{\bar{n}S} \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + 0\left(\frac{\log^2 S}{S^2}\right) \qquad \text{(A12-8)}$$

$$e^{-F} = \frac{\bar{e}}{2\bar{n}S} + 0\left(\frac{\log S}{S^2}\right) \qquad \text{(A12-9)}$$

Substituting (A12-8) and (A12-9) back into (A12-3) gives

$$e^{F} = \frac{2\bar{n}S}{\bar{e}} + \frac{2}{\bar{e}} \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + \frac{2}{\bar{n}\bar{e}S} \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + \frac{\bar{e}}{2\bar{n}S} + 0\left(\frac{\log^2 S}{S^2}\right) \qquad \text{(A12-10)}$$

$$F = \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + \frac{1}{\bar{n}S} \log\left(\frac{2\bar{n}S}{\bar{e}}\right) - \frac{1}{4\bar{n}^2S^2}\left[2\log^2\left(\frac{2\bar{n}S}{\bar{e}}\right)\right.$$

$$\left. - 4\log\left(\frac{2\bar{n}S}{\bar{e}}\right) - \bar{e}^2\right] + 0\left(\frac{\log^2 S}{S^3}\right) \qquad \text{(A12-13)}$$

$$e^{-F} = \frac{\bar{e}}{2\bar{n}S} - \frac{\bar{e}}{2\bar{n}^2S^2} \log\left(\frac{2\bar{n}S}{\bar{e}}\right) + 0\left(\frac{\log^2 S}{S^3}\right) \qquad \text{(A12-14)}$$

(A12-13) is the explicit form of (A12-1). Two additional useful expressions are

$$\sinh F = (\bar{n}S + F)/\bar{e} \qquad \text{(A12-15)}$$

$$\cosh F = \sinh F + e^{-F} \qquad \text{(A12-16)}$$

For $F \to -\infty$, (A12-2) is rewritten as

$$e^{-F} = -(2\bar{n}S + 2F - \bar{e}e^{F}) / \bar{e} \qquad \text{(A12-17)}$$

The same sequence of approximations as used for F positive leads to

$$e^{-F} = -\frac{2\bar{n}S}{\bar{e}} + \frac{2}{\bar{e}} \log\left(-\frac{2\bar{n}S}{\bar{e}}\right) - \frac{2}{\bar{n}\bar{e}S} \log\left(-\frac{2\bar{n}S}{\bar{e}}\right) - \frac{\bar{e}}{2\bar{n}S} + 0\left(\frac{\log|S|}{S^2}\right) \qquad \text{(A12-18)}$$

$$F = -\log\left(-\frac{2\bar{n}S}{\bar{e}}\right) + \frac{1}{\bar{n}S} \log\left(-\frac{2\bar{n}S}{\bar{e}}\right) + \frac{1}{4\bar{n}^2S^2}\left[2\log^2\left(-\frac{2\bar{n}S}{\bar{e}}\right)\right.$$

$$\left. - 4\log\left(-\frac{2\bar{n}S}{\bar{e}}\right) - \bar{e}^2\right] + 0\left(\frac{\log^2|S|}{S^3}\right) \qquad \text{(A12-19)}$$

$$e^{-F} = -\frac{\bar{e}}{2\bar{n}S} - \frac{\bar{e}}{2\bar{n}^2S^2} \log\left(-\frac{2\bar{n}S}{\bar{e}}\right) + 0\left(\frac{\log|S|}{S^3}\right) \qquad \text{(A12-20)}$$

The function sinhF is still defined by (A12-15) but coshF can now be written

$$\cosh F = -\sinh F + e^{F} \qquad \text{(A12-21)}$$

The departure asymptote with $F > 0$ must overlap with the outer solution where $t - t_k > 0$. Likewise, the approach asymptote with $F < 0$ must overlap with the outer solution where $t - t_k < 0$. Introducing $Q_k$ defined by (A11-34) makes it possible to write one expression for F:

$$F_k = Q_k \log\left(\frac{2Q_k\bar{n}_kS_k}{\bar{e}_k}\right) + \frac{1}{\bar{n}_kS_k} \log\left(\frac{2Q_k\bar{n}_kS_k}{\bar{e}_k}\right)$$

$$- \frac{Q_k}{4\bar{n}_k^2S_k^2}\left[2\log^2\left(\frac{2Q_k\bar{n}_kS_k}{\bar{e}_k}\right) - 4\log\left(\frac{2Q_k\bar{n}_kS_k}{\bar{e}_k}\right) - \bar{e}_k^2\right]$$

$$+ 0\left(\frac{\log^2|S_k|}{S_k^3}\right) \qquad \text{(A12-22)}$$

Using (A12-22) in (A12-15) gives

$$\sinh F_k = \frac{\bar{n}_k S_k}{\bar{e}_k} + \frac{Q_k}{\bar{e}_k} \log\left(\frac{2Q_k \bar{n}_k S_k}{\bar{e}_k}\right) + \frac{1}{\bar{n}_k \bar{e}_k S_k} \log\left(\frac{2Q_k \bar{n}_k S_k}{\bar{e}_k}\right)$$

$$- \frac{Q_k}{4\bar{n}_k^2 \bar{e}_k S_k^2}\left[2 \log^2\left(\frac{2Q_k \bar{n}_k S_k}{\bar{e}_k}\right) - 4 \log\left(\frac{2Q_k \bar{n}_k S_k}{\bar{e}_k}\right) - \bar{e}_k^2\right]$$

$$+ 0\left(\frac{\log^2 |S_k|}{S_k^3}\right) \tag{A12-23}$$

Using (A12-14) in (A12-15) and (A12-20) in (A12-21) gives

$$\cosh F_k = Q_k \sinh F_k + \frac{Q_k \bar{e}_k}{2n_k S_k} - \frac{\bar{e}_k}{2\bar{n}_k^2 S_k^2} \log\left(\frac{2Q_k \bar{n}_k S_k}{\bar{e}_k}\right)$$

$$+ 0\left(\frac{\log^2 |S_k|}{S_k^3}\right) \tag{A12-24}$$

(A12-22) through (A12-24) represents the first step in the development of the behavior of the inner solution in the overlap domain.

Now consider a rectangular $\bar{x}', \bar{y}'$ coordinate system in the inner orbital plane defined by $\underline{R}_{ko}$. If the $\bar{x}'$ axis lies along the line of nodes between the orbital plane and the $\bar{x}$, $\bar{y}$ reference plane with $\bar{x}'$ positive toward the ascending node then the motion is given by[8]

$$\bar{x}' = A' - \frac{A'}{\bar{e}} \cosh F - \frac{B'C'}{\bar{e}} \sinh F \tag{A12-25}$$

$$\bar{y}' = B' - \frac{B'}{\bar{e}} \cosh F + \frac{A'C'}{\bar{e}} \sinh F \tag{A12-26}$$

where

$$A' = \bar{e}\,\bar{a} \cos \bar{\omega} \tag{A12-27}$$

$$B' = \bar{e}\,\bar{a} \sin \bar{\omega} \tag{A12-28}$$

$$C' = (\bar{e}^2 - 1)^{1/2} \tag{A12-29}$$

Here $\bar{\omega}$ is the argument of pericenter measured in the orbital plane from the ascending node. The subscript k has again been temporarily dropped.

Using (A12-24) $\bar{x}'$ and $\bar{y}'$ become

$$\bar{x}' = -\frac{Q}{\bar{e}} (A' + QB'C') \sinh F$$

$$+ A' \left[ 1 - \frac{Q}{2\bar{n}S} + \frac{1}{2\bar{n}^2 S^2} \log\left(\frac{2Q\bar{n}S}{\bar{e}}\right) + 0\left(\frac{\log^2 |S|}{S^3}\right)\right] \tag{A12-30}$$

$$\bar{y}' = -\frac{Q}{\bar{e}} (B' - QA'C') \sinh F$$

$$+ B' \left[ 1 - \frac{Q}{2\bar{n}S} + \frac{1}{2\bar{n}^2 S^2} \log\left(\frac{2Q\bar{n}S}{\bar{e}}\right) + 0\left(\frac{\log^2 |S|}{S^3}\right)\right] \tag{A12-31}$$

Differentiating with respect to S gives

$$\frac{d\bar{x}'}{dS} = -\frac{Q}{\bar{e}} (A' + QB'C') \cosh F \frac{dF}{dS} + 0\left(\frac{1}{S^2}\right) \tag{A12-32}$$

$$\frac{d\bar{y}}{dS} = -\frac{Q}{\bar{e}} (B' - QA'C') \cosh F \frac{dF}{dS} + 0\left(\frac{1}{S^2}\right) \tag{A12-33}$$

Differentiating (A9-2) gives

$$\bar{n}\,\frac{dS}{dF} = \bar{e}\cosh F - 1 \tag{A12-34}$$

or

$$\frac{dF}{dS} = \frac{\bar{n}}{\bar{e}\cosh F - 1} \tag{A12-35}$$

Then

$$\cosh F\,\frac{dF}{dS} = \frac{\bar{n}\cosh F}{e\cosh F - 1} \tag{A12-36}$$

As $F \rightarrow \infty$, $S \rightarrow \infty$ and

$$\cosh F\,\frac{dF}{dS} \rightarrow \frac{\bar{n}\,Q\sinh F}{\bar{e}\,Q\sinh F - 1}$$

$$\rightarrow \frac{\bar{n}}{\bar{e}} \tag{A12-37}$$

because of (A12-23). Therefore the velocity components as $S \rightarrow \infty$ are given by

$$\bar{U}' \equiv \frac{d\bar{x}'}{dS} = -Q\,\bar{n}\,(A' + QB'C')\,/\,\bar{e}^2$$

$$= -\bar{n}\,(B'C' + QA')\,/\,\bar{e}^2 \tag{A12-38}$$

$$\bar{V}' \equiv \frac{d\bar{y}'}{dS} = -Q\,\bar{n}\,(B' - QA'C')\,/\,\bar{e}^2$$

$$= \bar{n}\,(A'C' - QB')\,/\,\bar{e}^2 \tag{A12-39}$$

The components $\overline{U}'$ and $\overline{V}'$ are the orbital plane components of the hyperbolic excess velocity, $\underline{V}_\infty$.

(A12-30) and (A12-31) can now be written

$$\overline{x}' = \frac{\overline{U}'\overline{e}}{\overline{n}} \sinh F + A' \left[ 1 - \frac{Q}{2\overline{n}S} + \frac{1}{2\overline{n}^2 S^2} \log\left(\frac{2Q\overline{n}s}{\overline{e}}\right) + 0\left(\frac{\log^2|S|}{S^3}\right)\right] \quad \text{(A12-40)}$$

$$\overline{y}' = \frac{\overline{V}'\overline{e}}{\overline{n}} \sinh F + B' \left[ 1 - \frac{Q}{2\overline{n}S} + \frac{1}{2\overline{n}^2 S^2} \log\left(\frac{2Q\overline{n}S}{\overline{e}}\right) + 0\left(\frac{\log^2|S|}{S^3}\right)\right] \quad \text{(A12-41)}$$

In the reference $\overline{x}$, $\overline{y}$, $\overline{z}$ coordinate system the motion is given by[9]

$$\overline{x} = \overline{x}' \cos\overline{\Omega} - \overline{y} \sin\overline{\Omega} \cos\overline{i} \quad \text{(A12-42)}$$

$$\overline{y} = \overline{x}' \sin\overline{\Omega} + \overline{y}' \cos\overline{\Omega} \cos\overline{i} \quad \text{(A12-43)}$$

$$\overline{z} = \overline{y}' \sin\overline{i} \quad \text{(A12-44)}$$

Where $\overline{\Omega}$ is the argument of ascending node and $\overline{i}$ is the inclination as shown in Figure A4.

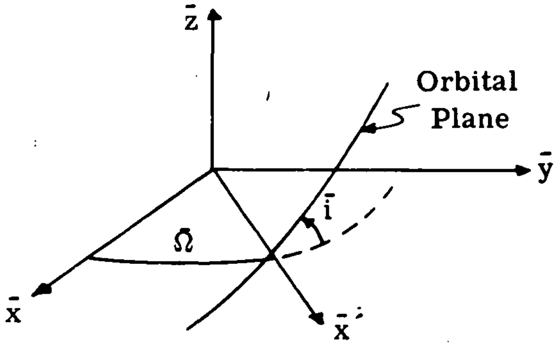


Figure A4. Inner Coordinates

Using (A12-40) and (A12-41) gives

$$\bar{x} = \frac{\bar{U}\bar{e}}{\bar{n}} \sinh F + \bar{A}\left[1 - \frac{Q}{2\bar{n}S} + \frac{1}{2\bar{n}^2S^2}\log\left(\frac{2Q\bar{n}S}{\bar{e}}\right) + O\left(\frac{\log^2|S|}{S^3}\right)\right] \tag{A12-45}$$

$$\bar{y} = \frac{\bar{V}\bar{e}}{\bar{n}} \sinh F + \bar{B}\left[1 - \frac{Q}{2\bar{n}S} + \frac{1}{2\bar{n}^2S^2}\log\left(\frac{2Q\bar{n}S}{\bar{e}}\right) + O\left(\frac{\log^2|S|}{S^3}\right)\right] \tag{A12-46}$$

$$\bar{z} = \frac{\bar{W}\bar{e}}{\bar{n}} \sinh F + \bar{C}\left[1 - \frac{Q}{2\bar{n}S} + \frac{1}{2\bar{n}^2S^2}\log\left(\frac{2Q\bar{n}S}{\bar{e}}\right) + O\left(\frac{\log^2|S|}{S^3}\right)\right] \tag{A12-47}$$

where

$$\bar{U} = \bar{U}' \cos\bar{\Omega} - \bar{V}' \sin\bar{\Omega}\cos\bar{i} \tag{A12-48}$$

$$\bar{V} = \bar{U}' \sin\bar{\Omega} + \bar{V}' \cos\bar{\Omega}\cos\bar{i} \tag{A12-49}$$

$$\bar{W} = \bar{V}' \sin\bar{i} \tag{A12-50}$$

$$\bar{A} = A' \cos\bar{\Omega} - B' \sin\bar{\Omega}\cos\bar{i} \tag{A12-51}$$

$$\bar{B} = A' \sin\bar{\Omega} + B' \cos\bar{\Omega}\cos\bar{i} \tag{A12-52}$$

$$\bar{C} = B' \sin\bar{i} \tag{A12-53}$$

$\overline{U}$, $\overline{V}$ and $\overline{W}$ are the components of $\underline{V}_\infty$ in the reference coordinate system, i.e.

$$\underline{V}_\infty = \overline{U}\underline{e}_1 + \overline{V}\underline{e}_2 + \overline{W}\underline{e}_3 \tag{A12-54}$$

Letting

$$\underline{L} = \overline{A}\underline{e}_1 + \overline{B}\underline{e}_2 + \overline{C}\underline{e}_3 \tag{A12-55}$$

and

$$\underline{R}_o = \overline{x}\underline{e}_1 + \overline{y}\underline{e}_2 + \overline{z}\underline{e}_3 \tag{A12-56}$$

gives

$$\underline{R}_{ko} = \frac{\overline{e}_k}{\overline{n}_k}\underline{V}_{\infty k} \sinh F_k + \underline{L}_k\left[1 - \frac{Q_k}{2\overline{n}_k S_k} + \frac{1}{2\overline{n}_k^2 S_k^2}\log\left(\frac{2Q_k\overline{n}_k S_k}{\overline{e}_k}\right) + O\left(\frac{\log^2|S_k|}{S_k^3}\right)\right] \tag{A12-57}$$

Replacing $\sinh F_k$ by (A12-23) gives

$$\underline{R}_{ko}(S_k) = \underline{A}_{ko} S_k + \underline{B}_{ko} \log Q_k S_k + \underline{C}_{ko} + \underline{D}_{ko} S_k^{-1} \log Q_k S_k + \underline{E}_{ko} S_k^{-1} + \underline{F}_{ko} S_k^{-2} \log^2 Q_k S_k + \underline{G}_{ko} S_k^{-2} \log Q_k S_k + \underline{H}_{ko} S_k^{-2} + O\left(S_k^{-3}\log^3|S_k|\right) \tag{A12-58}$$

where

$$\underline{A}_{ko} = \underline{V}_{\infty k} \tag{A12-59}$$

$$\underline{B}_{ko} = \frac{Q_k}{\overline{n}_k}\underline{V}_{\infty k} \tag{A12-60}$$

$$\underline{C}_{ko} = \frac{Q_k}{\bar{n}_k} \log\left(\frac{2\bar{n}_k}{\bar{e}_k}\right) \underline{V}_{\infty k} + \underline{L}_k \tag{A12-61}$$

$$\underline{D}_{ko} = \frac{1}{\bar{n}_k^2} \underline{V}_{\infty k} \tag{A12-62}$$

$$\underline{E}_{ko} = \frac{1}{\bar{n}_k^2} \log\left(\frac{2\bar{n}_k}{e_k}\right) \underline{V}_{\infty k} - \frac{Q_k}{2\bar{n}_k} \underline{L}_k \tag{A12-63}$$

$$\underline{F}_{ko} = - \frac{Q_k}{2\bar{n}_k^3} \underline{V}_{\infty k} \tag{A12-64}$$

$$\underline{G}_{ko} = - \frac{Q_k}{\bar{n}_k^3} \log\left(\frac{2\bar{n}_k}{\bar{e}_k}\right) \underline{V}_{\infty k} + \frac{Q_k}{\bar{n}_k^3} \underline{V}_{\infty k} + \frac{1}{2\bar{n}_k^2} \underline{L}_k \tag{A12-65}$$

$$\underline{H}_{ko} = - \frac{Q_k}{2\bar{n}_k^3} \log^2\left(\frac{2\bar{n}_k}{\bar{e}_k}\right) \underline{V}_{\infty k} + \frac{Q_k}{\bar{n}_k^3} \log\left(\frac{2\bar{n}_k}{\bar{e}_k}\right) \underline{V}_{\infty k}$$

$$+ \frac{\bar{e}_k^2}{4\bar{n}_k^3} \underline{V}_{\infty k} + \frac{1}{2\bar{n}_k^2} \log\left(\frac{2\bar{n}_k}{\bar{e}_k}\right) \underline{L}_k \tag{A12-66}$$

(A12-58) gives the behavior of $\underline{R}_{ko}(S_k)$ in the overlap domain. The singular behavior for large $S_k$ must be matched with the singular terms in the outer solution. The matching must also relate the constants $\underline{V}_{\infty k}$ and $\underline{L}_k$ with constants of the outer solution. Note that once $\underline{V}_{\infty k}$ and $\underline{L}_k$ are specified the hyperbola $\underline{R}_{ko}$ is completely determined since $\bar{n}_k$ and $\bar{e}_k$ can be obtained from $\underline{V}_{\infty k}$ and $\underline{L}_k$.

### A12.2 Second Order

The exact form of the second order inner solution is given by (A9-7). It is an extremely difficult and laborious task to determine the behavior of (A9-7) in the overlap domain, i.e., for large $S_k$.

The integral form can be written

$$\underset{\sim}{R}_{k2}\,(S_k) = \int_{S_{ko}}^{S_k} \overline{B}(S_k,\ \sigma)\, G_k\, \underset{\sim}{R}_{ko}\,(\sigma)\, d\sigma \qquad \text{(A12-67)}$$

Carlson uses a Taylor series expansion[3] for $B(S_k,\ \sigma)$ but this approach is not satisfactory in a second order theory. The largest term in the Taylor series expansion is

$$\overline{B}(S_k,\ \sigma) = I(S_k - \sigma) + O\left((S_k - \sigma)^3\right) \qquad \text{(A12-68)}$$

From (A12-58)

$$\underset{\sim}{R}_{ko}(\sigma) = \underset{\sim}{A}_{ko}\sigma + O(\log|\sigma|) \qquad \text{(A12-69)}$$

Therefore the largest term in $\underset{\sim}{R}_{k2}$ is proportional to

$$\int_{S_o}^{S} (S - \sigma)\,\sigma\, d\sigma = \int_{S_o}^{S} (S - \sigma)\left[S - (S - \sigma)\right] d\,(S - \sigma)$$

$$= \frac{S^3}{2}\left(1 - \frac{S_o}{S}\right)^2 - \frac{S^3}{3}\left(1 - \frac{S_o}{S}\right)^3 \qquad \text{(A12-70)}$$

Since, in general, $S_o << S$

$$\int_{S_o}^{S} (S - \sigma)\,\sigma\, d\sigma = \frac{S^3}{6} + O(S) \qquad \text{(A12-71)}$$

Therefore the leading term in the expansion of $\underline{R}_{k2}$ is order $S_k^3$ and is easily found. The higher order terms cannot be found by this approach, however. In the expansion of $\overline{B}$ the higher order terms are proportional to

$$\frac{d^n}{dS^n} G(\underline{R}_o)\ (S - \gamma)^{n+3}$$

For n = 0, 1, 2

$$\frac{d^o}{dS^o} G(\underline{R}_o) = G(\underline{R}_o) \tag{A12-72}$$

$$\frac{d^1}{dS^1} G(\underline{R}_o) = \frac{d}{d\underline{R}_o} G(\underline{R}_o) \frac{d\underline{R}_o}{dS} = \underline{H}(\underline{R}_o) \frac{d\underline{R}_o}{dS} \tag{A12-73}$$

$$\frac{d^2}{dS^2} G(\underline{R}_o) = \frac{d}{dS} \underline{H}(\underline{R}_o) \frac{d\underline{R}_o}{dS} + \underline{H}(\underline{R}_o) \frac{d^2\underline{R}_o}{dS^2}$$

$$= \underset{\sim}{T}(\underline{R}_o) \left(\frac{d\underline{R}_o}{dS}\right)^2 + \underline{H}(\underline{R}_o) \frac{d^2\underline{R}_o}{dS^2} \tag{A12-74}$$

From (A12-58)

$$\underline{R}_o = \underline{A}_o S + O(\log |S|) \tag{A12-75}$$

Comparing (A12-72) - (A12-75) with (A11-28), (A11-36), A11-37) and (A11-89) gives

$$G(\underline{R}_o) = G(\underline{A}_o S) + O(\log |S|)$$

$$= \frac{1}{S^3} G(\underline{A}_o) + O\left(\frac{\log |S|}{S^4}\right) \tag{A12-76}$$

$$\underline{H}(\underline{R}_o) = \underline{H}(\underline{A}_o S) + O\,(\log\,|S|)$$

$$= \frac{1}{S^4}\,\underline{H}(\underline{A}_o) + O\left(\frac{\log\,|S|}{S^5}\right) \qquad \text{(A 12-77)}$$

$$\underset{\sim}{T}(\underline{R}_o) = \underset{\sim}{T}(\underline{A}_o S) + O\,(\log\,|S|)$$

$$= \frac{1}{S^5}\,\underset{\sim}{T}(\underline{A}_o) + O\left(\frac{\log\,|S|}{S^6}\right) \qquad \text{(A 12-78)}$$

Putting (A 12-76) - (A 11-78) into the derivatives of G leads to the conclusion that

$$\frac{d^n}{dS^n} G(\underline{R}_o) = O\left(\frac{1}{S^{n+3}}\right) \qquad \text{(A 12-79)}$$

The higher order terms in B are thus proportional to

$$(S - \sigma)^{n+3} / S^{n+3}$$

Multiplying each of these terms by (A 12-69) and integrating gives the contribution of the higher order terms in the Taylor series expansion of B, i.e.,

$$\int_{S_o}^{S} \frac{(S - \sigma)^{n+3}}{S^{n+3}}\,\sigma d\sigma = \frac{1}{S^{n+3}} \int_{S}^{S_o} (S - \sigma)^{n+3} \left[S - (S - \sigma)\right] d\,(S - \sigma)$$

$$= \frac{S^2}{(n + 4)} \left(1 - \frac{S_o}{S}\right)^{n+4} - \frac{S^2}{(n + 5)} \left(1 - \frac{S_o}{S}\right)^{n+5}$$

$$= \frac{S^2}{n^2 + 9n + 20} + O(1) \qquad \text{(A 12-80)}$$

Therefore every term in the Taylor series expansion of $\overline{B}$ contributes at least one term of order $S_k^2$. Eventually the denominator in (A12-80) would become large and after n = N the terms could be ignored. However, in order to obtain a reasonable approximation of the $S_k^2$ contribution to the expansion of $\underline{R}_{k2}$ N would have to be quite large. It will be shown that in order to complete the matching $\underline{R}_{k2}$ must be expanded out to order $S_k$ therefore the Taylor series expansion of $\overline{B}$ is unacceptable.

What is actually needed is an asymptotic expansion of $B(S_k, \sigma)$ for both $S_k$ and $\sigma$ large. Such an expansion can be shown to differ from the Taylor series expansion since no assumption is made that $S_k - \sigma$ is small. An attempt was made to expand $\overline{B}(S_k, \sigma)$ to such an order that (A12-67) would be represented by an expansion to order $S_k$. The amout of algebra encountered was horrendous and even stymied an attempt to use Formac. It was finally determined that such an approach could, at best, only approximate the $S_k$ term with an extremely complicated expansion. This approach was therefore also considered to be unacceptable.

Based on (A12-71) and the previously derived behavior of $\underline{R}_{ko}$, (A12-58), it can be assumed that $\underline{R}_{k2}$ has the following expansion:

$$\underline{R}_{k2}(S_k) = \underline{A}_{k2} S_k^3 + \underline{B}_{k2} S_k^2 \log Q_k S_k + \underline{C}_{k2} S_k^2$$

$$+ \underline{D}_{k2} S_k \log^2 Q_k S_k + \underline{E}_{k2} S_k \log Q_k S_k$$

$$+ \underline{F}_{k2} S_k + O\left(\log^3 |S_k|\right) \qquad \text{(A12-81)}$$

(That the expansion of $\underline{R}_{k2}$ would have this form was strongly indicated by the work done with an asymptotic expansion of $B(S_k, \sigma)$. Some of the terms were actually derived but each successive term required an exponentially increasing amount of algebra.) Differentiating (A12-81) twice gives

$$\frac{d^2 \underline{R}_{k2}}{dS_k^2} = 6 \underline{A}_{k2} S_k + 2 \underline{B}_{k2} \log Q_k S_k + 3 \underline{B}_{k2} + 2 \underline{C}_{k2}$$

$$+ 2 \underline{D}_{k2} \frac{\log Q_k S_k}{S_k} + \frac{2 \underline{D}_{k2} + \underline{E}_{k2}}{S_k} + O\left(\frac{\log |S_k|}{S_k^2}\right) \tag{A12-82}$$

Now using (A2-20) and (A12-58) gives

$$\begin{aligned} G(\underline{R}_{ko}) = {} & G(\underline{A}_{ko} S_k) + \underline{H}(\underline{A}_{ko} S_k) \underline{B}_{ko} \log Q_k S_k \\ & + \underline{H}(\underline{A}_{ko} S_k) \underline{C}_{ko} + O\left[\underline{H}(\underline{A}_{ko} S_k) \frac{\log |S_k|}{S_k}, \right. \\ & \left. \underset{\sim}{T}(\underline{A}_{ko} S_k) \log^2 |S_k| \right] \end{aligned} \tag{A12-83}$$

Using (A12-76) - (A12-78) in (A12-83) gives

$$\begin{aligned} G(\underline{R}_{ko}) = {} & \frac{G(\underline{A}_{ko})}{S_k^3} + \underline{H}(\underline{A}_{ko}) \underline{B}_{ko} \frac{\log Q_k S_k}{S_k^4} \\ & + \underline{H}(\underline{A}_{ko}) \underline{C}_{ko} \frac{1}{S_k^4} + O\left(\frac{\log^2 |S_k|}{S_k^5}\right) \end{aligned} \tag{A12-84}$$

Multiplying (A12-81) by (A12-84) gives

$$\begin{aligned} G(\underline{R}_{ko}) \underline{R}_{k2} = {} & G(\underline{A}_{ko}) \underline{A}_{k2} + \Big[G(\underline{A}_{ko}) \underline{B}_{k2} \\ & + \underline{H}(\underline{A}_{ko}) \underline{B}_{ko} \underline{A}_{k2}\Big] \frac{\log Q_k S_k}{S_k} + \Big[G(\underline{A}_{ko}) \underline{C}_{k2} \\ & + \underline{H}(\underline{A}_{ko}) \underline{C}_{ko} \underline{A}_{k2}\Big] \frac{1}{S_k} + O\left(\frac{\log^2 |S_k|}{S_k^2}\right) \end{aligned} \tag{A12-85}$$

Substituting (A12-58) and (A12-85) into (A8-5) gives

$$\frac{d^2}{dS_k^2}\underline{R}_{k2}(S_k) = G_k\,\underline{A}_{ko}\,S_k + G_k\,\underline{B}_{ko}\log Q_k S_k$$

$$+ G(\underline{A}_{ko})A_{k2} + G_k\,\underline{C}_{ko}$$

$$+ \left[G(\underline{A}_{ko})\underline{B}_{k2} + \underline{H}(\underline{A}_{ko})\underline{B}_{ko}\,\underline{A}_{k2} + G_k\,\underline{D}_{ko}\right]\frac{\log Q_k S_k}{S_k}$$

$$+ \left[G(\underline{A}_{ko})\underline{C}_{k2} + \underline{H}(\underline{A}_{ko})\underline{C}_{ko}\,\underline{A}_{k2} + G_k\,\underline{E}_{ko}\right]\frac{1}{S_k}$$

$$+ O\left(\frac{\log^2|S_k|}{S_k^2}\right) \qquad \text{(A12-86)}$$

Equating functions of $S_k$ in (A12-82) and (A12-86) gives

$$\underline{A}_{k2} = G_k\,\underline{A}_{ko}/6 \qquad \text{(A12-87)}$$

$$\underline{B}_{k2} = G_k\,\underline{B}_{ko}/6 \qquad \text{(A12-88)}$$

$$\underline{C}_{k2} = \left[G(\underline{A}_{ko})\underline{A}_{k2} + G_k\,\underline{C}_{ko} - 3\,\underline{B}_{k2}\right]/2 \qquad \text{(A12-89)}$$

$$\underline{D}_{k2} = \left[G(\underline{A}_{ko})\underline{B}_{k2} + \underline{H}(\underline{A}_{ko})\underline{B}_{k2} + G_k\,\underline{D}_{ko}\right]/2 \qquad \text{(A12-90)}$$

$$\underline{E}_{k2} = G(\underline{A}_{ko})\underline{C}_{k2} + \underline{H}(\underline{A}_{ko})\underline{C}_{ko}\,\underline{A}_{k2} + G_k\,\underline{E}_{ko} - 2\underline{D}_{k2} \qquad \text{(A12-91)}$$

Since $\underline{F}_{k2}$ appears as a linear term in $S_k$ in (A12-81) it vanishes when taking the second derivative. Thus it is not possible to get $\underline{F}_{k2}$ by direct substitution. However, (A12-81) can be rewritten in the form

$$\underline{D}_{k2} \log^2 Q_k S_k + \underline{E}_{k2} \log Q_k S_k + \underline{F}_{k2}$$

$$= \underline{R}_{k2}(S_k)/S_k - \underline{A}_{k2} S_k^2 - \underline{B}_{k2} S_k \log Q_k S_k$$

$$- \underline{C}_{k2} S_k + O\left(\frac{\log^3 |S_k|}{S_k}\right) \tag{A12-92}$$

Letting

$$S_k = Q_k/\mu_k \tag{A12-93}$$

in (A12-92) gives

$$\underline{D}_{k2} \log^2 \mu_k - \underline{E}_{k2} \log \mu_k + \underline{F}_{k2}$$

$$= \mu_k Q_k \underline{R}_{k2} (Q_k/\mu_k) - \left[\mu_k^{-2} \underline{A}_{k2} + \mu_k^{-1} Q_k \underline{B}_{k2} \log \mu_k\right.$$

$$\left. + \mu_k^{-1} Q_k \underline{C}_{k2}\right] + O\left(\mu_k \log^3 \mu_k\right) \tag{A12-94}$$

where

$$\underline{R}_{k2} (Q_k/\mu_k) = \int_{S_{ko}}^{Q_k/\mu_k} B(Q_k/\mu_k, \sigma) G_k \underline{R}_{ko} (\sigma)d\sigma \tag{A12-95}$$

The left-hand side of (A12-94) is exactly the expression which will appear in the matching. For very small $\mu_k$'s such as in interplanetary applications the $\log^2 \mu_k$ and log $\mu k$ terms will be much larger than $\underline{F}_{k2}$, and $\underline{F}_{k2}$ can probably be ignored. In such a case (A12-94) is not needed since $\underline{D}_{k2}$ and $\underline{E}_{k2}$ can be evaluated directly from (A12-90) and (A12-91). For the larger value of $\mu_k$ corresponding to the earth-moon system it may be necessary to evaluate (A12-94). This requires first evaluating (A12-95) by numerical quadrature.

