RELATIVISTIC STELLAR STABILITY: PREFERRED-FRAME EFFECTS*

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ABSTRACT

In a previous paper, the PPN (Parametrized Post-Newtonian) formalism was used to analyze relativistic influences on stellar stability in nearly all metric theories of gravity. That analysis omitted all "preferred-frame" terms. In this paper, possible preferred-frame effects on stellar stability are examined and no new instabilities are found. In particular, we show that: (i) Although terms linear in the preferred-frame velocity $\underset{\sim}{w}$ (time-odd terms, analogous to viscosity and energy generation) change the shapes of the normal modes, their symmetry properties prevent them from changing the characteristic frequencies. Thus, no new vibrational or secular instabilities can occur. (ii) Terms quadratic in $\underset{\sim}{w}$ do not change either the shapes of the normal modes or the characteristic frequencies for <u>radial</u> pulsations. Thus, they have no influence on radial stability. (iii) Terms quadratic in $\underset{\sim}{w}$ do change both the normal modes and the characteristic frequencies of <u>nonradial</u> pulsations; but in the limit of a neutral mode these changes vanish. Hence, there is no modification of the criterion for convective stability, i.e., the standard Schwarzschild criterion remains valid.

I. INTRODUCTION AND SUMMARY

In a recent paper [Ni (1973); hereafter referred to as Paper I; equations therein will be referred to simply as I (25), I (34), etc.], we used the PPN (Parametrized Post-Newtonian) formalism of Will and Nordtvedt (1972) to analyze stellar stability for spherically symmetric stars in a large class of metric theories of gravity. We found that (i) for "conservative theories of gravity," current solar-system experiments guarantee the existence of a dynamical relativistic instability; (ii) for "nonconservative theories," current experiments do not permit any firm conclusion about whether relativistic effects will actually stabilize or destabilize stars; (iii) the standard Schwarzschild criterion for convection is valid. The basic equations of Paper I were rigorous within the PPN framework; but when analyzing and manipulating those basic equations, we ignored all "preferred-frame terms" (all terms containing the velocity $\underset{\sim}{w}$ of the star relative to the mean rest frame of the universe). This paper completes the analysis of Paper I by analyzing the effects of the preferred-frame terms.

At the time of writing Paper I, we believed that the preferred-frame terms would drive vibrational instabilities of spherically symmetric stars, and that those instabilities, when combined with astronomical observations on white-dwarf pulsations, might lead to experimental limits on the "preferred-frame parameters" α_1, α_2, and α_3. But as a result of the present analysis, the above beliefs turn out to be incorrect.[1] No new instabilities of any type

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[1] The experimental limits on α_1 $(= 7\Delta_1 + \Delta_2 - 4\gamma - 4)$, α_2 $(= \Delta_2 + \zeta_2 - 1)$, and α_3 $(= 4\beta_1 - 2\gamma - 2 - \zeta)$ due to stability observations of white dwarfs, as quoted in Ni (1972) must be dropped in view of the present analysis. Although preferred-frame vibrational instabilities might still occur in asymmetric stars, they would not likely give such tight limits on the α's.

are produced by the preferred-frame terms. Hence, the stability conclusions of Paper I are completely general within the PPN formalism.

II. ANALYSIS

In Paper I, by calculating in the PPN rest frame of the star, and by describing the stellar pulsation by a Lagrangian displacement of the form $\xi(x)\, e^{i\Omega t}$, we obtained the following linearized, adiabatic pulsation equations [eqs. I (32) rewritten in slightly different format]:[2]

[2] The notations and conventions of this paper are the same as those in Paper I unless otherwise specified. For definitions of the various quantities undefined here, see Paper I also.

$$\mathcal{L}\xi \equiv (\mathcal{L}_o + \mathcal{L}_{w^2} + \mathcal{L}_{iw}\Omega + \mathcal{L}_{\Omega^2}\Omega^2)\ \xi = 0 \quad . \tag{1}$$

