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Suboptimal Stochastic Controller for an n-Body Spacecraft

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PREFACE

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.

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Abstract

Considerable attention, in the open literature, is being focused on the problem of developing a suitable set of deterministic dynamical equations for a complex spacecraft. This paper addresses the problem of determining a stochastic optimal controller for an n-body spacecraft. The approach used in obtaining the stochastic controller involves the application, interpretation, and combination of advanced dynamical principles and the theoretical aspects of modern control theory. The stochastic controller obtained herein for a complicated model of a spacecraft

1) Uses sensor angular measurements associated with the base body to obtain smoothed estimates of the entire state vector

2) Can be easily implemented

3) Enables system performance to be significantly improved.

INTRODUCTION

For deep-space missions, the requirements placed on antenna pointing, articulation control, science platform settling times, etc. tend to continually become more stringent. Moreover, to meet the objectives of the scientific experiments and to provide isolation from the radiation produced by the power source, booms are frequently employed. These facts dictate that a suitable spacecraft model must be determined and, in addition, that sensor noise and plant disturbances be accounted for!

Considerable attention, in the open literature, has been focused on the problem of developing a suitable set of deterministic dynamical equations for a spacecraft ([1] through [5]). Recently, a particularly elegant albeit complicated set of dynamical equations for an n-hinged rigid-body spacecraft has been developed [2]. The salient features of this set of dynamical equations are (1) constraint torques do not appear, and (2) the number of variables involved is equal to the number of degrees of freedom of the system. Stochastic control theory has also been given special attention in the open literature. Ref. [6] is devoted exclusively to linear stochastic optimal control of linear systems subject to the average value of a quadratic cost functional. Stochastic optimal control theory has been successfully applied to such analytical problems as

1) Obtaining a stochastic controller for a personalized rapid transit system [7]

2) Obtaining a stochastic controller for achieving docking between an Orbit-to-Orbit Shuttle (OOS) and a malfunctioned satellite [8]

3) Obtaining a stochastic controller for a single rigid-body model of a spacecraft [11].

In the present work, the objective is to determine a controller which makes use of the elaborate, deterministic model of the spacecraft and, in addition, accounts for sensor noise, disturbances, etc. In essence, an optimal stochastic controller is sought. However, because of the practical importance of ease of implementation, simplicity, and reliability, a suboptimal stochastic controller is determined.

In obtaining the dynamical model for the spacecraft it is assumed that (see Fig. 1):

1) The spacecraft (s/c) can be adequately modeled as n-hinged rigid bodies (with r degrees-of-freedom for the entire system)
2) Chains of connected bodies do not form closed loops
3) Only rotational motion at a joint is allowed
4) A joint can be dissipative and elastic
5) There is a vector constraint torque normal to the axis of rotation at a joint whenever the rotational motion has only one or two degrees of freedom.

The approach used in this paper for obtaining the suboptimal stochastic controller for an n-hinged rigid body spacecraft involves the application, interpretation, and combination of advanced dynamical principles and the theoretical aspects of modern control theory. It is known that a solution can be found to a linear stochastic optimization problem involving a quadratic cost functional ([9] and [10]). However, the plant representing the dynamics of the spacecraft is nonlinear. In addition, a single quadratic cost functional which accounts for all of the desired characteristics of the controller cannot be found. Nevertheless, a suboptimal stochastic controller is obtained by:

1) Appropriately linearizing the dynamical model of the spacecraft to obtain the plant
2) Determining an appropriate model for the measurement process
3) Invoking the "certainty-equivalence" principle of modern control theory
4) Generating the suboptimal controller by passing the optimal controller for the linear stochastic problem through appropriate and desirable nonlinearities.

The contributions of this paper include the casting of the dynamical equations for the spacecraft in a form suitable for optimal stochastic control theory and the development of a suboptimal stochastic controller. To the writer's knowledge, a stochastic controller based on a realistic model of a complex spacecraft has not been previously obtained. This paper shows how knowledge of an elaborate model of the spacecraft can be effectively used in the design of a suboptimal stochastic controller to improve performance.†

DYNAMICAL EQUATIONS FOR AN n-HINGED RIGID-BODY SPACECRAFT

In this section, the dynamical equations for an n-hinged rigid-body spacecraft are provided. Emphasis is placed on the procedure used to obtain the results rather than on a detailed and lengthy derivation of the results. Consider an n-hinged rigid-body spacecraft having r degrees of freedom. The

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† The stochastic controller presented in this paper is to be used in analyzing the cruise, the thrust vector control (TVC), and articulation control (ARTC) modes of the Mariner Jupiter Saturn (MJS'77) Spacecraft.

number of scalar constraint torques for such a system is $n_c = 3n - r$. A set of r dynamical equations in which the constraint torques do not appear is given below. The pivotal steps involved in obtaining this canonical set of equations involve (see [2], [3] and [5]):

1) Recognizing that if the vector dynamical equations of all the bodies are summed, then the constraint torques cancel in pairs

2) Noting that a vector constraint torque at a typical joint j can be isolated by summing the vector dynamical equations over all bodies that lie to one side of joint j (the constraint torques on this set of bodies all cancel in pairs, except for the one at joint j)

3) Observing that the constraint torque (isolated in step 2) at joint j is orthogonal to the gimbal axis at joint j.

Effectively, 3 scalar equations result from the projection of the vector equations summed over all the bodies onto a suitable reference frame. Moreover, r - 3 additional scalar equations result from the dot products of the r - 3 gimbal axes and the constraint torques associated with these axes. The salient advantage associated with the elimination of the constraint torques is the accompanying reduction of the computer time required for integrating the equations (this is especially true for large n)!