Then the subtraction indicated on the right hand side of (A12-94) must be performed with significant care since it involves the difference of two large numbers, i.e., the difference of two numbers which are both order $\mu_k^{-2}$.

The behavior of $\underline{R}_{k2}$ in the overlap domain is therefore given by (A12-81). It has stronger singularities than does $\underline{R}_{ko}$ but it is also multiplied by $\mu_k^2$ so the effect of the singularities is somewhat diminished. The coefficients of the expansion (A12-81) can be found as functions of the coefficients in the expansion of $\underline{R}_{ko}$.

### A12.3 Third Order

Only the largest term in the expansion of $\underline{R}_{k3}$ is of interest. Since the differential equation for $\underline{R}_{k3}$, (A8-6), is similar to that for $\underline{R}_{k2}$ it is assumed to have a similar expansion. Let

$$\underline{R}_{k3} = \underline{A}_{k3} S_k^m + O\left(S_k^{m-1} \log Q_k S_k\right) \tag{A12-96}$$

where m is unknown. The second derivative is

$$\frac{d^2 \underline{R}_{k3}}{dS_k^2} = m(m-1)\underline{A}_{k3} S_k^{m-2} + O\left(S_k^{m-3} \log|S_k|\right) \tag{A12-97}$$

Multiplying (A12-96) by (A12-84) gives

$$G(\underline{R}_{ko})\underline{R}_{k3} = G(\underline{A}_{ko})\underline{A}_{k3} S_k^{m-3} + O\left(S_k^{m-4} \log|S_k|\right) \tag{A12-98}$$

According to (A8-8)

$$\begin{aligned} \underline{P}_3(\underline{R}_{ko}, \underline{R}_{k2}) &= \frac{1}{2} \underline{H}_k \underline{R}_{ko}^2 + O(\underline{R}_{ko}) \\ &= \frac{1}{2} \underline{H}_k \underline{A}_{ko}^2 S_k^2 + O\left(S_k \log|S_k|\right) \end{aligned} \tag{A12-99}$$

Substituting (A12-98) and (A12-99) into (A8-6) gives

$$\frac{d^2 \underline{R}_{k3}}{dS_k^2} = \frac{1}{2} \underline{H}_k \underline{A}_{ko}^2 S_k^2 + O\left(S_k \log|S_k|\right) + O\left(S_k^{m-3}\right) \tag{A12-100}$$

Comparing (A12-97) and (A12-100) gives

$$m = 4 \tag{A12-101}$$

$$\underline{A}_{k3} = \underline{H}_k \underline{A}_{ko}^2 / 24 \tag{A12-102}$$

Therefore, in the overlap domain the behavior of $\underline{R}_{k3}$ is given by

$$\underline{R}_{k3}(S_k) = \frac{1}{24} \underline{H}_k \underline{A}_{ko}^2 S_k^4 + O\left(S_k^3 \log|S_k|\right) \tag{A12-103}$$

## A12.4 Fourth Order

Like the third order outer solution, the fourth order inner solution is not actually used but its behavior in the overlap domain is important. This behavior can be deduced without rigorous proof. It has already been shown that in the overlap domain

$$\underline{R}_{ko} = O\left(S_k\right)$$

$$\underline{R}_{k2} = O\left(S_k^3\right)$$

$$\underline{R}_{k3} = O\left(S_k^4\right)$$

From this sequence it is assumed that in the overlap domain

$$\underline{R}_{k4} = O\left(S_k^5\right) \tag{A12-104}$$

## A13 INTERMEDIATE LIMIT

The overlap domain has been defined as the region where $t-t_k$ is small in the outer solution and $S_k$ is large in the inner solution. In this region a new independent variable $\sigma_k$ is defined by

$$\sigma_k = (t-t_{pk})/\mu_k^\alpha, \quad O \le \alpha_o < \alpha < \alpha_1 \le 1 \tag{A13-1}$$

Note that if $\alpha = O$, $\sigma_k$ simply shifts the time scale without any scaling. The limit $\alpha = O$ is then the outer limit. If $\alpha = 1$, (A13-1) reduces to (A7-27) giving the inner limit. The variable range $\alpha_o < \alpha < \alpha_1$ will be defined as the intermediate limit and $\alpha_o$ and $\alpha_1$ must be defined by the matching.

From (A7-12) with $\beta = 1$

$$t_{pk} = t_k + \mu_k \tau_k \tag{A13-2}$$

Substituting (A13-2) into (A13-1) gives the outer variable

$$t-t_k = \mu_k^\alpha \sigma_k + \mu_k \tau_k \tag{A13-3}$$

Using (A5-4)

$$t-t_k = \mu^\alpha M_k^\alpha \sigma_k + \mu M_k \tau_k \tag{A13-4}$$

The last result gives $t-t_k$ in terms of $\sigma_k$ and shows explicity that $t-t_k$ is small, i.e., order $\mu^\alpha$. Substituting (A13-1) into (A7-27) gives

$$S_k = \mu_k^{\alpha-1} \sigma_k$$

$$= \mu^{\alpha-1} M_k^{\alpha-1} \sigma_k \tag{A13-5}$$

(A13-5) gives $S_k$ in terms of $\sigma_k$ and shows explicity that $S_k$ is large, i.e., order $\mu^{\alpha-1}$, since $\alpha-1$ is always negative.

(A13-4) and (A13-5) must now be used to transform the outer and inner solutions into functions of a common variable $\sigma_k$. Then the matching can be carried out.

## A14 MATCHING

It has been assumed that both the outer solution, (A10-1), and the inner solution, (A12-2), are valid in the overlap domain. The matching of these two solutions is most simply stated by requiring their difference to be small in some appropriate limit. Cole[4] states this limit has

$$\lim_{\substack{\mu \to 0 \\ \sigma_k \text{ constant}}} \left[ \frac{(\text{outer solution}) - (\text{inner solution})}{\varepsilon(\mu)} \right] = O \qquad \text{(A14-1)}$$

where $\varepsilon(\mu)$ is a guage function. For a second order theory $\varepsilon(\mu)$ is most easily chosen to be $\mu^2$. The limit (A14-1) can be written

$$\lim_{\substack{\mu \to 0 \\ \sigma_k \text{ constant}}} \left[ \frac{\underline{r}_{ko}}{\mu^2} + \frac{\underline{r}_1}{\mu} + \underline{r}_2 + \mu \underline{r}_3 - \frac{\underline{p}_k}{\mu^2} - \left( \frac{M_k \underline{R}_{ko}}{\mu} + \mu\, M_k^3\, \underline{R}_{k2} + \mu^2\, M_k^4\, \underline{R}_{k3} + \mu^3\, M_k^5\, \underline{R}_{k4} + \dots \right) \right] = O \qquad \text{(A14-2)}$$

In order for this limit to hold all terms which are singular or constant as $\mu \to 0$ must vanish. For the singular terms, the coefficients of the different functions of $\sigma_k$ must vanish identically since the limit must hold independently of the value of $\sigma_k$.

In each of the expansions for $\underline{r}_{ko}$, $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$, $\underline{R}_{ko}$, $\underline{R}_{k2}$, $\underline{R}_{k3}$ and $\underline{R}_{k4}$ certain terms have been ignored and entered only as order something. These terms must all vanish in the limit given by (A14-2). In general these terms will vanish in the limit if the exponent of $\mu$ is positive. That is, $\mu^x$ vanishes if x>0, is constant if x = 0, and is singular if x<0, as $\mu \to 0$. In order to make these terms vanish the $\alpha$ of (A13-1) must be restricted as follows:

$$\frac{\underline{r}_{ko}}{\mu^2} \sim O\left(\frac{(t-t_k)^5}{\mu^2}, \frac{(t-t_k)^4}{\mu}\right) = O\left(\mu^{5\alpha-2}, \mu^{4\alpha-1}\right)$$

Both terms vanish if $\alpha > 2/5$.

$$\frac{\underline{r}_1}{\mu} \sim O\left(\frac{(t-t_k)^3}{\mu}\right) = O\left(\mu^{3\alpha-1}\right)$$

This vanishes if $\alpha > 1/3$.

$$\underline{r}_2 \sim O\left((t-t_k)^2, \mu\right) = O\left(\mu^{2\alpha}, \mu\right)$$

This vanishes if $\alpha > 0$.

$$\mu \underline{r}_3 \sim O\left(\frac{\mu}{(t-t_k)^2}\right) = O\left(\mu^{1-2\alpha}\right)$$

The effect of $\underline{r}_3$ vanishes if $\alpha < 1/2$.

$$\frac{\underline{R}_{ko}}{\mu} \sim O\left(\frac{1}{\mu S_k^3}\right) = O\left(\mu^{2-3\alpha}\right)$$

This vanishes if $\alpha < 2/3$.

$$\mu \underline{R}_{k2} \sim O(\mu)$$

This vanishes for any $\alpha$.

$$\mu^2 \underline{R}_{k3} \sim O\left(\mu^2 S_k^3\right) = O\left(\mu^{3\alpha-1}\right)$$

This vanishes if $\alpha > 1/3$.

$$\mu^3 \, \underline{R}_{k4} \sim O\left(\mu^3 \, S_k^5\right) = O\left(\mu^{5\alpha-2}\right)$$

This vanishes if $\alpha > 2/5$. All of these restrictions are satisfied if

$$\frac{2}{5} < \alpha < \frac{1}{2} \tag{A14-3}$$

Therefore

$$\alpha_o = \frac{2}{5} \tag{A14-4}$$

$$\alpha_1 = \frac{1}{2} \tag{A14-5}$$

It is assumed from this point on that (A14-3) is satisfied and that all of the above terms vanish in the limit. Therefore these terms need no longer be considered in the limit (A14-2).

## A15 INTERMEDIATE FORM OF THE OUTER SOLUTION

### A15.1 Zeroth Order

From (A13-4)

$$\frac{(t-t_k)}{\mu^2} = \mu^{\alpha-2} \, M_k \, \sigma_k + \mu^{-1} \, M_k \, \tau_k \tag{A15-1}$$

$$\frac{(t-t_k)^2}{\mu} = \mu^{2\alpha-1} \, M_k^{2\alpha} \, \sigma_k^2 + 2\mu^{\alpha} \, M_k^{1+\alpha} \, \tau_k \, \sigma_k + O(\mu) \tag{A15-2}$$

$$\frac{(t-t_k)^3}{\mu} = \mu^{3\alpha-2} \, M_k^{2\alpha} \, \sigma_k^3 + 3\mu^{2\alpha-1} \, M_k^{1+2\alpha} \, \tau_k \, \sigma_k^2$$

$$+ 3\mu^{\alpha} \, M_k^{2+\alpha} \, \tau_k^2 \, \sigma_k + O(\mu) \tag{A15-3}$$

$$\frac{(t-t_k)^4}{\mu^2} = \mu^{4\alpha-2} M_k^{4\alpha} \sigma_k^4 + O\left(\mu^{3\alpha-1} \sigma_k^3\right)$$

$$+ O\left(\mu^{2\alpha} \sigma_k^2\right) + O\left(\mu^{1+\alpha} \sigma_k\right) + O\left(\mu^2\right) \qquad \text{(A15-4)}$$

Substituting these expansions into (A11-26) gives

$$\frac{\underline{r}_{ko}}{\mu^2} = \mu^2 \left(\underline{a}_{ok} - \underline{p}_{ko}\right) + \left[\mu^{-1} M_k \tau_k \underline{V}_k + O(\mu)\right]$$

$$+ \sigma_k \left[\mu^{\alpha-2} M_k^{\alpha} \underline{V}_k - \mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{p}_k^*\right.$$

$$\left. + \frac{1}{2} \mu^{\alpha} M_k^{2+\alpha} \tau_k^2 G_k \underline{V}_k + O\left(\mu^{1+\alpha}\right)\right]$$

$$+ \sigma_k^2 \left[- \frac{1}{2} \mu^{2\alpha-1} M_k^{2\alpha} \underline{p}_k^* + \frac{1}{2} \mu^{2\alpha-1} M_k^{1+2\alpha} \tau_k G_k \underline{V}_k\right.$$

$$\left. + O\left(\mu^{2\alpha}\right)\right] + \sigma_k^3 \left[\frac{1}{6} \mu^{3\alpha-2} M_k^{3\alpha} G_k \underline{V}_k + O\left(\mu^{3\alpha-1}\right)\right]$$

$$+ \sigma_k^4 \left[\left(\frac{\mu^{4\alpha-2}}{24} \underline{H}_k \underline{v}_o^2 (t_k) - \frac{\mu^{4\alpha-2}}{24} \underline{H}_k \dot{p}_k^2 (t_k)\right) M_k^{4\alpha}\right.$$

$$\left. + O\left(\mu^{4\alpha-1}\right)\right] \qquad \text{(A15-5)}$$

where the leading term has been carried explicitly in order to show that the matching verifies (A10-3) and (A11-21).

A15.2 <u>First Order</u>
From (A13-4)

$$\mu^{-1} \log Q_k (t-t_k) = \mu^{-1} \log \Sigma_k + \mu^{-\alpha} M_k^{1-\alpha} \tau_k / \sigma_k$$

$$+ O\left(\mu^{1-2\alpha} / \sigma_k^2\right) \qquad \text{(A15-6)}$$

$$(t-t_k) \log Q_k (t-t_k) = \mu^{\alpha} M_k^{\alpha} \sigma_k \log \Sigma_k + O\left(\mu^{2-\alpha}/\sigma_k\right)$$

$$+ O\left(\mu \log \Sigma_k\right) \qquad \text{(A15-7)}$$

$$\mu^{-1} (t-t_k) = \mu^{\alpha-1} M_k \sigma_k + M_k \tau_k \qquad \text{(A15-8)}$$

$$\mu^{-1} (t-t_k)^2 \log Q_k (t-t_k) = \mu^{2\alpha-1} M_k^{2\alpha} \sigma_k^2 \log \Sigma_k$$

$$+ 2\mu^{\alpha} M_k^{1+\alpha} \tau_k \sigma_k \log \Sigma_k$$

$$+ \mu^{\alpha} M_k^{1+\alpha} \tau_k \sigma_k$$

$$+ O\left(\mu^{2-\alpha}/\sigma_k\right) + O\left(\mu \log \Sigma_k\right) \qquad \text{(A15-9)}$$

where

$$\Sigma_k = \mu^{\alpha} Q_k M_k^{\alpha} \sigma_k \qquad \text{(A15-10)}$$

Substituting these expansions and (A15-2) into (A11-78) gives

$$\frac{\underline{r}_1}{\mu} = \left[\mu^{-1} \underline{b}_{1k} + M_k \tau_k \underline{d}_{1k} + O(\mu)\right]$$

$$+ \sigma_k\left[\mu^{\alpha-1} M_k^{\alpha} \underline{d}_{1k} + \mu^{\alpha} M_k^{\alpha} \underline{e}_{1k} + \mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{f}_{1k}\right.$$

$$\left. + 2\mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{g}_{1k} + O\left(\mu^{1+\alpha}\right)\right]$$

$$+ \sigma_k^2 \left[\mu^{2\alpha-1} M_k^{2\alpha} \underline{g}_{1k} + O\left(\mu^{2\alpha}\right)\right]$$

$$+ \log \Sigma_k \left[\mu^{-1} \underset{\sim}{a}_{1k} + O(\mu)\right]$$

$$+ \sigma_k \log \Sigma_k \left[\mu^{\alpha} M_k \underset{\sim}{c}_{1k} + 2\mu^{\alpha} M_k^{1+\alpha} \tau_k \underset{\sim}{f}_{1k} + O\left(\mu^{1+\alpha}\right)\right]$$

$$+ \sigma_k^2 \log \Sigma_k \left[\mu^{2\alpha-1} M_k^{2\alpha} \underset{\sim}{f}_{1k} + O\left(\mu^{2\alpha}\right)\right]$$

$$+ \frac{1}{\sigma_k}\left[\mu^{-\alpha} M_k^{1-\alpha} \tau_k \underset{\sim}{a}_{1k} + O\left(\mu^{2-\alpha}\right)\right]$$

$$+ O\left(\mu^{1-2\alpha}/\sigma_k^2\right) \tag{A15-11}$$

A15. 3 Second Order

From (A13-4)

$$(t-t_k)^{-1} \log Q_k (t-t_k)$$

$$= \mu^{-\alpha} M_k^{-\alpha}/\sigma_k \log \Sigma_k + O\left(\mu^{1-2\alpha} \sigma_k^{-2} \log \Sigma_k\right) \tag{A15-12}$$

$$(t-t_k)^{-1} = \mu^{-\alpha} M_k^{-\alpha}/\sigma_k + O\left(\mu^{1-2\alpha}/\sigma_k^2\right) \tag{A15-13}$$

$$\log Q_k (t-t_k) = \log \Sigma_k + O\left(\mu^{1-\alpha}/\sigma_k\right) \tag{A15-14}$$

$$(t-t_k) \log^2 Q_k (t-t_k)$$

$$= \mu^{\alpha} M_k^{\alpha} \sigma_k \log^2 \Sigma_k + O\left(\mu \log^2 \Sigma_k\right)$$

$$+ O\left(\mu^{2-\alpha}/\sigma_k\right) + O\left(\mu^{2-3\alpha} \sigma_k^{-2} \log \Sigma_k\right) \tag{A15-15}$$

(A11-131) can now be written

$$\underline{r}_2 = \left[\underline{d}_{2k} + O(\mu)\right]$$

$$+ \sigma_k \left[\mu^{\alpha} M_k^{\alpha} \underline{g}_{2k} + O\left(\mu^{1+\alpha}\right)\right]$$

$$+ \log \Sigma_k \left[\underline{c}_{2k} + O(\mu)\right]$$

$$+ \sigma_k \log \Sigma_k \left[\mu^{\alpha} M_k^{\alpha} \underline{f}_{2k} + O\left(\mu^{1+\alpha}\right)\right]$$

$$+ \sigma_k \log^2 \Sigma_k \left[\mu^{\alpha} M_k^{\alpha} \underline{e}_{2k} + O\left(\mu^{1+\alpha}\right)\right]$$

$$+ \frac{1}{\sigma_k} \left[\mu^{-\alpha} M_k^{-\alpha} \underline{b}_{2k} + O\left(\mu^{1-\alpha}\right)\right]$$

$$+ \frac{\log \Sigma_k}{\sigma_k} \left[\mu^{-\alpha} M_k^{-\alpha} \underline{a}_{2k} + O\left(\mu^{1-\alpha}\right)\right]$$

$$+ O\left(\mu^{1-2\alpha} \sigma_k^{-2} \log \Sigma_k\right) + O\left(\mu \log^2 \Sigma_k\right) \qquad \text{(A15-16)}$$

A15.4 Third Order

The effect of $\underline{r}_3$ is vanishingly small since $\alpha<1/2$.

## A16 INTERMEDIATE FORM OF THE INNER SOLUTION

### A16.1 Zeroth Order

From (A13-5)

$$\log Q_k S_k = \log \Sigma_k - \log \mu_k \qquad \text{(A16-1)}$$

(A12-58) can be written

$$\frac{M_k \underline{R}_{ko}}{\mu} = \mu^{\alpha-2} M_k^{\alpha} \underline{A}_{ko} \sigma_k + \mu^{-1} M_k \underline{B}_{ko} (\log \Sigma_k - \log \mu_k)$$

$$+ \mu^{-1} M_k \underline{C}_{ko} + \mu^{-\alpha} M_k^{2-\alpha} \underline{D}_{ko} \frac{(\log \Sigma_k - \log \mu_k)}{\sigma_k}$$

$$+ \mu^{-\alpha} M_k^{2-\alpha} \underline{E}_{ko} \frac{1}{\sigma_k} + O\left(\mu^{1-2\alpha} \sigma_k^{-2} \log^2 \Sigma_k\right) \qquad \text{(A16-2)}$$

A16.2 Second Order

(A12-81) can be written

$$\mu M_k^3 \underline{R}_{k2} = \mu^{3\alpha-2} M_k^{3\alpha} \underline{A}_{k2} \sigma_k^3$$

$$+ \mu^{2\alpha-1} M_k^{1+2\alpha} \underline{B}_{k2} (\log \Sigma_k - \log \mu_k) \sigma_k^2$$

$$+ \mu^{2\alpha-1} M_k^{1+2\alpha} \underline{C}_{k2} \sigma_k^2$$

$$+ \mu^{\alpha} M_k^{2+\alpha} \underline{D}_{k2} \left(\log^2 \Sigma_k - 2 \log \mu_k \log \Sigma_k\right.$$

$$\left. + \log^2 \mu_k\right) \sigma_k + \mu^{\alpha} M_k^{2+\alpha} \underline{E}_{k2} (\log \Sigma_k$$

$$- \log \mu_k) \sigma_k + \mu^{\alpha} M_k^{2+\alpha} \underline{F}_{k2} \sigma_k \qquad \text{(A16-3)}$$

A16.3 Third Order

(A12-96) can be written

$$\mu^2 M_k^4 \underline{R}_{k3} = \mu^{4\alpha-2} M_k^{4\alpha} \underline{A}_{k3} \sigma_k^4 \qquad \text{(A16-4)}$$

A16.4 Fourth Order

The effect of $\underline{R}_{k4}$ is vanishingly small since $\alpha<2/5$.

## A17 RESULTS OF THE MATCHING

From each of the terms in (A14-2) similar functions of $\sigma_k$ can be collected from (A15-5), (A15-11), (A15-16) (A16-2), (A16-3) and (A16-4). Since the limit in (A14-2) must be satisfied independently of the value of $\sigma_k$ the coefficient of each function of $\sigma_k$ must vanish. The following results are then obtained.

(a) Terms proportional to $\sigma_k$:

$$\mu^{\alpha-2} M_k^{\alpha} \underline{V}_k - \mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{p}_k^* + \frac{1}{2}\mu^{\alpha} M_k^{2+\alpha} \tau_k^2 G_k \underline{V}_k$$

$$+ \mu^{\alpha-1} M_k^{\alpha} \underline{d}_{1k} + \mu^{\alpha} M_k^{\alpha} \underline{e}_{1k} + \mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{f}_{1k}$$

$$+ 2\mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{g}_{1k} + \mu^{\alpha} M_k^{\alpha} \underline{g}_{2k} + O\left(\mu^{1+\alpha}\right)$$

$$-\mu^{\alpha-2} M_k^{\alpha} \underline{A}_{ko} - \mu^{\alpha} M_k^{2+\alpha} \underline{D}_{k2} \log^2 \mu_k$$

$$-\mu^{\alpha} M_k^{2+\alpha} \underline{E}_{k2} \log \mu_k - \mu^{\alpha} M_k^{2+\alpha} \underline{F}_{k2} = O \tag{A17-1}$$

or

$$\underline{A}_{ko} + \mu^2 M_k^2 \left(\underline{D}_{k2} \log^2 \mu_k - \underline{E}_{k2} \log \mu_k + \underline{F}_{k2}\right)$$

$$= \underline{V}_k + \mu \underline{d}_{1k} + \mu^2 \left(\underline{e}_{1k} - M_k \tau_k \underline{p}_k^* + \frac{1}{2} M_k^2 \tau_k^2 G_k \underline{V}_k \right.$$

$$\left. + M_k \tau_k \underline{f}_{1k} + 2M_k \tau_k \underline{g}_{1k} + \underline{g}_{2k}\right) + O(\mu^3) \tag{A17-2}$$

(b) Terms proportional to $\log \Sigma_k$:

$$\mu^{-1} \underline{a}_{1k} + \underline{c}_{1k} + O(\mu) - \mu^{-1} M_k \underline{B}_{ko} = O \tag{A17-3}$$

or

$$\underline{B}_{ko} = M_k^{-1} \underline{a}_{1k} + \mu M_k^{-1} \underline{c}_{2k} + O(\mu^2) \tag{A17-4}$$

(c) Constant terms:

$$\mu^{-2} (\underline{a}_{ko} - \underline{p}_{ko}) + \mu^{-1} M_k \tau_k \underline{V}_k E \mu^{-1} \underline{b}_{1k} + M_k \tau_k \underline{d}_{1k}$$

$$+ \underline{d}_{2k} + O(\mu) + \mu^{-1} M_k \underline{B}_{ko} \log \mu_k - \mu^{-1} M_k \underline{C}_{ko} = O \tag{A17-5}$$

or

$$\underline{C}_{ko} - \underline{B}_{ko} \log \mu_k = \mu^{-1}\left(\underline{a}_{ok} - \underline{p}_{ko}\right) + \tau_k \underline{V}_k + M_k^{-1} \underline{b}_{1k}$$

$$+ \mu\left(\tau_k \underline{d}_{1k} + M_k^{-1} \underline{d}_{2k}\right) + O(\mu^2) \tag{A17-6}$$

It is clear that in order to balance the left and right hand sides of (A17-6) that the order $\mu^{-1}$ term must vanish, i.e.,

$$\underline{a}_{ok} - \underline{p}_{ko} = O \tag{A17-7}$$

(A17-7) is equivalent to (A10-3) and (A11-22) and verifies the earlier assumption. Then (A17-6) becomes.

$$\underline{C}_{ko} = \underline{B}_{ko} \log \mu_k + \tau_k \underline{V}_k + M_k^{-1} \underline{b}_{1k}$$

$$+ \mu\left(\tau_k \underline{d}_{1k} + M_k^{-1} \underline{d}_{2k}\right) + O(\mu^2) \tag{A17-8}$$

(d) Terms proportional to $\sigma_k^{-1} \log \Sigma_k$:

$$\mu^{-\alpha} M_k^{-\alpha} \underline{a}_{2k} + O(\mu^{-\alpha}) - \mu^{-\alpha} M_k^{2-\alpha} \underline{D}_{ko} = O \tag{A17-9}$$

or

$$\underline{D}_{ko} = M_k^{-2} \underline{a}_{2k} + O(\mu) \tag{A17-10}$$

(e) Terms proportional to $\sigma_k^{-1}$:

$$\mu^{-\alpha} M_k^{1-\alpha} \tau_k \underline{a}_{1k} + \mu^{-\alpha} M_k^{-\alpha} \underline{b}_{2k} + O(\mu^{1-\alpha})$$

$$+ \mu^{-\alpha} M_k^{2-\alpha} \underline{D}_{ko} \log \mu_k - \mu^{-\alpha} M_k^{2-\alpha} \underline{E}_{ko} = O \tag{A17-11}$$

or

$$\underline{E}_{ko} = \underline{D}_{ko} \log \mu_k + M_k^{-1} \tau_k \underline{a}_{1k} + M_k^{-2} \underline{b}_{2k} + O(\mu) \tag{A17-12}$$

(f) Terms proportional to $\sigma_k^3$:

$$\frac{1}{6} \mu^{3\alpha-2} M_k^{3\alpha} G_k \underline{V}_k + O(\mu^{3\alpha-1}) - \mu^{3\alpha-2} M_k^{3\alpha} \underline{A}_{k2} = O \tag{A17-13}$$

or

$$\underline{A}_{k2} = \frac{1}{6} G_k \underline{V}_k + O(\mu) \tag{A17-14}$$

(g) Terms proportional to $\sigma_k^2 \log \Sigma_k$:

$$\mu^{2\alpha-1} M_k^{2\alpha} \underline{f}_{1k} + O(\mu^{2\alpha}) - \mu^{2\alpha-1} M_k^{1+2\alpha} \underline{B}_{k2} = O \tag{A17-5}$$

or

$$\underline{B}_{k2} = M_k^{-1} \underline{f}_{1k} + O(\mu) \tag{A17-16}$$

(h) Terms proportional to $\sigma_k^2$:

$$- \frac{1}{2} \mu^{2\alpha-1} M_k^{2\alpha} \underline{p}_k^* + \frac{1}{2} \mu^{2\alpha-1} M_k^{1+2\alpha} \tau_k G_k \underline{V}_k$$

$$+ \mu^{2\alpha-1} M_k^{2\alpha} \underline{g}_{1k} + O(\mu^{2\alpha}) + \mu^{2\alpha-1} M_k^{1+2\alpha} B_k \log \mu_k$$

$$- \mu^{2\alpha-1} M_k^{1+2\alpha} \underline{C}_{k2} = O \qquad \text{(A17-17)}$$

or

$$\underline{C}_{k2} = \underline{B}_{k2} \log \mu_k - \frac{1}{2} M_k^{-1} \underline{p}_k^* + \frac{1}{2} \tau_k G_k \underline{V}_k$$

$$+ M_k^{-1} \underline{g}_{1k} + O(\mu) \qquad \text{(A17-18)}$$

(i) Terms proportional to $\sigma_k \log^2 \Sigma_k$:

$$\mu^{\alpha} M_k \underline{e}_{2k} + O(\mu^{1+\alpha}) - \mu^{\alpha} M_k^{2+\alpha} \underline{D}_{k2} = O \qquad \text{(A17-19)}$$

or

$$\underline{D}_{k2} = M_k^{-2} \underline{e}_{2k} + O(\mu) \qquad \text{(A17-20)}$$

(j) Terms proportional to $\sigma_k \log \Sigma_k$:

$$\mu^{\alpha} M_k^{\alpha} \underline{c}_{1k} + 2\mu^{\alpha} M_k^{1+\alpha} \tau_k \underline{f}_{1k} + \mu^{\alpha} M_k^{\alpha} \underline{f}_{2k}$$

$$+ O(\mu^{1+\alpha}) + 2\mu^{\alpha} M_k^{2+\alpha} \underline{D}_{k2} \log \mu_k$$

$$- \mu^{\alpha} M_k^{2+\alpha} \underline{E}_{k2} = O \qquad \text{(A17-21)}$$

or

$$\underline{E}_{k2} = 2\,\underline{D}_{k2} \log \mu_k + M_k^{-2}\,\underline{c}_{1k} + 2\,M_k^{-1}\,\tau_k\,\underline{f}_{1k}$$

$$+ M_k^{-2}\,\underline{f}_{2k} + O(\mu) \qquad \text{(A17-22)}$$

(k) Terms proportional to $\sigma_k^4$:

$$\frac{1}{24}\left[\mu^{4\alpha-2}\,M_k^{4\alpha}\,\underline{H}_k\,\underline{v}_o^2\,(t_k) - \mu^{4\alpha-2}\,M_k^{4\alpha}\,\underline{H}_k\,\dot{\underline{p}}_k^2\,(t_k)\right]$$

$$+ O(\mu^{4\alpha-1}) - \mu^{4\alpha-2}\,M_k^{4\alpha}\,\underline{A}_{k3} = O \qquad \text{(A17-23)}$$

or

$$\underline{A}_{k3} = \underline{H}_k\left[\underline{v}_o^2\,(t_k) - \dot{\underline{p}}_k^2\,(t_k)\right]\Big/24 + O(\mu) \qquad \text{(A17-24)}$$

(1) The terms not accounted for so far are of the following orders:

$\mu^{4\alpha-1}$, $\mu^{1-2\alpha}$, $\mu$

These terms vanish identically in the limit $\mu \to 0$.

