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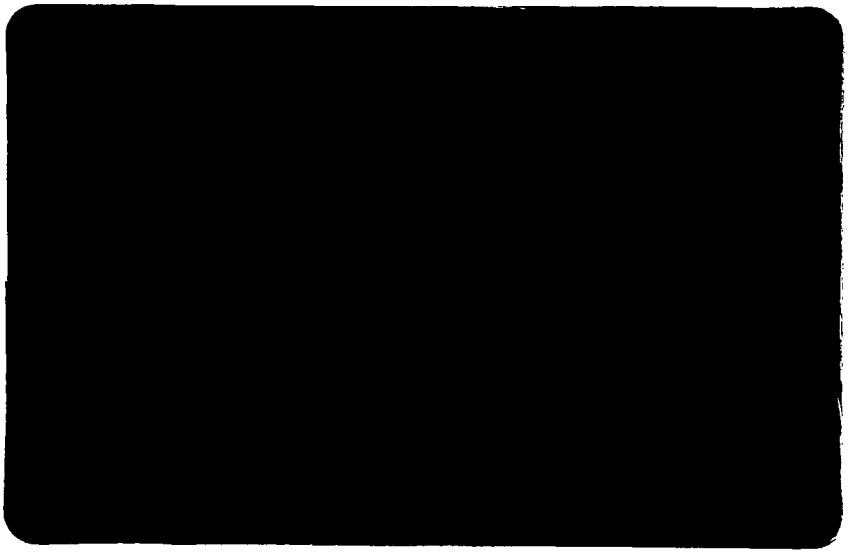

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A GEOMETRICAL INTERPRETATION OF THE 2n-th CENTRAL DIFFERENCE

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ABSTRACT

Many algorithms used for data smoothing, data classification and error detection require the calculation of the distance from a point to the polynomial interpolating its 2n neighbors (n on each side) as is clearly the case in [1]. This computation, if performed naively, would require the solution of a system of equations and could create numerical problems. It is the purpose of this note to show that if the data is equally spaced, then this calculation can be performed using a simple recursion formula.

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A GEOMETRICAL INTERPRETATION OF THE 2n-th CENTRAL DIFFERENCE*

R. A. Tapia**

1. Introduction. Many algorithms used for data smoothing, data classification and error detection require the calculation of the distance from a point to the polynomial interpolating its $2n$ neighbors (n on each side) as is clearly the case in [1]. This computation, if performed naïvely, would require the solution of a system of equations and could create numerical problems. It is the purpose of this note to show that if the data is equally spaced, then this calculation can be performed using a simple recursion formula.

2. The 2n-th Central Difference. Given $\{w_i : i = 0, \pm 1, \pm 2, \ldots\} \subset R^1$ and a scalar h we may define the second difference operator

$$D(w_i ; h) = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} .$$

Consider $Y = \{(x_i, y_i) \subset R^2 : i = 0, \pm 1, \pm 2, \ldots\}$ with $x_{i+1} - x_i = h$ for all i. Recall that by the $2n$-th central difference of Y at the i-th point we mean

$$r_i^{(2n)} = D(r_i^{(2n-2)} ; h), \quad n = 1,2,3,\ldots$$

with $r_i^{(0)} = y_i$ for all i.

By the $2n$-th canonical central difference $r^{(2n)}$ we mean the $2n$-th central difference of the set

$$\{(jh, \delta_{0j}) : j = 0, \pm 1, \pm 2, \ldots\}$$

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at $(0,1)$ (δ_{ij} is the Kronecker delta).

A well-known property of the $2n$-th central difference is that it is a good approximation to the $2n$-th derivative for small h, i.e., if $y_i = f(x_i)$ and f is $2n$ times differentiable, then $r_i^{(2n)} \to f^{(2n)}(x_i)$ as $h \to 0$. We offer the following alternate and very useful interpretation of the $2n$-th central difference.

Theorem. Let $P(x)$ be the unique polynomial of degree $2n - 1$ interpolating the points

$$\{(x_{i+j}, y_{i+j}): \quad j = \pm 1, \pm 2, \ldots, \pm n\} \quad (n \geq 1) .$$

Then

$$y_i - P(x_i) = \frac{r_i^{(2n)}}{r^{(2n)}} \quad \text{for any} \quad h \neq 0 .$$

Proof. We merely sketch a proof. Consider the following table.

TABLE 1

n	$\cdots$	y_{i-4}	y_{i-3}	y_{i-2}	y_{i-1}	y_i	y_{i+1}	y_{i+2}	y_{i+3}	y_{i+4}	$\cdots$
0		0	0	0	0	1	0	0	0	0	
1		0	0	0	1	-2	1	0	0	0	
2		0	0	1	-4	6	-4	1	0	0	
3		0	1	-6	15	-20	15	-6	1	0	
4		1	-8	28	-56	70	-56	28	-8	1	
					$\cdots$						

Table 1 is constructed by taking for the n-th row the second difference of the $(n-1)$-th row with $h = 1$, for $n = 1,2,\ldots$. Let a_{nj} denote the element in Table 1 in the n-th row and in the column corresponding to y_j.

From Table 1 we obtain $r_i^{(2n)} = \sum_j a_{nj} y_j$. Moreover $r^{(2n)} = a_{n1}$.

We now construct Table 2 from Table 1 by dividing by the column corresponding to y_i, i.e., by dividing the n-th row by $r^{(2n)}$.

TABLE 2

n	...	y_{i-4}	y_{i-3}	y_{i-2}	y_{i-1}	y_i	y_{i+1}	y_{i+2}	y_{i+3}	y_{i+4}	...
0		0	0	0	0	1	0	0	0	0	
1		0	0	0	-1/2	1	-1/2	0	0	0	
2		0	0	1/6	-2/3	1	-2/3	1/6	0	0	
3		0	-1/20	3/10	-3/4	1	-3/4	3/10	-1/20	0	
4		1/70	-4/35	4/10	-4/5	1	-4/5	4/10	-4/35	1/70	

As in Table 1 let b_{nj} denote a typical element of Table 2. It can be shown that $b_{n,i+j}$ for $-n \leq j \leq n$ is the value at x_i of the Lagrangian polynomial of degree $2n - 1$ which interpolates the n points on each side of x_i and is equal to 1 at x_{i+j} and 0 at the remaining points. Once this has been established, using Lagrange's formula it follows that

$$y_i - P(x_i) = \sum_j b_{nj} y_j \ ;$$

however $\dfrac{r_i^{(2n)}}{r^{(2n)}} = \sum_j b_{nj}\, y_j$. This proves the theorem .

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