The procedure used to arrive at the r scalar equations entails the following steps:

1) Writing Newton's and Euler's equations for each body λ

2) Eliminating the unknown interaction force F_{λ_j}

3) Evaluating the term

$$\sum_{j \epsilon J_\lambda} c_{\lambda_j} \times F_{\lambda_j}$$

which represents the sum of the moments about the center of mass of body λ due to interaction forces F_{λ_j} existing at joint j

4) Interpreting Euler's equations for body λ (after using the results of step 3)) as the equations for the augmented body λ relative to its barycenter B_λ

5) Expressing the interaction moment M_{λ_j} acting at joint j on body λ as a sum of a constraint torque $M^C_{\lambda_j}$ and a spring-damper torque $M^{SD}_{\lambda_j}$, i.e.,

$$M_{\lambda_j} = M^C_{\lambda_j} + M^{SD}_{\lambda_j} = M^C_{\lambda_j} + \tau_{\lambda_j}$$

6) Recognizing that if the vector dynamical equations for the augmented bodies λ are summed over all λ, then the constraint torques cancel in pairs and consequently disappear, i.e.,

$$\sum_{\lambda \epsilon S} \sum_{j \epsilon J_\lambda} M^C_{\lambda_j} = 0$$

7) Recognizing that the constraint torque at joint j acting on body λ can be isolated by summing over all bodies to one side of joint j

8) Recognizing that the gimbal axis g_j is orthogonal to the constraint torque $M^C_{\lambda_j}$ at joint j.

In vector-matrix notation, the equations are[†]:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\begin{pmatrix} \dot{\omega}_0 \\ --- \\ \dot{\omega}_R \end{pmatrix} = \begin{pmatrix} L_0 \\ --- \\ L_R \end{pmatrix} \tag{1}$$

or

$$A\dot{\omega} = L$$

In scalar form the equations are

$$a_{00} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{0k}\, \ddot{\gamma}_k = L_0 = \sum_{\lambda=0}^{n-1} E_\lambda \tag{2}$$

$$a_{i0} \cdot \dot{\omega}_0 + \sum_{k=1}^{r-3} a_{ik}\, \ddot{\gamma}_k = L_i = g_i \cdot \sum_{\lambda=0}^{n-1} \epsilon_{i\lambda} E^*_\lambda \quad , \quad i = 1, 2, \cdots, r-3$$

where

$$a_{00} = \sum_{\lambda=0}^{n-1} \sum_{\mu=0}^{n-1} \Phi_{\lambda\mu}$$

$$a_{0k} = \sum_{\lambda} \sum_{\mu} \epsilon_{k\mu}\, \Phi_{\lambda\mu} \cdot g_k$$

[†]See Definitions of Symbols for the definitions of all terms used in this paper.

$$a_{i0} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu}$$

$$a_{ik} = g_i \cdot \sum_{\lambda} \sum_{\mu} \epsilon_{i\lambda} \Phi_{\lambda\mu} \cdot g_k$$

$$E_\lambda = M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda \left[F_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu\lambda}) \right]$$

$$- \omega_\lambda \times \Phi_{\lambda\lambda} \cdot \omega_\lambda + \sum_{j \in J_\lambda} \tau_{\lambda_j}$$

$$E_\lambda^* = E_\lambda - \sum_{\mu} \Phi_{\lambda\mu} \cdot \sum_{k=1}^{r-3} \epsilon_{k\mu} \cdot \dot{g}_k \dot{\gamma}_k$$

A comparison of Eqs. (1) and (2) reveals that:

1) A_{11} is the 3 x 3 matrix representation of the operator $a_{00} \cdot$ (where a_{00} is a dyadic, and $\cdot$ represents the dot product operation).

2) A_{12} is the 3 x r - 3 matrix representation of the vectors a_{0k}.

3) A_{22} is the r - 3 x r - 3 matrix representation of the scalars a_{ik} (with i, k = 1, 2, $\cdots$, r-3).

Moreover, in Eqs. (1) and (2), $\dot{\omega}_0$ represents the angular acceleration of the base body, $\dot{\omega}_R$ represents the relative angular accelerations $\dot{\gamma}_k$ of the remaining n - 1 bodies, L_0 is a 3 x 1 matrix, and L_R is a r - 3 x 1 matrix.

LINEARIZED SET OF r DYNAMICAL EQUATIONS FOR AN n-HINGED RIGID-BODY SPACECRAFT

In this section, a linearized set of r dynamical equations for an n-hinged rigid-body spacecraft is provided. Linearization† is accomplished by retaining only terms of first order in ω_0, γ_k and their derivatives in the solution (i.e., products of ω_0 and γ_k with $k = 1, 2, \cdots, r-3$ and their derivatives are neglected). In addition, it is assumed that γ_k (with $k = 1, 2, \cdots, r-3$) and θ_i (with $i = 1, 2, 3$) are small angles -- hence the direction cosine matrices take a particularly simple form.

Typical direction cosine matrices for the linearized case become (see, e.g., Fig. 2)

$$C_1^2 = E - \gamma_2 \tilde{g}_2$$

$$C_3^4 = E - \gamma_4 \tilde{g}_4$$

$$C_0^1 = E - \gamma_1 \tilde{g}_1$$

$$C_0^2 = C_1^2 C_0^1 = [E - \gamma_2 \tilde{g}_2][E - \gamma_1 \tilde{g}_1] \cong E - \gamma_1 \tilde{g}_1 - \gamma_2 \tilde{g}_2$$

$$C_0^3 = [E - \gamma_3 \tilde{g}_3]$$

$$C_0^4 = C_3^4 C_0^3 = [E - \gamma_4 \tilde{g}_4][E - \gamma_3 \tilde{g}_3] \cong E - \gamma_3 \tilde{g}_3 - \gamma_4 \tilde{g}_4$$

$$C_N^0 = E - \tilde{\theta} = E - \begin{bmatrix} 0 & -\theta_1 & \theta_2 \\ \theta_1 & 0 & -\theta_3 \\ -\theta_2 & \theta_3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \theta_1 & -\theta_2 \\ -\theta_1 & 1 & \theta_3 \\ \theta_2 & -\theta_3 & 1 \end{bmatrix}$$

†Products such as $\dot{\omega}_0 \gamma$ and $\ddot{\gamma}\gamma$ are neglected in the linearization in this paper; such terms can be retained and included in the forcing function L if it is desirable!

where the vector θ consists of ordered rotations θ_3, θ_2, θ_1, E is a 3 x 3 identify matrix, and a $\sim$ over a vector represents the matrix representation of the cross-product operation.