## A18 SOLUTION OF THE INITIAL VALUE PROBLEM

The expressions presented in Section A18 contain the solution to the following problem: given a set of initial conditions at $t = t_o$ what are the parameters defining a close approach of the kth body at or near $t = t_k$?

$$\underline{r}(t_o) = \underline{r}_o(t_o) + \mu\underline{r}_1(t_o) + \mu^2\,\underline{r}_2(t_o) \qquad \text{(A6-13)}$$

$$\underline{v}(t_o) = \underline{v}_o(t_o) + \mu\,\underline{v}_1(t_o) + \mu^2\,\underline{v}_2(t_o) \qquad \text{(A6-14)}$$

Given these constants plus $t_o$ and $t_k$ defines all the constants of the outer solution (including integral constants).

The constants of the inner solution are the vectors $\underline{R}_{ko}(S_{ko})$ and $\underline{V}_{ko}(S_{ko})$ which appear in (A9-1). The higher order terms are set to zero in (A9-6) meaning that at $S_k = S_{ko}$ the zeroth order inner solution (an hyperbola) and the second order inner solution (a perturbed hyperbola) are equivalent. Therefore the complete inner solution is determined at $S_{ko}$ if the corresponding zeroth order hyperbola is known.

Rather than determining $\underline{R}_{ko}(S_{ko})$ and $\underline{V}_{ko}(S_{ko})$ as functions of $\underline{r}(t_o)$ and $\underline{v}(t_o)$ it is more convenient to determine the elements of the zeroth order hyperbola. They are

$$\left.\begin{array}{lll} \bar{a} & - & \text{semi-major axis} \\ \bar{e} & - & \text{eccentricity} \\ \bar{i}_k & - & \text{inclination} \\ \bar{\Omega}_k & - & \text{argument of the ascending node} \\ \bar{\omega}_k & - & \text{argument of pericenter} \\ t_{pk} & - & \text{time of pericenter passage.} \end{array}\right\} \tag{A18-1}$$

Usually it will be desired to determine the inner solution at closest approach. Then $S_{ko} = 0$ and the elements of the zeroth order hyperbola can be used directly to calculate the closest approach distance, etc.

In order to determine the relationship between the set of elements (A18-1) and the initial position and velocity it is necessary to rewrite (A17-2) and (A17-8) in a new form. From Battin[5] the unit normal to the plane of motion is

$$\underline{N}_k = (\sin\bar{\Omega}_k \sin\bar{i}_k, \; -\cos\bar{\Omega}_k \sin\bar{i}_k, \; \cos\bar{i}_k) \tag{A18-2}$$

The impact parameter vector is defined as the vector normal to $\underline{V}_{\infty k}$ with magnitude equal to the semi-minor axis $\bar{b}$ where

$$\bar{b}_k = \bar{a}_k\left(\bar{e}_k^2 - 1\right)^{1/2} \tag{A18-3}$$

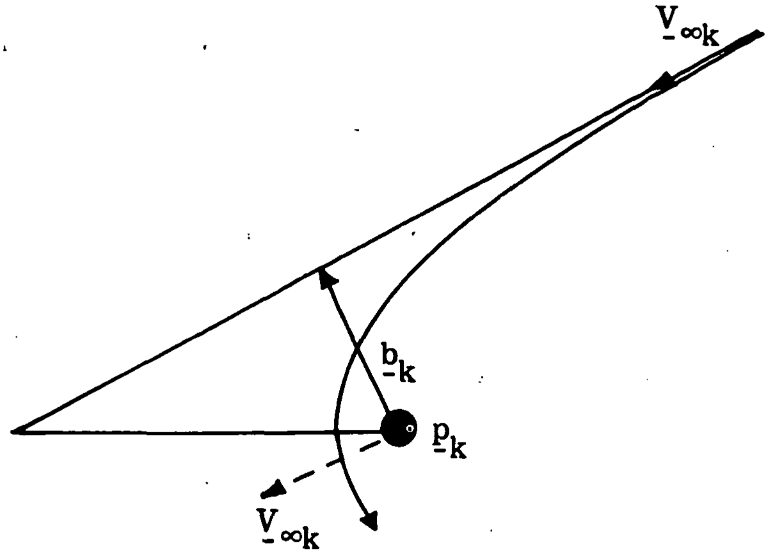


**Figure A5. Impact Parameter Vector**

For the trajectory shown in Figure A5 the unit normal $\underline{N}_k$ is out of the plane such that $\underline{b}_k$, $\underline{V}_{\infty k}$ and $\underline{N}_k$ form a right handed system.[1] Then

$$\underline{b}_k = \bar{b}_k \left( \frac{\underline{V}_{\infty k}}{V_{\infty k}} \right) \times \underline{N}_k \qquad \text{(A18-4)}$$

Substituting (A12-54) and (A18-2) into (A18-4) and then using (A12-38) - (A12-39), (A12-48) - (A12-53) and (A12-55) eventually gives

$$\underline{b}_k = \underline{L}_k + \frac{Q_k}{\bar{n}_k} \underline{V}_{\infty k} \qquad \text{(A18-5)}$$

where

$$\bar{n}_k = \bar{a}_k^{-3/2} \qquad \text{(A18-6)}$$

Then (A12-61) becomes

$$\underline{C}_{ko} = \frac{Q_k}{\bar{n}_k} \left[ \log \left( \frac{2\bar{n}_k}{\bar{e}_k} \right) - 1 \right] \underline{V}_{\infty k} + \underline{b}_k \qquad \text{(A18-7)}$$

The expressions for $\underline{A}_{ko}$ and $\underline{C}_{ko}$, (A17-2) and (A17-8) become

$$\underline{V}_{\infty k} = \underline{V}_k + \mu \underline{d}_{1k} + \mu^2 \Big[ \underline{e}_{1k} + \underline{g}_{2k} + M_k \tau_k \Big( \underline{f}_{1k} + 2\underline{g}_{1k}$$

$$- \underline{p}_k^* \Big) + \frac{1}{2} M_k^2 \tau_k^2 G_k \underline{V}_k + M_k^2 \Big( \underline{D}_{k2} \log^2 \mu_k$$

$$- \underline{E}_{k2} \log \mu_k + \underline{F}_{k2} \Big) \Big] + O(\mu^3) \tag{A18-8}$$

$$\underline{b}_k - \tau_k \underline{V}_{\infty k} = \frac{Q_k}{\bar{n}_k} \left[ \log \left( \frac{2\bar{n}_k}{\mu_k \bar{e}_k} \right) - 1 \right] \underline{V}_{\infty k}$$

$$+ M_k^{-1} (\underline{b}_{1k} + \mu \underline{d}_{2k}) + O(\mu^2) \tag{A18-9}$$

Since (A18-9) is only accurate to order $\mu$ the $\mu^2$ term in $\underline{V}_{\infty k}$ can temporarily be ignored. Also, since $\underline{b}_k$ is normal to $\underline{V}_{\infty k}$, taking the inner product of (A18-9) with $\underline{V}_{\infty k}$ gives

$$\tau_k = \frac{Q_k}{\bar{n}_k} \left[ \log \left( \frac{2\bar{n}_k}{\mu_k \bar{e}_k} \right) - 1 \right] - \frac{\bar{a}_k}{M_k} (\underline{b}_{1k} + \mu \underline{d}_{2k}) \cdot \underline{V}_{\infty k} + O(\mu^2) \tag{A18-10}$$

Subtracting this result from (A18-9) gives

$$\underline{b}_k = \frac{1}{M_k} \left( I - \frac{V_{\infty k} \underline{V}_{\infty k}^T}{V_{\infty k}^2} \right) (\underline{b}_{1k} + \mu \underline{d}_{2k}) + O(\mu^2) \tag{A18-11}$$

where I is the unit diagonal matrix and $\underline{V}_{\infty k}^T$ is the transpose of $\underline{V}_{\infty k}$. The transpose is introduced through the identity

$$(\underline{x} \cdot \underline{y}) \underline{y} \equiv (\underline{y}\, \underline{y}^T) \underline{x} \tag{A18-12}$$

The solution is now possible. The first two terms of (A18-8) are used to evaluate $\underline{V}_{\infty k}$. Then

$$\bar{a}_k = 1 \big/ V_{\infty k}^2 \tag{A18-13}$$

and $\bar{n}_k$ is found from (A18-6). Next (A18-11) is used to determine $\underline{b}_k$ and

$$\bar{b}_k = \left|\underline{b}_k\right| \tag{A18-14}$$

$$\bar{e}_k = \left(1 + \bar{b}_k^2/\bar{a}_k^2\right)^{1/2} \tag{A18-15}$$

Since $\bar{\Omega}_k$ and $\bar{i}_k$ are as yet unknown, the unit normal is determined from

$$\underline{N}_k = (\underline{b}_k \times \underline{V}_{\infty k})/(\bar{b}_k V_{\infty k}) \tag{A18-16}$$

Then from (A18-2)

$$\cos \bar{i}_k = \underline{N}_k \cdot \underline{e}_3, \quad 0 \leq \bar{i}_k \leq \pi \tag{A18-17}$$

$$\sin \bar{\Omega}_k = (\underline{N}_k \cdot \underline{e}_1)/\sin \bar{i}_k \tag{A18-18}$$

$$\cos \bar{\Omega}_k = -(\underline{N}_k \cdot \underline{e}_2)/\sin \bar{i}_k \tag{A18-19}$$

Finally, inverting (A12-48) and (A12-49) gives

$$\bar{U}_k' = \bar{U}_k \cos \bar{\Omega}_k + \bar{V}_k \sin \bar{\Omega}_k \tag{A18-20}$$

$$\bar{V}_k' = (\bar{V}_k \cos \bar{\Omega}_k - \bar{U}_k \cos \bar{\Omega}_k)/\cos \bar{i}_k \tag{A18-21}$$

and using (A12-27), (A12-28), (A12-38) and (A12-39) gives

$$\sin \bar{\omega}_k = -\frac{\bar{a}_k^{1/2}}{\bar{e}_k}\left[\left(\bar{e}_k^2 - 1\right)^{1/2} \bar{U}_k' + Q_k \bar{V}_k'\right] \tag{A18-22}$$

$$\cos \bar{\omega}_k = \frac{\bar{a}_k^{1/2}}{\bar{e}_k}\left[\left(\bar{e}_k^2 - 1\right)^{1/2} \bar{V}_k' - Q_k \bar{U}_k'\right] \tag{A18-23}$$

Enough information is now available to evaluate (A12-94) and this, along with (A18-10), allows the order $\mu^2$ term in (A18-8) to be evaluated.

The time of pericenter passage is found from (A13-2), i.e.,

$$t_{pk} = t_k + \mu \tau_k \tag{A18-24}$$

The complete set of orbital elements defining the zeroth order hyperbola can be evaluated from the expressions for $\underline{A}_{ko}$ and $\underline{C}_{ko}$ obtained through the matching. The remaining expressions obtained from the matching in Section A17 give redundant information which can be used as an analytical check on the matching process. For example, from (A12-60)

$$\underline{B}_{ko} = - Q_k \underline{f}(\underline{V}_{\infty k}) \tag{A18-25}$$

After some manipulation, (A17-4) can be reduced to

$$\underline{B}_{ko} = - Q_k \left[ \underline{f}(\underline{V}_k) + \mu \, G(\underline{V}_k) \, \underline{d}_{1k} + O(\mu^2) \right] \tag{A18-26}$$

Equating (A18-25) and (A18-26) gives

$$\underline{f}(\underline{V}_{\infty k}) = \underline{f}(\underline{V}_k) + \mu \, G(\underline{V}_k) \, \underline{d}_{1k} + O(\mu^2) \tag{A18-27}$$

Comparing (A18-27) with (A2-13) and (A2-14) leads to

$$\underline{V}_{\infty k} = \underline{V}_k + \mu \, \underline{d}_{1k} + O(\mu^2) \tag{A18-28}$$

Since (A18-28) is contained in (A18-8) no new information has been obtained.

As another example, (A12-62) gives

$$\underline{D}_{ko} = \underline{V}_{\infty k} / V_{\infty k}^6 \tag{A18-29}$$

while (A17-10) can be reduced to

$$\underline{D}_{ko} = \dot{\underline{V}}_k \Big/ V_k^6 + O(\mu) \tag{A18-30}$$

using the identity

$$G(\underline{V}_k)\, \underline{V}_k \equiv 2\underline{V}_k \Big/ V_k^3 \tag{A18-31}$$

Comparing (A18-29) and (A18-30) gives

$$\underline{V}_{\infty k} = \underline{V}_k + O(\mu) \tag{A18-32}$$

a result which is also contained in (A18-8).

It can also be shown that $\underline{E}_{ko}$ reproduces information contained in $\underline{C}_{ko}$. The proof involves replacing $\underline{L}_k$ by $\underline{b}_k$ using (A18-5) in (A12-61) and (A12-63) then taking scalar products with $\underline{V}_{\infty k}$ and $\underline{b}_k$. The algebra is long and tedious and requires certain identities such as

$$G(\underline{x})\, \underline{y} \cdot \underline{x} \equiv \frac{2}{x^3} (\underline{x} \cdot \underline{y}) \tag{A18-33}$$

Since the algebra is so lengthy and complex no further proofs will be given here but only the statement that such proofs have been worked out and the matching appears to give a consistent set of equations relating the constants of motion of the inner and outer solutions.

Although the notation is different the results obtained by Carlson are contained in (A18-8) - (A18-11) except that he has one order less accuracy in each expression. In terms of a common notation the results are identical to first order for $\underline{V}_{\infty k}$ and to zeroth order for $\tau_k$ and $\underline{b}_k$. The idea of decomposing (A18-9) into orthogonal components to solve for $\tau_k$ and $\underline{b}_k$ independently comes from Carlson who introduced $\underline{b}_k$ directly into the development of the inner solution prior to matching. In Section A12 the vector $\underline{L}_k$ was introduced and, from (A18-5),

$$\underline{L}_k = \underline{b}_k - \frac{Q_k}{\overline{n}_k} \underline{V}_{\infty k} \tag{A18-34}$$

The representation of $\underline{L}_k$ in terms of $\underline{b}_k$ and $\underline{V}_{\infty k}$ is certainly advantageous for solving the initial value problem, as presented in this section, but it is not unique. Other representations may also be useful, in particular for two-point boundary value problems. Therefore the solution developed in this study and contained in the expressions for $\underline{A}_{ko}$ and $\underline{C}_{ko}$ in the preceding section is somewhat more general than that of Carlson as well as being of a higher order. The development of the boundary value solution from $\underline{A}_{ko}$ and $\underline{C}_{ko}$ appears in Section B.

## Section B
## SECOND ORDER TWO-POINT BOUNDARY VALUE SOLUTIONS

### B1 FUNDAMENTAL SOLUTION

The fundamental relationships between the constants of the outer and inner solutions in the second order asymptotic solution of the problem of N bodies were derived in Section A17. In Section A18 it was shown how these equations can be used to formulate an initial value solution. In this section it will be shown how these same equations are used to formulate certain boundary value solutions of practical interest.

The fundamental equations resulting from the matching were shown in Section A17 to be (A17-2) and (A17-8). They can be rewritten as

$$\underline{A}_{ko} = C(t_k, t_o)\left[\mu \underline{r}_1(t_o) + \mu^2 \underline{r}_2(t_o)\right] + D(t_k, t_o)\left[\mu \underline{v}_1(t_o) + \mu^2 \underline{v}_2(t_o)\right]$$

$$+ \underline{V}_k + \mu \underline{d}^*_{1k} + \mu^2\left[\underline{g}^*_{2k} + \underline{e}_{1k} - \underline{A}^*_{k2} + M_k \tau_k(\underline{f}_{1k} + 2\underline{g}_{1k} - \underline{p}^*_k)\right.$$

$$\left. + \frac{1}{2} M_k^2 \tau_k^2 G_k \underline{V}_k\right] + 0(\mu^3) \qquad \text{(B1-1)}$$

$$M_k \underline{C}_{ko} - M_k\left(\tau_k + \frac{Q_k}{A^3_{ko}} \log \mu_k\right)\underline{A}_{ko} = A(t_k, t_o)\left[\underline{r}_1(t_o) + \mu \underline{r}_2(t_o)\right]$$

$$+ B(t_k, t_o)\left[\underline{v}_1(t_o) + \mu \underline{v}_2(t_o)\right]$$

$$+ \underline{b}^*_{1k} + \mu \underline{d}^*_{2k} + 0(\mu^2) \qquad \text{(B1-2)}$$

where

$$\underline{b}_{1k}^* = \underline{b}_{1k} - A(t_k, t_o)\, \underline{r}_1(t_o) - B(t_k, t_o)\, \underline{v}_1(t_o) \tag{B1-3}$$

$$\underline{d}_{1k}^* = \underline{d}_{1k} - C(t_k, t_o)\, \underline{r}_1(t_o) - D(t_k, t_o)\, \underline{v}_1(t_o) \tag{B1-4}$$

$$\underline{d}_{2k}^* = \underline{d}_{2k} - A(t_k, t_o)\, \underline{r}_2(t_o) - B(t_k, t_o)\, \underline{v}_2(t_o) \tag{B1-5}$$

$$\underline{g}_{2k}^* = \underline{g}_{2k} - C(t_k, t_o)\, \underline{r}_2(t_o) - D(t_k, t_o)\, \underline{v}_2(t_o) \tag{B1-6}$$

$$\underline{A}_{k2}^* = M_k^2 \left( \underline{D}_{k2} \log^2 \mu_k - \underline{E}_{k2} \log \mu_k + \underline{F}_{k2} \right) \tag{B1-7}$$

Given initial conditions along the outer solution, (B1-1) and B1-2) give the values of $\underline{A}_{ko}$, $\underline{C}_{ko}$ and $\tau_k$, the constants of the inner solution. However, using (A3-13) and (A3-31), (B1-1) and (B1-2) may be inverted to give the six component state vector.

$$\begin{pmatrix} \underline{r}_1(t_o) + \mu \underline{r}_2(t_o) \\ \underline{v}_1(t_o) + \mu \underline{v}_2(t_o) \end{pmatrix} = \Phi(t_o, t_k) \left\{ \begin{pmatrix} M_k \underline{C}_{ko} - M_k \left( \tau_k + \dfrac{Q_k}{A_{ko}^3} \log \mu_k \right) \underline{A}_{ko} \\ (\underline{A}_{ko} - \underline{V}_k)/\mu \end{pmatrix} + \begin{pmatrix} \underline{\gamma}_k \\ \underline{\delta}_k \end{pmatrix} + \mu \begin{pmatrix} \underline{\zeta}_k \\ \underline{\eta}_k \end{pmatrix} \right\} \tag{B1-8}$$

where

$$\underline{\gamma}_k = -\underline{b}_{1k}^* \tag{B1-9}$$

$$\underline{\delta}_k = -\underline{d}_{1k}^* \tag{B1-10}$$

$$\underline{\zeta}_k = -\underline{d}_{2k}^* \tag{B1-11}$$

$$\underline{\eta}_k = -\underline{g}^*_{2k} - \underline{e}_{1k} + \underline{A}^*_{k2} - M_k\tau_k\left(\underline{f}_{1k} + 2\underline{g}_{1k} - \underline{p}^*_k + \frac{1}{2} M_k\tau_k G_k \underline{V}_k\right) \tag{B1-12}$$

Given initial conditions along the inner solution (B1-8) gives the perturbations to the zeroth order ellipse at $t = t_o$. Thus (B1-8) is an inverted form of (B1-1) and (B1-2). Either form may be considered as the fundamental solution from the matching.

## B2 ASYMPTOTIC BOUNDARY VALUE SOLUTIONS

The fundamental solution from the matching is essentially an initial value solution. However, with a little algebraic manipulation the solution can be transformed into a boundary value solution where some variables are prescribed at each end of the trajectory and other variables at each end are dependent on the prescribed values. The goal then is to write (B1-8) in such a form that the dependent variables appear as explicit functions of the independent variables with the independent variables chosen to represent a realistic boundary value problem.

It is first necessary to derive some expressions for the inner hyperbola. Suppose the excess velocity, $\underline{V}_{\infty k}$, the inclination, $\bar{i}_k$, the pericenter radius, $\bar{\rho}_k$, and the time of pericenter passage, $t_{pk}$, are assumed to be known for a close approach to the $k^{th}$ body. The excess velocity has cartesian components defined by

$$\underline{V}_{\infty k} = (\overline{U}_k, \overline{V}_k, \overline{W}_k) \tag{B2-1}$$

The vector $\underline{V}_{\infty k}$ in the $(\bar{x}, \bar{y}, \bar{z})$ coordinate system is shown in Figure B1.

From Figure B1

$$\sin \bar{\lambda}_k = \overline{V}_k \Big/ \left(\overline{U}_k^2 + \overline{V}_k^2\right)^{1/2} \tag{B2-2}$$

$$\cos \bar{\lambda}_k = \overline{U}_k \Big/ \left(\overline{U}_k^2 + \overline{V}_k^2\right)^{1/2} \tag{B2-3}$$

$$\tan \bar{\alpha}_k = \overline{W}_k \Big/ \left(\overline{U}_k^2 + \overline{V}_k^2\right)^{1/2}, \quad -\frac{\pi}{2} \le \alpha_k \le \frac{\pi}{2} \tag{B2-4}$$

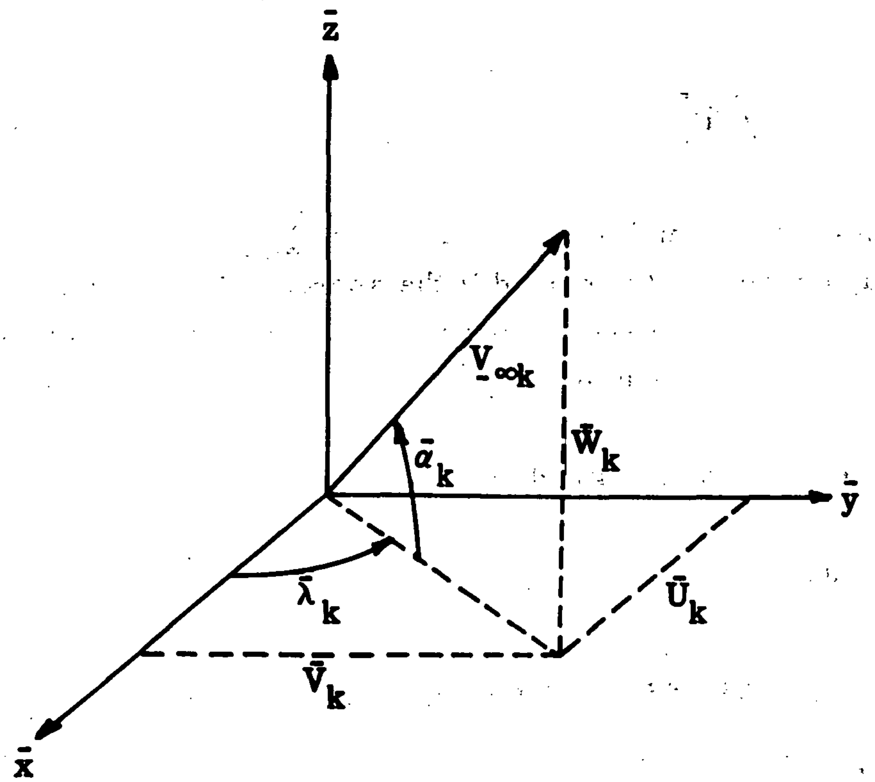


**Figure B1. Hyperbolic Excess Velocity $\underline{V}_{\infty k}$**

Since the plane of motion must contain $\underline{V}_{\infty k}$, the ascending node must satisfy (cf Battin[5], P. 179)

$$\overline{\Omega}_k = \overline{\lambda}_k \pm \overline{\sigma}_k + (1 \pm 1)\,\pi/2 \tag{B2-5}$$

where

$$\overline{\sigma}_k = \tan \overline{\alpha}_k / \tan \overline{i}_k\,, \quad 0 \le \overline{\sigma}_k \le \pi/2 \tag{B2-6}$$

With some manipulation (B2-5) becomes

$$\cos \overline{\Omega}_k = \left(\overline{U}_k^2 + \overline{V}_k^2\right)^{-1} \operatorname{ctn} \overline{i}_k \left\{ \overline{V}_k \overline{W}_k \mp \overline{U}_k \left[\left(\overline{U}_k^2 + \overline{V}_k^2\right) \tan^2 \overline{i}_k - \overline{W}_k^2\right]^{\frac{1}{2}} \right\} \tag{B2-7}$$

$$\sin \overline{\Omega}_k = -\left(\overline{U}_k^2 + \overline{V}_k^2\right)^{-1} \operatorname{ctn} \overline{i}_k \left\{ \overline{U}_k \overline{W}_k \pm \overline{V}_k \left[\left(\overline{U}_k^2 + \overline{V}_k^2\right) \tan^2 \overline{i}_k - \overline{W}_k^2\right]^{\frac{1}{2}} \right\} \tag{B2-8}$$

It is obvious that

$$\tan^2 \bar{i}_k \geq \bar{W}_k^2 \Big/ (\bar{U}_k^2 + \bar{V}_k^2) \qquad \text{(B2-9)}$$

giving a minimum $\bar{i}_k$ which is compatible with $\underline{V}_{\infty k}$. In (B2-5), (B2-7) and (B2-8) the upper sign is to be used if the approach or departure is to be over the kth body ($\underline{b}_k \cdot \underline{e}_3 > 0$) and the lower sign is to be used if the approach or departure is under the body ($\underline{b}_k \cdot \underline{e}_3 < 0$).

