# A CRITICAL SURVEY OF

# WAVE PROPAGATION AND IMPACT

# IN COMPOSITE MATERIALS

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16. Abstract

A review of the field of stress waves in composite materials is presented covering the period up to December 1972. The major properties of waves in composites are discussed and a summary is made of the major experimental results in this field. Various theoretical models for analysis of wave propagation in laminated, fiber and particle reinforced composites are surveyed. The anisotropic, dispersive and dissipative properties of stress pulses and shock waves in such materials are reviewed. A review of the behavior of composites under impact loading is presented along with the application of wave propagation concepts to the determination of impact stresses in composite plates.

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## I. INTRODUCTION

Stress waves in composite materials are of interest to the engineer both for their constructive application and for the potential damage that can occur when short duration stress pulses propagate in a structure. Stress waves have a constructive use as a diagnostic tool to measure elastic properties, search for flows and transmit information. Such application usually involves waves in the form of pulses or ultrasonic sinusoidal pulses. Seismologists have long been interested in this application of stress waves, particularly the study of waves in layered media (see e.g. Ewing et al., 1957; Brekhovskikh, 1960). Early studies of laminated media were aimed in fact at geophysical applications (e.g. Anderson, 1961).

Structural engineers however usually rely on composite materials to sustain forces or loads. When these forces are a result of shock or impact on the structure, the forces will be transmitted through the structure in the form of stress waves. While the prediction of stress distribution for static or quasi-static loads (vibrations) can usually be predicted by structural engineers, routine methods for predicting the path of stress pulses through a complicated structure are not readily available ,even for homogeneous materials. The anisotropy and inherent inhomogeneity in composite materials further complicates this problem.

The importance of impact stresses in composite structural design can best be illustrated by the application of these materials to jet engine fan blades (see Goatham, 1970). In addition to the load requirements imposed by centrifugal and vibratory forces, these blades must be designed to withstand the stresses due to impact with foreign objects such as birds,

hailstones, stones, and nuts and bolts. The relative velocity of the impacting body to the blade can be in the order of 450 meters per second (1500 ft/sec). The high speed impact of small objects results in very small impact times (< 50 $\mu$ sec) and the initial transmission of the total energy into a local region of the blade. The impact not only induces local cratering or splitting but long range damage away from the impact area can result from the reflection of stress waves (spalling) from boundaries and focusing effects due to changes in blade geometry. Solutions of the problem of foreign object impact involve considerable ingenuity, such as embedded high strength meshes and leading edge impact protection.

Impact loads involve two factors which are not considered in static stress analysis. One is the speed of propagation of the stress pulse in the material. In static problems the deformation energy can be distributed throughout the structure, but in impact loading the volume of energy storage is limited by the speed of propagation of the waves in the material. For short time impact loads, a small amount of energy in a small volume can result in stresses which can fracture or otherwise damage the material.

The speeds of propagation of stress waves for a number of composites are shown in Table I with comparable data for conventional structural materials. These speeds depend on the direction in which the wave propagates, and when the elastic limit is exceeded, depend also on the stress level. These wave speeds are motions averaged over a local region of the composite involving many layers, fibers or particles whichever is the case. Within each constituent, of course, the stresses propagate as in the respective homogeneous materials.

The second difference between impact loading and static loads in design is the rate of change of strain. Composites under high rates of strain have been shown to exhibit different strength properties (Sierakowski et al., 1970). Often this results in higher ultimate strength with increasing strain rate.

While the factors of finite wave time and rate dependent properties are common to impact problems in all structural materials, the anisotropy and inhomogeneity inherent in composites requires special attention in the design of an impact resistant composite structure.

Anisotropic waves in solids are familiar to those in crystal physics and seismology, however, these effects are not well known in structural design where conventional isotropic materials such as aluminum and steel are often used. Composite materials have the unique feature that the degree of anisotropy can be varied in the material and hence the analyst can change the directional distribution of stress waves in an impact zone and perhaps avoid serious failure or fracture (perhaps by a judicious choice of ply lay-up angles).

The effects of boundaries or discontinuities in material properties on stress waves are well known (see Ewing et al., 1957). When a stress wave encounters a boundary, normal to the wave front, separating materials of different densities and wave speeds, $\rho$ , $v$ , the stress at the surface is changed to

$$\sigma = \sigma_o \, \rho_2 \, v_2 / (\rho_1 \, v_1 + \rho_2 \, v_2) \tag{1}$$

where $\sigma_o$ would be the stress in material "1" if the boundary were
not present. The product $\rho v$ is called the acoustic impedence and
depends on the type of wave (e.g. shear or dilatational in isotropic
solids).

Thus a wave originating in a "softer" material i.e. $\rho_1 v_1 < \rho_2 v_2$
always suffers a stress increase at a boundary. This is indeed the case
for many composites especially those involving a compliant matrix, such
as epoxy, and a stiff fiber such as graphite, glass or boron.

Another effect of inhomogeneity is dispersion. Dispersion of the
average composite motion results in a distortion of the stress pulse
as it propagates. The effects of dispersion increase as the duration,
rise time, or period of the stress pulse decrease. Thus a pulse initially
containing compressional stresses can develop tensile stresses
as the wave propagates and perhaps induce micro-cracking in the composite.

The literature on the subject of waves in composites has expanded
enormously in the past few years and new theoretical and experimental
results are still being reported. This review then can only summarize
the work to the date of this writing. Also several good reviews have
appeared at this writing in which the various theoretical models for
waves in composites have been discussed, (Peck, 1971, 1972, Achenbach,
1972).

This chapter will be somewhat tutorial in nature rather than a
critical review of the various theories to date. Instead I will try
to summarize the results to date which seem to be accepted in the field
and which might be of use to the structural dynamics analyst.

In the following sections I will discuss

 i) anisotropic waves in composite structures (without dispersion)

 ii) dispersion effect on waves

iii) scattering and absorption of waves

 iv) shock waves in composites

  v) experimental results

 vi) the effects of impact

For a review of stress waves in conventional structural material see Miklowitz (1966).

## II. ANISOTROPIC WAVES IN COMPOSITES

In this section I will review those aspects of elastic wave propagation in anisotropic materials which are relevant to composites. When the scale of the changes in stress level, (rise distance, wavelength, etc.), is much larger than the sizes of the constituents of composites (fiber or particle diameter, fiber spacing, ply spacing, etc.) the material may be treated as an equivalent homogeneous elastic material as a first approximation.[*] In a homogeneous medium, the wave speeds are related to the elastic constants and density by relations of the type $\rho v^2 = C$, where $C$ is an elastic constant. This relation has led to the use of wave theory to determine the effective elastic moduli of composite materials when the wavelength becomes larger than the size of the scale of inhomogeneity. Thus the definition

$$C_{eff} = \lim_{\lambda/a \to \infty} \rho_{eff} \, v^2 (\lambda/a)$$

where $\lambda$ is the wavelength; "a" is a size associated with the composite elements (e.g. fiber spacing ,and $v(\lambda/a)$ is the phase velocity for a given harmonic wavelength. This method has been used by White and Angona (1955) for a laminated medium and by Behrens (1967a) (1967b).

In the case of particulate composites or dispersion strengthened composites the equivalent model may be considered as isotropic. But for fiber composites, laminates, and unidirectional eutectoids, the

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[*] The exception is the case of a composite plate with bending-extensional coupling.

equivalent stress-strain relation will be anisotropic i.e.

$$t_{ij} = C_{ijk\ell} \, e_{k\ell} \tag{2}$$

where $t_{ij}$ is the stress tensor and $e_{k\ell}$ the strain tensor.

A. Wave Speeds

The simplest wave to consider is a plane wave with no external boundaries present. For such a wave the displacement has the form

$$\underset{\sim}{u} = \underset{\sim}{A} \, f(\underset{\sim}{n} \cdot \underset{\sim}{x} - vt) \tag{3}$$

The vector $\underset{\sim}{n}$ defines a plane relative to the material axes and $v$ is the speed of the wave. When Eqs. (2), (3) are put into the equations of motion for the material, the following eigenvalue problem results

$$(C_{ijk\ell} \, n_\ell \, n_j - \rho v^2 \delta_{ij}) A_k = 0 \tag{4}$$

where $\rho$ is the equivalent density and $\delta_{ij}$ the Kronecker delta. (See, e.g. Musgrave (1954, 1970),and Kraut (1963)). In summary, for each wave direction there are three different waves

$$(v^{(i)}, A_k^{(i)}) \, , \, i = 1, 2, 3$$

When the $v^{(i)}$ are distinct, the three polarization vectors $\underset{\sim}{A}^{(i)}$ are orthogonal. For isotropic materials it is well known that only two speeds are distinct

$$v^{(1)} = \left[\frac{\lambda + 2\mu}{\rho}\right]^{1/2} \quad , \quad \underset{\sim}{u} \times \underset{\sim}{A}^{(1)} = 0$$

(5)

$$v^{(2)} = v^{(3)} = \left[\frac{\mu}{\rho}\right]^{1/2} \quad , \quad \underset{\sim}{u} \cdot \underset{\sim}{A}^{(2)} = 0$$

These are respectively the longitudinal and transverse (shear) waves. For anisotropic waves however such characterization is not possible except along symmetry directions.

For many composites, orthotropic symmetry suffices to describe the material and nine elastic constants are required. The stress-strain relation for this case is given by

$$
\begin{bmatrix} t_{11} \\ t_{22} \\ t_{33} \\ t_{23} \\ t_{13} \\ t_{12} \end{bmatrix}
=
\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}
\begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{bmatrix}
$$

(6)

For structural applications composites are usually used in the form of rods or plates. Consider, for example, the in-plane motion of a plate with the $x_2$ axis normal to the midsurface, i.e. $u_2 = 0$. For wavelengths much larger than the plate thickness we neglect the

effects of dispersion. For this case the equations of motion for the in-plane motion in the lowest approximation became (see Figure 1)

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \hat{C}_{11} \frac{\partial^2 u_1}{\partial x_1^2} + C_{55} \frac{\partial^2 u_1}{\partial x_3^2} + (C_{55} + \hat{C}_{13}) \frac{\partial^2 u_3}{\partial x_1 \partial x_3}$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \hat{C}_{33} \frac{\partial^2 u_3}{\partial x_3^2} + C_{55} \frac{\partial^2 u_3}{\partial x_1^2} + (C_{55} + \hat{C}_{13}) \frac{\partial^2 u_1}{\partial x_1 \partial x_3}$$

(7)

where

$$\hat{C}_{11} = C_{11} - C_{12}^2/C_{22}$$

$$\hat{C}_{33} = C_{33} - C_{23}^2/C_{22}$$

$$\hat{C}_{13} = C_{13} - C_{12}C_{23}/C_{22}$$

The constants $C_{ij}$ can be determined from the properties and geometric arrangement of the composite constituents.

For plane waves in the plate, two wave speeds exist for each direction $\underset{\sim}{n} = (\cos \phi, \sin \phi)$ (see Figure 1) and are determined by

$$(A_{11} - \rho v^2)(A_{22} - \rho v^2) - A_{12}^2 = 0$$

(8)

where

$$A_{11} = \hat{C}_{11} \cos^2\phi + C_{55} \sin^2\phi$$

$$A_{22} = \hat{C}_{33} \sin^2\phi + C_{55} \cos^2\phi$$

$$A_{12} = A_{21} = (C_{55} + C_{13}) \sin\phi \cos\phi$$

The ratio $A_1/A_3$ is determined by substituting each root $v^2$ into Eq. (4), (k = 1, 3).

These values have been calculated for a number of composites (see Moon, 1972a) and are shown in Figure 2 for 55% graphite fiber/epoxy matrix composite. As the fiber lay-up angle is changed for the same composite, the properties of the waves are seen to change dramatically.

The direction of particle motion relative to the wave normal is shown in Figure 3 for 55% graphite fiber/epoxy matrix four ply lay-up angles. For the $0^O$, $\pm 45^O$ lay-up angle cases, the direction of particle motion tends to lie close to the fiber directions for most wave normals.

The $\pm 15^O$ and $\pm 30^O$ lay-up angle cases present another departure from the isotropic case. In both cases $C_{55} > \hat{C}_{33}$ . This means that for waves traveling in the $x_3$ direction the faster wave becomes transverse and the slower longitudinal. This behavior is also found in pine wood.

## B. Wave Surfaces

The relation $v(\phi)$, in Figure 3, is called the <u>velocity surface</u>. However if the waves originated from some point in the plate, to an observer at position $(r_o, \Theta_o)$, the first signal to arrive may not be that corresponding to the wave normal $\phi = \Theta_o$ . If the arrival time is t = 1, the first plane wave, $\underset{\sim}{n}$ , to arrive at the point $\underset{\sim}{r}$ must satisfy

$$\underset{\sim}{r} \cdot \underset{\sim}{n}(\phi) = v(\phi) \tag{9}$$

or $\qquad \underset{\sim}{r} \cdot \underset{\sim}{s} = 1 \qquad$ where $\underset{\sim}{s} \equiv \underset{\sim}{n}/v$ ;

$\underset{\sim}{s}$ is called the slowness vector and $1/v(\phi)$ the slowness surface (see

e.g. Kraut, 1963). The equation $\underset{\sim}{r} \cdot \underset{\sim}{s} = 1$, then represents a line in

the slowness plane $(s_1, s_2)$ and $\underset{\sim}{r}$ is normal to that line. In addition

$\underset{\sim}{s} \cdot \underset{\sim}{r} = 1$ must be tangent to the locus of values $1/v(\phi)$ given by

$$g(\underset{\sim}{s}) = 0$$

Thus

$$\underset{\sim}{r} = \alpha \nabla g$$

where $\alpha$ is a constant.

This coupled with Eq. (9) gives the position, r ,to the first arrival

of a plane of normal $n(\phi)$ generated at the origin,

$$r = \frac{v \, \nabla \, g(\underset{\sim}{s})}{\underset{\sim}{n} \cdot \nabla g} \tag{10}$$

This locus $r(\underset{\sim}{s})$ is called the wave surface and for in-plane plate

motion, there exists two such surfaces. These are shown in Figure 4,5

for the system 55% graphite fiber/epoxy matrix (Moon, 1972a).

The equivalent elastic constants for fiber-matrix systems at various

lay-up angles were obtained by Chamis (1971). These constants, which are

listed in Table II,are based on a static analysis of an eight-ply plate

using the known properties of each fiber-matrix ply.

The graphite-epoxy systems contrasts with other composite systems

because of its high stiffness ratio; $C_{11}/C_{33} = 24$ (zero lay-up angle).

The velocity surfaces for lay-up angles of $\pm 0°$, $\pm 15°$, $\pm 30°$, and $\pm 45°$ are

shown in Figure 2.

Examining the wave surfaces for graphite-epoxy, as shown in Figures 4, 5, one sees that the inner surfaces show peculiar cusps and nonconvexity. This behavior is also characteristic of crystal systems such as zinc. Unlike the natural crystals, we can change the wave properties, without changing the material constituents, by varying the fiber lay-up angle. It becomes clear that, as the anisotropy in the outer wave is reduced , the cusped behavior of the inner waves increases. This is due to the increase in shear wave anisotropy (Figure 2).

Another peculiar property of wave propagation in this composite system can be noted by examination of the $\pm 45^{\circ}$ fiber lay-up case (Figure 5). On the outer wave surface, the angle of the wave normal of the first arrival plane wave is listed. One can see that the distribution of plane wave normals is heavily concentrated at positions on the wave surface close to the fiber directions. This might imply a focusing of waves along the fiber directions. For the other fiber orientations, the distribution of wave normals is also concentrated at those points on the wave surface close to the fiber directions but not as densely as in the $\pm 45^{\circ}$ lay-up case.

Similar results for the glass fiber-epoxy composite system have been calculated (Moon, 1971). The ratio of stiffnesses for this case $C_{11}/C_{33} = 3.1$ (zero lay-up angle). The wave surfaces for this system show features similar to the graphite-epoxy case.

The velocity and wave surfaces for a boron fiber/aluminum composite were also calculated (Moon, 1971). However, the shear velocity is almost isotropic and no cusps appear on the wave surface.

Weitsman (1972) and (1973) has recently studied waves in a transversely isotropic composite with a rigid fiber constraint.

## C. Flexural Waves in Orthotropic Plates

For the case when the motion includes displacements out of the plane of the plate, Mindlin (1961) and co-workers have formulated an approximate theory to describe flexural waves in anisotropic plates. In that theory the plate motion is expressed in a series in the thickness parameter i.e.

$$
u_1 = u_1^0 (x_1, x_3, t) + \frac{x_2}{b} u_1^1 (x_1, x_3, t) + \ldots
$$

$$
u_2 = u_2^0 (x_1, x_3, t) + \ldots \tag{11}
$$

$$
u_3 = u_3^0 (x_1, x_3, t) + \frac{x_2}{b} u_3^1 (x_1, x_3, t) + \ldots
$$

The average in-plane motion is governed by Eqs. (7) while the functions $u_2^0$, $u_1^1$, $u_3^1$ are coupled together in the equations;

$$
\rho \frac{\partial^2 u_2^0}{\partial t^2} = C_{66} \frac{\partial^2 u_2^0}{\partial x_1^2} + C_{44} \frac{\partial^2 u_2^0}{\partial x_3^2} + C_{66} \frac{1}{b} \frac{\partial u_1^1}{\partial x_1} + C_{44} \frac{1}{b} \frac{\partial u_3^1}{\partial x_3}
$$

$$
\rho \frac{\partial^2 u_1^1}{\partial t^2} = \hat{C}_{11} \frac{\partial^2 u_1^1}{\partial x_1^2} + C_{55} \frac{\partial^2 u_1^1}{\partial x_3^2} + (C_{55} + \hat{C}_{13}) \frac{\partial^2 u_3^1}{\partial x_1 \partial x_3}
$$

$$
- \frac{3}{b} C_{66} \left( \frac{\partial u_2^0}{\partial x_1} + \frac{u_1^1}{b} \right)
$$

$$\rho \; \frac{\partial^2 u_3^1}{\partial t^2} = \hat{C}_{33} \; \frac{\partial^2 u_3^1}{\partial x_3^2} + C_{55} \; \frac{\partial^2 u_3^1}{\partial x_1^2} + (C_{55} + \hat{C}_{13}) \; \frac{\partial^2 u_1^1}{\partial x_1 \partial x_3}$$

$$\tag{12}$$

$$- \frac{3}{b} \; C_{44} \left( \frac{\partial u_2^0}{\partial x_3} + \frac{u_3^1}{b} \right)$$

It should be noted that in the procedure used by Mindlin the coefficients $C_{44}$ and $C_{66}$ in Eqs. (12) are replaced by, $\kappa_3 C_{44}$ and $\kappa_1 C_{44}$, respectively. The correction constants $\kappa_1$ and $\kappa_3$ were adjusted in order to match the thickness shear vibration mode.

Consider the flexural plane waves. One can show that the only plane wave solutions of the form that satisfy Eqs. (12) are harmonic functions, that is,

$$\begin{bmatrix} u^1 \\ u_3^1 \\ u_2^0 \end{bmatrix} = \begin{bmatrix} -b\psi_3 \\ b\psi_1 \\ U_2 \end{bmatrix} e^{ik(\underset{\sim}{n} \cdot \underset{\sim}{x} - vt)}$$

For bending motion, the phase velocity $v$ depends on the frequency, $\omega = kv$, as well as the wave normal $\underset{\sim}{n}$. Mindlin (1961) has examined the dependence of $v$ on $\omega$ for various material anisotropies.

Thus the behavior of the bending motion at the wave fronts cannot be determined in the same manner as was the extensional motion. Consider the motion at the wave front only. Across this front, one imagines that certain quantities have discontinuities. The displacement

and the stress are assumed to be continuous across the wave front but discontinuities in the second derivatives of U are assumed. Such waves are called acceleration waves.

It can be shown (Moon, 1972) that the wave fronts associated with a jump in the bending accelerations $\partial^2 u_1^1/\partial t^2$ and $\partial^2 u_3^1/\partial t^2$ travel at the same speeds as the wave front associated with the extensional motion. There is another wave front corresponding to a jump in the quantitity $\partial^2 u_2^0/\partial t^2$ . The speed for this wave is governed by the equation

$$\rho v^2 = C_{66} \cos^2\phi + C_{44} \sin^2\phi \tag{13}$$

For the case of a composite with symmetric ply orientation about the midplane,

$$C_{66} = C_{44}$$

The bending wave front associated with the jump $[\partial^2 u_2^0/\partial t^2]$ is directionally isotropic.

If both extensional and bending motions are generated simultaneously by impact, the two extensional and two bending wave fronts will travel with the same wave speeds.

This conclusion does not hold if the laminate plate has only a few plys. For example, Sun (1972b) has shown that the bending and extensional wave fronts are different for a three-layered plate ($0^o$, $90^o$, $0^o$ fiber lay-up angle). Thus the use of the effective modulus theory to predict the speeds of the wave fronts would appear to be valid only when the number of plys is large (probably $\geq$ 10 layers).

The analysis presented here is not unique. The same results can be obtained if one considers the equations of motion from the method of characteristics.

For harmonic waves the relation between the frequency $\omega$ and wave number has three branches. For the lowest branch

$$\omega^{(1)}(k) \sim k^2 \qquad kb \to 0$$

$$\omega^{(1)}(k) \sim k \qquad kb \to \infty$$

Thus for low frequencies the phase velocity $v = \omega/k$ depends on the wavelength as in isotropic plates. For shorter wavelengths or higher frequencies, $v \to (C_{44}/\zeta)^{1/2}$ which is the isotropic velocity of the wave fronts as discussed above. For the other two branches, $\omega^{(2)}(k)$, $\omega^{(3)}(k)$ the phase velocities at high frequencies are constant and anisotropic and equal to the values calculated for in-plane plate waves.

The distinguishing feature about such waves in composites is that these dispersion relations depend on wave direction. In Figure 6, dispersion curves for flexural waves are given for the wave directions $0^o$, $90^o$ for $\pm 45^o$ lay-up angle, for 55% graphite fiber/ epoxy matrix composite using the data in Table 2 .

It should be noted that these mathematical models are approximate and will break down for those Fourier components of the wave with wavelengths of the order of the composite constituent dimensions. Such considerations induce additional dispersion in addition to that due to the plate surfaces.

D. Surface Waves

In contrast to the bulk waves discussed above, surface waves are motions with wave-like behavior along the surface or interface between two different materials, and exponential decay with distance from the

surface. Thus if $x_3$ is normal to the surface, a surface wave has the following form for harmonic waves

$$u \sim e^{-\gamma(\omega)x_3} e^{ik(\omega)[x_1 \cos\alpha + x_2 \sin\alpha - v(\omega)t]}$$

These waves are known as Rayleigh[*] waves (see e.g. Ewing et al., 1957) for a free surface and Stonely waves for an interface.

For an orthotropic material the velocity of such a wave traveling in the $x_1$ direction on the surface of a half space normal to $x_3$ (Figure 7) is given by the roots to the following equation.

$$\rho v^2 + \left[ \frac{C_{55} - \rho v^2}{C_{33} C_{55} (C_{11} - \rho v^2)} \right]^{1/2} [C_{13} - C_{33}(C_{11} - \rho v^2)] = 0 \quad (14)$$

Examination of this equation reveals that one real root lies in the internal,

$$0 < \rho v^2 < C_{55}$$

Thus the Rayleigh wave speed in this direction is less than the shear speed $[C_{55}/\zeta]^{1/2}$. The motion is planar, i.e. $u_2 = 0$, and can be

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[*]Another wave known as a Love wave exists for a surface with a layer of different material.

shown to be elliptical in a plane normal to the surface.

The Rayleigh wave speed, however, varies with direction in the plane of the surface and has been shown by Musgrave (1954) to give a wave surface with cusps for certain anisotropic materials similar to bulk shear waves. Also, the existence of such waves for all surfaces in the material has been vigorously debated in the literature. However, Lin and Farnell (1968) have found Rayleigh type solutions for all surfaces, though for certain planes and directions the variation of the motion from the surface combines exponential and harmonic functions. While this work has been applied to crystals, the application to composites should be obvious.

E. Edge Waves in Plates

Waves confined to the edges of plates should be important in the edge impact of plate-like structures e.g. jet engine fan blades. When the average plate motion lies in the plane of the plate, waves analogous to Rayleigh waves exist for low frequencies. For motion out of the plane of the plate, flexural edge waves may propagate but are dispersive even at low frequencies.

1. Extensional Edge Waves

As in the case of bulk waves, we may look for plane stress Rayleigh wave solutions in plates. By replacing $C_{11}$, $C_{33}$, $C_{13}$, by $\hat{C}_{11}$, $\hat{C}_{33}$ $\hat{C}_{13}$ in Eq. (14) values may be obtained for extentional edge waves in anisotropic plates.

The extension of Eq. 14 to plates must be made with caution since the plane stress approximation breaks down at frequencies approaching

that of the first thickness shear modes for which the waves became
dispersive. McCoy and Mindlin (1962) have used a higher mode analysis
to examine such waves for isotropic plates but the extension to
anisotropic plates does not seem to have been made at this writing.

2. Flexural Edge Waves

Flexural edge waves for isotropic plates using Mindlins plate
theory has been studied by Kane (1954). No reference to the problem
for anisotropic plates has been found by the author. As a brief
sketch of the procedure, let us consider the low frequency classical
anisotropic plate equation for the transverse displacement $u \equiv u_2(x_1, x_3, t)$

$$C_{11} \frac{\partial^4 u}{\partial x_1^4} + 2(C_{13} + 2C_{55}) \frac{\partial^4 u}{\partial x_1^2 \partial x_3^2} + C_{33} \frac{\partial^4 u}{\partial x_3^4} + \rho G \frac{\partial^2 u}{\partial t^2} = 0 \qquad (15)$$

where $G = 12/h^2$, $h$ is plate thickness. For an edge wave propagating
in the $x_1$ direction we look for solutions of the form

$$u = A e^{-\gamma x_3} e^{i(kx_1 - \omega t)}$$

When the frequency is given, $\gamma$ and $k$ are related by the equation

$$C_{33} \gamma^4 - 2(C_{13} + 2C_{55}) k^2 \gamma^2 + (C_{11} k^4 - \rho G \omega^2) = 0 \qquad (16)$$

The appropriate boundary conditions require the moment on the edge to
be zero, i.e.

$$C_{13} \frac{\partial^2 u}{\partial x_1^2} + C_{33} \frac{\partial^2 u}{\partial x_3^2} = 0 \qquad (17)$$

and the resultant shear contributions from shear force and edge torque gradient to vanish, i.e.

$$\frac{\partial}{\partial x_3}\left( C_{13}\frac{\partial^2 u}{\partial x_1^2} + C_{33}\frac{\partial^2 u}{\partial x_3^2} \right) + 4 C_{55}\frac{\partial^3 u}{\partial x_1^2 \partial x_3} = 0 \tag{18}$$

Choosing two values of $\gamma$ with negative real part, and applying the boundary conditions, we obtain an equation for the phase velocity of these waves $v = \omega/k$

$$\gamma_1^2 + \gamma_2^2 = 2(C_{13} + 2C_{55})k^2/C_{33} \quad , \gamma_1^2\gamma_2^2 = \left( \frac{C_{11}k^2}{\rho G} - v^2 \right)\frac{\rho Gk^2}{C_{33}}$$

Let $\tag{19}$

$$\beta^2 \equiv \left( \frac{C_{11}}{\rho G} - \frac{v^2}{k^2} \right)\rho GC_{33}$$

Then $\qquad \beta^2 + 4C_{55}\beta - C_{13}^2 = 0$, (choose $\beta > 0$)

## F. Waves in Coupled Composite Plates

Laminated plates made up of unidirectional plys can have coupling between the extensional or inplane motion and the flexural or out of plane displacements. Using an effective modulous theory the equations of motion for one dimensional waves in the $x_1$ direction, with $u_3 = 0$, assume the form (see Ashton, et al., 1969)

$$A_{11}\frac{\partial^2 u_1}{\partial x_1^2} - B_{11}\frac{\partial^3 u_2}{\partial x_1^3} = \rho h\frac{\partial^2 u_1}{\partial t^2}$$

$$\tag{20}$$

$$B_{11}\frac{\partial^3 u_1}{\partial x_1^3} - D_{11}\frac{\partial^4 u_2}{\partial x^1} = \rho h\frac{\partial^2 u_2}{\partial t^2}$$

In this case there is no pure flexural or extensional wave. Instead we have a coupled wave

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u \\ w \end{bmatrix} e^{i(kx-\omega t)}
$$

with a dispersion relation

$$
(\rho h \omega^2 - A_{11} k^2)(\rho h \omega^2 - D_{11} k^2) - B_{11}^2 k^2 = 0 \tag{21}
$$

For low frequencies, or when $B_{11}/A_{11}^2 \ll 1$, the wave with dominant extensional motion has the phase velocity

$$
v_1^2 \simeq v_0^2 \left[ 1 + \frac{B_{11}}{A_{11}^2} k^2 \right]
$$

where $v_0^2 = A_{11}/\rho h$ .

