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AN ITERATIVE SOLUTION OF AN INTEGRAL EQUATION FOR RADIATIVE TRANSFER BY USING VARIATIONAL TECHNIQUE

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16. Abstract

An effective iterative technique is introduced to solve a nonlinear integral equation frequently associated with radiative transfer problems. The problem is formulated in such a way that each step of an iterative sequence requires the solution of a linear integral equation. The advantage of a previously introduced variational technique which utilizes a stepwise constant trial function is exploited in this report to cope with the nonlinear problem. The method is simple and straightforward. Rapid convergence is obtained by employing a linear interpolation of the iterative solutions. Using absorption coefficients of the Milne-Eddington type, $\rho k_\nu = \alpha_\nu \beta(\rho,T)$, which are applicable to some planetary atmospheric radiation problems, solutions are found in terms of temperature and radiative flux. These solutions are presented numerically and show excellent agreement with other numerical solutions.

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NOMENCLATURE

A, A'	matrix defined in equations (16) and (17)
B	Planck's function (eq.(2b))
C	quantity defined in equations (15b) and (A12)
c	speed of light
E_n	exponential integral of order n
F	nondimensional radiative flux (eq. (6a))
f	boundary function
G	integral of the kernel function
g	boundary function defined in equation (7b)
H	integral of B over η (eqs. (15a) and (A15))
h	Planck's constant, or function defined in equation (7b)
I	one-half the total step number
J	extremal function
K	kernel function
k	Boltzmann's constant
L	physical length
M	number of iterations
N	number of spectral modes
q	radiative heat
S	source function
s	nondimensional source function, $\frac{S}{\sigma T_B^4}$
sgn	$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ - & \text{if } x < 0 \end{cases}$
T	temperature
x	dummy variable

y	independent variable, altitude
z	variable, $\frac{\bar{\nu}}{\theta}$
α, β	absorption coefficient defined in equation (3a)
δ	delta function, (eq. (6a)), or variational of the function
Δ	increment of step function
$\Delta\nu_k$	increment of frequency, $\nu_{k+1} - \nu_k$
η	independent distance variable, $-\ell + \int_0^y \beta \, dy$
η_k	optical thickness for the kth frequency, $\alpha_k \eta$
θ	normalized temperature, $\frac{T}{T_B}$
ℓ	one-half the thickness, $\frac{1}{2}\int_0^L \beta \, dy$
κ	mass absorption coefficient
λ	constant associated with integral equation
ν	frequency, or an arbitrary function (eq. (A7))
$\bar{\nu}$	nondimensional frequency, $\frac{h\nu}{kT_B}$
ρ	mass density
σ	Stefan-Boltzmann constant
τ_L	optical thickness defined in equation (4b)
ϕ	variational function
ψ	solution

Subscripts

A	at the lower boundary
B	at the upper boundary
I	at Ith point
i, j	indexes of summation, or at ith or jth step point
k	frequency interval corresponding to kth mode

L	at the boundary
ν	frequency
$\pm$	even or odd function

Superscript

$\pm$	positive or negative direction

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SUMMARY

An effective iterative technique is introduced to solve a nonlinear integral equation frequently associated with radiative transfer problems. The problem is formulated in such a way that each step of an iterative sequence requires the solution of a linear integral equation. The advantage of a previously introduced variational technique which utilizes a stepwise constant trial function is exploited in this report to cope with the nonlinear problem. The method is simple and straightforward. Rapid convergence is obtained by employing a linear interpolation of the iterative solutions. Using absorption coefficients of the Milne-Eddington type, $\rho k_\nu = \alpha_\nu \beta(\rho,T)$, which are applicable to some planetary atmospheric radiation problems, solutions are found in terms of temperature and radiative flux. These solutions are presented numerically and show excellent agreement with other numerical solutions.

INTRODUCTION

The variational approach is attractive for solving the integral equations of kinetic theory boundary-value problems and aero-gasdynamics. It is versatile, accurate, fast, and particularly useful in solving linear integral equations (refs. 1, 2, and 3). The variational function may be accurately represented by polynomials which lead to extremely difficult and complex evaluations. To alleviate these difficulties, reference 2 examined a simple step-function representation. For nonlinear integral equations (e.g., Fredholm's type) the number of solution techniques available is extremely limited and numerical calculation times are long (ref. 4).

In this report a new iterative technique is introduced to solve a nonlinear integral equation. The problem is formulated in such a way that each step of an iterative sequence requires the solution of a linear integral equation. In a previous paper, an efficient variational technique (using a stepwise constant trial function) was described for the solution of the second kind of linear Fredholm's equation. Consequently, the advantages of the variational method for linear equations can be exploited within the framework of an iterative sequence for the nonlinear problem. The results of the present method are compared with other numerical solutions and the significant advantages of the present method are demonstrated.

ANALYSIS

Radiative Transfer Equation

The basic transfer equation for one-dimensional atmospheric radiation is (e.g., refs. 3 and 4)

$$\frac{dq(y)}{dy} = -2\pi \int_0^\infty \rho\kappa_\nu\, q^+_{A\nu}\, E_2(\tau_\nu)d\nu - 2\pi \int_0^\infty \rho\kappa_\nu\, q^-_{B\nu}\, E_2(\tau_{\nu L} - \tau_\nu)d\nu$$
$$- 2\pi \int_0^\infty \rho\kappa_\nu\, d\nu \int_0^{\tau_{\nu L}} B_\nu(T) E_1(|\tau_\nu - t|)dt$$
$$+ 4\pi \int_0^\infty \rho\kappa_\nu B_\nu(T)d\nu \tag{1}$$

The net radiative heat flux q is the direct integral of equation (1),

$$q(y) = 2\pi \int_0^\infty q^+_{A\nu}\, E_3(\tau_\nu)d\nu - 2\pi \int_0^\infty q^-_{B\nu}\, E_3(\tau_{\nu L} - \tau_\nu)d\nu$$
$$+ 2\pi \int_0^\infty d\nu \int_0^{\tau_{\nu L}} B_\nu(T)\, sgn(\tau_\nu - t) E_2(|\tau_\nu - t|)dt \tag{2a}$$

where

$$\left.\begin{aligned}
\tau_\nu(y) &= \int_0^y \rho\kappa_\nu\, d_y \;;\; \tau_{\nu L} = \tau_\nu(L) \\
E_n(\tau) &= \int_0^1 x^{n-2}\, e^{-\tau/x}\, dx \\
B_\nu(T) &= (2h\nu^3/c^2)/(e^{-h\nu/kT} - 1)
\end{aligned}\right\} \tag{2b}$$

