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BOPACE THEORETICAL MANUAL

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1.0 INTRODUCTION

BOPACE is the acronym for the Boeing Plastic Analysis Capability for Engines. BOPACE was developed by Boeing/Huntsville to meet the evident need for an advanced thermal-elastic-plastic-creep structural analyzer. Although BOPACE development has been strongly influenced by the requirements for structural analysis of engines, in particular the space shuttle main engine, its capabilities have been kept quite general and it is applicable to many types of nonlinear structures.

The philosophy for program development was based on the following requirements.

1) Analysis of very high temperature and large plastic-creep effects.
2) Treatment of cyclic thermal and mechanical loads.
3) Improved material constitutive theory which closely follows actual behavior under variable temperature conditions.
4) A stable numerical solution approach which avoids cumulative errors.
5) Capability for handling up to 1000 degrees of freedom with moderate computation cost.

Although the finite-element method was first applied to plasticity in the early 1960's, and several good programs for nonlinear analysis have since been developed, numerous improvements were indicated in order

1.0 (Continued)

to satisfy the above requirements. For example, some other available programs assume linear plastic hardening, accumulate errors by failing to satisfy equilibrium at each step, or do not completely account for the effects of variable temperature on the elastic and plastic relations. The stated requirements have been effectively met by the current BOPACE program version. In addition, the research and development effort has led to an improved hardening theory for cyclic plasticity, a method for representing general cases of load reversal, and advanced techniques for improving the accuracy and controlling convergence of highly nonlinear solutions.

Two versions of the current BOPACE program are available. The first is a 300-DOF version developed for fast analysis of small size problems within moderate core-storage limitations. The second is the basic 1000-DOF version. In addition, a low-core modification of the 1000-DOF version has been accomplished through the use of overlays and dynamic storage of arrays. BOPACE is written in FORTRAN IV and has been extensively run on both the IBM 360 and UNIVAC 1108 computer systems. Documentation consists of three volumes: Theoretical Manual, User Manual (including example problems), and Programmer Manual.

The BOPACE development and programming effort has been performed at Boeing/Huntsville by Dr. R. G. Vos, with suggestions and review by W. H. Armstrong. A. H. Spring assisted with many analyses and program

1.0 (Continued)

checkout. J. L. Ballinger of Boeing Computer Services modified and programmed the Gauss wavefront solution method. Recognition is also due to N. L. Schlemmer, L. Salter and R. Hurford at the NASA Marshall Space Flight Center, and L. Johnston of Brown Engineering Co., for their suggestions and support of the program development.

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2.0 MATERIAL CONSTITUTIVE THEORY

The basic purpose of classical constitutive theory in an elasto-visco-plastic program such as BOPACE is to provide incremental relations between stresses and strains. BOPACE uses these relations with the finite-element stiffness method to provide a convenient and efficient approach for solution of an important class of nonlinear material problems.

BOPACE performs plane-stress and plane-strain analyses, as well as limited 3-dimensional analyses involving prescribed non-zero values of normal stress or strain. It accounts for elastic, plastic, thermal and creep deformations, and the nonlinear dependence of all deformations on temperature. New constitutive developments include a combined plastic-hardening theory, and a generalized approach to cyclic load reversal. The BOPACE constitutive theory is developed by a tensorial approach which provides all relations in a form which is invariant under coordinate transformations.

2.1 ELASTICITY EQUATIONS

This section defines the cumulative and incremental forms of the relations for temperature-dependent elasticity. The BOPACE formulations for plane stress and plane strain are developed, including the effects of prescribed non-zero values for normal stress or strain.

<u>General Concepts and 3-D Relations</u> - The basic cumulative stress-strain relation, for either temperature-dependent or temperature-independent elasticity, is

2.1 (Continued)

$$\sigma_{ij} = D^e_{ijk\ell}\ \varepsilon^e_{k\ell} \tag{2.1-1}$$

where σ and ε^e are the 3 x 3 tensors of stress and elastic (recoverable) strain, respectively, and D^e is the tensor of elastic coefficients which may depend on temperature. For convenience we will use the equivalent single-subscript notation

$$\sigma_i = D^e_{ij}\ \varepsilon^e_j \tag{2.1-2}$$

where subscripts i and j range over all nine of the tensor components. For 3-dimensional analysis the relation 2.1-2 is taken as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = \left[\begin{array}{c|c|c} D_{11} & 0 & 0 \\ \hline 0 & D_{22} & 0 \\ \hline 0 & & D_{33} \end{array}\right] \begin{Bmatrix} \varepsilon^e_{xx} \\ \varepsilon^e_{yy} \\ \varepsilon^e_{zz} \\ \varepsilon^e_{xy} \\ \varepsilon^e_{yz} \\ \varepsilon^e_{zx} \\ \varepsilon^e_{yx} \\ \varepsilon^e_{zy} \\ \varepsilon^e_{xz} \end{Bmatrix} \tag{2.1-3}$$

2.1 (Continued)

where

$$D_{11} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}$$

and

$$D_{22} = D_{33} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-2\nu & 0 & 0 \\ 0 & 1-2\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix}$$

Here E is Young's modulus and ν is Poisson's ratio.

Note that the elasticity matrix in Equation 2.1-3 is consistent with the tensorial definition of shear strains (e.g. $\varepsilon^e_{xy} = \gamma^e_{xy}/2$, where γ_{xy} is the engineering definition of shear strain). Tensorial definitions are used throughout the BOPACE program in order to easily formulate constitutive theory which is invariant with respect to coordinate transformations. Although such invariance can be achieved by a careful use of engineering strain definitions, such definitions are probably responsible for many invariance difficulties such as those described in Reference 1.

2.1 (Continued)

The last three of Equations 2.1-3 are somewhat redundant and may be discarded to give an abbreviated 6-component form (e.g. $\sigma_{yx} = \sigma_{xy}$ and $\varepsilon_{yx} = \varepsilon_{xy}$). It should be emphasized, however, that in performing later summations all non-zero values of the nine tensor components must be accounted for.

<u>Relations for Plane Stress and Plane Strain</u> - For plane stress, the abbreviated form of Equation 2.1-2 is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon^e_{xx} \\ \varepsilon^e_{yy} \\ \varepsilon^e_{xy} \end{Bmatrix} \tag{2.1-4}$$

and for plain strain it is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \varepsilon^e_{xx} \\ \varepsilon^e_{yy} \\ \varepsilon^e_{xy} \end{Bmatrix} \tag{2.1-5}$$

<u>Incremental Relations</u> - For the case of temperature-independent elasticity the incremental stress-strain relations are simply

2.1 (Continued)

$$\Delta\sigma_i = D^e_{ij} \quad \Delta\varepsilon^e_j \qquad (2.1\text{-}6)$$

where Δ denotes an incremental quantity and D^e is the appropriate elasticity matrix.

When temperature dependence is considered, the incremental relations become

$$\Delta\sigma_i = \Delta D^e_{ij}\varepsilon^{e0}_j + D^{e1}_{ij} \quad \Delta\varepsilon^e_j \qquad (2.1\text{-}7)$$

where ε^{e0} are the elastic strains at the beginning of the increment, D^{e1} is the elasticity matrix evaluated for the temperature at the end of the increment, and $\Delta D^e = D^{e1} - D^{e0}$ is the change in elasticity matrix from beginning to end of the increment. The first term in Equation 2.1-7 accounts for stress change due only to change in elastic properties, while the second term accounts for additional stress change due to the increment of elastic strain.

BOPACE Formulations - The BOPACE program allows the option for either a "plane-stress" analysis with prescribed non-zero values of normal stress σ_{zz}, or a "plane-strain" analysis with prescribed non-zero values of normal strain ε_{zz}.

In plane-strain problems, BOPACE uses the 3-D elasticity matrix of Equation 2.1-3 because, even though the total (physical) strain ε_{zz}

2.1 (Continued)

might vanish, the elastic strain ε^e_{zz} is in general not zero. If a non-zero value is prescribed by the user for ε_{zz}, no additional difficulties are introduced.

For plane-stress problems, BOPACE uses the elasticity matrix of Equation 2.1-4. For the general case when a non-zero value is prescribed by the user for σ_{zz}, an additional term is required in the stress-strain relation. The resulting general plane-stress equation is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon^e_{xx} \\ \varepsilon^e_{yy} \\ \varepsilon^e_{xy} \end{Bmatrix} + \frac{\nu}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \sigma_{zz} \qquad (2.1\text{-}8a)$$

or

$$\sigma_i = D^e_{ij}\, \varepsilon^e_j + \mu^e_i\, \sigma_{zz} \qquad (2.1\text{-}8b)$$

where $\mu^e = \frac{\nu}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$

The general incremental plane-stress relation for temperature-dependent elasticity then follows Equation 2.1-7:

$$\Delta\sigma_i = \Delta D^e_{ij}\, \varepsilon^{e0}_j + D^{e1}_{ij}\, \Delta\varepsilon^e_j + \Delta\mu^e_i\, \sigma^0_{zz} + \mu^{e1}_i\, \Delta\sigma_{zz} \qquad (2.1\text{-}9)$$

where σ^0_{zz} and $\Delta\sigma_{zz}$ are respectively the initial and incremental values of prescribed normal stress.

2.2 THERMAL STRAIN

Alternate Formulations - The conventional description of thermal strain is given by

$$\Delta \begin{Bmatrix} \varepsilon^t_{xx} \\ \varepsilon^t_{yy} \\ \varepsilon^t_{zz} \end{Bmatrix} = \gamma \, \Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{2.2-1}$$

where ε^t denotes thermal strain, T is the temperature, and γ is the thermal coefficient of expansion which may be a function of temperature.

An alternate integrated description of thermal strain is

$$\begin{Bmatrix} \varepsilon^t_{xx} \\ \varepsilon^t_{yy} \\ \varepsilon^t_{zz} \end{Bmatrix} = \bar{\varepsilon}^t \, (T) \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{2.2-2}$$

where here $\bar{\varepsilon}^t$ gives the thermal strain directly as a function of temperature. If only incremental thermal strains are of interest, $\bar{\varepsilon}^t$ may be taken as zero at any convenient reference temperature.

BOPACE Formulation - BOPACE uses the direct form 2.2-2. This form is preferred over that involving a thermal expansion coefficient because accumulated errors in thermal strain are not introduced. These errors could arise with the form 2.2-1, in case γ varied with temperature and the specified heating and cooling sequences used different temperature

2.2 (Continued)

increments. BOPACE takes the structural fabrication temperature as the reference temperature for zero thermal strains.

2.3 PLASTICITY

This section defines the incremental elasto-plastic relations used in the BOPACE program. BOPACE employs a new concept of combined isotropic and kinematic hardening, and accounts for temperature-dependent elasto-plastic behavior as well as a generalized form of cyclic load reversal. In order to develop the constitutive theory in a straightforward manner, discussion of the effects of temperature-dependent elasticity on the elasto-plastic relations is deferred until Section 2.5.

<u>Definitions</u> - The following nomenclature is defined.

σ	=	total stress
α	=	stress center (of yield surface in kinematic hardening)
s	=	deviatoric (total - hydrostatic) stress
a	=	deviatoric stress center
$\hat{s}$	=	$s - a$ = relative deviatoric stress
ε^e	=	elastic (recoverable) strain
ε^p	=	plastic (time independent non stress-inducing strain)
ε^{e+p}	=	$\varepsilon^e + \varepsilon^p$

2.3 (Continued)

<u>General Concepts</u> - The basic concepts in most elasto-plastic theories are those of a <u>yield surface</u>, the dependence of yield on only the <u>deviatoric</u> stress components, <u>incompressibility</u> under plastic strains, and <u>normality</u> of the incremental plastic-strain vector to the yield surface. The definition of a particular theory requires assumptions for three basic constituents:

1. a surface relating the stress components at yield
2. a flow rule defining a direction for the incremental plastic-strain vector
3. a hardening rule.

<u>Yield Surface</u> - BOPACE employs the Huber-Mises yield surface [2], defined by the relative deviatoric stresses as

$$F = \hat{s}_i \hat{s}_i - \hat{s}_i^\circ \hat{s}_i^\circ = 0 \qquad (2.3\text{-}1)$$

where the summation is again taken over all nine tensor components of s. The $\hat{s}_i^\circ$ are components of a point on the yield surface at a known condition of temperature and plastic deformation, e.g. from a uniaxial test. Equation 2.3-1 holds whenever the material is plastic, i.e. whenever the components of s are on the yield surface. Function F may be thought of geometrically as defining a hypersphere in 9-dimensional stress space. Alternatively, when expressed in the 3-D space of principal stresses, this yield surface can be shown to be an open-ended circular cylinder whose axis passes through the origin and makes equal angles with each of the

2.3 (Continued)

three principal stress axes. The Huber-Mises yield surface is generally used to describe plasticity in metals because it agrees reasonably well with test results and it gives a smooth surface which is convenient for calculations.

Flow Rule - BOPACE uses the Prandtl-Reuss flow rule, which is the usual rule associated with the Huber-Mises yield surface. The assumptions are that the material is incompressible under plastic flow, and that increments of plastic strain are normal to the yield surface at the stress point.

These provide the relation

$$\Delta\varepsilon_i^p = \lambda\hat{s}_i \tag{2.3-2}$$

where λ is a flow parameter.

Basic Hardening Concepts - An elastic-plastic material which work hardens in the plastic range is commonly analyzed using either of two classical hardening theories. Isotropic hardening [3], which assumes a uniform expansion of the yield surface during plastic flow, accounts for change in size of the hysteresis loop during cycling. Kinematic hardening [4], which assumes a rigid translation of the yield surface in the direction of the plastic strain increment, accounts for the pronounced Bauschinger effect which is evident in cyclic behavior of most metals. In general, the actual cyclic behavior can be more accurately described by a combination of isotropic and kinematic hardening. A combined hardening theory has been

2.3 (Continued)

given by Hodge [5] for materials which satisfy the Tresca yield condition. Because a better representation for most metals is provided by the Huber-Mises yield surface, a corresponding combined hardening theory [6] has been developed for the BOPACE program.