Here $\mathcal{L}_o$, $\mathcal{L}_{w^2}$, $\mathcal{L}_{iw}$, and $\mathcal{L}_{\Omega^2}$ are linear operators that are all independent of the frequency of pulsation Ω. In particular,

$$\mathcal{L}_o\xi = -\nabla\left\{[1 + (3\gamma - 1)\ U]\ \Delta p + (3\gamma - 1)\ p\Delta U\right\} + \frac{\Delta\rho}{\rho}\nabla\left\{[1 + (3\gamma - 1)\ U]p\right\}$$
$$+ \rho\nabla(\Delta U) + 2\rho[\Delta\bar{\Phi}\nabla U + \bar{\Phi}\nabla(\Delta U) + \nabla(\Delta\Phi)] \quad , \tag{2}$$

$$\mathcal{L}_{w^2}\xi = (\tfrac{1}{3}\alpha_2 + \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\alpha_1)\ w^2\rho\nabla(\Delta U) - \tfrac{1}{2}\alpha_2 w^\gamma w^\delta\rho\nabla(\Delta U_{\gamma\delta} - \tfrac{1}{3}\delta_{\gamma\delta}\Delta U) \quad ,^{3} \tag{3}$$

[3] In Paper I, there was an error in the $\Delta\rho/\rho$ term: i.e., $[1 + \frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1)\ w^2 + (3\gamma - 1)\ U]$ should be corrected to $[1 + (3\gamma - 1)\ U]$, as has been done above.

$$\mathcal{L}_{iw}\xi = i[\frac{1}{2}\alpha_1\rho(\Delta U - \xi \cdot \nabla U)\, w - \frac{1}{2}\rho(\alpha_1 - 2\alpha_3)\, w^\beta \nabla V_\beta$$
$$+ \alpha_2\rho(w \cdot \nabla)\, \nabla(\Delta X - \xi \cdot \nabla X) - \frac{1}{2}\alpha_1(w \cdot \xi)\, \rho\nabla U] \quad , \tag{4}$$

$$\mathcal{L}_{\Omega^2}\xi = \sigma\xi + \frac{1}{2}(\alpha_2 - \zeta_1 + 1)\,\rho(V - W) + (5\gamma - 1)\, U\xi - \frac{1}{2}(\alpha_1 + 4\gamma + 4)\, V \quad . \tag{5}$$

Also

$$\Delta\rho^* = -\rho^* \operatorname{div} \xi \quad . \tag{6}$$

Notice that the linear preferred-frame terms are embodied in the operator $\mathcal{L}_{iw}$, and the quadratic preferred-frame terms are embodied in $\mathcal{L}_{w^2}$. [Recall (Paper I) that w is the velocity of the star relative to the mean-rest frame of the universe.]

In Paper I, we derived stability criteria by analyzing the simplified pulsation equation $(\mathcal{L}_0 + \Omega^2\mathcal{L}_{\Omega^2})\, \xi = 0$. Here we look into possible modifications of those criteria by the presence of $\mathcal{L}_{iw}$ and $\mathcal{L}_{w^2}$. Terms in $\mathcal{L}_{iw}$ have imaginary coefficients; therefore they (like viscosity, energy generation, and radiative transport) might possibly affect the vibrational and secular stabilities of the star. Terms in $\mathcal{L}_{w^2}$ have real coefficients; therefore they might possibly affect the dynamical and convective stability of the star. But, in fact, no such effects occur. To see this, we proceed as follows.

Equation (1), when supplemented by the boundary conditions and the expressions for the various Lagrangian changes in terms of ξ, constitutes a characteristic value problem for Ω. In Paper I we have shown that (i) this characteristic value problem is self-adjoint if, and only if,

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \quad ; \tag{7}$$

but (ii) in the general case with $\underset{\sim}{w}$-terms deleted, although the characteristic value problem is not self-adjoint, a variational integral can still be constructed.

When the w-terms are included, one can use the same argument as in Paper I [the passage between I (43) and I (53)] to show the following: Take the full post-Newtonian characteristic equation (1); dot $\underset{\sim}{\xi}^*$ into it; and integrate over the star. The result

$$\Omega^2 \int \underset{\sim}{\xi}^* \cdot \mathcal{L}_{\Omega^2} \underset{\sim}{\xi} d^3x + \Omega \int \underset{\sim}{\xi}^* \cdot \mathcal{L}_{iw} \underset{\sim}{\xi} d^3x + \int \underset{\sim}{\xi}^* \cdot (\mathcal{L}_o + \mathcal{L}_{w^2}) \underset{\sim}{\xi} d^3x = 0 \quad , \qquad (8)$$

is a variational principle for the post-Newtonian normal modes.