The relationship between the attitude rate and the angular velocity of the base body 0 becomes (for the linearized case)

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & \theta_2 \\ 0 & 1 & -\theta_1 \\ 0 & \theta_1 & 1 \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_0 \cong \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{4}$$

when the small angle assumption is used and in addition products of θ_i and ω_i are neglected.

The elements $a_{\ell m}$ can be evaluated for the linear case by recognizing that products of γ_k and $\dot{\gamma}_k$ and γ_k and $\dot{\omega}_0$ can be frequently neglected (for $k = 1, 2, \cdots, r-3$). It is clear that only those portions of $a_{\ell m}$ that are not functions of γ_k are to be retained. Recall from Eqs. (1) and (2) that the $a_{\ell m}$'s are the multipliers of $\dot{\omega}_0$ and $\dot{\omega}_R$. Effectively, this implies that the direction cosine matrices C_k^0 (with $k = 1, 2, \cdots, r-3$) appearing in the expressions for the $a_{\ell m}$'s can be approximated by identity matrices. The matrix A which is composed of the elements $a_{\ell m}$ then becomes a constant.

The terms involved in the evaluation of the forcing function L are provided below for the linearized case. Recall that L is defined according to (see Eq. (1))

$$A\dot{\omega} = L \tag{5}$$

where $\dot{\omega}$ consists of $\dot{\omega}_0$ and $\dot{\omega}_R$ and L consists of L_0 and L_R.

Recall, too, that L_0 and L_R, the components of L, are given by

$$L_0 = \sum_{\lambda} E_\lambda$$

$$L_R = \begin{pmatrix} g_1 \cdot \sum_{\lambda} \epsilon_{1\lambda} E_\lambda^* \\ g_2 \cdot \sum_{\lambda} \epsilon_{2\lambda} E_\lambda^* \\ \vdots \\ g_{r-3} \cdot \sum_{\lambda} \epsilon_{r-3,\lambda} E_\lambda^* \end{pmatrix} \tag{6}$$

The linearized versions of E_λ and E_λ^* reduce to

$$E_\lambda = M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \epsilon J_\lambda} \tau_{\lambda j}$$

$$E_\lambda^* = E_\lambda \tag{7}$$

and consequently, the linearized versions of L_0 and L_R reduce to

$$L_0 = \sum_{\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \epsilon J_\lambda} \tau_{\lambda j} \right\} \tag{8}$$

$$
L_R = \begin{pmatrix}
g_1 \cdot \sum_{\lambda} \epsilon_{1\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \in J_\lambda} \tau_{\lambda j} \right\} \\
g_2 \cdot \sum_{\lambda} \epsilon_{2\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \in J_\lambda} \tau_{\lambda j} \right\} \\
\vdots \\
g_{r-3} \cdot \sum_{\lambda} \epsilon_{r-3,\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\mu \neq \lambda} D_{\lambda\mu} \times C_\mu^\lambda F_\mu + \sum_{j \in J_\lambda} \tau_{\lambda j} \right\}
\end{pmatrix}
$$

Note that the term

$$
\sum_{\lambda} \sum_{j \in J_\lambda} \tau_{\lambda j}
$$

is identically zero in the equation for L_0 (interaction moments cancel in pairs).

<u>Compact Form for the Linearized Set of r Dynamical Equations</u>

In this section, the linearized set of r dynamical equations for an n-hinged rigid-body spacecraft are expressed in compact form.

First, the term

$$
g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \sum_{j \in J_\lambda} \tau_{\lambda_j}
$$

is examined. By summing the dynamical equations over bodies λ, which are connected <u>beyond</u> gimbal axis g_i relative to the base body, the interaction moment at joint i on body λ can be isolated. This implies that

$$
g_i \cdot \sum_{\lambda} \epsilon_{i\lambda} \sum_{j \in J_\lambda} \tau_{\lambda_j} \equiv g_i \cdot \tau_{ii} = g_i \cdot \left[- K_i \gamma_i - B_i \dot{\gamma}_i \right] g_i \tag{10}
$$

where

$$\tau_{ii} = - K_i \gamma_i g_i - B_i \dot{\gamma}_i g_i$$

In Eq. (10), K_i and B_i are the stiffness and damping coefficients associated with joint i. Substitution of Eq. (10) into Eq. (8) yields

$$L_0 = \left[M_0 + D_0 \times F_0 + \sum_{\lambda=1}^{n-1} D_{\lambda 0} \times C_0^\lambda F_0 \right] + \sum_{\lambda=1}^{n-1} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \tag{11}$$

$$L_R = \begin{pmatrix} g_1 \cdot \sum_\lambda \epsilon_{1\lambda} D_{\lambda 0} \times C_0^\lambda F_0 - (K_1 \gamma_1 + B_1 \dot{\gamma}_1) \\ \quad + g_1 \cdot \sum_{\lambda=1}^{n-1} \epsilon_{1\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \\ \\ g_2 \cdot \sum_\lambda \epsilon_{2\lambda} D_{\lambda 0} \times C_0^\lambda F_0 - (K_2 \gamma_2 + B_2 \dot{\gamma}_2) \\ \quad + g_2 \cdot \sum_{\lambda=1}^{n-1} \epsilon_{2\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \\ \vdots \\ g_{r-3} \cdot \sum_\lambda \epsilon_{r-3,\lambda} D_{\lambda 0} \times C_0^\lambda F_0 - (K_{r-3} \gamma_{r-3} + B_{r-3} \dot{\gamma}_{r-3}) \\ \quad + g_{r-3} \cdot \sum_{\lambda=1}^{n-1} \epsilon_{r-3,\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \end{pmatrix}$$