The pericenter radius is defined as

$$\bar{\rho}_k = \bar{a}_k(\bar{e}_k - 1) \qquad \text{(B2-10)}$$

where the semi-major axis is given by

$$\bar{a}_k = V_{\infty k}^{\;-2} \qquad \text{(B2-11)}$$

Then the eccentricity is

$$\bar{e}_k = 1 + \bar{\rho}_k V_{\infty k}^{\;2} \qquad \text{(B2-12)}$$

From (A18-22) and (A18-23)

$$\sin \bar{\omega}_k = -\bar{a}_k^{\,1/2} \left[\left(\bar{e}_k^2 - 1\right)^{1/2} \bar{U}_k' + Q_k \bar{V}_k'\right] \Big/ \bar{e}_k \qquad \text{(B2-13)}$$

$$\cos \bar{\omega}_1 = \bar{a}_k^{-1/2} \left[\left(\bar{e}_k^2 - 1\right)^{1/2} \bar{V}_k' - Q_k \bar{U}_k'\right] \Big/ \bar{e}_k \qquad \text{(B2-14)}$$

where, from (A18-20) and (A18-21)

$$\bar{U}_k' = \bar{U}_k \cos \bar{\Omega}_k + \bar{V}_k \sin \bar{\Omega}_k \qquad \text{(B2-15)}$$

$$\bar{V}_k' = (\bar{V}_k \cos \bar{\Omega}_k - \bar{U}_k \sin \bar{\Omega}_k) \,/\, \cos \bar{i}_k \qquad \text{(B2-16)}$$

$$= \bar{W}_k / \sin \bar{i}_k \qquad \text{(B2-17)}$$

Equations (B2-7) - (B2-17) define the orbital elements (in addition to $\bar{i}_k$ which is assumed known) for the inner hyperbola. From (A12-27) and (A12-28)

$$A'_k = \bar{a}_k \bar{e}_k \cos \bar{w}_k \tag{B2-18}$$

$$B'_k = \bar{a}_k \bar{e}_k \sin \bar{w}_k \tag{B2-19}$$

and from (A12-51) - (A12-53)

$$\bar{A}_k = A'_k \cos \bar{\Omega}_k - B'_k \sin \bar{\Omega}_k \cos \bar{i}_k \tag{B2-20}$$

$$\bar{B}_k = A'_k \sin \bar{\Omega}_k + B'_k \cos \bar{\Omega}_k \cos \bar{i}_k \tag{B2-21}$$

$$\bar{C}_k = B'_k \sin \bar{i}_k \tag{B2-22}$$

Then, from (A12-55)

$$\underline{L}_k = (\bar{A}_k, \bar{B}_k, \bar{C}_k) \tag{B2-23}$$

The two vector constants of the inner hyperbola are given by (A12-59) and (A12-61), i.e.,

$$\underline{A}_{ko} = \underline{V}_{\infty k} \tag{B2-24}$$

$$\underline{C}_{ko} = \underline{L}_k + \frac{Q_k}{\bar{n}_k} \log\left(\frac{\partial \bar{n}_k}{\bar{e}_k}\right) \underline{V}_{\infty k} \tag{B2-25}$$

where

$$\bar{n}_k = V^3_{\infty k} \tag{B2-26}$$

The equations of this section show that known values of $\underline{V}_{\infty k}$, $\bar{i}_k$ and $\bar{\rho}_k$ are sufficient to define the inner hyperbola and its two constants, $\underline{A}_{ko}$ and $\underline{C}_{ko}$. It is now possible to proceed with the boundary value solution.

## B2.1 Midpoint-to-Target Body Solution

The basic midpoint-to-target body solution is shown in Figure B2. The initial time, $t_o$, and position, $\underline{r}(t_o)$, as well as the pericenter time, $t_{pT}$, radius, $\overline{\rho}_T$, and inclination, $\overline{i}_T$, for a close approach to the target body k = T are all specified. The initial velocity at $t = t_o$ is unknown and must be determined. An ephemeris is required giving positions and velocities of the N-2 perturbing bodies with respect to the primary body. The ephemeris fixes the coordinate system of the asymptotic solution.

The initial conditions for the asymptotic solution are given by (A6-13) and (A6-14). They are

$$\underline{r}(t_o) = \underline{r}_o(t_o) + \mu \underline{r}_1(t_o) + \mu^2 \underline{r}_2(t_o) \tag{B2-27}$$

$$\underline{v}(t_o) = \underline{v}_o(t_o) + \mu \underline{v}_1(t_o) + \mu^2 \underline{v}_2(t_o) \tag{B2-28}$$

From Figure B2

$$\underline{r}_o(t_o) = \underline{r}(t_o) \tag{B2-29}$$

therefore

$$\underline{r}_1(t_o) = \underline{r}_2(t_o) = 0 \tag{B2-30}$$

For simplicity let the initial velocity perturbation be defined by $\underline{\delta v}(t_o)$, i.e.,

$$\delta \underline{v}(t_o) = \underline{v}_1(t_o) + \mu \underline{v}_2(t_o) \tag{B2-31}$$

From Figure B2 the final position of $\underline{r}_o(t)$ is

$$\underline{r}_o(t_T) = \underline{p}_T(t_T) \tag{B2-32}$$

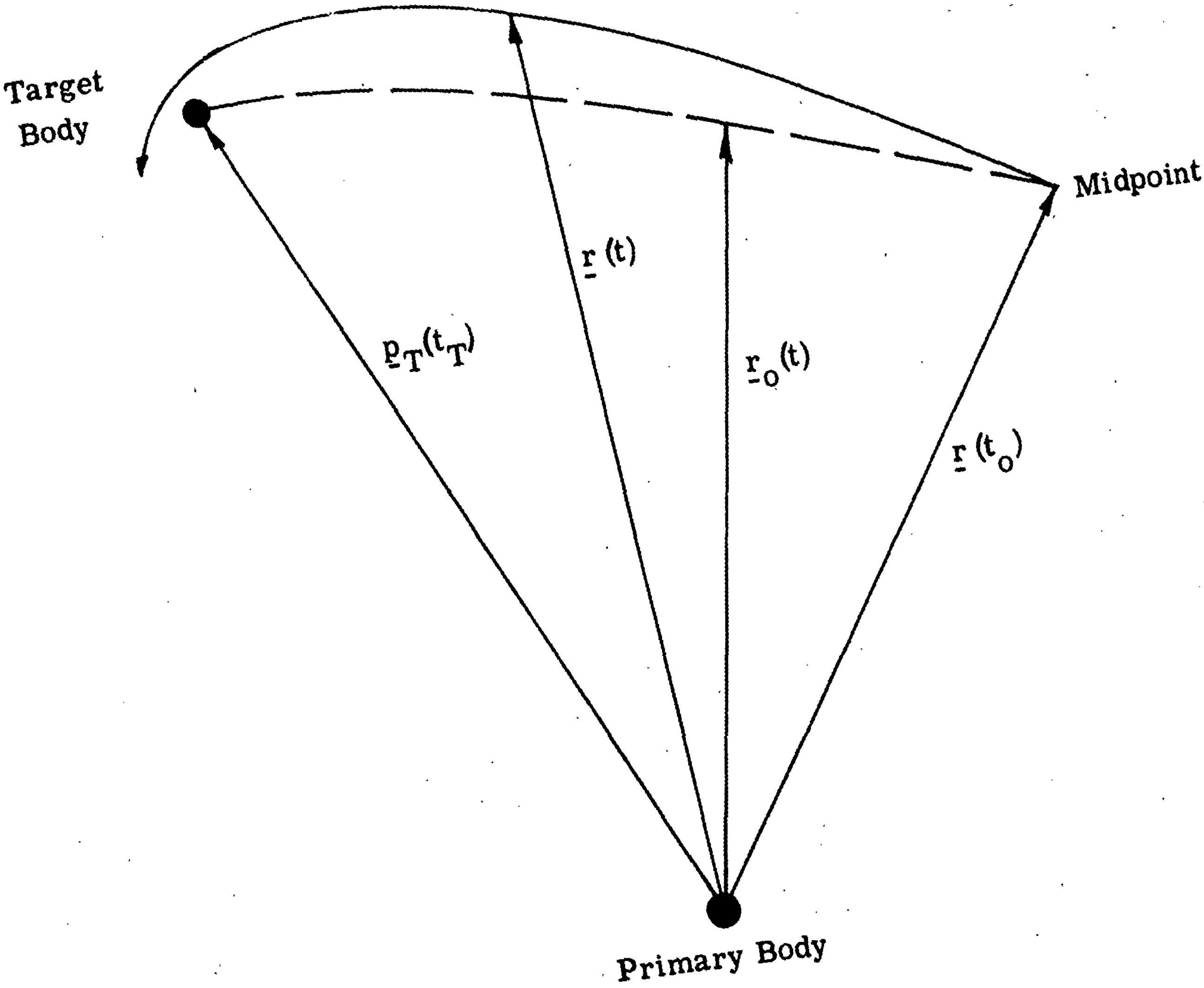


Figure B2. Midpoint-to-Target Body Solution where, from (A13-2)

$$t_T = t_{pT} - \mu_T \tau_T \tag{B2-33}$$

and $\underline{p}_T$ is the position of the target body obtained from the ephemeris. The parameter $\tau_T$ is not prescribed by the boundary conditions and can, without loss of generality, be set equal to zero.

The two position vectors given by (B2-29) and (B2-32) define a Lambert problem and the solution gives $\underline{r}_o(t)$, shown as the dashed line in Figure B2, and the velocities, $\underline{v}_o(t_o)$ and $\underline{v}_o(t_T)$.

Now let k = T and $Q_T = -1$ (cf. (A11-34)) with

$$\mu = \mu_T \tag{B2-34}$$

$$M_T = 1 \tag{B2-35}$$

and substitute (B2-24), (B2-25), (B2-30) and (B2-31) into (B1-8). Using (A3-31) gives

$$B(t_T, t_o)\, \underline{\delta v}(t_o) = \underline{L}_T - \left[\tau_T - \frac{1}{\bar{n}_T} \log\left(\frac{\mu_T \bar{e}_T}{2\bar{n}_T}\right)\right] \underline{V}_{\infty T} + \underline{\gamma}_T + \mu \underline{\zeta}_T \tag{B2-36}$$

$$D(t_T, t_o)\, \underline{\delta v}(t_o) = \mu^{-1}(\underline{V}_{\infty T} - \underline{V}_T) + \underline{\delta}_T + \mu \underline{\eta}_T \tag{B2-37}$$

Solving for $\underline{\delta v}(t_o)$ and $\underline{V}_{\infty T}$ gives

$$\underline{\delta v}(t_o) = B(t_T, t_o)^{-1} \left\{ \underline{L}_T - \left[\tau_T - \frac{1}{\bar{n}_T} \log\left(\frac{\mu_T \bar{e}_T}{2\bar{n}_T}\right)\right] \underline{V}_{\infty T} + \underline{\gamma}_T + \mu \underline{\zeta}_T \right\} \tag{B2-38}$$

$$\underline{V}_{\infty T} = \underline{V}_T + \mu\, D(t_T, t_o)\, \underline{\delta v}(t_o) - \mu \underline{\delta}_T - \mu^2 \underline{\eta}_T \tag{B2-39}$$

These two expressions are not explicit since $\underline{V}_{\infty T}$ appears in (B2-38) and $\delta\underline{v}(t_o)$ appears in (B2-39). However, if (B2-39) is substituted into (B2-38) an expression involving only $\delta\underline{v}(t_o)$ results. Rather than solving such an expression for $\delta\underline{v}(t_o)$ the solution can be obtained as follows:

1. Solve the Lambert problem defined by (B2-29) and (B2-32). This gives $\underline{v}_o(t_o)$ and $\underline{v}_o(t_T)$.

2. From (A11-22)

$$\underline{V}_T = \underline{v}_o(t_T) - \dot{\underline{p}}_T(t_T) \tag{B2-40}$$

   where $\dot{\underline{p}}_T(t_T)$ is obtained from the ephemeris.

3. Let $\underline{V}_{\infty T} = \underline{V}_T$. This gives the zeroth order excess velocity.

4. Evaluate (B2-1) - (B2-26).

5. Evaluate (B2-38) using the results of step 3 and 4 and with $\underline{\zeta}_T = 0$. This gives the first order velocity correction.

6. Evaluate (B2-39) using the results of step 5 and with $\underline{\eta}_T = 0$. This gives the first order excess velocity.

7. Repeat step 4 using the first order excess velocity.

8. Evaluate (B2-38) using the results of steps 6 and 7. This gives the second order velocity correction.

9. Evaluate (B2-39) using the results of step 8. This gives the second order excess velocity.

10. Using the results of Steps 1 and 8 in (B2-28) gives the second order initial velocity

$$\underline{v}(t_o) = \underline{v}_o(t_o) + \mu\delta\underline{v}(t_o) \tag{B2-41}$$

11. Repeat step 4. This gives second order orbital elements at the target body.

Throughout all the steps the prescribed values of $\underline{r}(t_o)$, $\bar{\rho}_T$, $\bar{i}_T$ and $t_{pT}$ remain constant. Steps 10 and 11 represent the second order solution to the boundary value problem.

## B2.2 Launch Body-to-Target Body Solution

This solution is shown in Figure B3. The pericenter time, $t_{pk}$, radius, $\bar{\rho}_k$, and inclination, $\bar{i}_k$ are prescribed at both the launch planet, $k = L$, and the target planet, $k = T$. The hyperbolic excess velocities at both planets are unknown and must be determined. An ephemeris is required giving positions and velocities of the N-2 perturbing bodies with respect to the primary body. The ephemeris fixes the coordinate system of the asymptotic solution.

From Figure B3 it can be seen that the zeroth order ellipse passes through the launch body at $t = t_L$ and through the target body at $t = t_T$ where

$$t_k = t_{pk} - \mu_k \tau_k \tag{B2-42}$$

where again $\tau_k$ can be set equal to zero without loss of generality. Then

$$\underline{r}_o(t_L) = \underline{p}_L(t_L) \tag{B2-43}$$

$$\underline{r}_o(t_T) = \underline{p}_T(t_T) \tag{B2-44}$$

where $\underline{p}_L$ and $\underline{p}_T$ are the positions of bodies $k = L$ and $k = T$ obtained from the ephemeris. The two position vectors given by (B2-43) and (B2-44) define a Lambert problem and the solution gives $\underline{r}_o(t)$, shown as the dashed line in Figure B3, and the velocities, $\underline{v}_o(t_L)$ and $\underline{v}_o(t_T)$.

Now let $k = L, T$ with

$$\mu = \mu_L \text{ or } \mu_T \tag{B2-45}$$

$$Q_L = +1 \tag{B2-46}$$

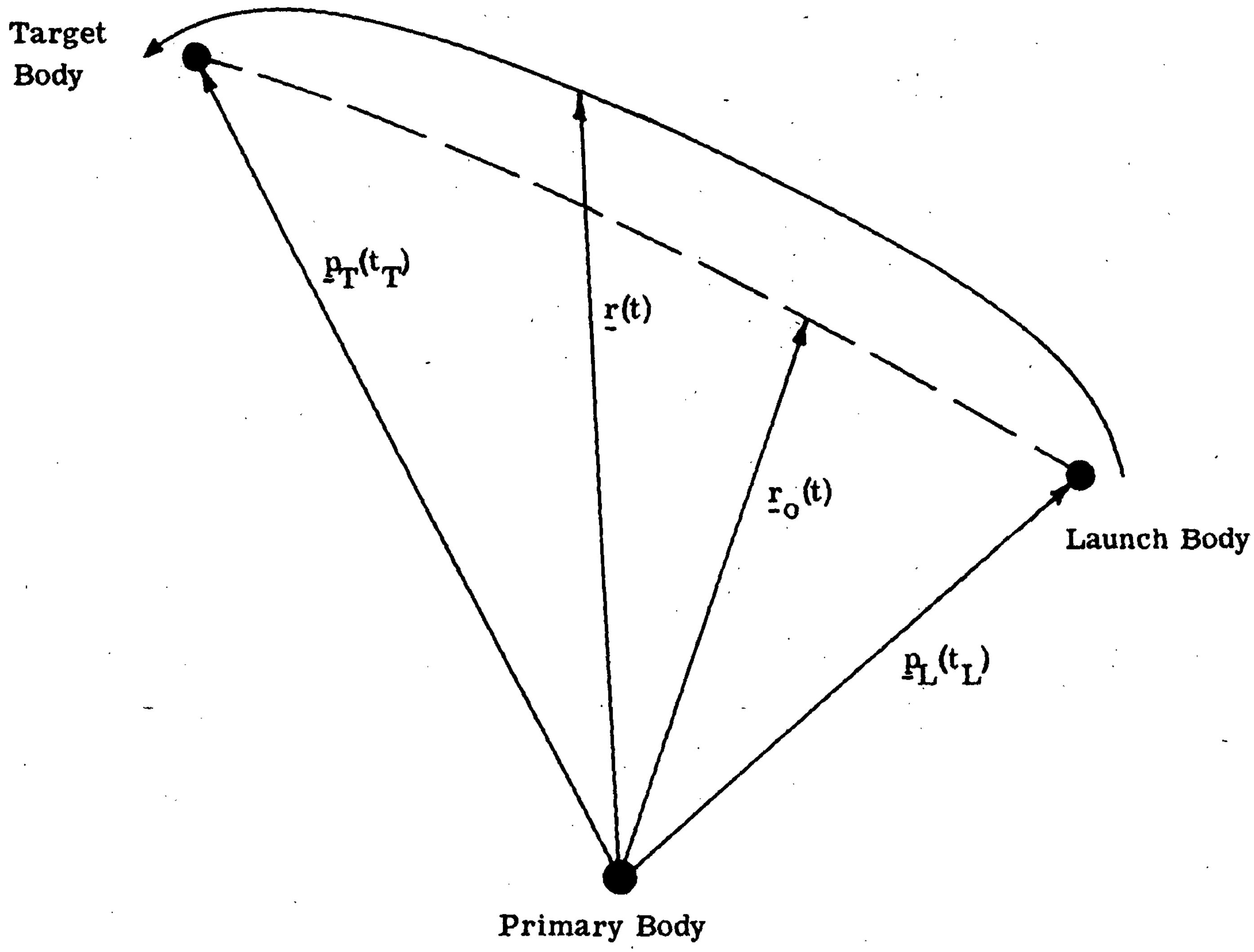


Figure B3. Launch Body-to-Target Body Solution

$$Q_T = -1 \qquad \text{(B2-47)}$$

(cf (A11-34)) and substitute (B2-24) and (B2-25) into (B1-8) for each k. This gives two equations each of which have the same left hand sides. Assuming the position and velocity continuous at $t = t_o$ allows the two right hand sides to be equated[10]. Using (A3-31) and (A3-32) gives

$$A(t_T, t_L) \left\{ M_L \underline{L}_L - M_L \left[ \tau_L + \frac{1}{\bar{n}_L} \log\left( \frac{\mu_L \bar{e}_L}{2\bar{n}_L} \right) \right] \underline{V}_{\infty L} + \underline{Y}_L + \mu \underline{\zeta}_L \right\} + B(t_T, t_L)$$

$$\cdot \left\{ \mu^{-1} (\underline{V}_{\infty L} - \underline{V}_L) + \underline{\delta}_L + \mu \underline{\eta}_L \right\} = M_T \underline{L}_T - M_T \left[ \tau_T - \frac{1}{\bar{n}_T} \log\left( \frac{\mu_T \bar{e}_T}{2\bar{n}_T} \right) \right] \underline{V}_{\infty T}$$

$$+ \underline{Y}_T + \mu \underline{\zeta}_T \qquad \text{(B2-48)}$$

$$C(t_T, t_L) \left\{ M_L \underline{L}_L - M_L \left[ \tau_L + \frac{1}{\bar{n}_L} \log\left( \frac{\mu_L \bar{e}_L}{2\bar{n}_L} \right) \right] \underline{V}_{\infty L} + \underline{Y}_L + \mu \underline{\zeta}_L \right\} + D(t_T, t_L)$$

$$\cdot \left\{ \mu^{-1} (\underline{V}_{\infty L} - \underline{V}_L) + \underline{\delta}_L + \mu \underline{\eta}_L \right\} = \mu^{-1} (\underline{V}_{\infty T} - \underline{V}_T) + \underline{\delta}_T + \mu \underline{\eta}_T \qquad \text{(B2-49)}$$

Solving for $\underline{V}_{\infty L}$ and $\underline{V}_{\infty T}$ gives

$$\underline{V}_{\infty L} = \underline{V}_L - \mu \underline{\delta}_L - \mu^2 \underline{\eta}_L + \mu\, B(t_T, t_L)^{-1} \left\{ M_T \underline{L}_T - M_T \left[ \tau_T \right.\right.$$

$$\left.\left. - \frac{1}{\bar{n}_T} \log\left( \frac{\mu_T \bar{e}_T}{2\bar{n}_T} \right) \right] \underline{V}_{\infty T} + \underline{Y}_T + \mu \underline{\zeta}_T \right\} - \mu\, B(t_T, t_L)^{-1} A(t_T, t_L)$$

$$\cdot \left\{ M_L \underline{L}_L - M_L \left[ \tau_L + \frac{1}{\bar{n}_L} \log\left( \frac{\mu_L \bar{e}_L}{2\bar{n}_L} \right) \right] \underline{V}_{\infty L} + \underline{Y}_L + \mu \underline{\zeta}_L \right\} \qquad \text{(B2-50)}$$

$$\underline{V}_{\infty T} = \underline{V}_T - \mu \underline{\delta}_T - \mu^2 \underline{\eta}_T + \mu C(t_T, t_L) \Big\{ M_L \underline{L}_L - M_L \Big[ \tau_L + \frac{1}{\bar{n}_L} \log$$

$$\cdot \left( \frac{\mu_L \bar{e}_L}{2\bar{n}_L} \right) \Big] \underline{V}_{\infty L} + \underline{\gamma}_L + \mu \underline{\zeta}_L \Big\} + D(t_T, t_L) \Big\{ \underline{V}_{\infty L} - \underline{V}_L + \mu \underline{\delta}_L + \mu^2 \underline{\eta}_L \Big\} \tag{B2-51}$$

If the excess velocities are used to calculate orbital elements then the position and velocity at pericenter are defined by the well known expressions

$$\underline{R}_{pk} = \frac{\bar{\rho}_k}{\bar{a}_k \bar{e}_k} \underline{L}_k \tag{B2-52}$$

$$\underline{V}_{pk} = \frac{1}{\bar{a}_k \bar{e}_k} \left( \frac{1 + \bar{e}_k}{\bar{\rho}_k} \right)^{1/2} \frac{\partial \underline{L}_k}{\partial \bar{\omega}_k} \tag{B2-53}$$

The two expressions for the excess velocities are not explicit but the solution can be obtained as follows:

1. Solve the Lambert problem defined by (B2-43) and (B2-44). This gives $\underline{v}_o(t_L)$ and $\underline{v}_o(t_T)$.

2. From (A11-22) with k = L, T

$$\underline{V}_k = \underline{v}_o(t_k) - \dot{\underline{p}}_k(t_k) \tag{B2-54}$$

   where $\dot{\underline{p}}_k(t_k)$ is obtained from the ephemeris.

3. Let $\underline{V}_{\infty k} = V_k$, k = L, M. This gives the zeroth order excess velocities.

4. Evaluate (B2-1) - (B2-26) with k = L, M.

5. Evaluate (B2-50) using the results of steps 3 and 4 and with $\underline{\eta}_L = \underline{\zeta}_T = \underline{\zeta}_L = 0$. This gives the first order $\underline{V}_{\infty L}$.

6. Evaluate (B2-51) using the results of steps 3, 4 and 5 and with $\underline{\eta}_T = \underline{\zeta}_L = \underline{\eta}_L = 0$. This gives the first order $\underline{V}_{\infty T}$.

7. Repeat step 4 using first order excess velocities.

8. Evaluate (B2-50) using the results of steps 5, 6 and 7. This gives the second order $\underline{V}_{\infty L}$.

9. Evaluate (B2-51) using the results of steps 7 and 8. This gives the second order $\underline{V}_{\infty T}$.

10. Repeat step 4 using second order excess velocities. This gives second order orbital elements.

11. Evaluate (B2-52) and (B2-53) using the results of step 10. This gives second order positions and velocities at the pericenters of the hyperbolic trajectories close to the launch and target bodies.

Throughout all the steps the prescribed values of $t_{pk}$, $\overline{\rho}_k$ and $\overline{i}_k$, $k = L, M$ remain constant. Steps 10 and 11 represent the second order solution to the boundary value problem.

Although the midcourse time $t_o$ does not appear explicitly in (B2-50) and (B2-51) it does enter implicitly since all of the constants are evaluated either between $t_L$ and $t_o$ or between $t_o$ and $t_T$. The midcourse time is not fixed and may be defined by

$$t_o = (t_L + t_T)/2 \tag{B2-55}$$

The first order position and velocity perturbations at $t = t_o$ are needed to evaluate the second order constants. They are obtained directly from (B1-8),

i. e.,

$$\begin{pmatrix} \underline{r}_1(t_o) \\ \underline{v}_1(t_o) \end{pmatrix} = \Phi(t_o, t_L) \left\{ \begin{pmatrix} M_L \underline{L}_L - M_L \left[ \tau_L + \dfrac{1}{\bar{n}_L} \log \left( \dfrac{\mu_L \bar{e}_L}{2\bar{n}_L} \right) \right] \underline{V}_{\infty L} \\ (\underline{V}_{\infty L} - \underline{V}_L)/\mu \end{pmatrix} + \begin{pmatrix} \underline{\gamma}_L \\ \underline{\delta}_L \end{pmatrix} \right\} \tag{B2-56}$$

B2.3 Non-linear Solutions

Evaluating (B2-38) and (B2-50) requires the inversion of the B-matrix. This inversion may tend to give inaccurate results in cases where the linear state transition matrix is not a good approximation due to large non-linear effects. Such cases arise if one of the endpoints is close to pericenter or apocenter of the zeroth order ellipse. The form of (B2-36) (B2-37) (B2-48) and (B2-49) suggests an alternative approach. This approach has been called the non-linear solution by Carlson.[3]

The zeroth order ellipse is defined by its position and velocity, $\underline{r}_o(t)$ and $\underline{v}_o(t)$. Suppose a neighboring ellipse is defined by

$$\underline{r}_o'(t) = \underline{r}_o(t) + \mu \Delta \underline{r}_o(t) \tag{B2-57}$$

$$\underline{v}_o'(t) = \underline{v}_o(t) + \mu \Delta \underline{v}_o(t) \tag{B2-58}$$

where $\Delta \underline{r}_o(t)$ and $\Delta \underline{v}_o(t)$ will be called the offset position and velocity. Since $\underline{r}_o'(t)$ and $\underline{v}_o'(t)$ also define a two-body trajectory the offset positions and velocities at any two times $t_1$ and $t_2$ are related by

$$\begin{pmatrix} \Delta \underline{r}_o(t_2) \\ \Delta \underline{v}_o(t_2) \end{pmatrix} = \Phi(t_2, t_1) \begin{pmatrix} \Delta \underline{r}_o(t_1) \\ \Delta \underline{v}_o(t_1) \end{pmatrix} \tag{B2-59}$$

Now define $\underline{X}_k$ and $\underline{Y}_k$ by

$$\underline{X}_k = M_k \underline{L}_k - M_k \left[\tau_k + \frac{Q_k}{\bar{n}_k} \log\left(\frac{\mu_k \bar{e}_k}{2\bar{n}_k}\right)\right] \underline{V}_{\infty k} + \underline{Y}_k + \mu \underline{\zeta}_k \tag{B2-60}$$

$$\underline{Y}_k = \mu^{-1} (\underline{V}_{\infty k} - \underline{V}_k) + \underline{\delta}_k + \mu \underline{\eta}_k \tag{B2-61}$$

Then the midcourse solution, (B2-36) and (B2-37), becomes

$$\begin{pmatrix} \underline{X}_T \\ \\ \underline{Y}_T \end{pmatrix} = \Phi(t_T, t_o) \begin{pmatrix} O \\ \\ \delta \underline{v}\ (t_o) \end{pmatrix} \tag{B2-62}$$

while (B2-48) and (B2-49) become

$$\begin{pmatrix} \underline{X}_T \\ \\ \underline{Y}_T \end{pmatrix} = \Phi(t_T, t_L) \begin{pmatrix} \underline{X}_L \\ \\ \underline{Y}_L \end{pmatrix} \tag{B2-63}$$

Now note the similarity between (B2-59) and (B2-62) and (B2-63). This suggests that (B2-62) and (B2-63) represent the propagation of offset end conditions for a new zeroth order solution.

For the midcourse solution, comparison of (B2-59) with (B2-62) gives

$$\Delta \underline{r}_o(t_o) = 0 \tag{B2-64}$$

$$\Delta \underline{r}_o(t_T) = \underline{X}_T \tag{B2-65}$$

Then from (B2-57)

$$\underline{r}_o'(t_o) = \underline{r}_o(t_o) \tag{B2-66}$$

$$\underline{r}_o'(t_T) = \underline{r}_o(t_T) + \mu \underline{X}_T \tag{B2-67}$$

The end points $\underline{r}_o'(t_o)$ and $\underline{r}_o'(t_T)$ define a new Lambert problem, the solution of which gives $\underline{r}_o'(t)$, shown as the dashed line in Figure B4, $\underline{v}_o'(t_o)$ and $\underline{v}_o'(t_T)$. Again comparing (B2-59) with (B2-62) gives

$$\Delta \underline{v}_o(t_o) = \delta \underline{v}(t_o) \tag{B2-68}$$

$$\Delta \underline{v}_o(t_T) = \underline{Y}_T \tag{B2-69}$$

By combining (B2-41), (B2-58) and (B2-68) the initial velocity becomes

$$\underline{v}(t_o) = \underline{v}_o'(t_o) \tag{B2-70}$$

By combining (B2-58), (B2-61) and (B2-69) the excess velocity at the target body becomes

$$\underline{V}_{\infty T} = \underline{V}_T' - \mu \underline{\delta}_T - \mu^2 \underline{\eta}_T \tag{B2-71}$$

where

$$\underline{V}_T' = \underline{v}_o'(t_T) - \dot{\underline{p}}_T(t_T) \tag{B2-72}$$

Equations (B2-70) and (B2-71) represent the non-linear solution. They replace (B2-38), (B2-39) and (B2-41) of the solution discussed in Section B2.1.