Hardening Parameters - A simple combined hardening theory such as that presented in Reference 6 makes two basic assumptions:

1) Size of the yield surface is a function of a cumulative hardening parameter, κ. This means that the isotropic hardening, i.e. the incremental change in size of the yield surface, depends on the initial value of κ and its change $\Delta\kappa$.

2) Yield surface translation is related (but only in an incremental manner) to a kinematic hardening parameter, κ^k. The kinematic hardening, i.e. the incremental translation of the yield surface, depends on the initial value of κ^k and its change $\Delta\kappa^k$.

It will be evident in the discussion to follow that isotropic hardening can be related to κ on either a cumulative or incremental basis, while kinematic hardening can be related to κ^k only on an incremental basis. In addition to the parameters κ and κ^k, hardening is also a function of temperature.

Figure 2.3-1 shows hysteresis loops for the first two strain-controlled cycles of a typical material which exhibits combined isotropic and

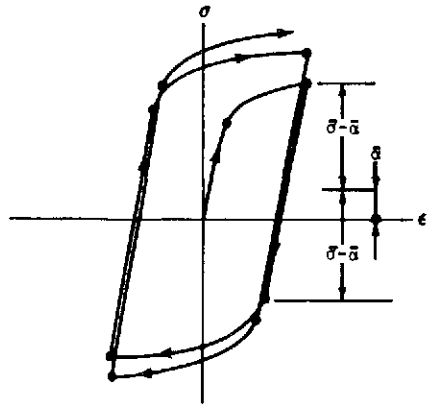

FIGURE 2.3-1 : COMBINED HARDENING BEHAVIOR (NON-STABILIZED)

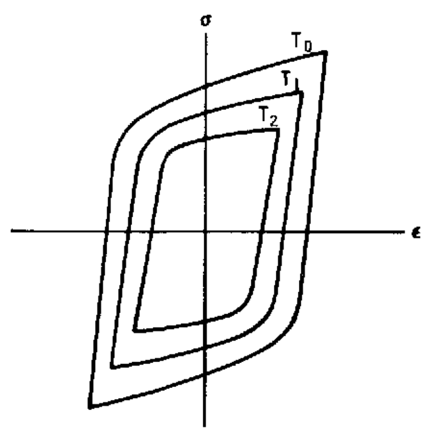

FIGURE 2.3-2 : VARIABLE TEMPERATURE HYSTERESIS LOOPS (STABILIZED)

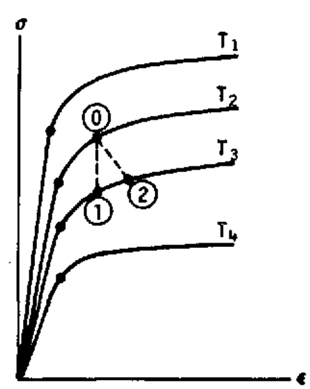

FIGURE 2.3-3 : VARIABLE TEMPERATURE HARDENING EFFECTS

2.3 (Continued)

kinematic hardening. The Bauschinger kinematic hardening effect is apparent in that the initial yielding in tension causes a reduced yield stress in compression, i.e. a shift of the yield center by an amount $\bar{\alpha}$. Successive yielding in compression causes a reduced yield stress in tension, and so forth. Isotropic hardening causes the increase in size of the hysteresis loop with continued cycling. The hysteresis loops for many materials become stabilized after a number of cycles, and then may begin to decrease in size as further deformation causes a softening effect.

Figure 2.3-2 shows the stabilized hysteresis loops for a material at various temperature levels. (Different strain ranges are used to separate the loops for purpose of illustration). The hysteresis loop of a material typically decreases in size with increasing temperature. Note that the size of the yield surface will vary in a similar manner with temperature. Also the rates of isotropic and kinematic hardening with respect to plastic deformation vary with temperature.

The isotropic hardening parameter κ may be appropriately taken as either the cumulative <u>plastic work</u>, or as the sum of increments of effective <u>plastic strain</u>. The kinematic hardening parameter κ^k must account for the Bauschinger effect in cyclic loading, and it may be taken as an <u>adjusted</u> value of κ. As long as no load reversal occurs and the loading is proportional, κ^k is simply equal to κ. However, κ^k must be set to zero at the start of each increment in which a complete load reversal occurs.

2.3 (Continued)

(A complete load reversal occurs when the incremental plastic strain vector has a direction exactly reversed from that of the previous plastic increment). For an incomplete load reversal, the BOPACE program computes the starting value for κ^k by multiplying the existing accumulated value of κ^k by the factor (1 + COSINE)/2, where COSINE is the Cosine of the angle between successive incremental plastic strain vectors. At the end of each increment, κ^k becomes $\kappa^k + \Delta\kappa$.

Because the Bauschinger effect varies with cumulative deformation in certain materials (e.g. it may become more pronounced as plastic cycling continues), BOPACE allows an additional option for the kinematic hardening to be defined as a product of two functions. The first is a function of κ^k and defines the <u>shape</u> of the kinematic hardening, while the second is an additional factor which depends on κ and defines the <u>magnitude</u> of the kinematic hardening.

In order to implement the BOPACE hardening theory, it must be determined how the size of the yield surface varies with temperature. In addition, the dependence of isotropic and kinematic hardening on the parameters κ and κ^k must be determined. This is accomplished through cyclic tests performed at various temperature levels. Because the cyclic hardening behavior is thus determined for a constant temperature, an assumption must be made for variable temperature cycling. The hardening effects of variable temperature are illustrated in Figure 2.3-3. As long as temperature remains constant, plastic hardening behavior is defined by

2.3 (Continued)

following the shape of a stress-strain curve at the given temperature, say to the point 0 on the T_2 curve. If temperature changes to T_3, and then plastic deformation continues, an initial point must be determined on the T_3 curve from which the new yield surface size and initial hardening slope may be determined. This transfer from curve T_2 to curve T_3 requires a definition of the basis for hardening, i.e. the definition of the parameters κ and κ^k. BOPACE allows the option of either plastic work or the sum of increments of effective plastic strain to be used as the hardening basis. The strain and work options correspond to the respective points 1 and 2 in Figure 2.3-3.

The hardening relationship determined from a series of cyclic tests may depend somewhat on the strain range used in a particular test. If strain range is a significant factor the test conditions should duplicate the approximate expected strain range for which an analysis is to be made. The choice between plastic strain and plastic work as a basis for the hardening parameters κ and κ^k may depend to a large extent on which basis provides the better overall representation of cyclic behavior at various strain ranges.

Multiaxial Hardening Rule - The kinematic hardening rule employed in BOPACE is that due to Prager [4]. It gives the increment of yield surface translation in terms of the incremental plastic strains, as

$$\Delta\alpha_i = C_{ij}\ \Delta\varepsilon_j^p = \frac{2}{3}\, c\ I_{ij}\ \Delta\varepsilon_j^p \qquad (2.3\text{-}3)$$

2.3 (Continued)

where c is the kinematic contribution to the slope of the uniaxial stress vs. plastic-strain curve, and I is the identity matrix. An alternate hardening rule due to Ziegler [7] is preferred by some plasticity analysts because the form of Ziegler's rule does not change with reduction in the number of spatial dimensions, and it is therefore supposed to simplify the calculations. Prager's rule is considered more acceptable from a physical point of view, however, and it presents no difficulty when all components of the required tensors are retained as in the BOPACE program. Note that for Prager's kinematic hardening rule, the deviatoric stress center is equal to the stress center, $a_i = \alpha_i$.

The isotropic hardening, i.e. change in size of the yield surface due to plastic deformation, is defined for a proportional test loading by

$$\Delta\hat{s}_i^\circ = R_{ij}^\circ \, \Delta\varepsilon_j^p = \frac{2}{3} \, r \, I_{ij} \, \Delta\varepsilon_j^p \tag{2.3-4}$$

where r is the isotropic contribution to the slope of the uniaxial stress vs. plastic-strain curve.

The necessary condition that stresses remain on the yield surface is satisfied by taking the differential of Equation 2.3-1. The condition is $\Delta F = 0$, which can be shown to give

$$\hat{s}_i \, \Delta\sigma_i - \hat{s}_i \, \Delta\alpha_i - \hat{s}_i^\circ \, \Delta\hat{s}_i^\circ = 0 \tag{2.3-5}$$

or

2.3 (Continued)

$$\hat{s}_i \, \Delta\sigma_i - A\lambda = 0 \tag{2.3-6}$$

where $A = \frac{1}{\lambda} \hat{s}_i \, \Delta\alpha_i + \frac{1}{\lambda} \hat{s}_i^\circ \, \Delta\hat{s}_i^\circ = C_{ij} \, \hat{s}_i \, \hat{s}_j + R_{ij}^\circ \, \hat{s}_i^\circ \, \hat{s}_j^\circ$ (2.3-7)

The key to a successful combined hardening theory is the proper determination of the hardening variable A. The BOPACE program uses hardening tables which give the yield-surface size and the surface translation as functions of the hardening parameters κ and κ^k. These are two-dimensional tables for each material whose ordinates and abscissas are, respectively, temperature and hardening parameter. Given the initial values of κ and κ^k at the beginning of an increment, and estimated values for $\Delta\kappa$ and $\Delta\kappa^k$, the corresponding increments of isotropic and kinematic stress increase are obtained from the hardening tables. (Hardening due to temperature change is included by adding it to the isotropic stress increment). The hardening slopes c and r are then computed, by dividing the incremental stress increases by the estimated increment of effective plastic strain. This procedure gives average values for the slopes c and r during the increment, and tends to produce a quite stable numerical iterative process. Note that it is the isotropic and kinematic stress increases, rather than the slopes c and r, which are directly relatable to the hardening parameters. The choice of a test value for $\hat{s}^\circ$ in Equation 2.3-7 is arbitrary, as long as it is a point on a yield surface of size corresponding to $\hat{s}$, i.e. a surface with equal values of temperature and parameter κ. It is convenient in BOPACE to take $\hat{s}^\circ$ equal to $\hat{s}$.

2.3 (Continued)

Incremental Stress-Strain Relation - The incremental stress-strain relation now follows the development of References 8 and 9. Take

$$\Delta\sigma_i = D^e_{ij}\,\Delta\varepsilon^e_j = D^e_{ij}\,\Delta\varepsilon^{e+p}_j - D^e_{ij}\,\hat{s}_j\lambda \qquad (2.3\text{-}8)$$

where D^e is the appropriate matrix of elastic constants.

Then

$$A\lambda = \hat{s}_i\,\Delta\sigma_i = \hat{s}_i\,D^e_{ij}\,\Delta\varepsilon^{e+p}_j - \hat{s}_i\,D^e_{ij}\,\hat{s}_j\lambda \qquad (2.3\text{-}9)$$

which gives

$$\lambda = \hat{s}_i\,D^e_{ij}\,\Delta\varepsilon^{e+p}_j/(A + \hat{s}_k\,D^e_{k\ell}\,\hat{s}_\ell) \qquad (2.3\text{-}10)$$

Substituting Equation 2.3-10 into Equation 2.3-8 provides the desired relation

$$\Delta\sigma_i = \left(D^e_{ij} - \frac{D^e_{ik}\,\hat{s}_k\,\hat{s}_\ell\,D^e_{\ell j}}{A + \hat{s}_m\,D^e_{mn}\,\hat{s}_n}\right)\Delta\varepsilon^{e+p}_j \qquad (2.3\text{-}11)$$

or

$$\Delta\sigma_i = (D^e_{ij} + D^p_{ij})\,\Delta\varepsilon^{e+p}_j = D_{ij}\,\Delta\varepsilon^{e+p}_j \qquad (2.3\text{-}12)$$

D is the elasto-plastic Jacobian (tangent-stiffness) matrix relating incremental stresses to incremental elastic+plastic strains. In effect, it separates the elastic and plastic strains and determines the incremental stress corresponding to the incremental elastic strain. D^p is the stiffness

2.3 (Continued)

reduction due to plastic flow, and becomes zero for the case of infinite hardening, i.e. $A = \infty$, or equivalently the total slope (c+r) of the stress vs. plastic-strain curve is infinite.

Yield Surface Correction - Because an incremental plasticity solution generally involves a variable stress-strain slope and a changing direction of the surface normal within each increment, the yield surface relationship given by Equation 2.3-1 may not be satisfied exactly. It is probably desirable to exactly satisfy this relationship, i.e. have $F = 0$, at the end of each plastic increment. This can best be accomplished by making a correction to the stress values, and holding constant all other quantities involved in Equation 2.3-1. Proceeding in a manner similar to that of Reference 10, the change in F due to a change in stress is

$$\Delta F = 2 \hat{s}_i \Delta\sigma_i \qquad (2.3\text{-}13)$$

We take the stress correction to be defined by

$$\Delta\sigma_i = \rho\eta_i \qquad (2.3\text{-}14)$$

where ρ is a scalar factor to be determined for the stress correction, while

$$\eta = \begin{Bmatrix} \hat{s}_{xx} \\ \hat{s}_{yy} \\ \hat{s}_{zz} \\ \hat{s}_{xy} \end{Bmatrix} \qquad (2.3\text{-}15a)$$

2.3 (Continued)

for plane strain, and

$$\eta = \begin{Bmatrix} \hat{s}_{xx} \\ \hat{s}_{yy} \\ 0 \\ \hat{s}_{xy} \end{Bmatrix} \tag{2.3-15b}$$

for plane stress. Thus the direction of the stress correction is taken normal to the yield surface, except that in plane stress the value of σ_{zz} is not corrected. Then

$$\Delta F = 2\, \eta_i \eta_i\, \rho \tag{2.3-16}$$

giving

$$\rho = \frac{\Delta F}{2\, \eta_i \eta_i} \tag{2.3-17}$$

and

$$\Delta\sigma_i = \rho\eta_i = \frac{\Delta F}{2\, \eta_i \eta_i}\, \eta_i \tag{2.3-18}$$

The stress correction is computed from Equation 2.3-18 using for ΔF the negative of the error evaluated for function F in Equation 2.3-1. Because the yield surface relation is to be satisfied at the end of the increment, all quantities in Equation 2.3-18 are also evaluated at the end of the increment.

