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A Theory for Scattering by Density Fluctuations Based on Three-Wave Interaction

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ABSTRACT

The theory of scattering by charged particle density fluctuations of a plasma is developed for the case or zero magnetic field. The source current is derived on the basis of, first, a three-wave interaction between the incident and scattered electromagnetic waves and one electrostatic plasma wave (either Langmuir or ion-acoustic), and second, a synchronous interaction between the same two electromagnetic waves and the discrete components of the charged particle fluctuations. Previous work is generalized by no longer making the assumption that the frequency of the electromagnetic waves is large compared to the plasma frequency. The general result is then applied to incoherent scatter, and to scatter by strongly driven plasma waves. An expansion is carried out for each of those cases to determine the lower order corrections to the usual high frequency scattering formulas.

Introduction

The scattering of electromagnetic waves by density fluctuations has been a topic of general interest for many years. The first derivations, given by Booker [1955], and Villars and Weisskopf [1955], were based on the idea that density fluctuations give rise to dipole-moment density fluctuations which in turn cause the familiar far-field electric dipole radiation. Most studies since then on scattering use the same basic idea. Rosenbluth and Weisskopf [1962] used a technique based on a far-field expansion of Maxwell's equations, and a source current consisting of a summation over discrete plasma particles. Birmingham et al. [1965], although not specifically addressing themselves to the far-field problem, showed that this scattering formula must be corrected by a factor equal to the refractive index of the scattered wave.

When the density fluctuations are excited by the random motion of charged particles, the scattering is referred to as incoherent scatter. The study of incoherent scattering of electromagnetic waves by a plasma has been given by a number of authors. Dougherty and Farley [1960], Salpeter [1960], and Fejer [1960] independently calculated the cross-section for random thermal fluctuations of the electron density. Hagfors [1961] extended the theory to include a static magnetic field. Rosenbluth and Rostoker [1962] generalized the theory to take into account departure from thermal equilibrium. The subject of scattering by density variations, and in particular, incoherent scattering, is thoroughly reviewed by Bekefi [1966].

To our knowledge, all of the previous work has been based on the high-frequency assumption, i.e. that the incident and scattered electromagnetic waves are much higher in frequency than the plasma frequency.

In this paper we generalize this previous work by dispensing with this assumption and derive a result which is valid for all frequencies. Of course, we still must assume that we are not so close to a resonance that we must include multiple scattering effects.

The source currents responsible for the scattering are determined on the basis of two types of interaction, one depending on collective effects and one on discrete particle effects. These two effects arise, in turn, from the fact that the charged particle distribution function may be resolved into two components. One is the spatially averaged part associated with plasma waves and collective effects, and the second is the spatially rapidly fluctuating component which vanishes when averaged over the macroscopic volume. It arises from the discrete motion of the particles and is basically a thermal fluctuation phenomenon.

The mechanism for the collective source current is basically no more than a three-wave plasma interaction between the incident and scattered electromagnetic waves on one hand, and a scattering electrostatic plasma wave on the other hand. The plasma wave may be either a Langmuir or ion-acoustic wave. A schematic of the process is shown in Figs. 1 and 2. In Fig. 1, the incoming wave $(\omega_\beta, \underset{\sim}{k}_\beta)$ mixes with the electrostatic plasma wave $(\omega_\gamma, \underset{\sim}{k}_\gamma)$ to produce a scattered electromagnetic wave $(\omega_\alpha = \omega_\beta + \omega_\gamma,\ \underset{\sim}{k}_\alpha = \underset{\sim}{k}_\beta + \underset{\sim}{k}_\gamma)$. In the second version of the process, shown in Fig. 2, the incoming electromagnetic wave $(\omega_\beta, \underset{\sim}{k}_\beta)$ decays into an electrostatic plasma wave $(\omega_\gamma, \underset{\sim}{k}_\gamma)$ and a scattered electromagnetic wave $(\omega_\alpha = \omega_\beta - \omega_\gamma,\ \underset{\sim}{k}_\alpha = \underset{\sim}{k}_\beta - \underset{\sim}{k}_\gamma)$. A synchronism diagram showing the dispersion curves of the interacting waves and the synchronism parallelogram for the conditions $\omega_\alpha = \omega_\beta \pm \omega_\gamma$, $\underset{\sim}{k}_\alpha = \underset{\sim}{k}_\beta \pm k_\gamma$ corresponding to Figs. 1 and 2, respectively, is shown in Figs. 3 and 4 for the case where the electrostatic wave is a Langmuir wave and an ion-acoustic wave, respectively

The mechanism for the source current arising from discrete particle effects is an interaction between the electromagnetic waves again, and the synchronous Fourier component of the fluctuating discrete component of the electron velocity distribution function. This source current is responsible for scattering by unscreened electrons, i.e. scattering which does not involve collective effects between the particles.

Our general mathematical approach is as follows. The far field is first determined in terms of an asymptotic expansion of Maxwell's equation (Lighthill, 1960). The effects of the two synchronous interactions mentioned above are then evaluated by solving the Vlasov equation to second order, and using the result to calculate the second order source currents. Once the source currents are evaluated, the far field and scattered power are determined in terms of products of certain fluctuating quantities. If the spectrum of the density fluctuations is known, the scattered power is determined by substituting the expressions for these products and carrying out the required mathematical manipulations.

In the case of incoherent scatter, where the density fluctuations are not externally driven, but are excited solely by the random motion of the plasma particles, it is possible to carry the problem forward to a final solution. In this paper we obtain expressions for the product of the fluctuating quantities under this assumption, and obtain a closed form solution for the incoherent scatter in terms of the unperturbed particle velocity distribution functions.

Since the resulting expression for the scattered power is somewhat involved, an expansion in inverse powers of the frequency of the incident electromagnetic waves is carried out to gain greater insight into the meaning of the results, and to provide a link with the results of previous workers.

The general theory is also applied to the case where the plasma waves are so strongly driven by an external source that one can neglect the effects of the random motions of the charged particles. Expansions are again derived for a high frequency incident electromagnetic wave.

The results in this paper are based on the assumption that the static magnetic field is zero, that the charged particle velocity distribution parameters are isotropic in velocity space, and that the medium is homogeneous.

Theory for Scattering in Terms of Current Sources

In this section we consider the scattering in general, without specifying the current sources responsible. Our system is described by Maxwell's equations,

$$\nabla \times \underset{\sim}{E} = - \mu_0 \frac{\partial \underset{\sim}{H}}{\partial t} , \tag{1}$$

$$\nabla \times \underset{\sim}{H} = \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} + \underset{\sim}{J}^{(1)} + \underset{\sim}{J}^{(2)} , \tag{2}$$

and the Vlasov equation,

$$\frac{\partial f}{\partial t} + (\underset{\sim}{v} \cdot \nabla) f + \eta \, (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0 , \tag{3}$$

where $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are the electric and magnetic fields, $\underset{\sim}{B}$ is the magnetic induction, $\underset{\sim}{J}_1$ and $\underset{\sim}{J}_2$ are the first and second order current densities, f is the electron velocity distribution function, $\underset{\sim}{v}$ is the velocity, and η is the electron charge-to-mass ratio.

Taking the Fourier transform of Eqs. (1) and (2), and then combining, yields the relation

$$\frac{c^2}{\omega_\alpha^2} \left[\underset{\sim}{k}_\alpha \left(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\alpha \right) - k_\alpha^2 \underset{\sim}{E}_\alpha \right] = - \underset{\sim}{E}_\alpha + \frac{j}{\omega_\alpha \epsilon_0} \left(\underset{\sim}{J}_\alpha^{(1)} + \underset{\sim}{J}_\alpha^{(2)} \right) . \tag{4}$$

The first order current is given by

$$\underset{\sim}{J}_\alpha^{(1)} = n_0 e \underset{\sim}{v}_\alpha = \frac{\epsilon_0}{j\omega_\alpha} \omega_p^2 \underset{\sim}{E}_\alpha , \tag{5}$$

where

$$\omega_p^2 = n_0 e^2 / m \epsilon_0 . \tag{6}$$

Equation (4) then becomes

$$\frac{c^2}{\omega_\alpha^2} \left[\underset{\sim}{k}_\alpha \left(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\alpha \right) - k_\alpha^2 \underset{\sim}{E}_\alpha \right] = - \epsilon_\alpha \underset{\sim}{E}_\alpha + \frac{j}{\omega_\alpha \epsilon_0} \underset{\sim}{J}_\alpha^{(2)} , \tag{7}$$

where

$$\epsilon_\alpha = 1 - \omega_p^2/\omega_\alpha^2 \ . \tag{8}$$

Taking the dot product of this equation with respect to $\underset{\sim}{k}_\alpha$ gives

$$\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\alpha = \frac{j\underset{\sim}{k}_\alpha \cdot \underset{\sim}{J}_\alpha^{(2)}}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \ . \tag{9}$$