For the launch body-to-target body solution comparison of (B2-59) with (B2-63) gives

$$\Delta \underline{r}_o(t_L) = \underline{X}_L \tag{B2-73}$$

$$\Delta \underline{r}_o(t_T) = \underline{X}_T \tag{B2-74}$$

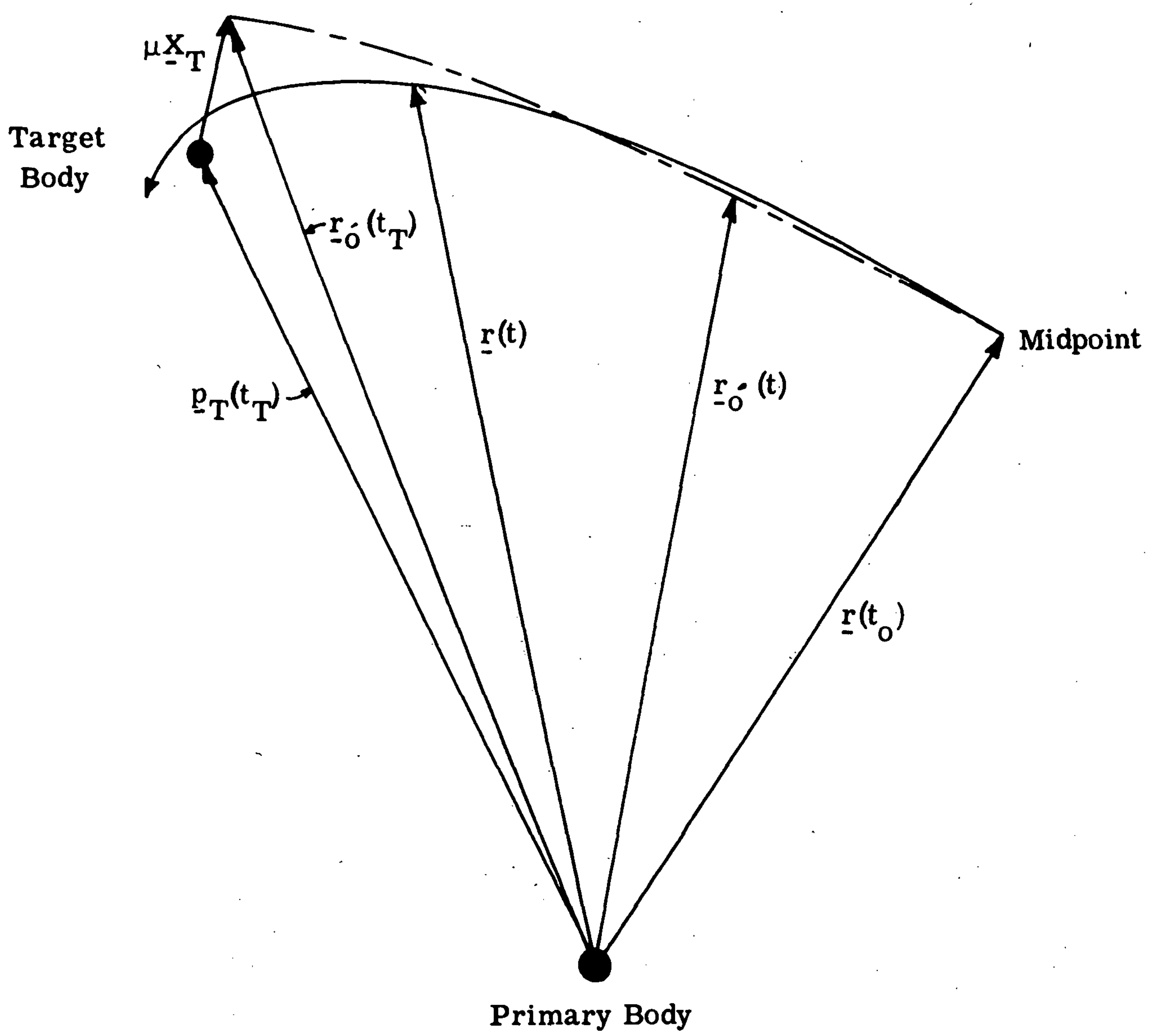


Figure B4. Non-linear Version of Midpoint-to-Target Body Solution

Then from (B2-57)

$$\underline{r}_o'(t_L) = \underline{r}_o(t_L) + \mu\underline{X}_L \qquad \text{(B2-75)}$$

$$\underline{r}_o'(t_T) = \underline{r}_o(t_T) + \mu\underline{X}_T \qquad \text{(B2-76)}$$

The end points $\underline{r}_o'(t_L)$ and $\underline{r}_o'(t_T)$ define a new Lambert problem, the solution of which gives $\underline{r}_o'(t)$, shown as the dashed line Figure B5, $\underline{v}_o'(t_L)$ and $\underline{v}_o'(t_T)$. Again comparing (B2-59) with (B2-63) gives

$$\Delta\underline{v}_o(t_L) = \underline{Y}_L \qquad \text{(B2-77)}$$

$$\Delta\underline{v}_o(t_T) = \underline{Y}_T \qquad \text{(B2-78)}$$

By combining (B2-58), (B2-61), (B2-77) and (B2-78) the excess velocities become

$$\underline{V}_{\infty L} = \underline{V}_L' - \mu\underline{\delta}_L - \mu^2\underline{\eta}_L \qquad \text{(B2-79)}$$

$$\underline{V}_{\infty T} = \underline{V}_T' - \mu\underline{\delta}_T - \mu^2\underline{\eta}_T \qquad \text{(B2-80)}$$

where

$$\underline{V}_L' = \underline{v}_o'(t_L) - \dot{\underline{p}}_L(t_L) \qquad \text{(B2-81)}$$

$$\underline{V}_T' = \underline{v}_o'(t_T) - \dot{\underline{p}}_T(t_T) \qquad \text{(B2-82)}$$

Equations (B2-79) and (B2-80) represent the non-linear solution. They replace (B2-50) and (B2-51) of the solution discussed in Section B2.2.

Since $\underline{X}_k$ is a function of $\underline{V}_{\infty k}$ the non-linear solutions, like those of Sections B2.1 and B2.2, must be evaluated in a sequence which uses the best available

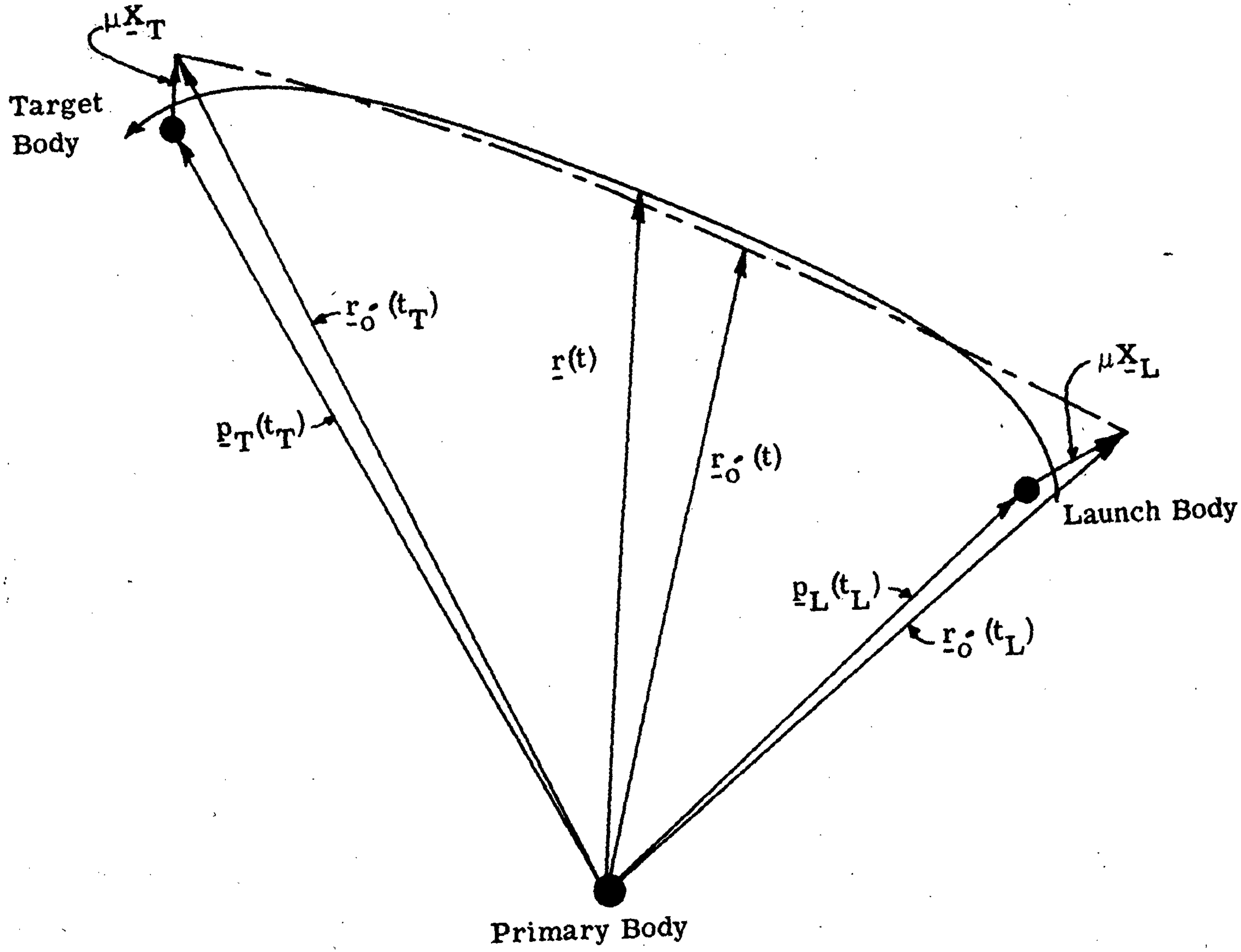


Figure B5. Non-linear Version of Launch Body-to-Target Body Solution approximation for $\underline{V}_{\infty k}$, i.e., the $n^{th}$ order solution for $\underline{V}_{\infty k}$ requires the $(n-1)^{th}$ order value of $\underline{V}_{\infty k}$ to evaluate the offset endpoint $\underline{X}_k$.

The non-linear solutions require more computation time than the linear or standard solutions since solving a second Lambert problem is, in general, more time consuming than inversion of the 3 x 3 B-matrix.

## B3 APPLICATIONS OF THE BOUNDARY VALUE SOLUTIONS

The solutions of Section 3 can be used to solve several boundary value problems. They are discussed in the following sections.

### B3.1 Earth-to-Moon

The simplest earth-to-moon boundary value problem is shown in Figure B6. The initial time, $t_o$, the initial position relative to the earth, $\underline{r}(t_o)$, and the pericynthion radius, $\overline{\rho}_T$, inclination $\overline{i}_T$, and time, $t_{pT}$, are all prescribed. The initial velocity relative to the earth, $\underline{v}(t_o)$, is unknown and must be determined.

This problem is solved using the midpoint-to-target body solution of Section B2.1. The primary body is the earth and an ephemeris is required giving the motions of the moon, sun and any other significant bodies in cartesian coordinates with the earth at the origin. Although the sun's mass is large compared to the moon's mass, its effect is diminished by its great distance from the earth. As discussed in Section A1.2 both the moon and sun contribute effects of order $\mu$ where $\mu$ is the dimensionless mass of the moon.

In this problem the subscript T of Section B2.1 refers to the target body which is the moon. The effects of the sun and any other bodies enter only through the constants $\underline{\gamma}_T$, $\underline{\delta}_T$, $\underline{\zeta}_T$ and $\underline{\eta}_T$.

### B3.2 Earth-to-Moon Midcourse

In the previous section the initial position, $\underline{r}(t_o)$, was implicitly assumed to be close to the earth. The same analysis may also be used for a midcourse maneuver where the position, $\underline{r}(t_o)$, represents a point between the earth and the moon as shown in Figure B7. The solution is identical to the Earth-to-Moon

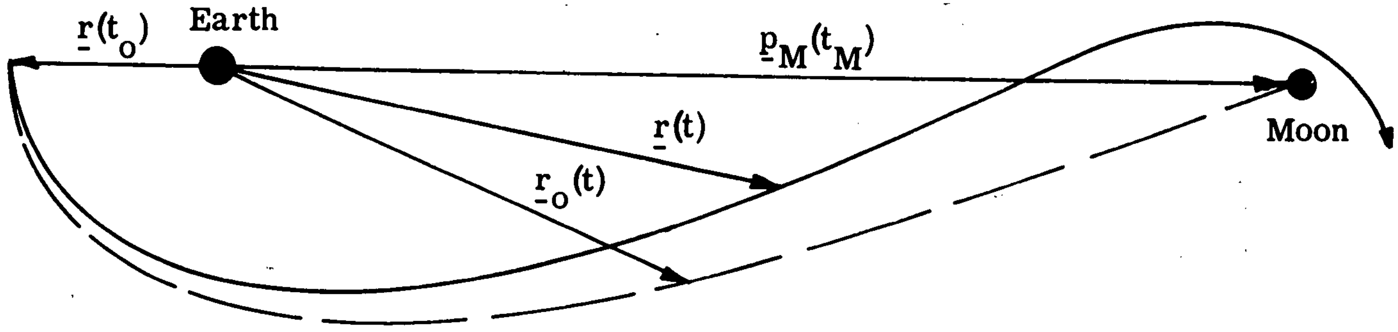


Figure B6. Earth-to-Moon Solution

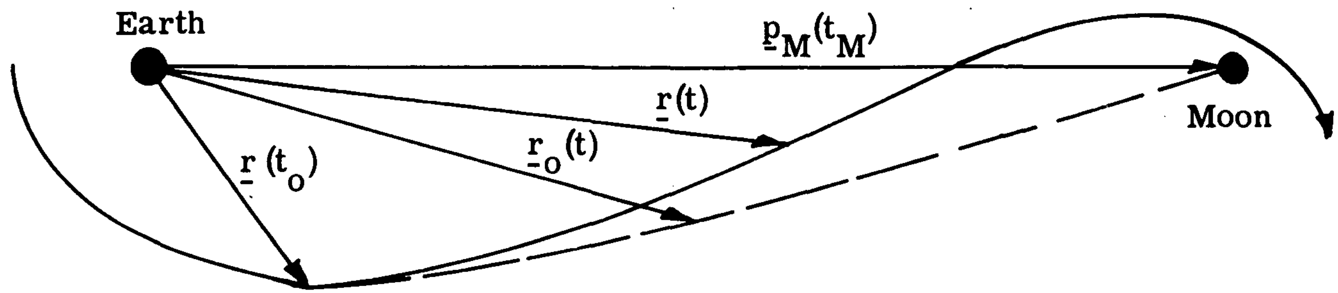


Figure B7. Earth-to-Moon Midcourse Solution

solution of the previous section. Assuming the velocity, $\underline{v}(t_o^-)$, just prior to the midcourse maneuver is known, the velocity correction is given by

$$\Delta\underline{v}(t_o) = \underline{v}(t_o) - \underline{v}(t_o^-) \qquad \text{(B3-1)}$$

where $\underline{v}(t_o)$ is determined from the asymptotic solution.

B3.3 Interplanetary Midcourse

The interplanetary midcourse problem is shown in Figure B8. The initial time, $t_o$, the initial position relative to the sun, $\underline{r}(t_o)$, and the pericenter radius, $\bar{\rho}_T$, inclination, $\bar{i}_T$, and time, $t_{pT}$, at the target planet are all prescribed. The initial velocity relative to the sun, $\underline{v}(t_o)$, is unknown and must be determined.

This problem is also solved using the midpoint-to-target body solution of Section B2.1. The primary body is the sun and an ephemeris is required giving the planetary motions in cartesian coordinates with the sun at the origin. In this problem the subscript T refers to the target planet and the effects of all other planets enter only through the constants $\underline{\gamma}_T$, $\underline{\delta}_T$, $\underline{\zeta}_T$ and $\underline{\eta}_T$.

B3.4 Interplanetary

The interplanetary boundary value problem is shown in Figure B9. The pericenter radius, $\bar{\rho}_k$, inclination, $\bar{i}_k$, and time, $t_{pk}$, are prescribed at both the launch planet, k = L, and at the target planet, k = T. The hyperbolic excess velocities at both planets are unknown and must be determined.

This problem is solved using the launch body-to-target body solution of Section B2.2. The primary body is the sun and the same ephemeris as is used in the interplanetary midcourse solution is required. In this solution the perturbing effects of the planets from $t = t_L$ to $t = t_o$ are included in the constants $\underline{\gamma}_L$, $\underline{\delta}_L$, $\underline{\zeta}_L$ and $\underline{\eta}_L$ while the effects from $t = t_o$ to $t = t_T$ are included in $\underline{\gamma}_T$, $\underline{\delta}_T$, $\underline{\zeta}_T$ and $\underline{\eta}_T$.

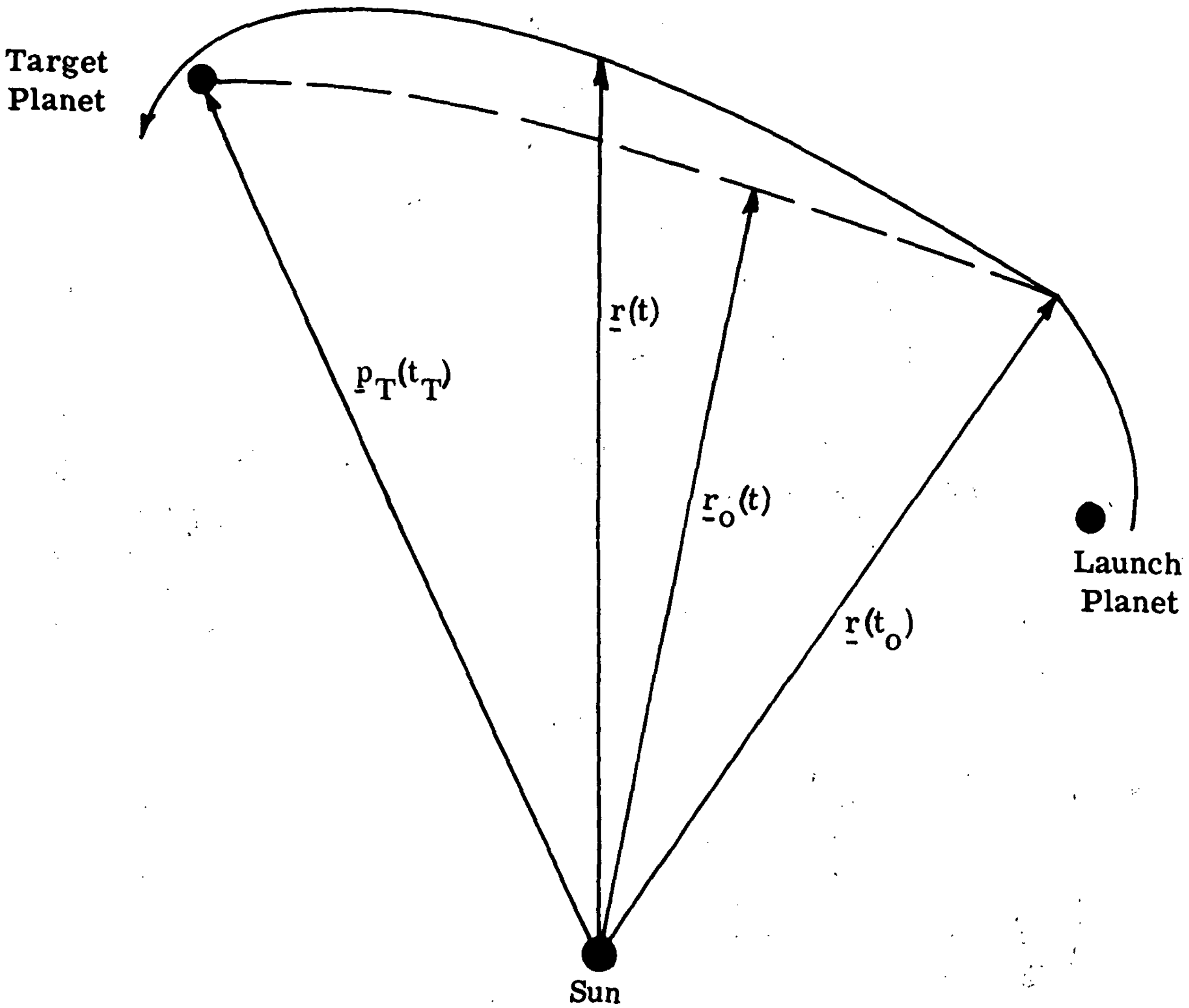


Figure B8. Interplanetary Midcourse Solution

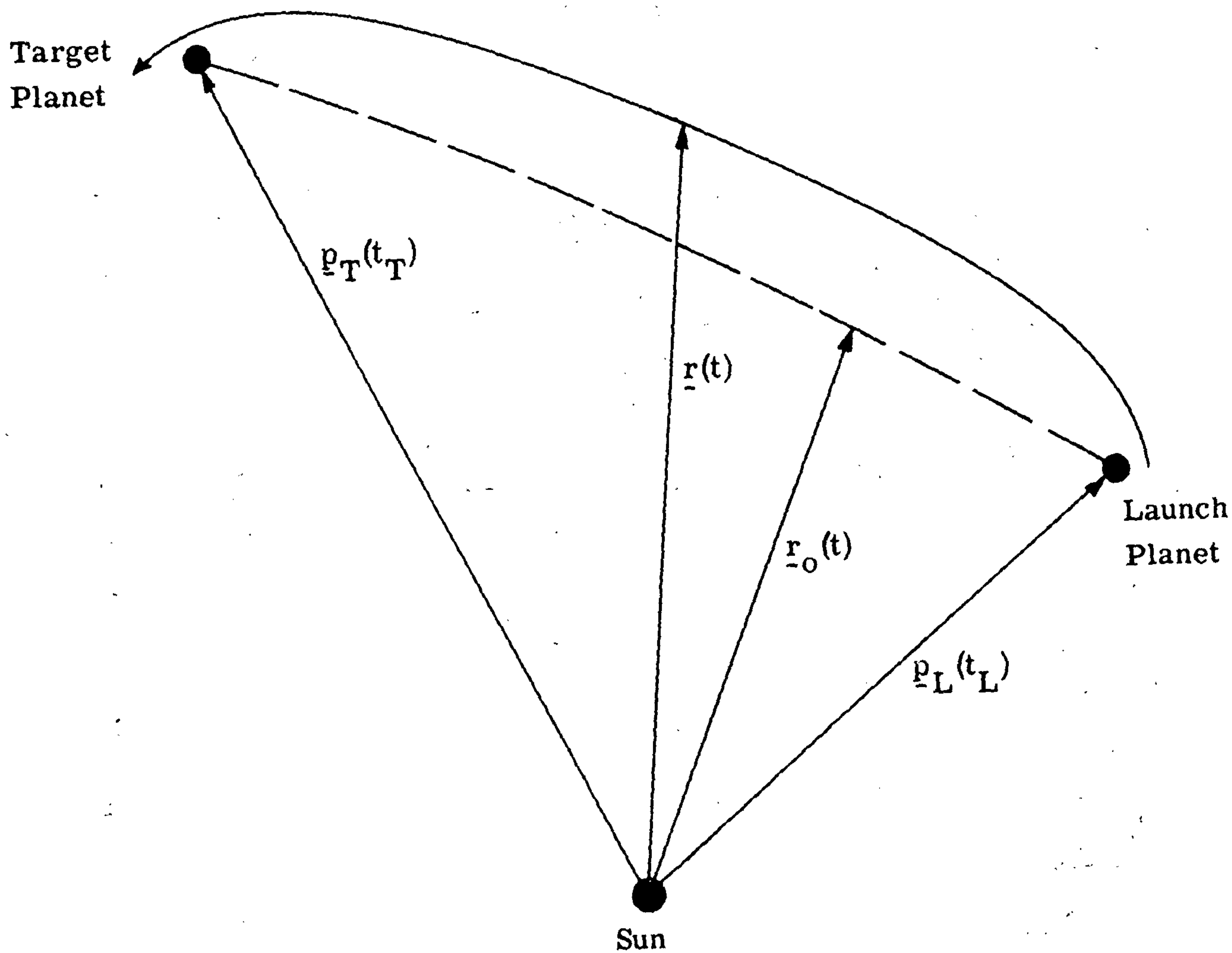


Figure B9. Interplanetary Solution

## B4 SPECIAL MOON-TO-EARTH SOLUTIONS

In this section two special moon-to-earth problems are considered. In both problems the boundary conditions are the initial and final times, $t_1$ and $t_e$, the initial position relative to the moon, $\underline{R}(t_1)$, the entry radius relative to the earth, $r_e$, the inclination relative to the earth, $i_e$, and the entry flight path angle, $\gamma_e$. Also in each problem the trajectory prior to $t = t_1$ is assumed to be an orbit about the moon with velocity $\underline{V}(t_1^-)$ just prior to $t_1$. The first problem involves finding the velocity $\underline{V}(t_1^+)$ which results in a trajectory satisfying the earth entry conditions. This constitutes a single impulse problem where

$$\Delta\underline{V}_1 = \underline{V}(t_1^+) - \underline{V}(t_1^-) \tag{B4-1}$$

is the impulsive velocity.

In the second problem the velocity after the impulse is assumed to be of the form

$$\underline{V}(t_1^+) = (1 + I_1)\,\underline{V}(t_1^-) \tag{B4-2}$$

which gives an impulsive velocity of

$$\Delta\underline{V}_1 = I_1\,\underline{V}(t_1^-) \tag{B4-3}$$

The new velocity does not necessarily result in a trajectory satisfying the earth entry conditions and a second impulse $\Delta\underline{V}_2$ is applied at $t = t_2$ where $(t_2 - t_1)$ is small compared to the total flight time, $(t_e - t_1)$. The second impulse must give a trajectory satisfying the entry conditions. This constitutes the two impulse problem.

### B4.1 Modified Lambert Problem

In the standard Lambert problem two position vectors are prescribed as well as the flight time from one position to the other. Solution of the problem gives the two-body trajectory connecting the two positions. Lambert's theorem is stated in functional form as

$$t_2 - t_1 = F(a, x_1 + x_2, c) \tag{B4-4}$$

where

$$c = (x_1^2 + x_2^2 + 2x_1x_2 \cos\theta)^{1/2} \tag{B4-5}$$

When the two position vectors are given

$$\cos\theta = (\underline{x}_1 \cdot \underline{x}_2)/(x_1x_2) \tag{B4-6}$$

Solution of the Lambert problem requires solving (B4-4) for a. Once a is known a set of equations is solved to eventually give $\underline{x}(t)$. Several forms of (B4-4) have been proposed, all of which require numerical techniques to solve for a.

For certain problems with prescribed entry conditions the final position $\underline{x}_2$ is not known, only its magnitude is given. However the inclination and entry flight path angle are also prescribed and allow for a solution. The angle $\theta$ between the two positions is defined as

$$\theta = f_2 - f_1 \tag{B4-7}$$

where $f_1$ and $f_2$ are the initial and final values of true anomaly given by

$$f_1 = \cos^{-1}\left[\frac{a(1-e^2) - x_1}{ex_1}\right], \quad \pi \le f_1 \le 2\pi \tag{B4-8}$$

$$f_2 = \cos^{-1}\left[\frac{a(1-e^2) - x_2}{ex_2}\right], \quad \pi \le f_2 \le 2\pi \tag{B4-9}$$

The range of $f_1$ and $f_2$ comes from restricting the flight time to be less than that for $\theta = \pi$ and having the entry before perigee. The eccentricity can be written as

$$e = \left[a^2 + x_2 (x_2 - 2a) \cos^2 \gamma_2\right]^{1/2} \Big/ a \qquad \text{(B4-10)}$$

where $\gamma_2$ is the flight path angle at $t_2$. This modified Lambert problem requires a simultaneous solution of (B4-4), (B4-5) and (B4-7) - (B4-10) for a, c, $\theta$, $f_1$, $f_2$ and e, i.e., six equations for six unknowns.