The symbols q^+_A and q^-_B are the boundary fluxes due to the surface emissivity (ε) and reflectivity (r) at boundaries A and B, respectively (see sketch (a)). The symbol κ_ν is the spectral absorption coefficient assumed to be of the Milne-Eddington type; that is,

$$\rho\kappa_\nu = \alpha_\nu \cdot \beta(\rho, T) \tag{3a}$$

where the strength α_ν is approximated by the "picket fence" model referring to N different rectangles,

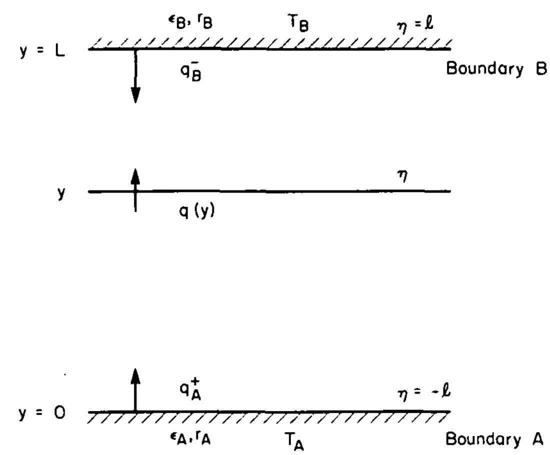
$$\alpha_\nu = \alpha_k \quad \text{for} \quad \nu_k \le \nu_{k+1} \;(k = 1, 2, \ldots N) \tag{3b}$$

The use of the Milne-Eddington type model, however, maybe limited to the planetary atmospheres where pressure-induced transition is important, as for example, in the atmospheres of Saturn, Uranus, and Neptune. The geometry and coordinate system under consideration are shown in sketch (a). To simplify the analysis, symmetric coordinates have been used, a process which leads to the following transformation for optical thickness

$$\tau_\nu = \tau_k = \ell_k + \eta_k \tag{4a}$$

where

$$\left.\begin{aligned} \eta_k &= -\ell_k + \alpha_k \int_0^y \beta \, dy \\ \ell_k &= \frac{1}{2} (\alpha_k \tau_L) \ ; \ \tau_L = \int_0^L \beta \, dy \end{aligned}\right\} \tag{4b}$$



Sketch (a).- Geometry and coordinate system.

The heating rate in equation (1), expressed in terms of the nondimensional internal source function $s(\rho,T)$, is assumed to be given by

$$\frac{dq}{dy} = \sigma T_B^4 \, s(\rho,T) \tag{4c}$$

so that equation (1), when combined with equations (3) and (4), can be rewritten as

$$\begin{aligned} \frac{1}{4} s + \sum_{k=1}^{N} \Bigg[& \frac{1}{2} \alpha_k F_{Ak} E_2(\ell_k + \eta_k) + \frac{1}{2} \alpha_k F_{Bk} E_2(\ell_k - \eta_k) \\ & + \frac{1}{2} \int_{-\ell_k}^{\ell_k} \alpha_k B_k(T) E_1(|\eta_k - \eta'|) d\eta' - \alpha_k B_k(T) \Bigg] = 0 \end{aligned} \tag{5}$$

where s, θ, η are the nondimensional internal source function, (assumed to be known), ratio of atmospheric temperature to the boundary temperature T/T_B (to be determined), and optical thickness, respectively, and

$$
\left.
\begin{aligned}
B_k(\theta) &= \pi\int_0^\infty \delta(\alpha_\nu)B_\nu(\theta)d\nu/(\sigma T_B^4) = \theta^4 I(\bar{\nu}/\theta) \\
F_{Ak} &= \pi\int_0^\infty \delta(\alpha_\nu)q_{A\nu}^+ \, d\nu/(\sigma T_B^4) \\
F_{Bk} &= \pi\int_0^\infty \delta(\alpha_\nu)q_{B\nu}^- \, d\nu/(\sigma T_B^4) \\
I(\bar{\nu}/\theta) &= 15/\pi^4\int_0^\infty \delta(\alpha_\nu)\cdot(\bar{\nu}/\theta)^3 d(\bar{\nu}/\theta)\Big/\left(e^{-\bar{\nu}/\theta} - 1\right) \\
\bar{\nu} &= h\nu/kT_B \\
\delta(\alpha_\nu) &= \begin{cases} 1 & \text{for the frequency } \alpha_\nu = \alpha_k \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\right\} \qquad (6a)
$$

Nondimensional radiative flux F is given by

$$
F(\eta) \equiv q/\sigma T_B^4 = \sum_{k=1}^{N} F_k(\eta_k) \qquad (6b)
$$

where

$$
\begin{aligned}
F_k(\eta_k) &= 2F_{Ak}E_3(\ell_k + \eta_k) - 2F_{Bk}E_3(\ell_k - \eta_k) \\
&\quad + 2\int_{-\ell_k}^{\ell_k} sgn(\eta_k - \eta')B_k(\theta)E_2(|\eta_k - \eta'|)d\eta'
\end{aligned} \qquad (6c)
$$

Note that the black body radiation has the relation of (which includes B_k for $a_k = 0$)

$$
\sum_{k=1}^{N} B_k(\theta) = \theta^4 \qquad (6d)
$$

where $\theta = T/T_B$.

Equation (5) is a nonlinear integral equation to be solved for $\theta = \theta(\eta)$. The description of the basic steps in solving the nonlinear equation by the present method may be briefly outlined as follows: First, the nonlinear integral equation is rearranged to take the standard form of Fredholm's integral equation (second kind). Second, with a pair of arbitrarily selected temperature distributions (stepwise-constant, initially isothermal distributions for the present problem) the nonhomogeneous part of the integral equation is evaluated analytically for the pair of temperature distributions. The two resulting linear integral equations are then solved by the variational method introduced previously, using a stepwise-constant trial function. Finally, an iterative method for calculating a new pair of temperature distributions is described; this method uses the linear interpolation (or extrapolation) between the pair of previous temperature distributions and its resulting linear solutions. The iterative procedure is continued until sufficient convergence is attained.