Equation (11) can thus be written as

$$L = \begin{pmatrix} L_{00} + L_{0R} \\ \hdashline L_{R0} + L_{RR} \end{pmatrix} + \begin{bmatrix} 0 \\ \hdashline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hdashline -B \end{bmatrix} \dot{\gamma} \tag{12}$$

or

$$L = \begin{pmatrix} \overline{L}_o \\ \hdashline \overline{L}_R \end{pmatrix} + \begin{bmatrix} 0 \\ \hdashline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hdashline -B \end{bmatrix} \dot{\gamma}$$

where

0 is an appropriately dimensional null matrix

K, B are r-3 x r-3 diagonal matrices composed of the stiffness coefficients K_i and the damping coefficients B_i

$$L_{00} = M_0 + D_0 \times F_0 + \sum_{\lambda=1}^{n-1} D_{\lambda 0} \times C_0^\lambda F_0 = M_0 + \overline{L}_{00}$$

$$L_{0R} = \sum_{\lambda=1}^{n-1} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\}$$

$$L_{R0} = \begin{pmatrix} g_1 \cdot \sum_\lambda \epsilon_{1\lambda} D_{\lambda 0} \times C_0^\lambda F_0 \\ g_2 \cdot \sum_\lambda \epsilon_{2\lambda} D_{\lambda 0} \times C_0^\lambda F_0 \\ \vdots \\ g_{r-3} \cdot \sum_\lambda \epsilon_{r-3,\lambda} D_{\lambda 0} \times C_0^\lambda F_0 \end{pmatrix}$$

$$L_{RR} = \begin{pmatrix} g_1 \cdot \sum_{\lambda=1}^{n-1} \epsilon_{1\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \\ g_2 \cdot \sum_{\lambda=1}^{n-1} \epsilon_{2\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \\ \vdots \\ g_{r-3} \cdot \sum_{\lambda=1}^{n-1} \epsilon_{r-3,\lambda} \left\{ M_\lambda + D_\lambda \times F_\lambda + \sum_{\substack{\mu \neq \lambda \\ \mu \neq 0}} D_{\lambda\mu} \times C_\mu^\lambda F_\mu \right\} \end{pmatrix}$$

$$\overline{L}_0 = L_{00} + L_{0R} = M_0 + \overline{L}_{00} + L_{0R}$$

$$\overline{L}_R = L_{R0} + L_{RR}$$

Collecting the above results, it follows that a set of r linearized dynamical equations for an n-hinged rigid-body spacecraft is given by

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{pmatrix} \dot{\omega}_0 \\ \hline \dot{\omega}_R \end{pmatrix} = \begin{pmatrix} L_{00} + L_{0R} \\ \hline L_{R0} + L_{RR} \end{pmatrix} + \begin{bmatrix} 0 \\ \hline -K \end{bmatrix} \gamma + \begin{bmatrix} 0 \\ \hline -B \end{bmatrix} \dot{\gamma} \qquad (13)$$

State Equations for an n-Hinged Rigid-Body Spacecraft

In this section, the linear model developed in the last section is cast in a form suitable for use in modern control theory. Essentially, the state equations are sought. As seen previously,

$$\dot{\theta} \cong \dot{\omega}_0$$

and

$$\dot{\gamma} = \omega_R$$

where the vectors θ and ω_0 are the attitude and angular velocity of the base body and the vectors γ and ω_R represent the relative attitude and angular velocities γ_k, $\dot{\gamma}_k$ for $k = 1, 2, \cdots, r-3$. The state can thus be defined as the $2r \times 1$ vector

$$x = \begin{pmatrix} \theta \\ \omega_0 \\ \gamma \\ \omega_R \end{pmatrix} .$$

The differential equations for θ and γ are given above and those for ω_0 and ω_R can be obtained from Eq. (13).

Manipulation of Eq. (13) yields

$$\begin{aligned} A_{11}\,\dot{\omega}_0 + A_{12}\,\dot{\omega}_R &= L_{00} + L_{0R} = \overline{L}_0 \\ A_{21}\,\dot{\omega}_0 + A_{22}\,\dot{\omega}_R &= L_{R0} + L_{RR} - K\gamma - B\dot{\gamma} = \overline{L}_R - K\gamma - B\dot{\gamma} \end{aligned} \tag{14}$$

Equation (14) can be written as

$$\begin{aligned} \dot{\omega}_0 &= \left[A_{11} - A_{12}\,A_{22}^{-1}\,A_{21}\right]^{-1} (L_{00} + L_{0R}) \\ &\quad - \left[A_{11} - A_{12}\,A_{22}^{-1}\,A_{21}\right]^{-1} A_{12}\,A_{22}^{-1}\,(L_{R0} + L_{RR} - K\gamma - B\dot{\gamma}) \\ \dot{\omega}_R &= A_{22}^{-1}\left\{L_{R0} + L_{RR} - K\gamma - B\dot{\gamma} - A_{21}\,\dot{\omega}_0\right\} \end{aligned} \tag{15}$$

Redefining the bracketed matrix in Eq. (15) as α, it follows that

$$
\begin{aligned}
\dot{\omega}_0 &= \alpha^{-1}\,(L_{00} + L_{0R}) - \alpha^{-1}\,A_{12}\,A_{22}^{-1}\left\{L_{R0} + L_{RR} - K\gamma - B\dot{\gamma}\right\} \\
\dot{\omega}_R &= A_{22}^{-1}\left\{L_{R0} + L_{RR} - K\gamma - B\dot{\gamma}\right\} - A_{22}^{-1}\,A_{21}\,\dot{\omega}_0
\end{aligned}
\tag{16}
$$

