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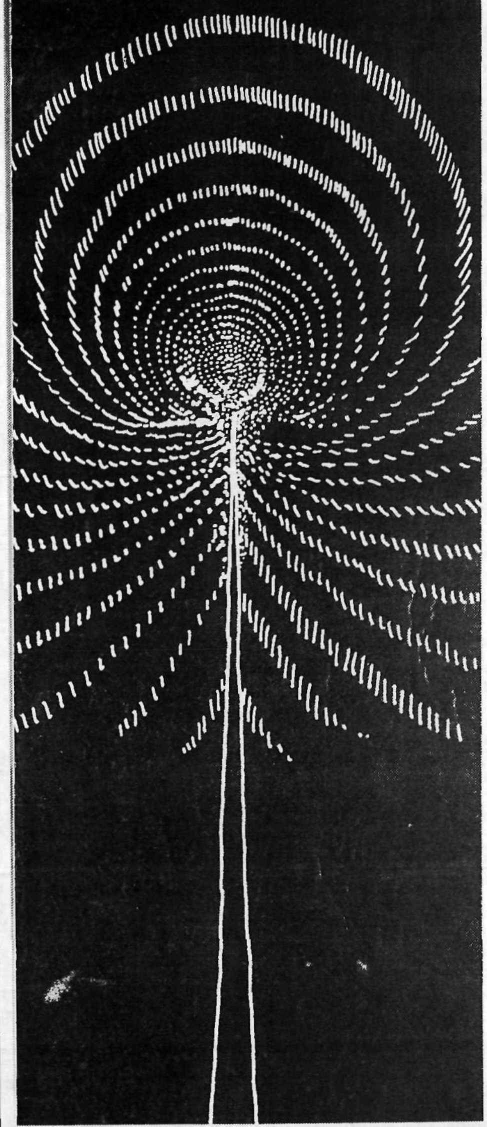

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A HALF PLANE AND A STRIP WITH AN ARBITRARILY LOCATED CRACK*

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Abstract

The paper introduces a technique to deal with the problem of an elastic domain containing an arbitrarily oriented internal crack. The problem is formulated as a system of integral equations for a fictitious layer of body forces imbedded in the plane along a closed smooth curve encircling the original domain. The problems of a half plane with a crack in the neighborhood of its free boundary and of an infinite strip containing a symmetrically located internal crack with an arbitrary orientation are considered as examples. In each case the stress intensity factors are computed and are given as functions of the crack angle.

1. INTRODUCTION

The plane elastostatic problem for a strip of finite width containing a crack perpendicular to the boundary was considered by Isida [1, 2], and more recently, by Sneddon and Srivastav [3]. In [2] and [3] the crack is symmetrically located, whereas in [1] its location with respect to the boundaries is arbitrary. In [1] and [2] the Kolosov-Muskhelishvili functions, and in [3] the integral transforms are used to solve the problem in which the external load is assumed to be uniform pressure on the crack surface.

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The plane problem of two bonded elastic half planes containing a crack perpendicular to, and terminating at, the interface was recently considered in [4]. The problem of a half plane with a crack perpendicular to the boundary is given in [4] as a special case. In [4] the singular integral equation governing the problem was obtained by using the Mellin Transforms.

In the solutions given in [1-4] symmetry in the geometry of the medium as well as in the external loads with respect to the plane of the crack is essential for the formulation of the problem. Generally, when the crack problem loses its geometric symmetry it becomes necessary to use techniques which are more and more numerically oriented such as the finite element or boundary collocation methods. The technique described in this paper is developed as an alternative to the boundary collocation method. The paper considers the two-dimensional elastostatic problem for a domain bounded by a smooth curve and containing a straight crack. The problem is formulated by imbedding a fictitious layer of body forces in the plane containing the crack along a closed curve which encircles the original boundary of the domain. By expressing the boundary conditions one obtains a relatively simple system of Fredholm integral equations for the unknown body forces. In limit when the outer curve is shrunk on the boundary the integral equations become singular. The technique is borrowed directly from the potential theory and may be traced back to the works of Betti, Somigliana, and Lauricella [5]. Its application to problems in elasticity has recently been revived and elaborated in [6-10]*.

2. FORMULATION OF THE PROBLEM

Let the smooth curve L be the boundary of a plane elastic domain D, containing a crack along $y=0$, $-a<x<a$ (Figure 1). Let the external loads acting on the body be the tractions (σ_n, τ_n) on L, (concentrated) body forces P_j, Q_j acting at z_j $(j=1,\ldots,m)$, and the tractions σ_y^+, τ_{xy}^+ and σ_y^-, τ_{xy}^- acting on the crack surfaces $x+i0$ and $x-i0$ respectively. The problem is the determination of the stress state in the body, particularly the evaluation of the stress intensity factors at the crack tips, $x=\mp a$, $y=0$.

Let $t=x+iy$ be a point on L which is defined by the following parametric equations:

$$t = x+iy\ , \qquad x = x(s)\ , \qquad y = y(s)\ , \qquad t = t(s) \tag{1}$$

where s is the arc length on L measured in the positive direction from a fixed point (Figure 1). Consider another smooth curve, L_0, in the plane enclosing the domain D. It is assumed that L and L_0 do not intersect. Let the parametric equations of L_0 be

$$t_0 = x_0+iy_0\ , \quad x_0 = x_0(s_0), \quad y_0 = y_0(s_0), \quad t_0 = t_0(s_0) \tag{2}$$

where s_0 is the arc length on L_0.

Consider now the infinite x-y plane (of the same material) containing a crack along $-a<x<a$, $y=0$, which is subjected to the

*After the work for the present manuscript was completed the application of the same technique to the plane elasticity problem for a truncated infinite wedge appeared in [11].

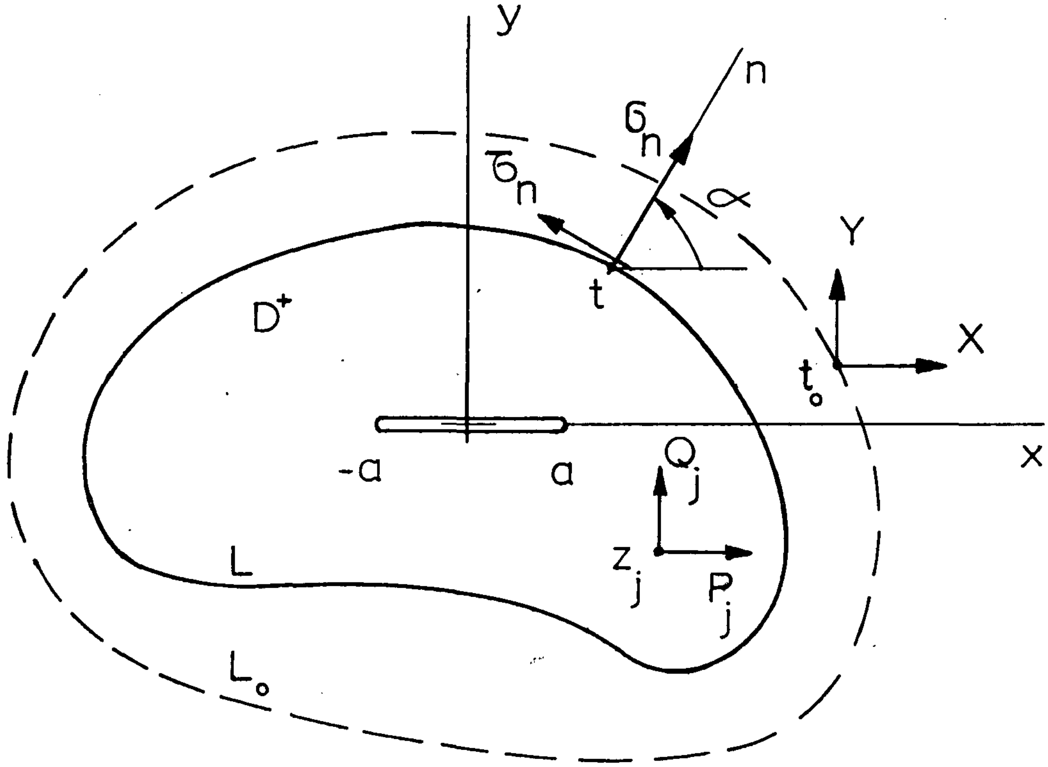

Figure 1. Geometry of the elastic medium.