In Section B2.1 the approach of Battin[5] was used to determine the ascending node of the inner solution. A similar approach is now used to determine the ascending node of the modified Lambert solution. Let the initial position $\underline{x}_1$ have components ($\xi_1$, $\eta_1$, $\zeta_1$) and, using Figure B2 with $\underline{V}_\infty$ replaced by $\underline{x}_1$, define

$$\sin \lambda = \eta_1 \Big/ (\xi_1^2 + \eta_1^2)^{1/2} \qquad \text{(B4-11)}$$

$$\cos \lambda = \xi_1 \Big/ (\xi_1^2 + \eta_1^2)^{1/2} \qquad \text{(B4-12)}$$

$$\tan \alpha = \zeta_1 \Big/ (\xi_1^2 + \eta_1^2)^{1/2} \qquad \text{(B4-13)}$$

Since the plane of motion must contain $\underline{x}_1$ the ascending node must satisfy

$$\Omega = \lambda \pm \sigma + (1 \pm 1)\, \pi/2 \qquad \text{(B4-14)}$$

where

$$\sigma = \tan \alpha / \tan i \qquad \text{(B4-15)}$$

and i is the prescribed inclination. The ± sign indicates two possible solutions satisfying the prescribed inclination. The unit normal is now defined by[5]

$$\underline{n} = (\sin \Omega \sin i,\ -\cos \Omega \sin i,\ \cos i) \qquad \text{(B4-16)}$$

Using standard formulas for elliptic motion (cf Battin) the initial position and velocity are

$$\underline{x}(t_1) = \underline{x}_1 \qquad \text{(B4-17)}$$

$$\underline{\dot{x}}(t_1) = \frac{1}{x_1}\left[V_r(t_1)\,\underline{x}_1 + V_\phi(t_1)\,\underline{n} \times \underline{x}_1\right] \tag{B4-18}$$

where

$$V_r(t_1) = e \sin f_1/(nab) \tag{B4-19}$$

$$V_\phi(t_1) = (1 + e \cos f_1)/(nab) \tag{B4-20}$$

and

$$n = a^{-3/2} \tag{B4-21}$$

$$b = a(1 - e^2)^{1/2} \tag{B4-22}$$

The final position and velocity are

$$\underline{x}(t_2) = \underline{x}_2$$

$$= \frac{x_2}{x_1}\left(\cos\theta\,\underline{x}_1 + \sin\theta\,\underline{n} \times \underline{x}_1\right) \tag{B4-23}$$

$$\underline{\dot{x}}(t_2) = \frac{1}{x_2}\left[V_r(t_2)\,\underline{x}_2 + V_\phi(t_2)\,\underline{n} \times \underline{x}_2\right] \tag{B4-24}$$

where

$$V_r(t_2) = e \sin f_2/(nab) \tag{B4-25}$$

$$V_\phi(t_2) = (1 + e \cos f_2)/(nab) \tag{B4-26}$$

The solution at any time t is given by (A6-2), (A6-5) and (A6-6), i.e.,

$$\underline{x}(t) = f(t)\,\underline{x}(t_1) + g(t)\,\underline{\dot{x}}(t_1) \tag{B4-27}$$

where

$$f(t) = 1 - a(1 - \cos \Delta E)/x_1 \tag{B4-28}$$

$$g(t) = (x_1 \sin \Delta E)/(na) + e \sin E_1 (1 - \cos \Delta E)/n \tag{B4-29}$$

and

$$\Delta E = E - E_1 \tag{B4-30}$$

The eccentric anomaly E is defined from

$$n(t - t_p) = E - e \sin E \tag{B4-31}$$

or by

$$\sin E = (1 - e^2)^{1/2} \sin f/(1 + e \cos f) \tag{B4-32}$$

$$\cos E = (\cos f + e)/(1 + e \cos f) \tag{B4-33}$$

Since e and $f_1$ are known (B4-32) and (B4-33) can be used to determine $E_1$. Then from (B4-31) the time of pericenter passage, $t_p$, is

$$t_p = t_1 - (E_1 - e \sin E_1)/n \tag{B4-34}$$

The equations derived in this section completely define a two-body ellipse satisfying the prescribed values of the initial position, $\underline{x}_1$, the inclination, i, the final radius, $x_2$, the final flight path angle, $\gamma_2$, and the time $t_2 - t_1$, where both $t_2$ and $t_1$ are given.

## B4.2 Single Impulse Solution

### B4.2.1 Outer Solution

The boundary conditions which are prescribed for the outer solution are the earth entry time, $t_e$, radius, $r_e$, inclination, $i_e$, and flight path angle, $\gamma_e$.

Thus in the modified Lambert problem the subscript 2 is replaced by e, i.e.,

$$t_2 = t_e \tag{B4-35}$$

$$x_2 = x_e \tag{B4-36}$$

$$i_2 = i_e \tag{B4-37}$$

$$\gamma_2 = \gamma_e \tag{B4-38}$$

An additional boundary condition for the outer solution is the initial time $t_1$. The initial position for the modified Lambert problem is taken as the position of the moon at $t = t_1$, i.e.,

$$\underline{x}_1 = \underline{p}_M(t_1) \tag{B4-39}$$

This forces the Lambert solution to pass through the center of the moon as in the earth-to-moon solutions.

Solution of the modified Lambert problem gives the zeroth order outer solution from (B4-27), i.e.,

$$\underline{r}_o(t) = \underline{x}(t) \tag{B4-40}$$

The position and velocity at $t_1$ are given by (B4-17) and (B4-18)

$$\underline{r}_o(t_1) = \underline{x}(t_1) \tag{B4-41}$$

$$\underline{v}_o(t_1) = \dot{\underline{x}}(t_1) \tag{B4-42}$$

and at $t_e$ by (B4-23) and (B4-24).

$$\underline{r}_o(t_e) = \underline{x}(t_2) \tag{B4-43}$$

$$\underline{v}_o(t_e) = \dot{\underline{x}}(t_2) \tag{B4-44}$$

The boundary conditions on the higher order outer solutions come from (B2-29) and (B2-30)

$$\underline{r}(t_e) = \underline{r}_o(t_e) + \mu \underline{r}_1(t_e) + \mu^2 \underline{r}_2(t_e) \tag{B4-45}$$

$$\underline{v}(t_e) = \underline{v}_o(t_e) + \mu \underline{v}_1(t_e) + \mu^2 \underline{v}_2(t_e) \tag{B4-46}$$

The effect of the higher order boundary conditions will be considered shortly. No further information about the outer solution is required at this point.

B4.2.2 Inner Solution

The only boundary condition prescribed for the inner solution is the initial position $\underline{R}_M$ at $t = t_1$. However $\underline{R}_M$ is actually a function of the inner variable

$$S_M = (t - t_{pM})/\mu \tag{B4-47}$$

where $t_{pM}$ is the time of pericenter passage of the inner solution and $\mu = \mu_M$, the dimensionless mass of the moon. From (A13-2)

$$t_{pM} = t_M + \mu \tau_M \tag{B4-48}$$

where $t_M$ and $\tau_M$ must be determined. (In the previous boundary value solutions it was stated that $\tau$ could be set equal to zero without loss of generality. The effect of a non-zero $\tau$ will be demonstrated in the next section.) When $t = t_1$ then $S_M = S_{M1}$ and the initial position is $\underline{R}(S_{M1})$.

From (A12-54) the hyperbolic excess velocity is

$$\underline{V}_{\infty M} = (\overline{U}_M, \overline{V}_M, \overline{W}_M) \tag{B4-49}$$

The excess velocity and the initial position are sufficient to determine the inner solution as follows:

The position and velocity at any time S are given by[3]

$$\underline{R}_M(S_M) = R_M(S_M) \cos \bar{f}_M(S_M)\, \underline{e}_a + R_M(S_M) \sin \bar{f}_M(S_M)\, \underline{e}_b \quad \text{(B4-51)}$$

$$\underline{V}_M(S_M) = -\frac{1}{\bar{\ell}_M} \left[ \sin \bar{f}_M(S_M)\, \underline{e}_a + \left[ \bar{e}_M + \cos \bar{f}_M(S_M) \right] \underline{e}_b \right] \quad \text{(B4-52)}$$

where

$$R_M = \left| \underline{R}_M \right| \quad \text{(B4-53)}$$

$$\bar{\ell}_M = \left[ \bar{a}_M (\bar{e}_M^2 - 1)^{1/2} \right]^{1/2} \quad \text{(B4-54)}$$

and $\bar{f}_M$ is the true anomaly. The unit vectors $\bar{e}_a$ and $\bar{e}_b$ lie in the orbital plane with $\bar{e}_a$ directed toward pericynthion as shown in Figure B10.

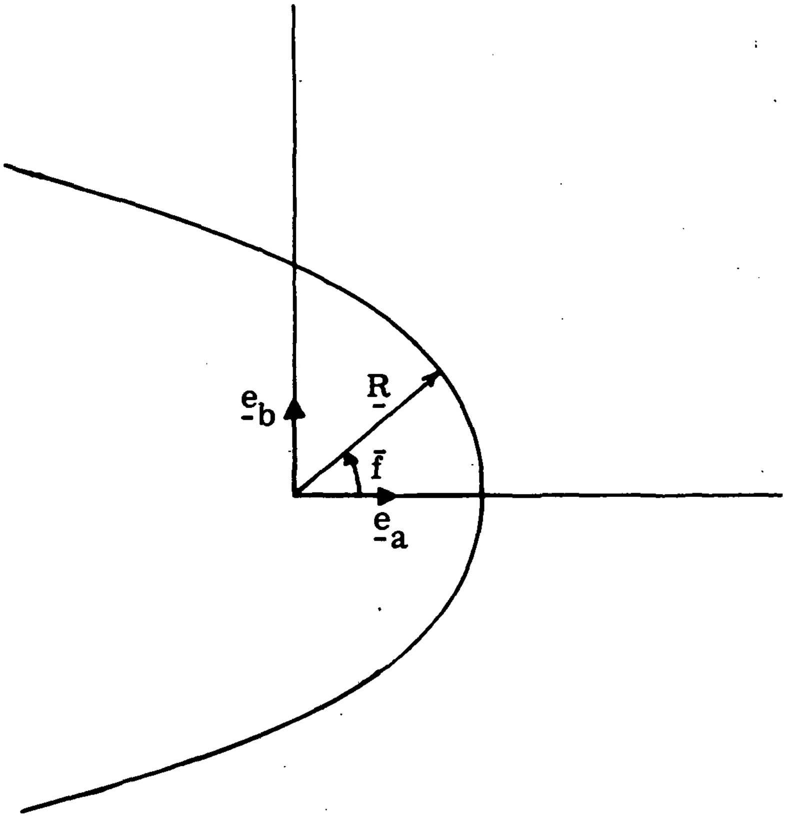


**Figure B10. Orbital Plane Coordinates for Inner Solution**

Now let

$$\underline{R}_{M1} = \underline{R}_M(S_{M1}) \tag{B4-55}$$

$$\underline{V}_{M1} = \underline{V}_M(S_{M1}) \tag{B4-56}$$

$$\overline{f}_{M1} = \overline{f}_M(S_{M1}) \tag{B4-57}$$

Then (B4-51) and (B4-52) may be solved for $\underline{e}_a$ and $\underline{e}_b$ giving

$$\underline{e}_a = \left(\frac{\overline{e}_M + \cos\overline{f}_{M1}}{\overline{\ell}_M^2}\right)\underline{R}_{M1} - \left(\frac{R_{M1}\sin\overline{f}_{M1}}{\overline{\ell}_M}\right)\underline{V}_{M1} \tag{B4-58}$$

$$\underline{e}_b = \left(\frac{\sin\overline{f}_{M1}}{\overline{\ell}_M^2}\right)\underline{R}_{M1} + \left(\frac{\underline{R}_{M1}\cos\overline{f}_{M1}}{\overline{\ell}_M}\right)\underline{V}_{M1} \tag{B4-59}$$

Substituting (B4-58) and (B4-59) into (B4-51) gives

$$\underline{R}_M = \left[\overline{\ell}_M^2 - R_M + R_M\cos(\overline{f}_M - \overline{f}_{M1})\right]\underline{R}_{M1}/\overline{\ell}_M^2$$

$$+ R_M R_{M1}\sin(\overline{f}_M - \overline{f}_{M1})\underline{V}_{M1}/\overline{\ell}_M \tag{B4-60}$$

Now define $\underline{e}_\infty$ by

$$\underline{e}_\infty = \lim_{R_M \to \infty}\left(\frac{\underline{R}_M}{R_M}\right) \tag{B4-61}$$

and $\phi$ by

$$\phi = \lim_{R_M \to \infty}(\overline{f}_M - \overline{f}_{M1}) \tag{B4-62}$$

Then, from (B4-60) and (B4-61)

$$\underline{e}_\infty = -\frac{1}{\bar{\ell}_M^2}(1 - \cos\phi)\,\underline{R}_{M1} + \frac{R_{M1}}{\bar{\ell}_M}\sin\phi\,\underline{V}_{M1} \tag{B4-63}$$

Solving for $\underline{V}_{M1}$ gives

$$\underline{V}_{M1} = \frac{\bar{\ell}_M}{R_{M1}\sin\phi}\,\frac{\underline{V}_{\infty M}}{V_{\infty M}} + \frac{(1 - \cos\phi)}{\bar{\ell}_M \sin\phi}\,\frac{\underline{R}_{M1}}{R_{M1}} \tag{B4-64}$$

where $\underline{e}_\infty$ has been replaced by $\underline{V}_{\infty M}$ using

$$\underline{e}_\infty = \underline{V}_{\infty M}/V_{\infty M} \tag{B4-65}$$

i.e., the radius vector becomes parallel to the velocity vector at infinity (cf. (A12-58)).

Now define $\gamma_1$ and the unit vectors $\underline{e}_\alpha$ and $\underline{e}_\beta$ as shown in Figure B11.

The velocity $\underline{V}_{M1}$ can be written

$$\underline{V}_{M1} = V_{M1}\cos\gamma_1\,\underline{e}_\alpha + V_{M1}\sin\gamma_1\,\underline{e}_\beta \tag{B4-66}$$

Also

$$\underline{e}_\infty = \cos\phi\,\underline{e}_\alpha + \sin\phi\,\underline{e}_\beta \tag{B4-67}$$

Eliminating $\underline{e}_\beta$ gives

$$\underline{V}_{M1} = V_{M1}(\cos\gamma_1 - \sin\gamma_1/\tan\phi)\,\underline{e}_\alpha + V_{M1}\sin\gamma_1/\sin\phi\,\underline{e}_\infty \tag{B4-68}$$

Using (B4-65) and

$$\underline{e}_\alpha = \underline{R}_{M1}/R_{M1} \tag{B4-69}$$

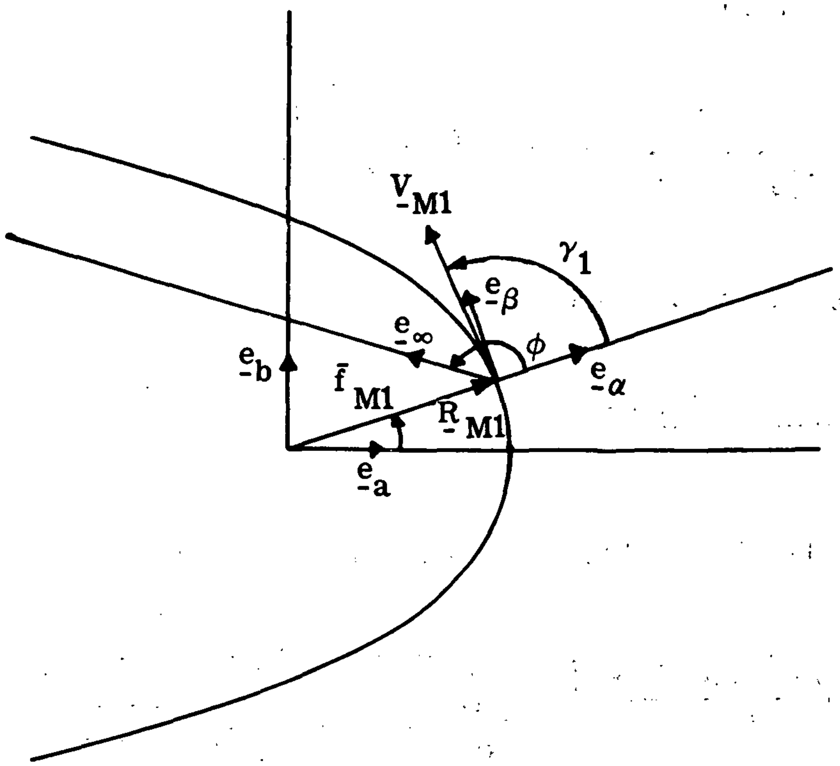


**Figure B11. Orbital Plane Coordinates at t = $t_1$**

gives

$$\underline{V}_M = V_{M1}\left(\cos\gamma_1 - \frac{\sin\gamma_1}{\tan\phi}\right)\frac{\underline{R}_{M1}}{R_{M1}} + V_{M1}\frac{\sin\gamma_1}{\sin\phi}\frac{\underline{V}_{\infty M}}{V_{\infty M}} \tag{B4-70}$$

Equating coefficients of $\underline{V}_{\infty M}$ in (B4-64) and (B4-70) gives

$$\bar{\ell}_M = R_{M1} V_{M1} \sin\gamma_1 \tag{B4-71}$$

which is simply the definition of the scalar angular momentum which agrees with (B4-54). Equating the coefficients of $\underline{R}_{M1}$ gives

$$V_{M1}\left(\cos\gamma_1 - \frac{\sin\gamma_1}{\tan\phi}\right) = \frac{(1 - \cos\phi)}{\bar{\ell}_M \tan\phi} \tag{B4-75}$$

Eliminating $\gamma_1$ using (B4-71) and solving for $V_{M1}$ gives

$$V_{M1}^2 = \frac{\bar{\ell}_M^2}{R_{M1}^2} + \left[\frac{\bar{\ell}_M}{R_{M1}}\frac{\cos\phi}{\sin\phi} + \frac{(1-\cos\phi)}{\bar{\ell}_M\sin\phi}\right]^2 \tag{B4-76}$$

From the conservation of energy

$$\frac{V_{M1}^2}{2} - \frac{1}{R_{M1}} = \frac{V_{\infty M}^2}{2} \tag{B4-77}$$

Substituting (B4-76) into (B4-77) and solving for $\bar{\ell}_M$ gives

$$\bar{\ell}_M = \frac{R_{M1}\, V_{\infty M}\sin\phi}{2}\left[1 + \sqrt{1 + \frac{4}{R_{M1}\, V_{\infty M}^2\,(1+\cos\phi)}}\right] \tag{B4-78}$$

where the plus sign is used in front of the radical in order to satisfy (B4-54).

Using (B4-78) in (B4-64) eventually gives

$$\underline{V}_{M1} = \frac{1}{2}\left[1 + \sqrt{1 + \frac{4}{R_{M1}\, V_{\infty M}^2\,(1+\cos\phi)}}\right]\underline{V}_{\infty M}$$

$$- \frac{V_{\infty M}}{2\, R_{M1}}\left[1 - \sqrt{1 + \frac{4}{R_{M1}\, V_{\infty M}^2\,(1+\cos\phi)}}\right]\underline{R}_{M1} \tag{B4-79}$$

This expression gives the initial velocity $\underline{V}_{M1}$ in terms of $\underline{V}_{\infty M}$ and $\underline{R}_{M1}$ since

$$\cos\phi = (\underline{R}_{M1}\cdot\underline{V}_{\infty M})\,/\,(R_{M1}\, V_{\infty M}) \tag{B4-80}$$

Finally the inclination is defined by

$$\cos \bar{i}_M = \underline{e}_3 \cdot \underline{H}_M \tag{B4-81}$$

where $\underline{H}_M$ is the unit normal to the orbit plane and is defined by

$$\underline{H}_M = (\underline{R}_{M1} \times \underline{V}_{M1}) / \bar{\ell}_M \tag{B4-82}$$

Therefore, combining (B4-81) and (B4-82) gives

$$\cos \bar{i}_M = (\underline{e}_3 \cdot \underline{R}_{M1} \times \underline{V}_{M1})/\bar{\ell}_M \tag{B4-83}$$

Substituting (B2-11) and (B4-54) into (B2-10) gives

$$\bar{\rho}_M = \left[\left(1 + V_{\infty M}^2 \bar{\ell}_M^2\right)^{1/2} - 1\right] / V_{\infty M}^2 \tag{B4-84}$$

It was shown in Section B2 that prescribed values of the hyperbolic excess velocity, the pericenter radius and the inclination are sufficient to determine the inner hyperbola and its two constants, $\underline{A}_{Mo}$ and $\underline{C}_{Mo}$. Equations (B4-83) and (B4-84) give the radius and inclination as functions of the excess velocity and the initial position vector. It has therefore been shown that the excess velocity and initial position do indeed determine the inner hyperbola as was stated at the beginning of this section.

A final parameter which must be determined before proceeding to the boundary value solution is $\tau_M$. In the previous boundary value solutions it was stated that $\tau_k$ was arbitrary. Such is not the case now, as can easily be shown. The hyperbolic eccentric anomaly, $\bar{F}$, at time $S_M = S_{M1}$ is given by the standard expressions (cf. Carlson[3]).

$$\cosh \bar{F}_1 = (\bar{a}_M + R_{M1})/(\bar{a}_M \bar{e}_M) \tag{B4-85}$$

$$\sinh \bar{F}_1 = (\underline{V}_{M1} \cdot \underline{R}_{M1})/(\bar{e}_M \bar{a}_M{}^{1/2}) \tag{B4-86}$$

Then from Kepler's equation, (A9-2),

$$S_{M1} = (\bar{e}_M \sinh \bar{F}_1 - \bar{F}_1) / \bar{n}_M \tag{B4-87}$$

and, from (B4-47) and (B4-48)

$$S_{M1} = (t_1 - t_M - \mu_M \tau_M) / \mu_M \tag{B4-88}$$

The time $t_M$ is the time at which the zeroth order outer solution passes through the center of the moon. But from (B4-39)

$$t_M = t_1 \tag{B4-89}$$

and therefore, from (B4-88)

$$\tau_M = - S_{M1} \tag{B4-90}$$

(Note: It would be possible to choose $t_M$ different from $t_1$ and still obtain a solution. However, the value of $t_M$ is required before $S_1$ can be calculated because the inner solution requires knowledge of the zeroth order outer solution to get the zeroth order approximation for $\underline{V}_{\infty M}$. This means that $t_1$, $t_M$ and $S_1$ are always fixed in (B4-88) and $\tau_M$ is then given by

$$\tau_M = - S_{M1} + (t_1 - t_M) / \mu \tag{B4-91}$$

Thus $\tau_M$ is never arbitrary. (The choice of $t_M = t_1$ gives the simplest result.)

B4.2.3 Boundary Value Solution

The single impulse boundary value problem is shown in Figure B12. It is similar to the earth-to-moon problem shown in Figure B6. The terminal conditions at the earth are given by (B2-27) and (B2-28) with $t_o = t_e$. From Figure B12

$$\underline{r}_o(t_e) = \underline{r}(t_e) \tag{B4-92}$$

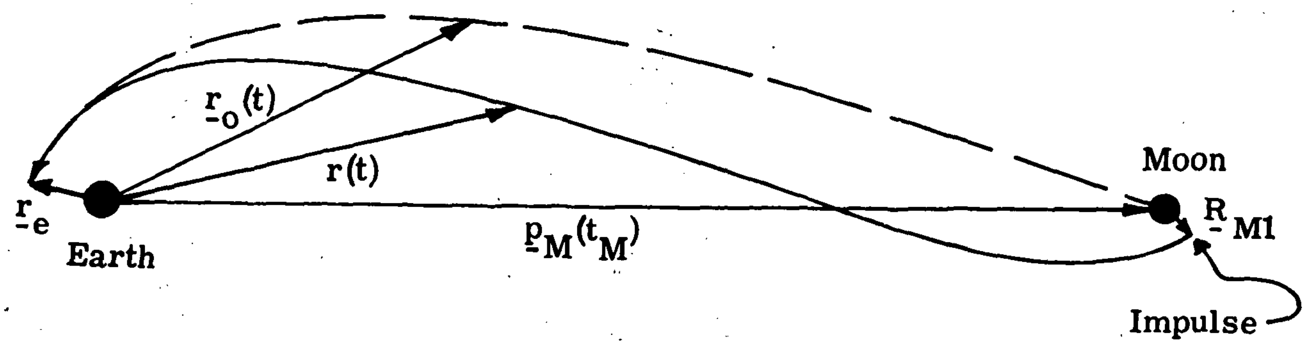


Figure B12. Single Impulse Moon-to-Earth Solution therefore

$$\underline{r}_1(t_e) = \underline{r}_2(t_e) = 0 \qquad \text{(B4-93)}$$

and, as in (B2-31), let

$$\delta \underline{v}(t_e) = \underline{v}_1(t_e) + \mu\, \underline{v}_2(t_e) \qquad \text{(B4-94)}$$

Now let k = M with

$$\mu = \mu_M \qquad \text{(B4-95)}$$

$$Q_M = +1 \qquad \text{(B4-96)}$$

and substitute (B2-24), (B2-25), (B4-93) and (B4-94) into (B1-8). Since $t_M = t_1$ (B1-8) reduces to

$$B(t_1, t_e)\, \delta \underline{v}(t_e) = \underline{L}_M - \left[\tau_M + \frac{1}{\bar{n}_M} \log\left(\frac{\mu_M \bar{e}_M}{2\bar{n}_M}\right)\right] \underline{V}_{\infty M} + \underline{\gamma}_M + \mu\, \underline{\zeta}_M \qquad \text{(B4-97)}$$

$$D(t_1, t_e)\, \delta \underline{v}(t_e) = \mu^{-1} (\underline{V}_{\infty M} - \underline{V}_M) + \underline{\delta}_M + \mu\, \underline{\phantom{x}}_M \qquad \text{(B4-98)}$$

where

$$\underline{V}_M = \underline{v}_o(t_1) - \dot{\underline{p}}_M(t_1) \qquad \text{(B4-99)}$$

should not be confused with $\underline{V}_M(S_M)$ defined by (B4-52). Solving for $\delta \underline{v}(t_e)$ and $\underline{V}_{\infty M}$ gives

$$\delta \underline{v}(t_e) = B(t_1, t_e)^{-1} \left\{ L_M - \left[\tau_M + \frac{1}{\bar{n}_M} \log\left(\frac{\mu_M \bar{e}_M}{2\bar{n}_M}\right)\right] \underline{V}_{\infty M} \right.$$

$$\left. + \underline{\gamma}_M + \mu\, \underline{\zeta}_M \right\} \qquad \text{(B4-100)}$$

$$\underline{V}_{\infty M} = \underline{V}_M + \mu D(t_1, t_e)\, \delta\underline{v}(t_e) - \mu\underline{\delta}_M - \mu^2 \underline{\eta}_M \tag{B4-101}$$

These two expressions are not explicit since $V_{\underline{\infty}M}$ and $\delta\underline{v}(t_e)$ appear on the right hand sides. The solution is obtained from the following steps:

1. Given the initial time $t_1$ solve (B4-39) and (B4-41) for $\underline{r}_o(t_1)$ using the ephemeris for $\underline{p}_M(t_1)$.

2. Using the prescribed terminal values of $t_e$, $r_e$, $\gamma_e$ and $i_e$ solve the modified Lambert problem given in Section B4. 1. This gives $\underline{v}_o(t_1)$, $\underline{r}_o(t_e)$ and $\underline{v}_o(t_e)$.

3. Solve (B4-99) for $\underline{V}_M$ where $\dot{\underline{p}}_M(t_1)$ is obtained from the ephemeris.

4. Let $\underline{V}_{\infty M} = \underline{V}_M$. This gives the zeroth order excess velocity.

5. Using the prescribed initial position $\underline{R}_{M1}$ evaluate (B4-78) - (B4-84).

6. Evaluate (B2-1) - (B2-26) and (B4-85) - (B4-90).

7. Evaluate (B4-100) using the results of step 6 and with $\underline{\zeta}_M = 0$. This gives the first order $\delta\underline{v}(t_e)$.

8. Evaluate (B4-101) using the results of step 7 and with $\underline{\eta}_M = 0$. This gives the first order excess velocity.

9. Repeat steps 5 and 6 using the first order excess velocity.

10. Evaluate (B4-100) using the results of step 9. This gives the second order $\delta\underline{v}(t_e)$.

11. Evaluate (B4-101) using the results of step 10. This gives the second order excess velocity.

12. Repeat step 5. This gives the second order initial velocity $\underline{V}_{M1}$.

13. The single impulse is given by

$$\Delta \underline{V}_{M1} = \underline{V}_{M1} - \underline{V}_M(S_{M1}^-) \qquad \text{(B4-102)}$$

where $\underline{V}_M(S_{M1}^-)$ is the velocity just prior to the impulse.

14. The entry position and velocities are given by (B4-92) and

$$\underline{v}(t_e) = \underline{v}_o(t_e) + \mu \underline{\delta v}\ (t_e) \qquad \text{(B4-103)}$$

Steps 12, 13 and 14 represent the solution of the single impulse problem using the standard technique. A nonlinear solution is also possible using

$$\underline{r}_o'(t_1) = \underline{r}_o(t_1) + \mu \left\{ \underline{L}_M - \left[ \tau_M + \frac{1}{\bar{n}_M} \log\left( \frac{\mu_M \bar{e}_M}{2\bar{n}_M} \right) \right] \underline{V}_{\infty M} \right.$$

$$\left. + \underline{\gamma}_M + \mu \underline{\zeta}_M \right\} \qquad \text{(B4-104)}$$

instead of (B4-39) - (B4-41) as the endpoint for a new modified Lambert problem. Solution of the new Lambert proglem gives $\underline{v}_o'(t_1)$ and $\underline{v}_o'(t_e)$. Then (B4-101) is replaced by

$$\underline{V}_{\infty M} = \underline{V}_M' - \mu \underline{\delta}_M - \mu^2 \underline{\eta}_M \qquad \text{(B4-105)}$$

where

$$\underline{V}_M' = \underline{v}_o'(t_1) - \underline{\dot{p}}_M(t_1) \qquad \text{(B4-106)}$$

and (B4-103) is replaced by

$$\underline{v}(t_e) = \underline{v}_o'(t_e) \qquad \text{(B4-107)}$$

The non-linear solution must also be evaluated in the sequence of the standard solution since the right hand side of (B4-104) is a function of $\underline{V}_{\infty M}$. In practical applications the entry will always be near perigee and the non-linear solution is probably preferable to the standard solution since it bypasses the errors introduced by the linear matrix $B(t_1, t_e)$. An even greater advantage of the non-linear solution over the standard solution is the fact that the latter does not satisfy the entry conditions exactly while the former does.

In the linear solution, the modified Lambert solution (the zeroth order solution) satisfies the entry conditions exactly but because of (B4-94) the first and second order entry velocities are slightly different. Thus the entry conditions remain satisfied only to order unity. In the non-linear solution the total entry velocity,

$$\underline{v}(t_e) = \underline{v}_o(t_e) + \mu\, \underline{v}_1(t_e) + \mu^2\, \underline{v}_2(t_e) \tag{B4-108}$$

is replaced by (B4-107) which comes from the modified Lambert solution. Since the Lambert solution satisfies the entry conditions exactly so does the total entry velocity and the entry conditions are satisfied to second order. The non-linear solution is shown in Figure B13.

### B4.3 Two Impulse Solution

The two impulse problem might be thought of as an extension of the single impulse problem. It is, however, a somewhat different problem since the impulse which gives a trajectory satisfying the earth entry conditions occurs not in the inner region, as in the single impulse case, but rather in a region which is more closely associated with the overlap domain. Since both the inner and outer solutions are valid in the overlap domain it should be possible to represent a trajectory in this region by either solution. This leads to two alternate approaches to solving the two-impulse problem. These approaches are developed in the following sections.