Linear Representation

For the present problem, equation (5) is recast into the standard form of Fredholm's integral equation (second kind) as

$$B_1(\eta_1) = \frac{1}{\alpha_1} f(\eta_1) + \frac{1}{2}\int_{-\ell_1}^{\ell_1} B_1(\eta')E_1(|\eta_1 - \eta'|)d\eta' \tag{7a}$$

where

$$\left.\begin{aligned} f(\eta_1) &= \frac{1}{4}(s) + \alpha_1 g_1(\eta_1) + \sum_{k=2}^{N} \alpha_k h_k(\eta_k) \\ h_k(\eta_k) &= g_k(\eta_k) + \frac{1}{2}\int_{-\ell_k}^{\ell_k} B_k(\theta)E_1(|\eta_k - \eta'|)d\eta' - B_k(\theta) \\ g_k(\eta_k) &= \frac{1}{2} F_{Ak}E_2(\ell_k + \eta_k) + \frac{1}{2} F_{Bk}E_2(\ell_k - \eta_k) \\ \eta_k &= (\alpha_k/\alpha_1)\eta_1 \end{aligned}\right\} \tag{7b}$$

Note that the f function in equations (7b) can be evaluated analytically since temperature is represented by a stepwise constant function. Initial guesses are two arbitrary isothermal approximations, for example, boundary temperature $\theta = \theta_A$ and $\theta = \theta_B$. The detailed iteration procedure is discussed in Iterative Calculations and Discussions (also in appendix B). The reference function B_1, a segment of B given by Eq. (6a), can be selected arbitrarily from the other spectral functions B_k. However, it is desirable to pick B_1 from the larger value of $\alpha_k B_k$, calculated for the isothermal conditions, so that more accurate solutions can be maintained.

The term containing the source function will be omitted from the present calculation. However, the general procedure will be exactly the same as for the rest of the terms in the f function.

Extremal Function

It can be shown (e.g., ref. 1) that the linear integral equation (Fredholm's equation of the second kind) containing the symmetric singular kernel function $K(\eta,\eta')$, that is,

$$\psi(\eta) = f(\eta) + \lambda \int_{-\ell}^{\ell} K(\eta,\eta')\psi(\eta')d\eta' \tag{8}$$

can be derived from the functional

$$J(\phi) = \frac{1}{2}\int_{-\ell}^{\ell} \phi(\eta)\left[\phi(\eta) - \lambda\int_{-\ell}^{\ell} K(\eta,\eta')\phi(\eta')d\eta' - 2f(\eta)\right]d\eta \tag{9}$$

where J takes a minimum value at $\phi = \psi$, which is the solution of equation (8).

For the simplicity of numerical calculations, the linear trial function ϕ, the boundary function f, and the extremal functional $J(\phi)$ are divided into even and odd functions, designated by subscripts + and - respectively, as

$$\phi = \phi_+ + \phi_- \tag{10a}$$

$$f = f_+ + f_- \tag{10b}$$

$$J(\phi) = J(\phi_+) + J(\phi_-) \tag{10c}$$

where

$$\left.\begin{aligned} f_+ &= \frac{1}{2}\left[f(\eta) + f(-\eta)\right] \\ f_- &= \frac{1}{2}\left[f(\eta) - f(-\eta)\right] \end{aligned}\right\} \tag{10d}$$

Note that the extremal function due to the cross product of an even and odd function has been eliminated since it is zero. Thus, the variational of the function J becomes

$$\delta J(\phi) = \delta J(\phi_+) = \delta J(\phi_-) = 0 \tag{11}$$

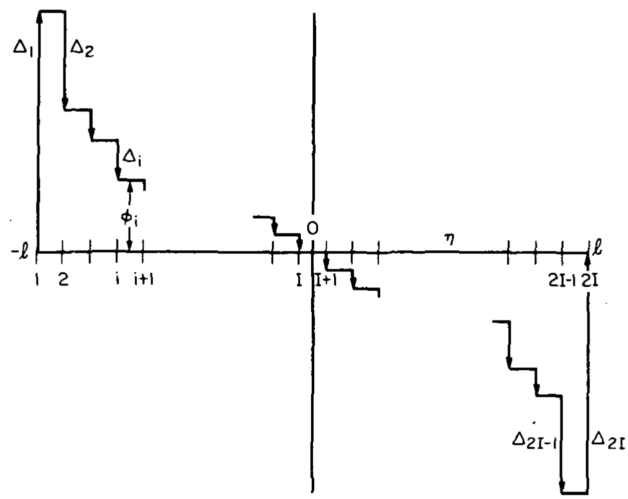
The number of points to be solved is significantly reduced (by one-half) by equation (11) since the even and odd function of ϕ can be determined independently. Following equation (9), the extremal function J associated with equations (7) can be written as

$$J(\phi) = \frac{1}{2}\left[\int_{-\ell_1}^{\ell_1} \phi^2(\eta_1)d\eta_1 - \frac{1}{2}\int_{-\ell_1}^{\ell_1} \phi(\eta_1)d\eta_1 \int_{-\ell_1}^{\ell_1} \phi(\eta_1)E_1(|\eta_1 - \eta'|)d\eta' \right.$$
$$\left. - 2\int_{-\ell_1}^{\ell_1} \phi(\eta_1)g_1(\eta_1)d\eta_1 - 2\sum_{k=2}^{N}\int_{-\ell_k}^{\ell_k} \phi\left(\frac{\alpha_1}{\alpha_k}\eta_k\right) h_k(\eta_k)d\eta_k\right] \tag{12}$$

where $B_1(\eta_1)$ is replaced by ϕ. For the conditions given by equation (10) and (11), even and odd functions of ϕ can be determined for the minimum value of J.