Similarly, for low frequencies $(k \to 0)$ or small $B_{11}$ , the mode with flexural motion dominant, has the phase velocity

$$
v_2^2 \simeq \frac{D_{11}}{\rho h} k^2 \left( 1 - \frac{B_{11}}{D_{11} A_{11}} k^2 \right)
$$

We might note here that the extensional shear wave in this direction is uncoupled from the flexural motion. Since the extensional-flexural coupling is unique to composites, it is suprising that such waves in laminated structures have not received as much attention to date as other topics in dynamics of composites. Sun (1972b) has studied the propagation of wave fronts in laminated plates with bending-extensional coupling. He also observes coupling of the inplane and flexural motions in the various waves.

## III.  DISPERSION IN COMPOSITES

One definition of wave dispersion is the distortion of the pulse shape as it propagates through the material.  It is to be distinguished from attenuation in which energy is scattered out of the waves or converted to heat.  A more precise definition of dispersion rests on the assumption of a linear material and the theorem that any wave pulse in the material can be expressed as a linear sum of harmonic waves, e.g. for a one dimensional wave the displacement might have the form

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega) \, e^{-i\omega(t-x/v)} d\omega \tag{22}$$

For a non-dispersive material the phase velocity of all the harmonic components are equal.  Examples of wave dispersion are common in structural dynamics of isotropic materials in the form of rods, plates and shells.  Although the bulk waves in elastic materials are non-dispersive, the introduction of bounding surfaces, which define the structural element, causes the reflection of these waves from the surface to depend on the wavelength ($\lambda = 2\pi/k = 2\pi v/\omega$) .. If "a" represents a length parameter (thickness, diameter) then $ka$ or $\omega a/v$ become critical parameters in the problem of wave dispersion.

It is only natural then to expect that inhomogeneities such as fibers, laminates or particles in a material matrix will result in a greater dispersion of waves as the wavelength approaches the size or spacing of the composite constituents.  It should be noted here that when a composite is used as a structural element there will be two

sources of dispersion; that associated with the lengths of material elements, e.g. fiber diameter, or ply thickness, and that associated with the structural dimensions. It should be expected that the latter will become important for wavelengths much longer than those of the order of the material elements.

There have been three general approaches to material dispersion in composites:

    i)   exact solutions of elastodynamic equations

    ii)   approximate solutions of elastodynamic equations

    iii)   micro continuum theories

Several reviews on waves in composites have appeared (Peck, 1971, 1972, Achenbach, 1972) in which the various models for wave dispersion have been discussed. The reader is referred to these reviews for detailed discussion of the various approaches. In this chapter I will try to summarize the principle conclusions of the work on dispersion published to date.

A.   Pulse Propagation and Dispersion

While theoretical descriptions of dispersion often employ an infinite train of harmonic waves, the engineer is more often interested in the propagation of stress pulses in a medium. Conceptually, calculation of the effects of dispersion is straightforward. One decomposes the stress pulse at a given time into a spectrum of harmonic waves and uses the phase velocity to translate each component wave, reconstructing the pulse at a later time using Eq. (22). Two developments in the last decade have made the execution of this procedure reasonably easy.

One tool is the digital computer and the development of efficient Fourier

summing algorithms such as the "fast Fourier transform". The other is

the analytical foundation developed      by Skalak (1957) and others

for using an approximate dispersion relation for the phase velocity

$$v \underset{\sim}{\sim} v_o \; (1 - \alpha k^2)$$

or                                                                                                    (23)

$$v \underset{\sim}{\sim} v_o \; (1 - \beta \omega^2)$$

for large times after impact or large distances from the wave source.

Peck (1971), Peck and Gurtman (1969) and others have explored in detail

the effects of dispersion in layered composites. They have demonstrated

that a wave with a stress discontinuity will be smoothed out, that stress

overshoot can occur, and that an initial compression pulse can develop

tensile stresses as the wave propagates. These effects can be heuristically

understood since the local inhomogeneities will reflect part of the

propagating stress discontinuity at each layer. Multiple reflections in

each layer will delay part of the pulse and effectively broaden the average

stress in the pulse. Further ,the local inhomogeneities can change the

sign of the reflected stress as well as raise the stress at a layer

interface. Examples of the effects of dispersion are shown in Figures 8, 9

where a fast Fourier computer routine was used. Case I (Fig. 8) involves the

example of decreasing phase velocity with frequency. It has been shown

(Peck, 1971) that when the input is a step in stress the response is

related to an integral of the Airy function

Smoothing of the stress jump can be seen in Figure 8, as well as the overshoot right after the arrival of the pulse. At the tail end, the stress is seen to change sign. This dispersion is characteristic of longitudinal waves propagating down the fibers or layers as well as across the layers.

Case II involves increasing phase velocity with frequency or wave number. This case is found for shear waves propagating down the fibers or layers. In Figure 9, the response seems to mirror the previous case, in that stress reversal obtains for early time.

B. Dispersion in Rods and Plates

Mechanicians have long been familiar with the effects of geometric dispersion when classical materials take the form of rods and plates. For isotropic cylindrical rods the long wavelength dispersion relation was given by Chree (1890) and others

$$v \simeq v_o [1 - \frac{1}{4} v^2 (ka)^2] \qquad , ka \ll 1$$

where $v$ is Poissons ratio, and $a$ is the rod radius. For anisotropic rods an equivalent dispersion relation for the phase velocity (neglecting material dispersion) was worked out by Pottinger (1970)

$$v \simeq v_o [1 - \frac{1}{4} v'^2 (ka)^2] \qquad , ka \ll 1$$

(24)

$$v' = \frac{1}{\sqrt{2} \, S_{11}} [S_{12}^2 + S_{13}^2 + \frac{1}{2} S_{14}^2 + S_{15}^2 + S_{16}^2]^{1/2}$$

where $v_o^2 = 1/S_{11}\rho$ , and $S_{ij}$ are the elastic compliances of the composite. Pottinger points out that the accuracy of this approximation depends on the values of $\nu'$ and $a/\lambda$. For 2% deviation from the exact dispersion relation, $2a/\lambda < 0.6$ for $\nu' = 0.1$, and $2a/\lambda < 0.16$ for $\nu' = 0.4$.

If the fiber direction of a unidirectional composite is varied relative to the rod axis, different dispersion relations are obtained for each angle. An example is shown in Figure 10 for a Boron-Aluminum composite.

For a longitudinal wave in a plate one can derive a similar dispersion relation for anisotropic plates incorporating the transverse inertia. For a wave propagating in the $x_1$ direction, and $x_2$ normal to the plate, the phase velocity is given for long wavelengths

$$ v \approx \sqrt{\frac{\hat{C}_{11}}{\rho}\left[1 - \frac{C_{12}^2}{C_{22}^2}\frac{(kd)^2}{24}\right]} \quad , \quad kd \ll 1 \tag{25}$$

where $\hat{C}_{11}$ was defined in Eq. (7), and d is the plate thickness.

The Eqs. (24), (25) neglect the material dispersion due to the inhomogeneous nature of the composite, which becomes increasingly important as the wavelength approaches the scale of the size of the constituents. Such effects are considered in the next section.

C. Dispersion in a Layered Composite

We now examine the propagation of elastic waves in a solid made up of alternating layers of different material stiffnesses and densities. This model has been used by many authors to examine the effects of dispersion in composites (Peck and Gurtman, 1969; Sun, Achenbach, Herrmann, 1968). It is also an important problem in seismology to which numerous authors have given attention (Rytov, 1955). The wave concept for this system is the same for connected discrete particle chains as described by Brillouin (1963).

A cell is defined as two adjacent layers. A local cell coordinate $\eta$ will be used to distinguish one cell particle from another. The position to any particle in the composite is given by $x = na + \eta$ where $a$ is the cell length, $a = d_1 + d_2$. A displacement wave in the direction normal to the layering has the form

$$u(\chi) = A(\eta)e^{i(kna - \omega t)}$$

We can consider either longitudinal waves for which $u$ represents a displacement normal to the layering or a transverse wave where $u$ represents the displacement parallel to the layering. In the former case let $\sigma$ be the normal stress on any plane parallel to the layering and $c$ the longitudinal speed of sound in the material. For the transverse case $\sigma$ will represent the shear stress and $c$ the speed of a shear wave in the material. The balance of momentum is given by the following equation

along with the stress-strain relation;

$$\frac{\partial \sigma}{\partial \chi} = \rho \frac{\partial^2 u}{\partial t^2} \quad , \qquad \sigma = c^2 \frac{\partial u}{\partial \chi}$$

These equations must be satisfied in each of the two materials and the solutions in each layer must satisfy boundary conditions of continuity of stress vector and displacement. Solutions which satisfy the above equations and the cell boundary conditions at $\eta = 0$ are given by

$$u_{1n}(\eta) = e^{ikna} \left[ A \cos \frac{\omega \eta}{c_1} + B \sin \frac{\omega \eta}{c_1} \right]$$

$$u_{2n}(\eta) = e^{ikna} \left[ A \cos \frac{\omega \eta}{c_2} + \frac{\rho_1 c_{01}}{\rho_2 c_2} B \sin \frac{\omega \eta}{c_2} \right]$$

(The factor $e^{-i\omega t}$ has been dropped for convenience). [The term $\rho c$ is called the acoustic impedence and is proportional to the ratio of stress/velocity and is analogous to the same concept in electrical systems.]

The boundary conditions at the $n^{th}$ cell - $(n + 1)^{th}$ cell interface yield two homogeneous algebraic equations for $A, B$ and also gives the dispersion relation.

$$\cos ka = \cos \frac{\omega d_1}{c_1} \cos \frac{\omega d_2}{c_2} - \frac{(1 + p^2)}{2p} \sin \frac{\omega d_1}{c_1} \sin \frac{\omega d_2}{c_2} \qquad (26)$$

where $\quad p = \rho_2 c_2 / \rho_1 c_1$

This relation is periodic in $k$ and symmetric about the $k = 0$ axis. For each $k$ in the Brillouin zone, $-\pi \leq k a \leq \pi$ , there are an infinite number of values for $\omega$ and hence an <u>infinite</u> <u>number of branches</u>. There is one acoustic branch for either transverse or longitudinal waves and may be obtained for long wave lengths by expanding Eq. (26) about $k = 0$, $\omega = 0$ ;

$$\omega = v_0 k$$

where

$$\frac{1}{v_0} = \left(\frac{d_1}{ac_1}\right)^2 + \left(\frac{d_2}{ac_2}\right)^2 + \frac{(1+p^2)}{2p} \frac{d_1 d_2}{a^2 c_1 c_2}$$

When the acoustic impedances are equal

$$\frac{a}{v_0} = \frac{d_1}{c_1} + \frac{d_2}{c_2}$$

which is just the sum of the times for a wave to traverse each of the layers in the cell.

For this case the waves are non-dispersive. This is true because there are no reflections at an interface of two materials when the acoustic impedances are matched.

If the successive branches are translated to successive zones, the dispersion relation takes the character of a continuous homogeneous medium, as shown in Figure 11. However there are stop bands proportional to the mismatch of impedance. This behavior is also characteristic of quantum electron waves in a periodic potential of a conducting solid.

For long wavelengths the dispersion relation Eq. (26) can be expanded about $\omega = 0$, $k = 0$, to obtain an approximate dispersion relation for the lowest or "acoustic" wave mode.

$$v \simeq v_o \, (1 - \alpha(kd)^2) \tag{27}$$

where

$$\alpha = \frac{1}{12} \left\{ 1 - v_0^4 \left[ \frac{\delta_1^4}{c_1^4} + \frac{\delta_2^4}{c_2^4} + \frac{6\delta_1^2\delta_2^2}{c_1^2 c_2^2} + \frac{2(1 + p^2)}{p} \left( \frac{\delta_1^2}{c_1^2} + \frac{\delta_2^2}{c_2^2} \right) \frac{\delta_1 \delta_2}{c_1 c_1} \right] \right\}$$

and

$$\delta_1 = d_1/a, \quad \delta_2 = d_2/a.$$

Note that when $p = 1$, $\alpha = 0$ .

The long wavelength phase velocity can be shown to be related to the equivalent static homogeneous elastic constant for the material, i.e. for longitudinal waves

$$v_o^2 = C_{11}/\rho$$

$$\rho = \rho_1 V_1 + \rho_2 V_2$$

and

$$C_{11} = 1/[\frac{V_1}{(\lambda_1 + 2\mu_1)} + \frac{V_2}{(\lambda_2 + 2\mu_2)}]$$

where
$$V_1 = d_1/(d_1 + d_2) \ , \ V_2 = d_2/(d_1 + d_2)$$

In the terminology of Herrmann and Achenbach (1968) $v_o$ is related to the effective modulus, $C_{11}$ whereas $v$ is related to an effective stiffness which is frequency or wavelength dependent.

More generally consider a composite made up of three dimensional repeating cells, such that $\underset{\sim}{L} = \ell\underset{\sim}{a}_1 + m\underset{\sim}{a}_2 + n\underset{\sim}{a}_3$ ($\ell$, m, n, integers) is a lattice vector between corresponding points in any two cells. The vectors are called a basis set for the material. One can think of the material properties $\rho$, $C_{ij}$ as periodic functions i.e. $\rho(\underset{\sim}{x}) = \rho(\underset{\sim}{x} + \underset{\sim}{L})$. It is well known that wave-like solutions exist for such a medium of the form

$$\mathcal{U}(\mathbf{x} + \mathbf{L}) = U_o(\mathbf{x}) e^{i(\mathbf{k} \cdot \mathbf{L} - \omega t)}$$

Thus the motion varies by a constant phase factor from cell to cell

and the problem is reduced to finding the motion in a single cell

i.e. $\mathcal{U}_o(\mathbf{x})$ defined by $\mathbf{L} = 0$. Further if one writes

$$\mathcal{U}_o(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} \mathcal{W}_o(\mathbf{x})$$

The function $\mathcal{W}(\mathbf{x})$ must be periodic,

$$\mathcal{W}_o(\mathbf{x} + \mathbf{L}) = \mathcal{W}_o(\mathbf{x})$$

For fiber composites in two dimensional periodic arrays one would

have only two basis vectors in the plane normal to the fiber directions

but similar properties on $\mathbf{u}$ would obtain.

Other work on the laminated composite includes that of Sve (1971a, 1971b) who

examined thermoelastic effects and waves oblique to the layering.

In addition to the layered or laminated composite, dispersion in

fiber or rod reinforced composites has been studied. Approximate

solutions for this problem were given by Puppo et al. (1968), Haener

and Puppo (1969), Jones (1970) and Ben-Amoz (1971). Jones shows the

phase velocity for longitudinal waves traveling down the fibers,

mode dispersed down for the lowest mode, as is indicated by experiments.

(Asay et al. (1968), Tauchert and Guzelsu (1972)). He also calculates

the cutoff frequency of the second mode for longitudinal waves, in terms of the matrix and fiber properties.

D.  Combined Material and Structural Dispersion

As mentioned above, dispersion due to structural geometry (e.g. in rods or plates) and dispersion due to material microgeometry (e.g. fiber size and spacing) have been studied separately but in actual structures both are present.  A class of problems in which both of these effects are examined simultaneously is the theory of laminated plates and shells.  Multilayer plates have been studied by Jones (1964), Sun and Whitney (1972), Biot (1972), Dong and Nelson (1972), Scott (1972), and Sun (1972a).  The study of waves in circularly laminated rods or shells of two materials has received attention from Lai (1968), McNiven et al. (1963), Armenakas (1965), (1967), Whittier and Jones (1967) and Chou and Achenbach (1970).

One can perhaps hazard a guess as to the comparison of the two effects on pulse propagation by an appeal to the head of the pulse approximation discussed above.  For longitudinal waves in a rod or plate the dispersive relation for a non-dispersive material has the form, for long wavelengths,

$$v \cong v_1 \ [1 - \alpha^2 (a/\lambda)^2]$$

where  a  is a structural thickness or diameter variable.  The constant $v_1$  is related to the square root of an elastic modulus.  If the material is itself dispersive, as occurs in a laminate or fiber composite,

the constant $v_1$ is related to the square root of an elastic stiffness which itself depends on the wavelength

$$v_1 \simeq v_0 [1 - \beta^2 (b/\lambda)^2]$$

where b is a fiber diameter or lamination thickness. The combined effects of both structural and material dispersion thus have the form

$$v \simeq v_0 [1 - (\alpha^2 + \beta^2 \frac{b^2}{a^2}) (a/\lambda)^2]$$

In most composites, $b/a \ll 1$, so that it would appear that the effect of material dispersion, where the structural geometries guide the waves, can only become important when $\beta \gg \alpha$.

The corresponding problem for a composite beam has been examined in detail by Sun (1972) in which he examines waves in a laminated beam using an effective stiffness continuum theory assuming that each layer obeys the Timoshenko beam assumptions. Sun compared his theory for a ten layered plate with both an exact analysis and an effective modulus Timoshenko beam theory. For alternating layers, of shear moduli in the ratio of 100, he found that the effective modulus model, based on Voight averaging of the constants, agreed with the exact analysis for $2\pi h/\lambda > 1$ where h is the total beam thickness. For waves of shorter wave length the effective modulus model deviated substantially from the microstructure and exact models.

E. Continuum Theories for Composites

When the dimensions of the constituents of a mixture (e.g. fiber

diameter, ply thickness) are much smaller than the structural dimen-
sions, the engineer is often satisfied with averages of the motions
of the constituents. In such cases a continuum model may suffice
to describe the motion in which the inhomogeneities are "smoothed
out". Examples of such are found in the use of classical elasticity
to describe conventional structural materials which have a hetero-
geneous grain structure. A similar model for laminated composites,
using an effective modulus theory has already been discussed above
which does not include material dispersion of waves (Chamis, 1971).

Attempts to construct continuum descriptions of dispersion in
composites have been varied, but there are,in general,two basic
approaches. The axiomatic method, is characterized by
the assumption of a stored energy function with certain functional
dependence on the deformation descriptors. The kinematic
variables include descriptors for the motion of the microconstituents,
e.g. fibers or particles, in addition to the average motion
at a point. Examples of this method are given by Mindlin
(1964), theory of "microstructure in elasticity", Eringen (1966, 1968)
Eringen and Suhubi (1964), theory of micropolar elasticity and micro-
morphic continua respectively, and also a mixture theory approach
by Green and Naghdi (1965). These theories attempt to characterize a
broad class of materials and in the linearized version of these theories,
contain a great number of material constants which must be determined
by experiment. Ozgur (1971) for example has used Eringen's micropolar
theory in an attempt to describe orthotropic fiber composites. His model

uses 30 material constants as compared to 9 constants for classical orthotropic elasticity. This theory results in the correct shear dispersion phenomena but does not predict dispersion for longitudinal waves.

In contrast to the first method the second approach starts from an assumption of a knowledge of the properties of each constituent, and by averaging, "smoothing" and energy methods tries to arrive at a continuum formulation in which the material constants are known in terms of the properties of the constituents. An example of this method has been given by Achenbach and Herrmann (1968a) (1968b) in which the microelements are fibers embedded in an elastic matrix. The fibers were assumed to behave as Timoshenko beams. Each point in the equivalent continuum is assigned two kinematic variables, the average displacement at a point $\underset{\sim}{\mu}$ , and the fiber rotation vector  which is independent of the vector $\underset{\sim}{\mu}$ . The resulting theory thus has six differential equations of motion to be satisfied at each point. These authors are able to predict dispersion for shear waves. For a wave normal along the fibers and motion transverse to the fibers the following dispersion relation is obtained

$$\rho^* \ \omega^2 \ \underset{\sim}{\cong} \ C_{44} \ k^2 (1 + \frac{\eta E_f}{C_{44}} \ (kr)^2)$$

where $\rho^*$ is the composite density, $C_{44}$ an effective shear modulus, $\eta$ the volume fraction of fiber, and $E_f$ , $r$ , the fiber modulus and

radius of gyration respectively. For a high modulus fiber in an organic-polymer matrix the ratio $E_f/C_{44}$ could be as high as $10^2$. Thus the dispersive effect of the reinforcement can become important even at wavelengths much larger than the fiber diameter. This model however does not predict dispersion for longitudinal waves. This problem was solved quite successfully in a series of papers by Sun et al. (1968), Achenbach et al. (1968) for the case of a laminated composite and by Achenbach and Sun (1972) for a fiber reinforced composite. The resulting dispersion relations predicted the correct phenomenon at low frequencies, Figure 12. The exact harmonic wave solution for alternating elastic isotropic layers was obtained by Sun et al. (1968) and compared with the continuum theory dispersion relation. These results (Figure 12) show good agreement at low frequencies and for materials whose moduli do not differ very much.

A similar method has been employed for fiber composites by Wu (1971). In these methods, one establishes a local cell at each point $\underset{\sim}{x}$ in the continuum containing a fiber and part of the matrix. Embedded in the cell is a local coordinate system $\underset{\sim}{\zeta}$ . There is assumed at each point $\underset{\sim}{x}$ , a local or microdisplacement field. In the case of Wu this takes the form

$$\underset{\sim}{u}(\underset{\sim}{x} , \underset{\sim}{\zeta}) = \underset{\sim}{u}(\underset{\sim}{x}) + \underset{\sim}{\phi}(\underset{\sim}{x}) \cdot \underset{\sim}{\zeta} \qquad (28)$$

for all points in the cell including fiber and matrix. In the paper of

Achenbach and Sun (1972) they assume different forms of displacement

fields for fiber and matrix material in each cell.

Thus for $r < a$ , $\zeta_1 = r \cos \theta$, $\zeta_2 = r \sin \theta$

$$u_i^f(\underset{\sim}{x}, \zeta) = u_i(\underset{\sim}{x}) + r \cos\theta \; \psi_{2i}^f(\underset{\sim}{x}) + r \sin\theta \; \psi_{3i}^f(\underset{\sim}{x})$$

and for $r > a$

$$u^m(\underset{\sim}{x}, \zeta) = u_i(\underset{\sim}{x}) + a \cos\theta \; \psi_{2i}^f(\underset{\sim}{x}) + a \sin\theta \; \psi_{3i}^f(\underset{\sim}{x})$$

$$+ (r-a) \cos\theta \; \psi_{2i}^m(\underset{\sim}{x}) + (r-a) \sin\theta \; \psi_{3i}^m$$

This procedure is repeated for neighboring cells and the average dis-

placements along the adjoining cell boundaries are matched. In the

model of Wu (1971) this resulted in constraint equations on the local

cell strains

$$u_{k,\ell} = \phi_{k\ell} \tag{29}$$

Similar relations are obtained in the model of Achenbach and Sun. The

concept of a local constraint was first introduced in these theories

in the earlier work of Sun, Achenbach and Herrmann (1968) for the laminated

continuum.

To obtain equations of motion in these methods the local displace-

ment Eq. (28) is put into constitutive equations for the fiber and

matrix. The resulting strain energy density in each cell is integrated over the cell coordinates (holding $\chi$ fixed) and divided by the cell volume to yield the strain energy density at the point $\chi$, $V(\underset{\sim}{u}, \underset{\sim}{\phi})$. A similar procedure is carried out for the kinetic energy density at $\chi$, $T(\underset{\sim}{u}, \underset{\sim}{\phi})$. Hamilton's principle is then used to find the differential equations of motion and boundary conditions

$$\delta \int_{t_0}^{t_1} \int_V (T - V) \, dt \, dv + \int_{t_0}^{t_1} \delta W \, dt = 0 \qquad (30)$$

subject to the constraints between $\underset{\sim}{u}$, $\underset{\sim}{\phi}$, (e.g. Eq. (29)) and where $W$ is the work done on the boundary.

Grot (1972) has recently completed similar work on the fiber composite continuum and has obtained very good agreement with the experiments of Tauchert and Guzelsu (1971), Achenbach (1972) in another review in this series discusses the continuum models of composites.

Other work of a similar nature includes Ben-Amoz (1968), Barker (1970), Bartholomew (1971), Bolotin (1965), Gurtman et al. (1971), Hegemier (1972), and Koh (1970). Several mixture theories have been developed in which the strain energy contains terms proprotional to the difference $(\underset{\sim}{u}^{(1)} - \underset{\sim}{u}^{(2)})$, between the average displacements of each constituent (e.g. fiber and matrix). Examples of this are the works of Bedford and Stern (1971), and Martin, Bedford and Stern (1971). In the latter work, applied to a fiber composite, estimates of the constitutive constants are given in

terms of the fiber and matrix properties and geometry. Other mixture

models have been given by Lempriere (1969) and Moon and Mow (1970) for

spherical particles in a matrix.

Two comments regarding continuum theories of composites are in

order before finishing this section. First, when the mathematical

structure of these ad hoc continuum models are examined, one notes a

simularity with the axiomatic theories discussed earlier. Thus the model

of Sun et al. compares with Mindlins (1964) microelasticity theory and

Wu's model compares with Eringen's micromorphic thoery.   Herrmann and

Achenbach (1968) have discussed the application of Cosserat theory of

continua to composite materials. While specialized, these ad hoc

theories however have the advantage of predicting the effective

material constants for the composite in terms of the material constants

of the constituents. This approach enables the analyst to quickly

check his predicted dispersion results,

while in the more general theories such confirmation is not built into

the theory.

The second remark concerns the usefulness of continuum theories.

While it is remarkable that the laminate continuum theory of Sun et al.

(1968) checked so very well with exact theory, what is more remarkable

is that insofar as wave propagation is concerned the digital computer

was sufficient to provide the exact dispersion relation. The efficacy

of analytic methods not withstanding ,continuum models of composites will

certainly find strong competition from computer oriented methods (such

as the finite element method) in the future.

F.  Variational Methods for Periodic Composites

When the constituents of a composite are arranged in a periodic

array, such as a laminated medium or a fiber composite with uniform

spacing, the displacements and stresses under harmonic waves can

be represented by periodic functions.  This problem is called

"Floquet theory" in the subject of differential equations.  Thus

the problem is reduced to finding a solution in one cell.  Such

problems have analogues in solid state theory of electron waves

in periodic potentials.  The solution of the Schroedinger equation for these

problems    by variational methods has been outlined by Kohn (1952).