2.3 (Continued)

Effective Stress-Strain and Plastic Work - The concepts of "effective stress" and "effective strain" are related to plastic work, and are used extensively in the literature on plasticity. These concepts are used in a limited way in the development of constitutive theory for the BOPACE program.

Because they can easily be misapplied, especially in the presence of kinematic hardening, the use and limitations of the concepts are briefly discussed here for the Mises plasticity theory.

Due to characteristics of the Prager hardening theory, the following statements of equivalence and proportionality should first be noted.

$$\Delta a_i = \Delta \alpha_i \approx \Delta \varepsilon_i^p \approx \hat{s}_i \not\approx s_i \tag{2.3-19}$$

Because of the incremental nature of kinematic hardening, s_i and $\hat{s}_i$ are in general not proportional.

The Mises effective stress $\bar{\sigma}$ is defined by

$$\bar{\sigma}^2 = \frac{3}{2} s_i \; s_i \tag{2.3-20}$$

The incremental and cumulative values for plastic work, W^p, are given by

$$\Delta W^p = \sigma_i \; \Delta \varepsilon_i^p \tag{2.3-21a}$$

2.3 (Continued)

and

$$W^p = \Sigma \sigma_i \Delta\varepsilon_i^p \tag{2.3-21b}$$

where Σ denotes summation over all increments. For the special case of proportional loading (i.e. loading in which all stress components are increased proportionately) followed by a constant stress level (i.e. no plastic hardening), the cumulative plastic work is given by

$$W^p = \sigma_i \varepsilon_i^p \tag{2.3-21c}$$

As a matter of convenience in computing plastic work, an increment of effective plastic strain, $\Delta\bar{\varepsilon}^p$, has historically been defined by

$$(\Delta\bar{\varepsilon}^p)^2 = \frac{2}{3} \Delta\varepsilon_i^p \Delta\varepsilon_i^p \tag{2.3-22}$$

At this point, however, care must be exercised in using the historical calculation for plastic work. If kinematic hardening were zero, then $\hat{s}_i = s_i$, and because $\Delta\varepsilon_i^p$ is proportional to $\hat{s}_i$ the use of Equations 2.3-20 and 2.3-22 would give plastic work as

$$\Delta W^p = \bar{\sigma} \Delta\bar{\varepsilon}^p \tag{2.3-23a}$$

and

$$W^p = \Sigma \bar{\sigma} \Delta\bar{\varepsilon}^p \tag{2.3-23b}$$

2.3 (Continued)

If in addition, the condition were one of proportional loading and constant stress, then by defining the cumulative effective plastic strain, $\bar{\varepsilon}^p$, in the same manner as $\Delta\bar{\varepsilon}^p$, we would have

$$W^p = \bar{\sigma}\,\bar{\varepsilon}^p \qquad (2.3\text{-}23c)$$

Of course the Equations 2.3-23 in general <u>are not valid</u>, because of the presence of kinematic hardening and non-proportional loading. Thus plastic work must be computed from Equation 2.3-21a and b, rather than from the product of effective stress and strain quantities.

The quantity $\bar{\varepsilon}^p$ serves little purpose in a general plasticity analysis, although it is a tensorially invariant quantity and does provide a measure of net residual deformation. For a rational measure of deformation history, either the plastic work, W^p, or the sum of increments of effective plastic strain, $\Sigma\,\Delta\bar{\varepsilon}^p$, is appropriate. The difference in concept between the quantities W^p and $\Sigma\,\Delta\bar{\varepsilon}^p$ should, however, be recognized.

2.4 CREEP

<u>Stages</u> - Metals characteristically exhibit the three stages of primary, secondary and tertiary creep. Figure 2.4-1 shows these stages in a typical creep history under conditions of constant temperature and stress. Because creep rate varies considerably during the different stages, the description of actual creep histories is considered to be essential for an accurate analysis. The BOPACE program accounts for the creep time

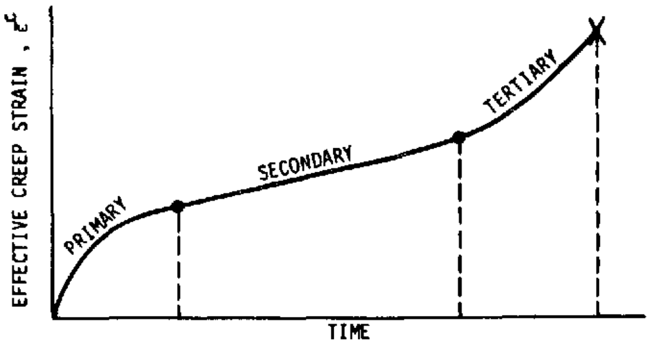

FIGURE 2.4-1 : TYPICAL CREEP STAGES

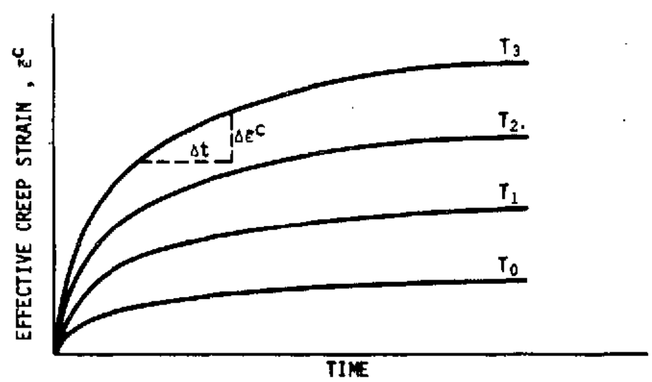

FIGURE 2.4-2 : BOPACE CREEP REPRESENTATION (EXAMPLE FOR VARIABLE TEMPERATURE AND CONSTANT STRESS)

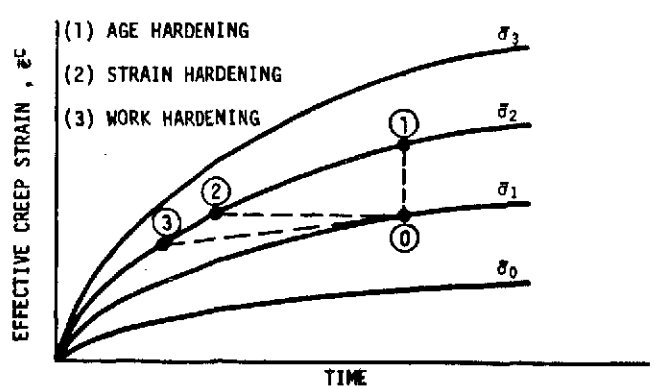

FIGURE 2.4-3 : BOPACE CREEP HARDENING OPTIONS (EXAMPLE FOR CONSTANT TEMPERATURE AND VARIABLE STRESS)

2.4 (Continued)

history by allowing the user to define, by a series of input points, the shape of the effective-creep-strain vs. time curve for each material.

Temperature and Stress Effects - Creep rate in most metals is very dependent upon temperature and stress level. The BOPACE approach to creep analysis provides a reasonable description of temperature and stress effects, while avoiding excessive storage and computational requirements. For each material, BOPACE requires a reference creep curve shape which gives the relative variation of effective-creep-strain vs. time for the various stages considered. This shape is assumed to be valid for all the temperatures and stress levels of the particular material. A table of creep factors for the material is then specified as a function of temperature and effective stress, and a portion of the actual creep curve is determined by multiplying the reference creep curve by the appropriate factor using the average temperature and stress during the increment. Figure 2.4-2 shows portions of typical creep curves for the special case of constant stress level and variable temperature. Note that according to BOPACE assumptions these curves have the same shape.

Hardening - As long as the temperature and stress level remain constant, an increment of creep is determined by following the corresponding creep curve for the given time increment. However, if temperature or stress level changes, an initial point must be identified on the corresponding new creep curve in order to determine the new creep rate. This transfer from one curve to another requires an assumption for creep hardening,

2.4 (Continued)

which in BOPACE is defined by a single hardening parameter, κ^c. BOPACE allows the option of either age, strain, or work hardening, for which κ^c is defined respectively as the accumulated time, sum of increments of effective creep strain, or creep work. These options are illustrated in Figure 2.4-3 for a case of constant temperature. Creep during the preceding increments has progressed to the point 0 on the $\bar{\sigma}_1$ curve. The average effective stress during the present increment is $\bar{\sigma}_2$, which gives the initial points 1, 2 and 3, respectively, for the options of age, strain and work hardening. Incremental creep for the current increment is then determined by continuing along the $\bar{\sigma}_2$ curve from the appropriate initial point, for a distance equal to the specified creep time increment. In the general case both temperature and stress will vary from one increment to the next, but the hardening option still determines in the same manner how the transfer is made between the creep curves.

Load Reversal - The main use of the creep-hardening parameter κ^c comes into play during a load reversal. When a complete reversal occurs, κ^c is set to zero and the initial point on the creep curve is taken as that corresponding to a zero value of κ^c. (A complete load reversal occurs if the incremental creep-strain vector has a direction exactly reversed from that of the preceding creep increment.) For an incomplete load reversal, the BOPACE program computes the starting value for κ^c by multiplying the existing value of κ^c by the factor (1 + COSINE)/2, where COSINE is the Cosine of the angle between successive incremental creep

2.4 (Continued)

strain vectors. Parameter κ^c then accumulates as before, i.e. at the end of each increment κ^c becomes $\kappa^c + \Delta\kappa^c$.

Multiaxial Flow Rule - The incremental creep-strain vector has historically been taken normal to a Mises type of surface which passes through the stress point. When kinematic plastic hardening is considered, this surface could be taken either as the actual translated yield surface, or as an untranslated surface which passes through the stress point but whose center remains at the origin. The appropriate choice of surface is not clear, and because the untranslated form of surface is somewhat simpler, further complications are perhaps unjustified. BOPACE therefore defines the multiaxial creep flow rule by

$$\Delta\varepsilon_i^c = \left(\frac{3}{2}\,\Delta\bar{\varepsilon}^c/\bar{\sigma}\right) s_i \tag{2.4-1}$$

where $\Delta\bar{\varepsilon}^c$ is the increment of effective creep strain defined by

$$(\Delta\bar{\varepsilon}^c)^2 = \frac{2}{3}\,\Delta\varepsilon_i^c\,\Delta\varepsilon_i^c \tag{2.4-2}$$

The increment of creep work is then given by either

$$\Delta W^c = \sigma_i\,\Delta\varepsilon_i^c \tag{2.4-3a}$$

or

$$\Delta W^c = \bar{\sigma}\;\Delta\bar{\varepsilon}^c \tag{2.4-3b}$$

2.5 COMPLETE STRESS-STRAIN RELATIONS

In Sections 2.1 to 2.4, the basic theory used in BOPACE for elasticity, thermal strains, plasticity and creep has been discussed. The present section describes the complete stress-strain relations, and the manner in which simultaneous elastic, plastic, thermal and creep strains are accounted for. The combined effects of temperature-dependent elasticity and plasticity are included, and the specializations are described for plane-stress and plane-strain problems.

General 3-D Relations - For temperature-dependent behavior, the required elasto-plastic incremental stress-strain relation follows from Equations 2.1-7 and 2.3-8:

$$\Delta\sigma_i = \Delta D^e_{ij}\,\varepsilon^{e0}_j + D^{el}_{ij}\,\Delta\varepsilon^{e+p}_j - D^{el}_{ij}\,s_j\lambda \qquad (2.5\text{-}1)$$

Here the first term accounts for stress change due to change in elastic properties, while the second and third terms account for stress change due to change in elastic strain. Following Equation 2.3-9,

$$A\lambda = \hat{s}_i\,\Delta\sigma_i = \hat{s}_i\,\Delta D^e_{ij}\,\varepsilon^{e0}_j + \hat{s}_i\,D^{el}_{ij}\,\Delta\varepsilon^{e+p}_j - \hat{s}_i\,D^{el}_{ij}\,s_j\,\lambda \qquad (2.5\text{-}2)$$

where again

$$A = C_{ij}\,\hat{s}_i\,\hat{s}_j + R^o_{ij}\,\hat{s}^o_i\,\hat{s}^o_j \qquad (2.5\text{-}3)$$

For the general case of temperature-dependent plasticity, R^o accounts for isotropic hardening due to both plastic deformation and temperature. Then

$$\lambda = \frac{\hat{s}_i\,\Delta D^e_{ij}\,\varepsilon^{e0}_j + \hat{s}_i\,D^{el}_{ij}\,\Delta\varepsilon^{e+p}_j}{A + \hat{s}_k\,D^{el}_{k\ell}\,\hat{s}_\ell} \qquad (2.5\text{-}4)$$

2.5 (Continued)

Substituting Equation 2.5-4 into 2.5-1 gives

$$\Delta\sigma_i = \left(\Delta D^e_{ij} - \frac{D^{el}_{ik}\,\hat{s}_k\,\hat{s}_\ell\,\Delta D^e_{\ell j}}{A + \hat{s}_m\,D^{el}_{mn}\,\hat{s}_n} \right) \varepsilon^{e0}_j + \left(D^{el}_{ij} - \frac{D^{el}_{ik}\,\hat{s}_k\,\hat{s}_\ell\,D^{el}_{\ell j}}{A + \hat{s}_m\,D^{el}_{mn}\,\hat{s}_n} \right) \Delta\varepsilon^{e+p}_j \tag{2.5-5}$$

or, using abbreviated notation

$$\Delta\sigma_i = (\Delta D^e_{ij} + \Delta D^p_{ij})\,\varepsilon^{e0}_j + (D^{el}_{ij} + D^{pl}_{ij})\Delta\varepsilon^{e+p}_j = \tag{2.5-6}$$

$$\Delta D_{ij}\,\varepsilon^{e0}_j + D^l_{ij}\,\Delta\varepsilon^{e+p}_j$$

Thus the increment of stress is given by the sum of two products: an incremental matrix times the initial elastic strains, plus an end-of-increment matrix times the incremental elastic+plastic strains.