Solution by Step Function

The even and odd functions of a trial function ϕ in equation (10a) are replaced by stepwise constant functions (see sketch (b)) given by



Sketch (b).- Notations for the odd step function ϕ_i.

$$\left.\begin{array}{ll} \left.\begin{array}{l} \phi_+(\eta) = \displaystyle\sum_{j=1}^{i}(\Delta_+)_j \\ \phi_-(\eta) = \displaystyle\sum_{j=1}^{i}(\Delta_-)_j \end{array}\right\} & \text{for } \eta_i \leq \eta < \eta_{i+1} \\ \phi_-(\eta) = 0 & \text{for } \eta_I \leq \eta < \eta_{I+1} \end{array}\right\} \tag{13a}$$

where the interval is equally divided by $2I$ points, $\Delta\eta = 2\ell/(2I - 1)$, and by definition

$$(\Delta_+)_j = -(\Delta_+)_{2I+1-j} \qquad (\Delta_-) = (\Delta_-)_{2I+1-j} \tag{13b}$$

$$\sum_{j=1}^{2I}(\Delta_{\pm})_j = 0 \; ; \; \sum_{j=1}^{I}(\Delta_{-})_j = 0 \; ; \text{or } (\Delta_{-})_I = -\sum_{j=1}^{I-1}(\Delta_{-})_j \tag{13c}$$

A major advantage of using the Δ function (rather than the direct θ_i function) is that many terms pertaining to the integral of the kernel function can be eliminated from the calculations by virtue of equation (13c) (see appendix A). Even though the system defined in sketch (b) is for only the odd function, similar notations apply for the even function, with the exception that the function is symmetric and the value at the center (ϕ_+) is not necessarily zero. After some algebraic manipulation, substitution of equations (13) into equation (12) yields the following equations (see appendix A).

$$\left.\begin{aligned}
J(\phi_{\pm}) = &\sum_{i=1}^{I}\left[\sum_{j=1}^{i}(\Delta_{\pm})_j\right]^2[(\eta_{i+1})_1 - (\eta_i)_1] \\
&- \frac{1}{2}\sum_{i=1}^{I}(\Delta_{\pm})_i\sum_{j=1}^{I}(\Delta_{\pm})_j[(G_{ij})_1 \mp (G_{i,2I+1-j})_1] \\
&- \frac{1}{2}\sum_{k=1}^{N}(F_{Ak} \pm F_{Bk})\sum_{i=1}^{I}(\Delta_{\pm})_i[(C_{1i})_k \mp (C_{1,2I+1-i})_k] \\
&- \frac{1}{2}\sum_{k=2}^{N}\left\{\sum_{i=1}^{I}(\Delta_{\pm})_i\sum_{j=1}^{2I}(\Delta B_j)_k[(G_{ij})_k \mp (G_{i,2I+1-j})_k]\right. \\
&\left. + 2\sum_{i=1}^{I}(\Delta_{\pm})_i[(H_i)_k \mp (H_{2I+1-i})_k]\right\}
\end{aligned}\right\} \tag{14}$$

where

$$\left.\begin{aligned}
(C_{1i})_k &= E_3[\ell_k + (\eta_i)_k] \\
(G_{ij})_k &= (G_{1,j-i+1})_k \\
&= \left[-E_3|(\eta_i)_k - (\eta_j)_k| - |(\eta_i)_k - (\eta_j)_k|\right] \qquad (i \leq j) \\
(H_i)_k &= (\Delta\eta)_k\sum_{j=1}^{i-1}(B_j)_k \\
(\Delta B_j)_k &= (B_j)_k - (B_{j-1})_k \; ; \; (B_o)_k = (B_{2I})_k = 0
\end{aligned}\right\} \tag{15a}$$

$$(B_j)_k = B_k(\theta_j)$$

$$= \theta_j^4 \left\{ I[(z_j)_k + (\Delta z_j)_k] - I[(z_j)_k] \right\} \qquad (15b)$$

and

$$\left. \begin{aligned} (z_j)_k &= \bar{\nu}_k/\theta_j \ , \quad (\Delta z_j)_k = \Delta\bar{\nu}_k/\theta_j \\ \bar{\nu}_k &= h\nu_k/kT_B \\ \Delta\bar{\nu}_k &= \bar{\nu}_{k+1} - \bar{\nu}_k \end{aligned} \right\} \qquad (15c)$$

Note that B_k in equation (15b) includes all frequency intervals for $\alpha_\nu = \alpha_k$. In equation (14), minimizing J with respect to Δ_i yields the following results:

<u>For the even function ϕ_+:</u>

$$\frac{\partial J(\phi_+)}{\partial (\Delta_+)_i} = \sum_{j=1}^{I} A_{ij}(\Delta_+)_j - A_i = 0 \qquad (i = 1,2,\ldots I) \qquad (16a)$$

where

$$\left. \begin{aligned} A_{ij} &= -4(\eta_j)_1 + 2[-(G_{ij})_1 + (G_{i,2I+1-j})_1] && \text{for } j \geq i \\ A_{ij} &= A_{ji} && \text{for } j < i \end{aligned} \right\} \qquad (16b)$$

$$A_i = (A_i)_1 + \sum_{k=2}^{N} (A_i)_k \qquad (16c)$$

$$(A_i)_1 = -(F_{A1} + F_{B1})[(C_{1i})_1 - (C_{1,2I+1-i})_1] \qquad (16d)$$

and

$$(A_i)_k = - \left\{ (F_{Ak} + F_{Bk})[(C_{1i})_k - (C_{1,2I+1-i})_k] \right.$$
$$+ \sum_{j=1}^{2I} (\Delta B_j)_k [G_{ij})_k - (G_{i,2J+1-j})_k]$$
$$\left. + 2[(H_i)_k - (H_{2I+1-i})_k] \right\} \qquad (16e)$$