#### B4.3.1 Inner Solution

The inner velocity prior to the first impulse is $\underline{V}_M(S_M^-)$. Then according to (B4-3) the first impulse is

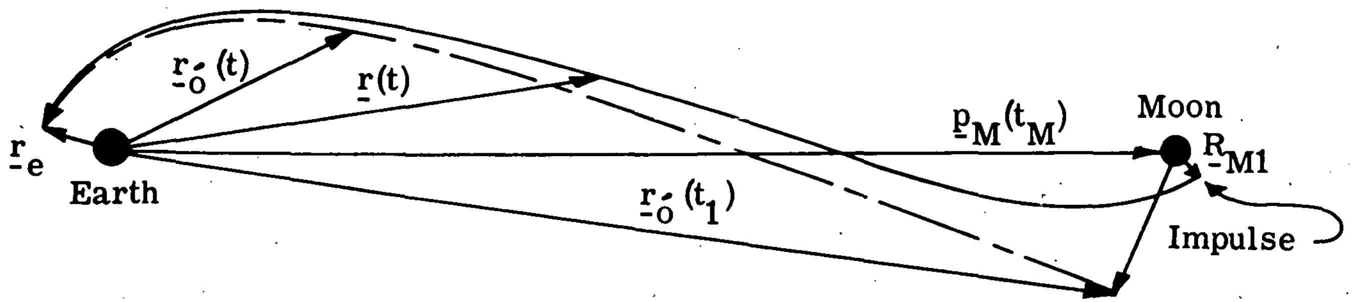


Figure B13. Non-Linear Version of Single Impulse Moon-to-Earth Solution

$$\Delta \underline{V}_1 = I_1 \, \underline{V}_M (S_1^-) \qquad \text{(B4-109)}$$

The value of $I_1$ must be chosen to give a hyperbolic trajectory after the impulse, thus

$$I_1 \geq \left(\sqrt{2/R_M(S_1)}\right) / V_M (S_1^-) - 1 \qquad \text{(B4-110)}$$

where $R_M(S_1)$ is the initial inner radius. Then the velocity following the impulse is

$$\underline{V}_M (S_1^+) = (1 + I_1) \, \underline{V}_M (S_1^-) \qquad \text{(B4-111)}$$

The velocity $\underline{V}_M (S_1^+)$, which is assumed known, uniquely defines the zeroth order inner hyperbola. The elements defining this trajectory can now be obtained. They are

$$\underline{\ell} = \underline{R}_M (S_1) \times \underline{V}_M (S_1^+)$$

$$= \text{angular momentum vector} \qquad \text{(B4-112)}$$

$$\overline{\ell} = \left| \underline{\ell} \right|$$

$$= \text{angular momentum} \qquad \text{(B4-113)}$$

$$\underline{N} = \underline{\ell}/\overline{\ell}$$

$$= \text{unit normal vector} \qquad \text{(B4-114)}$$

$$\overline{h} = V_M (S_1^+)^2 / 2 - 1/R_M (S_1)$$

$$= \text{energy} \qquad \text{(B4-115)}$$

$$\overline{a} = 1/(2\overline{k})$$

$$= \text{semi-major axis} \qquad \text{(B4-116)}$$

$$\bar{e} = (1 + 2\overline{h\ell}^2)^{1/2}$$

$$= \text{eccentricity} \tag{B4-117}$$

$$\bar{n} = \bar{a}^{-3/2}$$

$$= \text{mean motion} \tag{B4-118}$$

$$\bar{i} = \cos^{-1} (\underline{N} \cdot \underline{e}_3)$$

$$= \text{inclination} \tag{B4-119}$$

In addition, there are the argument of the ascending node,

$$\sin \bar{\Omega} = (\underline{N} \cdot \underline{e}_1) \Big/ \sin \bar{i} \tag{B4-120}$$

$$\cos \bar{\Omega} = - (\underline{N} \cdot \underline{e}_2) \Big/ \sin \bar{i} \tag{B4-121}$$

the initial eccentric anomaly,

$$\sinh \bar{F}_1 = \left[\underline{R}_M (S_1) \cdot \underline{V}_M (S_1^+)\right] \Big/ (\bar{a}^{1/2}\, \bar{e}) \tag{B4-122}$$

$$\cosh \bar{F}_1 = \left[\bar{a} + R_M (S_1)\right] \Big/ \bar{a}\, \bar{e} \tag{B4-123}$$

the initial true anomaly,

$$\sin \bar{f}_1 = \left[\bar{a}\, (\bar{e}^2 - 1)^{1/2} \sinh \bar{F}_1\right] \Big/ R_M (S_1) \tag{B4-124}$$

$$\cos \bar{f}_1 = \bar{a}(\bar{e} - \cosh \bar{F}_1) \Big/ R_M (S_1) \tag{B4-125}$$

and the argument of pericynthion

$$\bar{\omega} = \tilde{\omega}_1 - f_1 \tag{B4-126}$$

where

$$\sin \tilde{\omega} = \frac{\left[\underline{e}_3 \times \underline{N} \times \underline{R}_M (S_1)\right]}{R(S_1) \sin \bar{i}} \tag{B4-127a}$$

$$\cos \tilde{\omega} = \frac{\left[\underline{e}_3 \times \underline{N} \cdot \underline{R}_M (S_1)\right]}{R(S_1) \sin \bar{i}} \tag{B4-127b}$$

All of these expressions are derived from standard two-body relationships. Finally, from Kepler's equation for a hyperbola

$$S_1 = (\bar{e} \sinh \bar{F}_1 - \bar{F}_1) / \bar{n} \tag{B4-128}$$

while the inner time is defined by (A7-27) and (A13-2) as

$$S_1 = (t_1 - t_M - \mu\tau_M) / \mu \tag{B4-129}$$

Equating (B4-128) and (B4-129) gives

$$\tau_M = (t_1 - t_M) / \mu + (\bar{F}_1 - \bar{e} \sinh \bar{F}_1) / \bar{n} \tag{B4-131}$$

In this particular problem $t_1$ is fixed and $t_M$ is arbitrary. Putting $t_M = t_1$ gives

$$\tau_M = (\bar{F}_1 - \bar{e} \sinh \bar{F}_1) / \bar{n} \tag{B4-131}$$

Using the elements defined here it is possible to determine the behavior of the inner solution when S is large using the results of Section A12. The values of A', B' and C' are found from (A12-27) - (A12-29), the values of $\bar{U}'$ and $\bar{V}'$ are found from (A12-38) and (A12-39) with Q = 1, and the values of $\underline{V}_{\infty M}$ and $\underline{L}_M$ are found from (A12-48) - (A12-55).

The values of $\underline{V}_{\infty M}$ and $\underline{L}_M$ can be used in the fundamental solution to determine the position and velocity at some later time, i.e., the time of the second impulse. This approach would be followed if the second impulse were to be added strictly in the outer domain.

If the second impulse occurs in a time period approximately 11 to 18 hours after the first impulse then it occurs in the overlap domain where both the outer and inner solutions are valid. Therefore, rather than using the fundamental solution, which includes the matching of the inner and outer solutions, the position and velocity prior to the second impulse can be determined from an inner, perturbed hyperbola alone. Carlson has derived some simple formulas to achieve this.

The position and velocity at any time S, where S > 1, can be found from

$$\underline{R}_M (S) = \underline{R}_{Mo} (S) + \mu^2 \underline{R}_{M_2} (S) + \mu^3 \underline{R}_{M_3} (S) \tag{B4-132}$$

$$\underline{V}_M (S) = \underline{V}_{Mo} (S) + \mu^2 \underline{V}_{M_2} (S) + \mu^3 \underline{V}_{M_3} (S) \tag{B4-133}$$

where $\underline{R}_{Mo}$ and $\underline{V}_{Mo}$ are the two-body position and velocity,

$$\underline{R}_{M_2} (S) = G_M \left[ \underline{R}_{Mo} (S)\, S^2 / 2 - \underline{V}_{Mo} (S)\, S^3 / 3 \right] \tag{B4-134}$$

$$\underline{V}_{M_2} (S) = G_M \left[ \underline{R}_{Mo} (S)\, S - \underline{V}_{Mo} (S)\, S^2 / 2 \right] \tag{B4-135}$$

and

$$\underline{R}_{M_3} = \underline{A}_3\, S^4 \tag{B4-136}$$

$$\underline{V}_{M_3} = 4\, \underline{A}_3\, S^3 \tag{B4-137}$$

The two-body position can be obtained from any standard expression, (A9-1) is an example. The two-body velocity can then be obtained by differentiation with respect to S or by standard velocity expressions. The equations for $\underline{R}_{M2}$ and $\underline{V}_{M2}$ are simplifications of Carlson's expressions found by substituting $G_M$, which is defined by

$$G_M = G(\underline{p}_M(t_1)) \tag{B4-138}$$

for his time averaged gravity gradient matrix. The expressions for $\underline{R}_{M3}$ and $\underline{V}_{M3}$, found from (A12-102) and (A12-103), make a first order correction for the substitution of the constant matrix $G_M$ for the time averaged value which he used. Using $G_M$ simplifies his expressions considerably with very little loss of accuracy.

The expressions for $\underline{R}_{M2}$ and $\underline{V}_{M2}$ were derived using Taylor series expansions[3] while the expansion for $R_{M2}$ in Section A12 is an asymptotic expansion. The two forms are equivalent if the product $\bar{n}S \gg 1$, however the algebra necessary to show this is long and tedious. For moderate values of $\bar{n}S$, (A12-81) is not a good approximation and (B4-134) is preferred.

### B4.3.2 Overlap Solutions

The second impulse is to be applied at some prescribed time $t = t_2$. The impulse must result in a trajectory satisfying terminal constraints at a later time. In order to calculate the impulse it is necessary to determine the position and velocity at $t_2^-$, i.e., at the instant just prior to adding the impulse. There are two possible ways of achieving this.

The first method involves the use of the fundamental solution. This requires knowledge of a zeroth order outer solution. As in the single impulse solution, (B4-39) - (B4-41), let

$$\underline{r}_o(t_1) = \underline{p}_M(t_1) \tag{B4-139}$$

This gives the initial position. Now solve (B4-99) giving

$$\underline{v}_o(t_1) = \dot{\underline{p}}_M(t_1) + \underline{V}_M \tag{B4-140}$$

This is the initial velocity. In other solutions $\underline{V}_M$ has been used as the first approximation to $\underline{V}_{\infty M}$. Since $\underline{V}_{\infty M}$ was determined in the previous section let

$$\underline{V}_M = \underline{V}_{\infty M} \tag{B4-141}$$

Then at any time t the zeroth order solution is given by (A6-2) and its derivative

$$\underline{r}_o(t) = f_o(t)\,\underline{r}_o(t_1) + g_o(t)\,\underline{v}_o(t_1) \tag{B4-142}$$

$$\underline{v}_o(t) = \dot{f}_o(t)\,\underline{r}_o(t_1) + \dot{g}_o(t)\,\underline{v}_o(t_1) \tag{B4-143}$$

These expressions can be used to find the zeroth order position and velocity at $t = t_2$.

Since the moon corresponds to the launch body the perturbations at $t_2$ are found from (B2-56) by letting $t_o = t_2$, $L = M$ and adding the second order terms from (B1-8). The result is

$$\begin{pmatrix} \underline{r}_1(\bar{t}_2) + \mu\,\underline{r}_2(\bar{t}_2) \\ \underline{v}_1(\bar{t}_2) + \mu\,\underline{v}_2(\bar{t}_2) \end{pmatrix} = \Phi(t_2, t_1)\left\{\begin{pmatrix} \underline{L}_M - \left[\tau_M + \frac{1}{\bar{n}}\log\left(\frac{\mu\bar{e}}{2\bar{n}}\right)\right]\underline{V}_{\infty M} \\ 0 \end{pmatrix} + \begin{pmatrix} \underline{\gamma}_M \\ \underline{\delta}_M \end{pmatrix} + \mu\begin{pmatrix} \underline{\zeta}_M \\ \underline{\eta}_M \end{pmatrix}\right\} \tag{B4-144}$$

where the constants $\underline{\gamma}_M$, $\underline{\delta}_M$, $\underline{\zeta}_M$ and $\underline{\eta}_M$ are evaluated along $\underline{r}_o(t)$.

The position and velocity at $t = \bar{t_2}$ are found by combining (B4-142) - (B4-144) giving

$$\underline{r}(\bar{t_2}) = \underline{r}_o(\bar{t_2}) + \mu \underline{r}_1(\bar{t_2}) + \mu^2 \underline{r}_2(\bar{t_2}) \tag{B4-145}$$

$$\underline{v}(\bar{t_2}) = \underline{v}_o(\bar{t_2}) + \mu \underline{v}_1(\bar{t_2}) + \mu^2 \underline{v}_2(\bar{t_2}) \tag{B4-146}$$

By considering all of the error terms in Sections A12 and A13 it can be shown that in the region of interest the dominant error in (B4-145) is $0\left(\mu^3/(t_2 - t_1)^2\right)$. The following table for position error can be constructed:

| | | | | |
|---|---|---|---|---|
| $t_2 - t_1$, hours | 5 | 11 | 18 | 20 |
| $0(t_2 - t_1)$ | $\mu^{2/3}$ | $\mu^{1/2}$ | $\mu^{2/5}$ | $\mu^{3/8}$ |
| $0(\mu^3/(t_2 - t_1)^2)$ | $\mu^{1.67}$ | $\mu^{2.00}$ | $\mu^{2.20}$ | $\mu^{2.25}$ |

Thus as $t_2 - t_1$ increases the theoretical error decreases (i.e., the exponent of $\mu$ increases). Although the first term ignored in (B4-145) is order $\mu^3$ the error shown in the table is somewhat larger.

The second method does not use the fundamental solution but rather the perturbed hyperbola discussed in the previous section. Solving (A7-26) for $\underline{r}$ gives

$$\underline{r} = \underline{p}_M + \mu \underline{R}_M \tag{B4-147}$$

and differentiating with respect to t gives

$$\frac{d\underline{r}}{dt} = \dot{\underline{p}}_M + \frac{d\underline{R}_M}{dS} \tag{B4-148}$$

Letting $d\underline{r}/dt = \underline{v}(t)$ and $d\underline{R}_M/dS = \underline{V}_M(S)$ gives

$$\underline{r}(t_2^-) = \underline{p}_M(t_2^-) + \mu \underline{R}_M(S_2) \qquad \text{(B4-149)}$$

$$\underline{v}(t_2^-) = \dot{\underline{p}}_M(t_2^-) + \underline{V}_M(S_2) \qquad \text{(B4-150)}$$

where $\underline{p}_M$ and $\dot{\underline{p}}_M$ are the position and velocity of the moon at $t = t_2^-$ and $\underline{R}_M(S_2)$ and $\underline{V}_M(S_2)$ are obtained from (B4-132) - (B4-138). The value of $S_2$ is found from

$$S_2 = S_1 + (t_2 - t_1) / \mu \qquad \text{(B4-151)}$$

Based on Carlson's results[3] the dominant error in (B4-149) is $0\left(\mu(t_2 - t_1)^2\right)$ which results in the following table for position error:

| | | | | |
|---|---|---|---|---|
| $t_2 - t_1$, hours | 5 | 11 | 18 | 20 |
| $0\ (\mu(t_2 - t_1)^2)$ | $\mu^{2.33}$ | $\mu^{2.00}$ | $\mu^{1.80}$ | $\mu^{1.75}$ |

This solution has an increasing theoretical error as $t_2 - t_1$ increases (i.e., the exponent of $\mu$ decreases), which is just the opposite of the fundamental solution. At 5 hours the perturbed hyperbola is more accurate, at 11 hours the accuracy is the same for both solutions, and at 18 and 20 hours the fundamental solution is more accurate. It would therefore appear as if one solution or the other should be used depending on the value of $t_2 - t_1$. The following points, however, should be considered:

1. The perturbed hyperbola solution was evaluated for a hyperbola of moderate energy. At 5 hours the actual error was $\mu^{2.81}$ rather than the predicted $\mu^{2.33}$. At 20 hours the actual error was $\mu^{2.05}$ rather than the predicted $\mu^{1.75}$. Thus the actual errors are less than predicted.

2. Numerical analysis has shown that errors associated with the second order fundamental solution are usually larger than predicted.

3. The perturbed hyperbola solution is much easier and less time consuming to evaluate since it does not involve any definite integrals and uses basically two-body functions.

Although either (B4-145) and (B4-146) or (B4-149) and (B4-150) give the position and velocity at $t_2^-$ the latter is preferable.

B4.3.3 Outer Solution

The outer solution will be that part of the trajectory which follows the second impulse. The boundary conditions are the position $\underline{r}(t_2)$ and the entry conditions of time, $t_e$, radius, $r_e$, inclination, $i_e$, and flight path angle, $\gamma_e$. In the modified Lambert problem the subscript 1 is replaced by 2 and the subscript 2 replaced by e so that

$$t_1 = t_2 \tag{B4-152}$$

$$\underline{x}_1 = \underline{r}(t_2) \tag{B4-153}$$

$$t_2 = t_e \tag{B4-154}$$

$$x_2 = r_e \tag{B4-155}$$

$$i_2 = i_e \tag{B4-156}$$

$$\gamma_2 = \gamma_e \tag{B4-158}$$

Solution of the modified Lambert problem gives the zeroth order outer solution $\underline{r}_o(t)$ for $t \geq t_2$.

The first and second order outer solutions are obtained directly from (A6-10) giving

$$\begin{pmatrix} \underline{r}_1(t) + \mu \underline{r}_2(t) \\ \underline{v}_1(t) + \mu \underline{v}_2(t) \end{pmatrix} = \Phi(t, t_\varepsilon) \begin{pmatrix} \underline{r}_1(t_e) + \mu \underline{r}_2(t_e) \\ \underline{v}_1(t_e) + \mu \underline{v}_2(t_e) \end{pmatrix}$$

$$+ \begin{pmatrix} \underline{\Gamma}_{10}(t, t_e) \\ \underline{\Gamma}_{11}(t, t_e) \end{pmatrix} + \mu \begin{pmatrix} \underline{\Gamma}_{20}(t, t_e) \\ \underline{\Gamma}_{21}(t, t_e) \end{pmatrix} \tag{B4-159}$$

where

$$\underline{\Gamma}_{10}(t, t_\varepsilon) = \int_{t_e}^{t} B(t, \tau)\, \underline{F}_1(\tau)\, d\tau \tag{B4-160}$$

$$\underline{\Gamma}_{11}(t, t_\varepsilon) = \int_{t_e}^{t} D(t, \tau)\, \underline{F}_1(\tau)\, d\tau \tag{B4-161}$$

$$\underline{\Gamma}_{20}(t, t_\varepsilon) = \int_{t_e}^{t} B(t, \tau)\, \underline{F}_2(\tau)\, d\tau \tag{B4-162}$$

$$\underline{\Gamma}_{21}(t, t_\varepsilon) = \int_{t_e}^{t} D(t, \tau)\, \underline{F}_2(\tau)\, d\tau \tag{B4-163}$$

### B4.3.4 Boundary Value Solution

The two-impulse boundary value solution is shown in Figure B14. The terminal conditions at the earth are given by (B2-27) and (B2-28) with $t_o = t_e$. From Figure B14

$$\underline{r}_o(t_e) = \underline{r}(t_e) \tag{B4-164}$$

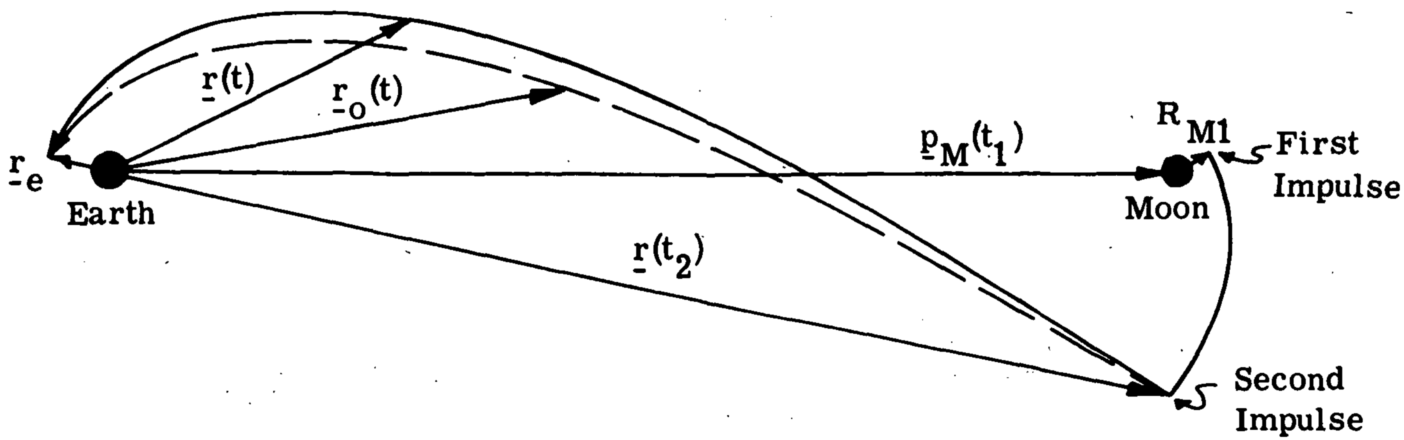


Figure B14. Two-Impulse Moon-to-Earth Solution so that

$$\underline{r}_1(t_e) = \underline{r}_2(t_e) = 0 \tag{B4-165}$$

Since the velocity perturbations do not vanish let

$$\delta \underline{v}(t_e) = \underline{v}_1(t_e) + \mu \underline{v}_2(t_e) \tag{B4-166}$$

The outer position and velocity at $t = t_2^+$ is then

$$\begin{pmatrix} \underline{r}(t_2^+) \\ \underline{v}(t_2^+) \end{pmatrix} = \begin{pmatrix} \underline{r}_o(t_2^+) \\ \underline{v}_o(t_2^+) \end{pmatrix} + \mu \begin{pmatrix} B(t_2^+, t_e) \\ D(t_2^+, t_e) \end{pmatrix} \delta \underline{v}(t_e)$$

$$+ \mu \begin{pmatrix} \underline{\Gamma}_{10}(t_2^+, t_e) \\ \underline{\Gamma}_{11}(t_2^+, t_e) \end{pmatrix} + \mu^2 \begin{pmatrix} \underline{\Gamma}_{20}(t_2^+, t_e) \\ \underline{\Gamma}_{21}(t_2^+, t_e) \end{pmatrix} \tag{B4-167}$$

However, from Figure B14

$$\begin{pmatrix} \underline{r}(t_2^+) \\ \underline{v}(t_2^+) \end{pmatrix} = \begin{pmatrix} \underline{r}(t_2^-) \\ \underline{v}(t_2^-) + \Delta \underline{V}_2 \end{pmatrix} \tag{B4-168}$$

and

$$\underline{r}(t_2^+) = \underline{r}_o(t_2^+) = \underline{r}(t_2^-) \tag{B4-170}$$

where $\underline{r}(t_2^-)$ is found from the inner solution. Then (B4-167) reduces to

$$\delta \underline{v}(t_e) = -B(t_2^+, t_e)^{-1} \left[ \underline{\Gamma}_{10}(t_2^+, t_e) + \mu \underline{\Gamma}_{20}(t_2^+, t_e) \right] \tag{B4-171}$$

$$\Delta \underline{V}_2 = \underline{v}_o(t_2^+) - \underline{v}(t_2^-) + \mu \left[ D(t_2^+, t_e)\, \delta \underline{v}(t_e) + \underline{\Gamma}_{11}(t_2^+, t_e) \right] + \mu^2 \underline{\Gamma}_{21}(t_2^+, t_e) \qquad \text{(B4-172)}$$

The solution is obtained from the following steps:

1. Given the initial position $\underline{R}_M(S_1)$, velocity $\underline{V}_M(S_1^-)$ and impulse $I_1$ evaluate (B4-111) through (B4-131).

2. Using the results of step 1 and the appropriate equations in Section A12 find the position and velocity at $t_2$ from either (B4-142) - (B4-146) or (B4-149) - (B4-151).

3. Using (B4-170) for the initial position and (B4-154) - (B4-157) for the terminal conditions solve the modified Lambert problem of Section B4.1. This gives $\underline{v}_o(t_2)$, $\underline{r}_o(t_e)$ and $\underline{v}_o(t_e)$.

4. Evaluate (B4-171). This gives the correction to the entry velocity.

5. Evaluate (B4-172). This gives the second order $\Delta \underline{V}_2$, the second impulse.

6. The entry position is given by (B4-164) and the entry velocity by

$$\underline{v}(t_e) = \underline{v}_o(t_e) + \mu \delta \underline{v}(t_e) \qquad \text{(B4-173)}$$

Steps 5 and 6 represent the standard solution for the two impulse problem. As in the single impulse solution the entry conditions are not satisfied exactly because of the non-zero $\delta \underline{v}(t_e)$. A non-linear solution which satisfies the entry conditions exactly to any order is obtained by first rewriting (B4-167) and (B4-168)

$$\begin{pmatrix} \underline{r}(t_2^-) \\ \underline{v}(t_2^-) \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\underline{V}_2 \end{pmatrix} - \mu \begin{pmatrix} \underline{\Gamma}_{10}(t_2^+, t_e) \\ \underline{\Gamma}_{11}(t_2^+, t_e) \end{pmatrix} - \mu^2 \begin{pmatrix} \underline{\Gamma}_{20}(t_2^+, t_e) \\ \underline{\Gamma}_{21}(t_2^+, t_e) \end{pmatrix}$$

$$= \begin{pmatrix} \underline{r}_o(t_2^+) \\ \underline{v}_o(t_2^+) \end{pmatrix} + \mu\Phi(t_2^+, t_e) \begin{pmatrix} 0 \\ \delta\underline{v}(t_e) \end{pmatrix} \tag{B4-174}$$

The right hand side of (B4-174) is the sum of a two-body solution plus the propagation of initial variations at $t_e$. Thus the right side is a pure two body function and can be replaced by

$$\begin{pmatrix} \underline{r}(t_2^-) \\ \underline{v}(t_2^-) \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\underline{V}_2 \end{pmatrix} - \mu \begin{pmatrix} \underline{\Gamma}_{10}(t_2^+, t_e) \\ \underline{\Gamma}_{11}(t_2^+, t_e) \end{pmatrix} - \mu^2 \begin{pmatrix} \underline{\Gamma}_{20}(t_2^+, t_e) \\ \underline{\Gamma}_{21}(t_2^+, t_e) \end{pmatrix}$$

$$= \begin{pmatrix} \underline{r}'_o(t_2^+) \\ \underline{v}'_o(t_2^+) \end{pmatrix} \tag{B4-175}$$

This defines a new modified Lambert problem for which the new initial position replacing (B4-170) is

$$\underline{x}'_1 = \underline{r}'_o(t_2^+)$$

$$= \underline{r}(t_2^-) - \mu\underline{\Gamma}_{10}(t_2^+, t_e) - \mu^2\underline{\Gamma}_{20}(t_2^+, t_e) \tag{B4-176}$$

Solving the new modified Lambert problem between $\underline{x}'_1$ and the entry conditions gives $\underline{v}'_o(t_2^+)$, $\underline{r}'_o(t_e)$ and $\underline{v}'_o(t_e)$. From (B4-175) the velocity impulse is

$$\Delta\underline{V}_2 = \underline{v}'_o(t_2^+) - \underline{v}(t_2^-) + \mu\underline{\Gamma}_{11}(t_2^+, t_e) + \mu^2\underline{\Gamma}_{21}(t_2^+, t_e) \tag{B4-177}$$

while the entry position and velocity are

$$\underline{r}(t_e) = \underline{r}'_o(t_e) \tag{B4-178}$$

$$\underline{v}(t_e) = \underline{v}'_o(t_e) \tag{B4-179}$$

Since (B4-176) and (B4-177) come from the modified Lambert solution they will satisfy the entry conditions exactly to any order. The non-linear solution is shown in Figure B15.

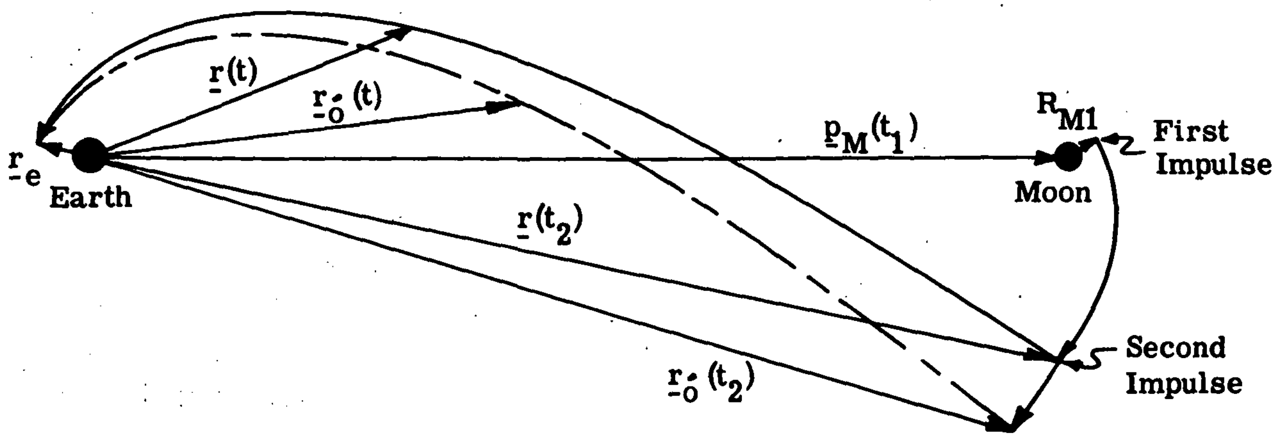


Figure B15. Non-Linear Version of Two-Impulse Moon-to-Earth Solution

## Section C
## EVALUATION OF PERTURBATION TERMS

### C1 TYPES OF PERTURBATION TERMS

The $n^{th}$ order outer solution is given by (A6-10). It consists of two types of terms: (1) two-body propagation of the $n^{th}$ order initial conditions, and (2) integrated effects of the $n^{th}$ order perturbations due to the N-2 perturbing bodies. Both of these effects enter into the boundary value solutions through the constants $\underline{\lambda}_k$, $\underline{\delta}_k$, $\underline{\zeta}_k$, and $\underline{\eta}_k$ which in turn are functions of partial derivative matrices and definite integrals (plus algebraic and trigonometric terms). Formulas for evaluating the partial derivative matrices and the definite integrals are given in the following sections.