same body forces P_j, Q_j $(j=1,...,m)$, and the same crack surface tractions σ_y^+, τ_{xy}^+, σ_y^-, τ_{xy}^-, as that acting on D, and a layer of unknown body forces, $X(t_o)$, $Y(t_o)$ distributed along L_o. The basic idea underlying the method used in this paper, which is directly borrowed from the potential theory, is that the body forces X and Y can be determined in such a way that the normal and tangential stresses, σ_n, τ_n, on the line L in the infinite plane are equal to the specified surface tractions on the boundary L of the original domain D. It is obvious that the stress state within the portion bounded by the line L of the infinite plane, which is subjected to the external loads P_j, Q_j, σ_y^+, τ_{xy}^+, σ_y^-, τ_{xy}^-, X, Y is identical to the stress state in D under the original system of external loads. Thus, once X and Y are determined, the problem is solved.

To derive the integral equations for the unknown functions X and Y, two elementary solutions for an infinite plane with a straight crack are needed. The first is the cracked plane under a concentrated body force (X,Y) acting at an arbitrary location t_o, and the second is the same plane subjected to the (arbitrary) crack surface tractions σ_y^+, τ_{xy}^+, σ_y^-, τ_{xy}^-. Let ϕ_1, Ω_1 and ϕ_2, Ω_2 be, respectively, the Kolosov-Muskhelishvili functions for these two problems. The solutions are given in [12]; here only the results will be stated.

$$\phi_1(z) = -\frac{S}{z-t_o} + \phi_o(z) ,$$

$$\Omega_1(z) = \frac{\kappa S}{z-\bar{t}_o} + \frac{(\bar{t}_o-t_o)\bar{S}}{(z-\bar{t}_o)^2} + \phi_o(z) ,$$

$$\phi_o(z) = \frac{1}{2(z^2-a^2)^{1/2}} \{\frac{S}{z-t_o} [I(z) - I(t_o)]$$

$$- \frac{\kappa S}{z-\bar{t}_o} [I(z) - I(\bar{t}_o)]$$

$$- (\bar{t}_o - t_o)\bar{S} [\frac{I(z) - I(\bar{t}_o)}{(z-\bar{t}_o)^2} - \frac{J(\bar{t}_o)}{z-\bar{t}_o}]\} , \qquad (3.a-c)$$

$$S = \frac{X + iY}{2\pi(1+\kappa)} ,$$

$$I(z) = (z^2-a^2)^{1/2} - z ,$$

$$J(z) = \frac{z}{(z^2-a^2)^{1/2}} - 1 . \qquad (4.a-c)$$

where X, Y are the components of the body force per unit thickness acting at t_o, and $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress, ν being the Poisson's ratio.

$$\phi_2(z) = \frac{C}{R(z)} + \frac{1}{2\pi i R(z)} \int_{-a}^{a} \frac{R(x)p(x)dx}{x-z} + \frac{1}{2\pi i} \int_{-a}^{a} \frac{q(x)dx}{x-z} ,$$

$$\Omega_2(z) = \frac{C}{R(z)} + \frac{1}{2\pi i R(z)} \int_{-a}^{a} \frac{R(x)p(x)dx}{x-z} - \frac{1}{2\pi i} \int_{-a}^{a} \frac{q(x)dx}{x-z} .$$

(5.a,b)

$$R(z) = (z^2-a^2)^{1/2} , \qquad R(x) = R^+(x) ,$$

$$p(x) = \frac{1}{2} [(\sigma_y^+ + \sigma_y^-) - i(\tau_{xy}^+ + \tau_{xy}^-)] ,$$

$$q(x) = \frac{1}{2} [(\sigma_y^+ - \sigma_y^-) - i(\tau_{xy}^+ - \tau_{xy}^-)] . \qquad (6.a-d)$$

where the constant C is determined from the following condition of single-valuedness of displacements [13],

$$\kappa \int_\Gamma \phi_2(z)dz - \int_\Gamma \Omega(\bar{z})d\bar{z} = 0 . \qquad (7)$$

In (7) Γ is a closed contour encircling the crack. Also, the stress states at a point t in the plane corresponding to the two basic solutions mentioned above are given in terms of the complex potentials ϕ_k and Ω_k, (k=1,2) by

$$\sigma_{kx} + \sigma_{ky} = 2[\phi_k(t) + \overline{\phi_k(t)}] ,$$

$$\sigma_{ky} - \sigma_{kx} + 2i\tau_{kxy} = 2[(\bar{t}-t)\phi_k'(t) - \phi_k(t) + \bar{\Omega}_k(t)] ,$$

$$(k=1,2). \quad (8.a,b)$$

In particular, for the infinite plane subjected to uniform pressure on the crack surface we have $\sigma_y^+ = \sigma_y^- = -p_o$, $\tau_{xy}^+ = \tau_{xy}^- = 0$, and

$$\sigma_{2x} + \sigma_{2y} = 2p_o \, Re[\frac{t}{(t^2-a^2)^{1/2}} - 1] ,$$

$$\sigma_{2y} - \sigma_{2x} + 2i\tau_{2xy} = - \frac{a^2 p_o(\bar{t}-t)}{(t^2-a^2)^{3/2}} . \quad (9.a,b)$$

At a point t on the line L the normal and tangential stresses $\sigma_n(t)$ and $\tau_n(t)$ are given in terms of the stress components $\sigma_x(t)$, $\sigma_y(t)$, $\tau_{xy}(t)$ by [6]

$$2(\sigma_n - i\tau_n) = (\sigma_x + \sigma_y) - (\sigma_y - \sigma_x + 2i\tau_{xy})e^{2i\alpha} , \quad (10)$$

where α is the angle between the outward normal and the x-axis, and is a known function of the arc length s (Figure 1). In this problem since the crack surface tractions are specified, the stress state (σ_{2x}, σ_{2y}, τ_{2xy}) or (σ_{2n}, τ_{2n}) is assumed to be known (see equations 5-8 and 10).

From (3), (4), (8), and (10) the stress components at a point t on L for a concentrated body force S at a point t_o on L_o

may be expressed as

$$\sigma_{1n}'(t,t_o) - i\tau_{1n}'(t,t_o) = Sk_1(t,t_o) + \bar{S}k_2(t,t_o) \qquad (11)$$

where k_1 and k_2 are known functions and are given in the Appendix. (11) constitutes the Green's function for $(\sigma_{1n} - i\tau_{1n})$. If we now assume that the body force S is a continuous function of the arc length s_o on L_o, and in the original problem the tractions $\sigma_n(t)$ and $\tau_n(t)$ on the boundary L of the domain D are known along with the crack surface tractions, p, q, and the concentrated body forces, P_j, Q_j, through superposition we obtain the following integral equation:

$$\int_{L_o} [S(s_o)k_1(t,t_o) + \bar{S}(s_o)k_2(t,t_o)]ds_o$$

$$+ \sum_1^m [R_j k_1(t,z_j) + \bar{R}_j k_2(t,z_j)] + [\sigma_{2n}(t) - i\tau_{2n}(t)]$$

$$= \sigma_n(t) - i\tau_n(t) \ , \qquad (t \in L) \qquad (12)$$

where $t = t(s)$, $t_o = t_o(s_o)$, and

$$R_j = \frac{P_j + iQ_j}{2\pi(1+\kappa)} \ , \qquad (j=1,\ldots,m) \qquad (13)$$

is the concentrated* body force at an internal point z_j. (12) gives two real integral equations of the first kind for the unknown functions $X(s_o)$ and $Y(s_o)$. These are Fredholm-type integral equations with bounded and continuous kernels and may be solved by the standard numerical techniques (see, for example, [14]).