Calculation of Strains - The total strain increment, $\Delta\varepsilon$, is computed from the incremental displacements. The strain $\Delta\varepsilon$ is the physical (measurable) strain in the structure, and is the sum of the elastic, plastic, thermal and creep strains. The elastic+plastic strain, $\Delta\varepsilon^{e+p}$, is needed for the calculation of the stress increment in Equation 2.5-6. It is computed by

$$\Delta\varepsilon^{e+p}_i = \Delta\varepsilon_i - \Delta\varepsilon^t_i - \Delta\varepsilon^c_i \tag{2.5-7}$$

where ε^t and ε^c are the thermal and creep strains, respectively. Because $\Delta\varepsilon^c$ depends on average stress level during the increment, and because the normal $\hat{s}$ to the yield surface as well as the plastic slope parameters c and r vary somewhat during the increment, BOPACE uses an iterative approach for solution. The solution approach is discussed in Section 4.

2.5 (Continued)

Plane Stress and Plane Strain - The development of Equations 2.5-1 through 2.5-6 is based on 3-dimensional relations in which all nine components of the various tensors are properly accounted for. For plane-strain problems, BOPACE uses the full 3-D relations and thus accounts automatically for prescribed non-zero values of the strains ε_{zz}.

For plane-stress problems, BOPACE uses the abbreviated elasticity matrix of Equation 2.1-4 in Equation 2.5-6 (except that both the xy and yx tensor components are accounted for). The hardening variable A must be computed from Equation 2.5-3 using all components of $\hat{s}$. In the case of a specified non-zero value of the normal stress σ_{zz}, the quantity $\Delta\mu_i^e \, \sigma_{zz}^0 + \mu_i^{el} \, \Delta\sigma_{zz}$ is added to the right-hand-side of Equation 2.5-1. Following through the incremental stress-strain derivation as before gives

$$\lambda = \frac{\hat{s}_i \, \Delta D_{ij}^e \, \varepsilon_j^{e0} + \hat{s}_i \, D_{ij}^{el} \, \Delta\varepsilon_j^{e+p} + \hat{s}_i \, \Delta\mu_i^e \, \sigma_{zz}^0 + \hat{s}_i \, \mu_i^{el} \, \Delta\sigma_{zz} + \hat{s}_{zz} \, \Delta\sigma_{zz}}{A + \hat{s}_k \, D_{k\ell}^{el} \, \hat{s}_\ell} \qquad (2.5\text{-}8)$$

The plane-stress equivalent of Equation 2.5-6 is then obtained in a similar manner as before.

3.0 FINITE ELEMENT REPRESENTATION

BOPACE uses the Constant-Strain-Triangle (CST) element in its thermal, elastic, plastic and creep analysis. The following sections describe the element stiffness matrices, the coordinate systems employed, and the various coordinate-related transformations.

3.1 CST ELEMENT STIFFNESS MATRICES

Element Coordinates and Displacement Functions - The CST element is shown in Figure 3.1-1. The element is defined using the x-y Cartesian coordinate system, which is arbitrarily oriented with respect to the element. (A coordinate system with origin at node 1, and x-axis along the nodal line 1-2, is used for output of element stresses and strains.) Displacements u and v correspond to the x and y coordinate directions, respectively.

Following the general development of Reference 11, the displacement functions are defined by

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad (3.1\text{-}1)$$

giving linear variation of displacements and constant strains within the element.

3.1 (Continued)

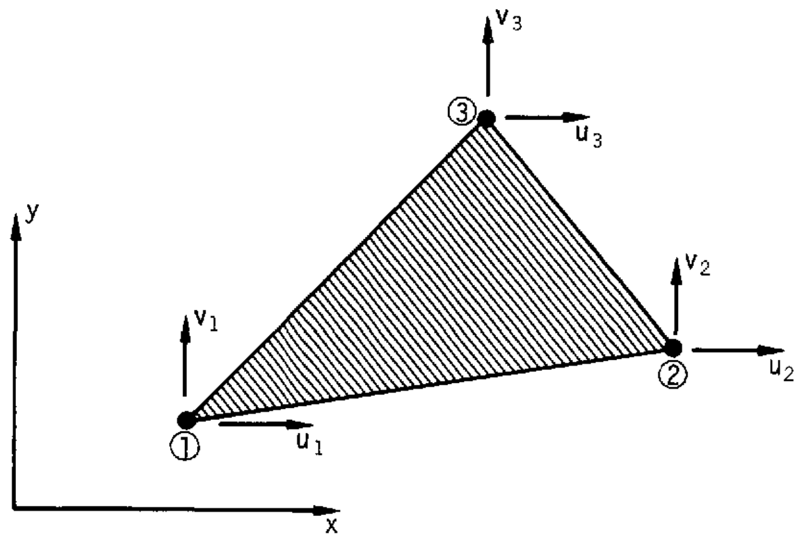

FIGURE 3.1-1: CST ELEMENT AND COORDINATES

Strain - Displacement Matrix - Evaluating the unknown coefficients b_1 through b_6 in terms of nodal displacements, substituting these values into Equation 3.1-1, and differentiating to obtain expressions for the strains, gives

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ (\partial u/\partial y+\partial v/\partial x)/2 \\ (\partial u/\partial y+\partial v/\partial x)/2 \end{Bmatrix} = [\Gamma] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad (3.1\text{-}2)$$

where the strain-displacement matrix, Γ, is given by

$$[\Gamma] = \frac{1}{2A} \begin{bmatrix} -y_{32} & 0 & y_{31} & 0 & -y_{21} & 0 \\ 0 & x_{32} & 0 & -x_{31} & 0 & x_{21} \\ x_{32}/2 & -y_{32}/2 & -x_{31}/2 & y_{31}/2 & x_{21}/2 & -y_{21}/2 \\ x_{32}/2 & -y_{32}/2 & -x_{31}/2 & y_{31}/2 & x_{21}/2 & -y_{21}/2 \end{bmatrix}$$

3.1 (Continued)

Here A is the area of the triangle, x_{ij} and y_{ij} are abbreviated notation for the differences x_i-x_j and y_i-y_j, respectively, and u_i and v_i are displacements at the ith node in the x and y directions, respectively.

Plane-Stress Stiffness Matrix - For plane-stress conditions it can be shown that the CST stiffness matrix, k, is given by

$$k_{ij} = A \, D_{mn} \, \Gamma_{mi} \, \Gamma_{mj} \tag{3.1-3}$$

The elasticity matrix D is

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 1-\nu \end{bmatrix}$$

where E is Young's modulus and ν is Poisson's ratio.

The stiffness matrix defines element generalized forces, p_i, in terms of displacements, q_i, by

$$p_i = k_{ij} \, q_j \tag{3.1-4}$$

where q is the right-hand-side nodal-displacement vector in Equation 3.1-2, and p is the vector of corresponding nodal forces.

The stiffness matrix can be written as the sum of the stiffnesses due to normal and shear components of the elastic properties

$$k = k_n + k_s \tag{3.1-5}$$

3.1 (Continued)

where

$$k_n = \frac{Et}{4A(1-\nu^2)} \begin{bmatrix} y_{32}^2 & & & & & \text{Symmetric} \\ -\nu y_{32}x_{32} & x_{32}^2 & & & & \\ -y_{32}y_{31} & \nu x_{32}y_{31} & y_{31}^2 & & & \\ \nu y_{32}x_{31} & -x_{32}x_{31} & -\nu y_{31}x_{31} & x_{31}^2 & & \\ y_{32}y_{21} & -\nu x_{32}y_{21} & -y_{31}y_{21} & \nu x_{31}y_{21} & y_{21}^2 & \\ -\nu y_{32}x_{21} & x_{32}x_{21} & \nu y_{31}x_{21} & -x_{31}x_{21} & -\nu y_{21}x_{21} & x_{21}^2 \end{bmatrix}$$

and

$$k_s = \frac{Et}{8A(1+\nu)} \begin{bmatrix} x_{32}^2 & & & & & \text{Symmetric} \\ -x_{32}y_{32} & y_{32}^2 & & & & \\ -x_{32}x_{31} & y_{32}x_{31} & x_{31}^2 & & & \\ x_{32}y_{31} & -y_{32}y_{31} & -x_{31}y_{31} & y_{31}^2 & & \\ x_{32}x_{21} & -y_{32}x_{21} & -x_{31}x_{21} & y_{31}x_{21} & x_{21}^2 & \\ -x_{32}y_{21} & y_{32}y_{21} & x_{31}y_{21} & -y_{31}y_{21} & -x_{21}y_{21} & y_{21}^2 \end{bmatrix}$$

Plane-Strain Stiffness Matrix - The plane-stress and plane-strain matrices of elastic coefficients are defined by Equations 2.1-4 and 2.1-5, respectively. If in the plane-stress elasticity matrix we substitute $\bar{E}$ for E, and $\bar{\nu}$ for ν, where

$$\bar{E} = E/(1-\nu^2) \qquad (3.1\text{-}6a)$$

$$\bar{\nu} = \nu/(1-\nu) \qquad (3.1\text{-}6b)$$

we obtain the elasticity matrix for plane strain. This means that the element stiffness matrix for plane-strain conditions will be defined by Equation 3.1-5, providing that the substitutions $\bar{E}$ for E, and $\bar{\nu}$ for ν are made in the matrices k_n and k_s. BOPACE therefore uses the same subroutine to generate stiffness matrices for both plane-stress and plane-

3.1 (Continued)

strain problems, by using the appropriate values for Young's modulus and Poisson's ratio in the plane-stress matrix generation.

Elasto-Plastic Stiffness Matrices - The complete incremental force-displacement relation for an element corresponds to the stress-strain relation 2.5-6. Because strains are constant within an element and are directly relatable to nodal displacements, the relations are identical in form:

$$\Delta p_i = (\Delta k_{ij}^{e} + \Delta k_{ij}^{p})\, q_j^{e0} + (k_{ij}^{el} + k_{ij}^{pl})\, \Delta q_i^{e+p} \tag{3.1-7}$$

where each of the force-displacement matrices k is obtained directly from the corresponding stress-strain matrix D, by using Equation 3.1-3. For example,

$$k_{ij}^{pl} = A\, D_{mn}^{pl}\, \Gamma_{mi}\, \Gamma_{mj} \tag{3.1-8}$$

3.2 COORDINATE SYSTEMS

Coordinate systems are used in BOPACE to define the locations of nodes and to define the directions of nodal displacements.

Location Coordinate Systems - BOPACE allows the user to define nodal locations by either of two global coordinate systems. These are the X-Y Cartesian system and the R-θ cylindrical system, shown in Figure 3.2-1. The X-Y and R-θ systems have a common origin, O, and the angle θ is measured counter-clockwise from the X axis. The global X-Y system

3.2 (Continued)

also serves as the local element x-y system which was discussed in Section 3.1.

Direction Coordinate Systems - BOPACE allows the user to define directions of the nodal displacements by three types of coordinate systems. These are local Cartesian systems, with origins at each particular node. The first two types are defined by the X-Y and R-θ systems, respectively, and the third type consists of special user-defined systems. The three direction-coordinate types are shown in Figure 3.2-2 for a particular node, and are denoted by the respective subscripts 1, 2 and 3. The first system measures displacements parallel to the global X-Y axes. The second system measures displacements parallel and perpendicular to the radius passing through the particular node. The user-defined systems are at given orientations, defined by the counter-clockwise angle from the global X axis to the X_3 axis. The special user-defined systems are useful in cases where nodal displacements are to be prescribed in particular directions.

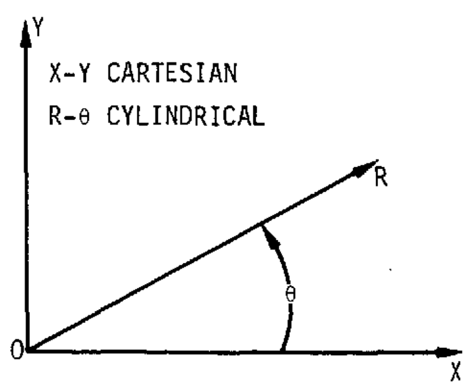

FIGURE 3.2-1: LOCATION COORDINATE SYSTEMS

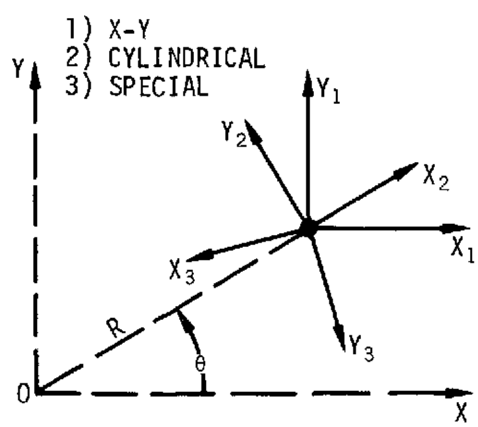

FIGURE 3.2-2: DIRECTION COORDINATE SYSTEMS

3.3 COORDINATE-RELATED TRANSFORMATIONS

This section gives the necessary transformations which convert BOPACE quantities defined in one coordinate system to the equivalent quantities in another coordinate system. The quantities considered are coordinates, displacements, forces, stiffnesses, strains and stresses.