### C2 PARTIAL DERIVATIVE MATRICES

The partial derivative matrices are evaluated on the $zero^{th}$ order outer solution $\underline{r}_o(t)$. The formulas given here were derived by Goodyear[11] and the notation is similar to that of Carlson.[3] Two forms are presented; the first can be used for direct calculation of the matrices and the second is useful in evaluating the definite integrals.

#### C2.1 Goodyear Formulas

The state transition matrix $\Phi(t_o, t)$ is a function of the two times, $t_o$ and t. The eccentric anomaly difference

$$u = E_o - E \tag{C2-1}$$

will be chosen as the independent variable. The Goodyear formulas then require the following functions of u:

$$f(t_o, t) = 1 - \frac{a}{r_o(t_o)} (1 - \cos u) \tag{C2-2}$$

$$\dot{f}(t_o, t) = \frac{na^2 \sin u}{r_o(t_o)\, r_o(t)} \tag{C2-3}$$

$$g(t_o, t) = \frac{1}{na}\, r_o(t_o)\, \left[\sigma - (1-\cos u) - \sin u\right] \tag{C2-4}$$

$$\dot{g}(t_o, t) = 1 - \frac{a(1-\cos u)}{r_o(t)} \tag{C2-5}$$

$$P(t_o, t) = 3(\sin u - u) + e \sin E_o (1-\cos u)^2$$

$$+ (1-\cos u)\, e \cos E_o \sin u \tag{C2-6}$$

$$A_{RR}(t_o, t) = \frac{a}{r_o^{\,2}(t)} \left[\frac{(1-\cos u)}{r_o(t)} + \frac{\sin^2 u}{r_o(t_o)}\right] \tag{C2-7}$$

$$A_{RV}(t_o, t) = \frac{(1-\cos u)\sin u}{n\, r_o(t)\, r_o(t_o)} \tag{C2-8}$$

$$A_{VR}(t_o, t) = \frac{a\, P(t_o, t)}{n\, r_o^{\,3}(t)} - \frac{(1-\cos u)\sin u}{n\, r_o^{\,2}(t)} \tag{C2-9}$$

$$A_{VV}(t_o, t) = -\frac{(1-\cos u)^2}{n^2\, a\, r_o(t)} \tag{C2-10}$$

$$B_{RR}(t_o, t) = A_{RV}(t_o, t) \tag{C2-11}$$

$$B_{RV}(t_o, t) = \frac{(1-\cos u)^2}{n^2\, a\, r_o(t_o)} \tag{C2-12}$$

$$B_{VR}(t_o, t) = A_{VV}(t_o, t) \tag{C2-13}$$

$$B_{VV}(t_o, t) = \frac{1}{n^3 a^2} \left[3(\sin u - u) + (1-\cos u) \sin u\right] \tag{C2-14}$$

$$C_{RR}(t_o, t) = \frac{na^2}{r_o(t)\, r_o(t_o)} \left[\frac{\cos u}{r_o(t)\, r_o(t_o)} + \frac{1}{r_o^{\,2}(t)} + \frac{1}{r_o^{\,2}(t_o)}\right] \sin u - \frac{na^4\, P(t_o, t)}{r_o^{\,3}(t)\, r_o^{\,3}(t_o)} \tag{C2-15}$$

$$C_{RV}(t_o, t) = \frac{a}{r_o^{\,2}(t_o)} \left[\frac{\sin^2 u}{r_o(t)} + \frac{(1-\cos u)}{r_o(t_o)}\right] \tag{C2-16}$$

$$C_{VR}(t_o, t) = - A_{RR}(t_o, t) \tag{C2-17}$$

$$C_{VV}(t_o, t) = - A_{RV}(t_o, t) \tag{C2-18}$$

$$D_{RR}(t_o, t) = C_{RV}(t_o, t) \tag{C2-19}$$

$$D_{RV}(t_o, t) = \frac{(1-\cos u)\sin u}{n\, r_o^{\,2}(t_o)} - \frac{a\, P(t_o, t)}{n\, r_o^{\,3}(t_o)} \tag{C2-20}$$

$$D_{VR}(t_o, t) = - A_{RV}(t_o, t) \tag{C2-21}$$

$$D_{VV}(t_o, t) = - B_{RV}(t_o, t) \tag{C2-22}$$

where $r_o(t_o)$ and $r_o(t)$ are the radii of the zero$^{th}$ order outer solution at $t_o$ and t; a, e and n are the semi-major axis, eccentricity and mean motion of $\underline{r}_o(t)$; and

$$\sigma = \frac{a\, e}{r_o(t_o)} \sin E_o \qquad \text{*C2-23)}$$

After some modification Goodyear's results can be expressed in the form of the transition matrix, i.e.,

$$\begin{pmatrix} A(t_o, t) & B(t_o, t) \\ C(t_o, t) & D(t_o, t) \end{pmatrix} = \begin{pmatrix} A^*(t_o, t) & B^*(t_o, t) \\ C^*(t_o, t) & D^*(t_o, t) \end{pmatrix} \begin{pmatrix} E^*(t_o, t) & F^* \\ F^* & E^*(t_o, t) \end{pmatrix} \tag{C2-24}$$

where

$$A^*(t_o, t) = \left[ \dot{g}(t_o, t) \; A_{RR}(t_o, t) \; A_{RV}(t_o, t) \; A_{VR}(t_o, t) \; A_{VV}(t_o, t) \right] \tag{C2-25}$$

$$B^*(t_o, t) = \left[ -g(t_o, t) \; B_{RR}(t_o, t) \; B_{RV}(t_o, t) \; B_{VR}(t_o, t) \; B_{VV}(t_o, t) \right] \tag{C2-26}$$

$$C^*(t_o, t) = \left[ -\dot{f}(t_o, t) \; C_{RR}(t_o, t) \; C_{RV}(t_o, t) \; C_{VR}(t_o, t) \; C_{VV}(t_o, t) \right] \tag{C2-27}$$

$$D^*(t_o, t) = \left[ f(t_o, t) \; D_{RR}(t_o, t) \; D_{RV}(t_o, t) \; D_{VR}(t_o, t) \; D_{VV}(t_o, t) \right] \tag{C2-28}$$

$$E^*(t_o, t) = \begin{pmatrix} I \\ \underline{r}_o(t_o) \, \underline{r}_o(t)^T \\ \underline{r}_o(t_o) \, \underline{v}_o(t)^T \\ \underline{v}_o(t_o) \, \underline{r}_o(t)^T \\ \underline{v}_o(t_o) \, \underline{v}_o(t)^T \end{pmatrix} \tag{C2-29}$$

$$F^* = \begin{pmatrix} O \\ O \\ O \\ O \\ O \end{pmatrix} \tag{C2-30}$$

The elements of A*, B*, C*, and D* are all scalars while the elements of E* are all 3 x 3 matrices, I being the identity matrix and $\underline{x}\,\underline{y}^T$ being the outer product of $\underline{x}$ and $\underline{y}$ (the superscript T indicating the transpose of the vector). Thus, each partial derivative matrix is the sum of five terms, each of which is the product of a scalar times a matrix.

These formulas for the partial derivative matrices may be used either for a variable t, as in the definite integrals or for a fixed value of t. The unique characteristic of this formulation is that each partial derivative matrix can be written as a function of scalars and vectors (plus the identity matrix) which are easily obtained from the zero$^{th}$ order two-body solution.

C2. 2 Modified Goodyear Formulas

Although the independent variable u was introduced in the previous section the formulas still contained $\underline{r}_o(t)$, $\underline{v}_o(t)$, and $r_o(t)$. In the definite integrals it is advantageous to remove this dependence on t by introducing the modified derivative matrices[3]

$$B_r(t_o, t) = \frac{r_o(t)}{na} B(t_o, t) \tag{C2-31}$$

$$D_r(t_o, t) = \frac{r_o(t)}{na} D(t_o, t) \tag{C2-32}$$

and replacing $\underline{r}_o(t)$ and $\underline{v}_o(t)$ in the previous formulas by

$$\underline{r}_o(t) = f(t_o, t)\, \underline{r}_o(t_o) + g(t_o, t)\, \underline{v}_o(t_o) \tag{C2-33}$$

$$\underline{v}_o(t) = \dot{f}(t_o, t)\, \underline{r}_o(t_o) + \dot{g}(t_o, t)\, \underline{v}_o(t_o) \tag{C2-34}$$

The last two expressions are simply another form of (A6-2) and A6-3). The following functions of u are now defined:

$$u_1 = \cos u - K(1-\cos u) \tag{C2-35}$$

$$u_2 = \frac{r_o(t_o)}{na}\left[\sigma(1-\cos u) - \sin u\right] \tag{C2-36}$$

$$u_3 = \frac{a}{r_o(t_o)} \sin u \tag{C2-37}$$

$$u_4 = \frac{r_o(t_o)}{na}\left[\cos u - \sigma \sin u\right] \tag{C2-38}$$

$$u_5 = \frac{(1-\cos u)\sin u}{n\, r_o(t_o)} \tag{C2-39}$$

$$u_6 = -\frac{(1-\cos u)^2}{n^2 a} \tag{C2-40}$$

$$u_7 = \frac{a}{r_o^2(t_o)}\left[\sin^2 u + \frac{na(1-\cos u)\, u_4}{r_o(t_o)} + \frac{a(1-\cos u)^2}{r_o(t_o)}\right] \tag{C2-41}$$

$$u_8 = -u_5 \tag{C2-42}$$

$$u_9 = n\, u_4 + (1-\cos u) \tag{C2-43}$$

$$B_o = -u_2\, u_9/n \tag{C2-44}$$

$$B_1 = u_1\, u_5/(na) + B_{RV}(t_o, t)\, u_3 \tag{C2-45}$$

$$B_2 = u_2\, u_5/(na) + B_{RV}(t_o, t)\, u_4 \tag{C2-46}$$

$$B_3 = u_1\, u_6/(na) + B_{VV}(t_o, t)\, u_3 \tag{C2-47}$$

$$B_4 = u_2\, u_6/(na) + B_{VV}(t_o, t)\, u_4 \tag{C2-48}$$

$$D_0 = u_1 u_9/n \tag{C2-49}$$

$$D_1 = u_1 u_7/(na) + D_{RV}(t_o, t)\, u_3 \tag{C2-50}$$

$$D_2 = u_2 u_7/(na) + D_{RV}(t_o, t)\, u_4 \tag{C2-51}$$

$$D_3 = u_1 u_8/(na) + D_{VV}(t_o, t)\, u_3 \tag{C2-52}$$

$$D_4 = u_2 u_8/(na) + D_{VV}(t_o, t)\, u_4 \tag{C2-53}$$

After some manipulation (C2-31) and (C2-32) become

$$\begin{aligned} B_r(t_o, t) = {} & B_0 I + B_1 \underline{r}_o(t_o)\, \underline{r}_o(t_o)^T + B_2\, \underline{r}_o(t_o)\, \underline{v}_o(t_o)^T \\ & + B_2\, \underline{v}_o(t_o)\, \underline{r}_o(t_o)^T + B_4\, \underline{v}_o(t_o)\, \underline{v}_o(t_o)^T \end{aligned} \tag{C2-54}$$

$$\begin{aligned} D_r(t_o, t) = {} & D_0 I + D_1 \underline{r}_o(t_o)\, \underline{r}_o(t_o)^T + D_2\, \underline{r}_o(t_o)\, \underline{v}_o(t_o)^T \\ & + D_3\, \underline{v}_o(t_o)\, \underline{r}_o(t_o)^T + D_4\, \underline{v}_o(t_o)\, \underline{v}_o(t_o)^T \end{aligned} \tag{C2-55}$$

In these expressions for $B_r$ and $D_r$ the outer product matrices are functions only of $t_o$ and are therefore constant for fixed $t_o$. The scalar coefficients are explicit functions of u (or of t through (C2-1)). Thus, as t varies only the scalar coefficients need to be evaluated.

## C3 DEFINITE INTEGRALS

The four integral constants of the outer solution, (A11-86), (A11-87), (A11-139), and (A11-140), can be written in the form

$$\underline{K}_{10k}(t_k, t_o) = \int_{t_o}^{t_k} \left[ B(t_k, \tau)\, \underline{F}_1(\tau) + \underline{I}_{10}(t_k, \tau) \right] d\tau \tag{C3-1}$$

$$\underline{K}_{11k}(t_k, t_o) = \int_{t_o}^{t_k} \left[D(t_k, \tau)\, \underline{F}_1(\tau) + \underline{I}_{11}(t_k, \tau)\right] d\tau \qquad \text{(C3-2)}$$

$$\underline{K}_{20k}(t_k, t_o) = \int_{t_o}^{t_k} \left[B(t_k, \tau)\, \underline{F}_2(\tau) + \underline{I}_{20}(t_k, \tau)\right] d\tau \qquad \text{(C3-3)}$$

$$\underline{K}_{21k}(t_k, t_o) = \int_{t_o}^{t_k} \left[D(t_k, \tau)\, \underline{F}_2(\tau) + \underline{I}_{21}(t_k, \tau)\right] d\tau \qquad \text{(C3-4)}$$

where

$$\underline{I}_{10}(t_k, \tau) = \underline{\Phi}^s_{10k}(t_k, \tau) - B(t_k, \tau)\, \underline{F}_{1S}(\tau) \qquad \text{(C3-5)}$$

$$\underline{I}_{11}(t_k, \tau) = \underline{\Phi}^s_{11k}(t_k, \tau) - D(t_k, \tau)\, \underline{F}_{1S}(\tau) \qquad \text{(C3-6)}$$

$$\underline{I}_{20}(t_k, \tau) = \underline{\Phi}^s_{20k}(t_k, \tau) - B(t_k, \tau)\, \underline{F}_{2S}(\tau) \qquad \text{(C3-7)}$$

$$\underline{I}_{21}(t_k, \tau) = \underline{\Phi}^s_{21k}(t_k, \tau) - D(t_k, \tau)\, \underline{F}_{2S}(\tau) \qquad \text{(C3-8)}$$

The $\underline{\Phi}$'s and $\underline{F}$'s are defined in Section A11.

The integrals (C3-1) - (C3-4) cannot be integrated in closed form and must therefore be evaluated numerically. The integrands are finite over the entire range of $\tau$ giving each integral a finite value. At $\tau = t_k$, however, both the $\underline{F}$'s and the $\underline{I}$'s are singular and the value of the integrand, which is actually the difference between two large numbers, must be determined by a limit process. This problem is eliminated by using a numerical technique such as Gaussian quadrature, which does not require the value of the integrand at either endpoint of the interval of integration. Thus the integrand is evaluated only at interior points, $t_o < \tau < t_k$, where both the $\underline{F}$'s and $\underline{I}$'s as well as the total integrand are finite.

C3.1 Change of Independent Variable

The accuracy of the integration technique for a fixed number of sub-intervals within the interval $t_o \le \tau \le t_k$ can be improved[3] by introducing the eccentric anomaly through Kepler's equation

$$n\tau = E - e \sin E \tag{C3-9}$$

Then

$$d\tau = \frac{(1-\cos E)}{n} dE$$

$$= \frac{r_o}{na} dE$$

$$= -\frac{r_o}{na} du \tag{C3-10}$$

where (C2-1) has been introduced in the last step. Introducing (C3-10) into (C3-1) - (C3-4) gives

$$\underline{K}_{10k}(t_k, t_o) = \int_o^{u_o} \left[ B_r(t_k, \tau)\, \underline{F}_1(\tau) + \frac{r_o \underline{I}_{10}(t_k, \tau)}{na} \right] du \tag{C3-11}$$

$$\underline{K}_{11k}(t_k, t_o) = \int_o^{u_o} \left[ D_r(t_k, \tau)\, \underline{F}_1(\tau) + \frac{r_o \underline{I}_{11}(t_k, \tau)}{na} \right] du \tag{C3-12}$$

$$\underline{K}_{20k}(t_k, t_o) = \int_o^{u_o} \left[ B_r(t_k, \tau)\, \underline{F}_2(\tau) + \frac{r_o \underline{I}_{20}(t_k, \tau)}{na} \right] du \tag{C3-13}$$

$$\underline{K}_{21k}(t_k, t_o) = \int_o^{u_o} \left[ D_r(t_k, \tau)\, \underline{F}_2(\tau) + \frac{r_o \underline{I}_{21}(t_k, \tau)}{na} \right] du \tag{C3-14}$$

where

$$\tau = t_k - \frac{1}{n}\left[u - e(1-\cos u)\sin E_k - e \sin u \cos E_k\right] \qquad \text{(C3-15)}$$

$$r_o = a\left[1 - e\cos(E_k - u)\right] \qquad \text{(C3-16)}$$

$$u = E_k - E \qquad \text{(C3-17)}$$

and

$$u_o = E_k - E_o \qquad \text{(C3-18)}$$

Equations (C3-11) - (C3-14) have an additional advantage over (C3-1) - (C3-4) which is not necessarily one of accuracy. It is that $\underline{r}_o$ appears both explicitly as $r_o$ and implicitly in determining $\underline{F}_1$ and $\underline{F}_2$. With u (or E) as independent variable, $\underline{r}_o(u)$ can be evaluated directly. With $\tau$ as independent variable, $\underline{r}_o(\tau)$ requires an iterative solution of Kepler's equation. Therefore using u as the independent variable reduces the computational requirements.

## C3.2 Analytical Approximation for First Order Integrand

Even though Gaussian quadrature does not require the values of the integrands at the endpoints, the point nearest u = 0 in (C3-11) - (C3-14) at which the integrands are evaluated may cause computational difficulties. This is because the integrands represent the differences of two large numbers when u is close to zero.

Theoretically the zero$^{th}$ order outer solution passes through the center of the k$^{th}$ body at u = 0 (i.e., at $t = t_k$). Actual numerical solutions of the Lambert problem may not satisfy this requirement at $t = t_k$ but may contain a small residual difference between $\underline{r}_o$ and $\underline{p}_k$ at $t = t_k$. Experience has shown that this small residual is greatly magnified when taking the difference between two large numbers. Therefore it is advantageous to have the first quadrature point as far from u = 0 as possible. This may be accomplished by breaking the integration interval into two parts, i.e., let each of the integrals have the form

$$\underline{K}(t_k, t_o) = \int_{t_o}^{t_{sk}} + \int_{t_{sk}}^{t_k} \tag{C3-19}$$

$$= \int_{u_s}^{u_o} + \int_{o}^{u_s} \tag{C3-20}$$

where

$$u_s = E_k - E_{sk} \tag{C3-21}$$

and

$$t_s - t_{sk} << 1 \tag{C3-22}$$

Over the interval from $t_{sk}$ to $t_k$ the integrands can be replaced by analytical approximations obtained from Section A11. First recall that the force $\underline{F}_1$ can be divided into a singular part, $\underline{F}_{1s}$, and a non-singular part, $\underline{F}_{1n}$. Near $t_k$ the non-singular force can be expanded in a Taylor series

$$\underline{F}_{1n}(\tau) = \underline{F}_{1n}(t_k) + O(\tau - t_k) \tag{C3-23}$$

Then, using (A3-28) and (A3-30)

$$B(t_k, \tau)\, \underline{F}_{1n}(\tau) = \underline{F}_{1n}(t_k)\,(t_k - \tau) + O\left((\tau - t_k)^2\right) \tag{C3-24}$$

$$D(t_k, \tau)\, \underline{F}_{1n}(\tau) = \underline{F}_{1n}(t_k) + O(\tau - t_k) \tag{C3-25}$$

Substituting (C3-24) and (A11-45) into (C3-1) and integrating from $t_{sk}$ to $t_k$ gives

$$\int_{t_{sk}}^{t_k} \left[B(t_k, \tau)\, \underline{F}_1(\tau) + \underline{I}_{10}(t_k, \tau)\right] d\tau = \underline{\beta}_{2k}(t_{sk} - t_k) + \left[\underline{\beta}_{4k}(t_k) + \underline{F}_{1n}(t_k)\right](t_{sk} - t_k)^2/2 + O\left[(t_{sk} - t_k)^3,\ \mu(t_{sk} - t_k)^2\right] \quad \text{(C3-26)}$$

Substituting (C3-25) and the derivative with respect to t of (A11-45) into (C3-2) and integrating gives

$$\int_{t_{sk}}^{t_k} \left[D(t_k, \tau)\, \underline{F}_1(\tau) + \underline{I}_{11}(t_k, \tau)\right] d\tau = -\left[\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k) + \underline{F}_{1n}(t_k)\right](t_{sk} - t_k) + O\left[(t_{sk} - t_k)^2\right] \quad \text{(C3-27)}$$

A similar analysis of the second order integrals shows that over the interval $t_{sk} \le \tau \le t_k$ both contribute terms of order $(t_{sk} - t_k) \log |t_{sk} - t_k|$.

Therefore (C3-11) - C3-14) can be written

$$\underline{K}_{10k}(t_k, t_o) = \int_{u_s}^{u_o} \left[B_r(t_k, \tau)\, \underline{F}_1(\tau) + \frac{r_o \underline{I}_{10}(t_k, \tau)}{na}\right] du + \underline{\beta}_{2k}(t_{sk} - t_k) + \left[\underline{\beta}_{3k} + \underline{\beta}_{4k}(t_k) + \underline{F}_{1n}(t_k)\right] \cdot (t_{sk} - t_k)^2/2 + O\left[(t_{sk} - t_k)^3,\ \mu(t_{sk} - t_k)^2\right] \quad \text{(C3-28)}$$

$$\underline{K}_{11k}(t_k, t_o) = \int_{u_s}^{u_o} \left[ D_r(t_k, \tau)\, \underline{F}_1(\tau) + \frac{r_o\, \underline{I}_{11}(t_k, \tau)}{na} \right] du$$

$$- \underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k) + \underline{F}_{1n}(t_k)\ (t_{sk} - t_k)$$

$$+ O\left((t_{sk} - t_k)^2\right) \tag{C3-29}$$

$$\underline{K}_{20k}(t_k, t_o) = \int_{u_s}^{u_o} \left[ B_r(t_k, \tau)\, \underline{F}_2(\tau) + \frac{r_o\, \underline{I}_{20}(t_k, \tau)}{na} \right] du$$

$$+ O\left((t_{sk} - t_k) \log|t_{sk} - t_k|\right) \tag{C3-30}$$

$$\underline{K}_{21k}(t_k, t_o) = \int_{u_s}^{u_o} \left[ D_r(t_k, \tau)\, \underline{F}_2(\tau) + \frac{r_o\, \underline{I}_{21}(t_k, \tau)}{na} \right] du$$

$$+ O\left((t_{sk} - t_k) \log|t_{sk} - t_k|\right) \tag{C3-31}$$

The interval $(t_{sk} - t_k)$ must be chosen to make the effective error of each approximation smaller than order $\mu^2$. Letting

$$t_{sk} - t_k = O(\mu^n) \tag{C3-32}$$

the effective error of each integral is

$$\mu\, \underline{K}_{10} - O\left(\mu^{1+3n},\ \mu^{2+2n}\right) \qquad \mu\, \underline{K}_{20} - O\left(\mu^{2+n} \log \mu^n\right)$$

$$\mu\, \underline{K}_{11} - O\left(\mu^{1+2n}\right) \qquad \mu\, \underline{K}_{21} - O\left(\mu^{2+n} \log \mu^n\right)$$

These error are all smaller than $O(\mu^2)$ if $n > 1/2$. In actual numerical studies the value used was

$$\left| t_{sk} - t_k \right| = \mu^{2/3} \qquad \text{(C3-33)}$$

This eliminated the problems associated with the residual error in the Lambert solution. If only a first order solution is desired then any $n > 0$ will suffice, $n = 1/3$ would be a likely choice.

C3.3 Analytical Approximation for First Order Solution

The second order integrals are functions of the force $\underline{F}_2$ which, according to (A5-10) is a function of $\underline{r}_1$ and $\underline{r}_1^2$. When integrating over the interval $t_o \geq \tau \geq t_k$, $\underline{r}_1(\tau)$ must be determined from (A6-11) which itself involves an integral function. Thus the second order integrals are actually double integrals and difficult to evaluate efficiently by numerical techniques without using a large number of quadrature points.

An alternative approach is to approximate $\underline{r}_1$ with an analytical function. Such an approximation, valid for t near $t_k$, is given by (A11-78). The range of this approximation can be extended by adding additional terms and fixing the coefficients to give $\underline{r}_1$ the correct magnitude and slope at $t = t_o$. Therefore (A11-78) is rewritten as

$$\begin{aligned}
\underline{r}_1(t) = {} & \underline{r}_{10}(t) - \underline{\beta}_{1k} \log Q_k (t - t_k) + \underline{\beta}_{1k} \left[\log Q_k(t_o - t_k) + 1\right] \\
& + \underline{K}_{10k}(t_k, t_o) + \underline{\beta}_{2k} (t - t_k) \log Q_k (t - t_k) + \Big[\underline{\beta}_{1k}/(t_o - t_k) \\
& + \underline{K}_{11k}(t_k, t_o) - \underline{\beta}_{2k} \left[\log Q_k(t_o - t_k) + 1\right]\Big] (t - t_k) \\
& - 3\underline{\beta}_{4k}(t_k)(t - t_k)^2 \log Q_k (t - t_k) + \Big[\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k) \\
& + 6\underline{\beta}_{4k}(t_k) \log Q_k (t_o - t_k) + G_k \underline{K}_{10k}(t_k, t_o) \\
& + \underline{p}_k^*\Big] (t - t_k)^2/2 + \underline{\phi}_{1K} (t - t_k)^3 + \underline{\phi}_{2k} (t - t_k)^4
\end{aligned} \qquad \text{(C3-34)}$$

where

$$\underline{r}_{10}(t) = A(t, t_o)\, \underline{r}_1(t_o) + B(t, t_o)\, \underline{v}_1(t_o) \tag{C3-35}$$

Taking the derivative with respect to t gives

$$\begin{aligned}
\underline{v}_1(t) = {} & \underline{v}_{10}(t) - \underline{\beta}_{1k}/(t - t_k) + \underline{\beta}_{2k}\left[\log Q_k(t - t_k) + 1\right] \\
& + \underline{\beta}_{1k}/(t_o - t_k) + \underline{K}_{11k}(t_k, t_o) - \underline{\beta}_{2k}\left[\log Q_k(t_o - t_k) + 1\right] \\
& - 3\underline{\beta}_{4k}(t - t_k)\left[2 \log Q_k\, (t - t_k) + 1\right] + \left[\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k)\right. \\
& \left. + 6\underline{\beta}_{4k}(t_k) \log Q_k(t_o - t_k) + G_k\, \underline{K}_{10k}(t_k, t_o) + \underline{p}_k^*\right](t - t_k) \\
& + 3\underline{\phi}_{1k}(t - t_k)^2 + 4\underline{\phi}_{2k}(t - t_k)^3
\end{aligned} \tag{C3-36}$$

where

$$\underline{v}_{10}(t) = C(t, t_o)\, \underline{r}_1(t_o) + D(t, t_o)\, \underline{v}_1(t_o) \tag{C3-37}$$

At $t = t_o$ (C3-35) and (C3-37) reduce to

$$\underline{r}_{10}(t_o) = \underline{r}_1(t_o) \tag{C3-38}$$

$$\underline{v}_{10}(t_o) = \underline{v}_1(t_o) \tag{C3-39}$$

Putting $t = t_o$ in (C3-34) and (C3-36) and solving for $\underline{\phi}_{1k}$ and $\underline{\phi}_{2k}$ gives

$$\underline{\phi}_{1k} = \frac{4\underline{\phi}_{1k}^*}{(t_o - t_k)^3} - \frac{\underline{\phi}_{2k}^*}{(t_o - t_k)^2} \tag{C3-40}$$

$$\underline{\phi}_{2k} = \frac{\underline{\phi}_{2k}^*}{(t_o - t_k)^3} - \frac{3\underline{\phi}_{1k}^*}{(t_o - t_k)^4} \tag{C3-41}$$

where

$$\underline{\phi}^*_{1k} = -2\underline{\beta}_{1k} - \underline{K}_{10k}(t_k, t_o) - \left[\underline{K}_{11k}(t_k, t_o) + \underline{\beta}_{2k}\right](t_o - t_k)$$

$$- \left[\underline{\beta}_{3k} + 3\underline{\beta}_{4k}(t_k) + G_k \underline{K}_{10k}(t_k, t_o)\right.$$

$$\left. + \underline{p}^*_k\right](t_o - t_k)^2/2 \qquad \text{(C3-42)}$$

$$\underline{\phi}^*_{2k} = - \underline{K}_{11k}(t_k, t_o) - \left[\underline{\beta}_{3k} + G_k \underline{K}_{10k}(t_k, t_o)\right.$$

$$\left. + \underline{p}^*_k\right](t_o - t_k) \qquad \text{(C3-43)}$$

Equations (C3-34), (C3-35) and (C3-40) - (C3-43) give an analytical approximation for $\underline{r}_1$ which has the correct behavior at both endpoints of the interval $t_o \leq t \leq t_k$. It is in these regions that the second order integrals are most susceptible to errors in $\underline{r}_1$ and therefore the effect of errors in the middle of the interval is minimized.

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