The extenstion of these methods to periodic composites has been made

by Kohn et al. (1972), who applied the theory to a laminated composite.

Wu (1971) has applied the variational method of the above authors to

a wave propagation normal to a periodic fiber composite material.

Wheeler and Mura (1972) and Tobón (1971), have looked at similar

problems.

According to the Floquet theory, wave like solutions to the

equations of motion in a periodic medium are themselves represented

in terms of periodic functions

$$\underset{\sim}{u} = \underset{\sim}{U}(\underset{\sim}{x})e^{-i\omega t} = U_0(\underset{\sim}{x})e^{i(k \cdot x - \omega t)}$$

$$\underset{\sim}{U}_0(\underset{\sim}{x}) = \underset{\sim}{U}_0(\underset{\sim}{x}+\underset{\sim}{R})$$

where $\underset{\sim}{R}$ is a lattice vector. If one writes the stresses in the form

$$t_{ij} = \sigma_{ij} e^{-i\omega t}$$

the equations of motion become

$$\sigma k\ell,\ell + \rho \, \omega^2 U_k = 0$$

A statement of a variational theorem is as follows; (Kohn et al.,

The problem of finding solutions to the equations of motion in terms of the functions $U_0(\underset{\sim}{\chi})$ which are periodic in the lattice vectors, and satisfy the displacement and stress vector continuity conditions across the cell and cell constituent boundaries, is tantamount to finding the stationary value of the functional

$$I[\underset{\sim}{U}] = \int_{cell} \{-\sigma_{k\ell} \, e^*_{k\ell} + \rho\omega^2 \underset{\sim}{U} \cdot U^*\} \, dv$$

with respect to a complete set of functions $\{\underset{\sim 0}{U}\}$ which are period$^{ic}$ in the lattice vectors, continuous and have continuous first derivatives in the cell ($e_{k\ell}$ is the strain tensor and $(\ )^*$ indicates complex conjugate).

This theorem allows one to choose a linear combination of functions from $\{U_0\}$ to approximate the wave in the cell. The amplitudes of each of the functions are chosen so as to extremize the functional $I[\underset{\sim}{U}]$ . This procedure leads to a homogeneous set of algebraic equations from which one obtains the dispersion relation between $\omega$ and $\underset{\sim}{k}$ .

There will be as many branches to the relation $\omega(\underset{\sim}{k})$ as there are approximating functions.

While the method can produce a reasonable approximation to the dispersion relation, the stresses in the cell may not be as accurate and lead to discontinuous stress vectors at the constituent boundaries, (Kohn et al., 1972) Bevilacqua, Lee (1971). However more general variational schemes can achieve better stress determination as well as obtaining the dispersion relation. (See e.g. Nemat-Nasser (1972)).

Lee (1972) has recently reviewed such methods for periodic composites Krumhansl (1970) has applied Floquet theory to the propagation of transient stress pulses in a layered medium and similar work has appeared by Krumhansl and Lee (1971).

## IV.  ATTENUATION AND SCATTERING

Attenuation of a propagating wave represents loss of energy, in contrast to dispersion in which the wave energy is conserved but redistributed in a deformed stress pulse.  Loss of energy during dynamic motion in composites can be attributed to at least four phenomena; i) viscoelastic or anelastic effects of the constituents, ii) wave scattering, iii) microfracture, iv) friction between poorly bonded constituents.  One important use of viscoelastic damping has been the concept of constrained layer damping of beams and plates (see e.g. Kerwin (1959), Yan (1972).  In this application a three layer laminate has a highly viscoelastic layer constrained by two stiffer elastic layers.  A continuum theory for a viscoelastic laminated composite has been given by Grot and Achenbach (1970), Biot (1972), as well as Bedford and Stern (1971) using a continuum mixture theory.  The former work does not treat waves, whereas Bedford and Stern calculate the attenuation coefficient in terms of the viscoelastic properties for a wave traveling along the layers.

As in acoustics, the effect of inhomogeneities in a solid is to scatter energy out of an incident wave.  If there is some order to the inhomogeneities e.g. a periodic array of fibers or particles, this scattered energy can be rescattered back into the wave (i.e. dispersion) or reflected back to the wave source.  To the extent that the inhomogeneity is random, elastic energy will be scattered out of the incident wave thus attenuating the pulse.  Thus a mixture of elastic solids can appear in its averaged properties to be inelastic.  Krumhansl (1972) has

some general remarks on this problem in comparing composites with the theory of crystal lattices. Knopoff and Hudson (1967) studied a randomly heterogeneous elastic medium with a plane harmonic wave incident on it. At low frequencies the scattered energy shows the familiar Rayleigh dependence on frequency i.e. $\sim \omega^2$ . Mok (1969) has constructed a model for the scattering of waves propagating normal to the fibers, when both fiber and matrix are elastic, and indicates the possible existence of dissipation in the composite under dynamic loadings. Recently Christensen (1972) and McCoy (1972) have examined attenuation due to scattering and disorder in composites.

Chow and Hermans (1971) have examined the intensity of scattered waves in a composite by considering the density and elastic constants to be random variables independent of an axial coordinate. The authors calcuate the scattering cross section (which is a measure of the energy of the scattered field)and find the cross-section proportional to $\omega^2$ (two dimensional Rayleigh scattering ). Theoretical data on the cross-section for longitudinal and shear waves propagating in a glass fiber-epoxy matrix composite are presented.

Moon and Mow (1970) presented a theoretical model for attenuation in dilute particulate composites using the dynamics of a single particle in an elastic medium. When the inhomogeneities are dilute (volume fraction, $V_f < .10$) and random, a first approximation to the calculation of scattered energy can often be obtained from the mechanics of a single scatterer, (The diffraction of elastic waves by single scatterers has been reviewed by Mow and Pao (1971)). When the density of a rigid inclusion $\rho_2$ , embedded in an elastic matrix, is greater

than that of the matrix, i.e. $\rho_2 > \rho_1$ the equation of translational motion of the sphere U can be found to be

$$\rho_2 \frac{d^2U}{dt^2} + \frac{9\rho_1}{\tau_0} \frac{(2\kappa^3 + 1)}{(2\kappa^3 + 1)^2} \left(\frac{dU}{dt} - \frac{du}{dt}\right) + \frac{9\rho_1}{\tau_0 (2\kappa^2 + 1)} (U-u) = 0 \quad (31)$$

where u, is the average motion of the matrix without the inclusion

κ, is the ratio of dilatational to shear speed in the martix, $C_L/C_S$

$\tau_0 = a/C_L$

a, radius of the sphere

The form of this equation suggests a mixture theory in which the elastic energy depends on $(U-u)^2$, and a dissipation function proportional to $(\dot{U}-\dot{u})^2$ . The dependence on the velocity $\dot{U}$ accounts for the radiation of elastic energy when the particle vibrates in the matrix. The dependence on the matrix velocity $\dot{u}$ accounts for scattered waves if the particle were motionless. The resulting equations for the particulate composite, of volume fraction $V_f$, were found to be

$$\rho_1 (1 - V_f) \frac{\partial^2 u}{\partial t^2} - Y \frac{\partial^2 u}{\partial z^2} = \rho_2 V_f \frac{\partial^2 U}{\partial t^2}$$

$$(32)$$

$$\frac{\partial^2 U}{\partial t^2} + 2C \left(\frac{\partial U}{\partial t} - \frac{\partial u}{\partial t}\right) + \Omega_0 (U-u) = 0$$

where

$$C = \frac{9\rho_1 \ (2\kappa^3 + 1)}{2\rho_2 \tau_0 \ (2\kappa^2 + 1)^2} \quad , \quad \Omega_0^2 = \frac{9\rho_1}{\rho_2 \tau_0^2 \ (2\kappa^2 + 1)}$$

If the damping were neglected the medium would exhibit a natural frequency of $\Omega_0$ (e.g. $\rho_2/\rho_1 = 10$, $\kappa = 2$, $a = 10^{-4}$m, $C_L = 4 \ 10^3$m/sec., $\Omega_0/2\pi \sim 2 \ 10^6$ HZ). For $N$ particles per unit volume, a sine wave pulse ( wavelength $\lambda \gg a$, and initial intensity $I_0$, at $Z = 0$, proportional to $(\partial u/\partial z)^2$), will decay as

$$I = I \ e^{-N \ z}$$

where

$$\gamma = \left(\frac{\rho_2}{\rho_1}\right)^2 \ \frac{2\pi}{27} \ (2\kappa^3 + 1) \ a^6 \left(\frac{\omega}{C_L}\right)^4 \quad , \quad (\rho_2 \gg \rho_1) \tag{33}$$

The scattering cross-section $\gamma$ follows the well known Rayleigh behavior at low frequencies.

While this model is clearly limited in application by the assumptions made, it serves to make a simple connection between attenuation in a composite mixture of elastic solids and the mechanics

of the individual constituents. Further work on attenuation in composites is needed. The extension of the model of Moon and Mow to dilute fiber composites could be made using the work of Mow and Pao (1971) on the dynamics of a cylindrical inclusion in an elastic matrix. For volume fractions above 10%, multiple scattering should be taken into account as was done by Mok, and Chow and Hermans. In a recent paper Sve (1973) constructs an equivalent viscoelastic model from the scattering of waves by cavities in a porous laminated composite.

The above model for scattering of waves is based on the interaction of harmonic waves of wavelengths long compared with the size of the scatterer. The analyses of problems of this type have been summarized by Mow and Pao (1971). When the scatterer encounters a stress wave with a very short rise distance, a wave front analysis based on ray theory may be more efficient. This method has been employed by Achenbach et al. (1968), (1970) and by Ting and Lee (1969). One should keep in mind that the time for the stress to rise from zero to a given value, in the distance of a fiber diameter (< .005 inch or .1 mm) is of the order $10^{-7} - 10^{-8}$ sec. Such waves only occur in shock waves or in ultrasonic pulses of frequency greater than 10 HZ.

## V.   SHOCK WAVES

The previous discussion has assumed that the deformation in
the propagating waves was small and that the material behaved in
a linear elastic manner.   Nonlinear elastic wave analyses in com-
posites are few, as e.g. that  of Ben-Amoz (1971) who studied finite
amplitude waves in a fiber composite for waves along the fibers.
Nor has much work been published to date on plastic waves in composites.
Wlodarczyk (1971) has examined shock waves in plastic layered media
with linear unloading behavior.   Calculation of plane waves in anisotropic
elastic-plastic solid$^s$ has been discussed by Johnson (1972) but was
not applied to composites.

Shock waves in composites, however, have received a great
deal of attention.   In this class of wave phenomena, the pressures
in the solid are assumed to be so high, that the material can be
treated as a hydrodynamic fluid.   This means that the shear or
deviatoric stresses are assumed to be small compared with the mean
stress or pressure.   This occurs for pressures much greater than the
yield or elastic limit stress.

A plane shock wave is defined as a thin planar region pro-
pagating relative to the material, across which the velocity has a
discontinuity.   When the medium is homogeneous, continuity and
momentum conditions across the shock surface yield the following
relations between the density  $\rho$, normal particle velocity  $v$ , shock
speed  U, and pressure  P ,

$$\| \rho (v-U) \| = 0$$

$$\| \rho v(v-U) \| = - \| P \|$$

When the conditions ahead of the wave are such that $P_0 = 0$, $v_0 = 0$, the conditions behind the shock require that

$$v_1 = U \left(1 - \frac{\rho_0}{\rho_1}\right)$$

$$U^2 = P_1/\rho_0 \left(1 - \frac{\rho_0}{\rho_1}\right)$$

In addition one must prescribe a constitutive relation for the pressure and satisfy an energy balance across the shock.

Munson and Schuler (1970, 1971) have extended this analysis to laminated composites and mechanical mixtures. In their model they neglect the thermodynamics and assume constitutive relations for all n constituents in the composite, $P_n = P_n(\rho_n)$, and require the pressures in all the constituents to be equal at any position x, i.e.

$$P_n(x) = P_1(x) \quad \text{for all} \quad n$$

Applying this theory to laminates for waves travelling both along and normal to the layers, Munson and Schuler (1970) conclude that the shock speed is independent of the direction , under certain assumptions on the strain in each constituent. The shock speed obtained has the form

$$U^2 = P_1/(\Sigma \alpha_n^0 \rho_n^0) \left[ 1 - \Sigma \frac{\alpha_n^0 \rho_n^0}{\rho_n} \right] \tag{34}$$

where $\alpha_n^0$ is the initial volume fraction of the $n^{th}$ constituent

and $\rho_n^0$ the initial density. They also conclude that the model

is not limited to laminates, and can be used for any mechanical

mixture.

The particle velocity immediately behind the shock is assumed

to be equal in all layers and given by

$$v_1 = U \ (1 - \Sigma \ \frac{\alpha_n^0 \rho_n^0}{\rho_n} )$$

Thus when the constitutive relations for each constituent are known,

the shock velocity can be found as a function of the particle speed.

This relation is called a Hugoniot curve. Munson and Schuler applied

this model to a mixture of $Al_2O_3$ particles in an epoxy matrix and

compared their calcuations with experimental points (Figure 13). For

this mixture the compressibility is shown to behave much like the

softer component.

Identical results were obtained by Torvik (1970). Tsou and

Chou (1970) used a similar model but included the thermodynamics in

the analysis. Bedford (1971) has reported a theory for Hugoniot

relations for a multi-continuum.

Measurements of shock waves and shock Hugoniot curves for

quartz-phenolic have been performed by Isbell et al. (1967), Charest

and Jenrette (1969), and Munson et al. (1971). Studies in shock waves

in aluminum-polymethyl methacrylate (PMMA) have been reported by

Barker and Hollenbach (1970), and Schuler (to appear), and Schuler

and Walsh (to appear). Other references to the study of shock waves in

composites include Gary and Kirsch (1971) and Holmes and Tsou (1972).

The independence of the shock speed on the direction of propagation in an elastically anisotropic composite can only hold at high pressures. Munson and Schuler (private communication) have indicated that such dependence on direction has been observed for some composite systems below pressures of 6 kilobars.

The construction of theoretical models in the region between elastic wave theory and hydrodynamic shock model will present a great challenge to the analyst.

## VI.  EXPERIMENTS

The generation and measurement of stress waves in composite

materials has, in general, been based on"state of the art"techniques

in applied science used to study waves in solids. These involve

the use of air guns, explosive charges, exploding foil flyer plates,

shock tubes and piezoelectric ultrasonic generators.  To measure the

stress waves, strain gages, piezoelectric crystals, capacitance gages,

optical interferometer, holographic and pholoelastic techniques are

used.  Experimental work in this area, while not as copious as the

theoretical efforts, has  provided a steady stream of experimental

data with which to check the mathematical models.  For a variety of

materials, including fiber, laminate and woven fiber composite, data

has been reported on measured wave speeds, attenuation and dispersion

of stress pulses, shock wave behavior, stress wave induced fracture

and impact.

The experiments can be categorized by the type of stress pulse

used.  The monotonic compressional pulse has been used,from long rise

time, half sine like pulses induced by projectible impact, to short

rise time waves induced by explosive flyer plate impact.  The Fourier

content of these pulses have a large zero frequency component.  Ultra-

sonic or pulsed sine wave tests have a narrow spectrum centered about

a particular frequency or wavelength.  The latter waves are thus

ideally suited to map the dispersion relation directly,by measuring

the group velocities of the pulses, while in the monotonic pulse method

the dispersion is indirectly manifested by a change in pulse shape with

passage through the material.

One of the problems associated with using pulsed sine waves is measuring a wave velocity. As has been pointed out in many texts (see e.g. Brillouin, 1960), the shape or envelope of the pulsed sine wave travels at the group velocity of the spectrum center frequency and

$$v_g \equiv \frac{d\omega(k)}{dk}$$

is not equal to the phase velocity $v_p = \omega/k$ when dispersion is present. The velocities are related however

$$v_g = v_p / [1 - \frac{\omega}{v_p} \frac{dv_p}{d\omega}]$$

Brillouin also describes two other velocities, the wave front velocity and the signal velocity. The latter is associated with the first arrival of signals with the spectrum center frequency. The signal velocity is sometimes equal to the group velocity (Brillouin, 1960). The message however is clear; careful definition and interpretation of ultrasonic wave velocity measurements are required in order to construct the dispersion relation $\omega(k)$.

Abbott and Broutman (1966)   demonstrated the use of a monotonic pulse to measure the equivalent elastic constants of steel/ glass and "S" glass/epoxy composites. This method is valid as long as the stress rise length and total pulse length are large compared

with the size of the fibers, fiber spacing and the transverse structural dimensions of the specimen (e.g. rod diameter or plate thickness). Potapov (1966) used pulsed ultrasound to measure the elastic constants of fiberglass plates. He concluded that orthotropic elasticity gave a sufficiently accurate description of the elastic properties so determined by these tests. Markham (1970) also used pulsed ultrasound in an ultrasonic tank to measure the elastic constants of a carbon fiber epoxy resin composite.

Tauchert and Moon (1970) used the monotonic pulse method and compared the results with data from resonance tests and static moduli. The dynamically and statically determined moduli for boron/epoxy and glass/epoxy were within 2% for waves along the fibers. It was found that the wave attenuation could be predicted from vibration resonance tests of the materials. Tauchert (1971a, 1971b) has used ultrasonic waves to measure all the elastic constants of a variety of composites. Tauchert (1972) has also measured ultrasonic attenuation in composites and observed increases in damping due to initial tensile stress.

Pottinger (1970) used a similar method in glass/epoxy and boron/aluminum and found agreement between statically and dynamically determined moduli to within 3% for waves in bars at various angles to the fibers. Also Nevill, Sierakowski et al. (1972) used the same method on steel/epoxy bars with waves along the steel fiber direction. The increase in wave speed with volume fraction of steel checked very closely the rule of mixtures (Figure 14 ). The attenuation was found to decrease with increase in steel. Tensile waves generated on reflection from a free end were found to propagate at a slightly

higher rate than the compressional pulse. Also wave broadening and a shift of peak stress to the rear of the pulse with time was observed. The attenuation of stress waves due to a transverse fiber composite lamina in a rod has been studied by Ross et. al. (1972). Other studies in which the elastic constants of composite materials were determined using waves include Kvasnikov (1967), Tuong (1970) and Cost and Zimmer (1970a). Elastic constants of filled elastomers were determined using ultrasonics by Waterman (1966) and showed the effects of temperature and percent filler on the properties of the two phase materials. Also White and Van Vlack (1970) have used an acoustic resonance technique to determine the properties of open-pore polymer foams with higher-moduli infiltrating matrices.

Using a gas dynamic shock (70 psi) to induce a short use time pulse, Whittier and Peck (1969) studied the effects of dispersion in graphite and boron reinforced carbon phenolic composites with the wave in the fiber directions. The transmitted pulse showed a smoothed pulse rise in place of the wave front, overshoot, and oscillations in the stress plateau region, which checked the prediction of Peck and Gurtman (1969). This technique is described in a paper by Cummerford and Whittier (1970). Lundergan and Drumheller (1971a) performed a similar experiment on a laminated composite of steel and epoxy. They used a flyer plate technique to generate compressional pulses (9 kilobar) normal to the layering. The wave front smoothing and oscillations were again observed. The authors compared the results with a numerical solution of dilatational wave propagation in a periodically laminated medium and obtained a satisfactory comparison with their experimental data. The calculations also showed that a steady wave was formed by about the time the pulse left the third bilaminar plate.

In another work Lundergan and Drumheller (1971b) studied the impact of an obliquely laminated composite of steel and polymethyl methacrylate (PMMA). Although theory predicts a two step wave corresponding to a quasi longitudinal and a quasi shear pulse, the experiments showed a three step sloping wave. The authors conclude that further work is needed to explain the discrepancy.

With somewhat different motives Schuster and Reed (1969) used a flyer plate technique to generate shock waves in a boron/aluminum composite at pressures up to 76 kbar and impact duration of less than 0.2 micro sec. The impact velocity of the flyer plates were increased until damage occurred. Increased fiber crushing with impact velocity was observed and the spalling velocity was measured for aluminum and two boron/aluminum composites. The spall velocity for the plasma sprayed, diffusion bonded composite showed a three fold increase in velocity over the spall velocity for the aluminum specimens, while the plasma sprayed, brazed composite showed a slight decrease in the spall velocity compared with aluminum. This dramatic effect is attributed to the two different geometrical arrangements of the fibers produced during fabrication. As shown in Figure 15b, in the diffusion bonded specimens the fibers are not touching and hence are able to attenuate the shock wave by multiple scattering. In the brazed specimen (Figure 15a) one can see that the fibers are contacting in the direction of the wave. Thus a boron path is created through the medium with less of an increase in attenuation, resulting in a spall velocity no greater than that for aluminum.

One may conclude from this experiment that the fiber geometry will be an increasingly important factor in stress wave failure as the stress rise distance or pulse length approaches the fiber

dimensions. In the experiments of Schuster and Reed the fiber diameter and spacing are about 0.1 mm and the pulses of the order of 1.0 mm in length in the aluminum.

Several important papers have examined the dispersive nature of composites directly with the use of ultrasonic waves. Asay et al. (1968), demonstrated a decrease in phase velocity with frequency as predicted by several theories (Peck and Gurtman, 1969) for waves along the fibers of thornel and Boron reinforced carbon-phenolic composites. The carbon-phenolic without the fibers showed no change in phase velocity with frequency out to 4 MHZ but a change in the rein-forced specimens of $\Delta v/v \sim 0.20$ at 3 MHZ (Figure 16).

Tauchert and Guzelsu (1972) performed similar experiments on boron/ epoxy and examined a variety of wave normal-fiber orientations for both longitudinal and shear waves. In addition to the decrease in group velocity of longitudinal with frequency both across and along the fibers, shear waves traveling along the fibers and polarized normal to the fibers showed a 25% increase in group velocity at about 1 MHZ (Figure 17). The wavelength in epoxy at this frequency is about 2.6 mm compared with a fiber diameter and spacing distance of about 0.1 mm. This behavior had been predicted by Achenbach and Herrmann (1968) in an early theoretical model and by other authors (e.g. Sun et al., 1968) in later works. Shear waves propagating across the fibers showed a slight decrease in group velocity with frequency.

Sutherland and Lingle (1972) performed a similar ultrasonic experi-ment with pulsed sine waves on tungsten fibers in an aluminum matrix. They also observed a decrease in wave velocity with frequency, for longitudinal waves across the fibers, of about, $\Delta v/v \sim 0.083$ at 3 MHZ. They also claim to have observed a cutoff band in frequency as well as

the second branch of the dispersion relation (optical branch) (Figure 18).
In addition, a frequency shift in transmitted pulse lower than the in-
cident pulse frequency was observed by these authors as the frequency
approached the cutoff region. This is attributed to the filtering of
the higher frequency components in the pulse lying in the stop bond of
frequencies, effectively shifting the observed frequency of the trans-
mitted pulse. It should be noted that Sutherland and Lingle (1972) claim
to have measured the phase velocity in their report. However precise
definition of the velocity measurement is lacking and the present
author suspects that the data represent group velocities.

Rowlands and Daniel (1972) have used interferometric holography
to measure the transverse displacement in vibrating laminated aniso-
tropic plates. This method may hold some promise for observing two
dimensional waves in anisotropic plates due to transient impact loads.

Dally, Link and Prabhakaran (1971) were able to observe two
dimensional waves in orthotropic fiber reinforced plates using photo-
elasticity. This development was made possible by the development
of orthotropic-birefringent materials which were sufficiently trans-
parent for photoelastic analysis (see Prabhakaran, 1970). In this
study the authors examined both the transient loading of a half plane
with edge loading and the full plane problem with a hole loaded with
an explosive charge of lead ozide, as shown in Figure 19. The anisotropic
nature of the stress wave propagation is clear from the figure (moduli
ratio $E_L/E_T \sim 3.0$). A cusp-like fringe seen in Figure 19 might
represent the effects of shear wave anisotropy. By measuring the

wave surface of the outer fringe the authors were able to reconstruct the velocity surface for the quasi longitudinal wave of the plate material. This velocity was within 10% of that determined from the static effective moduli of the plate material.

Another photoelastic study of stress waves is reported by Hunter (1970), who used an explosive strip along the specimen edge. Using alternating layers of different material, fringe patterns accompanying a plane wave in the layered direction were observed. Rose and Chow (1971) used a similar method to observe the build-up of a steady wave front in a alternately layered composite of different photoelastic materials.

Other experimental work on the propagation of waves in composites includes Benson et al. (1970), Berkowitz and Gurtman (1970), Berkowitz and Cohen (1970), Lord (1972), Cohen and Berkowitz (in press) have studied dynamic fracture in composites due to stress waves.

Sierakowski et. al. (1970a, 1970b) have measured dynamic stress strain relations for various composites and strain rates up to $10^3$ sec$^{-1}$. Up to a 85% increase in ultimate failure stress was observed at these rates (Figure 20).

## VII. IMPACT PROBLEMS IN COMPOSITES

### A. Introduction

The study of impact of isotropic solids has a large literature, part of which is documented in the book by Goldsmith (1960). The study of similar problems for composite structures has received very little attention at this writing. To be realistic, one should include the nonelastic and nonlinear aspects of the problem, since the object of such studies usually concerns the prediction or avoidance of failure due to impact. Also the material properties under high rates of strain become important, and the inhomogeneity and the anisotropy in composite materials invite a wider set of fracture or failure modes. Impact failure modes in isotropic materials include indentation, spalling, and penetration of the projectile through the structure. In composites one must add to this list fiber crushing, fiber pullout, splitting, and delamination (see Figure 21). Even with no visible damage, micro failure in the composite could produce a local stress riser, change the natural frequencies, and decrease the fatigue life.

Both empirical and analytic studies have been made but rarely are theory and experiment integrated. Much of this work has been motivated by the need for bird and hailstone impact protection of jet engine composite fan blades. Empirical studies of this problem, conducted at great cost, have produced results in the form of leading edge protection schemes and interleaving steel wire mesh between the plys, (Anon, 1971), while analytical studies have only begun to explore the problem (e.g. Moon, 1972).

One of the first areas of interest in bird impact problems was the design of aircraft transparent structures such as windows and wind shields to resist bird strikes. One such discussion is given by McNaughton (1964). A summary of tests in this article on vinyl sandwich panels reveals that the penetration velocity $V$, of one to eight pound birds on aircraft wind screens decreased as the cube root of the mass of the bird $M$; for a given set of structural conditions $MV^3$ = constant.

Research related to bird damage in aircraft engines has been reported by Allock and Collin (1968). Impact by chicken carcases, wax, wood and gelatine dummies have been investigated for target shapes resembling basic geometries. The authors constructed a momentum transfer model for the average impact force $F$ due to a spherical impactor

$$F = \frac{MV \sin^2\theta}{\Delta t}$$

where $M$ is bird mass, $V$ bird velocity, $\theta$ angle of deflection from line of flight, $\Delta t$ the duration of impact given by ratio of projectile diameter to velocity. In terms of bird density $\rho$

$$F = MV^2 \sin^2\theta \left(\frac{\pi\rho}{6M}\right)^{1/3}$$

Measurements showed that the assumed impact time was too long and the theoretical force too low. Deflections of the flat plate knife edge and round nose targets elastically mounted showed that the target deflection was proportional to the bird momentum.

Research on hail impact damage to typical aircraft structures has been presented by Hayduk (1973). Comparison is made of experimental and an analytical model for denting type hail damage in aluminum fuselage panels or dome segments (spherical cap).

