SEPARATION OF VARIABLES IN THE SPECIAL DIAGONAL HAMILTON-JACOBI EQUATION - APPLICATION TO THE DYNAMICAL PROBLEM OF A PARTICLE CONSTRAINED ON A MOVING SURFACE

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16. Abstract

For the time-dependent, n-dimensional, special diagonal Hamilton-Jacobi equation

$$\sum_{i=1}^{n} a_{ii}^{*}(\mathbf{q}, t)\left(\frac{\partial S}{\partial q_i}\right)^2 + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0$$

a necessary and sufficient condition for the separation of variables to yield a complete integral of the form

$$S(\mathbf{q}, t) = \sum_{i=1}^{n} X_i(q_i; \alpha) - T(t; \alpha)$$

is established by specifying the admissible forms of a_{ii}^{*} and V in terms of $(n + 1)^2$ arbitrary functions. A complete integral is then expressed in terms of these $(n + 1)^2$ arbitrary functions and also the n irreducible constants α.

As an application of the results obtained for the two-dimensional Hamilton-Jacobi equation

$$a_{11}^{*}(\xi, \eta, t)\left(\frac{\partial S}{\partial \xi}\right)^2 + a_{22}^{*}(\xi, \eta, t)\left(\frac{\partial S}{\partial \eta}\right)^2 + V(\xi, \eta, t) + \frac{\partial S}{\partial t} = 0$$

analysis is made for a comparatively wide class of dynamical problems involving a particle moving in Euclidean three-dimensional space under the action of external forces but constrained on a moving surface. All the possible cases in which this equation has a complete integral of the above form are obtained and these are tabulated for reference.

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FOREWORD

Part of this work is taken from a doctoral dissertation submitted by David L. Blanchard, and directed by F. K. Chan, to the Department of Space Science and Applied Physics, Catholic University of America, Washington, D.C. The work is an outgrowth of the course "Advanced Celestial Mechanics" taught by the second author during the academic year 1968-69.

Basic notation is as follows:

(1) V always denotes a column vector and V^T denotes the row vector that is the transpose of V.

(2) A^T, A^{-1}, and A^+ always denote the transpose, inverse, and adjoint, respectively, of the matrix A. (Neither B^* nor B' denotes the complex conjugate, adjoint, or transpose of the matrix B.)

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INTRODUCTION

Historical Background

The birth of a systematic approach to mechanics essentially occurred with Newton's statement of the fundamental laws of motion in his work *Philosophiae Naturalis Principia Mathematica* (1687). Aristotle (384-322 B.C.) was aware of the principle of the lever but erred in thinking of "virtual velocities" rather than "virtual displacements." Archimedes (287-212 B.C.) was the first to state the law of statics correctly; however, in the formulation of dynamics, there was virtually no progress for the next 2000 years because all the scholars of mechanics worked with the false concepts of the Aristotelian school that associated forces with all motion. It was Galileo (1564-1642) who first became aware of the principle of inertia when he quite accidentally studied motion of a body down an inclined plane and out of a horizontal plane.

Newton (1642-1726) stated his three basic laws as follows:

(1) Every body continues in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces imposed upon it.

(2) The rate of change of momentum is proportional to the impressed force and is in the direction in which the force acts:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{1}$$

(3) To every action there is always opposed an equal reaction.

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These three laws have provided the basis for the subsequent development of (Newtonian) mechanics. (For an informative account of the history of mechanics, see ref. 1.) A branch of mechanics developed strictly from Newton's laws of motion is now usually referred to as "vectorial mechanics" and its basic viewpoint is to account for all the forces acting on a body and its consequent motion under these forces. The basic formulation of mechanics remained unchanged for about 100 years after its introduction by Newton.

It was Lagrange (1736-1813) who gave mechanics a second great forward step when he introduced the concept of generalized coordinates and a formulation that (under fairly general conditions) depends only on a knowledge of the kinetic and potential energy. This branch of mechanics is usually called analytical mechanics. Partial credit for its development must also be given to Leibniz (1646-1716), Euler (1707-83), and D'Alembert (1717-83). Of course, both analytical mechanics and vectorial mechanics result in the same equations of motion. Lagrange's formulation is independent of physical or geometrical considerations once the two scalar quantities, the kinetic and potential energy, are known in analytical form. As he so modestly says in the preface of his book, *Mecanique Analytique* (1788), "The reader will find no figures in the work. The methods I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedures." This was the springboard of analytical mechanics and is summarized by his differential equations for a system with n degrees of freedom:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_i}\right) - \frac{\partial E_k}{\partial q_i} = Q_i \qquad i = 1, 2, \ldots, n \tag{2}$$

where

E_k = kinetic energy

q_i = ith generalized coordinate

Q_i = ith component of the generalized force

About 50 years later (1834-35), the third great jump in the development of mechanics was made by William Rowan Hamilton (1805-65). (Some work was also done by Poisson (1781-1840) and Lagrange.) By a Legendre transformation he converted (refs. 2 and 3) the set of Lagrange's second-order differential equations (2) into a set of $2n$ first-order differential equations. These new equations are called Hamilton's canonical equations and they treat momenta and position on an equivalent basis as the independent variables. The well-known Hamilton equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad i = 1, 2, \ldots, n \tag{3a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \ldots, n \tag{3b}$$

where

$H = H(\mathbf{q}, \mathbf{p}, t)$ = the Hamiltonian

$p_i = \partial L/\partial \dot{q}_i$ = generalized momentum associated with the ith generalized coordinate

$L = E_k - E_p$ = the Lagrangian

E_p = potential energy

Next, for the case of a time-independent Hamiltonian, by considering the action integral and constant energy surfaces $H = h$ in the $(\mathbf{q}, \mathbf{p})$ space, Hamilton (refs. 2 and 3) showed that

$$p_i = \frac{\partial}{\partial q_i} W(\mathbf{q}, \mathbf{q}') \qquad (4a)$$

$$p_i' = -\frac{\partial}{\partial q_i'} W(\mathbf{q}, \mathbf{q}') \qquad (4b)$$

where $(\mathbf{q}', \mathbf{p}')$ denotes the initial values of $(\mathbf{q}, \mathbf{p})$ at time $t = t'$, and that the function $W(\mathbf{q}, \mathbf{q}')$ satisfies the two simultaneous partial differential equations

$$H\left(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}\right) - h = 0 \qquad (5a)$$

$$H\left(\mathbf{q}, -\frac{\partial W}{\partial \mathbf{q}'}\right) - h = 0 \qquad (5b)$$

He called the function W the principal function. For the time-dependent case, by considering the extended action integral and the condition that the extended Hamiltonian function $H + p_t$ equal zero for all t, Hamilton then showed that equations (4) and (5) are to be replaced by

$$p_i = \frac{\partial W}{\partial q_i} \qquad (6a)$$

$$p_t = \frac{\partial W}{\partial t} \qquad (6b)$$

$$p_i' = -\frac{\partial W}{\partial q_i'} \qquad (7a)$$

$$p_t' = -\frac{\partial W}{\partial t'} \qquad (7b)$$

and

$$H\left(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}, t\right) + \frac{\partial W}{\partial t} = 0 \qquad (8a)$$

$$H\left(\mathbf{q}', -\frac{\partial W}{\partial \mathbf{q}'}, t'\right) - \frac{\partial W}{\partial t'} = 0 \qquad (8b)$$

where $W = W(\mathbf{q}, \mathbf{q}', t, t')$, which he still referred to as the principal function. Hamilton observed that if $W(\mathbf{q}, \mathbf{q}', t, t')$ is known, then by simple differentiation and eliminations in equations (6a) and (7a), the following equations are obtained

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$$q_i = q_i(\mathbf{q}', \mathbf{p}', t'; t) \tag{9a}$$

$$p_i = p_i(\mathbf{q}', \mathbf{p}', t'; t) \tag{9b}$$

(Equations (6b) and (7b) are automatically satisfied by virtue of the condition that $H + p_t = 0$ for all t and equations (8).) The solution of the dynamical problem is then known in terms of the initial conditions $(\mathbf{q}', \mathbf{p}', t')$ and time t. This is one of Hamilton's outstanding discoveries (ref. 4), even though he did not give an account of how to determine the principal function.

Jacobi (1804-51), by considering canonical transformations and generating functions, gave the fourth and probably the most gigantic step in the development of mechanics. He showed (ref. 5) in 1837 that one need only consider one partial differential equation in the generating function $S(\mathbf{q}, \alpha, t)$, which is presently known as Hamilton's principal function

$$H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{10}$$

Moreover, he proved that any complete integral of equation (10); i.e., a solution with the property that

$$\left| \frac{\partial^2 S}{\partial \alpha_i \, \partial q_j} \right| \neq 0$$

will provide the solution of Hamilton's equations (3). (See subsection entitled "Formulation of the Hamilton-Jacobi Equation.") However, Jacobi did not give any systematic rule for finding complete integrals of first-order partial differential equations or even, in particular, for equation (10). He did seek solutions of equation (10) that could be expressed as the sum of functions each of which is a function of only one coordinate; i.e., solutions of the form

$$S(\mathbf{q}, t; \boldsymbol{\alpha}) = \sum_{i=1}^{n} X_i(q_i; \boldsymbol{\alpha}) - T(t; \boldsymbol{\alpha}) \tag{11}$$

By seeking solutions of such a form, he meant the "separation of variables." As a tribute to the contributions of Jacobi, the partial differential equation (10) is now referred to as the "Hamilton-Jacobi" equation.

Much of the subsequent work on the separation of variables of the Hamilton-Jacobi equation involved the case for which the Hamiltonian is a constant. For this case it is easy to show that there exists a solution of the form

$$S(\mathbf{q}, t; \boldsymbol{\alpha}) = W(\mathbf{q}; \alpha_1, \alpha_2, \ldots, \alpha_n) - \alpha_1 t \tag{12}$$

where the function W is now known as Hamilton's characteristic function which satisfies the partial differential equation

$$H\left(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}\right) = \alpha_1 \tag{13}$$

In the period 1891-93, Stäckel (refs. 6 and 7) established a theorem giving a necessary and sufficient condition on the coefficients $a_{ii}^*(\mathbf{q})$ and $V(\mathbf{q})$ for the separation of variables for the Hamilton-Jacobi equation, if the Hamiltonian is a constant and of the special form

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$$H = \sum_{i=1}^{n} a_{ii}^{*}(\mathbf{q})p_i^2 + V(\mathbf{q}) \tag{14}$$

Shortly after this, in 1904, Levi-Civita (in a letter to Stäckel; also ref. 8) extended the results to include a more general Hamiltonian of the form

$$H = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{*}(\mathbf{q})p_i p_j + V(\mathbf{q}) \tag{15}$$

However, Levi-Civita's results are not expressed in an explicit form as Stäckel's are, but are in terms of a system of partial differential equations to be satisfied by the coefficients $a_{ii}^{*}(\mathbf{q})$ and $V(\mathbf{q})$. Consequently, they are of very little direct practical use.

For the Hamilton-Jacobi equation, there are infinitely many coordinate systems that have coefficients $a_{ii}^{*}(\mathbf{q})$ that satisfy the conditions obtained by Stäckel. Many researchers next used the results of Stäckel to study the Schrödinger equation. In 1927 Robertson (ref. 9) established a necessary and sufficient condition for separation of variables for the Schrödinger equation in the form of a product of functions, each of which is a function of only one coordinate. He found that to effect this separation, the coefficients in the Schrödinger equation have to satisfy Stäckel's condition on the coefficients in the Hamilton-Jacobi equation and also an additional condition which is usually referred to as Robertson's condition. In 1934 Eisenhart (ref. 10) proved that there are only 11 real coordinate systems that are of the Stäckel form in Euclidean three-dimensional space; moreover, these 11 systems automatically satisfy Robertson's condition.

Formulation of the Hamiltonian

To make this work self-contained, the next step will be to set up the Hamiltonian function under rather general conditions as a function of n generalized coordinates, the corresponding momenta, and time. This explicit form allows one to write out the Hamiltonian function once the kinetic and potential energy have been expressed analytically. Furthermore, it is quite easy to relate the Hamiltonian in this form to the energy of the system. In the process, the basic principles of mechanics that lead to this very general form of the Hamiltonian will be reviewed and nomenclature to be used later will also be introduced. A system of N particles will be considered; other systems, such as those consisting of many finite parts connected by complicated constraints, can also be included without much difficulty.

The time-dependent Hamiltonian will be developed with only the following assumptions:

(1) The k constraints are holonomic and the N particles have $n(= 3N - k)$ degrees of freedom.

(2) The generalized forces Q_i are derivable from a scalar work function $\cup$ through an equation of the form

$$Q_i(\mathbf{q}, \dot{\mathbf{q}}, t) = -\frac{\partial \cup}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial \cup}{\partial \dot{q}_i}\right) \tag{16}$$

where

$$\cup(\mathbf{q}, \dot{\mathbf{q}}, t) = E_p(\mathbf{q}, t) + \sum_{i=1}^{n} b_i'(\mathbf{q}, t)\dot{q}_i \qquad (17)$$

Forces described by equations (16) and (17) are said to be monogenic, and it is obvious that conservative forces are a special case for which $\cup = E_p(\mathbf{q})$ and $b'(\mathbf{q}, t) = 0$ in equation (17). (In contrast to the definition of monogenic forces here, Lanczos (ref. 4) defines them by eq. (16) only. The additional restriction given by equation (17) is included here to eliminate forces dependent on accelerations, which do not seem possible in Newtonian mechanics.)

One could proceed here and use vectorial mechanics and Newton's laws (eq. (1)); however, when the coordinate system is not Cartesian or when constraints are present, Newton's equations are quite cumbersome. It turns out that the approach of analytical mechanics is more suitable. Beginning with assumption (1), the equations of transformation to generalized coordinates are given by

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \dots, q_n, t) \qquad k = 1, 2, \dots, N \qquad (18)$$

From equation (18), the velocity is obtained by the rules of partial differentiation as follows:

$$\mathbf{v}_k = \frac{d\mathbf{r}_k}{dt}$$

$$= \sum_{i=1}^{n} \frac{\partial \mathbf{r}_k}{\partial q_i} \frac{\partial q_i}{\partial t} + \frac{\partial \mathbf{r}_k}{\partial t} \qquad (19)$$

Kinetic energy is defined by

$$E_k = \frac{1}{2} \sum_{k=1}^{N} m_k \mathbf{v}_k \cdot \mathbf{v}_k$$

$$= \frac{1}{2} \sum_{k=1}^{N} m_k \frac{d\mathbf{r}_k}{dt} \cdot \frac{d\mathbf{r}_k}{dt} \qquad (20)$$

Combining equations (19) and (20) yields the following expression for the kinetic energy in terms of the generalized coordinates, generalized velocities, and time:

$$E_k = a + \sum_{i=1}^{n} a_i'\dot{q}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\dot{q}_i\dot{q}_j \qquad (21)$$

where

$$a = a(\mathbf{q}, t)$$

$$= \frac{1}{2} \sum_{k=1}^{N} m_k \frac{\partial \mathbf{r}_k}{\partial t} \cdot \frac{\partial \mathbf{r}_k}{\partial t} \qquad (22a)$$

$$a_i' = a_i'(\mathbf{q}, t)$$

$$= \sum_{k=1}^{N} m_k \frac{\partial \mathbf{r}_k}{\partial t} \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} \tag{22b}$$

$$a_{ij} = a_{ij}(\mathbf{q}, t)$$

$$= \frac{1}{2} \sum_{k=1}^{N} m_k \frac{\partial \mathbf{r}_k}{\partial q_i} \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} \tag{22c}$$

Lagrange's equations may be derived in several ways, one of which is through D'Alembert's principle. (See ref. 11, 12, or 13.) In one of their more general forms, Lagrange's equations may be written as

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_i}\right) - \frac{\partial E_k}{\partial q_i} = Q_i \qquad i = 1, 2, \dots, n \tag{2}$$

where the components of the generalized forces are defined by

$$Q_i = \sum_{j=1}^{N} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i} \tag{23}$$

From equation (16) in assumption (2), it follows that Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, \dots, n \tag{24}$$

where

$$L = E_k - U \tag{25}$$

New variables p_i, which are called the generalized momenta and are conjugate to the generalized coordinates q_i, are introduced and defined by the equation

$$p_i = \frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t) \qquad i = 1, 2, \dots, n \tag{26}$$

A new function $H(\mathbf{q}, \mathbf{p}, t)$ is introduced and defined by

$$H(\mathbf{q}, \mathbf{p}, t) = \left[\sum_{i=1}^{n} p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t) \right]_{\dot{\mathbf{q}}=\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t)} \tag{27}$$

where $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$ denotes the set of equations obtained from the inversion of the set of equations in equation (26). Then, it can be shown (for example, see refs. 11 to 13) that the equations of motion are given by

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(\mathbf{q}, \mathbf{p}, t) \tag{28a}$$

$$\dot{p}_i = - \frac{\partial}{\partial q_i} H(\mathbf{q}, \mathbf{p}, t) \tag{28b}$$

which are known as Hamilton's equations.