Substitution in Eq. (7), and solution for $\underset{\sim}{E}_\alpha$, yields

$$\underset{\sim}{E}_\alpha = \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \ \frac{\underset{\sim}{G}(\omega_\alpha, \underset{\sim}{k}_\alpha)}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)} \ , \tag{10}$$

where

$$\underset{\sim}{G} = \underset{\sim}{k}_\alpha \times \left(\underset{\sim}{k}_\alpha \times \underset{\sim}{J}_\alpha^{(2)}\right) \ , \tag{11}$$

$$k_\alpha(\omega_\alpha) = \frac{\omega_\alpha}{c} \epsilon_\alpha^{1/2} \ . \tag{12}$$

Determination of Far-field Power Flux Density

We now determine the electric fields in the far-field zone, by taking the inverse Fourier transform, and then apply essentially an asymptotic expansion technique [Lighthill, 1960]. The inverse transform of Eq. (10) is given by

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \; d\underset{\sim}{k}_\alpha \; d\omega_\alpha \exp j(\omega_\alpha t - j\underset{\sim}{k}_\alpha \cdot \underset{\sim}{r}) \frac{\underset{\sim}{G}(\omega_\alpha, \underset{\sim}{k}_\alpha)}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)} \; . \tag{13}$$

If we replace $\underset{\sim}{G}$ by its spatial transform, we obtain

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \; \frac{\exp[-j\underset{\sim}{k}_\alpha \cdot (\underset{\sim}{r}-\underset{\sim}{r}')]}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)} \; \underset{\sim}{G}(\omega_\alpha, \underset{\sim}{r}') \exp(j\omega_\alpha t) d\underset{\sim}{k}_\alpha \; d\omega_\alpha \; d\underset{\sim}{r}' \; . \tag{14}$$

Since $\underset{\sim}{r} \gg \underset{\sim}{r}'$, the integral over $\underset{\sim}{k}_\alpha$ can be evaluated in the form

$$\int_{-\infty}^{\infty} \frac{\exp[-i\underset{\sim}{k}_\alpha \cdot (\underset{\sim}{r}-\underset{\sim}{r}')]}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)} \, d\underset{\sim}{k}_\alpha = \frac{2\pi^2 \exp[-ik_\alpha(\omega_\alpha)|\underset{\sim}{r}-\underset{\sim}{r}'|]}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \simeq \frac{2\pi^2[-ik_\alpha(\omega_\alpha)\underset{\sim}{e}_r \cdot (\underset{\sim}{r}-\underset{\sim}{r}')]}{r} \; , \tag{15}$$

where $\underset{\sim}{e}_r = \underset{\sim}{r}/r$. Finally, the integration over $\underset{\sim}{r}'$ yields

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \; \frac{\exp[j\omega_\alpha t - j\underset{\sim}{k}_\alpha(\omega_\alpha) \cdot \underset{\sim}{r}]}{r} \; \underset{\sim}{G}[\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)] d\omega_\alpha \; , \tag{16}$$

where $\underset{\sim}{k}_\alpha(\omega_\alpha) = k_\alpha(\omega_\alpha)\underset{\sim}{e}_r$. From Eq. (2), the corresponding magnetic field is given by

$$\underset{\sim}{H}(\underset{\sim}{r},t) = \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} \frac{c^2 j}{\omega_\alpha^2 \epsilon_\alpha} \; \frac{\exp[j\omega_\alpha t - j\underset{\sim}{k}_\alpha(\omega_\alpha) \cdot \underset{\sim}{r}]}{r} \; \underset{\sim}{k}_\alpha(\omega_\alpha) \times \underset{\sim}{G}[\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)] d\omega_\alpha \; . \tag{17}$$

The time-averaged power flow is given by

$$\underset{\sim}{P}(\underset{\sim}{r}) = \lim_{T \to \infty} \frac{2}{T} \int_{-T/2}^{T/2} dt \ \mathrm{Re}\ \underset{\sim}{E}(\underset{\sim}{r},t) \times \underset{\sim}{H}^*(\underset{\sim}{r},t), \tag{18}$$

therefore

$$\underset{\sim}{P}(\underset{\sim}{r}) = \lim_{T \to 0} \frac{1}{2T(2\pi)^4} \int_{-T/2}^{T/2}\int_{-\infty}^{\infty} dt\ d\omega_\alpha d\omega'_\alpha\ \mathrm{Re}\left\{ \frac{c^2 \exp\{j(\omega_\alpha - \omega'_\alpha)t - j[\underset{\sim}{k}_\alpha(\omega_\alpha) - \underset{\sim}{k}'_\alpha(\omega'_\alpha)]\cdot\underset{\sim}{r}\}}{r^2 \omega_\alpha (\omega'_\alpha)^2 \epsilon_0 \epsilon_\alpha \epsilon'_\alpha} \right.$$

$$\left. \cdot\ \underset{\sim}{G}\left(\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)\right) \times \left[\underset{\sim}{k}'_\alpha (\omega'_\alpha) \times \underset{\sim}{G}\left(\omega'_\alpha, \underset{\sim}{k}'_\alpha(\omega'_\alpha)\right)\right]^* \right\} . \tag{19}$$

If T is very large, we may take the limit

$$\int_{-T/2}^{T/2} \exp[j(\omega_\alpha - \omega'_\alpha)t\ dt = 2\pi\delta(\omega_\alpha - \omega'_\alpha)\ , \tag{20}$$

and Eq. (19) becomes

$$\underset{\sim}{P}(\underset{\sim}{r}) = \lim_{T \to \infty} \frac{1}{2T(2\pi)^3} \int_{-\infty}^{\infty} d\omega_\alpha \frac{c^2}{r^2 \omega_\alpha^3 \epsilon_0 \epsilon_\alpha^2} \underset{\sim}{G}\left(\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)\right) \times \left[\underset{\sim}{k}_\alpha(\omega_\alpha) \times \underset{\sim}{G}\left(\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)\right)\right]^* . \tag{21}$$

Upon substitution of Eqs. (11) and (12), and simplification, we obtain finally

$$\underset{\sim}{P}(\underset{\sim}{r}) = \lim_{T \to \infty} \frac{1}{T(2\pi)^3} \int_0^\infty \frac{\epsilon_\alpha^{1/2} \omega_\alpha^2}{k_\alpha^4 r^2 c^3 \epsilon_0} \left| \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{J}_\alpha^{(2)}\left(\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)\right) \right|^2 \underset{\sim}{e}_r\ d\omega_\alpha\ . \tag{22}$$

An extra factor of 2 has appeared in Eq. (22) because the integration is carried out over positive frequencies only.

In what follows, it will prove more convenient to write Eq. (22) in its differential form

$$\frac{\partial^2 P(\underset{\sim}{r})}{\partial\Omega\, \partial\omega_\alpha} = \lim_{TV \to \infty} \frac{\epsilon_\alpha^{1/2}\, \omega_\alpha^2\, V}{(2\pi)^3 TV c^3 \epsilon_0 k_\alpha^4} \left| \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{J}_\alpha^{(2)}\left(\omega_\alpha, \underset{\sim}{k}_\alpha(\omega_\alpha)\right) \right|^2 \underset{\sim}{e}_r \ , \tag{23}$$

where Ω is the solid angle into which Wave α is scattered and V is the scattering volume.