The range of structural impact problems includes other phenomena besides bird and hailstone impacts. These include micrometeorite damage on spacecraft, dust, sand and rain erosion, and cavitation erosion of solids which involves dynamic stresses due to collapsing bubbles. A discussion of impact erosion by dust particles for metal surfaces is given by Smeltzer et. al. (1970). The mechanics of a liquid drop impact with a solid surface has been given by Heyman (1969) and Peterson (1972). Rain erosion of composites is reported by Schmitt (1970). Ballistic problems of high velocity penetrations of plastic-aluminum laminates by steel projectiles have been analyzed by Kreyenhagen et. al. (1970) using numerical computer modes which illustrate several damage modes.

The testing of composite materials under impact forces encompasses a variety of load and specimen conditions. Classical Izod and Charpy impact tests use relatively small beam-like specimens, (less than 3 inches long) under a transverse point force. The duration of the impact is usually long compared with the time of a stress wave to traverse the specimen. For example, using a wave speed of $v \sim$ 5mm/$\mu$sec, a length $L$ = 50mm, and pulse time $\tau = 10^{-3}$sec, the nondimensional number $v\tau/L = 10^2$ is a measure of the number of reflections occurring during such tests. In shock wave impact testing of composites, high compressive stress pulses of extremely short duration, ($\sim 0.2 \ 10^{-6}$sec) in thin specimens ($\sim$2 mm thick) are used producing a nondimensional number, (using the same wave speed as above), $v\tau/L \sim .5$ . Also the NASA

and the Air Force in the United States are sponsoring ballistic impact
tests on composite plates, as well as full size jet engine fan blades,
using hailstones and liquid objects to simulate bird impact.

Almond et al. (1969) have reviewed the literature on classical
impact testing on laminated composites. Embury et al. (1967) conducted
Charpy V-notch tests on soft solder laminated steel specimens both
with the impact force normal to the laminated surface (crack arrester
configuration) and parallel to the laminate surfaces (crack divider
configuration). In the latter case the ductile-brittle transition
temperature was reduced and the specimens showed higher impact energy
absorption over homogeneous steel specimens.

Also Chamis et al. (1971) have performed miniature-Izod impact
tests on fiber composites of glass and graphite fibers in an epoxy
resin matrix (specimen size 7.9x7.9x37.6 mm). The tests included
specimens with the fibers either parallel or transverse to the cantilever
longitudinal axis. The tests show failure modes of cleavage, cleavage
with fiber pullout, and cleavage with delamination. In the transverse
mode the cleavage included matrix fracture, fiber debonding and
fiber splitting. The transverse impact strength was found
by these authors to be correlated with the intralaminar shear strength
of the various composites tested.

In similar work, Novak and DeCrescente (1972) report the results
of Charpy impact tests for unidirectional graphite, boron, and glass
fibers in a resin matrix. They conclude that the toughness of the
resin matrix is not an important factor in impact energy absorption.
"S glass" composites showed a higher impact strength than boron/resin

and graphite/resin composites. They also evaluate the energy-absorption mechanisms such as filament pullout, shear delamination, ect. They conclude that the impact strength is correlated with the area under the fiber stress-strain curve.

In a recent paper Peck (1972) has reviewed the literature on spall fracture in composites using one dimensional shock waves. In addition to the work of Schuster and Reed (1969) on fiber composites discussed above, Warnica and Charest (1967) have used 1-2 µsec compression pulses on laminated quartz phenolic to determine spall stress thresholds. Similar work by Cohen and Berkowitz (1972), and Barbee et al. (1970) are also discussed.

It is useful to compare the merits of these different tests. In spall tests and dynamic stress-strain tests (e.g. Sierakowski, et al., 1971) the stress waves are one dimensional. Thus, clearly defined stress states are used to measure the material strength properties. However, in foreign object damage, the conditions of impact failure involve the contact of blunt objects with a surface, thus producing a complicated stress state. Izod and Charpy tests appear to simulate actual impact, since a knife edge on a pendulum encounters a beam-like specimen. Similarly,ballistic tests involve a locally inhomogeneous stress state in the region of projectile contact which is found in actual impact problems. However, the ad hoc nature of these stress states does not allow comparison with other tests. Thus Izod and Charpy ratings often cannot be compared. Also, because of the small size of the specimens and the long contact time (e.g. $\sim 10^{-3}$ sec) many reflections occur during the impact thus obscuring the wave like nature of impact, which might be present in a larger specimen or in the actual structure. In ballistic tests, however, contact times of $5 \ 10^{-5}$ sec or less are obtained for high velocity projectiles of the

order of one inch diameter. For a wave speed of 6 mm/μsec the impact

of a plate would result in the total energy of impact contained in

a circle of radius less than 30 cm. If the plate is part of a large

structure fewer reflections might obtain than for a small test

specimen. It is the opinion of this writer that scale effects are

of importance in impact tests. Thus if test data is to be extrapo-

lated to larger structures the nondimensional numbers

$$\frac{v\tau}{L}$$

where $\tau$ is the contact time, $v$ a wave speed, and $L$ a repre-

sentative length should be matched in addition to other variables.

B. Analytical Models for Impact

The total problem description involves the local deformation

at the impact site and the simultaneous determination of the motion

of the structure during and after impact. When the overall motion

of the structure takes place over a time period much larger than the

impact contact time, and the size of the impactor is much smaller

than the structural dimensions, the problem may be split into two

distinct parts. I) The local mechanics of impact with a deformable

half space, II) The response of the structure to a prescribed local

impact force as determined in Part I. The errors involved in such

a scheme appear to be on the conservative side since the procedure

will underestimate the contact time and overestimate the contact force

(Goldsmith, 1960).

Discussed in the next section is the impact of solid objects on solid surfaces. As already mentioned liquid or rain drop impact erosion is also an important problem. For sufficiently high impact velocities, solids may be treated by a hydrodynamic model and a liquid drop model may be useful.

1. Impact of a Half Space-Hertz Theory

The problem of an impulsive line force on an anisotropic half space has been given by Kraut (1963) for a transversely elastic isotropic material. In particular, a line source on the surface normal to the symmetry axis produces two wave surfaces as shown in Figure 22 corresponding to the wave surfaces, discussed in an earlier section. The extension of this work to dynamical contact with another elastic body has not been given to date. The waves generated during a point impact on an isotropic half space have been studied by Pekeris (1955) where it was shown that on the surface large stresses propagate at the Rayleigh surface wave speed. But the dynamics of an elastic sphere hitting an elastic half space are not known.

Thus without even considering anelastic effects, the analytic literature on dynamic impact is limited even for isotropic materials. Instead, what has been used is a quasi-static theoretical model called the Hertz theory (Goldsmith, 1960). This is based on the static deformation produced by a point force on a surface. When the force, F , is between a sphere of radius R, and a half space, F

is related to the relative approach of sphere and half space, $\alpha$ , by

$$F = \kappa \, \alpha^{3/2} \tag{35}$$

where

$$\kappa = \frac{4}{3} \frac{R^{1/2}}{(\delta_1 + \delta_2)} \quad , \quad \delta_i = \frac{1 - \nu_i^2}{E_i}$$

($\nu$ is Poisson's ratio, and E is Young's modulus).

This relation is nonlinear since the contact area $\pi a^2$ depends on the force.

$$a^2 \sim R\alpha$$

Equating this force to the change of momentum of a sphere during impact with initial velocity $V_o$ , this theory gives the following expressions for the contact time and force history,

$$\tau = \frac{2.94 \alpha_m}{V_o} \tag{36}$$

$$\alpha_m = \left| \frac{5}{4} \frac{M V_o^2}{\kappa} \right|^{2/5}$$

$$\left\{ \begin{array}{l} F = \dfrac{1.140 \; M V_o^2}{\alpha_m} \; \sin 1.068 \; V_o t \quad t \leq \tau \\[3mm] F = 0 \quad , \quad t > \tau \end{array} \right. \tag{37}$$

where $\alpha_m$ is the maximum approach, and M is the mass of the sphere.

Extension of the Hertz theory of impact to anisotropic bodies has been made by Chen (1969) and Willis (1966). The contact region has been shown by Willis to be elliptic for an anisotropic half space in contrast to a circle for the isotropic case. However he obtains a force deflection relation similar to Eq. (35), where $\kappa$ depends in a complicated way on the elastic constants. The determination of the ellipse parameters must be done numerically and no examples have been given to date for typical composite anisotropy.

A simple model for estimating the contact time for isotropic spheres on composites has been suggested by the author Moon (1972d), which assumes a circular contact area. Experiments on the contact of a steel sphere on unidirectional fiber composite plates, with the fibers parallel to the surface, show the contact area to be ellipical with the large axis normal to the fibers, but only slightly deviating from a circle.

Thus in Eq. (35 the half space constants $(1 - \nu^2)/E$ are replaced by a transverse elastic constant for the composite. Chen* has suggested using the compliance $S_{33}$ where the "3" axis is normal to the surface. The author has used $1/C_{33}$ to replace $(1 - \nu^2)/E$ in Figure 23 to estimate the contact time for hailstones and granite spheres on 55% graphite fiber in epoxy. For impact speeds in the range 100-500 m/sec the contact times range from 15-85 $\mu$sec.

In summary these formulae reveal the following dependence of contact time and peak pressure on impact velocity

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*Private communication

$$\tau \sim 1/V^{1/5} \qquad , \qquad P_o \sim V^{2/5}$$

Such results, however, should only be used as guidelines, since the theory uses assumptions which break down at high velocities. Goldsmith (1960) has made the following summary of the Hertz theory of contact.

1) At high velocities the Hertz contact time is a lower bound on the contact time.

2) When a sphere strikes a beam, the motion of the beam decreases the force, but the contact time remains about the same.

In another reference Goldsmith and Lyman (1960) have shown the Hertz theory to be remarkably valid insofar as contact time and peak force for the impact of hard steel spheres (1/2 inch diameter) onto a hard steel surface for velocities up to 300 ft/sec ($\sim$91.5 m/sec). The data in Figure 23 for graphite epoxy can only be used as a rough guide for contact times, until experimental data becomes available.

2. Non Hertzian Impact

The Hertz theory of impact rests on the contact law $F = \kappa \alpha^{3/2}$. For boron/aluminum and graphite fiber/epoxy composite plates this force law was tested under 1/4 inch and 3/8 inch steel balls in a static testing machine. The preliminary results in Figures 24,25 show clearly that a more general law is required and that for moderate forces (less than 100 lbf) the deformation is inelastic, requiring a different law for approach and rebound.

A more general contact law was given by Meyer (see Goldsmith, 1960)

$$F = \kappa_1 \alpha^n$$

If such a law holds for both approach and rebound, formulas similar to the Hertz theory can be obtained, (see Goldsmith, 1960, p. 91).

Clearly the state of knowledge about the impact of composite or inhomogeneous bodies is unsatisfactory. In addition to the lack of a good quasi-static theory which can account for anelastic effects, a truly dynamical impact model for composites is needed.

For isotropic materials computer codes employing finite difference methods have been developed for dynamical impact and penetration projectiles and deformable bodies, (e.g. Wilkins, 1969, Kreyenhagen et. al. 1970). These models apply anelastic constitutive equations and can predict permanent deformation. The extension of these codes to composites will no doubt be available in the near future as well as codes based on finite element methods. H wever there is a need for analytical solution for impact phenomena; first for their simplicity and accessibility to the designer, and second to check the computer codes which will certainly appear in the near future.

In developing analytical models for impact, the use of an equivalent anisotropic material is questionable if one desires to explain stresses in the contact region. When a composite material is indented by another body of convex surface the area of contact goes to zero as the contact pressure decreases. Thus for small forces this area is necessarily of the order of the dimensions of the fibers or lamina. One would expect a force-deflection law to exhibit periodic changes in slope as the contact area engages each successive fiber (Figure 25).

This is the tentative explanation for the waviness in the experimental relation shown in Figure 25 for 50% boron fibers in aluminum matrix. The periodic plateau appear to occur at deflections corresponding to contact radii differing by the fiber spacing ($\sim$.004 inches). Further experimental work on this problem is needed.

## C. Structural Response to Impact

### 1. The Coupled Problem

When the impact force and duration depend on the structural motion the above procedure cannot be used. The coupled response of an isotropic plate and a spherical impactor was teated by Eringen (1953) and others, (see Goldsmith, 1960). Conceptually the extension to composite structures is similar. Let two coordinate systems be embedded in the two bodies (see Figure 26) and let the axes $x_3$, $x_3'$ be directed into the surfaces of structure and impactor respectively. Relative to these coordinates $\omega_3$, $\omega_3'$ represent surface deflections, $W_3$, the deflection of the plate or shell neutral surface, and $W_3'$ the displacement of the impactor center of mass. If the surface shapes of both structure and impactor are given by $x_3 = S(x_1, x_2)$, $x_3' = S'(x_1', x_2')$ then the boundary condition to be satisfied over the contact region is

$$(\omega_3 + \omega_3') = W_3' - W_3 - S - S',$$
$$\text{on} \quad x_3, x_3' = 0$$

(38)

The deflections $\omega_3$, $\omega_3'$ are determined from a three dimensional analysis, such as a Hertz analysis, (as e.g. Willis, 1966). The

displacement $W_3$ is governed by a two dimensional plate or shell theory, while the impactor displacement $W'_3$ is governed by Newtons law for the body under initial conditions

$$W'_3 = 0, \quad \frac{dW'_3}{dt} = V_o \quad , \quad \text{at} \quad t = 0$$

The solution of such a problem for a composite structure is not known to the author, though the problem seems fairly straightforward.

2. Transient Load Problems

There has been, however, a number of studies made of the response of a composite body to short duration or impact-like forces. Already mentioned is the work of Peck and Gurtman (1969) on the response of a laminated half space to a compressive stress on the surface in the direction of the layering. Sve (1972) has also treated the laminated half space under impulsive heating of the surface, (e.g. from a laser), with thermoelastic coupling. This work uses the approximate continuum theory of Sun et al. (1968). In another work Sve and Whittier (1970) have applied this theory to the pressure loading of an obliquely laminated half plane to determine the effects of lamination angle and dispersion on the stresses.

Voelker and Achenbach (1969) treated an infinite laminated body under a step body force in a plane normal to the layering using an exact modal analysis. The interface shear stress wave shows a slow rise to a static value, while the normal interface stress is found to be oscillatory. Also Sameh (1971) has used a discrete element model to calculate the elastic-plastic response of a layered half space.

The one dimensional impact loading of a laminated plate has been discussed by Hutchinson (1969) where the pressure is normal to the layering. This problem can be solved exactly using the reflection and transmission coefficients for a stress pulse when it encounters a discontinuity. For example, the transmitted stress across a plane boundary separating two different materials with normal stress incident on the surface is given by

$$\sigma = T\sigma_o \; , \qquad T = 2 \, Z_1 Z_2 / (Z_1 + Z_2)$$

where $Z^1$, $Z^2$ are the acoustic impedances of the two materials. (Note, that $T$ is independent of the direction of the incident stress). Thus a pressure discontinuity of intensity $p_o$ propagating normal to a laminated medium of alternating acoustic impedances suffers an attenuation at the head of the pulse of

$$[\sigma] = - P^o \, T^{2n} \tag{39}$$

after encountering $n$ pairs of layers. Analysis using the reflected and transmitted waves in each layer reveals the stress history behind the wave front.

3. Transient Edge Loading of a Plate

As noted earlier, when the pulse duration is long enough, dispersive effects can be neglected as a first approximation and an equivalent anisotropic model can be used (Eqs. (7), (12)). One of the effects of anisotropy

is revealed in the one dimensional edge impact of an orthotropic

plate with the impact force in the plane of the plate and the edge

oblique to a symmetry axis . Neglecting

structural and material dispersion, we can use Eqs. ( 7), with

the boundary conditions on the edge

$$t^{nn} = - P_o f(t), \quad t^{ns} = 0$$

For an isotropic material a compressional wave would be generated.

However, for an edge oblique to the symmetry axis, two waves are

propagated into the plate with wave speeds corresponding to those

on the velocity surface with wave normal $(\cos\phi, \sin\phi)$. Also dis-

placements normal and parallel to the edge will be excited. The

displacements will take the form $(x_n$ normal to edge, $x_s$ along

the edge)

$$U^n = U_1 [\cos\phi - \alpha_1 \sin\phi] \, f(t - \frac{x^n}{v_1})$$

$$+ U_2 [\cos\phi - \alpha_2 \sin\phi] \, f(t - \frac{x^n}{v_2})$$

$$U^s = U_1 [\sin\phi + \alpha_1 \cos\phi] \, f(t - \frac{x^n}{v_1})$$

$$+ U_2 [\sin\phi + \alpha_2 \cos\phi] \, f(t - \frac{x^n}{v_2})$$

(40)

The vectors $(1, \alpha_1)$ and $(1, \alpha_2)$ are the eigenvectors corresponding

to $v_1$ , $v_2$ respectively and depend on the angle $\phi$.

The quasi shear wave is generated through the coupling of the

normal stress $t^{nn}$ with the shear strain, $e_{ns}$ , in the constitutive

equations written in the $x^n - x^s$ coordinate system, i.e. on $x^n = 0$

$$t^{nn} = A^{11} e^{nn} + A^{12} e^{ss} + 2A^{16} e^{ns} = -P^0 f(t)$$

$$t^{ns} = A^{16} e^{nn} + A^{26} e^{ss} + 2A^{66} e^{ns} = 0$$

(41)

The constants $A^{11}$, $A^{16}$, etc. are related to the elastic constants and the angle $\phi$ , (see Ashton et al. 1969). Determination of the constants $U_1$, $U_2$ result from substitution of Eqs. (41) into these boundary conditions and is left for the reader.

4. Impact Generated Flexural Waves

Flexural waves generated by impact forces transverse to isotropic plates has been reviewed by Mikowitz (1960). The one dimensional line impact of anisotropic plate using both the Mindlin Eqs. (12) and the classical theory, Eq. (15), has recently been treated by Moon (1972d) In this work the line force is transverse to the plate surface and oblique to the composite symmetry axes. In the context of the Mindlin theory extensional waves are generated by a transverse force as well as a flexural wave. The importance of shear deformation and rotary inertia, as reflected in Mindlins theory, is shown to become important when the width of the contact force distribution is comparable to the plate thickness.

The calculation of the two dimensional stress wave response to central impact forces has recently been studied by Chow (1971) and Moon (1972b,c). Using a Timoshenko theory for laminated orthotropic plates

Chow (1971) treats the transient response of a rectangular plate to normal impact.

The author (Moon 1972b,c) uses a Mindlin plate model to examine the stress contours after impact in an infinite plate. Again,both extensional and flexural waves are shown to be generated under transverse impact.

Solutions to the equations which govern the central impact of anisotropic plates were found for impact-like pressures using an analytical/computational method. The impact pressure distribution used was the following

$$q_2 = - P_o \left(1 - 2 \left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4\right) \sin \frac{\pi t}{\tau_o}$$

$$\text{for } r < a, \ (r^2 = x_1^2 + x_3^2) \text{ and } t < \tau_o \tag{42}$$

$$q_2 = 0, \text{ for } r \geq a \text{ or } t > \tau_o$$

The three stress measures chosen were the average membrane stress $(t_{11} + t_{33})/2$, the average flexural stress $(t_{11} + t_{33})/2$ at the surface of the plate, and the maximum interlaminar shear stress given by $(t_{21}^2 + t_{23}^2)^{1/2}$.

The stresses were calculated in a quarter plane of the plate for a specific time after the initiation of impact and were normalized with respect to the maximum impact pressure as calculated in the above section. The data is presented for various times and lay-up angles in the form of stress contour plots (Figures 27, 28). Superimposed on these curves

are the theoretical wave front for the particular wave in question

and the radius of the circle which bounds the impact pressure.

The significant stress levels all lie within the surface bounded

by the theoretical wave surface. In Figure 27, the average or membrane

mean stress contours $1/2(t_{11} + t_{33})$ for graphite fiber/epoxy matrix

laminate plates are shown for lay-up angles of $0°$, $±45°$.

The flexural or bending motion has three waves associated with

it. The largest stresses however were found in the lowest flexural

wave which travels at an isotropic speed given by

$$v_3 = [C_{66} \; \kappa/\rho]^{1/2}$$

($\kappa = \pi^2/12$ , is Mindlin's correction factor). Stress contours for the

mean flexural stress $1/2(t_{11} + t_{33})$ in this wave are shown in

Figure 28 for graphite fiber/epoxy matrix laminate plates ($±15°$, $±45°$

lay-up angles) under the transverse impact pressure Eq. (42). Note that

the wave front is circular since $v_3$ is isotropic for laminate plates.

Stresses in the second and third flexural waves were found to be small.

A three dimensional computer plot is shown in Figure 29 for the flexural

stress for the $±45°$ lay-up angle composite plate.

The maximum stress levels were found to occur immediately after

the end of impact and appeared to propagate along the fiber directions,

given by the lay-up angles.

These results show the effect of the change of fiber lay-up angles

on the stress distributions. For the flexural stresses, the optimum

lay-up angle to be $\pm15^{\circ}$, showing a 34% lower stress level than the $\pm45^{\circ}$ case. However, regarding the interlaminar shear stresses, for the same impact conditions, there seems to be little difference in the maximum stress level with lay-up angle despite significant changes in stress distribution in space with lay-up angle.

Another result of these calculations is that the induced stresses depend on the impact circle radius to plate thickness ratio.

Of course, to evaluate the possibility of fracture or failure of the composite under impact, the complete stress matrix at a point must be known, as well as the failure criteria for the material.

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CAPTIONS FOR FIGURES

1        Geometry of a two dimensional wave in a multi-ply plate

2        Velocity surfaces versus wave direction for various

ply lay-up angles; 55% graphite fiber/epoxy matrix

(Moon, 1972)

3        Direction of particle motion versus wave normal for

various ply lay-up angles; 55% graphite fiber/epoxy

matrix (Moon, 1972)

4        Wave surfaces for multi-ply plates; a) $0^{\circ}$ fiber lay-up

angle, b) $\pm 15^{\circ}$ fiber lay-up angle (Moon, 1972)

5        Wave surfaces for multi-ply plates; a) $\pm 30^{\circ}$ fiber lay-up

angle, b) $\pm 45^{\circ}$ lay-up angle (Moon, 1972)

6        Flexural wave dispersion relations in an anisotropic plate

(Mindlin's theory); 55% graphite fiber/epoxy matrix multi-

ply plate, $\pm 45^{\circ}$ fiber lay-up angle.

7        Surface waves and edge waves in solids

8        Distortion of an initially shaped trapezoidal pulse

due to wave dispersion e.g. longitudinal waves pro-

pagating across or down the fibers of a unidirectional

fiber composite material

9           Distortion of an initially shaped pulse due to wave

dispersion e.g. shear wave propagating down the fibers

of a unidirectional fiber composite material

10          Approximate dispersion relations for longitudinal waves

in boron fiber/aluminum matrix rods for various orientations

of the fibers to the rod axis (Pottinger, 1970), material

dispersion not included.

11          Sketch of dispersion relation for longitudinal or shear waves

propagating normal to the layers of a composite of alter-

nating isotropic layers, Eq. (26).

12          Comparison of exact dispersion relations (solid lines) with

the microcontinuum theory of Sun et al. (1968) for various

shear modulus ratios; a) shear waves propagating in the

direction of the layering, b) longitudinal waves propa-

gating in the direction of the layering

13          Shock wave speed versus particle velocity (Hugoniot curve)

for a mixture of $Al_2O_3$ particles in an epoxy matrix

(Munson and Schuler, 1970)

14          Experimental speeds of longitudinal waves in steel fiber/epoxy

matrix rods for various volume fractions of steel, (Nevill

et al., 1972); waves travelling along the fibers

15          The effect of fabrication on shock wave spall damage

in a boron fiber/aluminum matrix composite (Schuster

and Reed, 1969); a) brazed composite, b) diffusion

bonded composite

16          Experimental dispersion relation for longitudinal waves

propagating down the fibers (Asay et al., 1968); a) graphite

fiber (Thornel) reinforced carbon phenolic composite,

b) boron fiber reinforced carbon phenolic composite.

17          Experimental dispersion relations for waves in boron fiber/

epoxy matrix composite (Tauchert and Guzelsu, 1972)

upper figure - longitudinal waves normal to the fibers;

lower figure - shear waves, $x_3$ axis is along the fibers

18          Experimental dispersion relation for longitudinal waves

in a tungsten fiber/aluminum matrix composite (Sutherland

and Lingle, 1972) lower curve shows second branch and a

cutoff frequency around 4 MHZ

19          Orthotropic photoelasticity experiment showing an anisotropic

extensional wave in a plate loaded with a lead azide charge

in the center (Dally, et al., 1971), compare with Figures

4, 27.

27    Stress contours for the membrane stress $1/2(t_{11} + t_{33})$ after impact for a 55% graphite fiber/epoxy matrix plate. Comparison of a) $0^{\circ}$, and b) $\pm45^{\circ}$ ply lay-up angle cases (Moon, 1972)

28    Stress contours for the lowest flexural wave stress $1/2(t_{11} + t_{33})$ after impact for a 55% graphite fiber/ epoxy matrix composite plate. Comparison of a) $\pm15^{\circ}$, and b) $\pm45^{\circ}$ ply lay-up angle cases (Moon, 1972)

29    Three dimensional plot of the lowest flexural wave, $1/2(t_{11} + t_{33})$, and a quarter plane of the plate for a 55% graphite fiber/epoxy matrix composite plate with $\pm45^{\circ}$ ply lay-up angles (fibers along diagonals) (Moon, 1972)

TABLE I

| MATERIAL[2] | WAVE SPEED mm/μsec | WAVE TYPE | DIRECTION OF PROPAGATION | SOURCE OF VALUE |
|---|---|---|---|---|
| 69% "S" Glass Fiber/Epoxy Matrix | 4.90[1] | Longitudinal | Parallel to Fibers | Abbott and Broutman (1966) |
| 69% "S" Glass/Epoxy | 2.92[1] | Longitudinal | Perpendicular to Fibers | Abbott and Broutman (1966) |
| 54% Boron/Epoxy | 10.2 | Longitudinal | Parallel to Fibers | Tauchert and Guzelsu (1972) |
| 54% Boron/Epoxy | 3.8 | Longitudinal | Perpendicular to Fibers | Tauchert and Guzelsu (1972) |
| 54% Boron/Epoxy | 1.9 | Shear | Parallel to Fibers | Tauchert and Guzelsu (1972) |
| 54% Boron/Epoxy | 1.9 | Shear | Perpendicular to Fibers | Tauchert and Guzelsu (1972) |
| 10% Steel/Epoxy | 3.6[1] | Longitudinal | Parallel to Fibers | Nevill et al. (1972) |
| 22.1% Tungsten/Aluminum | 4.85 | Longitudinal | Perpendicular to Fibers | Sutherland and Lingle (1972) |
| 11% Graphite/Carbon Pehnolic (3% Porosity) | 5.4 | Longitudinal | Parallel to Fibers | Asay et al. (1968) (see also Whittier and Peck (1969)) |
| 10% Boron/Carbon Phenolic (9% Porosity) | 6.1 | Longitudinal | Parallel to Fibers | Asay et al. (1968) |
| Boron/Aluminum | 9.2[1] | Longitudinal | Parallel to Fibers | Pottinger (1970) |
| Boron/Aluminum | 6.9[1] | Longitudinal | Perpendicular to Fibers | Pottinger (1970) |
| Aluminum | 6.3 | Longitudinal | | |
| Boron | 12.[1] | Longitudinal | Parallel to Fiber | |
| Carbon Cloth-Phenolic Resin | 4.1 | Longitudinal | Parallel to the Cloth Layers | Whittier and Peck (1969) |
| Epoxy (Epon 828) | 2.6 | Longitudinal | | |
| Steel | 5.9 | | | Lundergan and Drumheller (1970) |

(1) Indicates velocity in a bar.