From equations (17), (21), and (25), the Lagrangian can be written explicitly as

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = a + \sum_{i=1}^{n} a_i' \dot{q}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j - E_p - \sum_{i=1}^{n} b_i' \dot{q}_i \tag{29}$$

or, in matrix form,

$$L = \dot{\mathbf{q}}^T \mathbf{A} \dot{\mathbf{q}} + \mathbf{b}^T \dot{\mathbf{q}} - U \tag{30}$$

where

$$\mathbf{A} = [a_{ij}] \tag{31a}$$

$$\mathbf{b} = \mathbf{a}' - \mathbf{b}' \tag{31b}$$

$$U = E_p - a \tag{31c}$$

and the superscript T denotes the transpose of a vector or a matrix. From equations (22), it is seen that $a_{ij} = a_{ji}$ or

$$\mathbf{A} = \mathbf{A}^T \tag{32}$$

From equations (26) and (30) to (32), it follows that

$$\mathbf{p} = 2\mathbf{A}\dot{\mathbf{q}} + \mathbf{b} \tag{33}$$

Solving equation (33) for $\dot{\mathbf{q}}$ yields

$$\dot{\mathbf{q}} = \frac{1}{2} \mathbf{A}^{-1} (\mathbf{p} - \mathbf{b}) \tag{34}$$

where the superscript -1 denotes the inverse of a matrix. Using equations (30) and (33), equation (27) becomes

$$H(\mathbf{q}, \mathbf{p}, t) = (\dot{\mathbf{q}}^T \mathbf{A} \dot{\mathbf{q}} + U)_{\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)} \tag{35}$$

Then substitution of equations (34) and (32) into equation (35) gives

$$H(\mathbf{q}, \mathbf{p}, t) = \frac{1}{4} (\mathbf{p} - \mathbf{b})^T \mathbf{A}^{-1} (\mathbf{p} - \mathbf{b}) + U \tag{36a}$$

or

$$H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p}^T \mathbf{A}^* \mathbf{p} + \mathbf{b}^{*T} \cdot \mathbf{p} + V \tag{36b}$$

where by definition

$$[a_{ij}^*] = \mathbf{A}^* = \frac{1}{4} \mathbf{A}^{-1} = \frac{1}{4} \frac{[a_{ij}]^+}{|\mathbf{A}|} \tag{37a}$$

$$\mathbf{b}^{*T} = - \frac{1}{2} \mathbf{b}^T \mathbf{A}^{-1} \tag{37b}$$

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$$V = U + \frac{1}{4} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b} \tag{37c}$$

and the superscript + denotes the adjoint of a matrix.

Equations (36) are the most general form of the Hamiltonian that are possible under the two assumptions stated in this section. One can write the Hamiltonian directly after the equations of transformation (18) are used to obtain the coefficients a_{ij} given in equations (22) and the scalar work function $\cup$ of equation (17) is specified. In general, this Hamiltonian may contain both quadratic and first-order terms in the momenta. It is interesting to note in passing that even if the scalar work function is assumed to include second-degree terms in $\dot{q}_i$, then the Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ would still be of the form given by equations (36) except that the matrix $\mathbf{A}$ would include terms from the scalar work function $\cup$.

Finally, a comparison is made of the Hamiltonian H and the total energy h defined to be the sum of the kinetic energy and the potential energy. To discuss meaningfully the usual concept of potential energy, it is noted from equations (16) and (17) that $\mathbf{b}'$ must equal zero and the potential energy can then be taken to be E_p. Consequently, from the expression of the kinetic energy E_k given by equation (21)

$$h = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} a_i' \dot{q}_i + a + E_p \tag{38}$$

Substitution of equations (29) and (33) into (27) results in

$$[H(\mathbf{q}, \mathbf{p}, t)]_{\mathbf{p}=\mathbf{p}(\mathbf{q}, \dot{\mathbf{q}}, t)} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j - a + E_p \tag{39}$$

which remains valid regardless of any requirement on $\mathbf{b}'$. Hence, from these equations, it follows that the total energy is identical with the Hamiltonian if and only if

$$\sum_{i=1}^{n} a_i' \dot{q}_i + 2a = 0 \tag{40}$$

where a and $\mathbf{a}'$ are defined by equations (22a) and (22b). It is obvious that a sufficient condition is given by

$$a = \frac{1}{2} \sum_{k=1}^{N} m_k \frac{\partial \mathbf{r}_k}{\partial t} \cdot \frac{\partial \mathbf{r}_k}{\partial t} = 0 \tag{41a}$$

$$a_i' = \sum_{k=1}^{N} m_k \frac{\partial \mathbf{r}_k}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial q_i} = 0 \tag{41b}$$

These conditions are satisfied when the equations of transformation (18) are time independent; i.e., for coordinate systems at rest.

Formulation of the Hamilton-Jacobi Equation

Having obtained Hamilton's equations

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(\mathbf{q}, \mathbf{p}, t) \qquad (28a)$$

$$\dot{p}_i = - \frac{\partial}{\partial q_i} H(\mathbf{q}, \mathbf{p}, t) \qquad (28b)$$

the next step is to introduce new coordinates and momenta $(\mathbf{Q}, \mathbf{P})$ defined by

$$\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{p}, t) \qquad (42a)$$

$$\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p}, t) \qquad (42b)$$

These new variables must be such that there exists a function $K(\mathbf{Q}, \mathbf{P}, t)$ for which the new equations of motion are given by

$$\dot{Q}_i = \frac{\partial}{\partial P_i} K(\mathbf{Q}, \mathbf{P}, t) \qquad (43a)$$

$$\dot{P}_i = - \frac{\partial}{\partial Q_i} K(\mathbf{Q}, \mathbf{P}, t) \qquad (43b)$$

where $K(\mathbf{Q}, \mathbf{P}, t)$ is known as the new Hamiltonian. Transformations (42) for which equations (43) hold are known as canonical transformations. (In older literature the term "contact transformation" de-notes the case for which time does not appear explicitly in equations (42) and the term "canonical transformation" denotes the case for which it does. Such unnecessary differentiation will not be made here.) Equations (28) constitute the set of Euler equations corresponding to the condition

$$\delta \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p}, t) \right] dt = 0 \qquad (44)$$

whereas equations (43) correspond to

$$\delta \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} P_i \dot{Q}_i - K(\mathbf{Q}, \mathbf{P}, t) \right] dt = 0 \qquad (45)$$

Hence, given equation (44) or (28), it is sufficient (but not necessary) that there exists an arbitrary function F such that

$$\sum_{i=1}^{n} p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{n} P_i \dot{Q}_i - K(\mathbf{Q}, \mathbf{P}, t) + \frac{dF}{dt} \qquad (46)$$

in order for equation (45) or (43) to hold. That is, transformations (42) for which equation (46) holds are canonical, but not all canonical transformations are given by equation (46). The fact that equation

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(46) is not a necessary condition is easily seen by noting that given equation (44), then transformations (42) for which

$$\sum_{i=1}^{n} p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p}, t) = c \left[\sum_{i=1}^{n} P_i \dot{Q}_i - K(\mathbf{Q}, \mathbf{P}, t) \right] \tag{47}$$

where c is an arbitrary constant, are also canonical. The problem of finding a necessary and sufficient condition for canonical transformations as defined by equations (43) is not important to the solution of dynamical problems. All that is needed is a sufficient condition such as that given by equation (46). (There is some confusion in the literature about equation (46) being a sufficient or necessary condition for canonical transformations. For instance, Goldstein (ref. 11) erroneously states that it is necessary, whereas Nordheim and Fues (ref. 14) and Born (ref. 15) state that it is necessary and sufficient. Gelfand and Fomin (ref. 16) state it correctly and so does Lanczos (ref. 4) except that at the end of his discussion he calls it the most general condition for a canonical transformation, and thus causes undue confusion. Another important point of confusion is that some authors, e.g., Pars (ref. 12) and Whittaker (ref. 13), define canonical transformations for which equation (46) is valid and equations (43) are not. In this case, there is no question to begin with as to whether equation (46) is a necessary or a sufficient condition for a canonical transformation.) The function F appearing in equation (46) is called the generating function. To effect a transformation between the new variables $(\mathbf{Q}, \mathbf{P}, t)$ and the old variables $(\mathbf{q}, \mathbf{p}, t)$, F must be a function of one of the following forms (see ref. 11 or 15):

$$F = F_1(\mathbf{q}, \mathbf{Q}, t) \tag{48a}$$

$$F = F_2(\mathbf{q}, \mathbf{P}, t) \tag{48b}$$

$$F = F_3(\mathbf{p}, \mathbf{Q}, t) \tag{48c}$$

$$F = F_4(\mathbf{p}, \mathbf{P}, t) \tag{48d}$$

In any of these cases, the new Hamiltonian $K(\mathbf{Q}, \mathbf{P}, t)$ and the old $H(\mathbf{q}, \mathbf{p}, t)$ are related by

$$K = H + \frac{\partial F}{\partial t} \tag{49}$$

The next step is determination of canonical transformations such that the new Hamiltonian K is a constant, which without any loss of generality can be taken to be zero. Therefore, it follows that the generating function F must satisfy the equation

$$H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial F}{\partial t} = 0 \tag{50}$$

For all mechanical systems H (as given by equation (36)) is known in explicit form for the momenta $\mathbf{p}$ but not for the coordinates $\mathbf{q}$. Consequently, either the transformation of equation (48a) or (48b) is the best choice of F. For equation (48a)

$$p_i = \frac{\partial F_1}{\partial q_i} \tag{51a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \tag{51b}$$

and for equation (48b)

$$p_i = \frac{\partial F_2}{\partial q_i} \tag{52a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \tag{52b}$$

In either case, by combining equations (50) and (51) or (50) and (52), it follows that F must satisfy the partial differential equation

$$H\left(\mathbf{q}, \frac{\partial F}{\partial \mathbf{q}}, t\right) + \frac{\partial F}{\partial t} = 0$$

If F is replaced by S so as to conform to common usage, the equation becomes

$$H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{53}$$

which is known as the Hamilton-Jacobi equation. Some authors prefer to use $S = F_2(\mathbf{q}, \mathbf{P}, t)$ and others $S = F_1(\mathbf{q}, \mathbf{Q}, t)$, but, as previously demonstrated, there is no real difference between these choices. In view of equation (36b), the most general Hamilton-Jacobi equation (53) may also be written explicitly as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^*(\mathbf{q}, t) \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} + \sum_{i=1}^{n} b_i^*(\mathbf{q}, t) \frac{\partial S}{\partial q_i} + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0 \tag{54}$$

Jacobi (ref. 5) showed that the solution of the dynamical problem defined by Hamilton's equations (28) can also be obtained by finding any one complete integral, denoted by $S = S(\mathbf{q}, \alpha, t)$, of equation (53).

Using equation (48a) and letting new coordinates $\mathbf{Q}$ be the arbitrary constants α and the new momenta $\mathbf{P}$ be the constants β, one obtains from equations (51)

$$p_i = \frac{\partial}{\partial q_i} S(\mathbf{q}, \alpha; t) \tag{55a}$$

$$\beta_i = -\frac{\partial}{\partial \alpha_i} S(\mathbf{q}, \alpha; t) \tag{55b}$$

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Equation (47) can then be solved to yield

$$q_i = q_i(\alpha, \beta, t) \qquad (56a)$$

$$p_i = p_i(\alpha, \beta, t) \qquad (56b)$$

The constants α and β are obtained from the initial values $(\mathbf{q'}, \mathbf{p'})$ at time t', but they are not necessarily identical with the initial values. This is the main difference between Hamilton's partial differential equations (8) and Jacobi's partial differential equation (10). It is also noted that to invert equation (55b) to obtain equation (56a), the following property of complete integrals is a necessary and sufficient condition:

$$\left| \frac{\partial^2 S}{\partial \alpha_i \, \partial q_j} \right| \neq 0 \qquad (57)$$

Similarly, using equation (48b) and letting $\mathbf{P} = \alpha$ and $\mathbf{Q} = \beta$, from equations (52) one obtains

$$p_i = \frac{\partial}{\partial q_i} S(\mathbf{q}, \alpha, t) \qquad (58a)$$

$$\beta_i = \frac{\partial}{\partial \alpha_i} S(\mathbf{q}, \alpha, t) \qquad (58b)$$

which can be solved to yield a set of equations of the form of equations (56). Hence, the next step is to find complete integrals of the Hamilton-Jacobi equation. It follows from this discussion that once this is done the evolution of the whole dynamical system will be known.

Outline of Present Work

In the next section, analysis is made of the time-dependent Hamilton-Jacobi equation:

$$\sum_{i=1}^{n} a_{ii}^*(\mathbf{q}, t) \left(\frac{\partial S}{\partial q_i} \right)^2 + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0 \qquad (59)$$

for which necessary and sufficient conditions on the coefficients $a_{ii}^*(\mathbf{q}, t)$ and $V(\mathbf{q}, t)$ are sought so that there exists a complete integral of the form

$$S(\mathbf{q}, t; \alpha) = \sum_{i=1}^{n} X_i(q_i; \alpha) - T(t; \alpha) \qquad (11)$$

Because equation (59) does not contain cross terms $(\partial S / \partial q_i)(\partial S / \partial q_j)$ or first-degree terms $(\partial S / \partial q_i)$, for convenience it will be referred to as the special diagonal Hamilton-Jacobi equation in contrast to

$$\sum_{i=1}^{n} a_{ii}^*(\mathbf{q}, t) \left(\frac{\partial S}{\partial q_i} \right)^2 + \sum_{i=1}^{n} b_i^*(\mathbf{q}, t) \frac{\partial S}{\partial q_i} + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0 \qquad (60)$$

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and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{*}(\mathbf{q}, t) \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} + \sum_{i=1}^{n} b_i^{*}(\mathbf{q}, t) \frac{\partial S}{\partial q_i} + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0 \tag{54}$$

which are referred to as the general diagonal Hamilton-Jacobi equation and the general Hamilton-Jacobi equation, respectively.

In the section on solutions, the results of this analysis with $n = 2$ will be applied to the solution of a comparatively wide class of dynamical problems involving a particle moving in a Euclidean three-dimensional space under the action of external forces but constrained on a moving surface.

SEPARATION OF THE SPECIAL DIAGONAL HAMILTON-JACOBI EQUATION

In this section, necessary and sufficient conditions will be established for the separation of the special diagonal Hamilton-Jacobi equation in the form of a sum of functions, each of which is a function of only one variable together with the arbitrary constants α. The results are a generalization of those obtained by Stäckel (refs. 6 and 7) because the Hamiltonian is now time dependent. The case of two degrees of freedom will be considered specifically before the general case of n degrees of freedom.

Theorem on Separation of Variables for Two Degrees of Freedom

Consider the equation

$$a_{11}^{*}(x, y, t) \left(\frac{\partial S}{\partial x}\right)^2 + a_{22}^{*}(x, y, t) \left(\frac{\partial S}{\partial y}\right)^2 + V(x, y, t) + \frac{\partial S}{\partial t} = 0 \tag{61}$$

where, for convenience, the following notation has been introduced:

$$q_1 = x \tag{62a}$$

$$q_2 = y \tag{62b}$$

It is desired to investigate the existence of a complete integral of the form

$$S(x, y, t; \alpha_1, \alpha_2) = X(x; \alpha_1, \alpha_2) + Y(y; \alpha_1, \alpha_2) - T(t; \alpha_1, \alpha_2) \tag{63}$$

i.e.; a solution with the property that

$$\begin{vmatrix} \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial x} & \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial y} \\[2ex] \dfrac{\partial^2 S}{\partial \alpha_2 \, \partial x} & \dfrac{\partial^2 S}{\partial \alpha_2 \, \partial y} \end{vmatrix} \neq 0 \tag{64}$$

where $X(x; \alpha_1, \alpha_2)$, $Y(y; \alpha_1, \alpha_2)$, and $T(t; \alpha_1, \alpha_2)$ are arbitrary functions of x, y, and t, respectively, and α_1 and α_2 are arbitrary constants. Substituting equation (63) into (61) yields

$$a_{11}^* \left(\frac{dX}{dx}\right)^2 + a_{22}^* \left(\frac{dY}{dy}\right)^2 + V - \frac{dT}{dt} = 0 \tag{65}$$

For convenience, the symbols θ, ϕ, and ψ are introduced and defined by

$$\theta(x; \alpha_1, \alpha_2) = \left(\frac{dX}{dx}\right)^2 \tag{66a}$$

$$\phi(y; \alpha_1, \alpha_2) = \left(\frac{dY}{dy}\right)^2 \tag{66b}$$

$$\psi(t; \alpha_1, \alpha_2) = \frac{dT}{dt} \tag{66c}$$

Then equation (65) becomes

$$a_{11}^* \theta + a_{22}^* \phi + V = \psi \tag{67}$$

A partial differentiation of this equation with respect to α_1 and α_2 yields

$$a_{11}^* \theta_1 + a_{22}^* \phi_1 = \psi_1 \tag{68a}$$

$$a_{11}^* \theta_2 + a_{22}^* \phi_2 = \psi_2 \tag{68b}$$

where the subscripts 1 and 2 on θ, ϕ, and ψ only refer to partial derivatives with respect to α_1 and α_2, respectively. Next, from equations (63) and (66),

$$
\begin{vmatrix} \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial x} & \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial y} \\[3mm] \dfrac{\partial^2 S}{\partial \alpha_2 \, \partial x} & \dfrac{\partial^2 S}{\partial \alpha_2 \, \partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial \alpha_1}\left(\dfrac{dX}{dx}\right) & \dfrac{\partial}{\partial \alpha_1}\left(\dfrac{dY}{dy}\right) \\[3mm] \dfrac{\partial}{\partial \alpha_2}\left(\dfrac{dX}{dx}\right) & \dfrac{\partial}{\partial \alpha_2}\left(\dfrac{dY}{dy}\right) \end{vmatrix}
$$

$$
= \frac{1}{4\dfrac{dX}{dx}\dfrac{dY}{dy}} \begin{vmatrix} \dfrac{\partial}{\partial \alpha_1}\left(\dfrac{dX}{dx}\right)^2 & \dfrac{\partial}{\partial \alpha_1}\left(\dfrac{dY}{dy}\right)^2 \\[3mm] \dfrac{\partial}{\partial \alpha_2}\left(\dfrac{dX}{dx}\right)^2 & \dfrac{\partial}{\partial \alpha_2}\left(\dfrac{dY}{dy}\right)^2 \end{vmatrix} = \frac{\begin{vmatrix} \theta_1 & \phi_1 \\ \theta_2 & \phi_2 \end{vmatrix}}{4\dfrac{dX}{dx}\dfrac{dY}{dy}} \tag{69}
$$

Hence, in view of property (64), it follows that

$$\begin{vmatrix} \theta_1 & \phi_1 \\ \theta_2 & \phi_2 \end{vmatrix} \neq 0 \tag{70}$$

Consequently, equations (68) may be solved for a_{11}^* and a_{22}^* so as to obtain

$$a_{11}^*(x,y,t) = \frac{\psi_1(t;\alpha_1,\alpha_2)\phi_2(y;\alpha_1,\alpha_2) - \psi_2(t;\alpha_1,\alpha_2)\phi_1(y;\alpha_1,\alpha_2)}{\theta_1(x;\alpha_1,\alpha_2)\phi_2(y;\alpha_1,\alpha_2) - \theta_2(x;\alpha_1,\alpha_2)\phi_1(y;\alpha_1,\alpha_2)} \tag{71a}$$

$$a_{22}^*(x,y,t) = \frac{-\psi_1(t;\alpha_1,\alpha_2)\theta_2(x;\alpha_1,\alpha_2) + \psi_2(t;\alpha_1,\alpha_2)\theta_1(x;\alpha_1,\alpha_2)}{\theta_1(x;\alpha_1,\alpha_2)\phi_2(y;\alpha_1,\alpha_2) - \theta_2(x;\alpha_1,\alpha_2)\phi_1(y;\alpha_1,\alpha_2)} \tag{71b}$$

Because these relations must hold for all values of α_1 and α_2, it follows that a_{11}^* and a_{22}^* must have the forms

$$a_{11}^*(x,y,t) = \frac{T_1'(t)Y_2(y) - T_2(t)Y_1(y)}{X_1(x)Y_2(y) - X_2(x)Y_1(y)} \tag{72a}$$

$$a_{22}^*(x,y,t) = \frac{-T_1(t)X_2(x) + T_2(t)X_1(x)}{X_1(x)Y_2(y) - X_2(x)Y_1(y)} \tag{72b}$$

where $X_1(x)$, $X_2(x)$, $Y_1(y)$, $Y_2(y)$, $T_1(t)$, and $T_2(t)$ are arbitrary functions of their arguments.