Solution of Vlasov Equation

In the previous section, we derived an expression for the scattered power as a function of the source current $\underset{\sim}{J}_{\alpha}^{(2)}$. In this section, we determine the latter quantity. In our derivation, we assume an isotropic unperturbed electron velocity distribution function, and the absence of a static magnetic field. The Fourier transform of Eq. (3) has the form

$$j(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}) f_\alpha + \eta(\underset{\sim}{E}_\alpha + \underset{\sim}{v} \times \underset{\sim}{B}_\alpha) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} + \frac{\eta}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega_\delta \, d\omega_\epsilon \, d\underset{\sim}{k}_\delta \, d\underset{\sim}{k}_\epsilon (\underset{\sim}{E}_\delta + \underset{\sim}{v} \times \underset{\sim}{B}_\delta) \cdot \frac{\partial f_\epsilon}{\partial \underset{\sim}{v}}$$

$$\cdot \, \delta(\underset{\sim}{k}_\alpha - \underset{\sim}{k}_\delta - \underset{\sim}{k}_\epsilon) \delta(\omega_\alpha - \omega_\delta - \omega_\epsilon) , \qquad (24)$$

where f_0 is the unperturbed electron velocity distribution, and the subscripts α, δ, and ϵ refer to waves with frequency-wavenumber pairs $(\omega_\alpha, \underset{\sim}{k}_\alpha)$, $(\omega_\delta, \underset{\sim}{k}_\delta)$, and $(\omega_\epsilon, \underset{\sim}{k}_\epsilon)$, respectively. Since the incoming wave is plane and monochromatic, it has a spectrum of the form

$$\underset{\approx}{E} = (2\pi)^4 \underset{\sim}{E}_\beta \, \delta(\omega - \omega_\beta) \, \delta(\underset{\sim}{k} - \underset{\sim}{k}_\beta) , \qquad (25)$$

and Eq. (24) becomes

$$j(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}) f_\alpha + \eta(\underset{\sim}{E}_\alpha + \underset{\sim}{v} \times \underset{\sim}{B}_\alpha) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} + \sum_{\delta, \epsilon} \eta(\underset{\sim}{E}_\delta + \underset{\sim}{v} \times \underset{\sim}{B}_\delta) \cdot \frac{\partial f_\epsilon}{\partial \underset{\sim}{v}} = 0 , \qquad (26)$$

where δ, ϵ in the summation run over the values

$$\delta = \beta \; , \; \epsilon = \gamma \; , \qquad (27)$$

$$\delta = \gamma \; , \; \epsilon = \beta \; , \qquad (28)$$

and γ refers to the wave for which the synchronism conditions

$$\begin{aligned} \omega_\alpha &= \omega_\beta + \omega_\gamma \, , \\ \underset{\sim}{k}_\alpha &= \underset{\sim}{k}_\beta + \underset{\sim}{k}_\gamma \, , \end{aligned} \qquad (29)$$

hold.

We may solve Eq. (26) iteratively. The first order solution is given by

$$f_\alpha^{(1)} = \frac{j\eta \underline{E}_\alpha \cdot \partial f_{0e}/\partial \underset{\sim}{v}}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} + f_{u\alpha e} , \tag{30}$$

where $f_{u\alpha}$ is the fluctuating part of the solution [Kadomtsev, 1965], which vanishes when averaged over the macroscopic volume, and satisfies the equations

$$(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}) f_{u\alpha} = 0 , \tag{31}$$

$$\langle f_{u\alpha}(v) f_{u\alpha'}(\underset{\sim}{v}')\rangle = (2\pi)^5 \, \delta(\underset{\sim}{v}-\underset{\sim}{v}')\delta(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})\delta(\omega_\alpha - \omega_{\alpha'})\delta(\underset{\sim}{k}_\alpha - \underset{\sim}{k}_{\alpha'}) f_0(\underset{\sim}{v}) . \tag{32}$$

Substituting in Eq. (26), we obtain the second order solution

$$\begin{aligned} f_\alpha^{(2)} = {} & \frac{j\eta}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} \sum_{\delta,\epsilon} (\underline{E}_\delta + \underset{\sim}{v} \times \underline{B}_\delta) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \left(\frac{j\eta \underline{E}_\epsilon \partial f_{0e}/\partial \underset{\sim}{v}}{\omega_\epsilon - \underset{\sim}{k}_\epsilon \cdot \underset{\sim}{v}} \right) \\ & + \frac{j\eta}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} \sum_{\delta,\epsilon} (\underline{E}_\delta + \underset{\sim}{v} \times \underline{B}_\delta) \cdot \frac{\partial f_{u\epsilon}}{\partial \underset{\sim}{v}} . \end{aligned} \tag{33}$$

Second Order Source Currents

The first order currents, obtained by substitution of Eq. (30) into the expression

$$\underset{\sim}{J}_{\alpha}^{(1)} = e\int_{-\infty}^{\infty} f_{\alpha}^{(1)}\underset{\sim}{v}\, dv \ , \tag{34}$$

need not be considered further, since the contribution from the first term on the RHS of Eq. (30) has already been accounted for by Eq. (5); the second term does not contain $\underset{\sim}{E}_{\beta}$ as a factor, and therefore does not contribute to the scattering.

Substituting Eq. (33) into the expression

$$\underset{\sim}{J}_{\alpha}^{(2)} = e\int f_{\alpha}^{(2)}\ \underset{\sim}{v}\ d\underset{\sim}{v} \tag{35}$$

gives a second order current

$$\underset{\sim}{J}_{\alpha}^{(2)} = \underset{\sim}{J}_{s\alpha}^{(2)} + \underset{\sim}{J}_{u\alpha}^{(2)} \ , \tag{36}$$

where

$$\underset{\sim}{J}_{s\alpha}^{(2)} = -\sum_{\delta,\epsilon}\int_{-\infty}^{\infty} \frac{\eta^2 e}{\omega_\alpha - \underset{\sim}{k}_\alpha\cdot\underset{\sim}{v}}\ \underset{\sim}{v}\ d\underset{\sim}{v}\ (\underset{\sim}{E}_\delta + \underset{\sim}{v}\times\underset{\sim}{B}_\delta)\cdot\frac{\partial}{\partial\underset{\sim}{v}}\left(\frac{\underset{\sim}{E}_\epsilon\cdot\partial f_{0e}/\partial\underset{\sim}{v}}{\omega_\epsilon - \underset{\sim}{k}_\epsilon\cdot\underset{\sim}{v}}\right), \tag{37}$$

$$\underset{\sim}{J}_{u\alpha}^{(2)} = \sum_{\delta,\epsilon}\int_{-\infty}^{\infty} \frac{j\eta e}{\omega_\alpha - \underset{\sim}{k}_\alpha\cdot\underset{\sim}{v}}\ \underset{\sim}{v}\ d\underset{\sim}{v}\ (\underset{\sim}{E}_\delta + \underset{\sim}{v}\times\underset{\sim}{B}_\delta)\cdot\frac{\partial f_{u\epsilon e}}{\partial\underset{\sim}{v}}\ . \tag{38}$$

Source Current from Collective Effects

We will concentrate first on evaluating $\underset{\sim}{J}_{s\alpha}^{(2)}$, the source current due to collective effects. A partial integration reduces this to

$$\underset{\sim}{J}_{s\alpha}^{(2)} = \sum_{\delta,\epsilon} \eta^2 e \int_{-\infty}^{\infty} \left(\frac{\underset{\sim}{E}_\epsilon \cdot \partial f_{0e}/\partial \underset{\sim}{v}}{\omega_\epsilon - \underset{\sim}{k}_\epsilon \cdot \underset{\sim}{v}} \right) \left\{ \frac{(\underset{\sim}{E}_\delta + \underset{\sim}{v} \times \underset{\sim}{B}_\delta) \cdot \underset{\sim}{k}_\alpha}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \underset{\sim}{v} + \frac{\underset{\sim}{E}_\delta + \underset{\sim}{v} \times \underset{\sim}{B}_\delta}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} \right\} d\underset{\sim}{v} \ . \tag{39}$$

A second partial integration, followed by expansion of the summation according to Eqs. (27) and (28), yields

$$\begin{aligned}
\underset{\sim}{J}_{s\alpha}^{(2)} = - \eta^2 e \int_{-\infty}^{\infty} f_{0e} d\underset{\sim}{v} \Bigg\{ & \frac{\underset{\sim}{E}_\gamma}{\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}} \ \frac{(\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta) \cdot \underset{\sim}{k}_\alpha}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \qquad (40) \\
& + \frac{2\, \underset{\sim}{E}_\gamma \cdot \underset{\sim}{k}_\alpha}{\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}} \ \frac{(\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta) \cdot \underset{\sim}{k}_\alpha}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^3} \underset{\sim}{v} + \frac{\underset{\sim}{E}_\gamma \times \underset{\sim}{B}_\beta \cdot \underset{\sim}{k}_\alpha}{(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \underset{\sim}{v} \\
& + \frac{(\underset{\sim}{E}_\gamma \cdot \underset{\sim}{k}_\alpha)(\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta)}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2 (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} + \frac{\underset{\sim}{E}_\gamma \times \underset{\sim}{B}_\beta}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} \\
& + \frac{\underset{\sim}{k}_\gamma \cdot \underset{\sim}{E}_\gamma}{(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})^2} \left[\frac{(\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta) \cdot \underset{\sim}{k}_\alpha}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \underset{\sim}{v} + \frac{\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} \right] \\
& + \frac{\underset{\sim}{E}_\beta}{\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v}} \ \frac{\underset{\sim}{E}_\gamma \cdot \underset{\sim}{k}_\alpha}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} + \frac{2(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\beta)(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\gamma)}{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^3} \underset{\sim}{v} + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\beta) \underset{\sim}{E}_\gamma}{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \Bigg\} \ .
\end{aligned}$$

In obtaining this equation, we have used the relations

$$\underset{\sim}{k}_\alpha \cdot \underset{\sim}{E}_\alpha = 0 \ , \qquad \underset{\sim}{k}_\beta \cdot \underset{\sim}{E}_\beta = 0 \ , \qquad \underset{\sim}{B}_\gamma = 0 \ , \tag{41}$$

which follow from the transverse and longitudinal character of the linearized electromagnetic and electrostatic waves, respectively.