The changes in squared frequency due to preferred-frame terms can be obtained, to post-Newtonian accuracy, by inserting the Newtonian characteristic functions into the variational principle (8) and solving for Ω:

$$(\delta\Omega^2)_{\text{preferred frame}} = - \frac{\int \underset{\sim}{\xi}^* \cdot (\mathcal{L}_{w^2} + \Omega\mathcal{L}_{iw}) \underset{\sim}{\xi} d^3x}{\int \rho |\xi|^2 d^3x} \quad . \qquad (9)$$

The change $\delta\Omega$ due to $\mathcal{L}_{iw}$ is proportional to

$$\int \underset{\sim}{\xi}^* \cdot \mathcal{L}_{iw} \underset{\sim}{\xi} d^3x \quad . \qquad (10)$$

Being linear in the constant vector $\underset{\sim}{w}$, $\mathcal{L}_{iw}$ raises and lowers the spherical-harmonic index of a characteristic function by 1 — thereby producing a new function orthogonal to the original one. (For proof, please see the Appendix.):

$$\delta\Omega \propto \int (\ell - \text{mode})^* \cdot \mathcal{L}_{iw} (\ell - \text{mode})\, d^3x$$
$$= \int (\ell - \text{mode})^* \cdot \Big| a[(\ell - 1) - \text{mode}] + b[(\ell + 1) - \text{mode}] \Big| d^3x = 0 \quad . (11)$$

Thus, the $\mathcal{L}_{iw}$ term has no effect whatsoever on the pulsation frequencies of the star to post-Newtonian accuracy. Hence, to the same accuracy the stability criteria are not affected either — and, in particular, there is no post-Newtonian vibrational or secular instability.

Might the preferred-frame velocity $\underset{\sim}{w}$ drive a vibrational or secular instability at higher orders in the post-Newtonian expansion (post-post-Newtonian, etc.)? There are two ways that it might do so: (i) by higher-order corrections to the imaginary operator $\mathcal{L}_{iw}$, and (ii) by higher-order effects of the original $\mathcal{L}_{iw}$ (eq. [4]). Although one is not now in a position to evaluate the first way, it is clear that the second way <u>cannot</u> produce an instability: Use higher-order perturbation theory to solve equation (1) to higher-order accuracy in $\mathcal{L}_{iw}$. Since $\mathcal{L}_{iw}$ is purely imaginary and $\mathcal{L}_0$, $\mathcal{L}_{w^2}$, $\mathcal{L}_{\Omega^2}$ are all real, every term in a perturbation series expansion for $\mathrm{Im}(\Omega)$ must contain $\mathcal{L}_{iw}$ an odd number of times. [This can be proved by an actual expansion calculation despite the nonself-adjoint nature of the problem.] But $\mathcal{L}_{iw}$ only raises and lowers the spherical-harmonic index of a characteristic function by 1, while $\mathcal{L}_{w^2}$, $\mathcal{L}_0$, and $\mathcal{L}_{\Omega^2}$ only raise and lower the spherical-harmonic index by 2 or do not change it [for proof, see the Appendix]. Hence every term must contain an integration of the form

$$\int (\ell - \text{mode})^* \, (\ell' - \text{mode}) \, d^3x \quad , \tag{12}$$

where one of ℓ and ℓ' is even while the other is odd. By orthogonality of the different spherical-harmonic modes, (12) vanishes and

$$\mathrm{Im}(\Omega) = 0 \quad . \tag{13}$$

Therefore no higher-order vibrational or secular instabilities can occur.

By equation (9), the change in squared frequency due to the relativistic $\mathcal{L}_{w^2}$ operator is

$$\delta\Omega \propto \int \underset{\sim}{\xi}^* \mathcal{L}_{w^2} \underset{\sim}{\xi} d^3x \quad . \tag{14}$$

The first term in equation (3) for $\mathcal{L}_{w^2} \underset{\sim}{\xi}$ can be combined with the term $\delta\underset{\sim}{\nabla}(\Delta U)$ in $\mathcal{L}_o \underset{\sim}{\xi}$ [eq. (2)] to give a renormalization of the gravitational constant

$$G \rightarrow [1 + (\frac{1}{3}\alpha_2 + \frac{1}{2}\alpha_3 - \frac{1}{2}\alpha_1) w^2] G \quad .$$