Next, note that equation (65) may also be written as

$$V(x,y,t) = \psi(t;\alpha_1,\alpha_2) - a_{11}^*(x,y,t)\theta(x;\alpha_1,\alpha_2) - a_{22}^*(x,y,t)\phi(y;\alpha_1,\alpha_2) \tag{73}$$

Again, because this relation must hold for all values of α_1 and α_2, it follows that V must have the form

$$V(x,y,t) = T_0(t) - a_{11}^*(x,y,t)X_0(x) - a_{22}^*(x,y,t)Y_0(y) \tag{74}$$

where $X_0(x)$, $Y_0(y)$, and $T_0(t)$ are also arbitrary functions of their arguments.

Finally, it is seen that equations (72) and (74) constitute necessary conditions on a_{11}^*, a_{22}^*, and V for equation (61) to have a complete integral of the form of equation (63).

Next, it is shown that these conditions are also sufficient. That is, if a_{11}^*, a_{22}^*, and V are of the forms of equations (72) and (74), then equation (61) has a solution of the form of equation (63). To do this, first consider the equation

$$a_{11}^* \left(\frac{dX}{dx}\right)^2 + a_{22}^* \left(\frac{dY}{dy}\right)^2 + V - \frac{dT}{dt} = 0 \tag{65}$$

where

$$X = X(x;\alpha_1,\alpha_2) \tag{75a}$$

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$$Y = Y(y; \alpha_1, \alpha_2) \tag{75b}$$

$$T = T(t; \alpha_1, \alpha_2) \tag{75c}$$

Substituting a_{11}^*, a_{22}^*, and V from equations (72) and (74) into (65) yields

$$\frac{T_1 Y_2 - T_2 Y_1}{X_1 Y_2 - X_2 Y_1}\left[\left(\frac{dX}{dx}\right)^2 - X_0\right] - \frac{T_1 X_2 - T_2 X_1}{X_1 Y_2 - X_2 Y_1}\left[\left(\frac{dY}{dy}\right)^2 - Y_0\right] = \frac{dT}{dt} - T_0 \tag{76}$$

Next, multiply by

$$\frac{X_1 Y_2 - X_2 Y_1}{(T_1 Y_2 - T_2 Y_1)(T_1 X_2 - T_2 X_1)}$$

The result is

$$\frac{1}{T_1 X_2 - T_2 X_1}\left[\left(\frac{dX}{dx}\right)^2 - X_0\right] - \frac{1}{T_1 Y_2 - T_2 Y_1}\left[\left(\frac{dY}{dy}\right)^2 - Y_0\right]$$

$$= \frac{(X_1 Y_2 - X_2 Y_1)(dT/dt - T_0)}{(T_1 Y_2 - T_2 Y_1)(T_1 X_2 - T_2 X_1)} \tag{77}$$

It is noted that

$$\frac{X_1 Y_2 - X_2 Y_1}{(T_1 Y_2 - T_2 Y_1)(T_1 X_2 - T_2 X_1)} = \frac{X_1/T_1}{T_1 X_2 - T_2 X_1} - \frac{Y_1/T_1}{T_1 Y_2 - T_2 Y_1} \tag{78}$$

so that equation (77) becomes

$$\frac{X_1}{T_1 X_2 - T_2 X_1}\left\{\frac{1}{X_1}\left[\left(\frac{dX}{dx}\right)^2 - X_0\right] - \frac{1}{T_1}\left(\frac{dT}{dt} - T_0\right)\right\}$$

$$= \frac{Y_1}{T_1 Y_2 - T_2 Y_1}\left\{\frac{1}{Y_1}\left[\left(\frac{dY}{dy}\right)^2 - Y_0\right] - \frac{1}{T_1}\left(\frac{dT}{dt} - T_0\right)\right\} \tag{79}$$

The left-hand side of this equation is a function of x and t only whereas the right-hand side is a function of y and t only. This can only be true if each side is equal to the same function of t because $x, y,$ and t are all independent variables.

Consequently, from equation (79),

$$\frac{X_1}{T_1 X_2 - T_2 X_1}\left\{\frac{1}{X_1}\left[\left(\frac{dX}{dx}\right)^2 - X_0\right] - \frac{1}{T_1}\left(\frac{dT}{dt} - T_0\right)\right\} = f(t) \tag{80a}$$

$$\frac{Y_1}{T_1 Y_2 - T_2 Y_1} \left\{ \frac{1}{Y_1} \left[\left(\frac{dY}{dy} \right)^2 - Y_0 \right] - \frac{1}{T_1} \left(\frac{dT}{dt} - T_0 \right) \right\} = f(t) \tag{80b}$$

On subtracting these two equations, one obtains

$$\frac{1}{X_1} \left[\left(\frac{dX}{dx} \right)^2 - X_0 \right] - \frac{1}{Y_1} \left[\left(\frac{dY}{dy} \right)^2 - Y_0 \right] = f(t) T_1 \frac{X_2 Y_1 - Y_2 X_1}{X_1 Y_1} \tag{81}$$

or

$$\frac{Y_1}{X_2 Y_1 - Y_2 X_1} \left[\left(\frac{dX}{dx} \right)^2 - X_0 \right] - \frac{X_1}{X_2 Y_1 - Y_2 X_1} \left[\left(\frac{dY}{dy} \right)^2 - Y_0 \right] = f(t) T_1 \tag{82}$$

In equation (82), the left-hand side is a function of x and y only, whereas the right-hand side is a function of t only. This can only be true if each side is equal to the same constant; therefore,

$$f(t) T_1 = \alpha_2 \tag{83}$$

and

$$\frac{1}{X_1} \left[\left(\frac{dX}{dx} \right)^2 - X_0 \right] - \frac{1}{Y_1} \left[\left(\frac{dY}{dy} \right)^2 - Y_0 \right] = \frac{\alpha_2 (X_2 Y_1 - Y_2 X_1)}{X_1 Y_1} \tag{84}$$

From equation (84),

$$\frac{1}{X_1} \left[\left(\frac{dX}{dx} \right)^2 - X_0 \right] - \alpha_2 \frac{X_2}{X_1} = \frac{1}{Y_1} \left[\left(\frac{dY}{dy} \right)^2 - Y_0 \right] - \alpha_2 \frac{Y_2}{Y_1} \tag{85}$$

Again, the left-hand side is a function of x only and the right-hand side is a function of y only; therefore, they must both equal the same constant. Consequently,

$$\left(\frac{dX}{dx} \right)^2 - X_0 - \alpha_2 X_2 = \alpha_1 X_1 \tag{86a}$$

$$\left(\frac{dY}{dy} \right)^2 - Y_0 - \alpha_2 Y_2 = \alpha_1 Y_1 \tag{86b}$$

Finally, substituting equations (83) and (86a) into equation (80) yields

$$\frac{X_1}{T_1 X_2 - T_2 X_1} \left[\alpha_2 \frac{X_2}{X_1} + \alpha_1 - \frac{1}{T_1} \left(\frac{dT}{dt} - T_0 \right) \right] = \frac{\alpha_2}{T_1} \tag{87}$$

which reduces to

$$\frac{dT}{dt} - T_0 - \alpha_1 T_1 = \alpha_2 T_2 \tag{88}$$

Hence, it is seen that if a_{11}^*, a_{22}^*, and V are given by equations (72) and (74), then equation (65) has a solution for which the functions X, Y, and T are given by

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$$\left(\frac{dX}{dx}\right)^2 = X_0(x) + \alpha_1 X_1(x) + \alpha_2 X_2(x) \tag{89a}$$

$$\left(\frac{dY}{dy}\right)^2 = Y_0(y) + \alpha_1 Y_1(y) + \alpha_2 Y_2(y) \tag{89b}$$

$$\frac{dT}{dt} = T_0(t) + \alpha_1 T_1(t) + \alpha_2 T_2(t) \tag{89c}$$

Consequently, it follows that if a_{11}^*, a_{22}^*, and V are given by equations (72) and (74), then equation (61) has a solution of the form of equation (63).

The preceding results are summarized in the following theorem.

Theorem 1: The equation

$$a_{11}^*(x, y, t)\left(\frac{\partial S}{\partial x}\right)^2 + a_{22}^*(x, y, t)\left(\frac{\partial S}{\partial y}\right)^2 + V(x, y, t) + \frac{\partial S}{\partial t} = 0 \tag{61}$$

has a complete integral of the form

$$S(x, y, t; \alpha_1, \alpha_2) = X(x; \alpha_1, \alpha_2) + Y(y; \alpha_1, \alpha_2) - T(t; \alpha_1, \alpha_2) \tag{63}$$

if and only if a_{11}^*, a_{22}^*, and V have the forms

$$a_{11}^*(x, y, t) = \frac{T_1(t)Y_2(y) - T_2(t)Y_1(y)}{X_1(x)Y_2(y) - X_2(x)Y_1(y)} \tag{72a}$$

$$a_{22}^*(x, y, t) = \frac{-T_1(t)X_2(x) + T_2(t)X_1(x)}{X_1(x)Y_2(y) - X_2(x)Y_1(y)} \tag{72b}$$

$$V(x, y, t) = T_0(t) - a_{11}^*(x, y, t)X_0(x) - a_{22}^*(x, y, t)Y_0(y) \tag{74}$$

where $X_i(x)$, $Y_i(y)$, and $T_i(t)$ for $i = 0, 1, 2$ are arbitrary functions of their arguments. The functions $X(x; \alpha_1, \alpha_2)$, $Y(y; \alpha_1, \alpha_2)$, and $T(t; \alpha_1, \alpha_2)$ are obtained from the equations

$$\left(\frac{dX}{dx}\right)^2 = X_0(x) + \alpha_1 X_1(x) + \alpha_2 X_2(x) \tag{89a}$$

$$\left(\frac{dY}{dy}\right)^2 = Y_0(y) + \alpha_1 Y_1(y) + \alpha_2 Y_2(y) \tag{89b}$$

$$\frac{dT}{dt} = T_0(t) + \alpha_1 T_1(t) + \alpha_2 T_2(t) \tag{89c}$$

so that the complete integral is given by

$$S(x, y, t; \alpha_1, \alpha_2) = \int (X_0 + \alpha_1 X_1 + \alpha_2 X_2)^{1/2} \, dx$$

$$+ \int (Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2)^{1/2} \, dy - \int (T_0 + \alpha_1 T_1 + \alpha_2 T_2) \, dt \qquad (90)$$

Theorem on Separation of Variables for *n* Degrees of Freedom

In this section the equation

$$\sum_{i=1}^{n} a_{ii}^*(\mathbf{q}, t) \left(\frac{\partial S}{\partial q_i}\right)^2 + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0 \qquad (59)$$

is considered and an investigation is made of the existence of a complete integral of the form

$$S(\mathbf{q}, t; \alpha) = \sum_{i=1}^{n} X_i(q_i; \alpha) - T(t; \alpha) \qquad (11)$$

i.e., a solution with the property that

$$\left| \frac{\partial^2 S}{\partial \alpha_i \, \partial q_j} \right| \not\equiv 0 \qquad (57)$$

where $X_i(q_i; \alpha)$ and $T(t; \alpha)$ are arbitrary functions of q_i and t, respectively, and also of

$$\alpha \equiv [\alpha_1, \alpha_2, \ldots, \alpha_n] \qquad (91)$$

(As a rule, the vector **A** will always denote a column vector even though it may be written horizontally as above for the sake of convenience and conservation of space.)

Substituting equation (11) into (59) yields

$$\sum_{i=1}^{n} a_{ii}^* \left(\frac{dX_i}{dq_i}\right)^2 + V - \frac{dT}{dt} = 0 \qquad (92)$$

For convenience, the symbols θ_i, ψ, $\boldsymbol{\theta}$, and $\boldsymbol{\gamma}$ are introduced and defined:

$$\theta_i(q_i; \alpha) = \left(\frac{dX_i}{dq_i}\right)^2 \qquad i = 1, 2, \ldots, n \qquad (93a)$$

$$\psi(t; \alpha) = \frac{dT}{dt} \qquad (93b)$$

$$\boldsymbol{\theta} = [\theta_1(q_1; \alpha), \theta_2(q_2; \alpha), \ldots, \theta_n(q_n; \alpha)] \qquad (93c)$$

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$$\gamma = [a_{11}^*(\mathbf{q}, t), a_{22}^*(\mathbf{q}, t), \ldots, a_{nn}^*(\mathbf{q}, t)] \tag{93d}$$

Then, equation (92) becomes

$$\sum_{i=1}^{n} \gamma_i \theta_i + V = \psi \tag{94a}$$

or

$$\gamma^T \cdot \theta + V = \psi \tag{94b}$$

where the superscript T denotes the transpose of a vector (or a matrix).

Differentiation of this equation with respect to α_i for $i = 1, 2, \ldots, n$ yields

$$\sum_{j=1}^{n} \gamma_j \frac{\partial \theta_j}{\partial \alpha_i} = \frac{\partial \psi}{\partial \alpha_i} \qquad i = 1, 2, \ldots, n \tag{95}$$

or

$$\Theta \cdot \gamma = \Psi \tag{96}$$

where by definition

$$\Theta = \Theta_{ij}(q_j; \alpha)$$

$$= \frac{\partial}{\partial \alpha_i} \theta_j(q_j; \alpha) \tag{97a}$$

$$\Psi = [\Psi_1(t; \alpha), \Psi_2(t; \alpha), \ldots, \Psi_n(t; \alpha)]$$

$$= \left[\frac{\partial \psi(t; \alpha)}{\partial \alpha_1}, \frac{\partial \psi(t; \alpha)}{\partial \alpha_2}, \ldots, \frac{\partial \psi(t; \alpha)}{\partial \alpha_n} \right] \tag{97b}$$

From equation (11) it follows that

$$\left| \frac{\partial^2 S}{\partial \alpha_i \, \partial q_j} \right| = \begin{vmatrix} \frac{\partial}{\partial \alpha_1}\left(\frac{dX_1}{dq_1}\right) & \cdots & \frac{\partial}{\partial \alpha_1}\left(\frac{dX_n}{dq_n}\right) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \frac{\partial}{\partial \alpha_n}\left(\frac{dX_1}{dq_1}\right) & \cdots & \frac{\partial}{\partial \alpha_n}\left(\frac{dX_n}{dq_n}\right) \end{vmatrix} = \frac{1}{2^n \prod_{i=1}^{n} \frac{dX_i}{dq_i}} \begin{vmatrix} \frac{\partial}{\partial \alpha_1}\left(\frac{dX_1}{dq_1}\right)^2 & \cdots & \frac{\partial}{\partial \alpha_1}\left(\frac{dX_n}{dq_n}\right)^2 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \frac{\partial}{\partial \alpha_n}\left(\frac{dX_1}{dq_1}\right)^2 & \cdots & \frac{\partial}{\partial \alpha_n}\left(\frac{dX_n}{dq_n}\right)^2 \end{vmatrix}$$

$$= \frac{\left| \dfrac{\partial \theta_j}{\partial \alpha_i} \right|}{2^n \prod_{i=1}^{n} \dfrac{dX_i}{dq_i}} = \frac{|\Theta|}{2^n \prod_{i=1}^{n} \dfrac{dX_i}{dq_i}} \tag{98}$$