We will find it more convenient to write Eq. (39) in the form

$$\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{J}^{(2)}_{s\alpha} = - \frac{\eta^2 e E_\beta E_\gamma k_\gamma}{\omega_\alpha} \int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) dv \, \underset{\sim}{V}_s(\underset{\sim}{v}) \tag{42}$$

where

$$\underset{\sim}{V}_s(\underset{\sim}{v}) = \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \frac{\omega_\alpha}{k_\gamma^2} \left\{ \frac{\underset{\sim}{k}_\gamma [(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha) + (\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})]}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2 \, (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} \right.$$

$$+ 2 \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\gamma)[(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})(e_\beta \cdot \underset{\sim}{k}_\alpha) + (\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})]}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^3 (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} \underset{\sim}{v} + \frac{[(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha) - (\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)(\underset{\sim}{k}_\beta \cdot \underset{\sim}{k}_\gamma)]\underset{\sim}{v}}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2 \omega_\beta (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})}$$

$$+ \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\gamma)[(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})\underset{\sim}{e}_\beta + \underset{\sim}{k}_\beta (\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})]}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2 \omega_\beta (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} + \frac{[\underset{\sim}{k}_\beta (\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha) - \underset{\sim}{e}_\beta (\underset{\sim}{k}_\beta \cdot \underset{\sim}{k}_\gamma)]}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})(\omega_\gamma \cdot \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})}$$

$$+ \frac{k_\gamma^2}{(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})^2} \left[\frac{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha) + (\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \underset{\sim}{v} + \frac{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})\underset{\sim}{e}_\beta + (\underset{\sim}{e}_\beta \cdot \underset{\sim}{v}) \, \underset{\sim}{k}_\beta}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})} \right]$$

$$\left. + \frac{\underset{\sim}{e}_\beta}{\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v}} \quad \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\gamma)}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} + \frac{2(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)(\underset{\sim}{k}_\alpha \cdot k_\gamma)\underset{\sim}{v}}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^3 (\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})} + \frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha) \, \underset{\sim}{k}_\gamma}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2 (\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})} \right\} . \tag{43}$$

We know from the synchronism conditions [Eqs. (29)] that

$$(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v}) = (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}) - (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) \, . \tag{44}$$

$$\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha = \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times (\underset{\sim}{k}_\beta + \underset{\sim}{k}_\gamma) = 0 \, . \tag{45}$$

Substituting these into Eq. (43), and collecting terms, yields

$$
\begin{aligned}
\underset{\sim}{V}_s(\underset{\sim}{v}) = {} & \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \frac{\omega_\alpha}{\omega_\beta k_\gamma^2} \left\{ \frac{1}{(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} \left[\frac{[\underset{\sim}{k}_\alpha \cdot (\underset{\sim}{k}_\beta - \underset{\sim}{k}_\gamma)](\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \right. \right. \\
& \left. + \frac{k_\alpha^2 (\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)\underset{\sim}{v}}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} + 2\,\frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\gamma)(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{v}}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^3} \right] \\
& + \frac{k_\gamma^2}{(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})^2} \left[\frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)\underset{\sim}{v}}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{v}}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} + \underset{\sim}{e}_\beta - \frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} \right] \\
& \left. + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\gamma)(\underset{\sim}{k}_\beta \cdot \underset{\sim}{v})\underset{\sim}{e}_\beta}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})} + 2\,\frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\gamma)(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\gamma)(\underset{\sim}{k}_\beta \cdot \underset{\sim}{v})\underset{\sim}{v}}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})} + \frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\gamma)(\underset{\sim}{k}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})} \right\} .
\end{aligned}
\tag{46}
$$

Source Current from Discrete Particle Effects

The source current due to discrete particle effects is obtained in the same manner as the source current from collective effects. We expand the summation in Eq. (38), using only the term corresponding to Eq. (27); this is the only term which is dependent on the incoming electromagnetic wave, and therefore represents scattering. We obtain

$$\underset{\sim}{J}_{u\alpha}^{(2)} = \int_{-\infty}^{\infty} \frac{j\eta e}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} \underset{\sim}{v} \, d\underset{\sim}{v} \, (\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta) \cdot \frac{\partial f_{u\gamma e}}{\partial \underset{\sim}{v}} \, . \tag{47}$$

A partial integration reduces this to the form

$$\underset{\sim}{J}_{u\alpha}^{(2)} = - j\eta e \int_{-\infty}^{\infty} f_{u\gamma e} d\underset{\sim}{v} \left[\frac{\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta}{\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v}} + \frac{(\underset{\sim}{E}_\beta + \underset{\sim}{v} \times \underset{\sim}{B}_\beta) \cdot \underset{\sim}{k}_\alpha}{(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot v)^2} \underset{\sim}{v} \right] \, . \tag{48}$$

Here again we find it more useful to write this as

$$\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{J}_{u\alpha}^{(2)} = - \frac{j\eta e \, E_\beta}{\omega_\alpha} \int_{-\infty}^{\infty} f_{u\gamma e} \, d\underset{\sim}{v} \, \underset{\sim}{V}_u(\underset{\sim}{v}) \tag{49}$$

where

$$\underset{\sim}{V}_u(\underset{\sim}{v}) = \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \omega_\alpha \left\{ \frac{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v}) \underset{\sim}{e}_\beta + \underset{\sim}{k}_\beta (\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})} + \frac{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha) + (\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})}{\omega_\beta (\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})^2} \underset{\sim}{v} \right\} . \tag{50}$$

Scattering Formula:

We are now in a position to obtain the final scattering formula. Substituting Eqs. (42) and (49) into Eq. (23) gives the equation

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \lim_{TV\to\infty} \frac{\epsilon_\alpha^{1/2}\eta^2 e^2 |E_\beta|^2 V}{(2\pi)^3 TVc^3\epsilon_0 k_\alpha^4} \left| \left\{ \eta E_\gamma k_\gamma \int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) d\underset{\sim}{v}\, V_s(\underset{\sim}{v}) \right.\right. \tag{51}$$

$$\left.\left. + j\int f_{u\gamma e}\, d\underset{\sim}{v}\, \underset{\sim}{V}_u(\underset{\sim}{v}) \right|^2 \right\} .$$

Since the incoming flux is given by

$$S_\beta = 2\epsilon_0\, \epsilon_\beta^{1/2}\, c|E_\beta|^2 \quad , \tag{52}$$

and the classical electron radius by

$$r_0 = e^2/(4\pi\, \epsilon_0 mc^2) \quad , \tag{53}$$

the scattering formula can simply be written as

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \lim_{TV\to\infty} \frac{r_0^2\, S_\beta V}{\pi TV\, k_\alpha^4}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2} \left|\eta\, E_\gamma k_\gamma \int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) d\underset{\sim}{v}\, V_s(\underset{\sim}{v}) + j\int_{-\infty}^{\infty} f_{u\gamma e}\, d\underset{\sim}{v}\, \underset{\sim}{V}_u(\underset{\sim}{v})\right|^2 . \tag{54}$$

Expanding the squared term gives the equation

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \lim_{TV\to\infty} \frac{r_0^2\, S_\beta V}{\pi TV\, k_\alpha^4}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2} \left\{ \eta^2\, k_\gamma^2 [E_\gamma E_\gamma^*] \left|\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) \underset{\sim}{V}_s(\underset{\sim}{v}) d\underset{\sim}{v}\right|^2 \right.$$

$$- 2\,\mathrm{Re}\; j\eta k_\gamma \left(\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) d\underset{\sim}{v}\, V_s(\underset{\sim}{v})\right) \cdot \left(\int_{-\infty}^{\infty} \left[f_{u\gamma e}(\underset{\sim}{v})^*\, E_\gamma\right] \underset{\sim}{V}_u^*(\underset{\sim}{v}) d\underset{\sim}{v}\right) \tag{55}$$

$$\left. + \int_{-\infty}^{\infty} d\underset{\sim}{v}\, d\underset{\sim}{v}' \left[f_{u\gamma e}(\underset{\sim}{v})\, f_{u\gamma e}^*(\underset{\sim}{v}')\right] \underset{\sim}{V}_u(\underset{\sim}{v}) \cdot \underset{\sim}{V}_u^*(\underset{\sim}{v}') \right\} .$$

This equation is the general scattering formula we have sought to derive. If one knows the spectrum corresponding to $|E_\gamma|^2$, $f^*_{u\gamma e} E_\gamma$, and $|f^*_{u\gamma e}|^2$, and the unperturbed velocity distributions then the scattered power is determined.