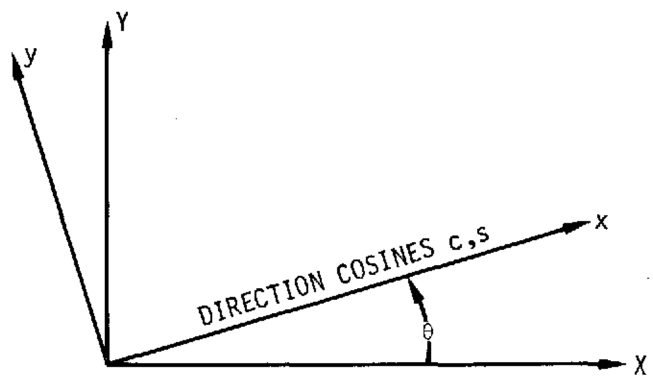

FIGURE 3.3-1: COORDINATE TRANSFORMATION SYSTEMS

<u>Basic Transformation Matrix</u> - Referring to Figure 3.3-1, the transformation relations will be developed between the x-y and X-Y coordinate systems. The angle θ is the angle, measured counter-clockwise from the X axis to the x axis. Using the shorthand notation c and s to denote respectively the Cosine and Sine of the angle θ (i.e., the direction Cosines of x given in the X-Y system), we define the 2-dimensional transformation matrix, Γ, by:

$$\Gamma = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \qquad (3.3\text{-}1a)$$

3.3 (Continued)

Because of the orthogonality properties of this matrix, the inverse of Γ is equal to its transpose:

$$\Gamma^{-1} = \Gamma^{T} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \tag{3.3-1b}$$

In the following transformations, upper and lower case letters will be used to denote quantities which are referred to the X-Y and x-y coordinate systems, respectively.

<u>Coordinates</u> - The coordinate transformations are given by

$$x_i = \Gamma_{ij} X_j \tag{3.3-2a}$$

and

$$X_i = \Gamma_{ij}^{-1} x_j = \Gamma_{ij}^{T} x_j \tag{3.3-2b}$$

where

$$x = \begin{Bmatrix} x \\ y \end{Bmatrix} \text{ and } X = \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

<u>Displacements</u> - The displacement relations are

$$q_i = \Gamma_{ij} Q_j \tag{3.3-3a}$$

and

$$Q_i = \Gamma_{ij}^{-1} q_i = \Gamma_{ij}^{T} q_i \tag{3.3-3b}$$

where

$$q = \begin{Bmatrix} u \\ v \end{Bmatrix} \text{ and } Q = \begin{Bmatrix} U \\ V \end{Bmatrix}$$

3.3 (Continued)

Forces - The generalized nodal force relations are

$$P_i = \Gamma^T_{ij} \, p_j \qquad (3.3\text{-}4a)$$

and

$$p_i = (\Gamma^T)^{-1} \, P_j = \Gamma_{ij} \, P_j \qquad (3.3\text{-}4b)$$

Stiffnesses - The transformation relations for a nodal (2x2) stiffness matrix partition are

$$K_{ij} = k_{mn} \, \Gamma_{mi} \, \Gamma_{mj} \qquad (3.3\text{-}5a)$$

and

$$k_{ij} = K_{mn} \, \Gamma^{-1}_{mi} \, \Gamma^{-1}_{mj} = K_{mn} \, \Gamma^T_{mi} \, \Gamma^T_{mj} \qquad (3.3\text{-}5b)$$

Strains - Transformation of the strain tensor is described by

$$E_{ij} = \varepsilon_{mn} \, \Gamma_{mi} \, \Gamma_{mj} \qquad (3.3\text{-}6a)$$

and

$$\varepsilon_{ij} = E_{mn} \, \Gamma^{-1}_{mi} \, \Gamma^{-1}_{nj} = E_{mn} \, \Gamma^T_{mi} \, \Gamma^T_{nj} \qquad (3.3\text{-}6b)$$

where here ε and E denote the strain tensor expressed in the x-y-z and X-Y-Z coordinate systems, respectively, and Γ is the transformation matrix (of x-y-z direction cosines) which gives x-y-z coordinates in terms of X-Y-Z coordinates.

The 2-dimensional (x-y plane) strain transformations are written as

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{xy} \end{Bmatrix} = \begin{bmatrix} cc & ss & -2cs \\ ss & cc & 2cs \\ cs & -cs & cc-ss \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \qquad (3.3\text{-}6c)$$

3.3 (Continued)

and

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} cc & ss & 2cs \\ ss & cc & -2cs \\ -cs & cs & cc-ss \end{bmatrix} \begin{Bmatrix} E_{XX} \\ E_{YY} \\ E_{XY} \end{Bmatrix} \tag{3.3-6d}$$

where, for example, cs denotes the product of the Cosine and Sine of the angle θ.

Stresses - The stresses transform between the Cartesian coordinate systems x-y-z and X-Y-Z in exactly the same manner as do the strains.

4.0 SOLUTION METHOD

4.1 BASIC SOLUTION REQUIREMENTS

The exact elasto-plastic-creep analysis of a structure requires the satisfaction, at all points in the structure, of three requirements:

1. satisfaction of constitutive theory, which is summarized by the appropriate incremental stress-strain relation

2. compatibility of strains

3. equilibrium of stresses

The following paragraphs summarize the BOPACE solution approach as it relates to satisfying these three requirements.

Stress-Strain Relation - The incremental stress-strain relation is defined by Equation 2.5-6, and it is satisfied exactly in the BOPACE solution procedure, provided the increment is sufficiently small so that incremental quantities can be treated in a differential manner. Because of the state of constant strain and stress which exists within a CST finite element, the stress-strain relation for each element holds over the entire area of that element.

Compatibility - Compatibility is satisfied exactly within each element as a result of the finite-element derivation. In the global sense, i.e., over the entire structure, compatibility is also satisfied exactly, by merging the element degrees of freedom into global degrees of freedom

4.1 (Continued)

and thereby establishing the equality of displacements at appropriate adjacent nodes.

Equilibrium - Equilibrium is satisfied exactly within each CST element because of its constant stress state (as long as there are no distributed loads). Stresses are not necessarily in equilibrium between adjacent elements, although stress equilibrium is satisfied there in the limit as the finite-element mesh is refined. For any mesh representation of the structure, global equilibrium is satisfied in BOPACE in an average sense, because equilibrium is established between the generalized nodal forces defined according to the usual finite-element procedure.

4.2 COMPARISON OF COMMON SOLUTION METHODS

The common stiffness methods used for solution of elasto-plastic problems can be classified by three general types:

1. The pure "tangent stiffness" method
2. The "constant-stiffness residual-load" method
3. "Combined" methods

Tangent-Stiffness Method - The pure tangent-stiffness method obtains the solution for each load increment by a single solution of the incremental equilibrium equation:

$$\Delta P_i = K_{ij}^{J} \Delta Q_j \tag{4.2-1}$$

4.2 (Continued)

in which ΔP and ΔQ are the global incremental forces and displacements, respectively, and K^J is the Jacobian (tangent-stiffness) matrix. This is the type of solution used in NASTRAN's "piecewise linear analysis", for example. There is no equilibrium check, and no iteration is performed to improve the incremental solution. The matrix K^J is determined by evaluation or extrapolation at previous solution points. Because in an actual structure the stress-strain slopes, creep rates, direction of the incremental plastic and creep strain vectors, etc., will generally vary within an increment, the pure tangent-stiffness approach can result in a substantial departure from the true force-displacement path unless load increments are kept quite small.

Constant-Stiffness Residual-Load Method - This solution method [9] employs an iterative procedure. In each iteration the residual (unbalanced) forces are computed based on the current estimate for the incremental configuration, and are then applied to the constant elastic stiffness matrix in order to solve for displacement corrections. The approach is computationally efficient because it requires the formation and decomposition of only a single stiffness matrix, but it is not directly applicable to highly nonlinear structures because of convergence difficulties.

Combined Methods - Various combined methods have been employed for solution of elasto-plastic problems, for example that described in Reference 12. These involve the use of an equilibrium check through

4.2 (Continued)

the calculation of unbalanced forces, as well as various procedures for updating the approximate Jacobian matrix.

<u>BOPACE Approach</u> - BOPACE uses a combined approach for solution, with the iterative procedure consisting of two basic parts:

1. Calculation of residual nodal forces based on the estimated configuration and "exact" constitutive theory.

2. Improvement of the solution configuration by reducing the residual forces.

Several user controlled options are available in BOPACE for updating the Jacobian matrix.

4.3 CALCULATION OF UNBALANCED FORCES

It is assumed for the present discussion that the exact solution configuration is known at the start of a particular load increment. (Actually the BOPACE program takes any unbalanced forces which might remain from the previous increment and adds them to the present load increment, in order to achieve greater accuracy.) For a given iteration within the present increment, i.e., for a given estimate of the incremental solution, it is necessary to compute the corresponding unbalanced forces. This section describes the steps involved in computing these forces, including determination of element strains, element stresses, and forces. A flowchart for these calculations is given in Figure 4.3-1.

4.3 (Continued)

Element Strains - For the given estimate of incremental global displacements, ΔQ, the corresponding element displacements, Δq, are obtained by coordinate transformations of the type 3.3-3a. Element strains, $\Delta\varepsilon$, are then computed using the strain-displacement matrix of Equation 3.1-2:

$$\Delta\varepsilon_i = \Gamma_{ij}\,\Delta q_i \tag{4.3-1}$$

The strain $\Delta\varepsilon$ is the total (physical) strain increment in the element:

$$\Delta\varepsilon_i = \Delta\varepsilon_i^e + \Delta\varepsilon_i^p + \Delta\varepsilon_i^t + \Delta\varepsilon_i^c \tag{4.3-2}$$

The thermal strains, $\Delta\varepsilon^t$, and creep strains, $\Delta\varepsilon^c$, are determined as described in Sections 2.2 and 2.4, respectively. Subtracting these strains from the total strain, gives:

$$\Delta\varepsilon_i^{e+p} = \Delta\varepsilon_i^e + \Delta\varepsilon_i^p = \Delta\varepsilon_i - \Delta\varepsilon_i^t - \Delta\varepsilon_i^c \tag{4.3-3}$$

Element Stresses - Incremental stresses are then computed using the elasto-plastic relation 2.5-6 (see also Reference 13):

$$\Delta\sigma_i = \Delta D_{ij}\varepsilon_j^{e0} + D_{ij}^1\,\Delta\varepsilon_j^{e+p} \tag{4.3-4}$$

where ε^{e0} are the known cumulative elastic strains at the beginning of the increment. This relation holds only if the element is plastic throughout the load increment. Therefore, the calculation of $\Delta\sigma$ may need to be modified, depending on which of three conditions exists within the particular element:

4.3 (Continued)

Condition I — element is elastic at end of load increment, i.e., either the element remains elastic or unloading occurs. Compute stress and elastic strains. Plastic strains are zero.

Condition II — element is plastic throughout load increment. Compute stresses, elastic and plastic strains from Equation 4.2-4.

Condition III — element is initially elastic, but becomes plastic at some point during the load increment. Find intermediate point at which yielding occurs (this requires solving a simple quadratic equation). Compute stresses and elastic strains up to that point. Compute stresses and strains beyond yielding as for Condition II.

The condition at the beginning of the increment is known for each element. The condition at the end of the increment is assumed, for the first iteration, to be Condition I. The end condition is re-evaluated during each iteration, using either the yield point or the plastic-strain vector. For an elastic element, it is determined whether or not the current yield point has been exceeded. For a plastic element, the plastic strain vector (normal to the yield surface) is observed; an outward vector ($\lambda>0$) implies a plastic condition, while an inward vector ($\lambda\leq0$) implies elastic unloading.