Hence, from equations (57) and (98), it is seen that

$$|\Theta| \neq 0 \qquad (99)$$

Solving equation (96) for γ yields

$$\gamma = \Theta^{-1} \cdot \Psi \qquad (100)$$

Denoting the minor of Θ_{ij} by Φ_{ij} allows equation (100) to be written as

$$\gamma_j(\mathbf{q}, t) = \frac{\displaystyle\sum_{i=1}^{n} (-1)^{i+j} \Psi_i(t; \alpha) \Phi_{ij}}{\displaystyle\sum_{i=1}^{n} (-1)^{i+j} \Theta_{ij}(q_j; \alpha) \Phi_{ij}}$$

$$= \frac{\displaystyle\sum_{i=1}^{n} (-1)^{i+j} \Psi_i(t; \alpha) \sum_{P} (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j}}^{n} \Theta_{kl}(q_l; \alpha)}{\displaystyle\sum_{i=1}^{n} (-1)^{i+j} \Theta_{ij}(q_j; \alpha) \sum_{P} (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j}}^{n} \Theta_{kl}(q_l; \alpha)} \qquad (101)$$

where for convenience the following notation has been introduced:

$$\sum_{P} (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j}}^{n} \Theta_{kl}(q_l; \alpha) = \delta_{1,2,\ldots,n-1}^{i_1, i_2, \ldots, i_{n-1}} \omega_{1 i_1}(q_{i_1}; \alpha) \ldots \omega_{(n-1) i_{n-1}}(q_{i_{n-1}}; \alpha) \qquad (102)$$

in which the summation convention is used, $\delta_{1,2,\ldots,n-1}^{i_1, i_2, \ldots, i_{n-1}}$ is the generalized Kronecker delta, and $[\omega_{kl}]$ is the matrix of elements appearing in the minor Φ_{ij}; i.e.,

$$[\omega_{kl}] = \begin{bmatrix} \Theta_{11}(q_1; \alpha) & \cdots & \Theta_{1(j-1)}(q_{j-1}; \alpha) & \Theta_{1(j+1)}(q_{j+1}; \alpha) & \cdots & \Theta_{1n}(q_n; \alpha) \\ & & & & & \\ \Theta_{(i-1)1}(q_1; \alpha) & \cdots & \Theta_{(i-1)(j-1)}(q_{j-1}; \alpha) & \Theta_{(i-1)(j+1)}(q_{j+1}; \alpha) & \cdots & \Theta_{(i-1)n}(q_n; \alpha) \\ \Theta_{(i+1)1}(q_1; \alpha) & \cdots & \Theta_{(i+1)(j-1)}(q_{j-1}; \alpha) & \Theta_{(i+1)(j+1)}(q_{j+1}; \alpha) & \cdots & \Theta_{(i+1)n}(q_n; \alpha) \\ & & & & & \\ \Theta_{n1}(q_1; \alpha) & \cdots & \Theta_{n(j-1)}(q_{j-1}; \alpha) & \Theta_{n(j+1)}(q_{j+1}; \alpha) & \cdots & \Theta_{nn}(q_n; \alpha) \end{bmatrix} \qquad (103)$$

Because the relations in equation (101) must hold for all values of α, it follows that $\gamma(\mathbf{q}, t)$ must have the form

$$\gamma_j(\mathbf{q}, t) = \frac{\displaystyle\sum_{i=1}^n (-1)^{i+j} T_i(t) \sum_P (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j}}^n X_{kl}(q_l)}{\displaystyle\sum_{i=1}^n (-1)^{i+j} X_{ij}(q_j) \sum_P (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j \\ j=1,2,\ldots,n}}^n X_{kl}(q_l)} \tag{104}$$

or

$$\gamma = \mathbf{X}^{-1} \cdot \mathbf{T} \tag{105}$$

$$|\mathbf{X}| = \neq 0$$

$$\tag{106}$$

where

$$\mathbf{X} = [X_{ij}(q_j)] \qquad i, j = 1, 2, \ldots, n \tag{107a}$$

$$\mathbf{T} = [T_1(t), T_2(t), \ldots, T_n(t)] \tag{107b}$$

Consequently, the right-hand side of equation (101) must have numerator N and denominator $|\Theta|$ of the form

$$N = \lambda(\alpha) \sum_{i=1}^n (-1)^{i+j} T_i(t) \sum_P (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j}}^n X_{kl}(q_l) \tag{108b}$$

$$|\Theta| = \lambda(\alpha) \sum_{i=1}^n (-1)^{i+j} X_{ij}(q_j) \sum_P (\pm) \prod_{\substack{k, l=1 \\ k \neq i, l \neq j}}^n X_{kl}(q_l) = \lambda(\alpha) \cdot |\mathbf{X}| \tag{108a}$$

where $\lambda(\alpha)$ is some arbitrary function that does not vanish identically.

Equation (94a) may also be written as

$$V(\mathbf{q}, t) = \psi(t; \alpha) - \sum_{i=1}^n \gamma_i(\mathbf{q}, t)\theta_i(q_i; \alpha) \tag{109}$$

Again, because this relation must hold for all values of α, it follows that $V(\mathbf{q}, t)$ must be of the form

$$V(\mathbf{q}, t) = T_0(t) - \sum_{i=1}^n \gamma_i(\mathbf{q}, t)X_{0i}(q_i) \tag{110}$$

or

$$V = T_0 - \mathbf{X}_0^T \cdot \boldsymbol{\gamma} \qquad (111)$$

where

$$\mathbf{X}_0 = [X_{01}(q_1), X_{02}(q_2), \ldots, X_{0n}(q_n)] \qquad (112a)$$

$$T_0 = T_0(t) \qquad (112b)$$

Finally, it is seen that equations (104) and (109) constitute necessary conditions on $\boldsymbol{\gamma}$ and V for equation (59) to have a complete integral of the form of equation (11).

Next, to show that these conditions are also sufficient, it is noted that if $\boldsymbol{\gamma}$ is given to have the form of equation (105) where $X_{ij}(q_j)$ and $T_i(t)$ are arbitrary functions, then $\Theta_{ij}(q_j; \boldsymbol{\alpha})$ and $\Psi_i(t; \boldsymbol{\alpha})$ may be chosen such that

$$\left. \begin{aligned} \frac{\partial}{\partial \alpha_i} \theta_j(q_j; \boldsymbol{\alpha}) &= \Theta_{ij}(q_j; \boldsymbol{\alpha}) \\[2mm] &= X_{ij}(q_j) \end{aligned} \right\} \qquad i, j = 1, 2, \ldots, n \qquad (113a)$$

$$\left. \begin{aligned} \frac{\partial}{\partial \alpha_i} \psi(t; \boldsymbol{\alpha}) &= \Psi_i(t; \boldsymbol{\alpha}) \\[2mm] &= T_i(t) \end{aligned} \right\} \qquad i, j = 1, 2, \ldots, n \qquad (113b)$$

noting that $|\Theta| = \lambda(\boldsymbol{\alpha}) \cdot |\mathbf{X}|$ is satisfied with $\lambda(\boldsymbol{\alpha}) \equiv 1$. From this, it follows that

$$\theta_j(q_j; \boldsymbol{\alpha}) = X_{0j}^*(q_j) + \sum_{i=1}^{n} X_{ij}(q_j)\alpha_i \qquad j = 1, 2, \ldots, n \qquad (114a)$$

$$\psi(t; \boldsymbol{\alpha}) = T_0^*(t) + \sum_{i=1}^{n} T_i(t)\alpha_i \qquad (114b)$$

where $X_{0j}^*(q_j)$ and $T_0^*(t)$ are arbitrary functions.

Moreover, if V is given to have the form of equation (110) where $X_{0i}(q_i)$ and $T_0(t)$ are arbitrary functions, then from equations (94a) and (101) it is found that

$$\sum_{i=1}^{n} \gamma_i \left[X_{0i}^*(q_i) + \sum_{j=1}^{n} X_{ji}(q_i)\alpha_j \right] + T_0(t) - \sum_{i=1}^{n} \gamma_i X_{0i}(q_i) = T_0^*(t) + \sum_{j=1}^{n} T_j(t)\alpha_j \qquad (115)$$

Because $\boldsymbol{\gamma}(\mathbf{q}, t)$ is arbitrary (to the extent that $X_{ij}(q_j)$ and $T_i(t)$ are arbitrary), it follows that

$$X_{0i}^*(q_i) = X_{0i}(q_i) \qquad (116a)$$

$$T_0^*(t) = T_0(t) \qquad (116b)$$

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In passing, for the sake of completeness of discussion, it is noted that because α are arbitrary constants, the remaining terms in equation (115) lead to the equation

$$\sum_{i=1}^{n} \gamma_i X_{ji} = T_j \tag{117}$$

or

$$\mathbf{X} \cdot \boldsymbol{\gamma} = \mathbf{T} \tag{118}$$

or

$$\boldsymbol{\gamma} = \mathbf{X}^{-1} \cdot \mathbf{T} \tag{119}$$

which is automatically satisfied because γ is given by equation (104) or (105).

Next, from equations (114) and (116) it follows that

$$\theta_i(q_i; \boldsymbol{\alpha}) = X_{0i}(q_i) + \sum_{j=1}^{n} X_{ji}(q_i)\alpha_j \qquad i = 1, 2, \ldots, n \tag{120a}$$

$$\psi(t; \alpha) = T_0(t) + \sum_{j=1}^{n} T_j(t)\alpha_j \tag{120b}$$

which are expressions for $\theta_i(q_i; \alpha)$ and $\psi(t; \alpha)$ that satisfy equation (94).

Hence, in view of equations (94) and (120), it is seen that equation (59) has a solution of the form of equation (11) for which $X_i(q_i; \boldsymbol{\alpha})$ and $T(t; \boldsymbol{\alpha})$ are given by

$$\left(\frac{dX_i}{dq_i}\right)^2 = X_{0i}(q_i) + \sum_{j=1}^{n} X_{ji}(q_i)\alpha_j \qquad i = 1, 2, \ldots, n \tag{121a}$$

$$\frac{dT}{dt} = T_0(t) + \sum_{j=1}^{n} T_j(t)\alpha_j \tag{121b}$$

where $|\mathbf{X}| \neq 0$. The fact that this solution is a complete integral is seen from equation (98) from which is obtained

$$\left| \frac{\partial^2 S}{\partial \alpha_i\, \partial q_j} \right| = \frac{|\Theta|}{2^n \prod_{i=1}^{n} \dfrac{dX_i}{dq_i}}$$

$$= \frac{|\mathbf{X}|}{2^n \prod_{i=1}^{n} \dfrac{dX_i}{dq_i}} \neq 0 \tag{122}$$

The preceding results are summarized in the following theorem:

Theorem 2: The equation

$$\sum_{i=1}^{n} a_{ii}^{*}(\mathbf{q}, t) \left(\frac{\partial S}{\partial q_i}\right)^2 + V(\mathbf{q}, t) + \frac{\partial S}{\partial t} = 0 \qquad (59)$$

has a complete integral of the form

$$S(\mathbf{q}, t; \boldsymbol{\alpha}) = \sum_{i=1}^{n} X_i(q_i; \boldsymbol{\alpha}) - T(t; \boldsymbol{\alpha}) \qquad (11)$$

if and only if $\boldsymbol{\gamma} \equiv [a_{11}^{*}, a_{22}^{*}, \ldots, a_{nn}^{*}]$ and V have the forms

$$\boldsymbol{\gamma} = \mathbf{X}^{-1} \cdot \mathbf{T} \qquad (105)$$

$$|\mathbf{X}| \not\equiv 0 \qquad (106)$$

$$V = T_0(t) - \mathbf{X}_0^T \cdot \mathbf{X}^{-1} \cdot \mathbf{T} \qquad (123)$$

where $T_0(t)$ is an arbitrary function and $\mathbf{X}_0$, $\mathbf{T}$, and $\mathbf{X}$ are arbitrary vectors or matrix given by

$$\mathbf{X}_0 = [X_{01}(q_1), X_{02}(q_2), \ldots, X_{0n}(q_n)] \qquad (112a)$$

$$\mathbf{T} = [T_1(t), T_2(t), \ldots, T_n(t)] \qquad (108b)$$

$$\mathbf{X} = [X_{ij}(q_j)] \qquad i, j = 1, 2, \ldots, n \qquad (108a)$$

and the functions $X_i(q_i; \boldsymbol{\alpha})$ and $T(t; \boldsymbol{\alpha})$ are obtained from the equations

$$\left(\frac{dX_i}{dq_i}\right)^2 = X_{0i}(q_i) + \sum_{j=1}^{n} X_{ji}(q_i)\alpha_j \qquad i = 1, 2, \ldots, n \qquad (121a)$$

$$\frac{dT}{dt} = T_0(t) + \sum_{j=1}^{n} T_j(t)\alpha_j \qquad (121b)$$

so that the complete integral is given by

$$S(\mathbf{q}, t; \boldsymbol{\alpha}) = \sum_{i=1}^{n} \int \left[X_{0i}(q_i) + \sum_{j=1}^{n} X_{ji}(q_i)\alpha_j \right]^{1/2} dq_i - \int \left[T_0(t) + \sum_{j=1}^{n} T_j(t)\alpha_j \right] dt \qquad (124)$$

Remarks

In the transition from equation (101) to (104) it is not required that the arbitrary functions $\Theta_{ij}(q_j; \boldsymbol{\alpha})$ and $\Psi_i(t; \boldsymbol{\alpha})$ of q_j and t be equal correspondingly to $X_{ij}(q_j)$ and $T_i(t)$. The reason for this is that it is possible for additive terms or factors involving $\boldsymbol{\alpha}$ to be eliminated respectively when a sum or

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a product (or quotient) is considered on the right-hand side of equation (101). The error made by Stäckel when considering the time-independent Hamilton-Jacobi equation was in stating that $\Theta_{ij}(q_j; \alpha)$ must be equal to $X_{ij}(q_j)$.

It is seen that these results reduce, as one would expect, to those obtained by Stäckel for the time-independent Hamilton-Jacobi equation merely by letting $T_1(t) = 1, T_i(t) = 0$ for $i = 0, 2, 3, \ldots, n$, and $\alpha_1 = h$.

It is possible for the arbitrary constants to appear in a more complicated form in the complete integral instead of just the simple form. This is accomplished by taking new arbitrary constants α^* where

$$\alpha^* = [\alpha_1^*(\alpha), \alpha_2^*(\alpha), \ldots, \alpha_n^*(\alpha)] \tag{125}$$

such that

$$\left| \frac{\partial \alpha_i^*}{\partial \alpha_j} \right| \neq 0 \tag{126}$$

In this case, it is obvious that, in general, $\Theta_{ij}(q_j; \alpha)$ and $\Psi_i(t; \alpha)$ are not equal to $X_{ij}(q_j)$ and $T_i(t)$ and that $\lambda(\alpha^*)$ in equation (107) is explicitly dependent on α^*.

However, these additional insights do not alter the necessary and sufficient conditions for equation (59) to have a complete integral of the form of equation (11). Neither do they introduce any essential generalization to the complete integral given in theorem 2 where it is stated in its simplest form.

SOLUTIONS OF THE DYNAMICAL PROBLEM OF A PARTICLE CONSTRAINED ON A MOVING SURFACE

In the previous section, conditions were derived for separating the solution of the n-dimensional special diagonal time-dependent Hamilton-Jacobi equation in the form of a sum of functions. It is obvious that these results can be applied to the solution of problems in dynamics as long as the conditions of separation are satisfied. In this section, the comparatively wide class of problems involving a particle moving under the action of external forces in a Euclidean three-dimensional space, but constrained on a surface moving as a function of time, will be considered. For these problems, there are two generalized coordinates and the coefficients in the Hamilton-Jacobi equation can involve time explicitly. Hence, the results stated in theorem 1 may be used.

The Motion of a Particle Under the Action of Forces in Euclidean Three-Dimensional Space and Constrained on a Moving Surface

Consider a particle moving under the action of forces in a Euclidean three-dimensional space E such that its coordinates always satisfy a constraint given by

$$f(x, y, z, t) = 0 \tag{127}$$

This means that at the instant $t = t_0$, it lies on a surface Σ whose equation is $f(x, y, z, t_0) = 0$; for various instants of time Σ will assume different shapes and positions in E. For simplicity, it is assumed

that the continuous motion of Σ in E generates a family of smooth surfaces and that no member intersects with any other member of the family. (See fig. 1.) One way of relaxing the condition on the smoothness of Σ is by considering Σ to be a cone whose angle changes so as to generate a family of cones with a common axis and a common vertex. (See subsection entitled "Spherical Coordinates" in the "Results" section.) Two ways of relaxing the condition on nonintersecting members of the family are by making Σ be a plane rotating about an axis (see subsection entitled "Cylindrical Coordinates" in the "Results" section) or by requiring the existence of finite intervals (t_1, t_2), (t_2, t_3), etc., such that no member intersects any other member of the family in the same time interval. (See the section entitled "Results" with $f(t)$ a piecewise-monotonic continuous function.)

Let ξ and η be constants defining any two families of intersecting curves always lying on Σ. As Σ moves in E, these two sets of curves will also, in general, change their shape. Thus, the motion of the particle in E can be described by the equations

$$x = x(\xi, \eta, t) \tag{128a}$$

$$y = y(\xi, \eta, t) \tag{128b}$$

$$z = z(\xi, \eta, t) \tag{128c}$$

which can also be interpreted as the equations of transformation from (x, y, z) to generalized coordinates (ξ, η) as a result of eliminating the constraint of equation (127).