Since Newtonian stability is not sensitive to the value of the gravitational constant, this term does not affect stability (even though it does affect the values of nonzero pulsation frequencies). The second term in $\mathcal{L}_{w^2} \underset{\sim}{\xi}$ [eq. (3)] can be put into the form

$$-\frac{1}{2}\alpha_2 w^\gamma w^\delta \rho \underset{\sim}{\nabla}(\delta U_{\gamma\delta} - \frac{1}{3}\delta_{\gamma\delta}\delta U) \quad , \tag{15}$$

where

$$\delta U_{\gamma\delta}(\underset{\sim}{x}) = \int \delta\rho(\underset{\sim}{x}') \frac{(x_\gamma - x'_\gamma)(x_\delta - x'_\delta)}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}' \quad , \tag{16}$$

and

$$\delta U(\underset{\sim}{x}) = \int \delta\rho(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}' \quad . \tag{17}$$

The symbol δ denotes the Eulerian change in the quantity that it qualifies. For radial pulsations

$$\delta U_{\gamma\delta}(\underset{\sim}{x}) = \frac{1}{3}\delta_{\gamma\delta}\delta U(\underset{\sim}{x}) \quad ; \tag{18}$$

therefore equation (15) vanishes and there are no changes either in the shape of the normal modes or the characteristic frequencies. For nonradial pulsations, although the normal modes and characteristic frequencies are modified in general [see the Appendix for an explicit expression for $\delta\Omega^2$], at the onset of convection

$$\delta\rho(x') = 0 \quad , \tag{19}$$

so $\delta U_{\gamma\delta} = \delta U = 0$ and equation (15) vanishes. Hence by equation (14), neither spherical stability nor the onset of convection are affected by $\mathcal{L}_{w^2}$.

III. CONCLUSIONS

Preferred-frame terms have been shown to not affect stability criteria for stellar pulsations about spherical equilibrium, even though they do modify the characteristic functions and (in some cases) characteristic frequencies. This enables us to conclude that the stability criteria found in Paper I are completely general and can be used for all metric theories of gravity which fit into the PPN formalism in their post-Newtonian limits.

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APPENDIX

SOME DETAILS OF THE CALCULATION

In this appendix we will derive an explicit version of equation (9) for $\delta\Omega^2$, and will prove the statement ("S") that $\mathcal{L}_{iw}$ only raises and lowers the spherical-harmonic index of a characteristic function by 1, while $\mathcal{L}_{w^2}$, $\mathcal{L}_o$, and $\mathcal{L}_{\Omega^2}$ only raise and lower the spherical-harmonic index by 2 or do not change it.

To accomplish this, we analyze the Lagrangian displacement $\underset{\sim}{\xi}$ into normal modes belonging to different vector spherical harmonics, i.e.,

$$\xi_r = \frac{\psi(r)}{r^2} Y_{\ell m}(\Theta,\phi) \quad , \tag{A1}$$

$$\xi_\Theta = \frac{1}{\ell(\ell+1)\, r} \, \frac{d\chi(r)}{dr} \, \frac{\partial Y_{\ell m}(\Theta,\phi)}{\partial\Theta} \quad , \tag{A2}$$

and

$$\xi_\phi = \frac{1}{\ell(\ell+1)\, r \sin\Theta} \, \frac{d\chi(r)}{dr} \, \frac{\partial Y_{\ell m}(\Theta,\phi)}{\partial\phi} \quad , \tag{A3}$$

(ξ_r, ξ_Θ, ξ_ϕ are physical components, not covariant components). For convenience, we choose our coordinate system such that the z-axis is in the direction of the "preferred-frame" velocity, i.e., $\underset{\sim}{w} = w\hat{\underset{\sim}{z}}$ where $\hat{\underset{\sim}{z}}$ is the unit vector along the z-axis.

Without the preferred-frame terms $\mathcal{L}_{iw}$ and $\mathcal{L}_{w^2}$, the operator $\mathcal{L}_o + \Omega^2\mathcal{L}_{\Omega^2}$ is invariant under rotation; hence the characteristic value problem has m-degeneracy in this approximation. With the preferred-frame terms, the degeneracy in m is broken; but since the problem still possesses azimuth symmetry, normal modes have definite m-numbers. Therefore (A1)-(A3) are normal modes for the characteristic problem to the first approximation, and can be used to evaluate the changes in characteristic frequencies due to preferred-frame terms by equation (9).