For the odd function ϕ_-: the index of summation is reduced to $I - 1$ by substituting the relation given by equation (13c) into equation (14)

$$\frac{\partial J(\phi_-)}{\partial(\Delta_-)_i} = \sum_{j=1}^{I-1} A'_{ij}(\Delta_-)_j - A'_i = 0 \quad (i = 1,2,\ldots I-1) \tag{17a}$$

where

$$\left.\begin{aligned} A'_{ij} &= 4[(\eta_I)_1 - (\eta_j)_1] + 2[-(G_{ij})_1 - (G_{i,2I+1-j})_1 \\ &\quad + (G_{iI})_1 + (G_{i,I+1})_1 + (G_{jI})_1 + (G_{j,I+1})_1 \\ &\quad + \tfrac{1}{2} - (G_{I,I+1})_1] \quad \text{for } j \geq i \\ A'_{ij} &= A'_{ji} \qquad\qquad \text{for } j < i \end{aligned}\right\} \tag{17b}$$

$$A'_i = (A'_i)_1 + \sum_{k=2}^{N} (A'_i)_k \tag{17c}$$

$$(A'_i)_1 = -(F_{A1} - F_{B1})[(C_{1i})_1 + (C_{1,2I+1-i})_1 - (C_{1I})_1 - (C_{1,I+1})_1] \tag{17d}$$

and

$$\begin{aligned} (A'_i)_k = -\Big\{&(F_{Ak} - F_{Bk})[(C_{1i})_k + (C_{1,2I+1-i})_k - (C_{1I})_k \\ &- (C_{1,I+1})_k] + \sum_{j=1}^{2I} (\Delta B_j)_k[(G_{ij})_k + (G_{i,2I+1-j})_k - (G_{Ij})_k \\ &- (G_{I+1,j})_k] + 2[(H_i)_k + (H_{2I+1-i})_k - (H_I)_k - (H_{I+1})_k]\Big\} \end{aligned} \tag{17e}$$

Equations (16a) and (17a) and I and $(I - 1)$ simultaneous linear equations for Δ_i, where coefficients of these equations are calculated from a given temperature distribution. Successive approximations follow thereafter until the convergence in ϕ_i or θ_i is obtained (see appendix B). From equations (10a), (13a), (16a), and (17a)

$$\phi_i = (B_i)_1 = \sum_{j=1}^{i} [(\Delta_+)_j + (\Delta_-)_j] \tag{18}$$

Evaluation of the extremal function J given by equation (14) and the procedure for solving the essential equations (16) and (17), are systematic and straightforward, requiring minimal computing time (Computing time is roughly proportional to I^2NM, where N is total number of frequency intervals and M is a number of iterations.). Numerical calculations for I as low as 3 show surprisingly good accuracy, and the computing time on an IBM 360/67 for a general case using I = 13 is nominal [i.e., less than 1/10 sec per frequency interval per iteration (see appendix B)].

The radiative fluxes, from equations (6b) and (6c), are calculated by

$$F(-\ell) = F(\ell) = \sum_{k=1}^{N} F_k(\pm\ell), \quad F(0) = \sum_{k=1}^{N} F_k(0) \tag{19a}$$

where

$$\left.\begin{aligned} F_k(-\ell) &= F_{Ak} - 2F_{Bk}E_3(2\ell_k) - 2\sum_{j=1}^{2I} (\Delta B_j)_k\, E_3[\ell_k + (\eta_j)_k] \\ F_k(\ell) &= 2F_{Ak}E_3(2\ell_k) - 2\sum_{j=1}^{2I} (\Delta B_j)_k E_3[\ell_k - (\eta_j)_k] \end{aligned}\right\} \tag{19b}$$

and

$$F_k(0) \doteq 2(F_{Ak} - F_{Bk})E_3(\ell_k) - 2\sum_{j=1}^{I} [(\Delta B_j)_k + (\Delta B_{2I+1-j})_k]E_3[-(\eta_j)_k] \tag{19c}$$

However, as noted in reference 2, the radiative flux F computed by the present method is not exactly constant. The mean value, suggested in reference 2, is therefore employed here:

$$F = \frac{1}{2}\,[F(-\ell) + F(0)] \tag{20}$$

ITERATIVE CALCULATIONS AND DISCUSSIONS

For nonlinear equations, the successive substitution or successive approximation methods are extremely time consuming (ref. 4) and frequently experience stability problems if the initial approximation is not adequately selected. The efficiency of the calculation depends on simplifications made in the analysis, the desired order of accuracy, convergence rate, and the scope of the results of interest (i.e., whether interest lies in the macroscopic physical quantity or solution itself). The convergence rate also depends on the function to be solved; for example, in the present problem, the solution in terms of temperature θ converges much faster than the direct solution of the function ψ itself. For a faster convergence, the successive approximation method used for the present problem is therefore modified by considering a linearly weighted effect of two independent initial guesses and their corresponding solutions. That is, two sets of arbitrarily selected temperature distributions, θ_1 and θ_2, are assumed (Initially, isothermal distributions are used.). The integrated Planck's function, $B_1(\theta)$ and the solution function ϕ are then computed from equations (15b) and (18), respectively. The results are then used to obtain one new temperature distribution θ_3 by linear interpolation (or extrapolation) and a second new temperature distribution θ_4 by linear interpolation between θ_3 and either θ_1 or θ_2 (see appendix B). With these new data, the iteration procedure is repeated until the temperatures θ converge. The present results show that the fourth or fifth iterate provides sufficient convergence.

Sample calculations for several models of absorption coefficients have been selected for this study. Included in these is the model from reference 4 which is used to compare the present results with other numerical solutions. It is noted the number of spectrals N (where $N > 1$) in the absorption coefficient has no significant effect on the convergence of the solution in the present method. However, the selection of the reference Planck function (e.g., if much smaller B_1 is selected) will obviously affect the accuracy of the solution.

The results for temperature distribution and normalized flux using the simple spectral model from reference 4 are presented in tables 1 and 2. In these tables the optical thickness is selected such that $\tau_L = 1$ and the number of steps is varied from 3 to 13 in order to determine the effect of step size on the computed radiative flux. The present results show surprisingly good agreement with the numerical values of reference 4 even for the minimum number of steps, $I = 3$. (The midpoint values of each step function are tabulated in the illustrations.) Table 1 shows the results of reference 4 and the present results for $I = 3$ and 13, so that comparisons can be made for coincident optical distance η. The results of normalized radiative flux (F) are presented in table 2 for various step sizes. The step size for $I = 13$ is more than sufficient to retain good accuracy for the cases considered in this paper.

The present method has been used to calculate several other selected sample cases using absorption coefficients described by a larger number of spectral modes and a variety of optical thicknesses. The absorption coefficient and temperature θ for a spectral model consisting of 36 nonzero and nonrepeated values (model 3) are presented in tables 3 and 4. The optical thickness selected for table 4 is $\tau_L = 2$. Note that the calculation will be significantly reduced when the absorption coefficient contains zero or repeated values. The results for a three step mode ($I = 3$) shows good agreement with those of larger steps ($I = 13$) for temperature distribution and radiative flux (Note: The last decimal in table 4 may be not correct even though the solution converges to the fifth decimal place, since the computer carries only five significant figures.). Results of radiative flux using model 3 with several optical thicknesses are presented in table 5 ($I = 13$).