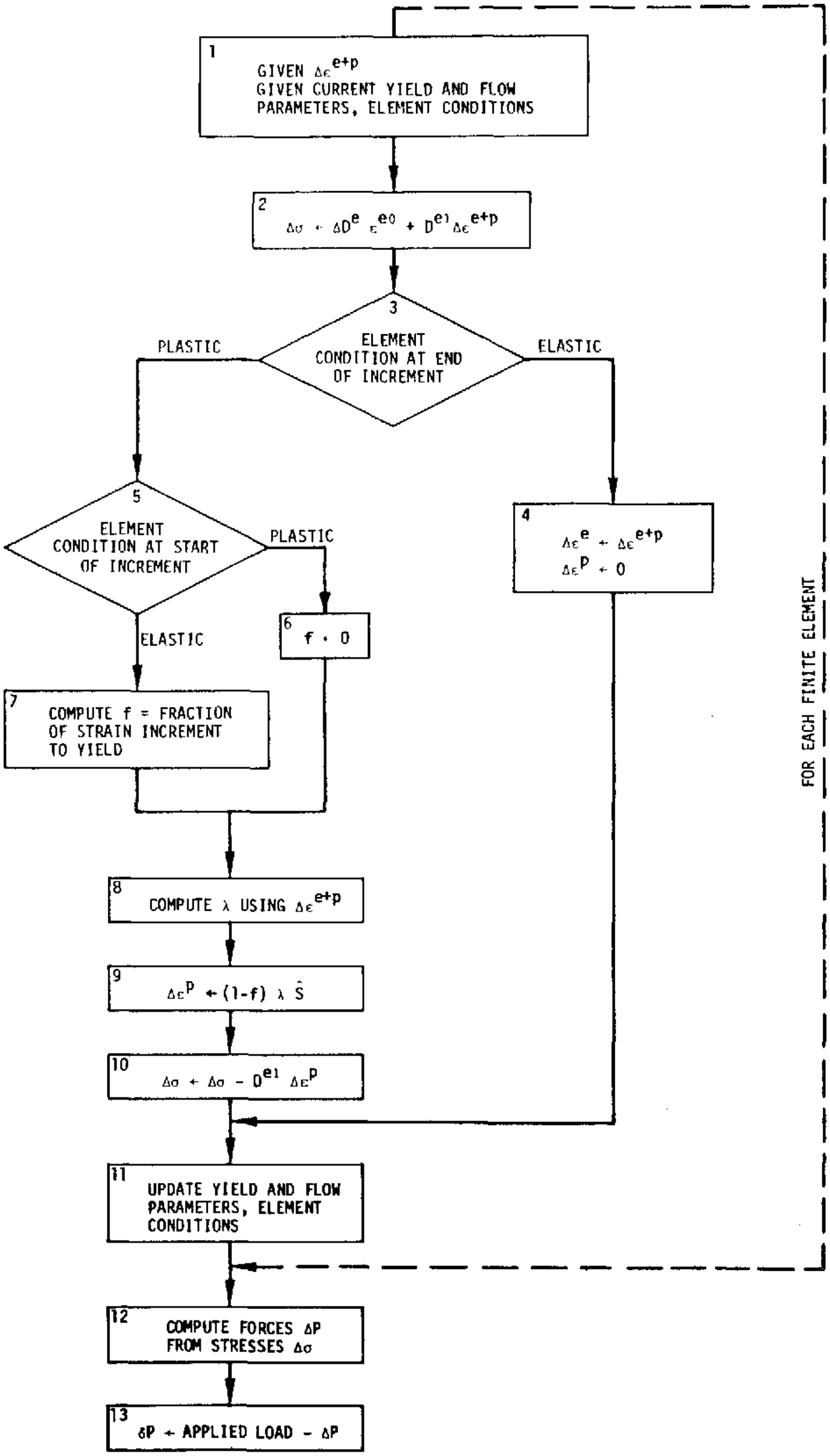

Figure 4.3-1: BOPACE UNBALANCED-FORCE CALCULATIONS

4.3 (Continued)

Element Forces - The element force-stress relation for the CST element is directly apparent from the expressions for the strain-displacement and stiffness matrices given in Equations 3.1-2 and 3.1-3. The relation is

$$\Delta p_i = A\, \Gamma^T_{ij}\, \Delta\sigma_j \tag{4.3-5}$$

where A is the element area, Γ is the strain-displacement matrix, and p_i are the element nodal forces.

Global Unbalanced Forces - Incremental global forces, ΔP, are obtained from the incremental element forces, by adding nodal contributions from all elements and applying transformations of the type 3.3-4a. The global unbalanced forces, δP, are then determined by subtracting these computed (internal) incremental forces from the applied (external) incremental loads:

$$\delta P_i = (\Delta\ \text{Load})_i - \Delta P_i \tag{4.3-6}$$

4.4 IMPROVING THE SOLUTION

The basic global relation for incremental forces and displacements corresponds to the element relation 3.1-7:

$$\Delta P_i = (\Delta K^e_{ij} + \Delta K^p_{ij})\, Q^{e0}_j + (K^{el}_{ij} + K^{pl}_{ij})\, \Delta Q^{e+p}_j \tag{4.4-1}$$

where the first term accounts for forces due to change in elastic properties, and the second term accounts for forces due to change in elastic strains.

4.4 (Continued)

In order to improve a given displacement configuration, the displacement corrections δQ corresponding to unbalanced forces δP, are obtained in BOPACE by solving a set of linear equations of the form

$$\delta P_i = K^J_{ij}\ \delta Q_j \tag{4.4-2}$$

The matrix K^J is the Jacobian (tangent-stiffness) matrix, or some approximation to the Jacobian. The purpose of this section is to discuss the procedure for relating Equations 4.4-1 and 4.1-2, and describe BOPACE options for updating the Jacobian.

Procedure - In the iterative BOPACE approach, the only global solution employed is the displacement-correction relation 4.4-2. The best approximation for the Jacobian is

$$K^J = K^{el} + K^{pl} \tag{4.4-3}$$

where K^{el} and K^{pl} are evaluated at the end of the current load increment. The effects of change in elastic properties (ΔK^e and ΔK^p), as well as the effects of thermal and creep strains, are computed on the elemental level and accounted for by the unbalanced forces. Thus Equation 4.4-1 is satisfied in an iterative fashion.

Updating the Jacobian - In order to account for possible large-scale elastic unloading of the structure under cyclic load conditions, one or more initial iterations are performed for each load increment using only the elastic portion, K^{el}, of the K^J matrix. Succeeding iterations use the total K^J matrix.

4.4 (Continued)

Initially the K^J matrix is taken to be the usual elastic stiffness matrix for the structure, with elastic properties evaluated at the fabrication temperature. Whenever convergence is not achieved within a specified number of iterations, the Jacobian matrix is updated. BOPACE allows five options for updating the matrix K^J and/or its component matrices K^{el} and K^{pl}.

1. Use only elastic matrix K^{el} with no updating. This option corresponds to the constant-stiffness residual-load method, and is most effective for problems with small plastic strains and elastic properties which do not vary much with temperature.

2. Update only K^{el}. This option is best for problems with small plastic strains and elastic properties which vary considerably with temperature.

3. Update only K^{pl}. This option is best for problems with large plastic strains and elastic properties which do not vary much with temperature.

4. Update total K^J matrix, but not its component matrices. This option may be used for problems with large plastic strains and elastic properties which vary somewhat with temperature.

5. Update both K^J, and K^{el} and K^{pl} matrices. This is the most effective option for problems with large plastic strains and elastic properties which vary considerably with temperature.

4.5 SUMMARY OF BOPACE SOLUTION METHOD

An outline of the BOPACE solution method is given in the flowchart of Figure 4.5-1. In step 1, the Jacobian is initialized to the elastic stiffness matrix, based on elastic properties at the fabrication temperature.

At the start of each load increment (step 2) the residual forces δP are set equal to the increment of applied loads. Also, if any residual forces remain from the previous load increment, these are added to δP. The estimate for incremental displacements, ΔQ, is set to zero.

The iteration loop involves successive improvement of the solution, by solving for displacement corrections using the unbalanced forces and the Jacobian, and then recomputing the unbalanced forces corresponding to the new displacement configuration. The displacement corrections δQ are determined in step 3, and in step 4 the improved incremental configuration ΔQ is updated by addition of δQ. Although convergence of this iterative process is usually quite good, BOPACE has a feature for modifying the process if convergence is not occurring. This involves using only a specified fraction of the computed correction, e.g., $\Delta Q \leftarrow \Delta Q + 0.5\ \delta Q$. This would increase the numerical stability but could tend to slow down convergence.

In step 5 the strain-displacement relations are used to compute the total incremental strains $\Delta\varepsilon$ from displacements ΔQ. The thermal and creep strains are then determined in step 6, and subtracted from total strains to give the elastic+plastic strains required for the calculation of

4.5 (Continued)

incremental stresses. Step 7 involves the major iteration algorithm, in which the strain is separated into elastic and plastic components. Incremental stresses are determined according to Equation 2.5-5, and the corresponding unbalanced forces are computed.

If the maximum allowable iterations have been exceeded, step 8 is used to update the Jacobian matrices according to the specified updating option. The Jacobian update is based on the current estimates of the yield surface and flow parameters for each element at the end of the present increment. Iteration is stopped when an error norm (determined by a ratio of residual forces to applied forces) is sufficiently small.

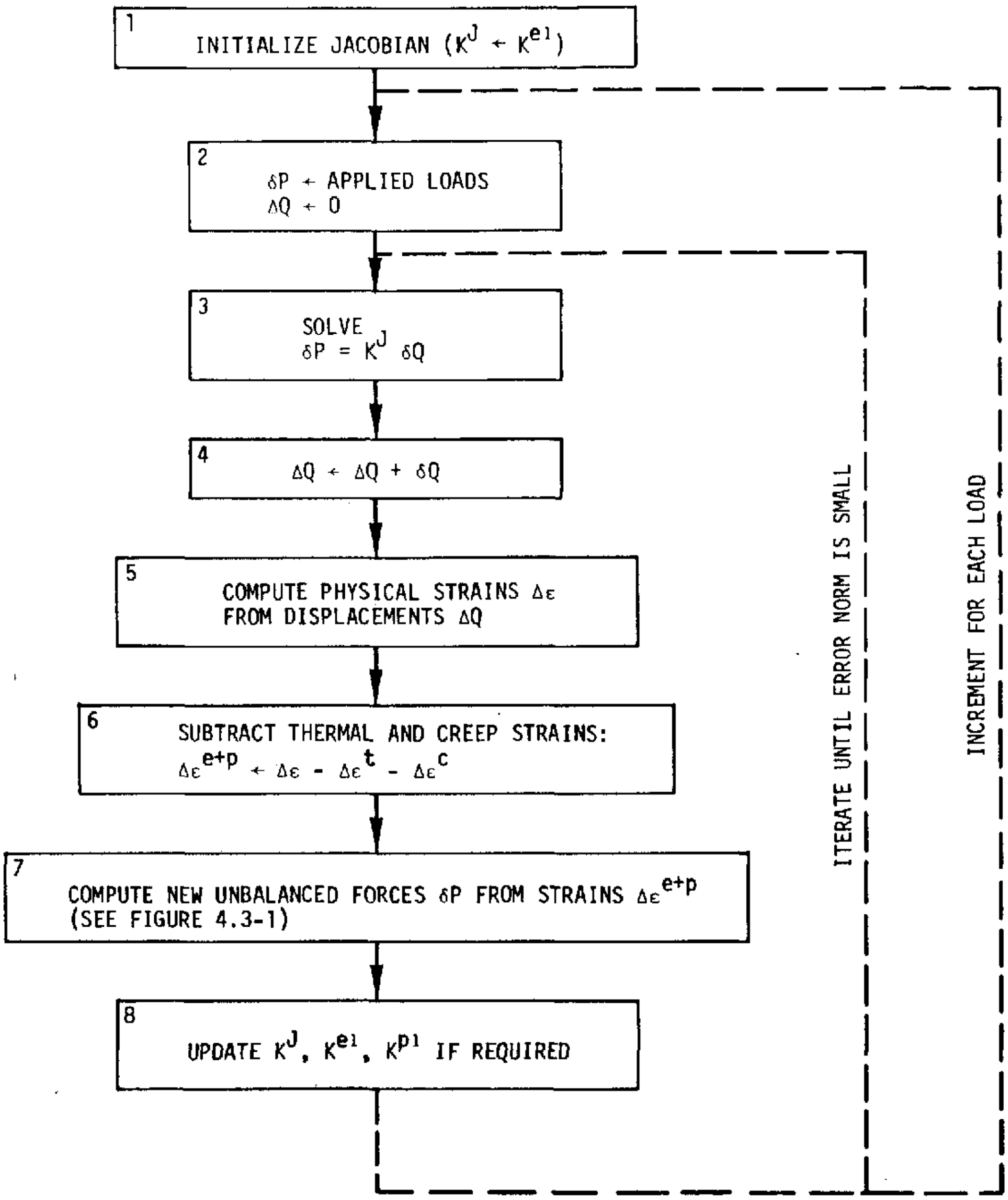

Figure 4.5-1: BOPACE SOLUTION

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5.0 LINEAR EQUATIONS

5.1 LINEAR EQUATION FORMATION

Single Element - Figure 5.1-1 shows a linearly elastic spring. It is defined by its nodal points 1 and 2, and its stiffness coefficient, k (force/displacement). Forces F_x and displacements u are assigned to the nodes as shown in the figure. It is then desired to determine the stiffness matrix for this element. To do this consider all possible displacement modes which the spring is capable of experiencing.

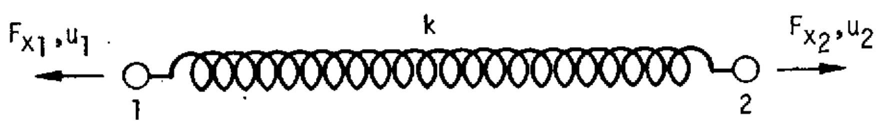


Figure 5.1-1: ELASTIC SPRING AS A FINITE ELEMENT

For example,

a. $u_1 = 0, u_2 \neq 0$

For this case it is clear that Fx_2 must be an applied load, while Fx_1 is a reaction. Also, u_2 is an unknown displacement. From the definition of k,

$$k = \frac{Fx2}{u_2} \qquad \text{or } Fx_2 = ku_2$$

Equilibrium of forces gives,

$$Fx_1 = -Fx_2$$

b. $u_1 \neq 0, u_2 = 0$

5.1 (Continued)

This is the second (and final) displacement state which the spring element can undergo. Conditions are now reversed from those which applied in the first case; that is, Fx_1 is now an applied load and u_1 are unknown displacement. The applicable equations are therefore,

$$k = \frac{Fx_1}{u_1} \qquad \text{or } Fx_1 = ku_1$$

and

$$Fx_2 = -Fx_1$$

Now assume the superposition of the individual displacement modes. Then $u_1 \neq 0$ and $u_2 \neq 0$. From cases (a) and (b) above it then follows that,

$$Fx_1 = ku_1 - ku_2$$

$$Fx_2 = -ku_1 + ku_2$$

or in matrix form,

$$\begin{Bmatrix} Fx_1 \\ Fx_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{5.1-1a}$$

or simply,

$$\{F\} = [K] \{u\} \tag{5.1-1b}$$

In this equation [K] represents the element stiffness matrix. It is seen to relate nodal displacements and corresponding nodal forces. The most important characteristics of a finite element are represented by its stiffness matrix.