From the introduction it is known that the most general Hamilton-Jacobi equation is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^* \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} + \sum_{i=1}^{n} b_i^* \frac{\partial S}{\partial q_i} + V + \frac{\partial S}{\partial t} = 0 \tag{54}$$
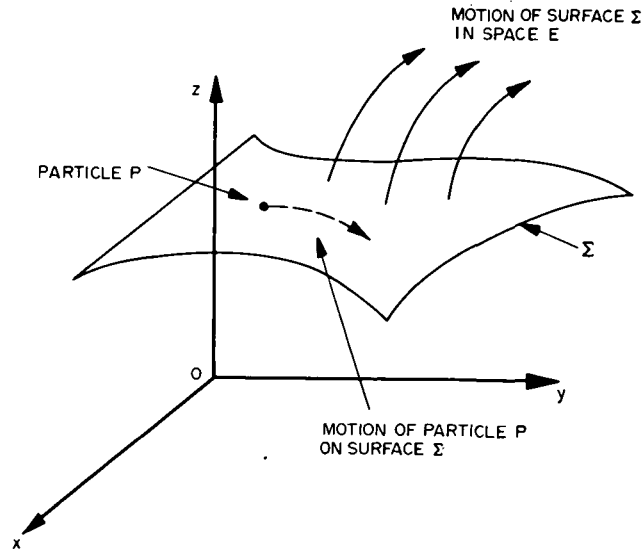

Figure 1.—Geometry of the particle.

where a_{ij}^*, b_i^*, and V are defined by equations (37). Hence, for the problem of the particle constrained on a surface moving in the space E,

$$a_{11}^* \left(\frac{\partial S}{\partial \xi}\right)^2 + a_{22}^* \left(\frac{\partial S}{\partial \eta}\right)^2 + 2a_{12}^* \frac{\partial S}{\partial \xi}\frac{\partial S}{\partial \eta} + b_1^* \frac{\partial S}{\partial \xi} + b_2^* \frac{\partial S}{\partial \eta} + V + \frac{\partial S}{\partial t} = 0 \qquad (129)$$

For simplicity, it is assumed that $\mathbf{b}' = 0$ in equations (16) and (17) so that the external force acting on the particle is derivable from a potential E_p. Then, from equations (18), (26), (31), and (128) it is seen that

$$a_{11} = \frac{1}{2}(x_\xi^2 + y_\xi^2 + z_\xi^2) \qquad (130a)$$

$$a_{22} = \frac{1}{2}(x_\eta^2 + y_\eta^2 + z_\eta^2) \qquad (130b)$$

$$a_{12} = a_{21} = \frac{1}{2}(x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta) \qquad (130c)$$

$$b_1 = a_1' - b_1' = a_1' = x_t x_\xi + y_t y_\xi + z_t z_\xi \qquad (130d)$$

$$b_2 = a_2' - b_2' = a_2' = x_t x_\eta + y_t y_\eta + z_t z_\eta \qquad (130e)$$

$$U = E_p - a = E_p - \frac{1}{2}(x_t^2 + y_t^2 + z_t^2) \qquad (130f)$$

From equations (37) and (129) it is found that

$$a_{11}^* = \frac{a_{11}^+}{4|A|} = \frac{(1/2)(x_\eta^2 + y_\eta^2 + z_\eta^2)}{4|A|} \qquad (131a)$$

$$a_{22}^* = \frac{a_{22}^+}{4|A|} = \frac{(1/2)(x_\xi^2 + y_\xi^2 + z_\xi^2)}{4|A|} \qquad (131b)$$

$$a_{12}^* = a_{21}^* = \frac{a_{12}^+}{4|A|} = \frac{-(1/2)(x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta)}{4|A|} \qquad (131c)$$

$$b_1^* = -\frac{1}{2}\left(b_1 \frac{a_{11}^+}{|A|} + b_2 \frac{a_{21}^+}{|A|}\right) \qquad (131d)$$

$$b_2^* = -\frac{1}{2}\left(b_1 \frac{a_{12}^+}{|A|} + b_2 \frac{a_{22}^+}{|A|}\right) \qquad (131e)$$

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$$V = E_p - \frac{1}{2}(x_t^2 + y_t^2 + z_t^2) + \frac{1}{4}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \tag{131f}$$

where

$$4|\mathbf{A}| = (x_\xi y_\eta - x_\eta y_\xi)^2 + (x_\xi z_\eta - x_\eta z_\xi)^2 + (y_\xi z_\eta - y_\eta z_\xi)^2 > 0 \tag{132}$$

If the results of theorem 1 are used and (ξ, η, t) are substituted for (x, y, t) in equations (72), a_{11}^* and a_{22}^* must be of the forms

$$a_{11}^* = \frac{T_1(t)N_2(\eta) - T_2(t)N_1(\eta)}{M_1(\xi)N_2(\eta) - M_2(\xi)N_1(\eta)} \tag{133a}$$

$$a_{22}^* = \frac{-T_1(t)M_2(\xi) + T_2(t)M_1(\xi)}{M_1(\xi)N_2(\eta) - M_2(\xi)N_1(\eta)} \tag{133b}$$

and

$$a_{12}^* = 0 \tag{134a}$$

$$b_1^* = 0 \tag{134b}$$

$$b_2^* = 0 \tag{134c}$$

It follows from equations (131), (132), and (134) that

$$-2a_{12}^+ = -2a_{21}^+ = x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta = 0 \tag{135a}$$

$$b_1 = 0 \tag{135b}$$

$$b_2 = 0 \tag{135c}$$

$$V = E_p - \frac{1}{2}(x_t^2 + y_t^2 + z_t^2) \tag{135d}$$

From equations (130) and (135), it is seen that

$$a_1' = x_t x_\xi + y_t y_\xi + z_t z_\xi = 0 \tag{136a}$$

$$a_2' = x_t x_\eta + y_t y_\eta + z_t z_\eta = 0 \tag{136b}$$

Consequently, to use the results of theorem 1, the equations of transformation (128) must satisfy the conditions

$$x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta = 0 \tag{137a}$$

$$x_t x_\xi + y_t y_\xi + z_t z_\xi = 0 \tag{137b}$$

$$x_t x_\eta + y_t y_\eta + z_t z_\eta = 0 \tag{137c}$$

$$\frac{x_\eta^2 + y_\eta^2 + z_\eta^2}{8|\mathbf{A}|} = a_{11}^* = \frac{T_1 N_2 - T_2 N_1}{M_1 N_2 - M_2 N_1} \tag{137d}$$

$$\frac{x_\xi^2 + y_\xi^2 + z_\xi^2}{8\,|\mathbf{A}|} = a_{22}^* = \frac{-T_1 M_2 + T_2 M_1}{M_1 N_2 - M_2 N_1} \tag{137e}$$

where

$$4\,|\mathbf{A}| = (x_\xi y_\eta - x_\eta y_\xi)^2 + (x_\xi z_\eta - x_\eta z_\xi)^2 + (y_\xi z_\eta - y_\eta z_\xi)^2 \tag{138}$$

and the potential energy E_p must be of the form

$$E_p - \frac{1}{2}(x_t^2 + y_t^2 + z_t^2) = V = T_0(t) - a_{11}^* M_0(\xi) - a_{22}^* N_0(\eta) \tag{139}$$

The conditions given by equations (137) to (139) are necessary and sufficient for the Hamilton-Jacobi equation describing the motion of a particle on a moving surface to be separable.

It is obvious that the set of equations (137) and (138) is highly nonlinear and that to use the results of theorem 1, one must be able to solve this set of coupled nonlinear first-order partial differential equations; actually, at the beginning it is not even known whether any solutions exist. First, let us prove the following lemma.

Lemma 1: Given a Euclidean three-dimensional space E and a set of Cartesian coordinates (x, y, z), let (ξ, η, t) be new coordinates defined by the transformation

$$x = x(\xi, \eta, t) \tag{128a}$$

$$y = y(\xi, \eta, t) \tag{128b}$$

$$z = z(\xi, \eta, t) \tag{128c}$$

Suppose that these equations can be inverted to yield

$$\xi = \xi(x, y, z) \tag{140a}$$

$$\eta = \eta(x, y, z) \tag{140b}$$

$$t = t(x, y, z) \tag{140c}$$

Then the set of equations

$$x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta = 0 \tag{137a}$$

$$x_t x_\xi + y_t y_\xi + z_t z_\xi = 0 \tag{137b}$$

$$x_t x_\eta + y_t y_\eta + z_t z_\eta = 0 \tag{137c}$$

holds if and only if the following set of equations holds

$$\xi_x \eta_x + \xi_y \eta_y + \xi_z \eta_z = 0 \tag{141a}$$

$$t_x \xi_x + t_y \xi_y + t_z \xi_z = 0 \tag{141b}$$

$$t_x \eta_x + t_y \eta_y + t_z \eta_z = 0 \tag{141c}$$

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Proof: Consider the line element

$$ds^2 = g_{ij} \, dx^i \, dx^j \qquad i, j = 1, 2, 3 \tag{142}$$

where

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z$$

Note that this may also be written as

$$ds^2 = \bar{g}_{kl} \, d\bar{x}^k \, d\bar{x}^l \qquad k, l = 1, 2, 3 \tag{143}$$

where

$$\bar{x}^1 = \xi$$

$$\bar{x}^2 = \eta$$

$$\bar{x}^3 = t$$

Because

$$\bar{g}_{kl} = g_{ij} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} \tag{144}$$

and

$$g_{ij} = \delta_{ij} \tag{145}$$

therefore,

$$\bar{g}_{12} = \frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \eta} \tag{146a}$$

$$\bar{g}_{13} = \frac{\partial x}{\partial \xi} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial t} + \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial t} \tag{146b}$$

$$\bar{g}_{23} = \frac{\partial x}{\partial \eta} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial t} + \frac{\partial z}{\partial \eta} \frac{\partial z}{\partial t} \tag{146c}$$

Hence, from equations (137) and (146), it is seen that

$$\bar{g}_{ij} = 0 \qquad i \neq j \tag{147}$$

so that the coordinate system (ξ, η, t) is orthogonal. That is, the three families of the surfaces given by

$$\xi(x, y, z) = c_1 \tag{148a}$$

$$\eta(x, y, z) = c_2 \tag{148b}$$

$$t(x, y, z) = c_3 \tag{148c}$$

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where c_1, c_2, and c_3 are arbitrary constants, are mutually orthogonal. Hence, it follows that

$$\nabla \xi \cdot \nabla \eta = 0 \qquad (149a)$$

$$\nabla \xi \cdot \nabla t = 0 \qquad (149b)$$

$$\nabla \eta \cdot \nabla t = 0 \qquad (149c)$$

which are precisely equations (141). Conversely, if equations (141) hold, then the three families of surfaces given by equations (148) are mutually orthogonal. Consequently, equation (147) holds. Then, from equations (144), (145), and (147), it follows that equations (137) hold. This completes the proof of the lemma.

From this lemma, it is seen that to use the results of theorem 1, the equations of transformation (128) must be such that one must construct $\xi(x, y, z) = $ constant and $\eta(x, y, z) = $ constant surfaces in the Euclidean three-dimensional space E such that they form a mutually orthogonal set with the $t(x, y, z) = $ constant surfaces obtained from the constraint (127). Moreover, these three sets of surfaces must also satisfy equations (137d) and (137e). Even though the statement of the conditions given by equations (137) has been simplified, the existence of such transformations remains to be proved and will be done with the following lemma.

Lemma 2: Consider a Euclidean three-dimensional space and a set of Stäckel coordinates (μ, ν, ω), which are therefore orthogonal, and let the line element be denoted by

$$ds^2 = \sum_{i=1}^{3} g_{ii} \, dx^i \, dx^i \qquad (150)$$

where

$$x^1 = \mu$$

$$x^2 = \nu$$

$$x^3 = \omega$$

Next, consider a particle moving under the action of external forces in the Euclidean three-dimensional space but constrained on a moving surface on which the coordinates are (ξ, η). If the surface is allowed to move in such a way that it assumes the forms of any one set of the Stäckel surfaces, and if the constant ξ and η surfaces correspond to the two remaining sets of Stäckel surfaces; e.g.,

$$\mu = \xi \qquad (151a)$$

$$\nu = \eta \qquad (151b)$$

$$\omega = f(t) \qquad (151c)$$

where $f(t)$ is a monotonic function (this condition may be relaxed by considering intervals (t_1, t_2), (t_2, t_3), etc., if $f(t)$ is not monotonic), then the Stäckel coordinate system satisfies the five conditions

in equations (137) if and only if the g^{33} component of the contravariant metric tensor is not a function of ω.

Proof: For a particle moving in a Euclidean three-dimensional space with a coordinate system (μ, ν, ω), it is well known that the coefficients a^*_{ij} in the Hamilton-Jacobi equation are related to the contravariant metric tensor g^{ij} by the equation

$$a^*_{ij} = \frac{1}{2} g^{ij} \tag{152}$$

This follows by noting that the kinetic energy E_k for a particle of unit mass can also be written as

$$E_k = \frac{1}{2} \left(\frac{ds}{dt}\right)^2$$

$$= \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \tag{153}$$

and then making use of equations (21), (30), (36), and (37). (For the case of dynamical problems with n degrees of freedom, and not merely those involving the motion of a particle in a three-dimensional space, it is also possible to consider an n space and associate with it a metric defined by equation (152).) Also, it is noted that for a particle moving in a Euclidean three-dimensional space, with a Stäckel coordinate system (μ, ν, ω), the coefficients a^*_{ij} in the Hamilton-Jacobi equation may be obtained as a special case of the results of theorem 2, $n = 3$, by letting $T_3 = 1$ and $T_1 = T_2 = \cdots = T_n = 0$ in equation (124). Then,

$$a^*_{11}(\mu, \nu, \omega) = \frac{1}{\Delta} (Q_1 R_2 - R_1 Q_2) \tag{154a}$$

$$a^*_{22}(\mu, \nu, \omega) = \frac{1}{\Delta} (R_1 P_2 - P_1 R_2) \tag{154b}$$

$$a^*_{33}(\mu, \nu, \omega) = \frac{1}{\Delta} (P_1 Q_2 - Q_1 P_2) \tag{154c}$$

where

$$\Delta = \begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix} \tag{155}$$

P_i, Q_i, and R_i are functions of μ, ν, and ω, respectively. Next, consider the motion of a particle constrained on a moving surface in the Euclidean three-dimensional space and choose (ξ, η, t) as described before by the Stäckel surfaces and expressed in equations (151). For this, the Hamilton-Jacobi equation is given by

$$a^*_{11}[\xi, \eta, f(t)] \left(\frac{\partial S}{\partial \xi}\right)^2 + a^*_{22}[\xi, \eta, f(t)] \left(\frac{\partial S}{\partial \eta}\right)^2 + V[\xi, \eta, f(t)] + \frac{\partial S}{\partial t} = 0 \tag{156}$$

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The conditions in equations (137a), (137b), and (137c) are automatically satisfied because all the Stäckel coordinate systems are orthogonal. Consequently, it remains to prove the following statement: The Stäckel coordinate system has the property that it results in having the coefficients $a_{11}^*[\xi, \eta, f(t)]$ and $a_{22}^*[\xi, \eta, f(t)]$ in equation (156) satisfy the two remaining conditions (137d) and (137e) if and only if g^{33} is independent of ω. To prove that it is necessary, it is observed that $a_{11}^*[\xi, \eta, f(t)]$ and $a_{22}^*[\xi, \eta, f(t)]$ have the form

$$a_{11}^*[\xi, \eta, f(t)] = \frac{T_1(t)N_2(\eta) - T_2(t)N_1(\eta)}{M_1(\xi)N_2(\eta) - M_2(\xi)N_1(\eta)} \tag{157a}$$

$$a_{22}^*[\xi, \eta, f(t)] = \frac{-T_1(t)M_2(\xi) + T_2(t)M_1(\xi)}{M_1(\xi)N_2(\eta) - M_2(\xi)N_1(\eta)} \tag{157b}$$

From studying equations (151), (154), (155), and (157) and noting that

$$\Delta \equiv R_1(P_2 Q_3 - Q_2 P_3) - R_2(P_1 Q_3 - Q_1 P_3) + R_3(P_1 Q_2 - Q_1 P_2) \tag{158}$$

it is concluded that $R_i(\omega)$ must be a constant for $i = 1, 2$, or 3 and the remaining terms vanish in equation (158). Hence, from equations (154) it is seen that $a_{33}^*(\mu, \nu, \omega)$ cannot contain ω explicitly. That is, g^{33} is independent of ω. Conversely, to prove that it is sufficient, it is observed that if g^{33} is independent of ω, then $a_{33}^*(\mu, \nu, \omega)$ does not contain ω explicitly. Without going into the details of Stäckel coordinate systems, it cannot be concluded at this point, observing from equations (154), (155), and (158), that $R_i(\omega)$ is constant for $i = 1, 2$, or 3 and that the remaining terms vanish in Δ so as to arrive at equations (157). However, if cases are constructed in the way described by equations (151) for all the 11 Stäckel coordinate systems (see section entitled "Results"), it is observed that if a_{33}^* is not a function of ω, then equations (157) are satisfied. This completes the proof of the lemma.

Next, it will be proved that the preceding admissible cases exhaust all possibilities for all coordinate systems besides those of Stäckel. That is, no other coordinate system can give rise to a choice of (ξ, η, t) such that the five conditions in equations (137) are satisfied. This leads us to the following lemma.

Lemma 3: Consider a Euclidean three-dimensional space and a set of orthogonal coordinates (μ, ν, ω). Next, consider a particle moving under the action of external forces in the Euclidean three-dimensional space but constrained on a moving surface on which the coordinates are (ξ, η). Let the surface move in such a way that it assumes the forms of a given set of the coordinate surfaces and also let the constant ξ and η surfaces correspond to the two remaining sets; e.g.

$$\mu = \xi \tag{151a}$$

$$\nu = \eta \tag{151b}$$

$$\omega = f(t) \tag{151c}$$

where $f(t)$ is a monotonic function. (This condition may be relaxed by considering intervals (t_1, t_2), (t_2, t_3), etc., if $f(t)$ is not monotonic.) If, by this construction, the five conditions in equations (137) are satisfied, then the given coordinate system (μ, ν, ω) must be of the Stäckel type.

Proof: By starting with an orthogonal system, the three conditions in equations (137a), (137b), and (137c) are automatically satisfied. If one now has $a_{11}^*[\xi, \eta, f(t)]$ and $a_{22}^*[\xi, \eta, f(t)]$ given by equations (157), then it is obvious in view of equations (151) that $P_i(\mu)$, $Q_i(\nu)$, and $R_i(\omega)$ for $i = 1, 2, 3$ can be chosen such that $a_{11}^*(\mu, \nu, \omega)$, $a_{22}^*(\mu, \nu, \omega)$, and $a_{33}^*(\mu, \nu, \omega)$ will have the forms given by equations (154). From these results and equation (152), the contravariant metric tensor g^{ij} may be obtained. Now, by the theorem of Eisenhart (ref. 10), it is known that the 11 Stäckel coordinate systems are the only real ones in Euclidean three-dimensional space that have g^{ij} of this form. This completes the proof of the lemma.