*If R_j is a distributed body force, then the summation in (12) must be replaced by a line or a surface integral, giving again a known function of t.

3. THE CASE OF $L_o \to L$

From the viewpoint of applications a more effective method to solve the problem would be the reduction of the integral equation (12) to a singular integral equation by shrinking L_o to L. As seen from the Appendix, if t and t_o are points on the same line L the kernel of the integral equation (12) will have a Cauchy type singularity. In this problem (12) with the Cauchy singularity separated as shown in the Appendix would not be the correct integral equation and the limiting process $L_o \to L$ requires some care. First, let us consider the infinite plane loaded by the crack surface tractions p and q (see (6)), the body forces P_j and Q_j, (j=1,...,m), and the distributed line load S acting on t_o which is now assumed to be a point on the boundary L of the given domain D. The left hand side of (12) then gives the expression for the stress vector $(\sigma_n - i\tau_n)$ at any point t inside the line L (i.e., in D^+, see Figure 1). This stress vector must approach the known tractions on the boundary, $\sigma_n(t) - i\tau_n(t)$, as the point t goes to L from inside (i.e., from the positive side). In this limiting process the Fredholm kernels $K_1(t,t_o)$ and $K_2(t,t_o)$ given in the Appendix will remain unchanged. The remaining part of the integral equation has a Cauchy kernel. Hence as t goes to L from inside the limiting process has to be treated as taking the boundary value of the holomorphic function represented by the Cauchy integral. Thus, by using the Plemelj Formulas [13] the integral equation (12) may now be expressed as

$$(1+\kappa)\pi e^{i\alpha}\bar{S}(s) + \int_L \bar{S}(s_0)\left(\frac{\kappa e^{2i\alpha}}{t_0-t} + \frac{1}{\bar{t}_0-\bar{t}}\right)ds_0$$

$$+ \int_L [K_1(t,t_0)S(s_0) + K_2(t,t_0)\bar{S}(s_0)]ds_0$$

$$+ \sum_1^m [R_j k_1(t,z_j) + \bar{R}_j k_2(t,z_j)] + \sigma_{2n}(t) - i\tau_{2n}(t)$$

$$= \sigma_n(t) - i\tau_n(t) , \qquad (t \in L), \tag{14}$$

where the bounded kernels $K_i(t,t_0)$ and $k_i(t,z_j)$, $(i=1,2)$ are given in the Appendix. Referring to the definition of S in (4.a), (14) gives two real singular integral equations in the unknown functions X(s) and Y(s).

4. THE HALF PLANE AND THE STRIP

If the domain D is bounded by simple curves such as a circle or infinite straight lines, the analysis given in the previous section is somewhat simplified. As an example, consider the cracked half plane, $x'<d$, $-\infty<y'<\infty$, shown in Figure 2. If the arc lengths s and s_0 are measured from the points B and B_0, respectively, in the positive direction shown in the figure, then $t = x+iy$ and $t_0 = x_0+iy_0$ give the equations of the lines L and L_0 and may be expressed as

$$t = e^{i\alpha}(d+is) , \qquad t_0 = e^{i\alpha}(d_0+is_0) , \tag{15.a,b}$$

where d, d_0, and α are constant and are shown in Figure 2. Here L and L_0 are infinite lines on which the real variables s and s_0 vary from $-\infty$ to $+\infty$. In the examples considered in this paper, namely the half plane and the strip of finite width, it is assumed that

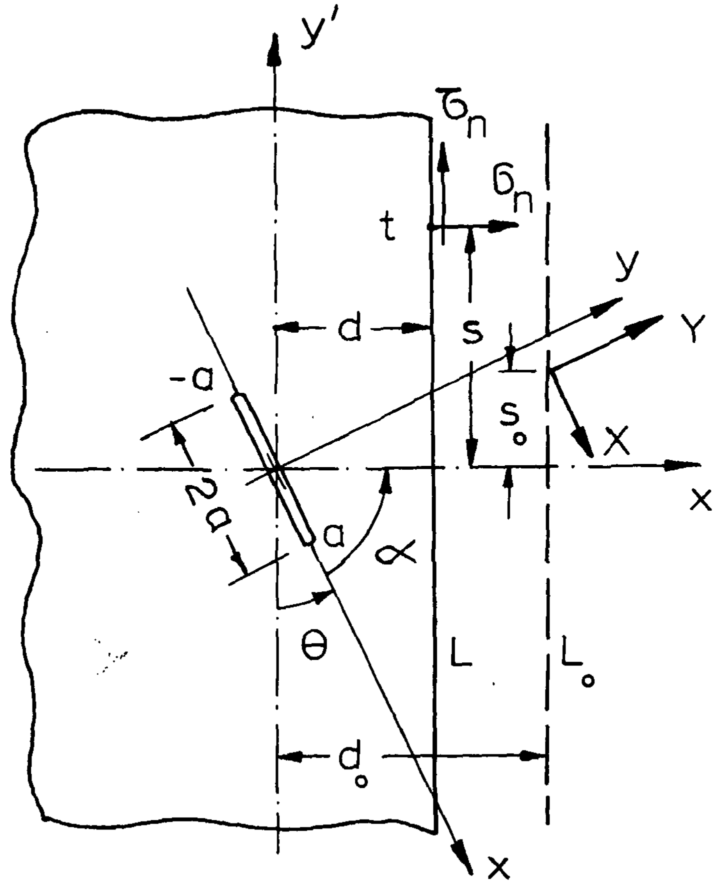

Figure 2. Half plane containing a crack.

$$\sigma_n(t) - i\tau_n(t) = 0 \ , \qquad R_j = 0 \ , \qquad (j=1,\ldots,m), \qquad (16.a,b)$$

that is, the only external load acting on the medium is the crack surface traction. Thus, separating the real and the imaginary parts, (12) may be expressed as

$$\int_{-\infty}^{\infty} [k_{11}(s_0,s)X(s_0) + k_{12}(s_0,s)Y(s_0)]ds_0 = -\sigma_{2n}(s) \ ,$$

$$\int_{-\infty}^{\infty} [k_{21}(s_0,s)X(s_0) + k_{22}(s_0,s)Y(s_0)]ds_0 = \tau_{2n}(s) \ ,$$

$$(-\infty<s<\infty), \qquad (17.a,b)$$

where k_{ij} are real bounded functions obtained from (12), (15) and the expressions given in the Appendix, and the input functions σ_{2n} and τ_{2n} are obtained from (5), (6), (8), (10) and (14) in terms of the given crack surface tractions.

In solving the system of integral equations (16), it is expedient to make a change in variables such that the range of integration is finite. An appropriate change in variables for this purpose is the following:

$$s = \tan\frac{\pi\phi}{2} \ , \qquad s_0 = \tan\frac{\pi\phi_0}{2} \ ,$$
$$-\infty < (s,s_0) < \infty \ , \qquad -1 < (\phi,\phi_0) < 1 \ . \qquad (18)$$

In the case of the strip, if there is no symmetry with respect to the y' axis (see Figure 2), there are four unknown functions and the system of integral equations giving these functions are obtained in a similar way.