An interesting feature of the variational method is that accurate values for a physical quantity, such as radiative flux F for the present case, can be obtained without having an accurate solution for θ. For example, a linear (or in some cases, only a one step constant mode) trial function provides remarkably accurate values for physical quantities even though the solutions themselves are not necessarily accurate (ref. 1). The present result substantiates this conclusion; in other words, the results for macroscopic physical quantities show much faster convergence than the solution itself. Consequently, a large number of steps and/or iterations are not usually necessary. Examples of this convergence on the solution (temperature at $\eta = -0.96$ and $\eta = 0.96$) and radiative flux are illustrated in table 6.

The boundary gas temperature, even though it is not considered in this paper, may be calculated in two ways: (1) by using a small step interval at the boundaries; and (2) evaluating the right-hand side of equation (7a) by substituting the present step solution for B_k at $\eta = \pm\ell$ (also at any point).

An iterative method used in conjunction with the variational technique which uses a step-function provides the following advantages:

1. Accurate descriptions of the solutions can be obtained with a simple iteration scheme (nonlinear case).

2. Rapid convergence of the physical quantities may be obtained without accurate solutions.

3. The procedure of numerical calculations is straightforward.

4. Nonlinear problems which are difficult to handle analytically by other methods (e.g., the polynomial trial function or successive approximation methods) may be exploited by a method similar to the present technique.

CONCLUDING REMARKS

With the aid of the variational method to the linear integral equation, an iterative technique has been applied to solve the nonlinear integral equation which is frequently involved in physical and gas dynamic problems. It has been demonstrated that the iterative technique, in conjunction with the variational method which uses the step function as a trial function, provides effective and accurate descriptions of the solutions to these problems and related average physical quantities.

The method is relatively simple, straightforward, and requires nominal computing time. It is expected that the method can be further extended for solving other nonlinear integral equations of radiative transfer or problems in kinetic theory.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., January 22, 1973

APPENDIX A

EVALUATION AND SIMPLIFICATION OF INTEGRATIONS INVOLVING THE KERNEL FUNCTION

Integrations of kernel functions using a step function are usually relatively simple. Further simplifications which are important in reducing the computation process are realized by eliminating all terms from the integrated function when these terms vanish in certain summations.

Integrals of Symmetric Kernel Functions

Some major results involving the integrals of the symmetric functions can be summarized. For the exponential integrals

$$\int_{\eta_i}^{\ell} E_1(|\eta - \eta'|)d\eta = sgn(\eta_i - \eta')[-1 + E_2(|\eta_i - \eta'|)] + 1 - E_2(\ell - \eta') \tag{A1}$$

Similarly

$$\int_{\eta_i}^{\ell} E_2(|\eta - \eta'|)d\eta = sgn(\eta_i - \eta')\left[-\frac{1}{2} + E_3(|\eta_i - \eta'|)\right] + \frac{1}{2} - E_3(\ell - \eta') \tag{A2}$$

$$\int_{\eta_i}^{\ell} sgn(\eta' - \eta_j)d\eta' = -|\eta_i - \eta_j| + (\ell - \eta_i) \tag{A3}$$

$$\int_{\eta_j}^{\ell} sgn(\eta' - \eta_i)E_2(|\eta' - \eta_i|)d\eta' = E_3(|\eta_i - \eta_j|) - E_3(\ell - \eta_i) \tag{A4}$$

Simplification of Integrated Functions

Integration of the symmetric kernel function with a singularity required in equation (12) if ϕ is a step function can be evaluated by equation (A1) through (A4) as follows:

$$\begin{aligned} G_{ij} &= \int_{\eta_i}^{\ell}\int_{\eta_j}^{\ell} E_1(|\eta - \eta'|)d\eta\, d\eta' \\ &= (\ell - \eta_i) + (\ell - \eta_j) - |\eta_i - \eta_j| - \frac{1}{2} - E_3(|\eta_i - \eta_j|) \\ &\quad + E_3(\ell - \eta_i) + E_3(\ell - \eta_j) \end{aligned} \tag{A5}$$

Since the symmetric matrix G_{ij} of equation (A5) will be summed over i and/or j for determining the extremal values of J (eq. (14)), all terms in G_{ij} except $(\eta_i - \eta_j)$ will vanish from the summation as a result of the conditions imposed for the step function by equation (13c). Thus, G can be redefined as

$$G_{ij} = -|\eta_i - \eta_j| - E_3(|\eta_i - \eta_j|) \tag{A6}$$

Integration of Symmetric Functions

Since the extremal function of equations (9) or (12) is evaluated for both even and odd boundary or trial functions (eqs. (9)), some simplifications with regard to the single and double integrations may be made; assuming K represents the symmetric kernel,

$$\int_{-\ell}^{\ell} \phi_{\pm}(\eta)d\eta \int_{-\ell}^{\ell} \nu(\eta')[K(\eta,\eta') \pm K(-\eta,\eta')]d\eta'$$

$$= \int_{-\ell}^{\ell} \nu(\eta')d\eta' \int_{-\ell}^{\ell} \phi_{\pm}(\eta)[K(\eta,\eta') \pm K(-\eta,\eta')]d\eta$$

$$= 2\int_{-\ell}^{\ell} \phi_{\pm}(\eta)d\eta \int_{-\ell}^{\ell} \nu(\eta')K(\eta,\eta')d\eta \tag{A7}$$

where $\nu(\eta)$ is an arbitrary function. Also

$$\int_{-\ell}^{\ell} \phi_{\pm}(\eta)[K(\ell,\eta) \pm K(\ell,-\eta)]d\eta = 2\int_{-\ell}^{0} \phi_{\pm}(\eta)[K(\ell,\eta) \pm K(\ell,-\eta)]d\eta \tag{A8}$$

Equations (A7) and (A8) have been incorporated in the results indicated by equation (14).