The discussion in lemmas 2 and 3 and a study of the section entitled "Results" lead to the following theorem.

Theorem 3: The Hamilton-Jacobi equation

$$a_{11}^*(\xi, \eta, t)\left(\frac{\partial S}{\partial \xi}\right)^2 + a_{22}^*(\xi, \eta, t)\left(\frac{\partial S}{\partial \eta}\right)^2 + V(\xi, \eta, t) + \frac{\partial S}{\partial t} = 0 \qquad (159)$$

having a complete integral of the form

$$S(\xi, \eta, t) = M(\xi, \alpha) + N(\eta, \alpha) - T(t, \alpha) \qquad (160)$$

describes the motion of a particle constrained on a moving surface in the Euclidean three-dimensional space if and only if the moving surface assumes the forms of a set of coordinate surfaces (e.g., $\omega = f(t)$) in a coordinate system $(x^1, x^2, x^3) \equiv (\mu, \nu, \omega)$ in which the component of the contravariant metric tensor g^{33} is independent of ω, and μ and ν are chosen such that $\mu = \xi$ and $\nu = \eta$. There are only five of these surfaces (see fig. 2):

(1) The plane in transverse motion

(2) The plane in rotation about a longitudinal axis

(3) The cylinder in radial motion

(4) The spherical surface in radial motion

(5) The cone whose angle is variable

In these cases, it is not necessary for the motion of the surface to be uniform.

Remarks: A study of the results in the next section reveals that in some cases it is possible to choose more than one set of coordinates (ξ, η) on the surface to have the desired form required for separation of variables. Every problem involving a particle constrained on a moving surface, satisfying only some very general conditions (eq. 127), has a Hamilton-Jacobi equation that can be expressed in the form of equation (159) by a suitable choice of an orthogonal set of coordinates (ξ, η) on the moving surface.

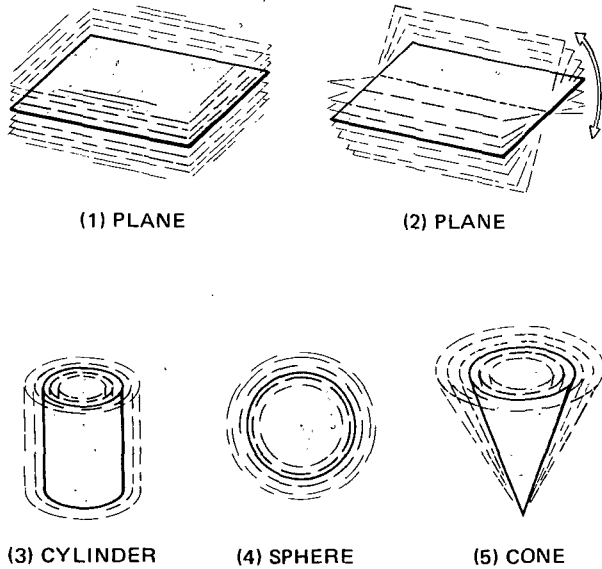(1) PLANE   (2) PLANE

(3) CYLINDER   (4) SPHERE   (5) CONE

Figure 2.—Permissible surfaces.

A Time-Dependent Problem

Consider a particle constrained to move on the surface of a sphere whose radius is $r = f(t)$, where $f(t)$ is an arbitrary continuous function of time. It is desirable to investigate in this case the form of the coefficients a_{11}^* and a_{22}^* and the potential energy that will permit the separation of the Hamilton-Jacobi equation.

For spherical coordinates (r, θ, ϕ), the components of the covariant metric tensor g_{ii} are given by

$$g_{rr} = 1 \tag{161a}$$

$$g_{\theta\theta} = r^2 \tag{161b}$$

$$g_{\phi\phi} = r^2 \sin^2 \theta \tag{161c}$$

The kinetic energy E_k of a particle of unit mass is given by equation (153) as

$$E_k = \frac{1}{2}(g_{rr}\,\dot{r}^2 + g_{\theta\theta}\,\dot{\theta}^2 + g_{\phi\phi}\,\dot{\phi}^2) \tag{162}$$

Because $r = f(t)$,

$$E_k = \frac{1}{2}(\dot{f}^2 + f^2\,\dot{\theta}^2 + f^2 \sin^2 \theta\,\dot{\phi}^2) \tag{163}$$

Suppose that $\mathbf{b}' = 0$ in equations (16) and (17). Hence,

$$\mathsf{U} = E_p \tag{164}$$

Next, from equations (26) and (27),

$$H = \frac{1}{2f^2(t)}p_\theta^2 + \frac{1}{2f^2(t)\sin^2\theta}p_\phi^2 - \frac{1}{2}[\dot{f}(t)]^2 + E_p \tag{165}$$

Consequently, the Hamilton-Jacobi equation is given by

$$\frac{1}{2f^2}\left(\frac{\partial S}{\partial\theta}\right)^2 + \frac{1}{2f^2\sin^2\theta}\left(\frac{\partial S}{\partial\phi}\right)^2 - \frac{1}{2}\dot{f}^2 + E_p + \frac{\partial S}{\partial t} = 0 \tag{166}$$

which may also be obtained more directly from equations (152) and (156). Note that the coefficients of $(\partial S/\partial\theta)^2$ and $(\partial S/\partial\phi)^2$ are of the forms required in equations (137), which are

$$a_{\theta\theta}^* = \frac{T_1(t)N_2(\phi) - T_2(t)N_1(\phi)}{M_1(\theta)N_2(\phi) - M_2(\theta)N_1(\phi)} \tag{167a}$$

$$a_{\phi\phi}^* = \frac{-T_1(t)M_2(\theta) + T_2(t)M_1(\theta)}{M_1(\theta)N_2(\phi) - M_2(\theta)N_1(\phi)} \tag{167b}$$

This can be seen by choosing

$$T_1 = \frac{1}{2f^2(t)} \tag{168a}$$

$$M_1 = 1 \tag{168b}$$

$$N_1 = 0 \tag{168c}$$

$$T_2 = 0 \tag{169a}$$

$$M_2 = -\frac{1}{\sin^2 \theta} \tag{169b}$$

$$N_2 = 1 \tag{169c}$$

Furthermore, E_p is chosen to satisfy equation (139) so that

$$E_p - \frac{1}{2}[\dot{f}(t)]^2 = T_0(t) - a^*_{\theta\theta} M_0(\theta) - a^*_{\phi\phi} N_0(\phi) \tag{170}$$

where $T_0(t)$, $M_0(\theta)$, and $N_0(\phi)$ are arbitrary functions of their respective arguments. Consequently, a complete integral of the form

$$S(\theta, \phi, t; \alpha_1, \alpha_2) = M(\theta, \alpha_1, \alpha_2) + N(\phi, \alpha_1, \alpha_2) - T(t, \alpha_1, \alpha_2) \tag{171}$$

is obtained for the Hamilton-Jacobi equation (166). From equation (11), the functions M, N, and T are determined from the equations

$$\left(\frac{dM}{d\theta}\right)^2 = M_0 + \alpha_1 - \frac{\alpha_2}{\sin^2 \theta} \tag{172a}$$

$$\left(\frac{dN}{d\phi}\right)^2 = N_0 + \alpha_2 \tag{172b}$$

$$\frac{dT}{dt} = T_0 + \frac{\alpha_1}{2f^2(t)} \tag{172c}$$

RESULTS

Based on the discussion in the previous section, results are presented summarizing the forms of a^*_{11}, a^*_{22}, and E_p, where
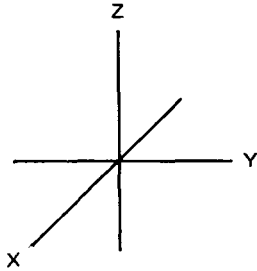
$$E_p = V - (x_t^2 + y_t^2 + z_t^2)$$

for which one has a separable Hamilton-Jacobi equation.

In the following subsections describing each of the 11 Stäckel coordinate systems, h_i for $i = 1, 2, 3$ denotes the scale factors, which are related to the components of the covariant metric tensor g_{ij}, defined by

$$ds^2 = \sum_{i=1}^{3} h_i^2 \, d\xi_i^2 \tag{173}$$

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and Δ_k for $k = 1, 2, \ldots, 11$ denotes the 11 Stäckel determinants defined by equations (154) and (155). These determinants are the transpose of those found in Morse and Feshbach (ref. 17), from which have been taken geometrical representations of the various coordinate systems.

Rectangular Coordinates (1)

$$h_1 = h_2 = h_3 = 1$$

$$x = \xi_1$$

$$y = \xi_2$$

$$z = \xi_3$$

Unconstrained Particle

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 1$$

$$a_{11}^* = a_{22}^* = a_{33}^* = 1$$

$$V = X_0(x) + Y_0(y) + Z_0(z)$$

$$= \frac{X_0(x)}{h_1^2} + \frac{Y_0(y)}{h_2^2} + \frac{Z_0(z)}{h_3^2}$$

Constrained Particle on a Moving Surface

(1) If

$$x = f_1(t)$$

$$y = y$$

$$z = z$$

then

$$a_{22}^* = a_{33}^* = 1$$

$$E_p = \dot{f}_1^2 + T_0(t) + Y_0(y) + Z_0(z)$$

(2) If

$$x = x$$

$$y = f_2(t)$$

$$z = z$$

then

$$a^*_{11} = a^*_{33} = 1$$

$$E_p = \dot{f}_2^2 + T_0(t) + X_0(x) + Z_0(z)$$
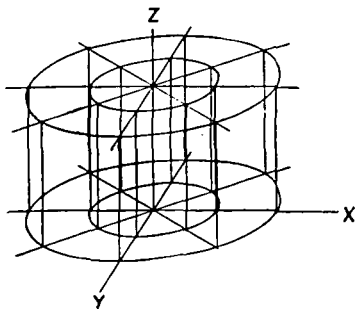
(3) If

$$x = x$$

$$y = y$$

$$z = f_3(t)$$

then

$$a^*_{11} = a^*_{22} = 1$$

$$E_p = \dot{f}_3^2(t) + T_0(t) + X_0(x) + Y_0(y)$$

Cylindrical Coordinates (2)

$$h_1^2 = h_3^2 = 1$$

$$h_2^2 = \xi_1^2$$

$$x = \xi_1 \cos \xi_2$$

$$y = \xi_1 \sin \xi_2$$

$$z = \xi_3$$

$$\xi_1 = r$$

$$\xi_2 = \theta$$

$$\xi_3 = z$$

$$\xi_1 = (x^2 + y^2)^{1/2}$$

$$\tan \xi_2 = \frac{y}{z}$$

$$\xi_3 = z$$

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Unconstrained Particle

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ -\dfrac{1}{r^2} & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 1$$

$$a_{11}^* = 1$$

$$a_{22}^* = \frac{1}{r^2}$$

$$a_{33}^* = 1$$

$$V = X_0(r) + \frac{Y_0(\theta)}{r^2} + Z_0(z)$$

$$= \frac{X_0(r)}{h_1^2} + \frac{Y_0(\theta)}{h_2^2} + \frac{Z_0(z)}{h_3^2}$$

Constrained Particle on Moving Surface

(1) If

$$r = f_1(t)$$

$$\theta = \theta$$

$$z = z$$

then

$$a_{22}^* = \frac{1}{f_1^2}$$

$$a_{33}^* = 1$$

$$E_p = \dot{f}_1^2 + T_0(t) + a_{22}^* Y_0(\theta) + a_{33}^* Z_0(z)$$

(2) If

$$r = r$$

$$\theta = f_2(t)$$

$$z = z$$

then

$$a^*_{11} = a^*_{33} = 1$$

$$E_p = r^2 f_2^{\cdot 2} + T_0(t) + X_0(r) + Z_0(z)$$

(3) If

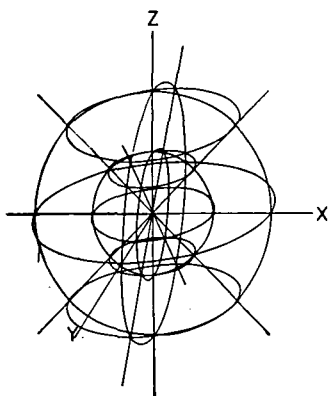$$r = r$$

$$\theta = \theta$$

$$z = f_3(t)$$

Then

$$a^*_{11} = 1$$

$$a^*_{22} = \frac{1}{r^2}$$

$$E_p = f_3^{\cdot 2} + T_0(t) + a^*_{11} X_0(r) + a^*_{22} Y_0(\theta)$$

Spherical Coordinates (3)

$$h_1^2 = 1$$

$$h_2^2 = \xi_1^2$$

$$h_3^2 = \xi_1^2 \sin^2 \xi_2$$

$$x = \xi_1 \sin \xi_2 \cos \xi_3$$

$$y = \xi_1 \sin \xi_2 \sin \xi_3$$

$$z = \xi_1 \cos \xi_2$$

$$\xi_1 = r$$

$$\xi_2 = \theta$$

$$\xi_3 = \phi$$

$$\xi_1 = r = (x^2 + y^2 + z^2)^{1/2}$$

$$\cos \xi_2 = \frac{z}{r}$$

$$\tan \xi_3 = \frac{y}{x}$$

Unconstrained Particle

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 0 \\ -\dfrac{1}{r^2} & 1 & 0 \\ 0 & \dfrac{-1}{\sin^2 \theta} & 1 \end{vmatrix} = 1$$

$$a_{11}^* = 1$$

$$a_{22}^* = \frac{1}{r^2}$$

$$a_{33}^* = \frac{1}{r^2 \sin^2 \theta}$$

$$V = X_0(r) + \frac{1}{r^2} Y_0(\theta) + \frac{1}{r^2 \sin^2 \theta} Z_0(\phi)$$

$$= \frac{X_0(r)}{h_1^2} + \frac{Y_0(\theta)}{h_2^2} + \frac{Z_0(\phi)}{h_3^2}$$

Constrained Particle on Moving Surface

(1)   If

$$r = f_1(t)$$

$$\theta = \theta$$

$$\phi = \phi$$

then

$$a_{22}^* = \frac{1}{f_1^2}$$

$$a^*_{33} = \frac{1}{f^2_1 \sin^2 \theta}$$

(2) If
$$E_p = f^2_1 + T_0(t) + a^*_{22} Y_0(\theta) + a^*_{33} Z_0(\phi)$$

$$r = r$$

$$\theta = f_2(t)$$

then
$$\phi = \phi$$

$$a^*_{11} = 1$$

$$a^*_{33} = \frac{1}{r^2 \sin^2 f_2}$$

(3) If
$$E_p = r^2 f^2_2 + T_0(t) + a^*_{11} X_0(r) + a^*_{33} Z_0(\phi)$$

$$r = r$$
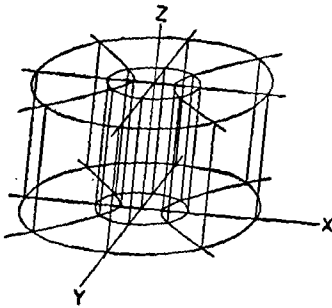
$$\theta = \theta$$

then
$$\phi = f_3(t)$$

$$a^*_{11} = 1$$

$$a^*_{22} = \frac{1}{r^2}$$

$$E_p = r^2 \sin^2 \theta\, f^2_3 + T_0(t) + a^*_{11} X_0(r) + a^*_{22} Y_0(\theta)$$

Elliptic Cylindrical Coordinates (4)

$$h^2_1 = h^2_2$$

$$= \frac{d^2}{2} (\cosh 2\xi_1 - \cos 2\xi_2)$$

$$h^2_3 = 1$$

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where d is a constant

$$x = d \cosh \xi_1 \; \cos \xi_2$$

$$y = d \sinh \xi_1 \; \sin \xi_2$$

$$z = \xi_3$$

$$\xi_1 = \xi$$

$$\xi_2 = \eta$$

$$\xi_3 = z$$

$$\sinh^2 \xi_1 = A + B$$

$$\sin^2 \xi_2 = -A + B$$

$$\xi_3 = z$$

where

$$A = \frac{1}{2d^2} (x^2 + y^2 - d^2)$$

$$B = \left(A^2 + \frac{y^2}{d^2} \right)^{1/2}$$

Unconstrained Particle

$$\Delta_4 = \begin{vmatrix} 0 & 0 & 1 \\ \dfrac{d^2}{2} \cosh 2\xi & -\dfrac{d^2}{2} \cos 2\eta & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \frac{d^2}{2} (\cosh 2\xi - \cos 2\eta)$$

$$a^*_{11} = a^*_{22} = \frac{2}{d^2 (\cosh 2\xi - \cos 2\eta)}$$

$$a^*_{33} = 1$$

$$V = \frac{2X_0(\xi)}{d^2(\cosh 2\xi - \cos 2\eta)} + \frac{2Y_0(\eta)}{d^2(\cosh 2\xi - \cos 2\eta)} + Z_0(z)$$

$$= \frac{X_0(\xi)}{h_1^2} + \frac{Y_0(\eta)}{h_2^2} + \frac{Z_0(z)}{h_3^2}$$

Constrained Particle on Moving Surface

(1) If

$$\xi_1 = f_1(t)$$

$$\xi_2 = \eta$$

$$\xi_3 = z$$

then not separable.

(2) If

$$\xi_1 = \xi$$

$$\xi_2 = f_2(t)$$

$$\xi_3 = z$$

then not separable.