In the alternate formulation of the problem $d_0 = d$, $t = e^{i\alpha}(d+is)$, $t_0 = e^{i\alpha}(d+is_0)$, $S = (X+iY)/2\pi(1+\kappa)$, and instead of the system of Fredholm-type integral equations, from (14) we obtain the following singular integral equation:

$$\frac{e^{i\alpha}}{2}[X(s) - iY(s)] + \frac{\kappa-1}{\kappa+1}\frac{e^{i\alpha}}{2\pi i}\int_{-\infty}^{\infty}\frac{X(s_0) - iY(s_0)}{s_0-s}ds_0$$

$$+ \frac{1}{2\pi(1+\kappa)}\int_{-\infty}^{\infty}\{H_1(s,s_0)[X(s_0) + iY(s_0)]$$

$$+ H_2(s,s_0)[X(s_0) - iY(s_0)]\}ds_0$$

$$= - \sigma_0[Re(t/(t^2-a^2)^{1/2}) - 1 - \frac{a^2(\bar{t}-t)e^{2i\alpha}}{2(t^2-a^2)^{3/2}}]$$

$$+ \tau_0[Im(t/(t^2-a^2)^{1/2}) + ie^{2i\alpha}\{1 - \frac{1}{2(t^2-a^2)^{1/2}}(\frac{a^2(\bar{t}-t)}{t^2-a^2} + 2t)\}],$$

$$(t = e^{i\alpha}(d+is), \quad -\infty<s<\infty), \qquad (19)$$

where

$$H_j(s,s_0) = K_j(t,t_0) \ , \qquad (j=1,2). \qquad (20)$$

In (19) it is assumed that the crack surface tractions, which are the only external loads acting on the half plane, are constant and are given by (see Figure 2)

$$\sigma_y + i\tau_{xy} = - (\sigma_0 + i\tau_0) \ , \qquad (y=0, \quad -a<x<a). \qquad (21)$$

5. THE STRESS INTENSITY FACTORS

Let $\phi(z)$ and $\Omega(z)$ be the Kolosov-Muskhelishvili functions for the infinite plane subjected to the body forces $X(s_0)$, $Y(s_0)$, P_j, Q_j, and the given crack surface tractions. From (3-6) it is seen that in the close neighborhood of the crack we can write

$$\phi(z) = \frac{F_1(z)}{\sqrt{z^2-a^2}} + F_2(z) \ ,$$

$$\Omega(z) = \frac{F_1(z)}{\sqrt{z^2-a^2}} + F_3(z) \ , \qquad (z = x+iy), \qquad (22.a,b)$$

where $F_k(z)$, $(k=1,2,3)$ is holomorphic in the entire z plane. Referring to [12] for details, the stress intensity factors at the crack tips $y=0$, $x = \mp a$ may be expressed as*

$$k_1(a) - ik_2(a) = \frac{2F_1(a)}{\sqrt{a}} \ ,$$

$$k_1(-a) - ik_2(-a) = -\frac{2F_1(-a)}{\sqrt{a}} \ . \qquad (23.a,b)$$

The constants $F_1(a)$ and $F_1(-a)$ are easily obtained from $\phi(z)$ as follows:

$$F_1(\mp a) = \lim_{z\to\mp a} [(z^2-a^2)^{1/2}\phi(z)] \qquad (24)$$

Using (3), (15), (22), (23) and (24), the stress intensity factors at x=a, y=0 for the half plane shown in Figure 2 may be obtained as

$$k_1(a) - ik_2(a) = k_1^o(a) - ik_2^o(a)$$

$$+ \int_{-\infty}^{\infty} [h_1(a,s_o)X(s_o) + h_2(a,s_o)Y(s_o)]ds_o \ , \qquad (25)$$

where the (Green's) functions h_1 and h_2 are easily obtained from (3), (23) and (24), and k_1^o and k_2^o are the stress intensity factors for the infinite plane subjected to the known crack surface tractions (and the body forces P_j, Q_j, if any). Stress intensity factors at $x = -a$, y=0 may be obtained in a similar way.

*Here the definition of k_1 and k_2 is such that at $y=0$, $x-a = r$ the asymptotic values of the stresses for small r are given by $\sigma_y \simeq \sigma_x \simeq k_1/\sqrt{2r}$, $\tau_{xy} \simeq k_2/\sqrt{2r}$.

6. ON THE SOLUTION OF THE INTEGRAL EQUATIONS

The solution of the system of Fredholm-type integral equations given by (17) is, in principle, straightforward. However, to improve the effectiveness of the numerical solution, it may be necessary to make a change in variables such as suggested in (18) and to use a Gaussian-type integration formula rather than one based on dividing the domain into equal subintervals as mentioned in [14]. In this type of problem an appropriate integration formula would be the following [15]:

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=1}^{n} w_i f(x_i) ,$$

$$P_n(x_i) = 0 ,$$

$$w_i = 2/(1-x_i)^2[P_n'(x_i)]^2 , \qquad (i=1,\dots,n) \qquad (26.a-c)$$

where the Legendre polynomials $P_n(x)$ are the related orthogonal polynomials.

Two singular integral equations obtained from (19) for the unknown functions X(s) and Y(s) may be considered as a special case of the following system:

$$Af(x) + \frac{B}{\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt + \int_{-\infty}^{\infty} K(x,t)f(t)dt = g(x) , \quad (-\infty<x<\infty) \qquad (28)$$

where

$$A = (a_{ij}) , \quad B = (b_{ij}) , \quad K(x,t) = (k_{ij}(x,t)) ,$$

$$f(x) = (f_i(x)) , \quad g(x) = (g_i(x)) , \quad (i,j=1,\dots,n). \qquad (29)$$

The matrices A and B are constant and are such that (A+B) and

(A-B) are nonsingular, the kernels k_{ij} and the input functions g_i are known bounded functions, and the functions f_i are unknown. Generally the quantities given in (29) may be complex. The system of singular integral equations given by (28) may formally be regularized as follows: Define

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt \ ,$$

$$H(z) = \begin{cases} F(z) \ , & (z \in S^{+}) \\ (A+B)^{-1}(A-B)F(z) \ , & (z \in S^{-}) , \end{cases} \qquad (30.a,b)$$

$$F(z) = (F_j(z)) \ , \quad H(z) = (H_j(z)) \ , \quad (j=1,...,n),$$

$$(A-B)^{-1} = C = (c_{ij}) \ , \quad (A+B)^{-1} = D = (d_{ij}) \ , \qquad (31.a,b)$$

In terms of the matrix of sectionally holomorphic functions, H(z) (28) may be expressed as

$$H^{+}(x) - H^{-}(x) = D[g(x) - \int_{-\infty}^{\infty} K(x,t)f(t)dt] \ , \qquad (32)$$

the solution of which is

$$H(z) = \frac{D}{2\pi i} \int_{-\infty}^{\infty} [g(s) - \int_{-\infty}^{\infty} K(s,t)f(t)dt] \frac{ds}{s-z} + P(z) \ , \qquad (33)$$

where P(z) is an arbitrary (matrix) polynomial. Now imposing the condition that F(z) should vanish as $|z| \to \infty$, it is clear that P(z) must be zero. From

$$f(x) = F^{+}(x) - F^{-}(x) = H^{+}(x) - C(A+B)H^{-}(x) \ , \qquad (34)$$

the solution of (28) may then be obtained as

$$f(x) + \int_{-\infty}^{\infty} M(x,t)f(t)dt = p(x) \ , \quad (-\infty<x<\infty), \qquad (35)$$

where

$$p(x) = \frac{D+C}{2} g(x) + \frac{D-C}{2\pi i} \int_{-\infty}^{\infty} \frac{g(s)}{s-x} ds ,$$

$$M(x,t) = (D+C)K(x,t) + \frac{D-C}{2\pi i} \int_{-\infty}^{\infty} \frac{K(s,t)}{s-x} ds , \qquad (36.a,b)$$

$$M(x,t) = (m_{jk}(x,t)), \quad p(x) = (p_j(x)), \quad (j,k = 1,...,n).$$

(35) is now a system of ordinary Fredholm integral equations of the second kind and may be solved numerically by using the transformation (18) and the integration formula (26). In (28) if $K(x,t) = 0$, then (35) gives the closed form solution of the system as $f(x) = p(x)$. This is the generalization of the Hilbert transforms for one unknown function [16].