In vector-matrix notation, the state equations become

$$
\begin{pmatrix} \dot{\theta} \\ \dot{\omega}_0 \\ \dot{\gamma} \\ \dot{\omega}_R \end{pmatrix} = \left[\begin{array}{c:c:c:c} 0 & E & 0 & 0 \\ \hdashline 0 & 0 & \alpha^{-1}A_{12}A_{22}^{-1}K & \alpha^{-1}A_{12}A_{22}^{-1}B \\ \hdashline 0 & 0 & 0 & E \\ \hdashline 0 & 0 & -A_{22}^{-1}\left(E + A_{21}\alpha^{-1}A_{12}A_{22}^{-1}\right)K & -A_{22}^{-1}\left(E + A_{21}\alpha^{-1}A_{12}A_{22}^{-1}\right)B \end{array}\right] \begin{pmatrix} \theta \\ \omega_0 \\ \gamma \\ \omega_R \end{pmatrix}
$$

$$
+ \left[\begin{array}{c} 0 \\ \hdashline \alpha^{-1}\overline{L}_0 - \alpha^{-1}A_{12}A_{22}^{-1}\,\overline{L}_R \\ \hdashline 0 \\ \hdashline A_{22}^{-1}\left(E + A_{21}\alpha^{-1}A_{12}A_{22}^{-1}\right)\overline{L}_R - A_{22}^{-1}A_{21}\alpha^{-1}\,\overline{L}_0 \end{array}\right] \tag{17}
$$

where E is a 3 x 3 identity matrix and 0 is an appropriately dimensioned null matrix. Note that Eq. (17) has the desired form

$$\dot{x} = Fx + Gu$$

where x is the state and u is the control vector (G is an identity matrix in this case).

Stochastic Controller for Multi-Hinged Rigid-Body Spacecraft

In this section, a stochastic controller based on the dynamical model presented above is given. It is known from stochastic optimal control theory [9] that an optimal stochastic controller can be obtained for a linear problem subject to a quadratic cost functional. For the general nonlinear case, however, one must be content with a suitable suboptimal stochastic controller. In this work, the form of the stochastic controller appropriate for a linear problem subject to a quadratic cost functional is retained; however, the form of the control function for this special case is passed through desirable nonlinearities peculiar to attitude control before being applied to the plant (see Fig. 3).

The models for the plant and measurement process are given by

$$\begin{aligned} \dot{x} &= Fx(t) + Gu + w \\ z &= Hx(t) + v \end{aligned} \tag{19}$$

where x is the state vector, u is the control vector, z is the measurement vector, H is the measurement matrix, F and G are the plant matrices, v is the measurement noise vector, and w is the plant noise vector. As is customary with Kalman filter theory and, without loss of generality, the noise processes w and v are assumed to be white and gaussian. The well-known Kalman-Bucy filter equations are given by [9]

$$\begin{aligned} \dot{\hat{x}} &= F\hat{x} + Gu + K\left[z - H\hat{x}\right] \\ K &= PH^T R^{-1} \\ \dot{P} &= FP + PF^T - KRK^T + Q \end{aligned} \tag{20}$$

where $\hat{x}$ is the estimate of the state, K is the Kalman gain matrix, P is covariance of the error in the estimate of the state, and the matrices R and Q are defined by

$$\begin{aligned} E\left[w(t) - \overline{w}(t)\right]\left[w(\tau) - \overline{w}(\tau)\right]^T &= Q(t)\ \delta(t - \tau) \\ E\left[v(t) - \overline{v}(t)\right]\left[v(\tau) - \overline{v}(\tau)\right]^T &= R(t)\ \delta(t - \tau) \end{aligned} \tag{21}$$

The values of the matrices Q, R used in the filter are based on engineering judgment; Q depends on disturbance torques, uncertainties in the dynamical model, etc.; R depends on sensor errors.

Generation of Suboptimal Controller

As stated above, the optimal control function u^* cannot, in general, be determined for the stochastic problem. For the linear case subject to the average value of a quadratic cost functional, the optimal u^* can be determined. The equations characterizing this problem and its solution are given by [9]

$$\left.\begin{aligned} \dot{x} &= F(t)\,x + G(t)\,u + w(t) \\ z &= H(t)\,x + v(t) \\ J &= E\left[\frac{1}{2}\,x^T(t_f)\,\overline{S}_f\,x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(x^T\,\overline{A}\,x + u^T\,\overline{B}\,u\right)dt\right] \end{aligned}\right\}\ \text{PROBLEM FORMULATION}$$

$$\left.\begin{aligned} u^* &= -C\hat{x} \\ C &= \overline{B}^{-1}\,G^T\,\overline{S} \\ \dot{\overline{S}} &= \overline{S}F - F^T\overline{S} + C^T\overline{B}C - \overline{A};\ \overline{S}(t_f) = \overline{S}_f \end{aligned}\right\}\ \text{OPTIMAL CONTROL}$$

$$
\left.
\begin{aligned}
\dot{\hat{x}} &= F\hat{x} + G u^{*} + K(t)\left[z(t) - H\hat{x}\right] \\
K &= PH^{T} R^{-1} \\
\dot{P} &= FP + PF^{T} - KRK^{T} + Q
\end{aligned}
\right\} \text{FILTER EQUATIONS}
$$

In this analysis, the suboptimal vector function u_0 for the base body is obtained by

1) Generating the actuating signal $\bar{u}$ used to fire the thrusters located on the base body according to

$$
\bar{u} = C_{\theta}\hat{\theta} + C_{\omega_0}\hat{\omega}_0 + C_{\gamma}\hat{\gamma} + C_{\omega_R}\hat{\omega}_R \tag{23}
$$

2) Passing the function $\bar{u}$ through a vector deadzone function to obtain the applied moment M_0 and the associated applied thrust F_0 acting on the base body; this contribution to u_0 is given by

$$
M_0 = -\left[\sum_{i=1}^{3} \left|M_{0_i}\right| b_i b_i^{T}\right] \cdot \text{DEZ}\,\bar{u}
$$

3) Generating the terms $\bar{L}_{00}$, L_{0R}, L_{R0}.