Incoherent Scatter

We will now take up the case of incoherent scatter, where it is possible to evaluate Eq. (55) explicitly. In this case the assumption that the charged particle motions are random allows one to evaluate the products of the fluctuating quantities in the equation. In the appendix we show that these products are given by

$$\lim_{TV \to \infty} \frac{1}{TV} f_{u\gamma e}(\underset{\sim}{v}) f^*_{u\gamma e}(\underset{\sim}{v}') = 2\pi\delta(\underset{\sim}{v}-\underset{\sim}{v}')\delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) f_{0e}(\underset{\sim}{v}) \ , \qquad (56)$$

$$\lim_{TV \to \infty} \frac{1}{TV} f^*_{u\gamma e}(\underset{\sim}{v}) E_\gamma = \frac{2\pi e j}{\epsilon_0 \epsilon_\gamma k_\gamma} f_{0e}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) \ , \qquad (57)$$

$$\lim_{TV \to \infty} \frac{1}{TV} |E_\gamma|^2 = \frac{2\pi e^2}{\epsilon_0^2 |\epsilon_\gamma|^2 k_\gamma^2} \left[\int_{-\infty}^{\infty} d\underset{\sim}{v} \ f_{0e}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) + \int_{-\infty}^{\infty} d\underset{\sim}{v} f_{0i}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) \right] . \qquad (58)$$

Substituting these expressions, and integrating over $\underset{\sim}{v}'$, reduces Eq. (55) to the form

$$\frac{\partial^2 P}{\partial\Omega \partial\omega_\alpha} = 2r_0^2 \, S_\beta V \left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2} \frac{1}{k_\alpha^4} \left\{ (\underset{\sim}{L}_1 \cdot \underset{\sim}{L}_1^*) \left[\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) d\underset{\sim}{v} + \int_{-\infty}^{\infty} f_{0i}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) d\underset{\sim}{v} \right] \right.$$

$$\left. + 2 \, \mathrm{Re} \, \underset{\sim}{L}_1 \cdot \underset{\sim}{L}_2 + \underset{\sim}{L}_3 \right\} , \qquad (59)$$

where

$$\underset{\sim}{L}_1 = \frac{\eta e}{\epsilon_0 \epsilon_\gamma} \int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) \underset{\sim}{V}_s(\underset{\sim}{v}) d\underset{\sim}{v} \ , \qquad (60)$$

$$\underset{\sim}{L}_2 = \int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) \delta(\omega_0 - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) \underset{\sim}{V}^*_u(\underset{\sim}{v}) dv \ , \qquad (61)$$

$$\underset{\sim}{L}_3 = \int_{-\infty}^{\infty} d\underset{\sim}{v} \ f_{0e}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) \underset{\sim}{V}_u(\underset{\sim}{v}) \cdot \underset{\sim}{V}_u(\underset{\sim}{v})^* . \qquad (62)$$

Because of the delta function and Eq. (44), we can replace the factor $(\omega_\alpha - \underset{\sim}{k}_\alpha \cdot \underset{\sim}{v})$ by $(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})$ in the definitions for $\underset{\sim}{L}_2$ and $\underset{\sim}{L}_3$. Carrying this out, along with the application of Eq. (45), yields the simpler equations

$$\underset{\sim}{L}_2 = \int_{-\infty}^{\infty} d\underset{\sim}{v}\; f_{0e}(\underset{\sim}{v})\,\delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})\; \underset{\sim}{W}_u(\underset{\sim}{v}) \;, \tag{63}$$

$$\underset{\sim}{L}_3 = \int_{-\infty}^{\infty} d\underset{\sim}{v}\; f_{0e}(\underset{\sim}{v})\,\delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})\,\underset{\sim}{W}_u(\underset{\sim}{v}) \cdot \underset{\sim}{W}_u^*(\underset{\sim}{v}) \;, \tag{64}$$

where

$$\underset{\sim}{W}_u = \frac{\omega_\alpha}{\omega_\beta}\, \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \left\{ \underset{\sim}{e}_\beta + \frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)\underset{\sim}{v} - (\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma}{\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v}} + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{v}}{(\omega_\beta - \underset{\sim}{k}_\beta \cdot \underset{\sim}{v})^2} \right\} . \tag{65}$$

Equation (59), along with the definition given by Eqs. (46), (60), (63-65), is our final result describing scattering by random density fluctuations.

High Frequency Expansion for Incoherent Scatter

Equation (59) is a final result in the sense that it specifies completely the scattered power once the unperturbed velocity distribution functions are known. It will be useful, however, to expand this equation in powers of $1/\omega_\beta$, in order to interpret the meaning of the result, and to compare with previous work.

Let us first expand $\underset{\sim}{L}_1$ in Eq. (60) to second order in ω_β^{-1}. This yields, upon application of Eq. (44), the relation

$$\underset{\sim}{V}_s(\underset{\sim}{v}) = \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \frac{\omega_\alpha}{\omega_\beta k_\gamma^2}\left\{ \frac{[\underset{\sim}{k}_\alpha(\underset{\sim}{k}_\beta - \underset{\sim}{k}_\gamma)](\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma}{\omega_\beta^2(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} + \frac{k_\alpha^2(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)\underset{\sim}{v}}{\omega_\beta^2(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})} \right.$$

$$+ \frac{k_\gamma^2}{(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})^2}\left[\left(\frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)\underset{\sim}{v} - (\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma}{\omega_\beta} \right)\left(1 + \frac{\underset{\sim}{k}_\beta \cdot \underset{\sim}{v}}{\omega_\beta} - \frac{\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}}{\omega_\beta} \right) \right.$$

$$\left.\left. + \underset{\sim}{e}_\beta + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{v}}{\omega_\beta^2} \right] \right\} . \tag{66}$$

If we substitute into this equation the vector identity

$$\underset{\sim}{v} = \frac{1}{k_\gamma^2}\left[\underset{\sim}{k}_\gamma(\underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) - \underset{\sim}{k}_\gamma \times \underset{\sim}{k}_\gamma \times \underset{\sim}{v}\right] , \tag{67}$$

and collect terms, we obtain

$$\underset{\sim}{V}_s(\underset{\sim}{v}) = \frac{\omega_\alpha}{\omega_\beta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})^2}\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \left\{ \underset{\sim}{e}_\beta + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)}{k_\gamma^4\,\omega_\beta^2}\left[\omega_\gamma^2 - (\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})^2\right]\underset{\sim}{k}_\gamma \right\}$$

$$+ \text{ terms in } (\underset{\sim}{k}_\gamma \times \underset{\sim}{k}_\gamma \times \underset{\sim}{v}) . \tag{68}$$

Finally, substituting into Eq. (60) gives the result

$$\underset{\sim}{L}_1 = -\frac{\omega_\alpha}{\omega_\beta \epsilon_\gamma} \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \left[\chi_{e\gamma} \left\{ \underset{\sim}{e}_\beta + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)}{k_\gamma^4} \left(\frac{\omega_\gamma}{\omega_\beta}\right)^2 \underset{\sim}{k}_\gamma \right\} + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)}{k_\gamma^4} \left(\frac{\omega_p}{\omega_\beta}\right)^2 \underset{\sim}{k}_\gamma \right], \tag{69}$$

where

$$\chi = -\frac{\eta e}{\epsilon_0} \int_{-\infty}^{\infty} \frac{f_0(\underset{\sim}{v})d\underset{\sim}{v}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2} \tag{70}$$

is the charged particle susceptibility. In obtaining this result we have ignored the terms in $\underset{\sim}{V}_s(\underset{\sim}{v})$ containing $(\underset{\sim}{k}_\gamma \times \underset{\sim}{k}_\gamma \times \underset{\sim}{v})$, since these give rise to terms of order $(v_e/c)^2$, and higher, when dealing with isotropic electron velocity distribution functions.