The foregoing technique would be highly recommended provided one can evaluate the kernels $m_{jk}(x,t)$, $(j,k = 1,...,n)$ in closed form. On the other hand if these kernels have to be evaluated numerically through the singular integrals given by (36.b), the technique could be quite laborious. In this case the following simpler and more direct approach may be preferable: Noting that

$$\int_{-\infty}^{\infty} \frac{dt}{t-x} = 0$$

(28) may be expressed as

$$Af(x) + \frac{B}{\pi i} \int_{-\infty}^{\infty} \frac{f(t)-f(x)}{t-x} dt + \int_{-\infty}^{\infty} K(x,t)f(t)dt = g(x) ,$$

$$(-\infty<x<\infty). \qquad (37)$$

Unlike the solution of the singular integral equations defined on arcs, the solution of the singular equations defined on infinite lines and smooth closed contours (e.g., (28) and (14)) are usually bounded and continuous functions. Hence for the purpose of numerical analysis, in these equations the singularity of the kernel can always be removed and the integral equation can be treated as a Fredholm equation. In (37) note that at t=x the integrand in the second term becomes the derivative of f(t) which is assumed to be bounded. To solve (37) again a transformation such as (18) would be very useful.

It should perhaps be pointed out that in ordinary applications one may prefer to work with the Fredholm equations (12) rather than the singular equations (14). One reason for this is that the advantage of working with an integral equation of the second kind (14) instead of that which is of the first kind (12) is somewhat eliminated by the necessity of evaluating the Cauchy principal value of singular integrals in (14). Another reason for preferring (12) over (14) may be the flexibility it offers for improving the accuracy of the results without exhaustive numerical work. For example, in solving (12) the results can be improved over that obtained through the conventional numerical techniques for solving Fredholm-type integral equations either by selecting the number of intervals, N on L_0 in the integrations (and hence, the number of unknowns) less than the number, M of the "collocation points", t_j, (j=1,...,M) and using a least square technique, or by selecting $N > M$ and solving the resulting system of equations in some optimal sense*.

*The "distance" between L and L_0 may also be considered as another parameter which could be varied to improve the convergence of the calculated results. For the straight boundaries $a < d_0 - d < 2a$ (see Figure 2) appears to give the fastest convergence. Final numerical results in this paper were obtained by selecting $d_0 - d = 1.5\,a$.

7. NUMERICAL RESULTS

As a first example we consider an elastic half plane containing an arbitrarily oriented internal crack near its free boundary. The results for uniform normal and shear tractions on the crack surfaces are given in Figures 3 and 4. For the special cases $\theta = 0$ and $\theta = \frac{\pi}{2}$ these results reduce to that given in [17] and [4], respectively, which are partly reproduced in Figures 5 and 6. The inclined crack results are given for two values of d/a, a and d being the half-length and center-to-boundary distance of the crack. Two general observations which could be made on the basis of these results are that the stress intensity factors increase with decreasing distance of the crack tip to the free boundary, and generally for the same crack surface tractions, the resistance of the medium to (brittle-type) fracture would be higher for crack angles around $\theta = \frac{\pi}{2}$ (i.e., for cracks nearly perpendicular to the boundary) than for $\theta \simeq 0$ (i.e., for cracks parallel to the boundary).

The results for the infinite strip with a symmetrically located internal crack are given in Figure 7. The results are obtained for $d = 3a$ and for uniform normal and shear tractions on the crack surfaces. In limit when $\theta = \frac{\pi}{2}$ and $\theta = 0$ these results reduce to (and agree with) that found in [2,3] and [18], respectively. In a superposition to obtain the results for other load combinations it should be noted that in the results given by Figures 3-7 the following crack surface tractions have been used:

$$\sigma_y(x,0) = -\sigma_0 , \qquad \tau_{xy}(x,0) = -\tau_0 , \qquad (-a<x<a).$$

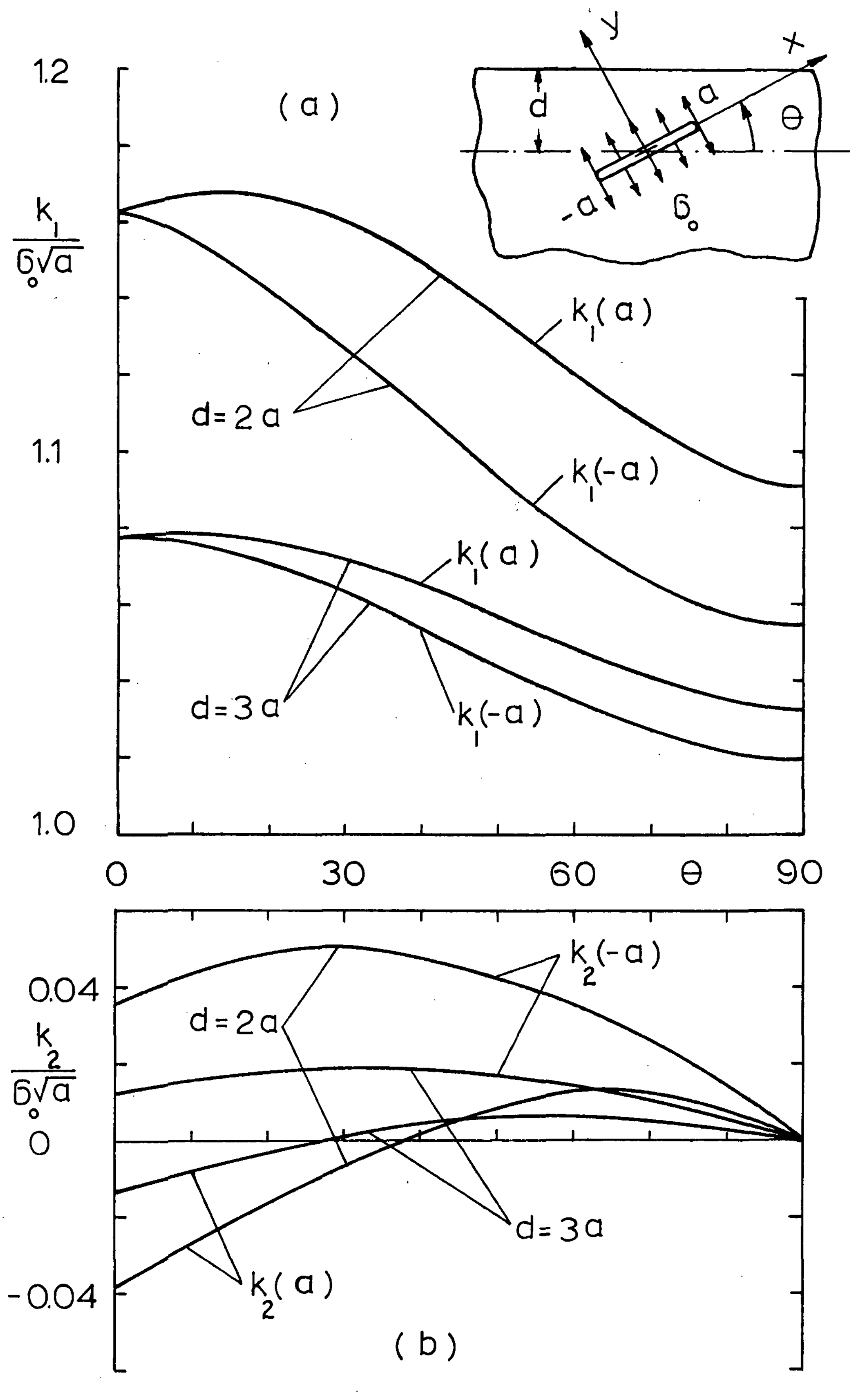

Figure 3. Normal (k_1) and the shear (k_2) components of the stress intensity factors for a crack in a half plane under uniform pressure σ_0.