The vector control function u_0 is given by

$$
\begin{aligned}
u_0 = {} & \alpha^{-1}\left[-\sum_{i=1}^{3} \left|M_{0_i}\right| b_i b_i^{T}\right] \cdot \text{DEZ}\,\bar{u} + \alpha^{-1}\bar{L}_{00} + \alpha^{-1} L_{0R} \\
& - \alpha^{-1} A_{12} A_{22}^{-1} \left(L_{R0} + L_{RR}\right)
\end{aligned} \tag{24}
$$

where $\bar{L}_{00}$, L_{0R}, L_{R0}, L_{RR} are defined in Eq. (12). Correspondingly, the vector control function for the remaining n - 1 bodies is given by

$$u_R = -A_{22}^{-1} A_{21} u_0 + A_{22}^{-1} (L_{R0} + L_{RR}) \tag{25}$$

Note that the time-varying functions $\bar{L}_{00}$, L_{0R}, L_{R0}, and L_{RR} are based on $\hat{\gamma}$.

Description and Uses of Suboptimal Stochastic Controller

From Fig. 3, it is seen that the stochastic controller consists of a Kalman filter adjoined to the generators of u_0 and u_R. The measurement vector z, in this paper, consists of the sensed attitude angles of the base body. The estimated state $\hat{x}$ consisting of $\hat{\theta}$, $\hat{\omega}_0$, $\hat{\gamma}$, $\hat{\omega}_R$ is used to generate the actuating signal $\bar{u}$ which fires the thrusters located on the base body. In essence, the applied moment M_0 tends to null a weighted combination of the attitude and angular velocity of the base body and the relative motion of the remaining n - 1 bodies. Note that, if it is not desirable to null a specific relative motion γ_k and its associated rate $\dot{\gamma}_k$, then the appropriate components of the control gain matrices C_λ and C_{ω_R} are zero. Note, too, that the system matrices A_{11}, A_{22}, A_{12}, A_{21}, α are constant. The Kalman gain matrices $\bar{K}_\theta$, $\bar{K}_{\omega_0}$, $\bar{K}_{\omega_R}$, $\bar{K}_\gamma$ and the control gain matrices C_θ, C_{ω_0}, C_{ω_R}, C_γ can be approximated by piecewise constant functions, if it is so desired!

This stochastic controller can be effectively used to study the effects of interactions of an articulated science platform on the base body motion, and to study the effects of interactions of booms on thrust vector control performance. In fact, this controller will be used to analyze the Mariner Jupiter Saturn (MJS'77) spacecraft in the cruise, the thrust vector control, and the articulation control modes. The MJS'77 stringent accuracy requirements and settling times associated with the articulated science platform dictates that an elaborate dynamical model be used and that disturbances and sensor noise be properly accounted for.

DEFINITION OF SYMBOLS

n	Number of rigid bodies in spacecraft model
n_c	Number of constraints
r	Number of degrees of freedom
j	Integer used to designate a joint
J_λ	Set of labels for joints j belonging to body λ
S	Set of labels of the ensemble of bodies λ, μ
c_{λ_j}	Vector from c. m. of body λ to joint j
F_{λ_j}, M_{λ_j}	Interaction force and moment on body λ due to joint j
m, m_μ	System mass; mass of body μ
$M^c_{\lambda_j}$	Constraint moment at joint j on body λ
$M^{SD}_{\lambda_j}$	Spring-damper interaction moment on body λ due to joint j
A_{11}, A_{12}, A_{21}, A_{22}	Elements of partitioned matrix A appearing in Eq. (1)
A	$3 + r - 3 \times 3 + r - 3$ matrix appearing in Eq. (1)
ω_0, ω_R, ω	Angular velocity vector of base body 0; relative angular velocity components $\dot{\gamma}_k$, $k = 1, 2, \cdots, r - 3$; ω has components ω_0 and ω_R
L_0, L_R, L	Vector forcing function for base body; vector forcing function for $n - 1$ remaining bodies; vector L has components L_0 and L_R
a_{00}, a_{0k}, a_{i0}, a_{ik}	Dyadics defined in Eq. (2)

γ_k, γ	Relative angular motion at joint k; γ is $r - 3 \times 1$ vector having components γ_k, $k = 1, 2, \cdots, r - 3$
$\epsilon_{i\lambda}$	$\begin{cases} 1 \text{ if gimbal axis } g_i \text{ is between body } \lambda \text{ and body } 0 \\ 0 \text{ otherwise} \end{cases}$
$\overline{L}_0$, $\overline{L}_R$	Vector forcing functions used in defining L (see Eq. (12)); $\overline{L}_0$ and $\overline{L}_R$ are formed from L_0 and L_R by not including the terms $K\gamma + B\dot{\gamma}$
L_{00}, L_{0R}	Vector forcing functions used in defining $\overline{L}_0$; L_{00} is the contribution to $\overline{L}_0$ due to forces F_0 and moments M_0 applied to the base body; L_{0R} is the contribution made by forces F_λ and moments M_λ with $\lambda \neq 0$
L_{R0}, L_{RR}	Vector forcing functions used in defining $\overline{L}_R$; L_{R0} is the contribution to $\overline{L}_R$ due to F_0; L_{RR} is the contribution made by F_λ and M_λ with $\lambda \neq 0$
$\overline{L}_{00}$	Vector forcing function used in defining L_{00}; $\overline{L}_{00}$ is formed from L_{00} by not including the moment M_0
b_i	Basis vectors for base body 0, $i = 1, 2, 3$
U	Unit dyadic
E_λ, $\dot{E}_\lambda^*$	Vectors used in defining terms L_0 and L_R of Eq. (2)
D_{λ_j}, $D_{\lambda\mu}$	D_{λ_j} is vector from barycenter of body λ to joint j of body λ; $D_{\lambda\mu} = D_{\lambda_j}$ for all bodies μ belonging to S_{λ_j} (the set of bodies connected to body λ via joint j)
$\Phi_{\lambda\mu}$	A dyadic defined by $\Phi_{\lambda\mu} = -m\left[D_{\lambda\mu} \cdot D_{\mu\lambda} U - D_{\mu\lambda} D_{\lambda\mu}\right]$