To evaluate equation (9) and to find the effects of $\mathscr{L}_{iw}$ and $\mathscr{L}_{w^2}$ on normal modes, we first derive a number of formulas involving spherical harmonics. From the integration formula involving three spherical hermonics [cf. e.g., Edmonds (1960) p. 63]

$$\int Y_{\ell_1 m_1}(\Theta,\varphi)\, Y_{\ell_2 m_2}(\Theta,\varphi)\, Y_{\ell_3 m_3}(\Theta,\varphi)\, d\Omega = \left[\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}\right]^{1/2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad \text{(A4)}$$

we have

$$\int Y^*_{\ell' m'}(\Theta,\varphi)\, Y_{\ell m}(\Theta,\varphi) \cos\Theta\, d\Omega = (-1)^{m'}[(2\ell'+1)(2\ell+1)]^{1/2} \begin{pmatrix} \ell' & \ell & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell' & \ell & 1 \\ -m' & m & 0 \end{pmatrix}, \quad \text{(A5)}$$

and

$$\int Y^*_{\ell' m'}(\Theta,\varphi)\, \frac{\partial Y_{\ell m}(\Theta,\varphi)}{\partial\Theta} \sin\Theta\, d\Omega = (-1)^{m'}[2(\ell-m)(\ell+m+1)(2\ell'+1)(2\ell+1)]^{1/2} \begin{pmatrix} \ell' & \ell & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell' & \ell & 1 \\ -m' & m+1 & -1 \end{pmatrix}, \quad \text{(A6)}$$

where $\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3-j symbol. $\begin{pmatrix} \ell' & \ell & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nonzero only if $\ell' = \ell + 1$ or $\ell' = \ell - 1$.

From the gradient formula [e.g., Edmonds (1960) p. 84],

$$\nabla[\Phi(r)\, Y_{\ell m}(\Theta,\varphi)] = -\left(\frac{\ell+1}{2\ell+1}\right)^{1/2}\left[\left(\frac{d}{dr} - \frac{\ell}{r}\right)\Phi(r)\right]\vec{Y}_{\ell\,\ell+1\,m}(\Theta,\varphi) + \left(\frac{\ell}{2\ell+1}\right)^{1/2}\left[\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\Phi(r)\right]\vec{Y}_{\ell\,\ell-1\,m}(\Theta,\varphi), \quad \text{(A7)}$$

and from the definition of vector spherical harmonics, we derive

$$\hat{\underset{\sim}{z}} \cdot \nabla [\Phi(r)\, Y_{\ell m}(\Theta,\varphi)] = \left[\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2} \left[\left(\frac{d}{dr} - \frac{\ell}{r}\right)\Phi\right] Y_{\ell+1m}$$

$$+ \left[\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} \left[\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\Phi\right] Y_{\ell-1m} \quad , \qquad \text{(A8)}$$

and

$$\int \left[\hat{\underset{\sim}{z}} \cdot \nabla\left(\Phi(r)\, Y^*_{\ell m}\right)\right] \left[\hat{\underset{\sim}{z}} \cdot \nabla\left(\Phi(r)\, Y_{\ell m}\right)\right] d\Omega$$

$$= \frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)} \left[\left(\frac{d}{dr} - \frac{\ell}{r}\right)\Phi\right] \left[\left(\frac{d}{dr} - \frac{\ell}{r}\right)\Psi\right]$$

$$+ \frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)} \left[\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\Phi\right] \left[\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\Psi\right] \quad . \qquad \text{(A9)}$$

From the formulas

$$\frac{\partial\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}}{\partial r} = \frac{r - r'\cos\gamma}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} \quad , \qquad \text{(A10)}$$

and

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\gamma), \quad (r < r') \qquad \text{(A11)}$$

we find

$$\sqrt{r^2 + r'^2 - 2rr'\cos\gamma} = \sum_{\ell=0}^{\infty} \left[\frac{1}{2\ell+3}\, \frac{r^{\ell+2}}{r'^{\ell+1}} - \frac{1}{2\ell-1}\, \frac{r^\ell}{r'^{\ell+1}}\right] P_\ell(\cos\gamma). \quad \text{(A12)}$$

By the addition theorem, we can then write equation (A12) as

$$|\underset{\sim}{x} - \underset{\sim}{x}'| = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left[\frac{1}{2\ell+3}\, \frac{r^{\ell+2}}{r'^{\ell+1}}\right.$$

$$\left. - \frac{1}{2\ell-1}\, \frac{r^\ell}{r'^{\ell-1}}\right] Y^*_{\ell m}(\Theta',\phi')\, Y_{\ell m}(\Theta,\phi) \quad . \qquad \text{(A13)}$$