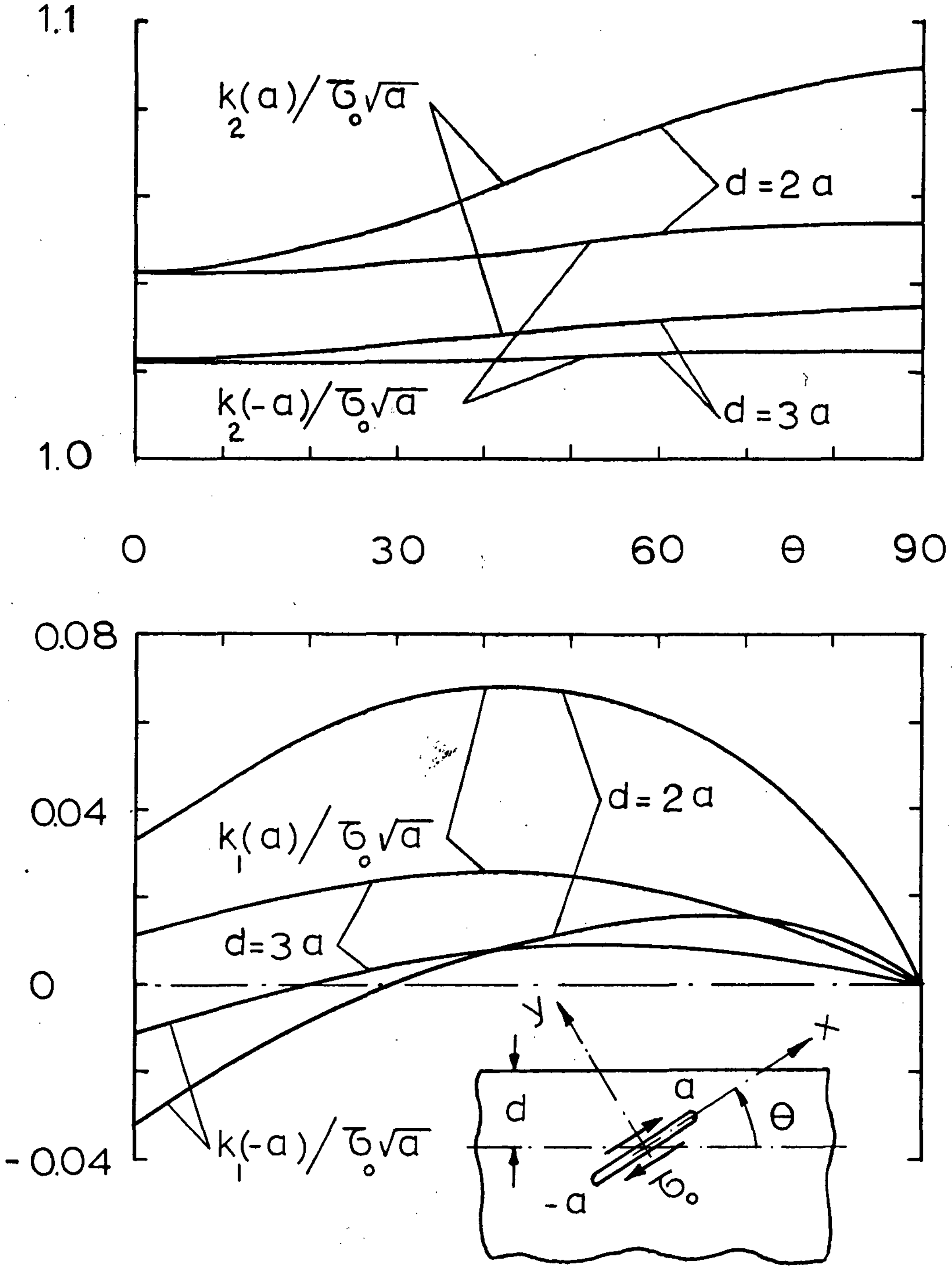

Figure 4. Normal and shear components of the stress intensity factors for a crack in a half plane under uniform crack surface shear traction $\tau_{xy} = -\tau_0$.

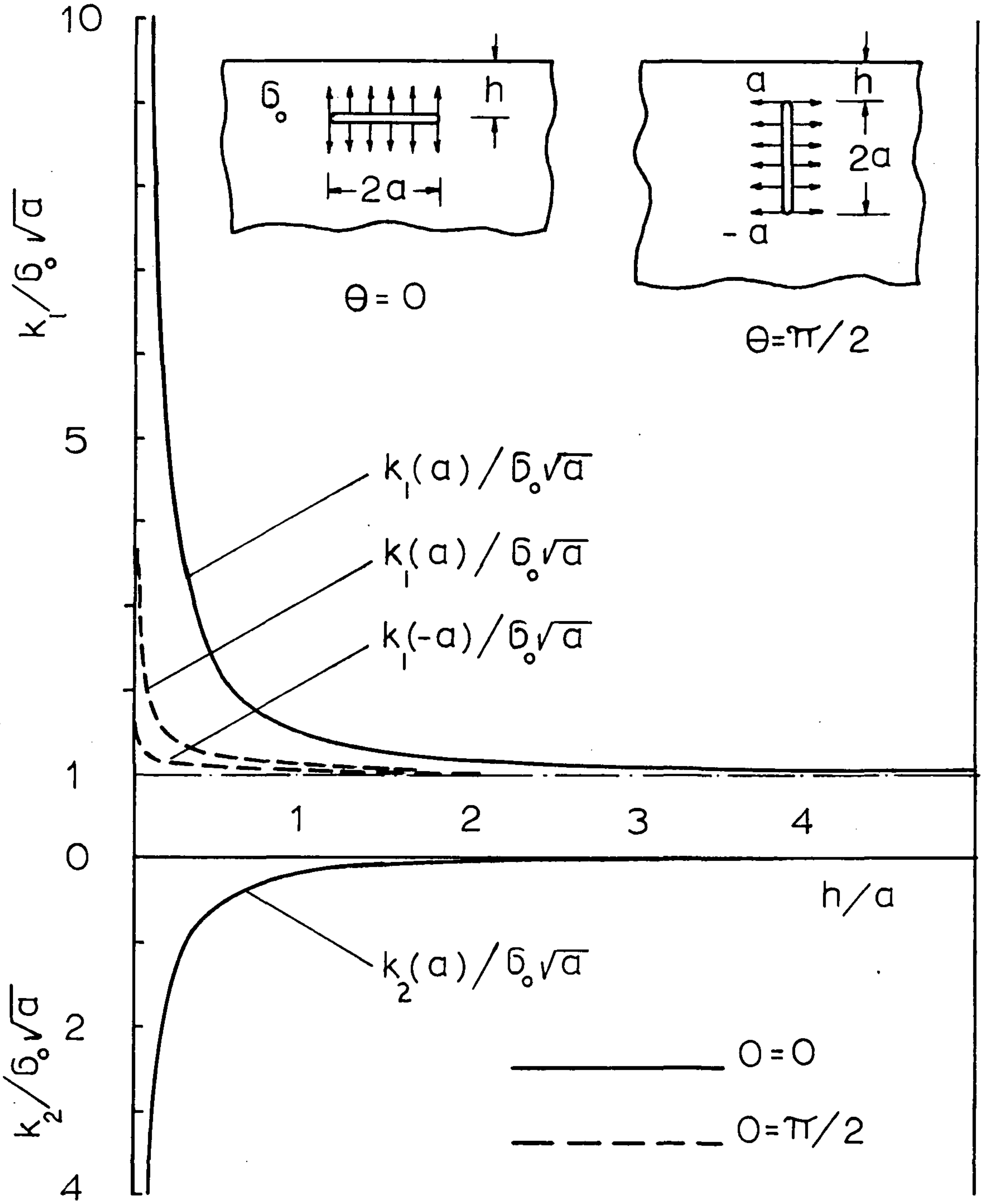

Figure 5. Stress intensity factors for a crack parallel and for that perpendicular to the free boundary; external load: crack surface pressure, σ_0.