Let us now expand $\underset{\sim}{L}_2$ in Eq. (63). To second order in ω_β^{-1} we obtain

$$\underset{\sim}{L}_2 = \int d\underset{\sim}{v}\, f_{0e}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) \frac{\omega_\alpha}{\omega_\beta} \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \left\{ \underset{\sim}{e}_\beta + \left[(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)\underset{\sim}{v} - (\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{k}_\gamma \right] \left[\frac{1}{\omega_\beta} + \frac{\underset{\sim}{k}_\beta \cdot \underset{\sim}{v}}{\omega_\beta^2} \right] + \frac{(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta \cdot \underset{\sim}{v})\underset{\sim}{v}}{\omega_\beta^2} \right\}. \tag{71}$$

We again expand $\underset{\sim}{v}$ according to Eq. (67), and ignore the component perpendicular to $\underset{\sim}{k}_\gamma$. This yields, after replacing $(\underset{\sim}{k}_\gamma \cdot \underset{\sim}{v})$ by ω_γ, the result

$$\underset{\sim}{L}_2 = \left(\frac{\omega_\alpha}{\omega_\beta}\right) \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \left[\underset{\sim}{e}_\beta + \left(\frac{\omega_\gamma}{\omega_\beta}\right)^2 \frac{(\underset{\sim}{e}_\beta \cdot \underset{\sim}{k}_\alpha)(\underset{\sim}{k}_\alpha \cdot \underset{\sim}{k}_\beta)\underset{\sim}{k}_\gamma}{k_\gamma^4} \right] \int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v}) \delta(\omega_\gamma - \underset{\sim}{k}_\gamma \cdot \underset{\sim}{v}) d\underset{\sim}{v}\,. \tag{72}$$

A similar procedure for $\underset{\sim}{L}_3$ yields the expression

$$\underset{\sim}{L}_3 = \left(\frac{\omega_\alpha}{\omega_\beta}\right)^2 (\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{e}_\beta)\cdot\left(\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \left[\underset{\sim}{e}_\beta + 2\left(\frac{\omega_\gamma}{\omega_\beta}\right)^2 \frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)\underset{\sim}{k}_\gamma}{k_\gamma^4}\right]\right)$$

$$\cdot\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\ (\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\ . \qquad (73)$$

We now substitute the expressions for $\underset{\sim}{L}_1$, $\underset{\sim}{L}_2$ and $\underset{\sim}{L}_3$ into Eq. (59) to obtain

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = r_0^2 S_\beta V \left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2} \frac{2\omega_\alpha^2}{|\epsilon_\gamma|^2 k_\alpha^4 \omega_\beta^2} (\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{e}_\beta)\cdot\left\{\left[\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \left(\underset{\sim}{e}_\beta + 2\frac{(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)}{k_\gamma^4}\frac{\omega_\gamma^2}{\omega_\beta^2}\underset{\sim}{k}_\gamma\right)\right]\right.$$

$$\cdot\left[\left(|\chi_{e\gamma}|^2 - 2\,\mathrm{Re}\,\chi_{e\gamma}\,\epsilon_\gamma^* + |\epsilon_\gamma|^2\right)\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v} + |\chi_{e\gamma}|^2\int f_{0i}(\underset{\sim}{v})\delta(\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right]$$

$$+ 2(\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\gamma)\ \frac{(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)}{k_\gamma^4}\frac{\omega_p^2}{\omega_\beta^2}\left[\mathrm{Re}\,\chi_{e\gamma}\int_{-\infty}^{\infty} f_{0i}(\underset{\sim}{v})\delta(\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right.$$

$$\left.\left. + \mathrm{Re}(\chi_{e\gamma} - \epsilon_\gamma)\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right]\right\}. \qquad (74)$$

Using the identities

$$\epsilon_\gamma = 1 + \chi_{e\gamma} + \chi_{i\gamma}\ , \qquad (75)$$

$$|\epsilon_\gamma|^2 - 2\,\mathrm{Re}\,\epsilon_\gamma\cdot\chi_{e\gamma}^* + |\chi_{e\gamma}|^2 = |1 + \chi_{i\gamma}|^2\ , \qquad (76)$$

and rearranging, gives the final form for the expanded incoherent scattering formula as

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \frac{2r_0^2 S_\beta V}{|\epsilon_\gamma^2|}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2}\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2 \left\{\left[|1+\chi_{i\gamma}|^2\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right.\right. \tag{77}$$

$$+\left.|\chi_{e\gamma}|^2\int_{-\infty}^{\infty} f_{0i}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right]\left[1-\frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2}{k_\alpha^2}+2\frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)^2}{k_\alpha^2\, k_\gamma^4}\frac{\omega_\gamma^2}{\omega_\beta^2}\right]$$

$$\left.+\,2\left[\mathrm{Re}\,\chi_{e\gamma}\int_{-\infty}^{\infty} f_{0i}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}-\mathrm{Re}(1+\chi_{i\gamma})\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right]\frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)^2}{k_\alpha^2\, k_\gamma^4}\frac{\omega_p^2}{\omega_\beta^2}\right\}.$$

When the frequency of the incident electromagnetic wave, ω_β, becomes very large compared to ω_γ and ω_p, the high frequency incoherent scattering formula [Bekefi, 1966],

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \frac{2r_0^2 S_\beta V}{|\epsilon_\alpha|^2}\left[|1+\chi_{i\gamma}|^2\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right. \tag{78}$$

$$+\left.|\chi_{e\gamma}|^2\int_{-\infty}^{\infty} f_{0i}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right]\left[1-\frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2}{k_\alpha^2}\right],$$

is retrieved. In the case of backscatter, where $(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)=0$, Eq. (77) reduces to the even simpler form

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \frac{2r_0^2 S_\beta V}{|\epsilon_\alpha|^2}\left[|1+\chi_{i\gamma}|^2\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}+|\chi_{e\gamma}|^2\int_{-\infty}^{\infty} f_{0i}(\underset{\sim}{v})\delta(\omega_\gamma-\underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})d\underset{\sim}{v}\right]. \tag{79}$$

Scattering in Case of Strongly Driven Plasma Waves

In the case where the coherent waves are so strongly driven by an external source that the random fluctuations of the charged particles can be ignored, we may ignore the terms involving $f_{u\gamma e}$ in Eq. (55), and the scattering is then given simply by

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \frac{r_0^2 S_\beta V}{\pi\, k_\alpha^4}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2} \eta^2\, k_\gamma^2 \left|\int_{-\infty}^{\infty} f_{0e}(\underset{\sim}{v})\underset{\sim}{V}_s(\underset{\sim}{v})\,d\underset{\sim}{v}\right|^2 \lim_{TV\to\infty} \frac{|E_\gamma|^2}{TV} . \tag{80}$$

As in the case of incoherent scatter, it is useful to determine the behavior for a high frequency incident wave. Substituting Eqs. (60) and (69) into this formula reduces it to the form

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \frac{r_0^2 S_\beta V}{\pi}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{1/2} \frac{k_\gamma^2\epsilon_0^2}{e^2}\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2 \left\{|\chi_{e\gamma}|^2\left[1 - \frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2}{k_\alpha^2} + 2\,\frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)^2}{k_\alpha^2\, k_\gamma^4}\left(\frac{\omega_\gamma}{\omega_\beta}\right)^2\right]\right.$$

$$\left. + 2\,\mathrm{Re}\,\chi_{e\gamma}\left[\frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2(\underset{\sim}{k}_\alpha\cdot\underset{\sim}{k}_\beta)^2}{k_\alpha^2\, k_\gamma^4}\left(\frac{\omega_p}{\omega_\beta}\right)^2\right]\right\} \lim_{TV\to\infty}\frac{|E_\gamma|^2}{TV} . \tag{81}$$

If we let $\omega_\beta \to \infty$ and note that

$$|n_\gamma|^2 = \left|\frac{\epsilon_0\chi_{e\gamma}k_\gamma E_\gamma}{e}\right|^2 , \tag{82}$$

we obtain the standard high frequency formula [Bekefi, 1966] given by

$$\frac{\partial^2 P}{\partial\Omega\partial\omega_\alpha} = \frac{r_0^2 S_\beta V}{\pi}\left[1 - \frac{(\underset{\sim}{e}_\beta\cdot\underset{\sim}{k}_\alpha)^2}{k_\alpha^2}\right] \lim_{TV\to\infty}\frac{|n_\gamma|^2}{TV} . \tag{83}$$

The $(\epsilon_\alpha/\epsilon_\beta)^{1/2}$ correction to Eq. (83), contained in Eq. (80), was previously derived by Birmingham et al. [1965] by a different method.

Summary

A general theory for scattering of electromagnetic waves by density fluctuations in a plasma has been carried out. The general scattering formula is given by Eqs. (55), (46), and (50). Its application to incoherent scatter is given by Eqs. (59), (46), (60), (63-65), and to scatter by strongly driven plasma waves by Eqs. (46) and (80). The theory generalizes previous high-frequency theories in that it is valid for all frequencies of the incident and scattered electromagnetic waves. It does assume, however, a zero magnetic field, isotropic unperturbed charged particle velocity distribution functions, and the absence of multiple scattering.