(2) Percentages indicate volume fraction of fiber material.

## TABLE II. – STRESS-STRAIN COEFFICIENTS FOR 55 PERCENT GRAPHITE FIBER-EPOXY MATRIX COMPOSITE

[All constants to be multiplied by $10^6$ psi; data obtained from ref. 7.]

### 0° Layup

| | | | | | |
|---|---|---|---|---|---|
| 27.95 | 0.3957 | 0.3957 | 0 | 0 | 0 |
| | 1.170 | 0.4601 | 0 | 0 | 0 |
| | | 1.170 | 0 | 0 | 0 |
| | | | 0.3552 | 0 | 0 |
| | | | | 0.7197 | 0 |
| | | | | | 0.3552 |

### ±15° Layup

| | | | | | |
|---|---|---|---|---|---|
| 24.56 | 0.4000 | 1.986 | 0 | 0 | 0 |
| | 1.170 | 0.4558 | 0 | 0 | 0 |
| | | 1.374 | 0 | 0 | 0 |
| | | | 0.3552 | 0 | 0 |
| | | | | 2.310 | 0 |
| | | | | | 0.3552 |

### ±30° Layup

| | | | | | |
|---|---|---|---|---|---|
| 16.48 | 0.4118 | 5.167 | 0 | 0 | 0 |
| | 1.170 | 0.4400 | 0 | 0 | 0 |
| | | 3.093 | 0 | 0 | 0 |
| | | | 0.3552 | 0 | 0 |
| | | | | 5.491 | 0 |
| | | | | | 0.3552 |

### ±45° Layup

| | | | | | |
|---|---|---|---|---|---|
| 8.197 | 0.4279 | 6.758 | 0 | 0 | 0 |
| | 1.170 | 0.4279 | 0 | 0 | 0 |
| | | 8.179 | 0 | 0 | 0 |
| | | | 0.3552 | 0 | 0 |
| | | | | 7.082 | 0 |
| | | | | | 0.3552 |

Fig. 1

φ = 0°

±15°

±30°

±45°

Velocity, mm/μsec

4   6   8

Fig. 2

Fig. 3

Fig. 4

Fig. 5
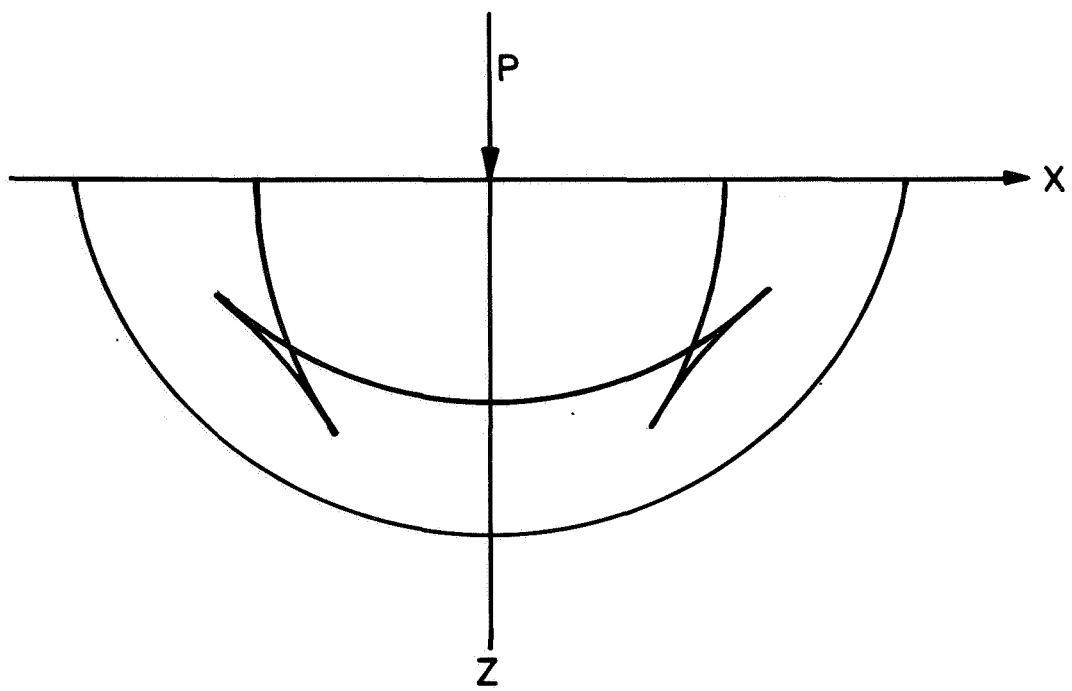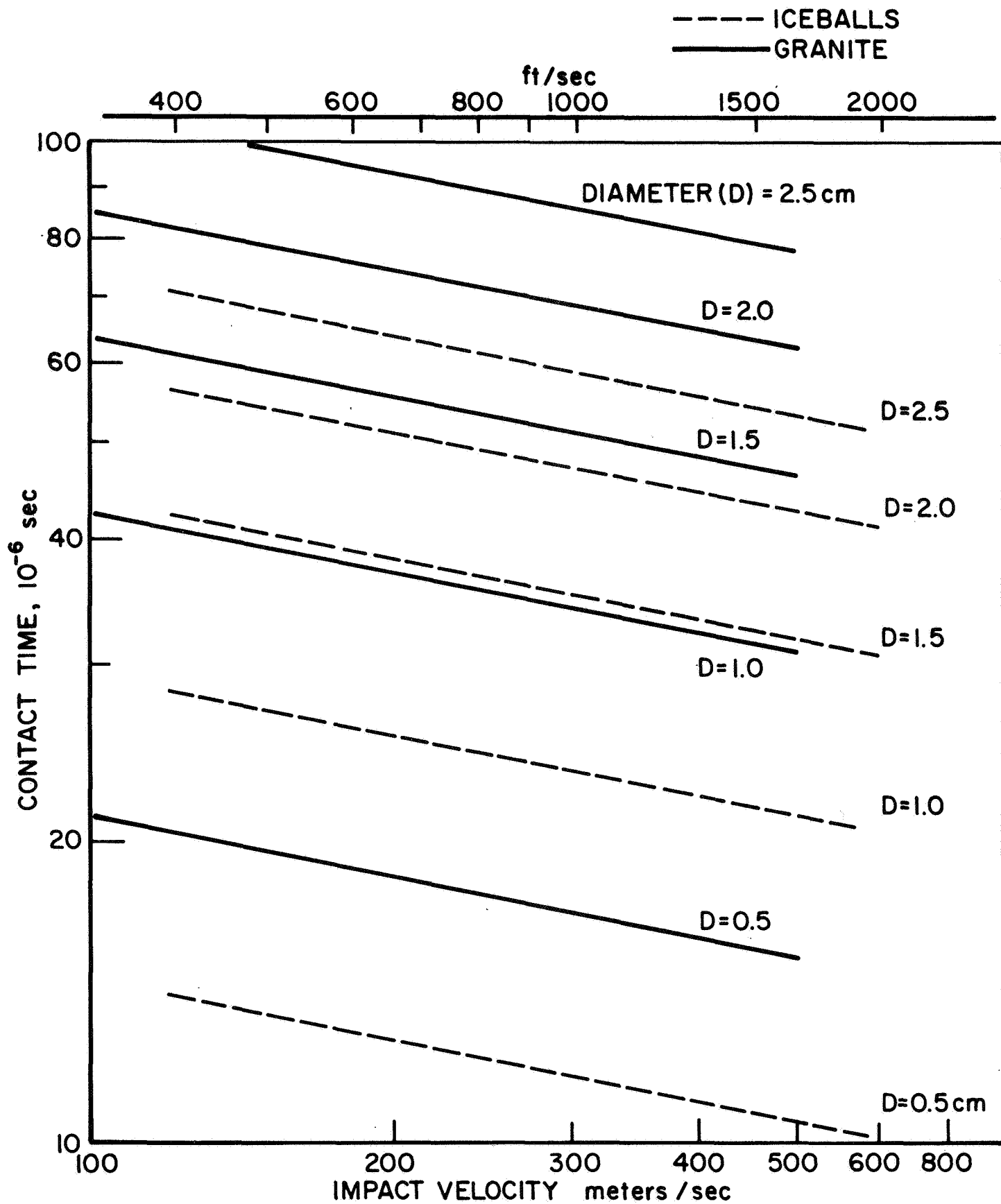
Fig. 6

$$U \sim e^{-\gamma x_3}$$

RAYLEIGH WAVE

IN PLANE PLATE
EDGE WAVE
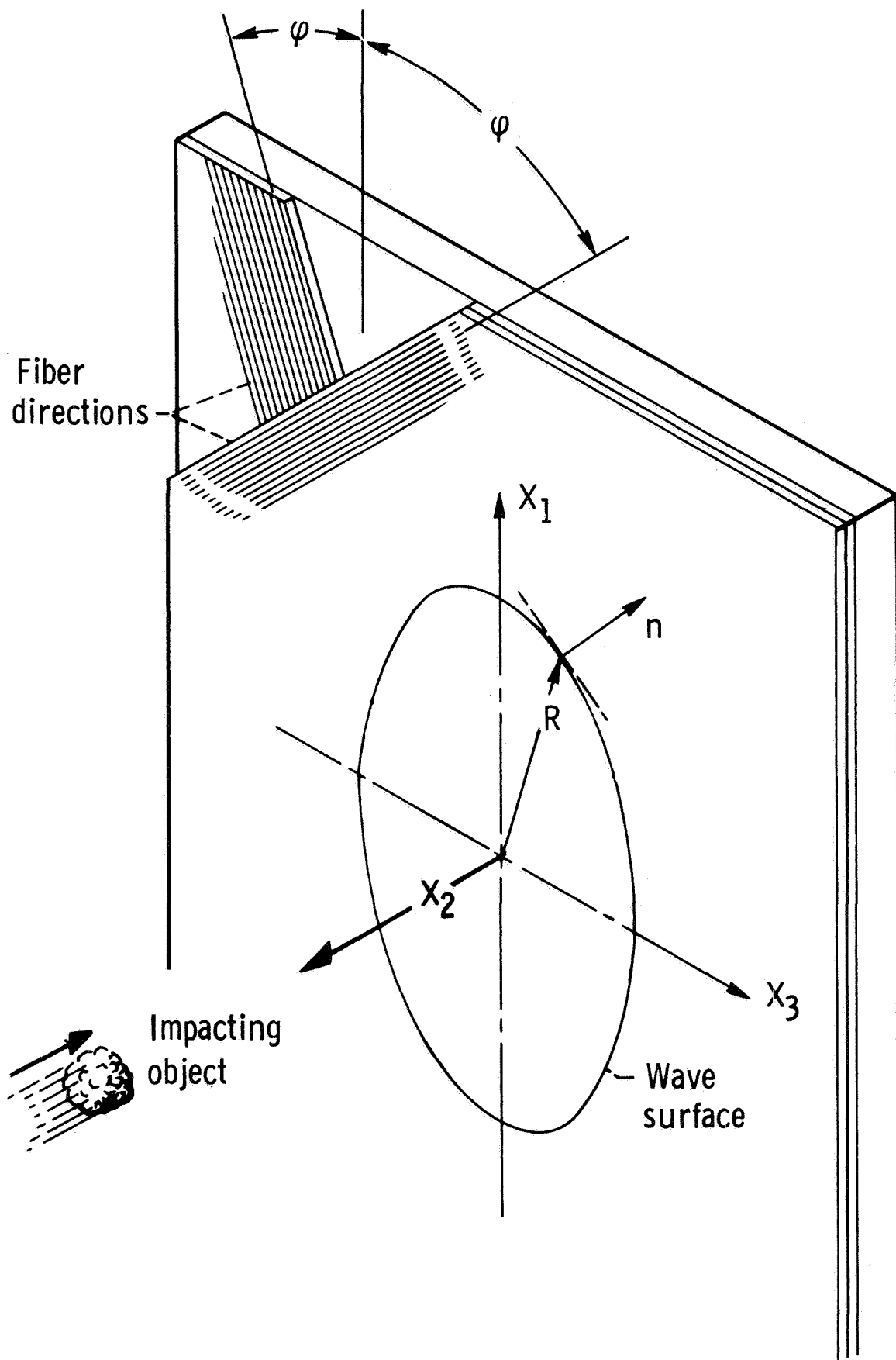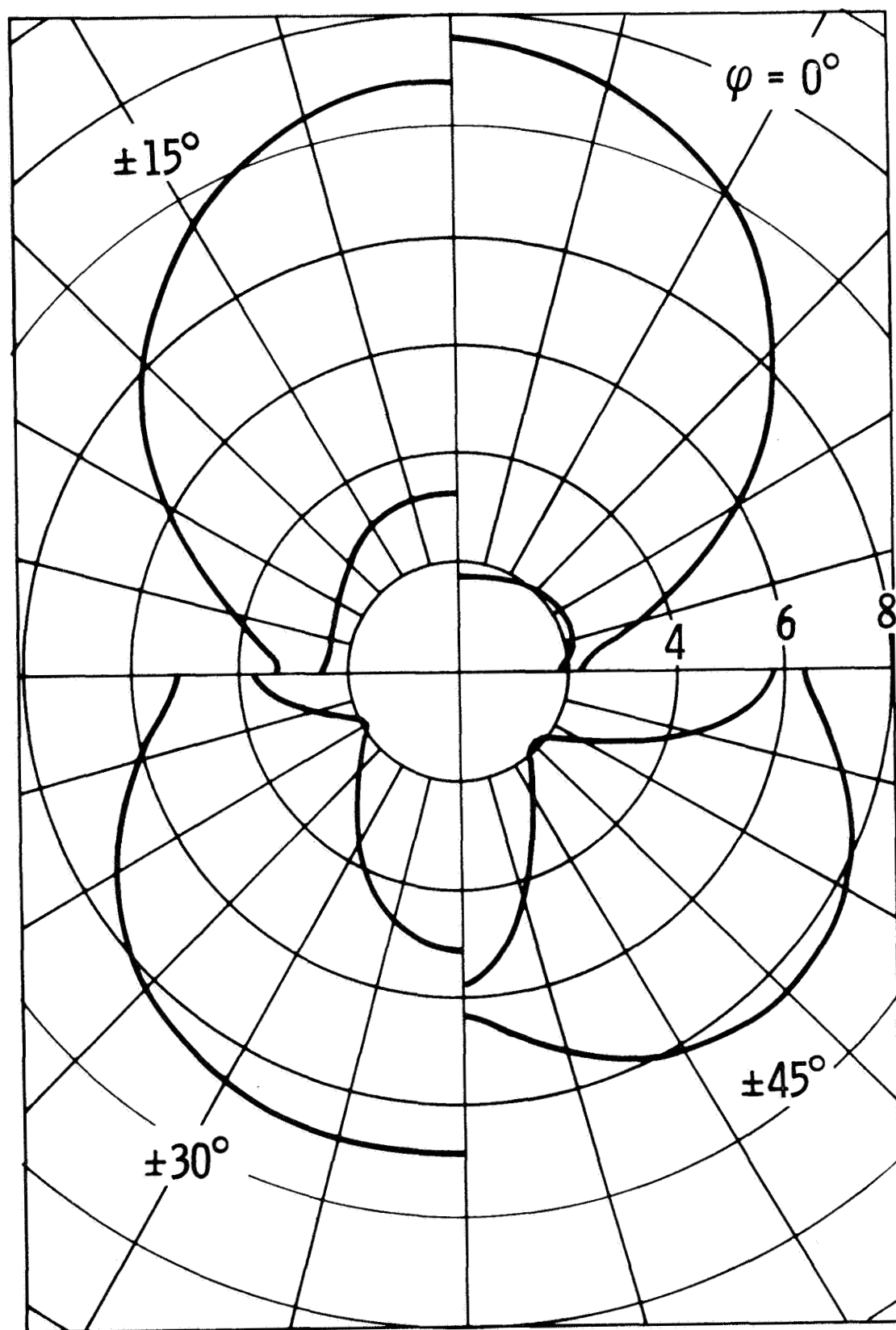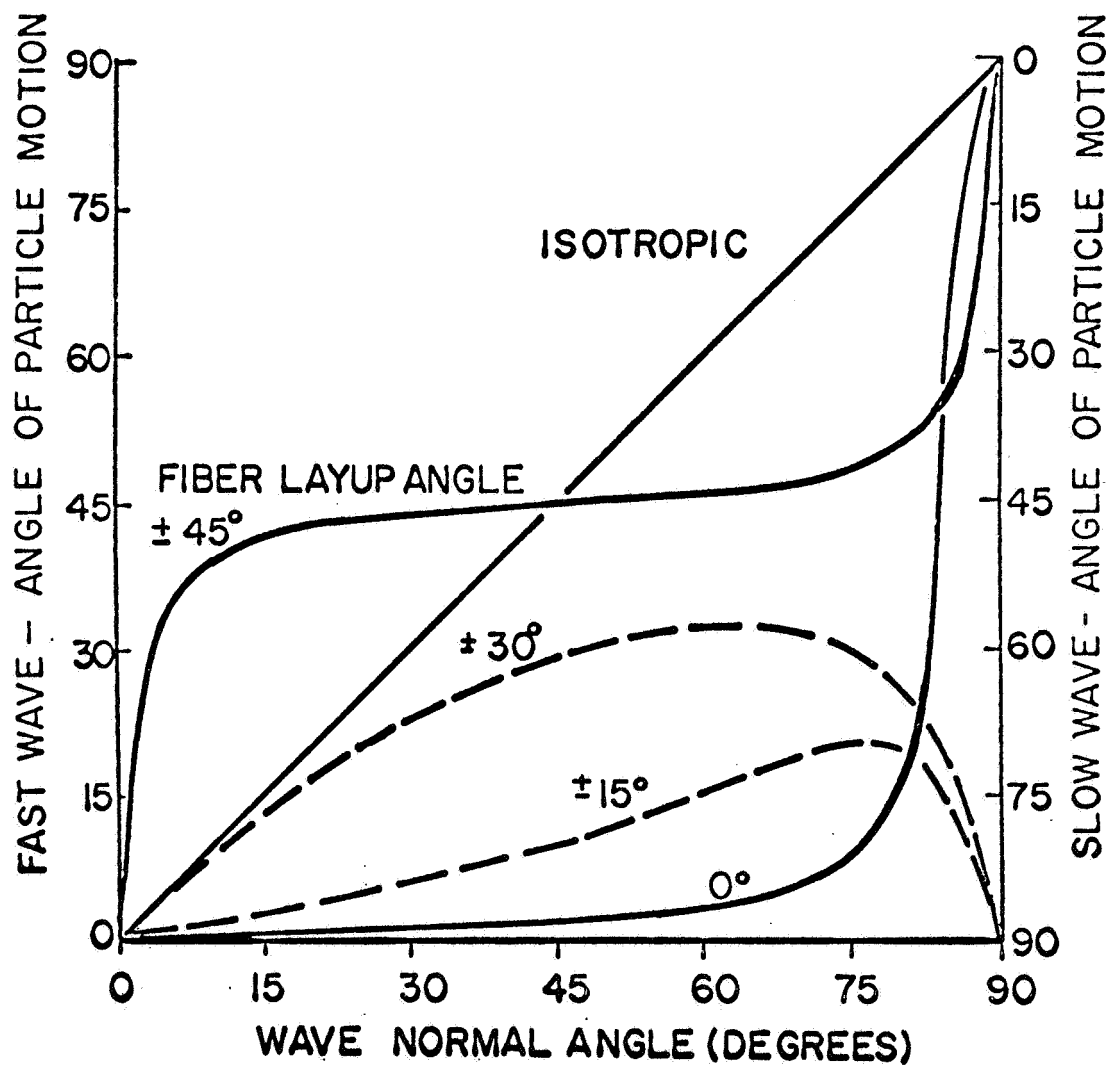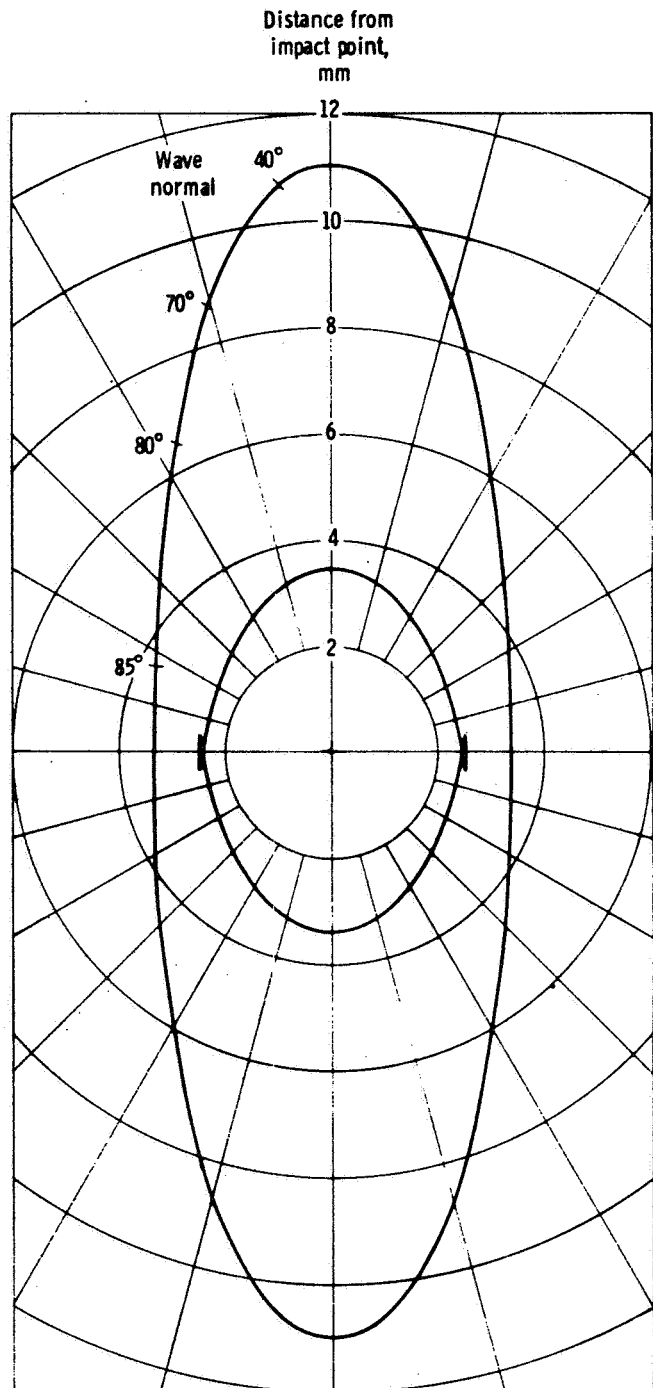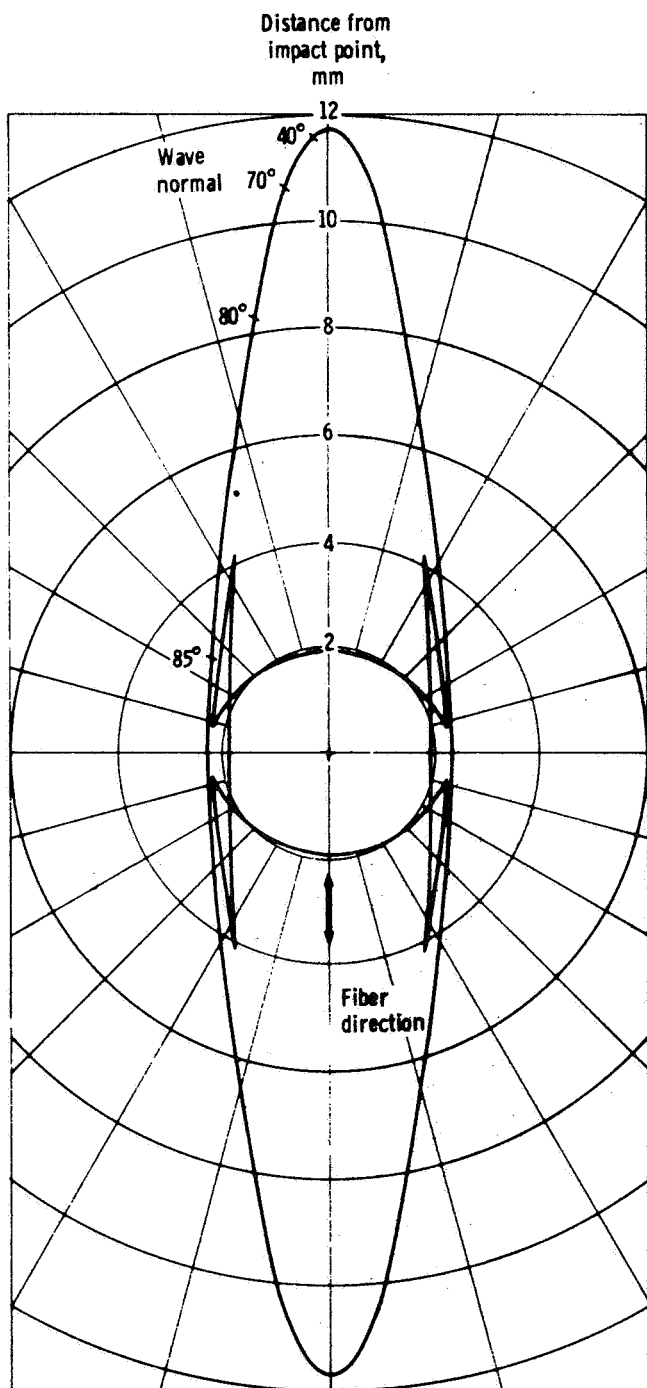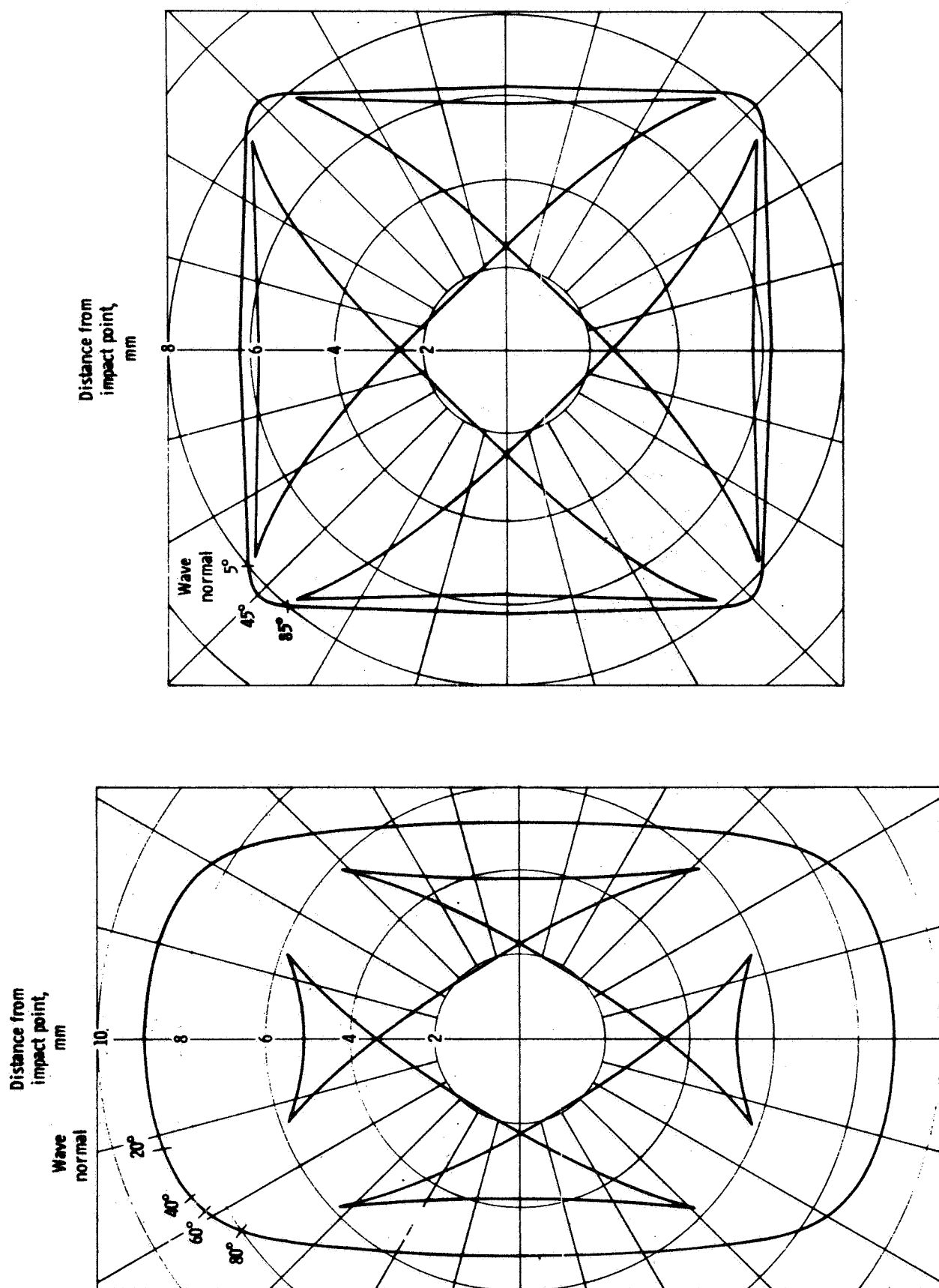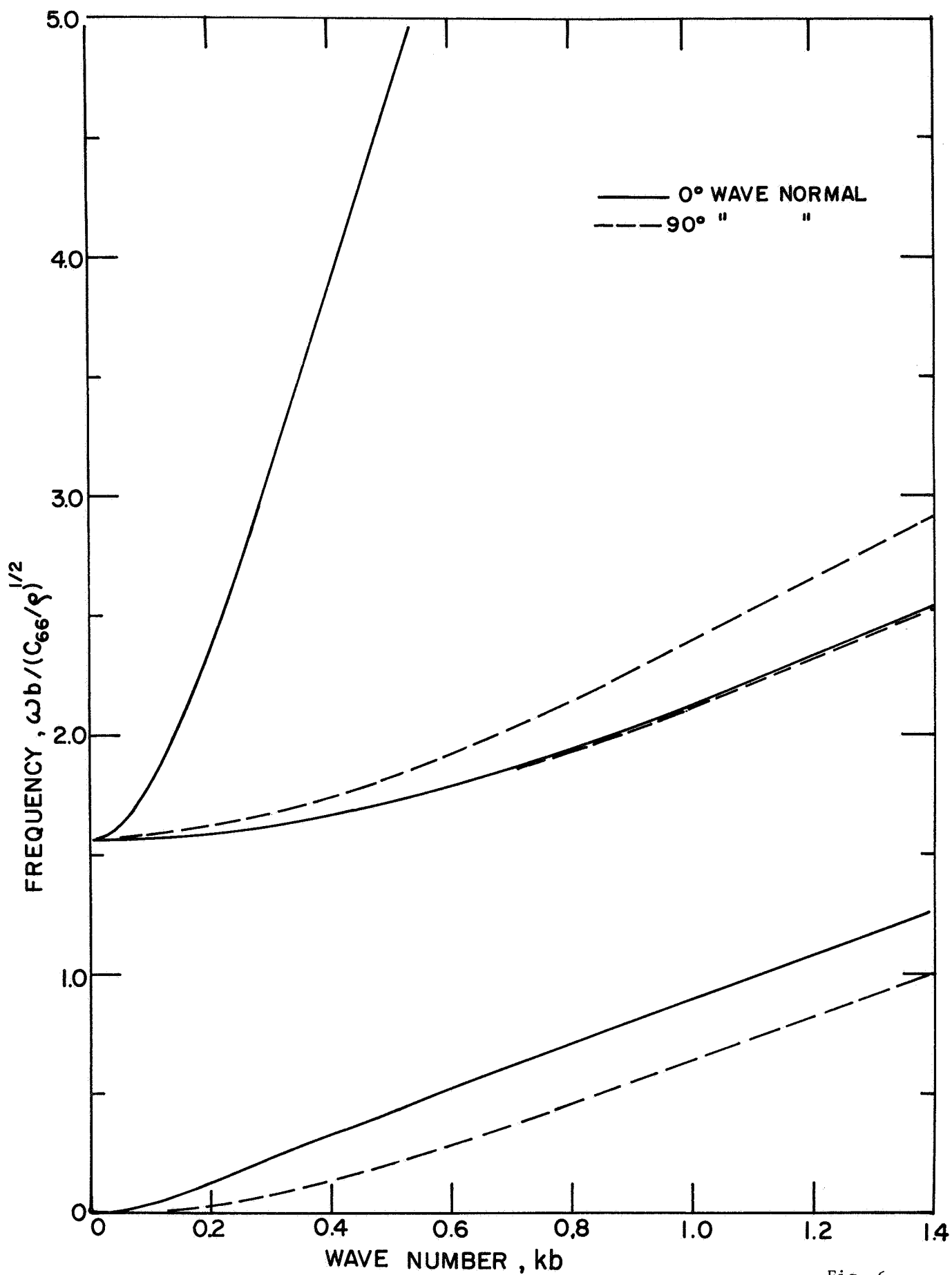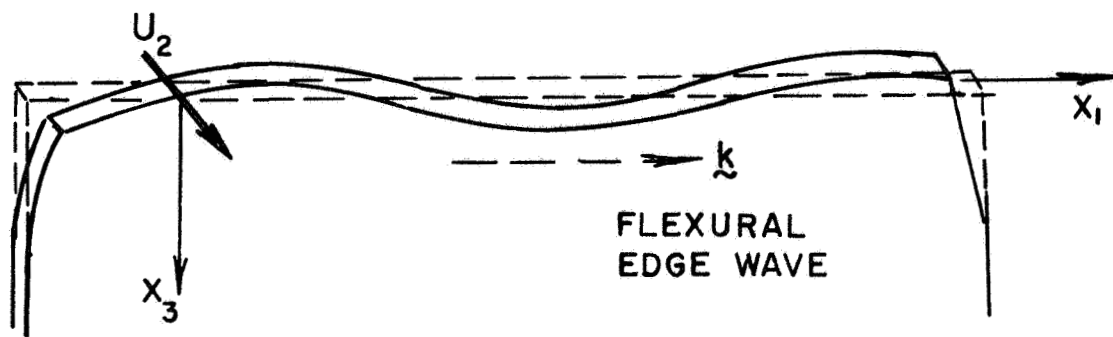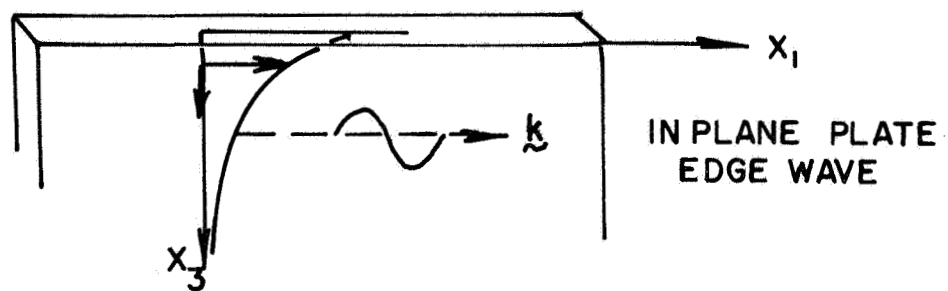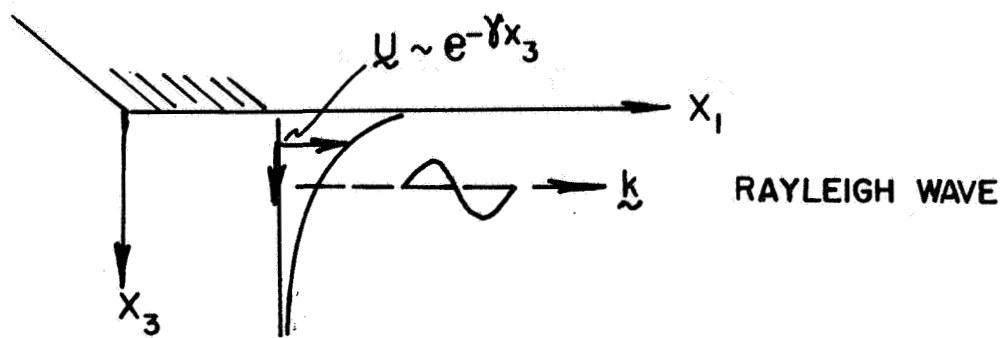
FLEXURAL
EDGE WAVE