(3) If
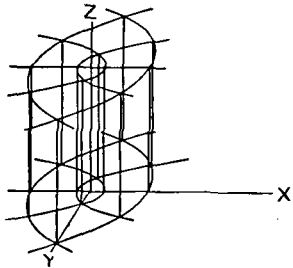
$$\xi_1 = \xi$$

$$\xi_2 = \eta$$

$$\xi_3 = f_3(t)$$

then

$$a_{11}^* = a_{22}^* = \frac{2}{d^2(\cosh 2\xi - \cos 2\eta)}$$

$$E_p = \dot{f}_3^2 + T_0 + a_{11}^*[X_0(\xi) + Y_0(\eta)]$$

Parabolic Cylindrical Coordinates (5)

$$h_1^2 = h_2^2 = \xi_1^2 + \xi_2^2$$

$$h_3^2 = 1$$

$$x = \frac{1}{2}(\xi_1^2 - \xi_2^2)$$

$$y = \xi_1 \xi_2$$

$$z = \xi_3$$

$$\xi_1^2 = x + (y^2 - x^2)^{1/2}$$

$$\xi_2^2 = -x + (y^2 - x^2)^{1/2}$$

$$\xi_3 = z$$

Unconstrained Particle

$$\Delta_5 = \begin{vmatrix} 0 & 0 & 1 \\ \xi_1^2 & \xi_2^2 & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \xi_1^2 + \xi_2^2$$

$$a_{11}^* = a_{22}^* = \frac{1}{\xi_1^2 + \xi_2^2}$$

$$a_{33}^* = 1$$

$$V = \frac{X_0(\xi_1)}{\xi_1^2 + \xi_2^2} + \frac{Y_0(\xi_2)}{\xi_1^2 + \xi_2^2} + Z_0(z)$$

$$= \frac{X_0(\xi_1)}{h_1^2} + \frac{Y_0(\xi_2)}{h_2^2} + \frac{Z_0(z_1)}{h_3^2}$$

$$= \frac{X_0(\xi_1)}{2(y^2 - x^2)^{1/2}} + \frac{Y_0(\xi_2)}{2(y^2 - x^2)^{1/2}} + Z_0(z)$$

Constrained Particle on Moving Surface

    (1) If

$$\xi_1 = f_1(t)$$

$$\xi_2 = \xi_2$$

$$\xi_3 = z$$

then not separable.

    (2) If

$$\xi_1 = \xi_1$$

$$\xi_2 = f_2(t)$$

$$\xi_3 = z$$

then not separable.

    (3) If
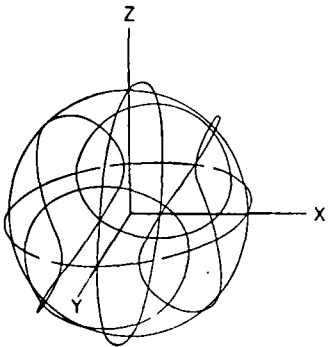
$$\xi_1 = \xi_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = f_3(t)$$

then

$$a_{11}^* = a_{22}^* = \frac{1}{\xi_1^2 + \xi_2^2}$$

$$E_p = \dot{f}_3^2(t) + T_0 + a_{11}^*[X_0(\xi_1) + Y_0(\xi_2)]$$

Conical Coordinates (6)

$$h_1^2 = h_2^2$$

$$= \xi_3^2[k^2 \, \mathrm{cn}^2 \, (\xi_1, k) + k'^2 \, \mathrm{cn}^2 \, (\xi_2, k')]$$

$$h_3^2 = 1$$

where k and k' are constants of the elliptic functions and

$$k^2 + k'^2 = 1$$

$$x = \xi_3 \, \mathrm{dn} \, (\xi_1, k) \, \mathrm{sn} \, (\xi_2, k')$$

$$y = \xi_3 \, \mathrm{sn} \, (\xi_1, k) \, \mathrm{dn} \, (\xi_2, k')$$

Because

$$z = \xi_3 \; cn \, (\xi_1, k) \; cn \, (\xi_2, k')$$

$$sn^2 = cn^2 = 1$$

$$dn^2 + k' \, sn^2 = 1$$

$$dn^2 = k^2 \, cn^2 = k$$

$$k^2 \, cn^2 \, (\xi_1, k) = A + B$$

$$k'^2 \, cn^2 \, (\xi_2, k') = -A + B$$

$$\xi_3^2 = r^2$$

where

$$A = \frac{1}{2r^2} \, [k^2 x^2 - k'^2 y^2 + (k^2 - k'^2)z^2]$$

$$B = \left(A^2 + \frac{k^2 k'^2 z^2}{r^2} \right)^{1/2}$$

Unconstrained Particle

$$\Delta_6 = \begin{vmatrix} 0 & 0 & \xi_3^2 \\ k^2 \, cn^2 \, (\xi_1, k) & k'^2 \, cn^2 \, (\xi_2, k') & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \xi_3^2 \, [k^2 \, cn^2 \, (\xi_1, k) + k'^2 \, cn^2 \, (\xi_2, k')]$$

$$a_{11}^* = a_{22}^* = \frac{1}{\xi_3^2 \, [k^2 \, cn^2 \, (\xi_1, k) + k'^2 \, cn^2 \, (\xi_2, k')]}$$

$$a_{33}^* = 1$$

$$V = a_{11}^* X_0(\xi_1) + a_{22}^* Y_0(\xi_2) + a_{33}^* Z_0(\xi_3)$$

$$= \frac{X_0}{h_1^2} + \frac{Y_0}{h_2^2} + Z_0$$

(1) If

$$\xi_1 = f_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = \xi_3$$

then not separable.

(2) If

$$\xi_1 = \xi_1$$

$$\xi_2 = f_2(t)$$

$$\xi_3 = \xi_3$$

then not separable.
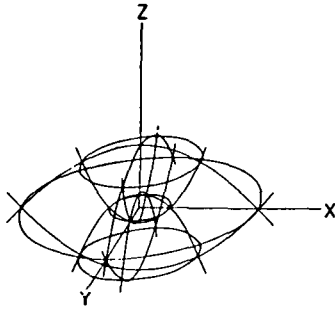
(3) If

$$\xi_1 = \xi_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = f_3(t)$$

then

$$a_{11}^* = a_{22}^* = \frac{1}{f_3^2 \,[k^2\ cn^2\ (\xi_1, k) + k'^2\ cn^2\ (\xi_2, k')]}$$

$$E_p = \dot{f}_3^2(t) + T_0(t) + a_{11}^* \,[X_0(\xi_1) + Y_0(\xi_2)]$$

Parabolic Coordinates (7)

$$h_1^2 = h_2^2 = \xi_1^2 + \xi_2^2$$

$$h_3^2 = \xi_1^2 \xi_2^2$$

$$x = \xi_1 \xi_2\ \cos \xi_3$$

$$y = \xi_1 \xi_2\ \sin \xi_3$$

$$z = \frac{1}{2}\,(\xi_1^2 - \xi_2^2)$$

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$$\xi_1^2 = r + z$$

$$\xi_2^2 = r - z$$

$$\tan \xi_3 = \frac{y}{x}$$

where

$$r = (x^2 + y^2 + z^2)^{1/2}$$

Unconstrained Particle

$$\Delta_7 = \begin{vmatrix} \xi_1^2 & \xi_2^2 & 0 \\ \dfrac{1}{\xi_1^2} & \dfrac{1}{\xi_2^2} & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \xi_1^2 + \xi_2^2$$

$$a_{11}^* = a_{22}^* = \frac{1}{\xi_1^2 + \xi_2^2}$$

$$a_{33}^* = \frac{1}{\xi_1^2 \xi_2^2}$$

$$V = \frac{X_0(\xi_1)}{\xi_1^2 + \xi_2^2} + \frac{Y_0(\xi_2)}{\xi_1^2 + \xi_2^2} + \frac{Z_0(\xi_3)}{\xi_1^2 \xi_2^2}$$

$$= \frac{X_0}{h_1^2} + \frac{Y_0}{h_2^2} + \frac{Z_0}{h_3^2}$$

Constrained Particle on Moving Surface

(1) If

$$\xi_1 = f_1(t)$$

$$\xi_2 = \xi_2$$

$$\xi_3 = \xi_3$$

then not separable.

(2) If

$$\xi_1 = \xi_1$$

$$\xi_2 = f_2(t)$$

$$\xi_3 = \xi_3$$

then not separable.

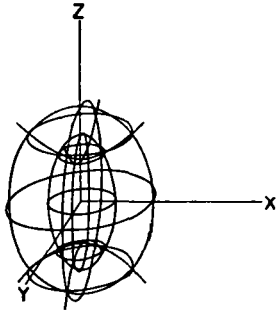(3) If

$$\xi_1 = \xi_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = f_3(t)$$

then

$$a^*_{11} = a^*_{22} = \frac{1}{\xi_1^2 + \xi_2^2}$$

$$E_p = \xi_1^2 \xi_2^2 \dot{f}_3^2(t) + T_0 + a^*_{11}[X_0(\xi_1) + Y_0(\xi_2)]$$

Prolate Spheroidal Coordinates (8)

$$h_1^2 = h_2^2 = d^2(\sinh^2 \xi_1 + \sin^2 \xi_2)$$

$$h_3^2 = d^2 \sinh^2 \xi_1 \sin^2 \xi_2$$

$$x = d \sinh \xi_1 \sin \xi_2 \cos \xi_3$$

$$y = d \sinh \xi_1 \sin \xi_2 \sin \xi_3$$

$$z = d \cosh \xi_1 \cos \xi_2$$

$$\sinh^2 \xi_1 = A + B$$

$$\sin^2 \xi_2 = B - A$$

$$\tan \xi_3 = \frac{y}{x}$$

where

$$A = \frac{1}{2d^2}(r^2 - d^2)$$

$$B = \left(A^2 + \frac{x^2 + y^2}{d}\right)^{1/2}$$

$$r^2 = x^2 + y^2 + z^2$$

Unconstrained Particle

$$\Delta_8 = \begin{vmatrix} d^2 \sinh^2 \xi_1 & d^2 \sin^2 \xi_2 & 0 \\[2mm] \dfrac{1}{\sinh^2 \xi_1} & \dfrac{1}{\sin^2 \xi_2} & -1 \\[2mm] -1 & 1 & 0 \end{vmatrix}$$

$$= d^2 (\sinh^2 \xi_1 + \sin^2 \xi_2)$$

$$a_{11}^* = a_{22}^* = \frac{1}{d^2 (\sinh^2 \xi_1 + \sin^2 \xi_2)}$$

$$a_{33}^* = \frac{1}{d^2 \sinh^2 \xi_1 \sin^2 \xi_2}$$

$$V = \frac{X_0(\xi_1)}{d^2 (\sinh^2 \xi_1 + \sin^2 \xi_2)} + \frac{Y_0(\xi_2)}{d^2 (\sinh^2 \xi_1 + \sin^2 \xi_2)} + \frac{Z_0(\xi_3)}{d^2 \sinh^2 \xi_1 \sin^2 \xi_2}$$

$$= \frac{X_0}{h_1^2} + \frac{Y_0}{h_2^2} + \frac{Z_0}{h_3^2}$$

Constrained Particle on Moving Surface

(1) If

$$\xi_1 = f_1(t) \qquad \xi_2 = \xi_2 \qquad \xi_3 = \xi_3$$

then not separable.

(2) If

$$\xi_1 = \xi_1 \qquad \xi_2 = f_2(t) \qquad \xi_3 = \xi_3$$
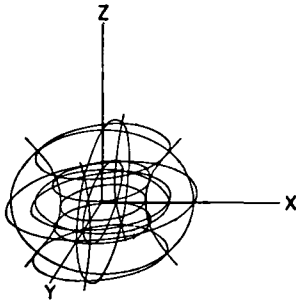
then not separable.

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(3) If

$$\xi_1 = \xi_1 \qquad \xi_2 = \xi_2 \qquad \xi_3 = f_3(t)$$

then

$$a^*_{11} = a^*_{22} = \frac{1}{d^2(\sinh^2 \xi_1 + \sin^2 \xi_2)}$$

$$E_p = d^2 \sinh^2 \xi_1 \sin^2 \xi_2 \, \dot{f}_3^2 + T_0(t) + a^*_{11}[X_0(\xi_1) + Y_0(\xi_2)]$$

Oblate Spheroidal Coordinates (9)

$$h_1^2 = h_2^2 = d^2(\cosh^2 \xi_1 - \sin^2 \xi_2)$$

$$h_3^2 = d^2 \cosh^2 \xi_1 \sin^2 \xi_2$$

$$x = d \cosh \xi_1 \sin \xi_2 \cos \xi_3$$

$$y = d \cosh \xi_1 \sin \xi_2 \sin \xi_3$$

$$z = d \sinh \xi_1 \cos \xi_2$$

$$\cosh^2 \xi_1 = A + B$$

$$\cos^2 \xi_2 = -A + B$$

$$\tan \xi_3 = \frac{y}{x}$$

where

$$A = \frac{1}{2d^2}(r^2 - d^2)$$

$$B = \left(A^2 + \frac{z^2}{d^2}\right)^{1/2}$$

Unconstrained Particle

$$\Delta_9 = \begin{vmatrix} d^2 \cosh^2 \xi_1 & d^2 \sin^2 \xi_2 & 0 \\ \dfrac{1}{\cosh^2 \xi_1} & \dfrac{1}{\sin^2 \xi_2} & -1 \\ -1 & 1 & 0 \end{vmatrix} = d^2(\cosh^2 \xi_1 + \sin^2 \xi_2)$$

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$$a_{11}^* = a_{22}^* = \frac{1}{d^2(\cosh^2 \xi_1 + \sin^2 \xi_2)}$$

$$a_{33}^* = \frac{1}{d^2 \cosh^2 \xi_1 \sin^2 \xi_2}$$

$$V = \frac{X_0(\xi_1) + Y_0(\xi_2)}{d^2(\cosh^2 \xi_1 + \sin^2 \xi_2)} + \frac{Z_0(\xi_3)}{d^2 \cosh^2 \xi_1 \sin^2 \xi_2}$$

Constrained Particle on Moving Surface

(1) If

$$\xi_1 = f_1(t)$$

$$\xi_2 = \xi_2$$

$$\xi_3 = \xi_3$$

then not separable.

(2) If

$$\xi_1 = \xi_1$$

$$\xi_2 = f_2$$

$$\xi_3 = \xi_3$$

then not separable.
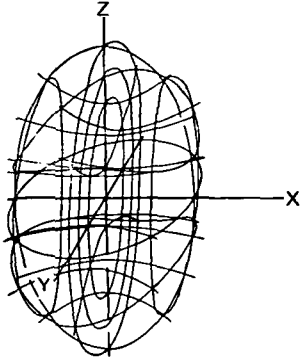
(3) If

$$\xi_1 = \xi_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = f_3(t)$$

then

$$a_{11}^* = a_{22}^* = \frac{1}{d^2(\cosh^2 \xi_1 + \sin^2 \xi_2)}$$

$$E_p = d^2 \cosh^2 \xi_1 \sin^2 \xi_2 \, \dot{f}_3^2 + T_0(t) + a_{11}^* [X_0(\xi_1) + Y_0(\xi_2)]$$

Ellipsoidal Coordinates (10)

$$h_1^2 = \frac{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}$$

$$h_2^2 = \frac{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}{(\xi_2^2 - a^2)(\xi_2^2 - b^2)}$$

$$h_3^2 = \frac{(\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)}{(\xi_3^2 - a^2)(\xi_3^2 - b^2)}$$

$$x = \left[\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}\right]^{1/2}$$

$$y = \left[\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}\right]^{1/2}$$

$$z = \frac{\xi_1 \xi_2 \xi_3}{ab}$$

Unconstrained Particle

$$\Delta_{10} = \begin{vmatrix} 1 & 1 & 1 \\ \dfrac{1}{\xi_1^2 - a^2} & \dfrac{1}{\xi_2^2 - a^2} & \dfrac{1}{\xi_3^2 - a^2} \\ \dfrac{1}{(\xi_1^2 - b^2)(a^2 - b^2)} & \dfrac{1}{(\xi_2^2 - b^2)(a^2 - b^2)} & \dfrac{1}{(\xi_3^2 - b^2)(a^2 - b^2)} \end{vmatrix}$$

$$= \frac{(\xi_3^2 - \xi_2^2)(\xi_3^2 - \xi_1^2)(\xi_2^2 - \xi_1^2)}{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}$$

$$a_{11}^* = \frac{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}{(\xi_3^2 - \xi_1^2)(\xi_2^2 - \xi_1^2)}$$

$$a_{22}^* = \frac{(\xi_2^2 - a^2)(\xi_2^2 - b^2)}{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}$$

$$a_{33}^* = \frac{(\xi_3^2 - a^2)(\xi_3^2 - b^2)}{(\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)}$$

$$V = a_{11}^* X_0(\xi_1) + a_{22}^* Y_0(\xi_2) + a_{33}^* Z_0(\xi_3)$$

Constrained Particle on Moving Surface

(1) If

$$\xi_1 = f_1(t)$$

$$\xi_2 = \xi_2$$

$$\xi_3 = \xi_3$$

then not separable.

(2) If

$$\xi_1 = \xi_1$$

$$\xi_2 = f_2(t)$$

$$\xi_3 = \xi_3$$

then not separable.

(3) If

$$\xi_1 = \xi_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = f_3(t)$$

then not separable.