An expansion for both incoherent scatter and scattering by strongly driven plasma waves in inverse powers of the frequency, ω_β^{-1}, of the incident electromagnetic wave has been carried out [Eqs. (77) and (81) respectively]. These expansions show that two types of lower order corrections must be applied to the high frequency theory as the incident electromagnetic wave frequency approaches the plasma frequency. The first type of correction is of order $(\omega_p/\omega_\beta)^2$, and must be applied irrespective of the value of the difference frequency, ω_γ, between the electromagnetic waves. The second correction is of order $(\omega_\gamma/\omega_\beta)^2$, and is clearly of importance only for scattering by the Langmuir waves. These lower order corrections disappear for the case of backscatter.

As the frequency, ω_β, of the electromagnetic wave comes closer to ω_p, then of course it is necessary to use the full theory [Eqs. (59), (46), (60), and (63-65), or Eqs. (46) and (80)]. It is important to note that the full theory has non-vanishing higher order corrections for the backscatter case, even though the lower order corrections mentioned in the previous paragraph disappear.

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Appendix

In this section will be derived the expressions for the space-time averages given in Eqs. (56-58). Our first step is to derive a relation for the averages in terms of the ensemble average of the Fourier components. According to Parseval's theorem, the average of the product of two variables, $A(\underset{\sim}{r},t)$ and $B(\underset{\sim}{r},t)$, is given by

$$\overline{A(\underset{\sim}{r},t)B(\underset{\sim}{r},t)} = \lim_{TV \to \infty} \frac{1}{(2\pi)^4 TV} \int_{-\infty}^{\infty} A_\gamma B^*_\gamma \, d\omega_\gamma \, d\underset{\sim}{k}_\gamma \ . \tag{A.1}$$

The ensemble average, on the other hand, is given by

$$\langle A(\underset{\sim}{r},t)B(\underset{\sim}{r},t)\rangle = \frac{1}{(2\pi)^8} \int_{-\infty}^{\infty} d\omega_\gamma \, d\omega_{\gamma'} \, d\underset{\sim}{k}_\gamma \, d\underset{\sim}{k}_{\gamma'} \langle A_\gamma B^*_{\gamma'}\rangle \exp[j(\omega_\gamma - \omega_{\gamma'})t - j(\underset{\sim}{k}_\gamma - \underset{\sim}{k}_{\gamma'})\cdot\underset{\sim}{r}] . \tag{A.2}$$

All of the cases studied in this paper have the property that

$$\langle A_\gamma B^*_{\gamma'}\rangle = C_\gamma(AB^*)\delta(\omega_\gamma - \omega_{\gamma'})\delta(\underset{\sim}{k}_\gamma - \underset{\sim}{k}_{\gamma'}) \tag{A.3}$$

and therefore

$$\langle A(\underset{\sim}{r},t)B(\underset{\sim}{r},t)\rangle = \frac{1}{(2\pi)^8} \int_{-\infty}^{\infty} d\omega_\gamma d\underset{\sim}{k}_\gamma \ C_\gamma \ (AB^*) \ . \tag{A.4}$$

Equating the two averages given by Eqs. (A.1) and (A.4) yields the desired relation

$$\lim_{TV \to \infty} \frac{1}{TV} \ A_\gamma B^*_\gamma = \frac{1}{(2\pi)^4} \ C_\gamma \ (AB^*) \ . \tag{A.5}$$

Equation (32) shows that

$$C_\gamma[f_u(\underset{\sim}{v})f^*_u(\underset{\sim}{v}')] = (2\pi)^5\delta(\underset{\sim}{v}-\underset{\sim}{v}')\delta(\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v})f_0(\underset{\sim}{v}) \ . \tag{A.6}$$

Substituting this into Eq. (A.5), with A and B equal to $f_u(\underset{\sim}{v})$ and $f_u(\underset{\sim}{v}')$, respectively, yields Eq. (56) immediately.

In order to prove Eqs. (57) and (58), we will need to use the linearized Poisson equation

$$\epsilon_\gamma k_\gamma E_\gamma = \frac{je}{\epsilon_0}\left[\int_{-\infty}^{\infty} f_{u\gamma e}(\underset{\sim}{v})d\underset{\sim}{v} + \int_{-\infty}^{\infty} f_{u\gamma i}(\underset{\sim}{v})d\underset{\sim}{v}\right] . \tag{A.7}$$

If we multiply this equation by $f^*_{u\gamma' e}(v')$, take the ensemble average, and assume that the ion and electron motions are uncorrelated, we obtain

$$\epsilon_\gamma k_\gamma \langle f^*_{u\gamma' e}(v')E_\gamma\rangle = \frac{je}{\epsilon_0}\int_{-\infty}^{\infty} \langle f^*_{u\gamma' e}(v')f_{u\gamma e}(v)\rangle dv . \tag{A.8}$$

Substituting Eq. (32) shows that

$$C_\gamma[E\, f^*_{ue}(v)] = \frac{(2\pi)^5 je}{\epsilon_0\epsilon_\gamma k_\gamma} f_{0e}(\underset{\sim}{v})\delta(\omega_\gamma - \underset{\sim}{k}_\gamma\cdot\underset{\sim}{v}) . \tag{A.9}$$

Substituting this result into Eq. (A.5), with A and B equal to E and f^*_{ue}, respectively, yields Eq. (57).

If we multiply Eq. (A.7) by its complex conjugate, and take the ensemble average, we obtain

$$\langle E_\gamma E^*_{\gamma'}\rangle = \frac{e^2}{k_\gamma k_{\gamma'}\epsilon_0^2\epsilon_\gamma\epsilon^*_{\gamma'}}\left[\int_{-\infty}^{\infty} \langle f_{ue\gamma}(\underset{\sim}{v})f_{ue\gamma'}(\underset{\sim}{v}')\rangle d\underset{\sim}{v}\, d\underset{\sim}{v}' + \int_{-\infty}^{\infty} \langle f_{ui\gamma}(\underset{\sim}{v})f_{ui\gamma}(\underset{\sim}{v}')\rangle d\underset{\sim}{v}\, d\underset{\sim}{v}'\right] . \tag{A.10}$$

Substituting Eq. (32) allows us to determine $C_\gamma(EE^*)$. If this is in turn substituted into Eq. (A.5), with A and B equal to E, we obtain Eq. (58).

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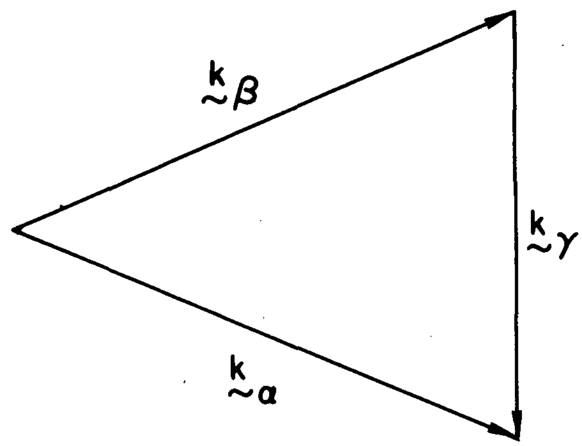

FIG. 1. Mixing of an incoming transverse wave ($\underset{\sim}{k}_\beta$) and an electrostatic wave ($\underset{\sim}{k}_\gamma$) to produce a scattered transverse wave ($\underset{\sim}{k}_\alpha$).

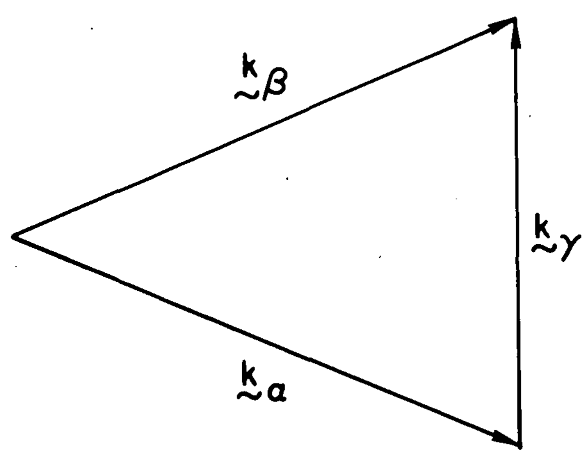

FIG. 2. Decay of an incoming transverse wave ($\underset{\sim}{k}_\beta$) into an electrostatic wave ($\underset{\sim}{k}_\gamma$) and a scattered transverse wave ($\underset{\sim}{k}_\alpha$).