Now by (A1)-(A13) and by the expressions [cf. I (38), (39), (40)]

$$\delta U = - \int \operatorname{div}[\rho(\underset{\sim}{x}') \, \underset{\sim}{\xi}(\underset{\sim}{x}')] \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' \quad , \tag{A14}$$

$$\delta U_{\gamma\delta} = - \int \operatorname{div}[\rho(\underset{\sim}{x}') \, \underset{\sim}{\xi}(\underset{\sim}{x}')] \frac{(x_\gamma - x_\gamma')(x_\delta - x_\delta')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}' \quad , \tag{A15}$$

$$\delta X = - \int \operatorname{div}[\rho(\underset{\sim}{x}') \, \underset{\sim}{\xi}(\underset{\sim}{x}')] \, |\underset{\sim}{x} - \underset{\sim}{x}'| \, d\underset{\sim}{x}' \quad , \tag{A16}$$

and

$$\delta U_{\gamma\delta} = \delta_{\gamma\delta} \delta U - \frac{\partial^2(\delta X)}{\partial x^\gamma \partial x^\delta} \quad , \tag{A17}$$

we can readily verify, by straightforward calculations, that

$$\mathcal{L}_{iw}(\ell - \text{mode}) = a[(\ell - 1) - \text{mode}] + b[(\ell + 1) - \text{mode}] \quad , \tag{A18}$$

and

$$\mathcal{L}_{w^2}(\ell - \text{mode}) = c[(\ell - 2) - \text{mode}] + d[\ell - \text{mode}] + e[(\ell + 2) - \text{mode}] \, . \tag{A19}$$

Equations (A18) and (A19) together with the invariance of $\mathcal{L}_0$ and $\mathcal{L}_{\Omega^2}$ under rotation prove the statement "S" made at the beginning of this appendix.

To derive an explicit formula for $\delta\Omega^2$ [eq. (9)], we note that

$$\int \underset{\sim}{\xi}^* \mathcal{L}_{iw} \underset{\sim}{\xi} \, d^3x = 0$$

[by (A18)]. Therefore the integral remaining to be evaluated is

$$\int \underset{\sim}{\xi}^* \cdot \mathcal{L}_{w^2} \underset{\sim}{\xi} \, d^3x \, . \, .$$

By (A1)-(A17) and the formula [by (A1)-(A3)]

$$\underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{\xi}) = \frac{1}{r^2} \left[\frac{d(\rho\psi)}{dr} - \rho \frac{d\bar{X}}{dr} \right] Y_{\ell m}(\theta, \varphi) \quad , \tag{A20}$$

we have, after some calculations and reductions,

$$\int \underset{\sim}{\xi}^* \cdot \mathcal{L}_{w^2} \underset{\sim}{\xi} d^3x = -\frac{2\pi}{2\ell + 1} \alpha_2 w^2 \left\{ \iint \left[\frac{d(\rho\psi)}{dr'} - \rho \frac{d\bar{\chi}}{dr'}\right]\left[\frac{d(\rho\psi)}{dr} - \rho \frac{d\bar{\chi}}{dr}\right]\left[\frac{2}{3} \frac{r_<^{\ell}}{r_>^{\ell+1}}\right.\right.$$

$$- \frac{(\ell + m + 1)(\ell - m + 1)}{(2\ell + 1)(2\ell + 3)}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell + 2)}{r^2}\right)\left(\frac{1}{2\ell + 3} \frac{r_<^{\ell+2}}{r_>^{\ell+1}}\right.$$

$$\left. - \frac{1}{2\ell - 1} \frac{r_<^{\ell}}{r_>^{\ell-1}}\right) - \frac{(\ell + m)(\ell - m)}{(2\ell + 1)(2\ell - 1)} \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right.$$

$$\left.\left. - \frac{(\ell - 1)(\ell + 1)}{r^2}\right)\left(\frac{1}{2\ell + 3} \frac{r_<^{\ell+2}}{r_>^{\ell+1}} - \frac{1}{2\ell - 1} \frac{r_<^{\ell}}{r_>^{\ell-1}}\right)\right] dr\, dr' \Bigg\} \quad . \tag{A21}$$

In general (A21) does not vanish. But for $\ell = 0$, (A21) vanishes; and for $\ell \neq 0$, at the onset of convection where, $\delta\rho = 0$, i.e., where

$$\frac{d(\rho\psi)}{dr} - \rho \frac{d\bar{\chi}}{dr} = 0 \quad , \tag{A22}$$

(A21) also vanishes, in agreement with the results in §II.

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