Reduction of the Index of the Summation

Since the solution is divided into the even and odd functions, the index of double summation appearing in equation (14) can be rearranged using equations (13) as follows:

$$\sum_{i=1}^{2I} \Delta i \sum_{j=1}^{2I} \Delta_j G_{ij} = \sum_{i=1}^{2I} \Delta_i \sum_{j=1}^{I} \Delta_j (G_{ij} \mp G_{i,2I+1-j})$$

$$= \sum_{i=1}^{I} \Delta_i \sum_{j=1}^{I} \Delta_j (G_{ij} + G_{2I+1-i,2I+1-j} \mp G_{i,2I+1-j} \mp G_{2I+1-i,j})$$

$$= 2\sum_{i=1}^{I} \quad \sum_{j=1}^{I} \Delta_j (G_{ij} \mp G_{i,2I+1-j}) \tag{A9}$$

where

$$\left.\begin{aligned} G_{ij} &= G_{ji} \\ G_{i,2I+1-j} &= G_{2I+1-i,j} \end{aligned}\right\} \qquad \text{(A10)}$$

Note that the minus sign in the last equation (A9) is for the even function (or Δ_+) while the plus sign is for the odd function (Δ_-). The results of equation (A9) have been used in obtaining equation (14) of the text.

From the condition given by equation (13c), the index of summation for the odd function in equation (A9) can be further reduced. (For simplicity, the minus sign indicating the odd function has been dropped from the following equation.)

$$\begin{aligned} \sum_{i=1}^{I} \Delta_i \sum_{j=1}^{I} \Delta_j (G_{ij} + G_{i,2I+1-j}) &= \sum_{i=1}^{I} \Delta_i \sum_{j=1}^{I-1} \Delta_j (G_{ij} + G_{i,2I+1-j} - G_{iI} - G_{i,I+1}) \\ &= \sum_{i=1}^{I-1} \Delta_i \sum_{j=1}^{I-1} \Delta_j (G_{ij} + G_{i,2I+1-j} - G_{iI} - G_{i,I+1} \\ &\quad - G_{jI} - G_{j,I+1} + G_{II} + G_{I,I+1}) \end{aligned} \qquad \text{(A11)}$$

The symbols G_{Ij} and $G_{I,2I+1-j}$ have been replaced by G_{jI} and $G_{j,I+1}$ (eq. (A10)). The result of equation (A11) is given by equation (17b) of the text.

Miscellaneous Integration

Other integrated functions appearing in equation (14) and (15) are derived and denoted as follows:

$$\begin{aligned} \int_{-\ell}^{\ell} \phi_{\pm}(\eta) E_2(\ell + \eta) d\eta &= \sum_{i=1}^{2I} (\Delta_{\pm})_i \int_{\eta_i}^{\ell} E_2(\ell + \eta) d\eta \\ &= \sum_{i=1}^{2I} (\Delta_{\pm})_i [E_3(\ell + \eta_i) - E_3(2\ell)] \\ &= \sum_{i=1}^{2I} (\Delta_{\pm})_i C_{1i} \end{aligned} \qquad \text{(A12)}$$

By equation (13c),

$$\sum_{i=1}^{2I} (\Delta_{\pm})_i \; E_3(2\ell) = E_3(2\ell) \sum_{i=1}^{2I} (\Delta_{\pm})_i = 0$$

it follows that

$$C_{1i} = E_3(\ell + \eta_i)$$

and similarly,

$$\int_{-\ell}^{\ell} \phi_\pm(\eta)B(\theta)d\eta = \sum_{i=1}^{2I} (\Delta_\pm)_i \int_{\eta_i}^{\ell} B(\theta)d\eta \tag{A13}$$

$$= \sum_{i=1}^{2I} (\Delta_\pm)_i \left[\int_{-\ell}^{\ell} B(\theta)d\eta - \int_{-\ell}^{\eta_i} B(\theta)d\eta\right]$$

$$= -\sum_{i=1}^{2I} (\Delta_\pm)_i \; H_i(\eta) \tag{A14}$$

Since the first term inside the square bracket in the last equation vanishes, it follows that

$$H_i(\eta) = \int_{-\ell}^{\eta_i} B(\theta)d\eta = \Delta\eta \sum_{j=1}^{i-1} B_j \tag{A15}$$

APPENDIX B

THE METHOD OF ITERATION

Since equation (5) and (7) are nonlinear integral equations, a modified successive substitution method using a pair of initial temperature distributions is introduced for solving the resulting equations; in other words, the boundary function f in equation (7) or matrices A and A' in equations (16) and (17) are first calculated from the initial set of arbitrarily selected temperature distribution $\theta(1)$ and solved for ϕ at each η_i.

The value ϕ obtained from equation (18) is matched with the value $(B)_1$ calculated by equation (6) for the given $\theta(1)$. The same procedure is repeated for the second set of arbitrarily selected temperature distribution $\theta(2)$. From this pair of initial temperature distributions, $\theta(1)$ and $\theta(2)$, and their resulting values, $(B)_1$ and ϕ, the first new temperature distribution $\theta(3)$ is linearly interpolated at each point. The second new temperature $\theta(4)$ is then calculated by the linear interpolation (or extrapolation) between $\theta(3)$ and either $\theta(1)$ or $\theta(2)$. The scheme of the linear interpolation is formulated as follows (for simplicity, the subscript 1 pertaining to the reference function has been omitted):

Temperature $\theta(3)$ is given by

$$\theta_i^4(3) = \theta_i^4(1) - \frac{[\theta_i^4(2) - \theta_i^4(1)][\phi_i(1) - B_i(1)]}{[\phi_i(2) - B_i(2) - \phi_i(1) + B_i(1)]} \tag{B1}$$

Temperature $\theta(4)$, depending on the conditions, is given by

(a) if

$$[\phi_i(3) - B_i(3)][\theta_i^4(3) - \theta_i^4(2)] > 0$$

$$\theta_i^4(4) = \theta_i^4(1) - \frac{[\theta_i^4(3) - \theta_i^4(1)][\phi_i(1) - B_i(1)]}{[\phi_i(3) - B_i(3) - \phi_i(1) + B_i(1)]} \tag{B2}$$

or

(b) (replacing all the indexes 1 in equation (B2) with the index 2),

$$\theta_i^4(4) = \theta_i^4(2) - \frac{[\theta_i^4(3) - \theta_i^4(2)][\phi_i(2) - B_i(2)]}{[\phi_i(3) - B_i(3) - \phi_i(2) + B_i(2)]} \tag{B3}$$

For the next iterations, $\theta(5)$ and $\theta(6)$, the indexes 1 and 2 in equations (B1) through (B3) are replaced by the indexes 3 and 4. Then the successive iteration M follows until ϕ converges to B, as

$$\frac{\phi_i(M) - B_i(M)}{\phi_i(M)} \leq \varepsilon \quad \text{or} \quad \frac{\theta_i(M) - \theta_i(M - 1)}{\theta_i(M)} \leq \varepsilon \tag{B4}$$

For the present problem, the fifth iteration ($M = 5$) provides sufficient accuracy. Note that the two initial temperatures selected for the present computations are the isothermal distributions $\theta(1) = \theta_A$ and $\theta(2) = \theta_B$, where θ_A and θ_B represent the respective temperatures at each of the boundaries.