$\Phi_{\lambda\lambda}$	Augmented inertia matrix for body λ relative to barycenter B_λ
g_i	Gimbal axis, $i = 1, 2, \cdots, r - 3$
D_λ	Vector from barycenter of body λ to c.m. of body λ
F_λ, M_λ	Vectors representing externally applied forces and moments to body λ
C_μ^λ	Direction cosine matrix transforming coordinates of body μ to coordinates of body λ
E	Unit (identity) matrix
θ_i, ω_{0_i}	Attitude angles of base body 0; angular velocity measure numbers of base body 0, $i = 1, 2, 3$
τ_{λ_j}	Vector representing spring-damper interaction torque on body λ at joint j (same as $M_{\lambda_j}^{SD}$)
K_i, B_i	Stiffness and damping coefficients for joint i
α	Matrix used in state equations (see Eq. (17))
K, B	Diagonal stiffness and damping matrices
F, u, w, x	F is matrix used in defining state equations (see Eq. (18)); u, w, x are vectors used in defining state equations (see Eq. (19))
z, H, v	z, v vectors used in defining measurement process; H is the measurement matrix
$\overline{A}$, $\overline{B}$, $\overline{S}$, $\overline{S}_f$	Matrices used in defining average value of quadratic cost functional

C	Gain matrix used in defining controller $\overline{u}$; C has components C_θ, C_{ω_0}, C_γ, C_{ω_R}
$\hat{x}$, P, R, Q, $\overline{K}$	$\hat{x}$ is estimate of state x; P, Q, R, $\overline{K}$ are matrices appearing in filter equations (see Eq. (20))
DEZ{ }	Vector deadzone function
u_0, u_R	Suboptimal control vector for base body 0; suboptimal control vector for $n - 1$ remaining bodies

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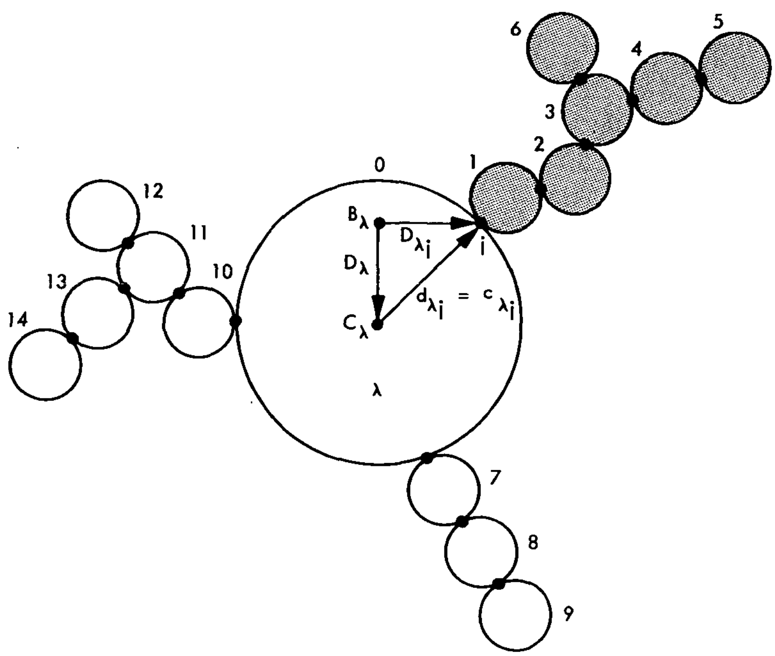
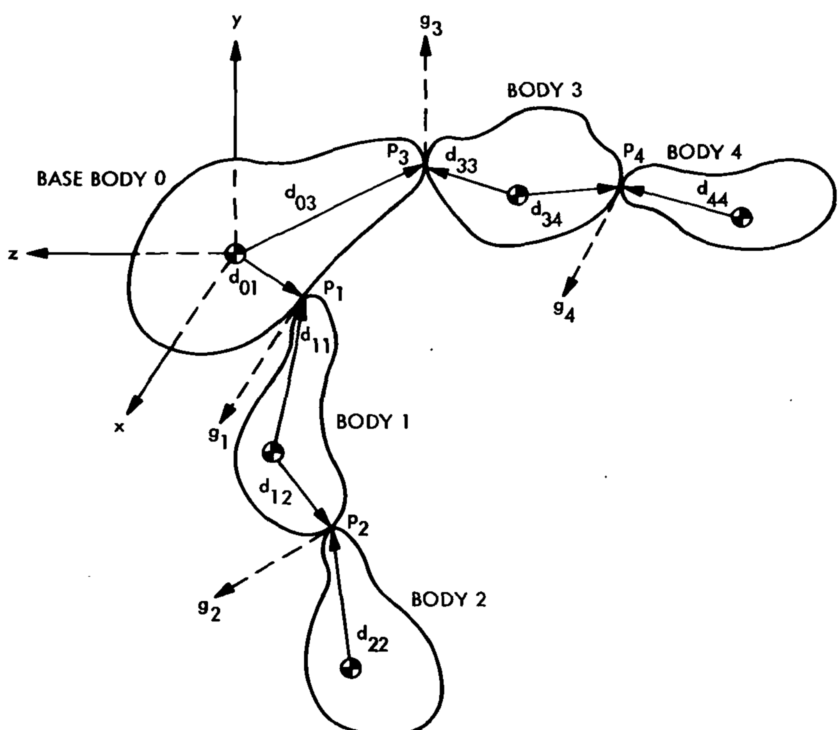