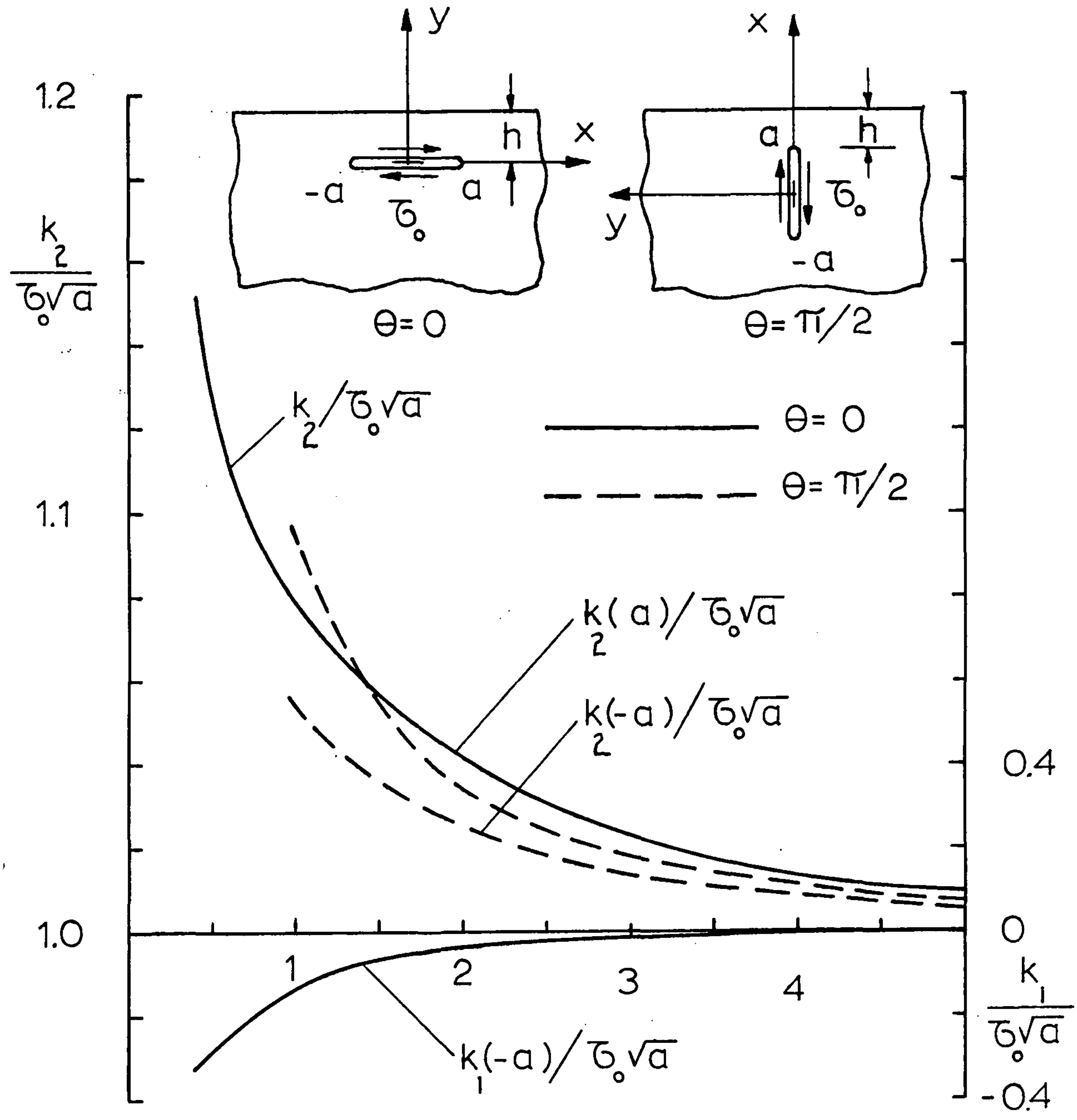

Figure 6. Stress intensity factors for a crack parallel and perpendicular to the free boundary; external load: crack surface shear traction, $\tau_{xy} = -\tau_o$.

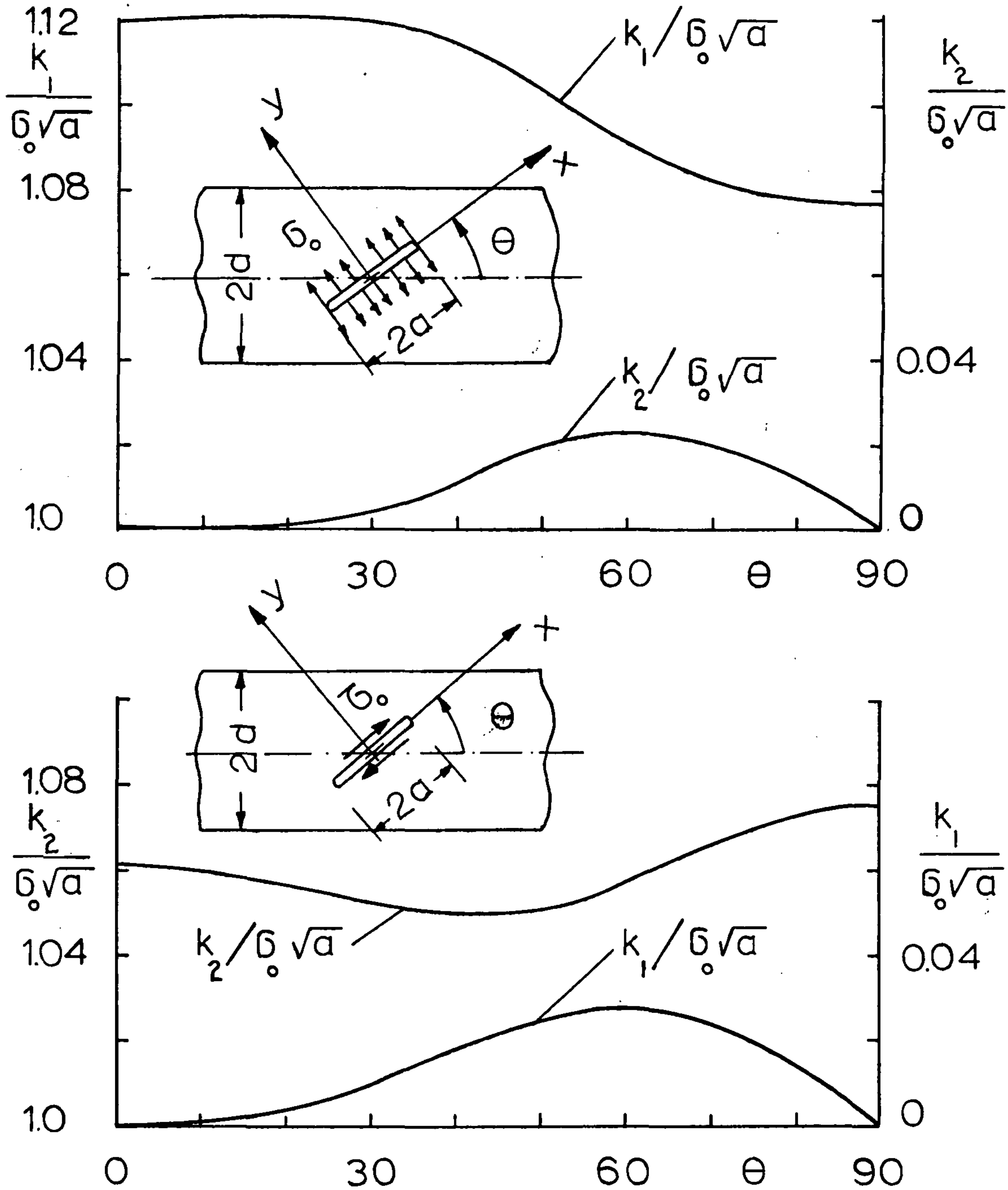

Figure 7. Normal and shear components of the stress intensity factors for a crack in an infinite strip; external loads: crack surface pressure, $\sigma_y = -\sigma_0$ or crack surface shear traction $\tau_{xy} = -\tau_0$.