Paraboloidal Coordinates (11)

$$h_1^2 = \xi_1^2 \frac{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}$$

$$h_2^2 = \xi_2^2 \frac{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}{(\xi_2^2 - a^2)(\xi_2^2 - b^2)}$$

$$h_3^2 = \xi_3^2 \frac{(\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)}{(\xi_3^2 - a^2)(\xi_3^2 - b^2)}$$

$$x = \left[\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2 - b^2} \right]^{1/2}$$

$$y = \left[\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2 - a^2} \right]^{1/2}$$

$$z = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 - a^2 - b^2)$$

Unconstrained Particle

$$\Delta_{11} = \begin{vmatrix} \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \dfrac{\xi_1^2}{\xi_1^2 - a^2} & \dfrac{\xi_2^2}{\xi_2^2 - a^2} & \dfrac{\xi_3^2}{\xi_3^2 - a^2} \\ \dfrac{\xi_1^2}{(\xi_1^2 - b^2)(a^2 - b^2)} & \dfrac{\xi_2^2}{(\xi_2^2 - b^2)(a^2 - b^2)} & \dfrac{\xi_3^2}{(\xi_3^2 - b^2)(a^2 - b^2)} \end{vmatrix}$$

$$= \xi_1^2 \xi_2^2 \xi_3^2 \frac{(\xi_3^2 - \xi_2^2)(\xi_3^2 - \xi_1^2)(\xi_2^2 - \xi_1^2)}{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}$$

$$a_{11}^* = \frac{1}{\xi_1^2} \frac{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}$$

$$a_{22}^* = \frac{1}{\xi_2^2} \frac{(\xi_2^2 - a^2)(\xi_2^2 - b^2)}{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}$$

$$a_{33}^* = \frac{1}{\xi_3^2} \frac{(\xi_3^2 - a^2)(\xi_3^2 - b^2)}{(\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)}$$

$$V = a_{11}^*(\xi_1, \xi_2, \xi_3)X_0(\xi_1) + a_{22}^*(\xi_1, \xi_2, \xi_3)Y_2(\xi_2) + a_{33}^*(\xi_1, \xi_2, \xi_3)Z_0(\xi_3)$$

Constrained Particle on Moving Surface

(1) If

$$\xi_1 = f_1(t)$$

$$\xi_2 = \xi_2$$

$$\xi_3 = \xi_3$$

then not separable.

(2) If

$$\xi_1 = \xi_1$$

$$\xi_2 = f_2(t)$$
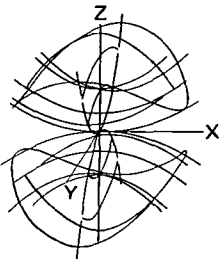
$$\xi_3 = \xi_3$$

then not separable.

(3) If

$$\xi_1 = \xi_1$$

$$\xi_2 = \xi_2$$

$$\xi_3 = f_3(t)$$

then not separable.

DISCUSSION

Application of Present Results to Other Classes of Dynamical Problems

In a previous section, the results of theorem 1 for the time-dependent two-dimensional Hamilton-Jacobi equation were applied to the solution of the class of problems describing the motion of a particle constrained on a moving surface in a Euclidean three-dimensional space.

Theorem 2, $n = 3$, may similarly be applied to study problems comprising two moving particles subject to three constraints. For example, consider the following class of problems: Let P_1 and P_2 respectively denote two particles with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2). Suppose these particles are constrained on the same moving surface and also the distance between them is constant. Then,

$$f(x_1, y_1, z_1, t) = 0 \qquad (174\text{a})$$

$$f(x_2, y_2, z_2, t) = 0 \qquad (174\text{b})$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = d^2 \qquad (174\text{c})$$

and the constraint may be used to reduce the given problem to one involving only three independent generalized coordinates (ξ, η, ζ). Using the conditions given in theorem 2, $n = 3$, a general class of moving surfaces may be found that leads to problems solvable by the separation of variables in the Hamilton-Jacobi equation.

In this way, the comparatively wide classes of problems that can be solved by the results of theorem 2 (which deals with n degrees of freedom) may be studied. There is no confinement to prob lems in Newtonian mechanics because the results are purely mathematical. Thus, problems arising in relativity and even any other non-Newtonian form of mechanics may be considered as long as one has a Hamilton-Jacobi type of equation.

More General Forms of the Hamilton-Jacobi Equation

The Hamilton-Jacobi equation studied in this work is diagonal and also does not have first-degree terms. Thus, one may consider the general diagonal Hamilton-Jacobi equation

$$\sum_{i=1}^{n} a_{ii}^* \left(\frac{\partial S}{\partial q_i}\right)^2 + \sum_{i=1}^{n} b_i^* \frac{\partial S}{\partial q_i} + V + \frac{\partial S}{\partial t} = 0 \tag{60}$$

or the most general Hamilton-Jacobi equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^* \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} + \sum_{i=1}^{n} b_i^* \frac{\partial S}{\partial q_i} + V + \frac{\partial S}{\partial t} = 0 \tag{54}$$

and seek complete integrals of the form

$$S(\mathbf{q}, t; \boldsymbol{\alpha}) = \sum_{i=1}^{n} X_i(q_i; \boldsymbol{\alpha}) - T(t; \boldsymbol{\alpha}) \tag{11}$$

where X_i and T are arbitrary functions of q_i and t. These have already been done by Chan.[1]

Other Forms of Separation

Ever since the time of Jacobi, complete integrals of the Hamilton-Jacobi equation have always been sought in the form of equation (11). In fact, the term "separation of variables" is defined specifically with reference to this form. It has long been known that many dynamical problems have Hamilton-Jacobi equations that do not possess complete integrals of this form. However, essentially all the problems illustrated in texts that are solvable by separating variables of the Hamilton-Jacobi equation can also be solved without the sophistication developed in the theory of Hamilton and Jacobi. For example, one may solve the problem of the linear harmonic oscillator in many steps fewer and more simply by starting from Newton's equations than by starting from the Hamilton-Jacobi equation. But this lack of usefulness of the Hamilton-Jacobi theory is only superficial.

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[1]F. K. Chan: "Classical and Non-Classical Separation of the Hamilton-Jacobi Equation," in preparation.

In accordance with terminology introduced by Chan,[2] separation of variables yielding complete integrals of the form of equation (11) shall be referred to as *classical separation* and separation of variables yielding complete integrals not of the form of equation (11) shall be referred to as *nonclassical separation*. Thus, as an example, he has considered the general Hamilton-Jacobi equation (54) and sought a complete integral of the form

$$S(\mathbf{q}, t; \alpha) = \sum_{i=1}^{n} X_i(q_i; \alpha) - T(t; \alpha) + F(\mathbf{q}, t) \qquad (175)$$

where $F(\mathbf{q}, t)$ is an arbitrary function specified a priori.

Disadvantages of Specifying Forms of Complete Integrals

Even though the form of equation (175) is more general than the form of equation (11), it turns out that it still does not yield the solutions of many interesting dynamical problems. In fact, this is the main disadvantage in the whole outlook of seeking complete integrals of some specified form. For, even after one has obtained necessary and sufficient conditions for such separation, one has then to find physical problems that satisfy these conditions. Whereas, in life, one is usually given a definite physical problem to which one has to find the answer.

Thus, after considering many forms of complete integrals, Chan[3] is still unable to handle problems involving forces resulting from rotating sources. Specifically, there is the time-honored circular restricted planar three-body problem described by the Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial W}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial \eta}\right)^2 + \omega\eta\,\frac{\partial W}{\partial \xi} - \omega\xi\,\frac{\partial W}{\partial \eta} - \left(\frac{GM_1}{\rho_1} + \frac{GM_2}{\rho_2}\right) = \alpha_1 \qquad (176)$$

where

$$\rho_1^2 = (\xi - a)^2 + \eta^2 \qquad (177a)$$

$$\rho_2^2 = (\xi + b)^2 + \eta^2 \qquad (177b)$$

for which a complete integral involving a finite number of terms is sought. However, he has been able to obtain solutions involving a finite number of terms for problems with rotating sources if the potential is given by

$$V = \sum_{i=1}^{n} \frac{1}{2} k_i^2 [(\xi - a_i)^2 + (\eta - b_i)^2] \qquad (178)$$

which is of the type described by Hooke's law.

Separation of Variables in Higher Order Partial Differential Equations

The technique of separation of variables has long been applied to the solution of the usual linear second-order partial differential equations of mathematical physics:

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[2] F. K. Chan: "Classical and Non-Classical Separation of the Hamilton-Jacobi Equation," in preparation.
[3] F. K. Chan: "Reformulation of Problems With Rotating Force Fields." Submitted to *Celestial Mechanics*.

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Heat equation:
$$\nabla^2 u = \frac{1}{a^2} \frac{\partial u}{\partial t} \tag{179}$$

Wave equation:
$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \tag{180}$$

Laplace equation:
$$\nabla^2 u = 0 \tag{181}$$

Helmholtz equation:
$$\nabla^2 u + \lambda u = 0 \tag{182}$$

Klein-Gordon equation:
$$\nabla^2 u - K^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \tag{183}$$

Schrödinger equation:
$$\frac{-\hbar^2}{2M} \nabla^2 u + V u = i\hbar \frac{\partial u}{\partial t} \tag{184}$$

Thus, one seeks a solution of the form

$$u(\mathbf{x}, t) = T(t;\alpha) \prod_{i=1}^{n} X_i(x_i;\alpha) \tag{185}$$

or

$$u(\mathbf{x}) = \prod_{i=1}^{n} X_i(x_i;\alpha) \tag{186}$$

and substitutes into the differential equation in question to verify that a solution of this form exists. In physical situations, the coordinate system is usually one of the 11 Stäckel forms and, consequently, the question of separability has been answered completely by Robertson (ref. 9) and Eisenhart (ref. 10).

However, the question of separability for the most general linear second-order partial differential equation has not been answered yet in the sense that no necessary and sufficient conditions equivalent to Robertson's for equations involving the Laplacian operator have been published. Thus for the sake of convenience, if one considers the forms

$$\sum_{i=1}^{n} a_i(\mathbf{x}, t) \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(\mathbf{x}, t) \frac{\partial u}{\partial x_i} + c(\mathbf{x}, t)u = \frac{\partial u}{\partial t} \tag{187}$$

$$\sum_{i=1}^{n} a_i(\mathbf{x}, t) \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(\mathbf{x}, t) \frac{\partial u}{\partial x_i} + c(\mathbf{x}, t)u = \frac{\partial^2 u}{\partial t^2} + d(\mathbf{x}, t) \frac{\partial u}{\partial t} \tag{188}$$

$$\sum_{i=1}^{n} a_i(\mathbf{x}) \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} + c(\mathbf{x})u = 0 \qquad (189)$$

many problems of physical interest are included, such as the three-dimensional heat equation with space- and time-dependent specific heat $c(\mathbf{x}, t)$ and density $\rho(\mathbf{x}, t)$ and also the equation for non-isotropic, space- and time-dependent thermal conductivity $k_i(\mathbf{x}, t)$:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(k_i(\mathbf{x}, t) \frac{\partial u}{\partial x_i} \right) = c(\mathbf{x}, t)\rho(\mathbf{x}, t) \frac{\partial u}{\partial t} \qquad (190)$$

the one-dimensional wave equation for a vibrating string with space- and time-dependent tension $p(x, t)$, elastic restoring force $q(x, t)$, and density $\rho(x, t)$:

$$\frac{\partial}{\partial x} \left[p(x, t) \frac{\partial u}{\partial x} \right] - q(x, t)u = \rho(x, t) \frac{\partial^2 u}{\partial t^2} \qquad (191)$$

and the two-dimensional wave equation for a vibrating membrane with space- and time-dependent density $\rho(\mathbf{x}, t)$ and nonisotropic, space- and time-dependent tension $p_i(\mathbf{x}, t)$:

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[p_i(\mathbf{x}, t) \frac{\partial u}{\partial x} \right] = \rho(\mathbf{x}, t) \frac{\partial^2 u}{\partial t^2} \qquad (192)$$

The problem of separation of variables in the forms of equation (185) or (186) for the most general linear second-order partial differential equation expressed for convenience in equations (187) to (189) has also been studied by Chan.[4]

CONCLUSIONS

Theorem 2 gives necessary and sufficient conditions for the separation of the special diagonal Hamilton-Jacobi equation. Theorem 3 gives the complete set of solutions of the dynamical problem of a particle constrained on a moving surface solvable by the separation of the special diagonal Hamilton-Jacobi equation. There are only five classes of such constraining surfaces:

(1) The plane in transverse motion

(2) The plane in rotation about a longitudinal axis

(3) The cylinder in radial motion

(4) The spherical surface in radial motion

(5) The cone whose azimuthal angle is variable

In these cases, it is not necessary for the motion of the surface to be uniform.

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  National Aeronautics and Space Administration
    Greenbelt, Maryland, July 11, 1972
      311-80-43-01-51

[4]F. K. Chan: "Separation of a General Second-Order Linear Partial Differential Equation of Mathematical Physics," in preparation.

REFERENCES

1. Dugas, R.: *A History of Mechanics.* Éditions du Griffon (Neuchâtel, Switzerland), 1955.

2. Hamilton, W. R.: "On a General Method in Dynamics." *Phil. Trans. Roy Soc.*, 1834, pp. 247-308.

3. Hamilton, W. R.: "Second Essay on a General Method in Dynamics." *Phil. Trans. Roy. Soc.*, 1835, pp. 94-144.

4. Lanczos, C.: *The Variational Principles of Mechanics.* Second ed., Univ. of Toronto Press, 1962.

5. Jacobi, C.: "Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf Integration eines einzigen Systemes gewohnlicher Differential-gleichung." *Crelle's J. Math.* 17(2): 97-162, 1837.

6. Stäckel, P.: *Integration der Hamilton-Jacobischen Gleichungen mittels Separation der Variabeln.* Habilitationsschrift Halle, 1891.

7. Stäckel, P.: "Über die Bewegung eines Punktes in einer n-fachen Mannigfastigkeit." *Math. Ann.* 42: 537-563, 1893.

8. Levi-Civita, T.: "Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili." *Math. Ann.* 59: 383-397, 1904.

9. Robertson, H. P.: "Bemerkung über separierbare Systeme in der Wellenmechanik." *Math. Ann.* 98: 749-752, 1927.

10. Eisenhart, L. P.: "Separable Systems of Stäckel." *Ann. Math.* 35(2): 284-305, 1934.

11. Goldstein, H.: *Classical Mechanics.* Addison-Wesley Pub. Co., Inc., 1953.

12. Pars, L. A.: *Analytical Dynamics.* Heinemann (London), 1965.

13. Whittaker, E. T.: *Analytical Dynamics.* Fourth ed., Cambridge Univ. Press, 1937.

14. Nordheim, L.; and Fues, E.: "Die Hamilton-Jacobische Theorie der Dynamik," Vol. V of *Handbuch der Physik,* Springer (Berlin), 1927, pp. 91-177.

15. Born, M.; *The Mechanics of the Atom.* George Bell & Sons, Ltd. (London), 1960.

16. Gelfand, I. M.; and Fomin, S. V.: *Calculus of Variations.* Prentice-Hall, Inc., 1963.

17. Morse, P. M.; and Feshbach, H.: *Methods of Theoretical Physics.* Vol. I, McGraw-Hill Book Co., Inc., 1953.

BIBLIOGRAPHY

Theory of the Hamilton-Jacobi Equation

Carathéodory, C.: *Calculus of Variations and Partial Differential Equations of the First Order.* Pt. I. Holden-Day, Inc. (San Francisco), 1965.

Courant, R.; and Hilbert, D.: *Methods of Mathematical Physics.* Vol. II. John Wiley & Sons, Inc., 1962.

Hermann, R.: *Differential Geometry and the Calculus of Variations.* Academic Press, Inc., 1968.

Rund, H.: *Hamilton-Jacobi Theory in the Calculus of Variations.* D. Van Nostrand Co., Inc., 1966.

Separation of Variables in the Hamilton-Jacobi Equation

Burgatti, P.: "Determinazione dell'equazioni di Hamilton-Jacobi integrabili mediante la separazione delle variabili." *Atti. Accad. Naz. Lincei Cl. Sci. Fis., Mat., Natur. Rend.* 20: 108-111, 1911.

Dall'Acqua, F.: "Sulle integrazione delle equazione di Hamilton-Jacobi per separazione di variabili." *Math. Ann.* 66: 389-415, 1908.

Dall'Acqua, F.: "Le equazioni di Hamilton-Jacobi che si integrano per separazione di variabili." *Rend. Palermo* 33: 341-351, 1912.

Demin, V. G.: "On a Particular Case of Integrability of the Hamilton-Jacobi Equation." *Vestn. Mosk. Univ., Fiz. Astron.* 15(1): 80-82, 1960.

Eisenhart, L. P.: *Riemannian Geometry.* Princeton Univ. Press, 1926.

Eisenhart, L. P.: "Separable Systems in Euclidean 3-Space." *Phys. Rev.* 45: 427-428, 1934.

Eisenhart, L. P.: "Enumeration of Potentials for Which One-Particle Schrödinger Equations Are Separable." *Phys. Rev.* 74(1): 87-89, 1948.

Eisenhart, L. P.: *Proc. Nat. Acad. Sci., U.S.* 35: 412, 490, 1949.

Fues, E.: *Z. Phys.* 34: 788, 1925.

Iarov-Iarovoi, M. S.: "On the Integration of the Equations of Motion of a Material Point by the Method of Separation of Variables." *Izv. Akad. Nauk SSSR,* 1963, pp. 64-69.

Iarov-Iarovoi, M. S.: "Integration of the Hamilton-Jacobi Equation by the Method of Separation of Variables." *J. Appl. Math. Mech.* 27(6): 973-987, 1963.

Imshenetskii, V. G.: "Integration of First and Second Order Partial Differential Equations." *Izd. Mosk. matem. obshchestva,* 1916, pp. 172-224, 227-229.

Kneser, H.: *Math. Ann.* **84**: 277, 1921.

Levi-Civita, T.: *Annal. Math. Pura Appl.* **24**: 255, 1896.

Liouville, R.: *Acta Math.* **19**: 251, 1895.

Morera, G.: "Separazione della variabili nell' equazione del moto d'un punto su una superficio. *Atti. Accad. Torino* **16**: 276-295, 1880.

Painlevé, P.: *Compt. Rend.* **124**: 221, 1897.

Ricci, A.; and Levi-Civita, T.: "Méthodes de Calcul différentiel absolu." *Math. Ann.* **54**: 183-184, 1901.

Stäckel, P.: "Sur des Problèmes de Dynamique, qui se Réduasent à des Quadratures." *Compt. Rend.* **116**: 1284-1286, 1893.

Stäckel, P.: "Sur l'Intégration de l'Équation Différentielle de Hamilton." *Compt. Rend.* **121**: 489-492, 1895.

Weinacht, J.: "Über die Bodingt-periodische Bewegung eines Massenpunktes." *Math. Ann.* **91**: 279-299, 1924.

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