NOTES:

1. B_λ IS BARYCENTER OF BODY λ ($\lambda = 0$ IN DRAWING).
2. C_λ IS CENTER OF MASS (C.M.) OF BODY λ.
3. d_{λ_j} IS VECTOR FROM C.M. OF BODY λ TO JOINT j.
4. S_{λ_j} IS SET OF BODIES μ CONNECTED TO BODY λ VIA JOINT j (SHOWN SHADED).
5. D_λ IS VECTOR FROM B_λ TO C_λ.
6. D_{λ_j} IS VECTOR FROM B_λ TO JOINT j.
7. $D_{\lambda\mu} = D_{\lambda_j}$ FOR ALL $\mu \epsilon S_{\lambda_j}$ (IN ABOVE SKETCH, $D_{0j} = D_{01} = D_{02} = D_{03} = D_{04} = D_{05} = D_{06}$).

Fig. 1. Pictorial Sketch of 15-Hinged Rigid Bodies



NOTES:

1. d_{λ_j} REPRESENTS VECTOR FROM C.M. OF BODY λ TO JOINT j.
2. g_j REPRESENTS GIMBAL AXIS AT JOINT j.
3. p_j REPRESENTS HINGE POINT AT JOINT j.

Fig. 2. Pictorial Sketch of a 5-Hinged Rigid-Body Spacecraft Showing Gimbal Axes

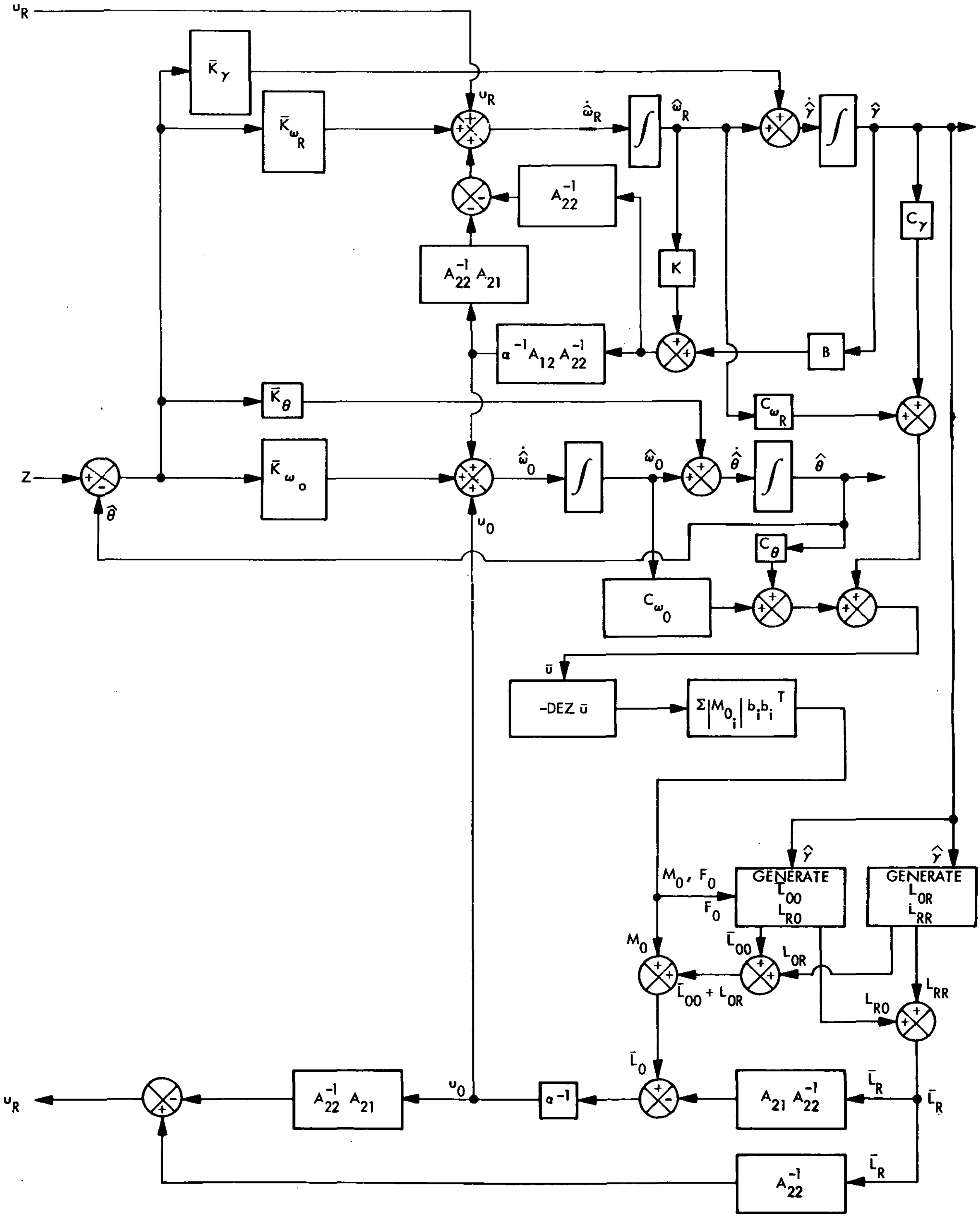


Fig. 3. Block Diagram of Suboptimal Stochastic Controller for n-Body Spacecraft

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