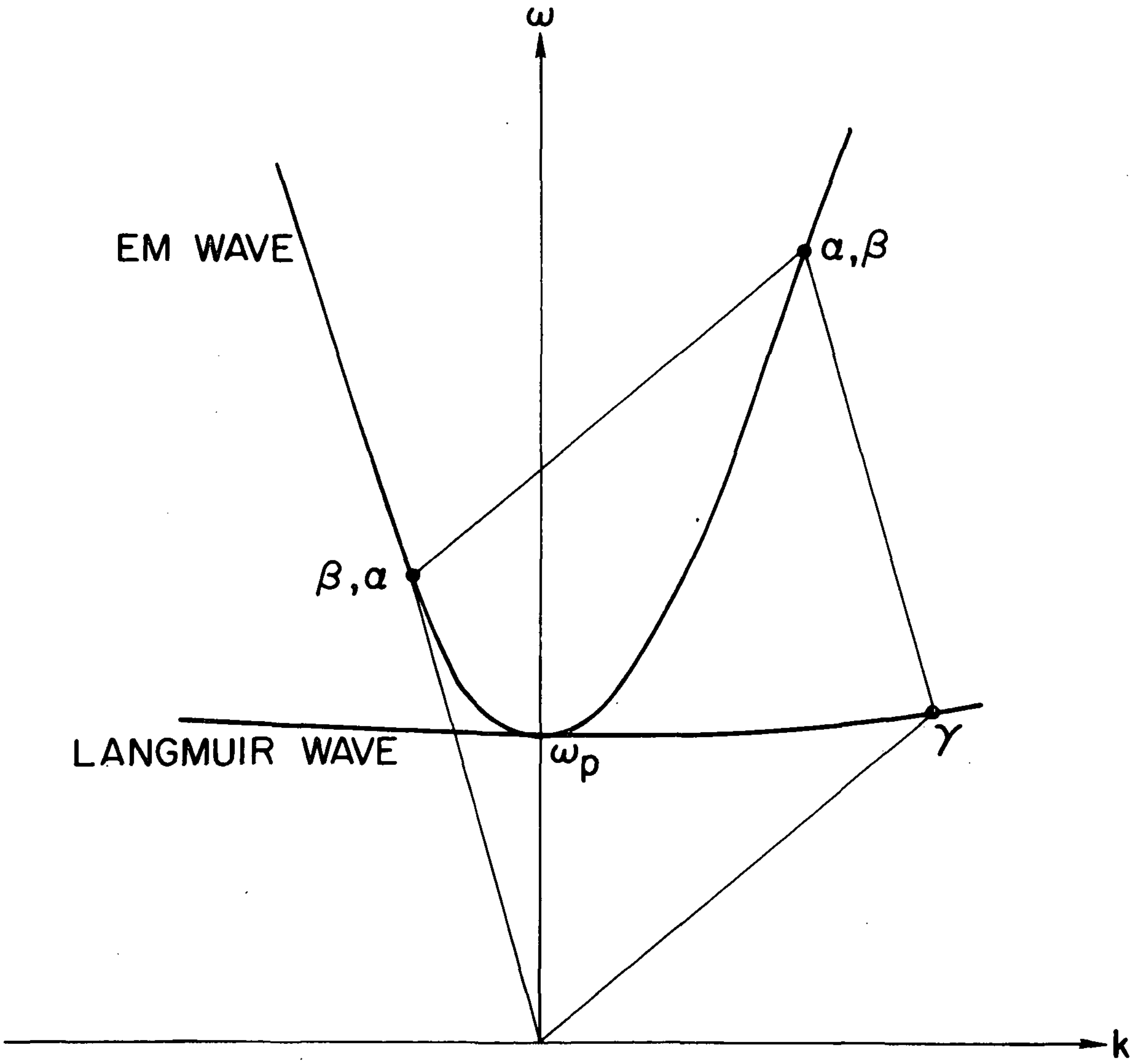

FIG. 3. Synchronism diagram for the interaction of two transverse waves and a Langmuir wave.

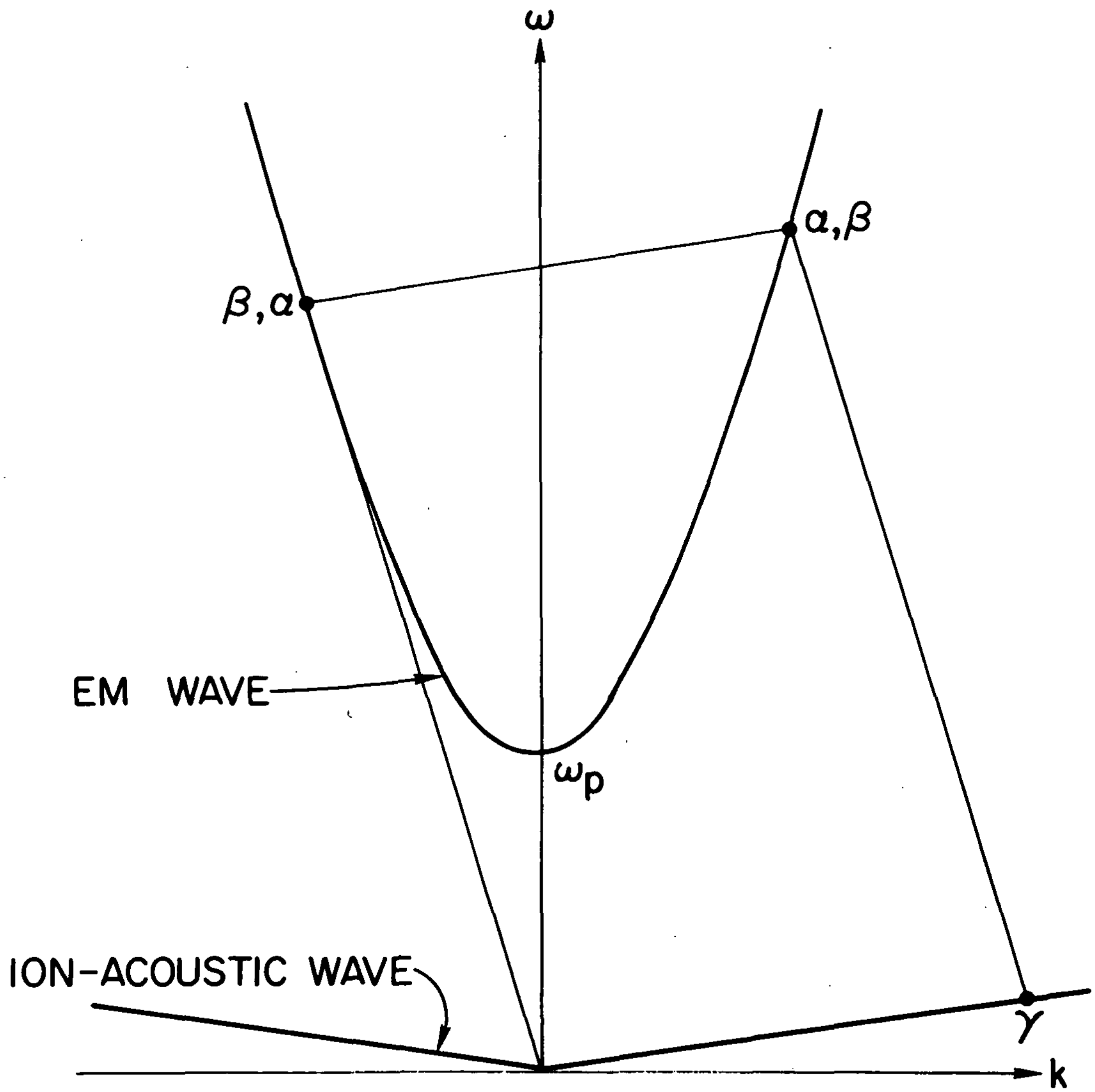

FIG. 4. Synchronism diagram for the interaction of two transverse waves and an ion-acoustic wave.

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13. ABSTRACT

The theory of scattering by charged particle density fluctuations of a plasma is developed for the case of zero magnetic field. The source current is derived on the basis of, first, a three-wave interaction between the incident and scattered electromagnetic waves and one electrostatic plasma wave (either Langmuir or ion-acoustic), and second, a synchronous interaction between the same two electromagnetic waves and the discrete components of the charged particle fluctuations. Previous work is generalized by no longer making the assumption that the frequency of the electromagnetic waves is large compared to the plasma frequency. The general result is then applied to incoherent scatter, and to scatter by strongly driven plasma waves. An expansion is carried out for each of those cases to determine the lower order corrections to the usual high frequency scattering formulas.

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