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

NORMALIZED WAVE NUMBER

$\gamma = \mu_f / \mu_m$

TH. OF ELASTICITY

EFF. STIFFNESS

NORMALIZED PHASE VELOCITY

Fig. 12

Fig. 13

Fig. 14

1.70 mm/μsec
IMPACT VELOCITY

DIFFUSION BONDED

INITIAL
CONDITION

0.77 mm/μsec
IMPACT VELOCITY

BRAZED

INITIAL
CONDITION

Fig. 15

Fig. 16

| SYMBOL | WAVE DIRECTION | PARTICLE DIRECTION |
|--------|----------------|---------------------|
| △ | $X_3$ | $X_1$ |
| ▲ | $X_1$ | $X_3$ |
| ○ | $X_3$ | $X_2$ |
| ● | $X_2$ | $X_3$ |
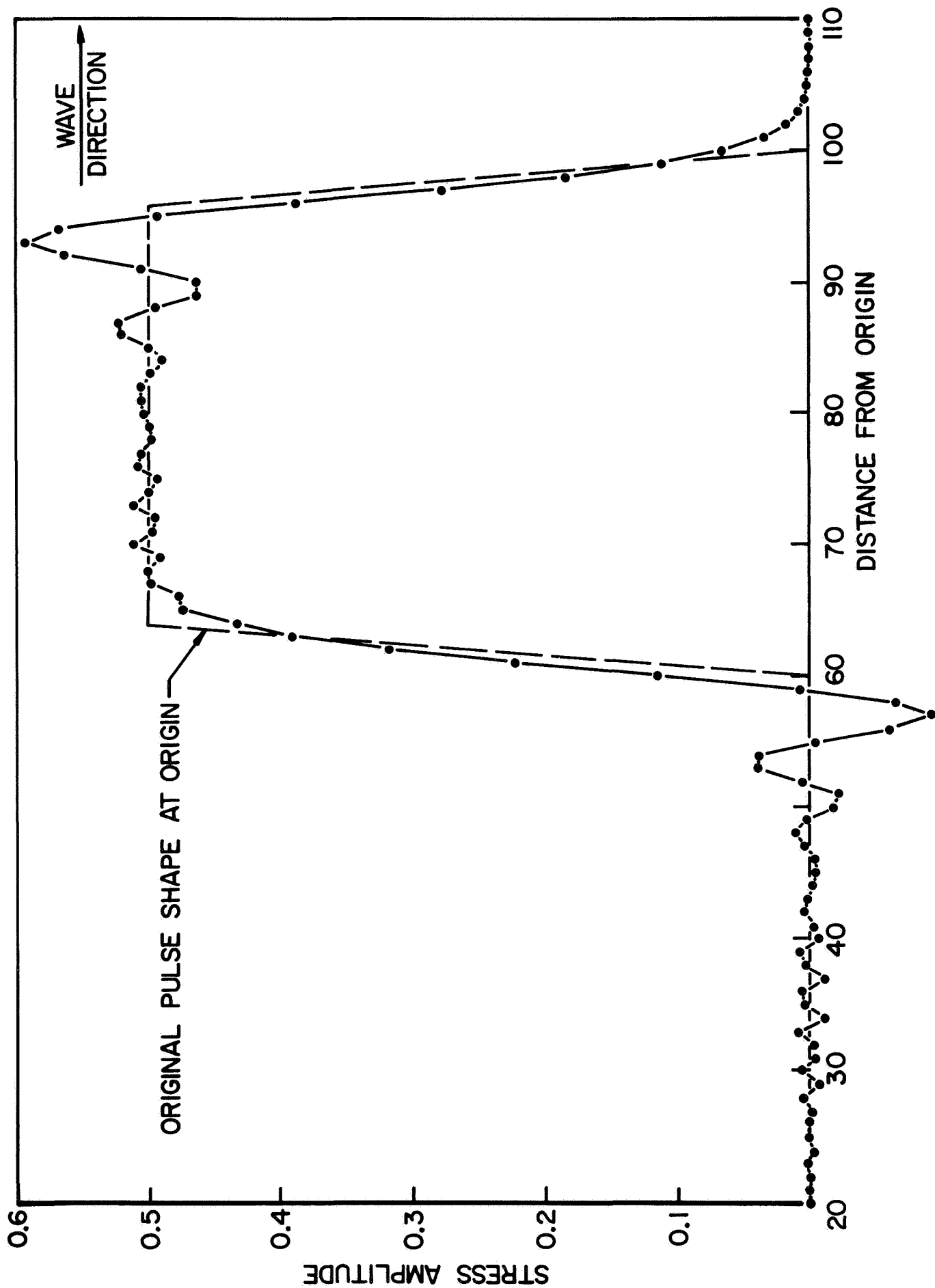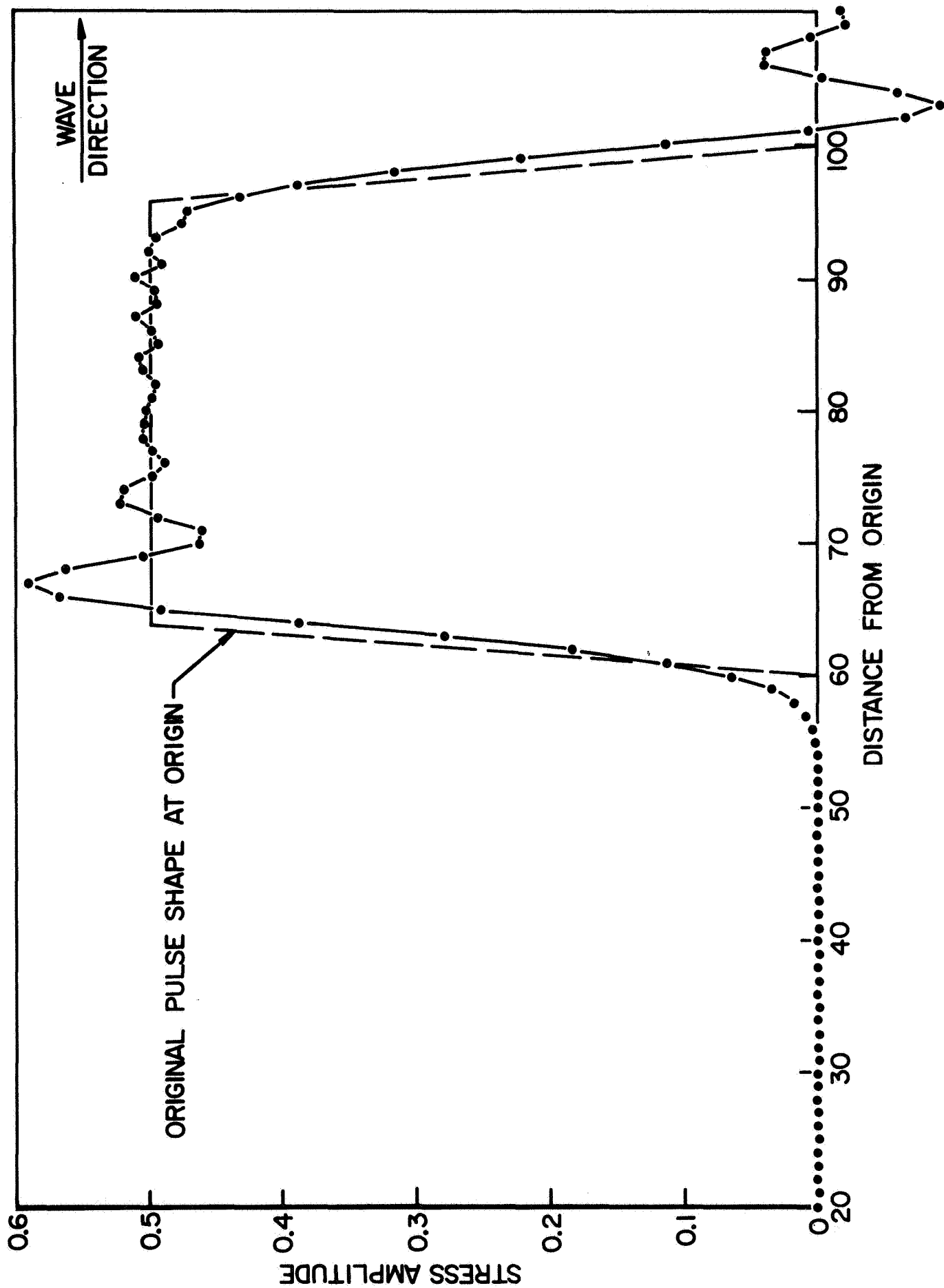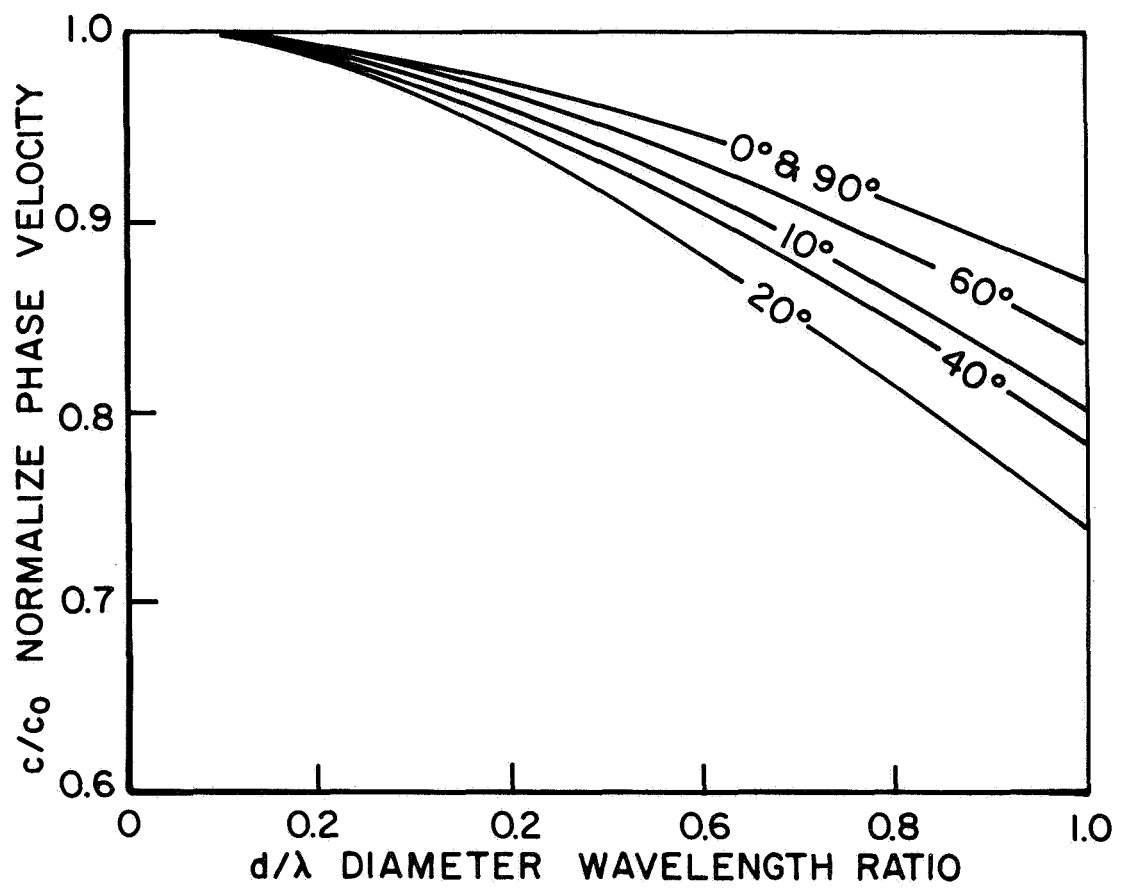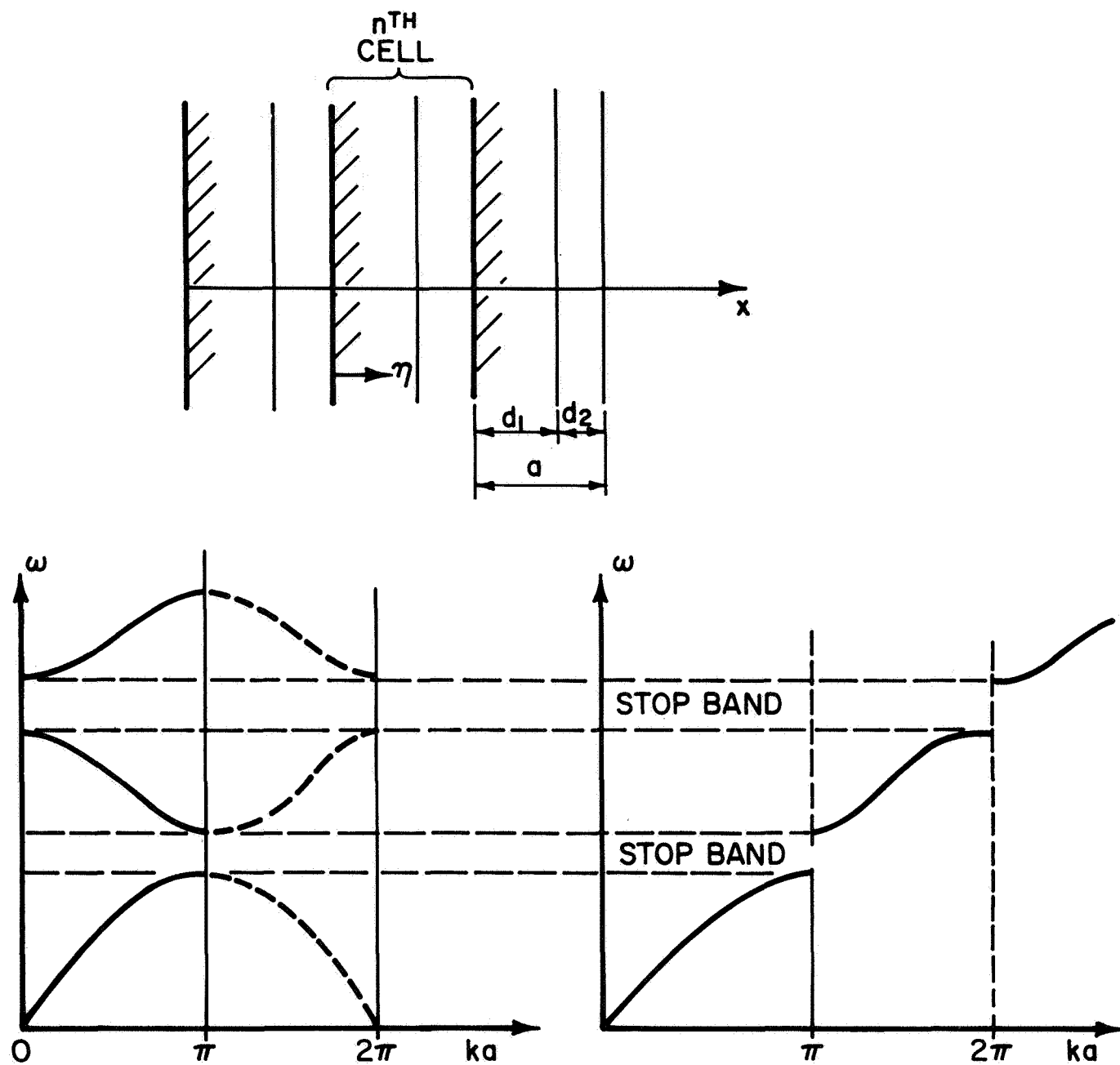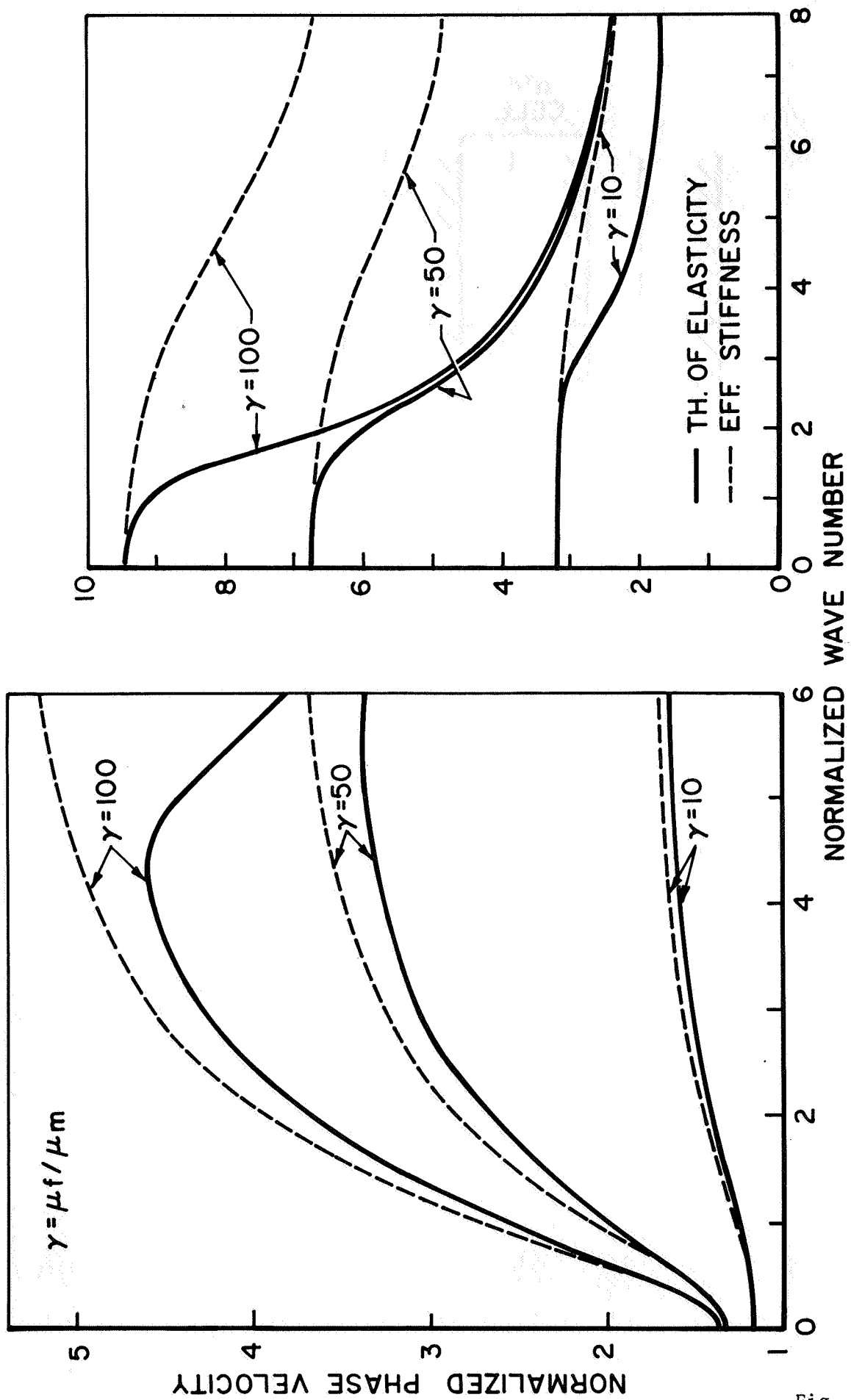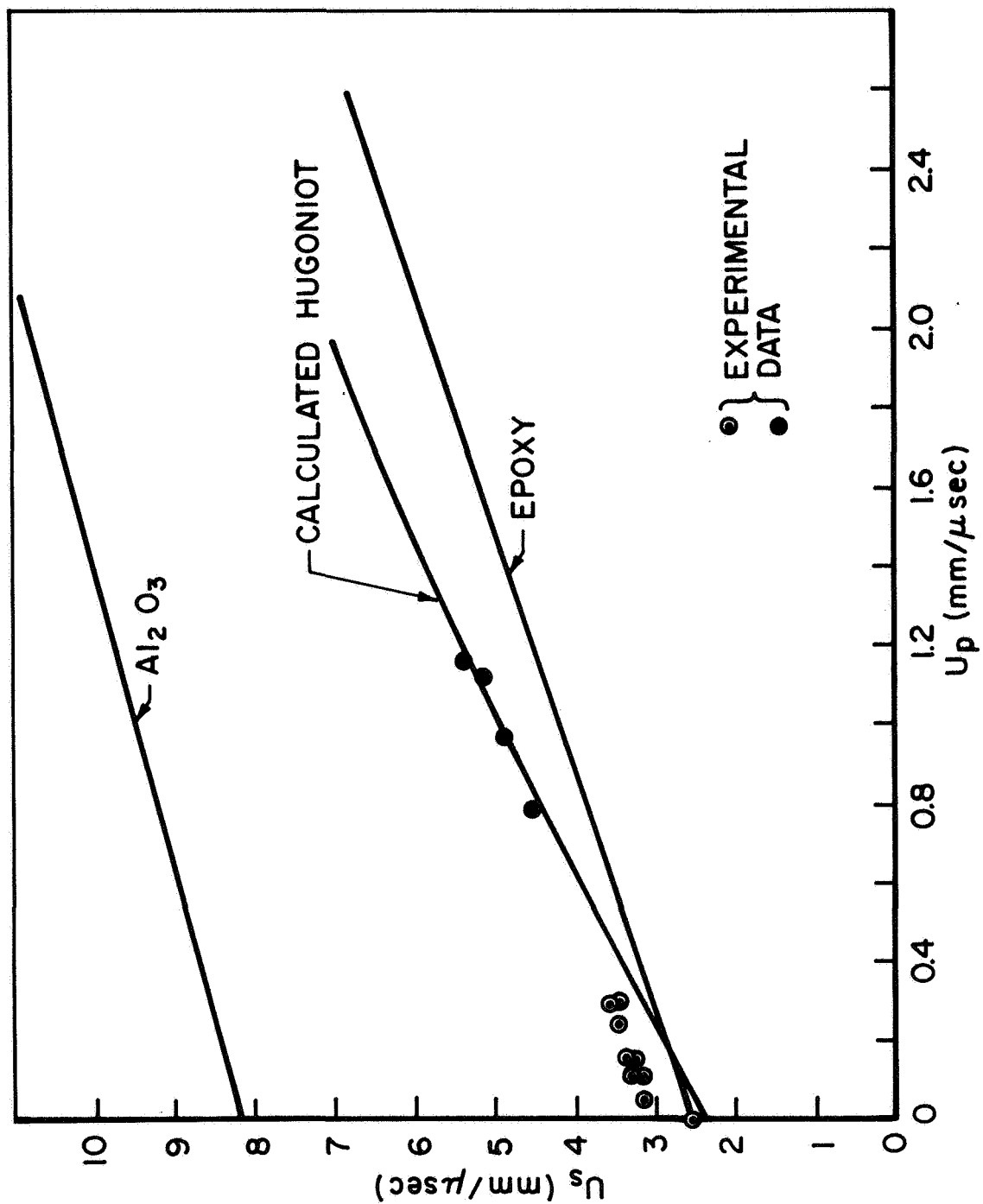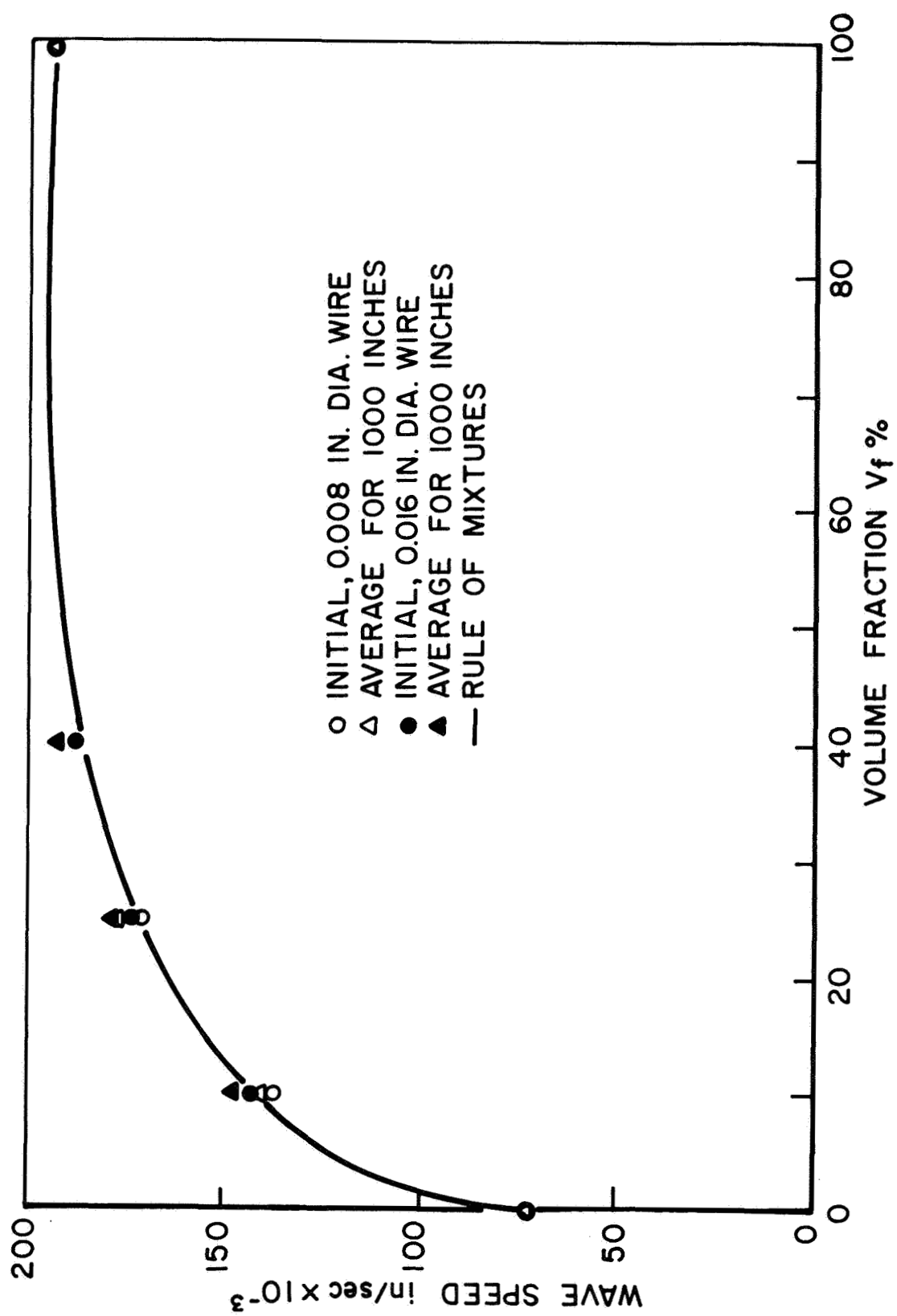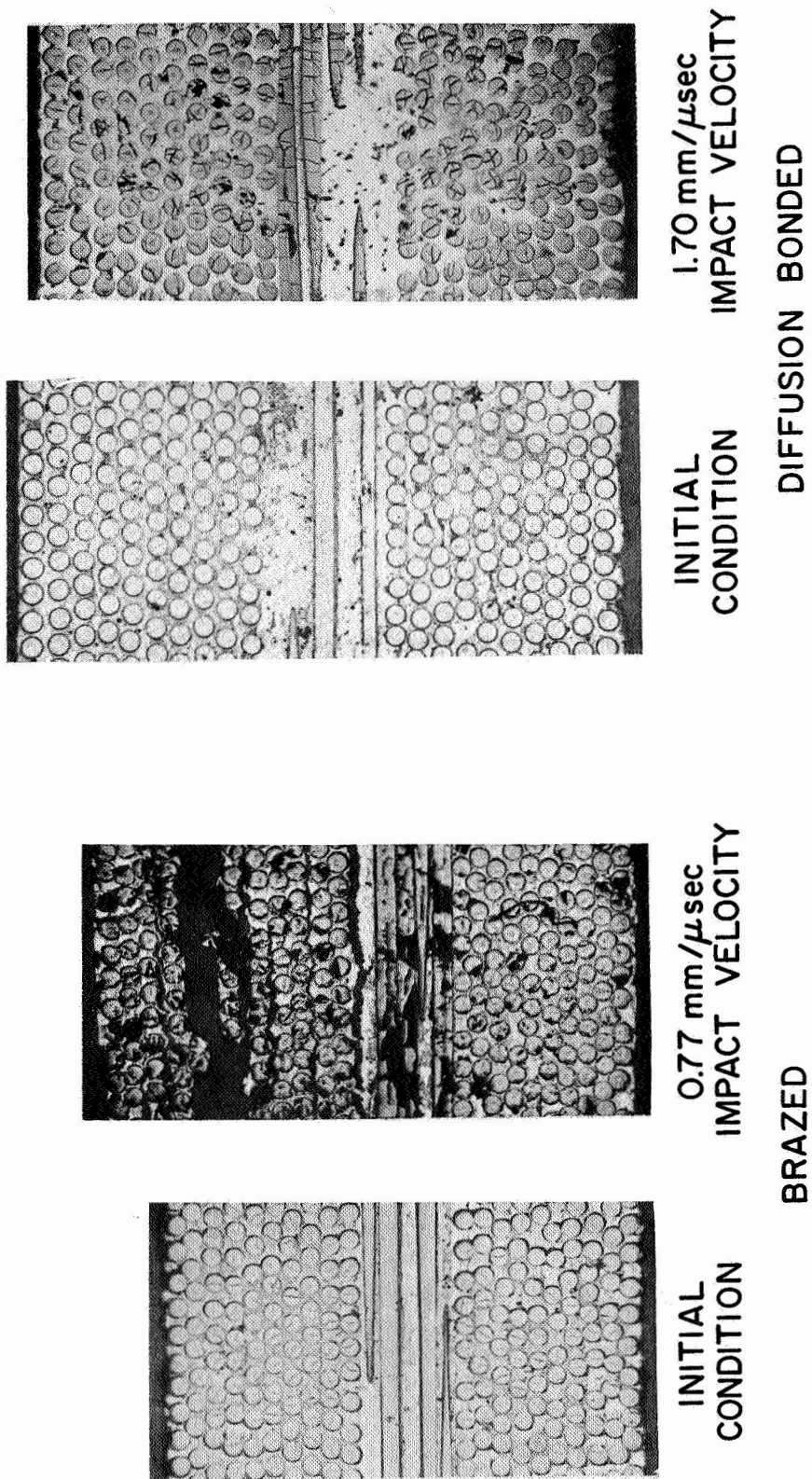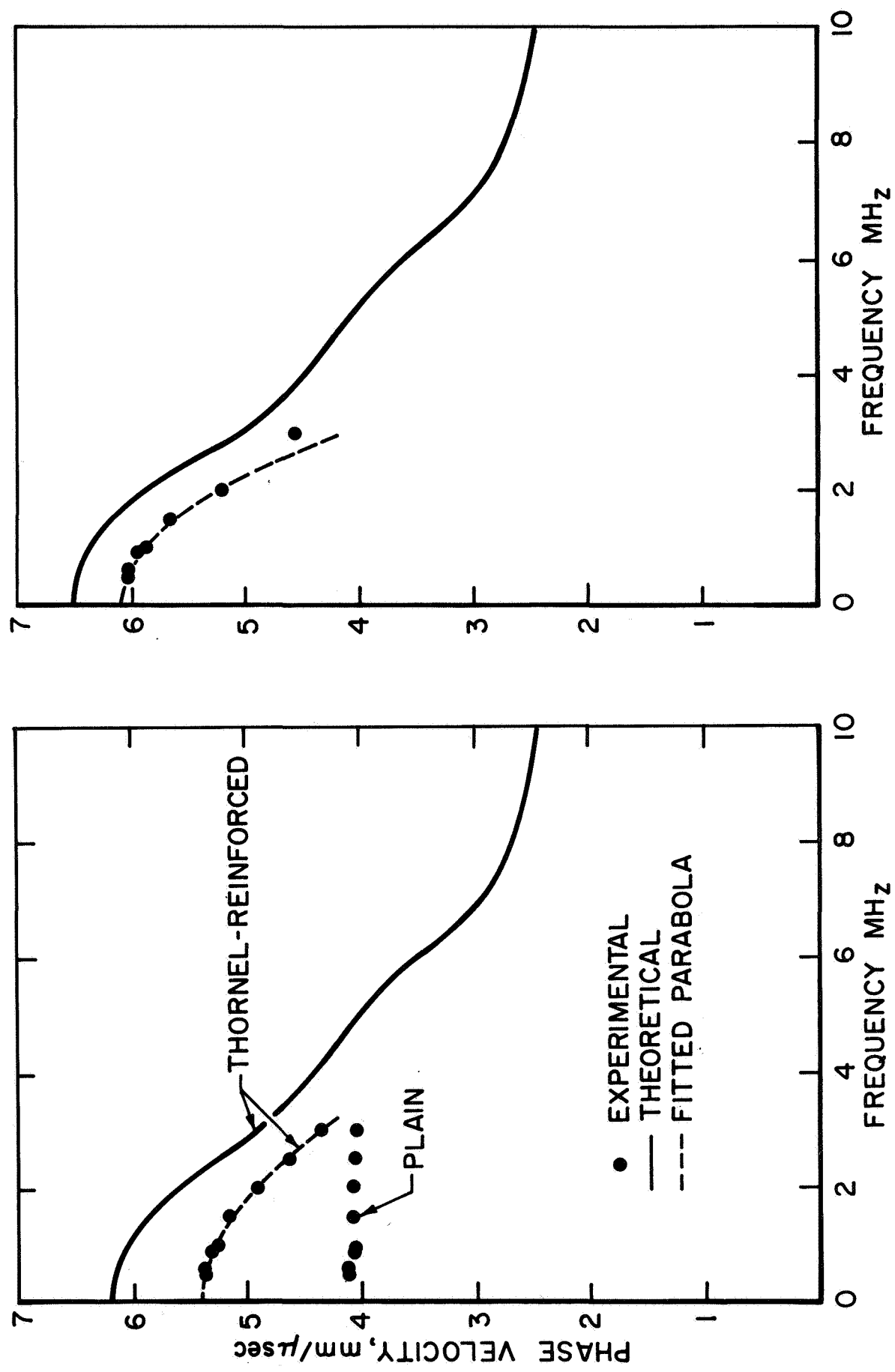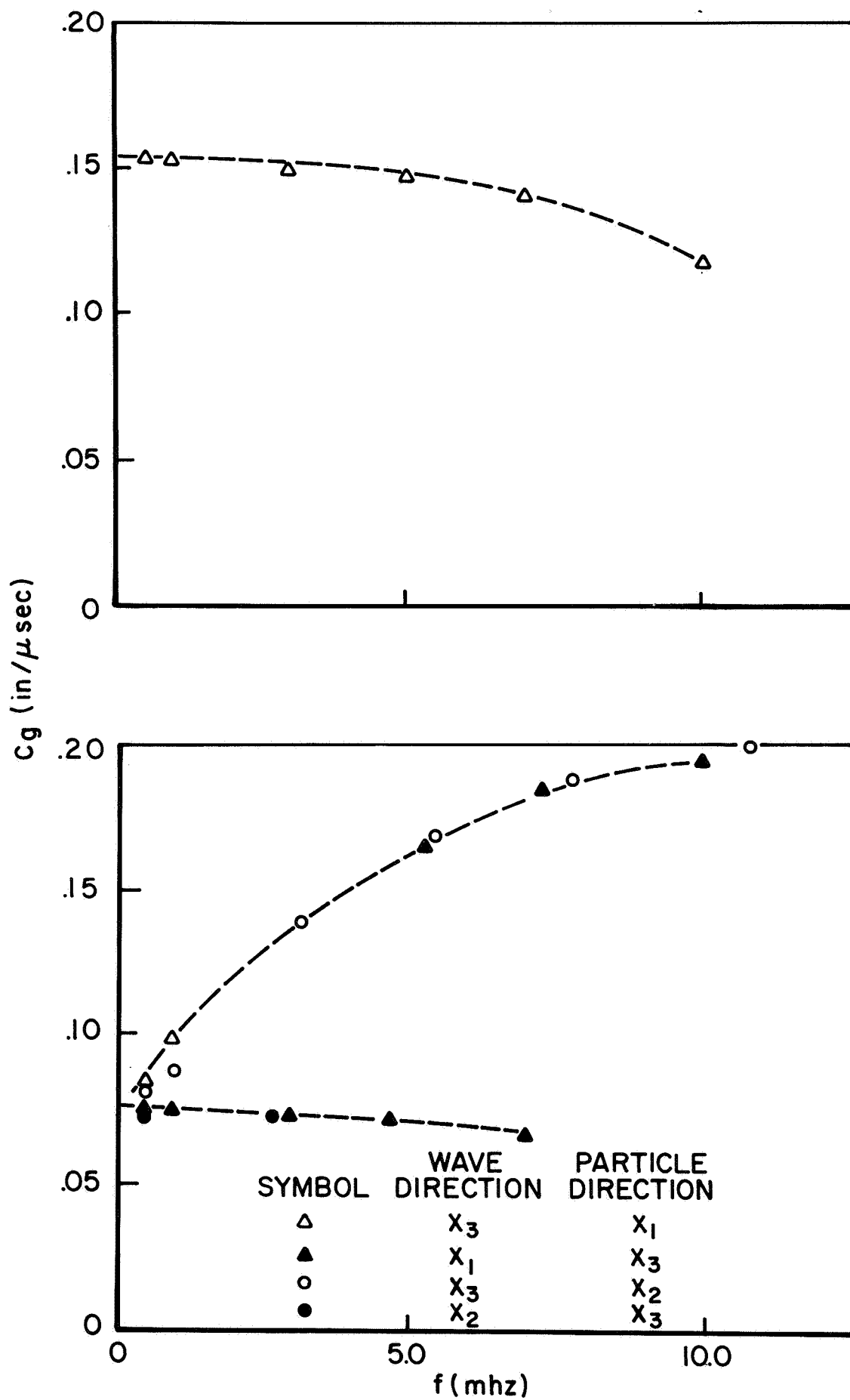
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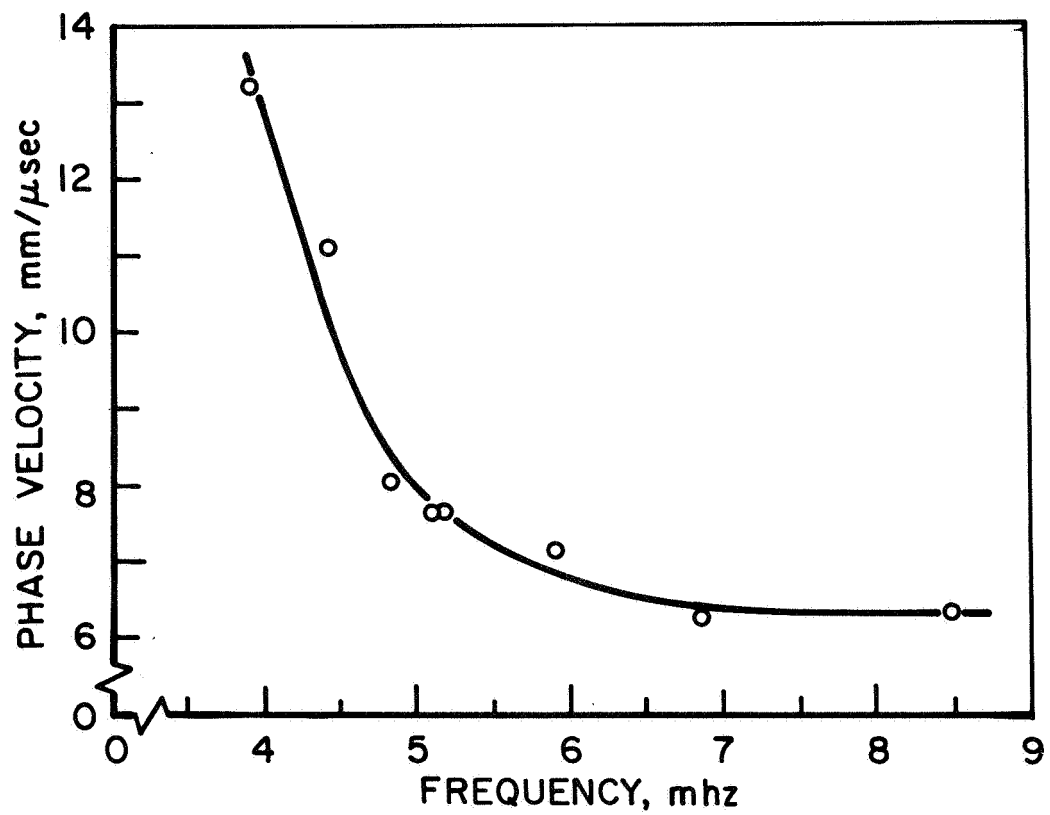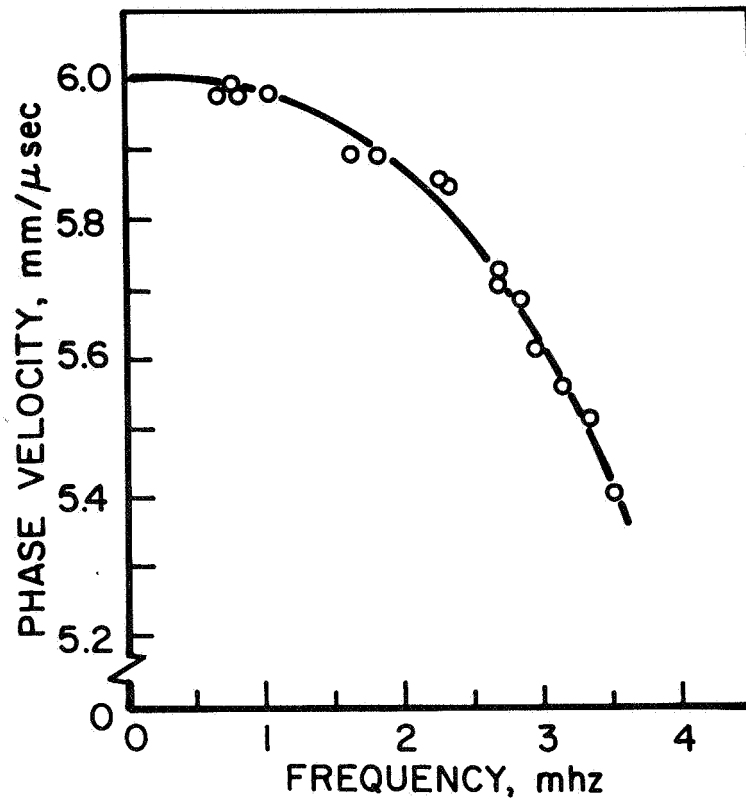
Fig. 18
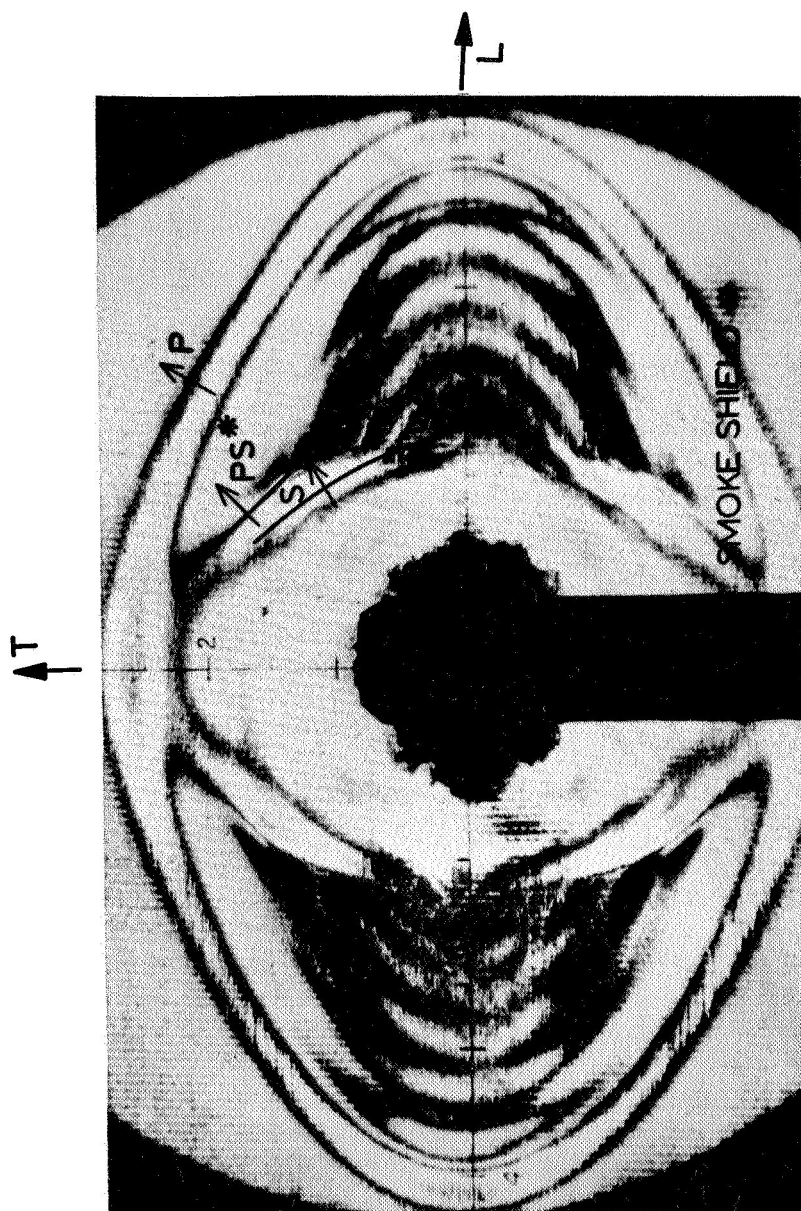
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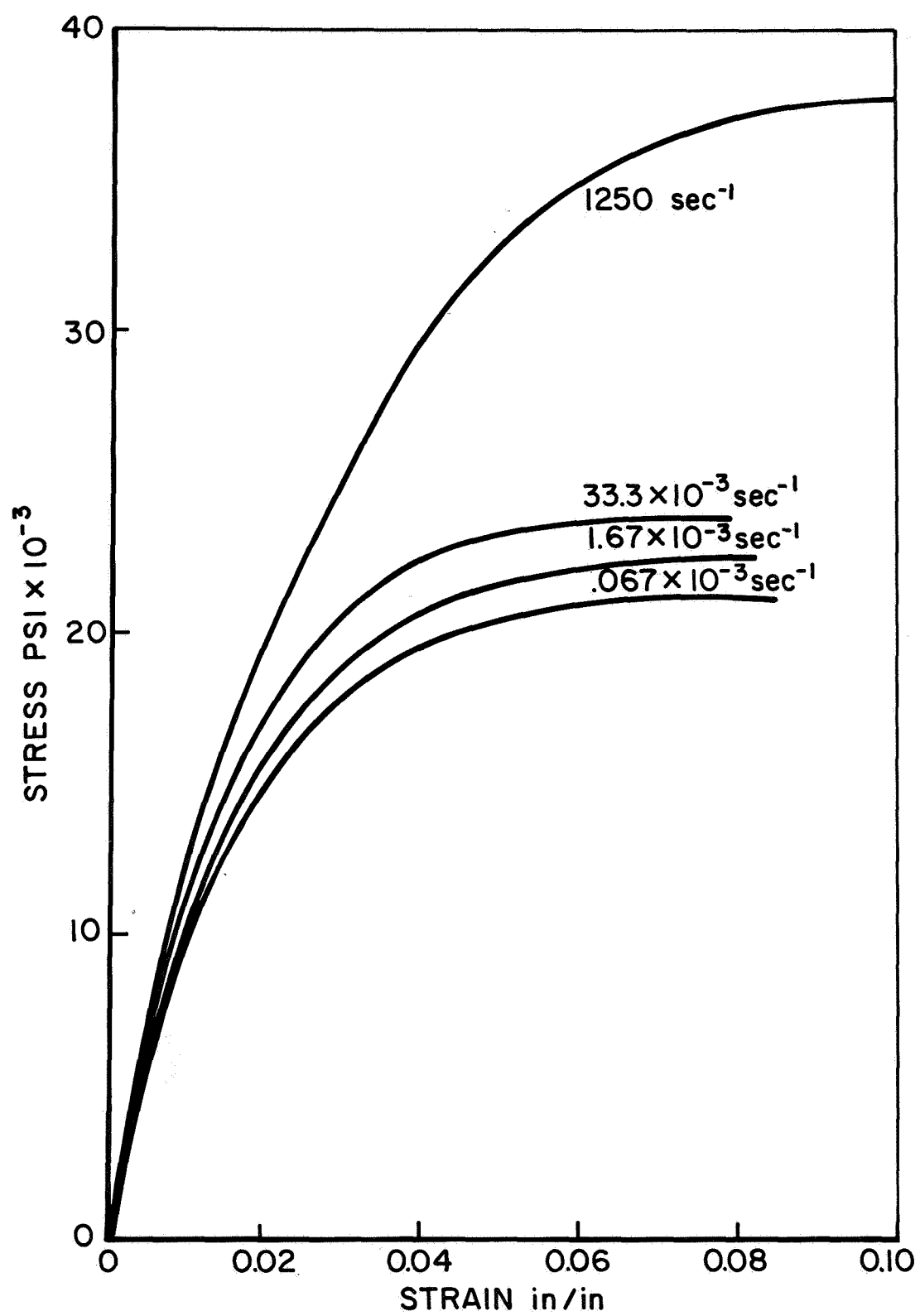
Fig. 20