5.1 (Continued)

Assemblage of Spring Elements - Consider now an assemblage of spring elements as shown in Figure 5.1-2. Nodes are given as 1, 2, 3, and 4. The elements are connected to each other at the nodes. Each spring has a different stiffness as designated by the constants k_a, k_b, and k_c. It is then desired to find the stiffness equation for this assemblage.

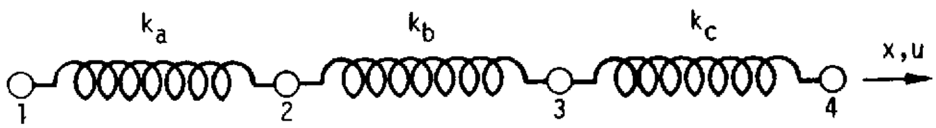


Figure 5.1-2: ASSEMBLAGE OF FINITE SPRING ELEMENTS

Nodal forces are Fx_i, i = 1, 2, 3, 4 and the corresponding displacements are simply u_i. All possible displacement configurations for this system can be obtained by superimposing the four basic states. These occur as each node is displaced in turn, the other nodes in the meantime being held fixed. As in the case of the single element these displacement states can be imposed on the assemblage. The relevant force-displacement expressions can then be written for each case as follows:

a. $u_1 \neq 0$, $u_i = 0$, $i = 2, 3, 4$

Only spring 1-2 is strained. Fx_1 can be considered an applied load and Fx_2 a reaction. No forces are carried through to nodes 3 and 4. Hence,

$$k_a = \frac{Fx_1}{u_1} \qquad \text{or} \qquad Fx_1 = k_a u_1$$

5.1 (Continued)

and by equilibrium,

$$Fx_2 = -Fx_1 = -k_a u_1$$

b. $u_2 \neq 0,\ u_i = 0,\ i = 1, 3, 4$

Both springs 1-2 and 2-3 are now strained. Fx_2 is the applied load. Reactions occur at nodes 1 and 3. No load carries through to node 4. Let,

$$Fx_2 = (Fx_2)_a + (Fx_2)_b$$

so that,

$$k_a = \frac{(Fx_2)_a}{u_2} \qquad , \qquad k_b = \frac{(Fx_2)_b}{u_2}$$

then,

$$Fx_2 = (k_a + k_b)\ u_2$$

while,

$$Fx_1 = -(Fx_2)_a = -k_a u_2$$

$$Fx_3 = -(Fx_2)_b = -k_b u_2$$

c. $u_3 \neq 0,\ u_i = 0,\ i = 1, 2, 4$

This is similar to case (b). By a similar calculation it follows that,

$$Fx_3 = (k_b + k_c)\ u_3$$

$$Fx_2 = -k_b u_3 \text{ and } Kx_4 = -k_c u_3$$

d. $u_4 \neq 0,\ u_i = 0,\ i = 1, 2, 3$

5.1 (Continued)

This is similar to case (a). Hence,

$$Fx_4 = k_c u_4 \text{ and } Fx_3 = -k_c u_4$$

e. $u_i \neq 0$, $i = 1, 2, 3, 4$

This is the general case. It is obtained by superimposing the first four cases. Collecting terms from the previous force-displacement equations yields,

$$Fx_1 = k_a u_1 - k_a u_2$$

$$Fx_2 = -k_a u_1 + (k_a + K_b)\ u_2 - k_b u_3$$

$$Fx_3 = -k_b u_2 + (k_b + k_c)\ u_3 - k_c u_4$$

$$Fx_4 = -k_c u_3 + k_c u_4$$

In matrix form,

$$\begin{Bmatrix} Fx_1 \\ Fx_2 \\ Fx_3 \\ Fx_4 \end{Bmatrix} = \begin{bmatrix} k_a & -k_a & 0 & 0 \\ -k_a & (k_a + k_b) & -k_b & 0 \\ 0 & -k_b & (k_b + k_c) & -k_c \\ 0 & 0 & -k_c & k_c \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad (5.1\text{-}2a)$$

or again,

$$\{F\} = [K]\ \{u\} \qquad (5.1\text{-}2b)$$

5.1.1 (Continued)

The assemblage stiffness matrix is shown in Equation 5.1-2a. Note that its order (4 x 4) is governed by the number of distinct nodal displacements which the assemblage can experience. Some other important features about [K] are as follows: (a) it is a square, symmetric matrix; (b) the stiffness matrix for an assemblage is only sparsely populated with non-zero elements and these can be banded along the main diagonal; (c) the sum of elements in any column (or row) is zero; (d) the stiffness matrix is singular or the determinant of [K] vanishes. These properties are important and except for (c) are generally true for any problem*. They will be referred to from time to time in this document.

A useful form for writing a stiffness matrix is shown in Equation 5.1-3. The matrix is the same as previously given in Equation 5.1-2a.

$$[K] = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} k_a & -k_a & 0 & 0 \\ -k_a & (k_a + k_b) & -k_b & 0 \\ 0 & -k_b & (k_b + k_c) & -k_c \\ 0 & 0 & -k_c & k_c \end{bmatrix} \end{array} \qquad (5.1\text{-}3)$$

Note that the column of displacements {u} in Equation 5.1-2a has simply been turned into a horizontal position and written above the columns making up [K]. Actually Equation 5.1-3 is sufficient for defining the total stiffness equation since the order of terms in the force column {F} must agree with that given for {u}. See Equation 5.1-2a.

* Vector sum of "forces" are zero; however moment terms require products of moment arms and forces in the moment equilibrium equations.

5.1 (Continued)

Fortunately a simple, direct procedure exists for forming the assemblage stiffness matrix. In other words it is not necessary to use the procedure just described for the spring assemblage. Instead the total assemblage stiffness matrix can be formed directly from the stiffness matrices for the individual elements.

Assemblage Stiffness by Superimposing Element Stiffnesses - It can be shown rigorously and is here demonstrated for the assemblage of Figure 5.1-2 that superimposing the stiffnesses of the component elements will yield the stiffness matrix for the assemblage. It is this process which characterizes the direct stiffness method.

Prior to superposition a simple preliminary step is necessary; namely, to increase the order of each element stiffness matrix to that applying to [K] for the total assemblage. This is done by simply adding columns of zeros (plus corresponding rows) for each displacement inapplicable to the given element. For example, for spring 1-2 of Figure 5.1-2, displacements u_3 and u_4 are irrelevant; hence,

$$
[Ka] \quad \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \left[\begin{array}{cc|cc} k_a & -k_a & 0 & 0 \\ -k_a & k_a & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}
$$

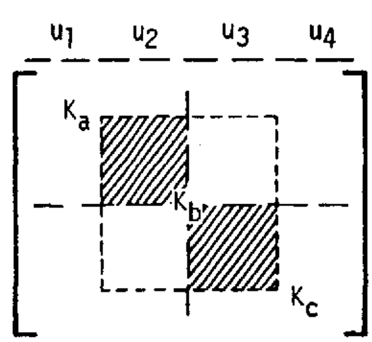
5.1 (Continued)

The columns and rows of zeros are seen to apply to u_3 and u_4. Likewise for the other two springs,

$$
[K_b] = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & k_b & -k_b & 0 \\ \hline 0 & -k_b & k_b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}
\qquad
[K_c] = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & k_c & -k_c \\ 0 & 0 & -k_c & k_c \end{array}\right] \end{array}
$$

Summing the element stiffness matrices is now seen to lead directly to [K] for the assemblage as given by Equation 5.1-2a or Equation 5.1-3. In carrying out this superposition the 2 x 2 non-vanishing parts of the element stiffness matrices are seen to lie along the main diagonal as illustrated by Equation 5.1-4. As a result the tendency of the gross stiffness matrix to be sparsely populated by non-zero elements, which are banded along the main diagonal, can be appreciated. This fact is important in the numerical calculations which lead to the solution for the unknown displacements, particularly for large problems.

[K] =



(5.1-4)

5.1 (Continued)

Shaded areas in Equation 5.1-4 represent regions where element stiffnesses overlap and hence jointly contribute to the gross stiffness. Non-zero terms lie inside the three boxes labeled Ka, Kb, and Kc respectively.

Forming the assemblage stiffness matrix may be regarded as the key to the stiffness method of solution. The procedure as illustrated above for carrying out this step is well suited to machine operation. The computer determines the individual element stiffness matrices and then combines these to form the gross stiffness matrix.

Special Equation Generation - Figure 5.1-3 shows two triangular plates connected at a single node.

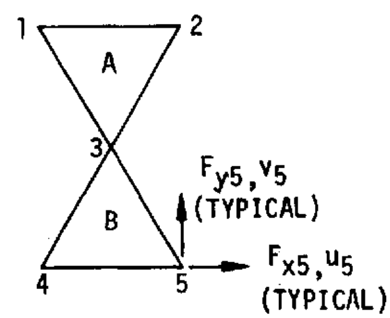


Figure 5.1-3: TWO PLATE ASSEMBLAGE

5.1 (Continued)

The two plates are defined by their nodal points 1, 2, and 3, and 3, 4, and 5. Forces Fx and Fy, and displacements u and v, are assigned to each node as shown for node 5 in the figure. Each plate has a 6 x 6 stiffness matrix as shown in Equation 5.1-5.

$$\begin{Bmatrix} Fx_1 \\ Fy_1 \\ Fx_2 \\ Fx_2 \\ Fx_3 \\ Fy_3 \end{Bmatrix} = \begin{bmatrix} {}^{a}k_{11}^{xx} & {}^{a}k_{11}^{xy} & {}^{a}k_{12}^{xy} & {}^{a}k_{12}^{xy} & {}^{a}k_{13}^{xx} & {}^{a}k_{13}^{xy} \\ & {}^{a}k_{11}^{yy} & {}^{a}k_{12}^{yx} & {}^{a}k_{12}^{yy} & {}^{a}k_{13}^{yx} & {}^{a}k_{13}^{yy} \\ & & {}^{a}k_{22}^{xx} & {}^{a}k_{22}^{xy} & {}^{a}k_{23}^{xx} & {}^{a}k_{23}^{xy} \\ & & & {}^{a}k_{22}^{yy} & {}^{a}k_{23}^{yy} & {}^{a}k_{23}^{yy} \\ & \text{(SYM)} & & & {}^{a}k_{33}^{xx} & {}^{a}k_{33}^{xy} \\ & & & & & {}^{a}k_{33}^{yy} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad (5.1\text{-}5a)$$

$$\begin{Bmatrix} Fx_3 \\ Fy_3 \\ Fx_4 \\ Fx_4 \\ Fx_5 \\ Fx_5 \end{Bmatrix} = \begin{bmatrix} {}^{b}k_{33}^{xx} & {}^{b}k_{33}^{xy} & {}^{b}k_{34}^{xx} & {}^{b}k_{34}^{xy} & {}^{b}k_{35}^{xx} & {}^{b}k_{35}^{xy} \\ & {}^{b}k_{33}^{yy} & {}^{b}k_{34}^{xx} & {}^{b}k_{34}^{yy} & {}^{b}k_{35}^{yx} & {}^{b}k_{35}^{yy} \\ & & {}^{b}k_{44}^{xx} & {}^{b}k_{44}^{xy} & {}^{b}k_{45}^{xx} & {}^{b}k_{45}^{xy} \\ & \text{(SYM)} & & {}^{b}k_{44}^{yy} & {}^{b}k_{45}^{yx} & {}^{b}k_{45}^{yy} \\ & & & & {}^{b}k_{55}^{xx} & {}^{b}k_{55}^{xy} \\ & & & & & {}^{b}k_{55}^{yy} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix} \qquad (5.1\text{-}5b)$$

$$
\begin{Bmatrix} Fx_1 \\ Fy_1 \\ Fx_2 \\ Fy_2 \\ Fx_3 \\ Fy_3 \\ Fx_4 \\ Fy_4 \\ Fx_5 \\ Fy_5 \end{Bmatrix}
=
\begin{bmatrix}
{}^{a}k^{xx}_{11} & {}^{a}k^{xy}_{11} & {}^{a}k^{xx}_{12} & {}^{a}k^{xy}_{12} & {}^{a}k^{xx}_{13} & {}^{a}k^{xy}_{13} & 0 & 0 & 0 & 0 \\
 & {}^{a}k^{yy}_{11} & {}^{a}k^{yx}_{12} & {}^{a}k^{yy}_{12} & {}^{a}k^{yx}_{13} & {}^{a}k^{yy}_{13} & 0 & 0 & 0 & 0 \\
 & & {}^{a}k^{xx}_{22} & {}^{a}k^{xy}_{12} & {}^{a}k^{xx}_{23} & {}^{a}k^{xx}_{23} & 0 & 0 & 0 & 0 \\
 & & & {}^{a}k^{yy}_{12} & {}^{a}k^{yx}_{23} & {}^{a}k^{yy}_{23} & 0 & 0 & 0 & 0 \\
 & & & & {}^{a}k^{xx}_{33}+{}^{b}k^{xx}_{33} & {}^{a}k^{xy}_{33}+{}^{b}k^{xy}_{33} & {}^{b}k^{xx}_{34} & {}^{b}k^{xy}_{34} & {}^{b}k^{xx}_{35} & {}^{b}k^{xy}_{35} \\
 & & & & & {}^{a}k^{yy}_{33}+{}^{b}k^{yy}_{33} & {}^{b}k^{yx}_{34} & {}^{b}k^{yy}_{34} & {}^{b}k^{yx}_{35} & {}^{b}k^{yy}_{35} \\
 & (SYM) & & & & & {}^{b}k^{xx}_{44} & {}^{b}k^{xy}_{44} & {}^{b}k^{xx}_{45} & {}^{b}k^{xy}_{45} \\
 & & & & & & & {}^{b}k^{yy}_{44} & {}^{b}k^{yx}_{45} & {}^{b}k^{yy}_{45} \\
 & & & & & & & & {}^{b}k^{xx}_{55} & {}^{b}k^{xy}_{55} \\
 & & & & & & & & & {}^{b}k^{44}_{55}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix}
\qquad (5.1\text{-}6)
$$