From the results of the present calculations (by $I = 13$), the average computing time required for the solution and flux is less than 1/10 sec per spectral mode per iteration on an IBM 360/67; in other words, for $N = 36$ and $M = 5$, the computing time was about 18 sec.

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TABLE 1.— TEMPERATURE DISTRIBUTION: $\theta_A = 0.5$, $\theta_B = 1.0$ $\varepsilon_A = \varepsilon_B = 1.0$, $r_A = r_B = 0$, AND $\tau_L = 1.0$

η	Model 1: $\alpha_1 = 1.0(\bar{\nu}_1 = 0)$, $\alpha_2 = 0.5(\bar{\nu}_2 = 3)$			Model 2: $\alpha_1 = 0.5(\bar{\nu}_1 = 0)$, $\alpha_2 = 1.0(\bar{\nu}_2 = 3)$		
	Reference 4	Present method		Reference 4	Present method	
		$I = 3$	$I = 13$		$I = 3$	$I = 13$
-0.5	0.7408	---	---	0.7095	---	---
-.4	.7726	0.7723	0.7726	.7432	0.7428	0.7432
-.2	.8148	.8148	.8148	.7882	.7881	.7882
0	.8484	.8483	.8484	.8247	.8245	.8247
.2	.8781	.8780	.8781	.8574	.8573	.8574
.4	.9066	.9066	.9065	.8899	.8899	.8899
.5	.9235	---	---	.9098	---	---

TABLE 2.— NORMALIZED FLUX ($-F$); BOUNDARY CONDITIONS SAME AS TABLE 1.

Models		Present method			Reference 4
α_1	α_2	$I = 3$	$I = 8$	$I = 13$	
1.0	0.0	0.7856	0.7860	0.7860	0.7860
1.0	.1	.7371	---	.7376	.7376
1.0	.3	.6637	---	.6644	.6644
1.0	.5	.6092	---	.6100	.6101
1.0	1.0	.5177	.5187	.5188	.5188
0.8	1.0	.5351	.5361	.5361	.5362
.5	1.0	.5693	.5701	.5702	.5702
.1	1.0	---	---	.6427	.6427
.0	1.0	.6696	.6703	.6703	.6703

TABLE 3.— ABSORPTION COEFFICIENT FOR MODEL 3.

$\bar{\nu}$	$\Delta\bar{\nu}$	$\alpha_{\bar{\nu}}$	$\bar{\nu}$	$\Delta\bar{\nu}$	$\alpha_{\bar{\nu}}$
0	0.25	0.65	4.05	0.04	0.82
.25	.75	.10	4.09	.01	.72
1.00	.50	.80	4.10	.05	.62
1.50	.25	.25	4.15	.03	.52
1.75	.05	.35	4.18	.01	.42
1.80	.02	.88	4.19	.01	.32
1.82	.03	.78	4.20	.55	.22
1.85	.05	.68	4.75	.25	.55
1.90	.10	.58	5.00	.25	.275
2.00	.10	.48	5.25	.25	.75
2.10	.15	.38	5.50	.50	.45
2.25	.25	.60	6.00	.25	.90
2.50	.50	.20	6.25	.25	.975
3.00	.25	1.00	6.50	.25	.70
3.25	.25	.30	6.75	.50	.15
3.50[a]	.50[a]	.85[a]	7.25	.25	.95
4.00	.03	.40	7.50	.25	.50
4.03	.02	.92	7.75	∞	.05

[a]Reference frequency and absorption coefficient selected for $k = 1$.

TABLE 4.— TEMPERATURE DISTRIBUTION (θ) AND NORMALIZED FLUX ($-F$): BOUNDARY CONDITIONS SAME AS IN TABLE 1. MODEL 3(N = 36), τ_L = 2.0.

Optical Thickness, τ_L	Temperature, θ	
	I = 3	I = 13
-0.96		0.73340
-.88		.74897
-.80	0.76202	.76184
-.72		.77330
-.64		.78380
-.56		.79358
-.48		.80281
-.40	.81174	.81159
-.32		.81997
-.24		.82804
-.16		.83583
-.08		.84338
.00	.85075	.85075
.08		.85792
.16		.86495
.24		.87185
.32		.87867
.40	.88530	.88541
.48		.89209
.56		.89878
.64		.90548
.72		.91228
.80	.91915	.91925
.88		.92657
.96		.93476
Flux, $-F$	.5706	.57124

TABLE 5.— NORMALIZED FLUX ($-F$) AS A FUNCTION OF OPTICAL THICKNESS (τ_L): BOUNDARY CONDITIONS SAME AS IN TABLE 1. MODEL 3: (N = 36, I = 13).

Optical thickness, τ_L	Normalized flux, $-F$
0	1.00000
0.5	0.79393
1.0	.69628
2.0	.57124
3.0	.49076
4.0	.43307
5.0	.38907
6.0	.35412
7.0	.32552
8.0	.30161

TABLE 6.— CONVERGENCE OF THE SOLUTION (TEMPERATURE AND FLUXES): BOUNDARY CONDITIONS SAME AS TABLE 4.

Number of iteration, M	Temperature, $\theta(\eta)$		Normalized flux, $-F$
	η = -0.96	η = 0.96	
1	0.5000	0.5000	---
2	1.0000	1.0000	---
3	.7493	.9298	0.5703
4	.7354	.9339	.5711
5	.7334	.9348	.5712
6	.7334	.9348	.5712
7	.7334	.9348	.5712
8	.7334	.9348	.5712
9	.7334	.9348	.5712

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