GRAPHITE / EPOXY



Fig. 21

Fig. 22

Fig. 23

STAINLESS STEEL BALL BETWEEN 2 SAMPLES OF GRAPHITE FIBER/EPOXY RESIN EPOXY (0.15" AND 0.13" THICK)

● INCREASING FORCE
○ DECREASING FORCE

FORCE (lb$_f$)

$\alpha$, DEFLECTION ($10^{-3}$ INCHES)

$n = 1.21$

Fig. 24

Fig. 25

Fig. 26

45° LAYUP

THEORETICAL WAVE FRONT

0° LAYUP

THEORETICAL WAVE FRONT

IMPACT CIRCLE

Fig. 27
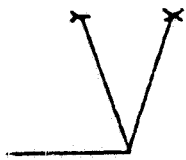
45° LAYUP

THEORETICAL WAVE FRONT

15° LAYUP

IMPACT CIRCLE

r/a
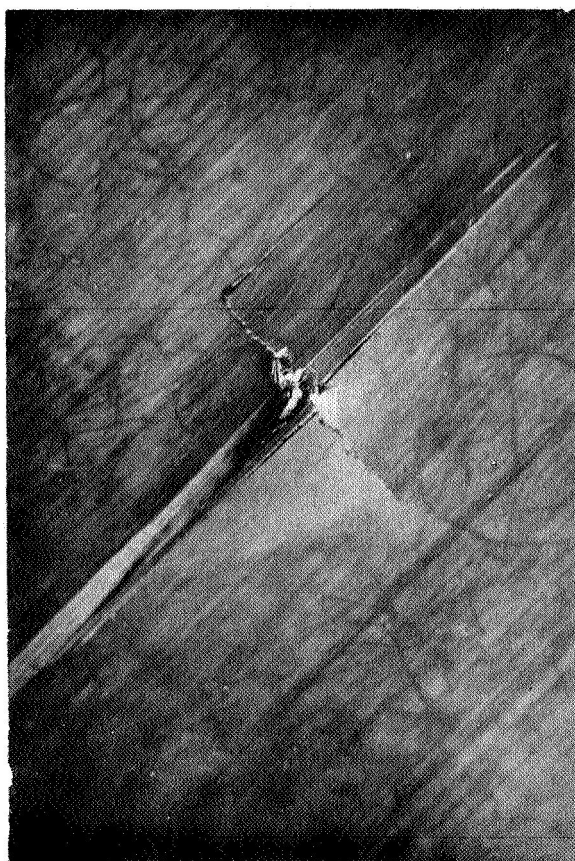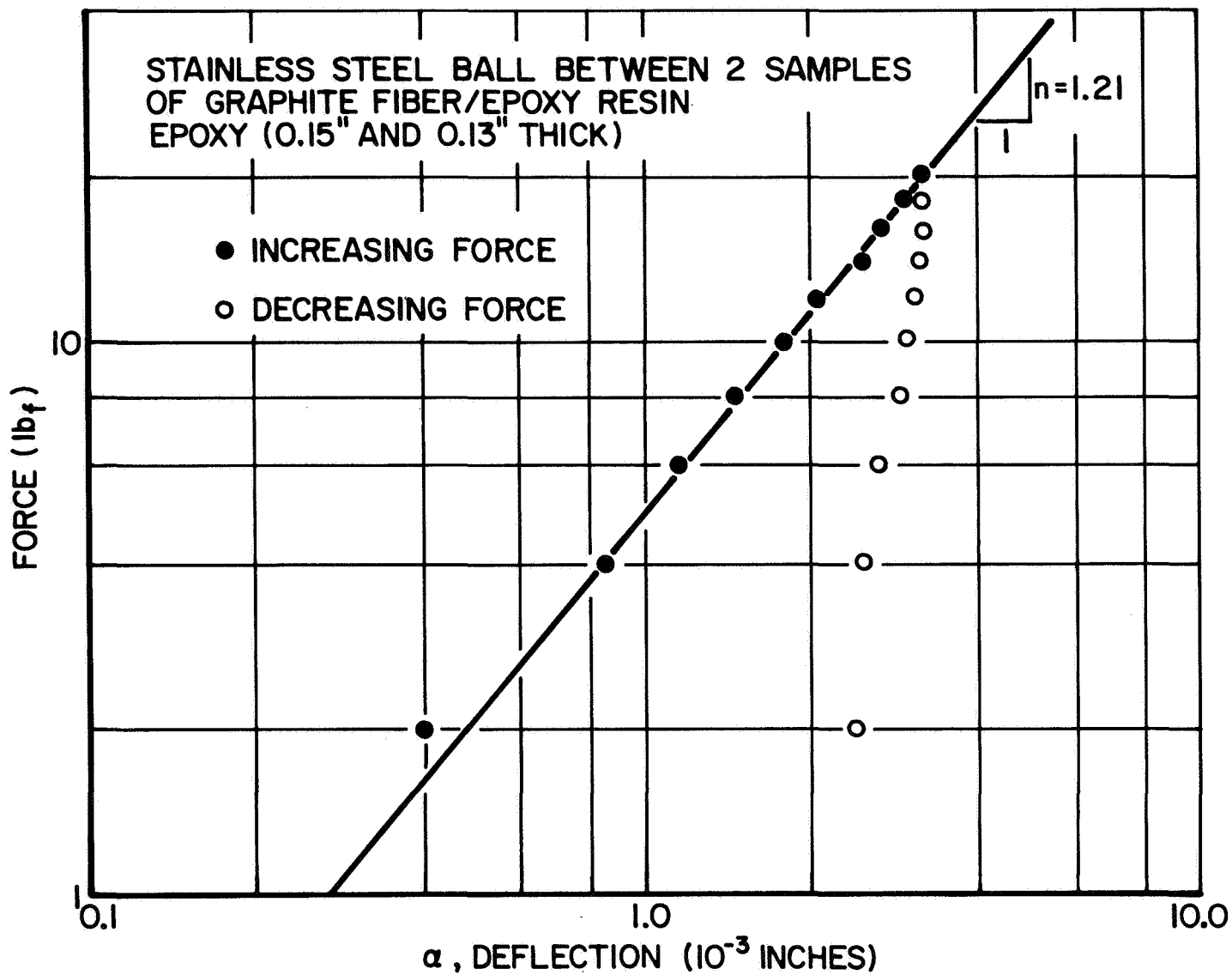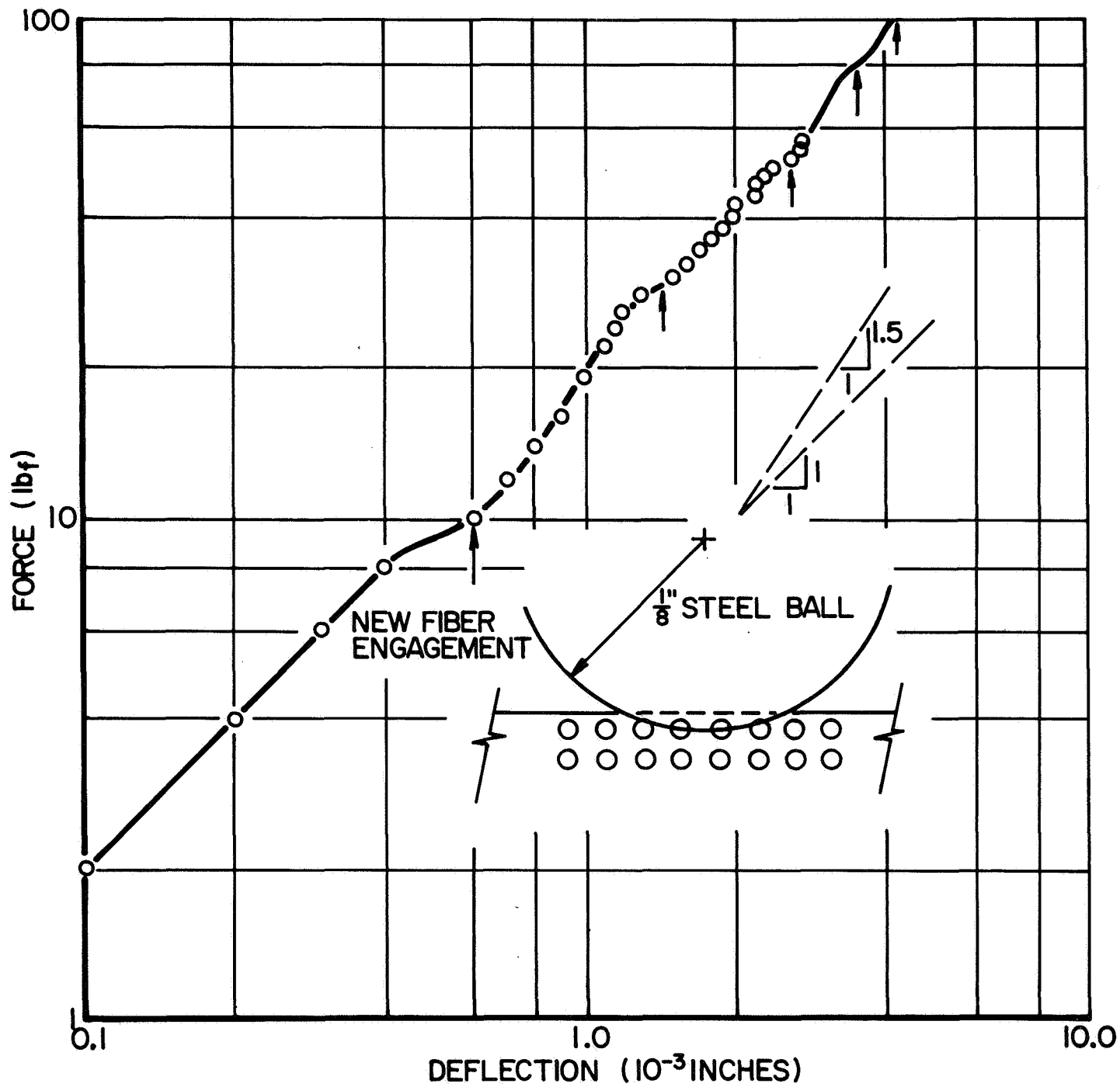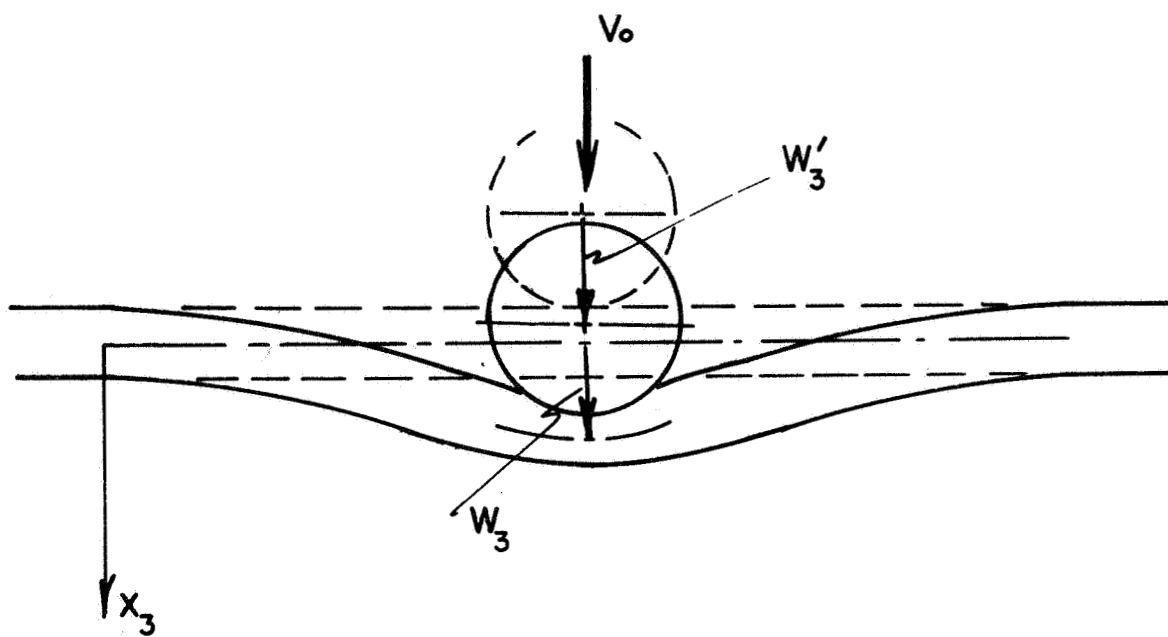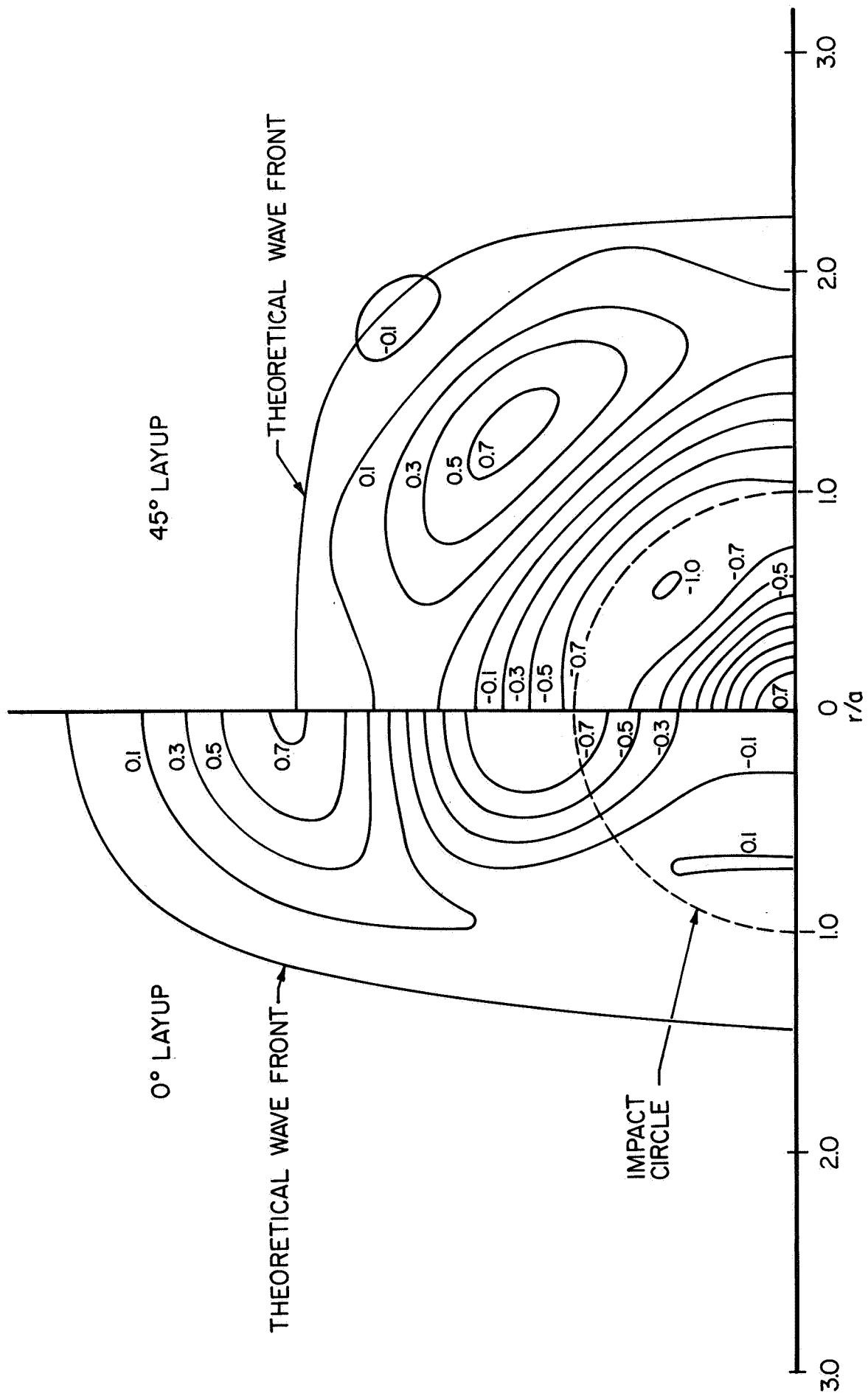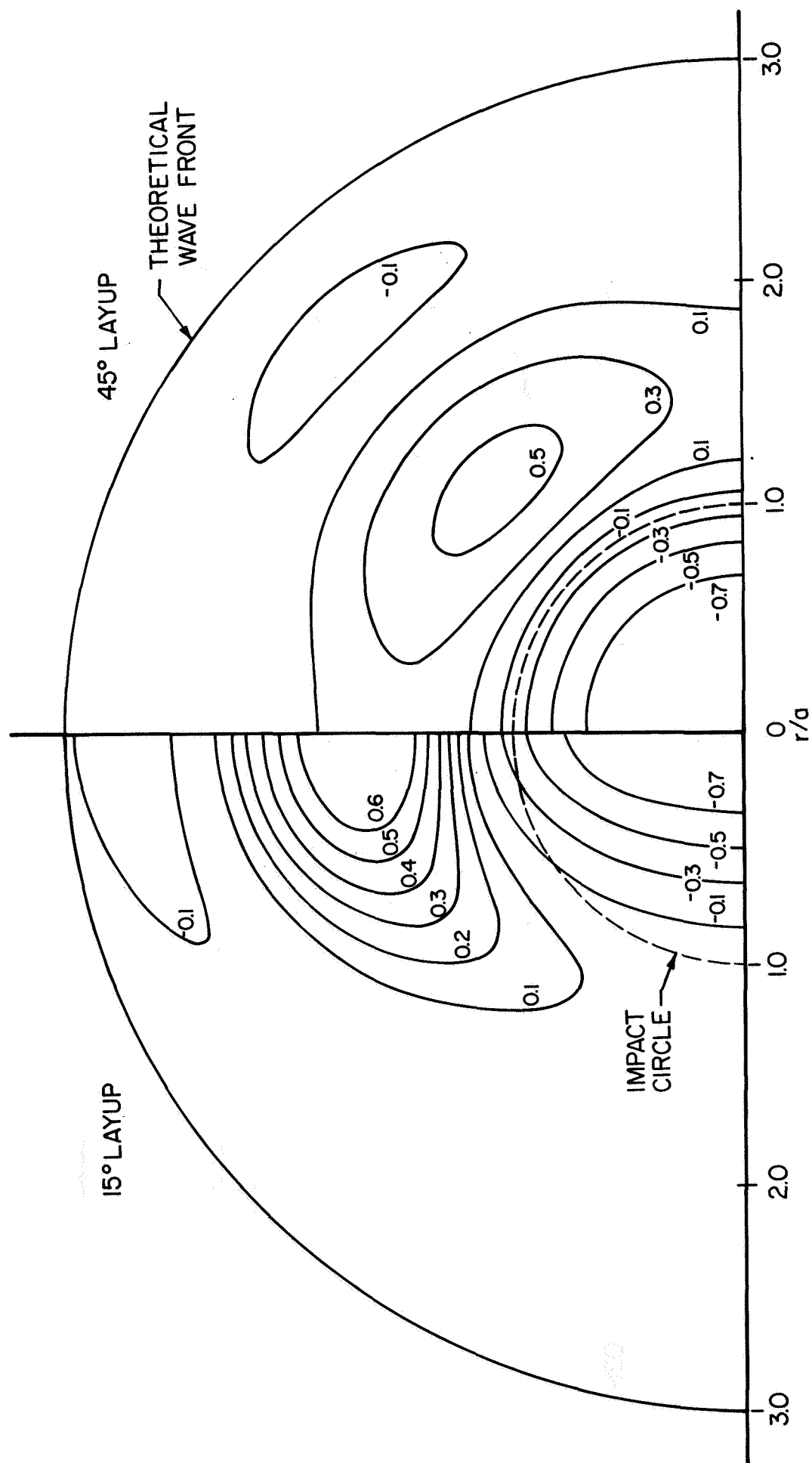
Fig. 28

FLEXURAL STRESS $\frac{1}{2}(t_{11}+t_{33})$
GRAPHITE FIBER/EPOXY MATRIX

LAYUP ANGLE : ± 45°
IMPACT RADIUS - HALF THICKNESS RATIO : 10
NORM. IMPACT TIME $\tau$ =1.0 , NORM. TIME : 20·$\tau$

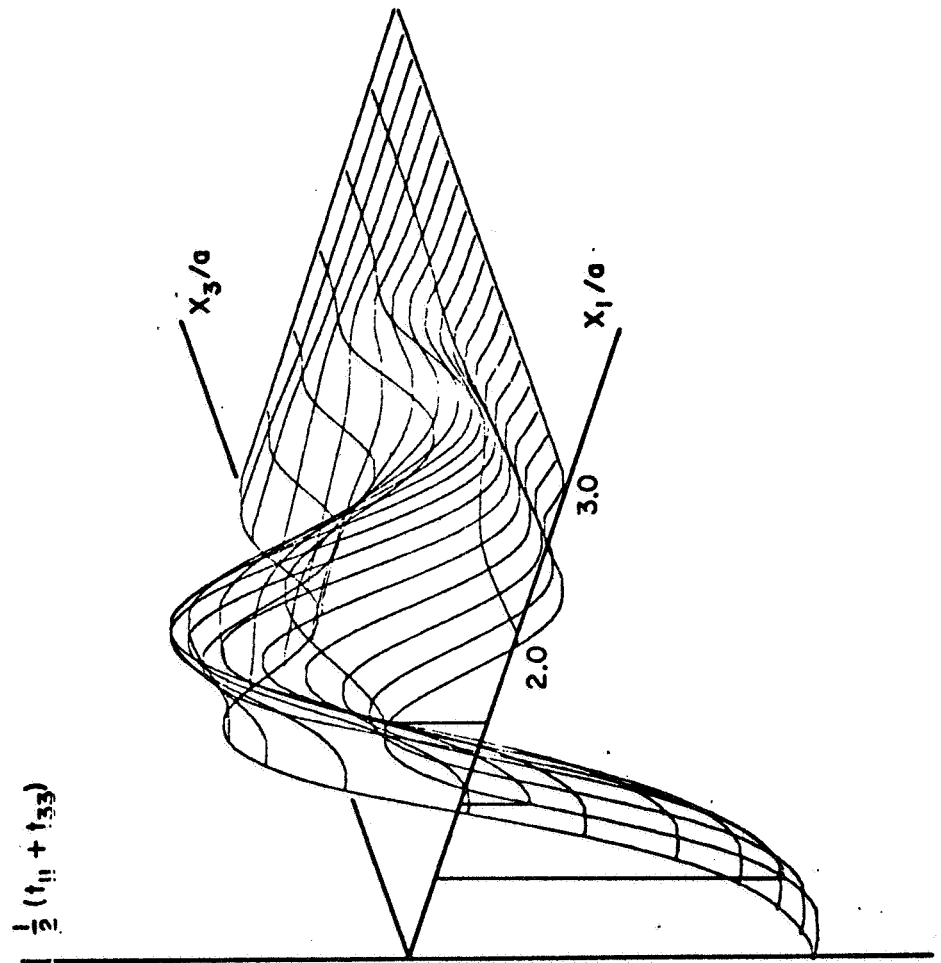$\frac{1}{2}(t_{11}+t_{33})$

$x_3/a$

$x_1/a$

3.0

2.0

Fig. 29