REFERENCES

1. M. Isida, "Stress Intensity Factors for the Tension of an Eccentrically Cracked Strip", J. Appl. Mech., vol. 33, Trans. ASME, Series E, pp. 674-675, (1966).

2. M. Isida, "On the Tension of a Strip with a Central Elliptical Hole", Trans. Japan Soc. Mech. Engrs., vol. 21, pp. 507-518, (1955).

3. I. N. Sneddon and R. P. Srivastav, "The Stress Field in the Vicinity of a Griffith Crack in a Strip of Finite Width", Int. J. Engineering Science, vol. 9, pp. 479-488, (1971).

4. T. S. Cook, "Flaws in Bonded Wedges", Ph.D. Dissertation, Lehigh University, (1971).

5. A. E. H. Love, The Mathematical Theory of Elasticity, Dover Publications, New York, (1944).

6. M. A. Jawson and A. R. Ponter, "An Integral Equation Solution of the Torsion Problem", Proc. Roy. Soc., Ser. A, 273, (1963).

7. F. J. Rizzo, "An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics", Q. Appl. Math., vol. 25, pp. 83-95, (1967).

8. T. A. Cruse and F. J. Rizzo, "A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem - I", J. Math. Anal. Applic., vol. 22, p. 244, (1968).

9. T. A. Cruse, "A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem - II", J. Math. Anal. Applic., vol. 22, p. 341, (1968).

10. F. J. Rizzo and D. J. Shippy, "A Formulation and Solution Procedure for the General Nonhomogeneous Elastic Inclusion Problem", Int. J. Solids, Structures, vol. 4, p. 1161, (1968).

11. B. Benjumea and D. L. Sikarskie, "On the Solution of Plane, Orthotropic Elasticity Problems by an Integral Method", J. Appl. Mech., vol. 39, Trans. ASME, pp. 801-808, (1972).

12. F. Erdogan, "On the Stress Distribution in Plate with Collinear Cuts Under Arbitrary Loads", Proc. 4th U.S. National Congress of Applied Mechanics, pp. 547-553, (1962).

13. N. I. Muskhelishvili, "Some Basic Problems of the Mathematical Theory of Elasticity", P. Noordhoff, Groningen, Holland, (1953).

14. L. V. Kantorovich and V. I. Krylov, "Approximate Methods of Higher Analysis", Interscience, New York, (1958).

15. M. Abramovitz and I. A. Stegun, "Handbook of Mathematical Tables", National Bureau of Standards, Appl. Math. Series, 55, (1964).

16. E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals", Oxford University Press, (1937).

17. F. Erdogan, "Bonded Dissimilar Materials Containing Cracks Parallel to the Interface", J. Engng. Fracture Mechanics, vol. 3, pp. 231-240, (1971).

18. F. Erdogan and G. D. Gupta, "Stress Analysis of Multi-Layered Composites with a Flaw", Int. J. Solids, Structures, vol. 7, pp. 39-61, (1971).

APPENDIX

The expressions for the kernels k_1 and k_2

$$k_1(t,t_o) = f_1(t,t_o) + \overline{f_2(t,t_o)} - f_3(t,t_o)e^{2i\alpha} , \qquad (A.1)$$

$$k_2(t,t_o) = \overline{f_1(t,t_o)} + f_2(t,t_o) - f_4(t,t_o)e^{2i\alpha} , \qquad (A.2)$$

$$f_1(t,t_o) = -\frac{1}{t-t_o} + \frac{1}{2R(t)}[f(t,t_o) - \kappa f(t,\overline{t}_o)] , \qquad (A.3)$$

$$f_2(t,t_o) = -\frac{1}{2R(t)}\left(\frac{\overline{t}_o-t_o}{t-\overline{t}_o}\right)[f(t,\overline{t}_o) + 1 - \frac{\overline{t}_o}{R(\overline{t}_o)}] , \qquad (A.4)$$

$$R(t) = (t^2-a^2)^{1/2} , \qquad (A.5)$$

$$f_3(t,t_o) = \frac{1}{2R(t)}\{\frac{2R(t)}{t-t_o}\left(\frac{\overline{t}-\overline{t}_o}{t-t_o}\right) - f(t,t_o)[\frac{t(\overline{t}-t)}{t^2-a^2}$$

$$+ \frac{\overline{t}-\overline{t}_o}{t-t_o}] + \kappa f(t,\overline{t}_o)[\frac{t(\overline{t}-t)}{t^2-a^2} + \frac{\overline{t}-\overline{t}_o}{t-\overline{t}_o}]$$

$$+ (\overline{t}-t)J(t)[\frac{1}{t-t_o} - \frac{\kappa}{t-\overline{t}_o}] + J(t_o)\left(\frac{t_o-\overline{t}_o}{t-t_o}\right)\} , \qquad (A.6)$$

$$f(t,t_o) = \frac{I(t) - I(t_o)}{t-t_o} , \qquad (A.7)$$

$$I(t) = R(t) - t , \qquad J(t) = \frac{t}{R(t)} - 1 , \qquad (A.8.a,b)$$

$$f_4(t,t_o) = \frac{1}{2R(t)}\{\left(\frac{\overline{t}_o-t_o}{t-\overline{t}_o}\right)[-\left(\frac{\overline{t}-t}{t-\overline{t}_o}\right)\left(J(t) + J(\overline{t}_o) - 2f(t,\overline{t}_o)\right)$$

$$+ \left(\frac{t(\overline{t}-t)}{t^2-a^2} + 1\right)\left(f(t,\overline{t}_o) - J(\overline{t}_o)\right)]$$

$$+ \frac{2\kappa R(t)}{t-t_o} + f(t,\overline{t}_o) - \kappa f(t,t_o)\} . \qquad (A.9)$$

As L_o shrinks on L (Figure 1), the kernels k_1 and k_2 contain Cauchy type singularity which may be separated as follows:

$$f_1(t,t_o) = \frac{1}{t_o - t} + F_1(t,t_o) \ ,$$

$$f_2(t,t_o) = F_2(t,t_o) \ ,$$

$$f_3(t,t_o) = \frac{e^{-2i\alpha}}{t_o - t} + F_3(t,t_o) \ ,$$

$$f_4(t,t_o) = \frac{\kappa}{t - t_o} + F_4(t,t_o) \ , \qquad ((t,t_o) \in L), \qquad \text{(A.10.a-d)}$$

giving

$$k_1(t,t_o) = K_1(t,t_o) \ ,$$

$$k_2(t,t_o) = \frac{1}{\bar{t}_o - \bar{t}} + \frac{\kappa e^{2i\alpha}}{t_o - t} + K_2(t,t_o) \ ,$$

$$K_1(t,t_o) = F_1(t,t_o) + \overline{F_2(t,t_o)} - e^{2i\alpha}F_3(t,t_o) \ ,$$

$$K_2(t,t_o) = \overline{F_1(t,t_o)} + F_2(t,t_o) - e^{2i\alpha}F_4(t,t_o) \ , \qquad \text{(A.11.a-d)}$$

where F_j, $(j=1,..,4)$ and K_j, $(j=1,2)$ are bounded functions on L obtained from (A.3 - A.11).