$$
\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ {}^{a}F_{x_3} \\ F_{y_3} \\ {}^{b}F_{x_3} \\ F_{x_4} \\ F_{y_4} \\ F_{x_5} \\ F_{y_5} \end{Bmatrix}
=
\begin{bmatrix}
{}^{a}k^{xx}_{11} & {}^{a}k^{xy}_{11} & {}^{a}k^{xx}_{12} & {}^{a}k^{xy}_{12} & {}^{a}k^{xx}_{13} & {}^{a}k^{xy}_{13} & 0 & 0 & 0 & 0 & 0 \\
 & {}^{a}k^{yy}_{11} & {}^{a}k^{yx}_{12} & {}^{a}k^{yy}_{12} & {}^{a}k^{yx}_{13} & {}^{a}k^{yy}_{13} & 0 & 0 & 0 & 0 & 0 \\
 & & {}^{a}k^{xx}_{22} & {}^{a}k^{xy}_{22} & {}^{a}k^{xx}_{13} & {}^{a}k^{xy}_{23} & 0 & 0 & 0 & 0 & 0 \\
 & & & {}^{a}k^{yy}_{22} & {}^{a}k^{yx}_{23} & {}^{a}k^{yy}_{23} & 0 & 0 & 0 & 0 & 0 \\
 & & & & {}^{a}k^{xx}_{23} & {}^{a}k^{xy}_{33} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & {}^{a}k^{yy}_{33}+{}^{b}k^{yy}_{33} & {}^{b}k^{yx}_{33} & {}^{b}k^{xx}_{34} & {}^{b}k^{yy}_{34} & {}^{b}k^{yx}_{35} & {}^{b}k^{yy}_{35} \\
 & & & & & & {}^{b}k^{xx}_{33} & {}^{b}k^{xx}_{34} & {}^{b}k^{xy}_{34} & {}^{b}k^{xx}_{35} & {}^{b}k^{xy}_{35} \\
 & \text{(SYM)} & & & & & & {}^{b}k^{xx}_{44} & {}^{b}k^{xy}_{44} & {}^{b}k^{xx}_{45} & {}^{b}k^{xy}_{45} \\
 & & & & & & & & {}^{b}k^{yy}_{44} & {}^{b}k^{yx}_{45} & {}^{b}k^{yy}_{45} \\
 & & & & & & & & & {}^{b}k^{xx}_{55} & {}^{b}k^{xy}_{55} \\
 & & & & & & & & & & {}^{b}k^{yy}_{55}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ {}^{a}u_3 \\ v_3 \\ {}^{b}u_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix}
\quad (5.1\text{-}7)
$$

5.1 (Continued)

Merging these matrices using superposition as described in the preceeding section yields a 10 x 10 matrix for the entire assemblage as shown in Equation 5.1-6. Note that the displacements u_3 and v_3 are constrained to be equal for the two elements.

Structural discontinuity is idealized by allowing one or more of the displacements at a connecting node to be unconstrained in the displacement of the attached elements, while the remaining displacements of the node are constrained to be equal.

The equation generation procedure is modified to handle this condition. In the problem of Figure 5.1-3 assume that the v displacements at node 3 of the two elements are constrained to be equal, and the u displacements are constrained. Merging the elemental stiffness matrices of Equations 5.1-5a and 5.1-5b for this condition yields an 11 x 11 matrix for the entire assemblage as shown in Equation 5.1-7.

5.2 SOLUTION OF EQUATIONS

Introduction - The stiffness equations can be written in matrix form as shown in Equation 5.2-1:

$$\{F\} = [K]\{Q\} \qquad (5.2\text{-}1)$$

where $\{F\}$ is the force vector

$[K]$ is the stiffness matrix

$\{Q\}$ is the displacement vector

5.2 (Continued)

The solution of Equation 5.2-1 generally involves a mixed set of known and unknown forces and displacements. Traditionally the approach has been to reorder the equations collecting the known and unknown forces and displacements such that Equation 5.2-1 is rewritten in partitioned matrix form as shown in Equation 5.2-2:

$$\left\{ \begin{array}{c} F_k \\ \hline F_u \end{array} \right\} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array} \right] \left\{ \begin{array}{c} Qu \\ \hline Qk \end{array} \right\} \qquad (5.2\text{-}2)$$

where the subscripts are

k = known

u = unknown

While this approach is efficient and yields accurate solution, the reordering necessary is a tedious detail which should be bypassed when possible. A somewhat different approach is the wavefront method, which is efficient and as accurate as any other method. The approach is to decompose the n x n stiffness matrix [K], then solve for the unknown forces $\{F_u\}$ and displacements $\{Q_u\}$ by a forward and backward substitution.

The solution steps are first shown here for a system of equations involving only known forces and unknown displacements. Then the modifications are described for the general case of mixed known and unknown forces and displacements.

5.2 (Continued)

Solution with Known Forces - The decomposed form of Equation 5.2-1 takes the form

$$[U]^T \quad [D]^{-1} \quad [U] \quad \{Q_u\} \quad = \quad \{F_K\} \qquad (5.2\text{-}3)$$

where [U] is an upper triangular matrix, and [D] is a diagonal matrix whose elements are the diagonal elements of [U]. The solution is in three steps.

1. Decomposition

The elements of K are

$$K_{ij} = \sum_{k=1}^{i-1} U_{ki} \quad D_{kk}^{-1} \quad U_{kj} \quad + \quad U_{ii} \, D_{ii}^{-1} \quad U_{ij} \qquad (5.2\text{-}4)$$

Since $D_{ii} = U_{ii}$, the elements of U are obtained successively by row, as

$$U_{ij} = K_{ij} \quad - \sum_{k=1}^{i-1} D_{kk}^{-1} \quad U_{ki} \quad U_{kj} \qquad (5.2\text{-}5)$$

2. Forward Substitution

Let $Y = D^{-1} \; U \; Q$

then

$$F_i = \sum_{k=1}^{i-1} U_{ki} \; Y_k + U_{ii} \; Y_i \qquad (5.2\text{-}6)$$

giving

$$Y_i = D_{ii}^{-1} \quad (F_i - \sum_{k=1}^{i-1} U_{ki} \; Y_k) \qquad (5.2\text{-}7)$$

5.2 (Continued)

3. Backward Substitution

$$Y_i = \sum_{k=i+1}^{n} D_{ii}^{-1} U_{ik} Q_k + D_{ii}^{-1} U_{ii} Q_i \tag{5.2-8}$$

giving

$$Q_i = Y_i - D_{ii}^{-1} \sum_{k=i+1}^{n} U_{ik} Q_k \tag{5.2-9}$$

Solution of Mixed Set - For the general case where there is a mixed set of known and unknown forces and displacements, the three solution steps must be modified. The procedure as described is for the case in which a single displacement (the r^{th} displacement) is known, with the corresponding r^{th} force unknown. Additional known displacements with corresponding unknown forces would be treated in the same manner.

The modified form of the decomposed matrix is given by Equation 5.2-3:

$$\begin{bmatrix} [U_{11}] & [U_{1r}] & [U_{1n}] \\ 0 & [U_{rr}] & [U_{rn}] \\ 0 & 0 & [U_{nn}] \end{bmatrix}^T \begin{bmatrix} [D_{11}]^{-1} & & \\ & [1] & \\ & & [D_{nn}^{-1}] \end{bmatrix} \begin{bmatrix} [U_{11}] & [U_{1r}] & [U_{1n}] \\ 0 & [U_{rr}] & [U_{rn}] \\ 0 & 0 & [U_{nn}] \end{bmatrix} \begin{Bmatrix} \{Q_1\} \\ \{\bar{Q}_r\} \\ \{Q_n\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{F}_1\} \\ \{F_r\} \\ \{\bar{F}_n\} \end{Bmatrix} \tag{5.2-10}$$

where $\bar{Q}_r$, and $\bar{F}_1$ and $\bar{F}_n$ denote known quantities. The elements of U are given by equation 5.2-4 except that no contribution from $[U_{rn}]$ is distributed to the elements of $[U_{nn}]$. Detailed steps are:

5.2 (Continued)

1) Decomposition

First rows: Compute each row (1 to r-1) according to Equation 5.2-4 and distribute contributions to later rows.

r^{th} row: Compute the r^{th} row of U according to Equation 5.2-4 but do not distribute contributions to later rows.

Last rows: Compute each row (r+1 to r) according to Equation 5.2-4 and distribute contributions to later rows.

2) Forward Substitution

First rows: Compute

$$\{Y_1\} = [U_{11}]^{-1} \{F_1\} \text{ using Equation 5.2-7} \tag{5.2-11}$$

and distribute to $\{Y_r\}$ and $\{Y_n\}$. This produces the vector

$$\{Y\} = \begin{Bmatrix} \{Y_1\} \\ \{Y_r\} = -[U_{1r}]^T \{Y_1\} \\ \{Y_n\} = -[U_{1n}]^T \{Y_1\} \end{Bmatrix} \tag{5.2-12}$$

r^{th} row:

The r^{th} row of Equation 5.2-3 gives the relationship

$$\left([U_{rr}] , [U_{rn}]\right) \begin{Bmatrix} \{\bar{Q}_r\} \\ \{Q_n\} \end{Bmatrix} = \left\{\{F_r\} - [U_{1r}]^T \{Y_1\}\right\} \tag{5.2-13}$$

where $-[U_{1r}]^T \{Y_1\}$ is from Equation 5.2-11 and replaces $\{F_r\}$, and $\{Y_r\}$ is set to $\{\bar{Q}_r\}$ and contributions are distributed forward as in Equation 5.2-6.

5.2 (Continued)

Last rows:

Continue forward substitution by Equation 5.2-6 to obtain

$$\{Y_2\} = [U^T_{nn}]^{-1} \left\{ \{F_n\} - [U_{rm}]^T \{\bar{Q}_n\} - [U_{1n}]^T \{Y_1\} \right\} \quad (5.2\text{-}14)$$

3. Backward Substitution

Last rows:

From Equation 5.2-8

$$\{Q_n\} = [U_{22}]^{-1} (D_{22}] \{Y_n\} \quad (5.2\text{-}15)$$

rth row:

From Equation 5.2-12

$$\{F_r\} = -[U_{1r}] \{Y_1\} + [U_{rr}] \{\bar{Q}_r\} + [U_{rm}] \{Q_n\} \quad (5.2\text{-}16)$$

First rows:

From Equation 5.2-8

$$\{Q_1\} = [U_{11}]^{-1} [D_{11}] \left\{ \{Y_1\} - [U_{1r}] \{\bar{Q}_r\} - [U_{1m}] \{Q_n\} \right\} \quad (5.2\text{-}17)$$

6.0 DEFINITIONS

This section defines symbols used in the BOPACE Theoretical Manual. Symbols used in Section 5 (on solution of linear equations) are not included here, because that section uses somewhat different notation and is considered to be self contained.

Variables

a	Deviatoric stress center
b	Displacement-function coefficient
c	Kinematic hardening slope
c,s	Cosine, Sine abbreviations
k	Element stiffness matrix
p,q	Local nodal forces, displacements
r	Isotropic hardening slope
s	Deviatoric (total - hydrostatic) stress
$\hat{s}$	s - a = relative deviatoric stress
u,v	Local displacements in X,Y directions
x,y	Local Cartesian coordinates
A	Elasto-plastic hardening parameter
C	Kinematic hardening matrix
D	Elasticity matrix
E	Young's modulus; global strain
F	Yield surface function
F^c	Creep hardening factor
F^k	Kinematic hardening factor
I	Identity matrix

6.0 (Continued)

K	Global stiffness matrix
K^J	Jacobian (tangent-stiffness) matrix
0	Origin of coordinates
P,Q	Global nodal forces, displacements
R	Isotropic hardening matrix
T	Temperature
W	Work
R,θ	Cylindrical coordinates
α	Stress center of yield surface
ε	Strain
η	Direction of yield surface stress correction
ρ	Scalar factor for yield surface stress correction
γ	Thermal coefficient of expansion
κ	Cumulative plastic hardening parameter
κ^c	Creep hardening parameter
κ^k	Kinematic hardening parameter
μ	Stress coefficients for prescribed normal stress
ν	Poisson's ratio
λ	Plastic flow parameter
σ	Stress
Γ	Transformation matrix

Subscripts

i,j,k,ℓ,m,n	General indices

6.0 (Continued)

Superscripts

0	Start-of-increment quantity
1	End-of-increment quantity
$^\circ$	Known test value
c	Creep quantity
e	Elastic quantity
p	Plastic quantity
e+p	Sum of elastic and plastic quantities
t	Thermal quantity

Special Symbols

$\delta(\ )$	Residual (corrective) quantity
$\Delta(\ )$	Incremental quantity
$(\ )^T$	Matrix transpose
$(\ )^{-1}$	Matrix inverse
$(\bar{\ })$	Effective quantity
Σ	Summation
$\approx$	Proportional
$\not\approx$	Non proportional

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