

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1589

*Signal-to-Noise Ratios in
Coherent Soft Limiters*

J. R. Lesh

CASE FILE
COPY

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

September 15, 1973

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1589

*Signal-to-Noise Ratios in
Coherent Soft Limiters*

J. R. Lesh

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

September 15, 1973

Preface

The work described in this report was performed by the DSN Operations Division of the Jet Propulsion Laboratory.

Contents

I. Introduction	1
II. System Model	1
III. Mean Value of z	3
IV. Second Moment of z	4
V. Signal-to-Noise Power Ratios	7
A. SNRs and the Parameter β	7
B. SNRs and the Parameter D	8
VI. Some Comments on Signal-to-Noise Spectral Densities	10
References	12

Figures

1. Coherent soft limiter model	1
2. Error function transfer characteristic	2
3. Ratio of signal-to-noise ratios, parameter β , $\theta = 0$	8
4. Ratio of signal-to-noise ratios, parameter β , $\theta = \pi/4$	8
5. Ratio of signal-to-noise ratios, parameter β , $\theta = \pi/2$	8
6. Ratio of signal-to-noise ratios, parameter D , $\theta = 0$	9
7. Ratio of signal-to-noise ratios, parameter D , $\theta = \pi/4$	9
8. Ratio of signal-to-noise ratios, parameter D , $\theta = \pi/2$	9
9. Ratio of signal-to-noise ratios, $D = 0.5, 1.0, 1.5$, $\theta = 0$	10

Abstract

Expressions for the output signal-to-noise power ratio of a bandpass soft limiter followed by a coherent detection device are presented and discussed. It is found that a significant improvement in the output signal-to-noise ratio (SNR) at low input SNRs can be achieved by such soft limiters as compared to hard limiters. This indicates that the soft limiter may be of some use in the area of threshold extension. Approximation methods for determining output signal-to-noise spectral densities are also presented.

Signal-to-Noise Ratios in Coherent Soft Limiters

I. Introduction

In his classic 1953 paper Davenport (Ref. 1) found that the ratio of the output to input signal-to-noise power ratio of a bandpass hard limiter asymptotically approached $\pi/4$ and 2 as the input signal-to-noise ratio approached zero and infinity, respectively. Since that time a number of papers have appeared in the literature on the hard bandpass limiter and its generalization to the soft limiter (Refs. 2-9). All of these papers produced results which were consistent with the original Davenport finding.

More recently, however, Tausworthe (Ref. 10) and Springett and Simon (Ref. 11) found that if a hard bandpass limiter is followed by a coherent detector, the strong signal asymptotic value no longer agrees with Davenport's (incoherent limiter) result. The reason for this deviation is that the relative phase relationship between components at the limiter input is not preserved as the signal passes through the limiter, and consequently quadrature components of the output noise do not necessarily have the same spectra (Ref. 10). It would be interesting to see if the results of these coherent limiter studies can be generalized to the case of the soft limiter, and this is the subject of investigation herein.

II. System Model

Consider the coherent soft limiter model shown in Fig. 1. The input to the system consists of a sinusoidal signal of known parameters and additive narrowband gaussian noise and is given by

$$x(t) = \sqrt{2}A \cos(\omega_0 t) + n(t) \quad (1)$$

The noise process is assumed to be created by passing white gaussian noise having a one-sided spectral density N_0 through an ideal narrowband filter having a bandwidth B , centered at ω_0 . Using the narrowband expansion for $n(t)$, we can express Eq. (1) as

$$x(t) = v(t) \cos[\omega_0 t + \gamma(t)] \quad (2)$$

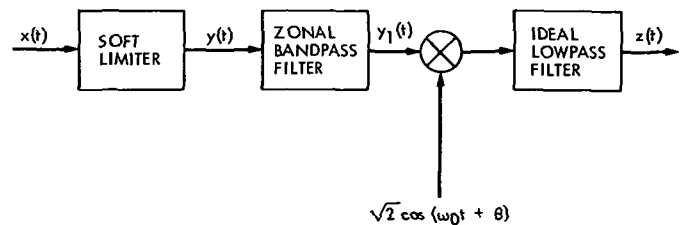


Fig. 1. Coherent soft limiter model

where

$$v(t) = \sqrt{2} \{ [A + n_c(t)]^2 + n_s^2(t) \}^{1/2} \quad (3)$$

and

$$\gamma(t) = \tan^{-1} \left[\frac{n_s(t)}{A + n_c(t)} \right] \quad (4)$$

where $n_c(t)$ and $n_s(t)$ are the orthogonal projections of $n(t)$ onto the cosine and sine axes, respectively. We assume that $n(t)$ is zero mean, so that

$$E[n_c(t)] = E[n_s(t)] = 0 \quad (5)$$

and

$$E[n_c^2(t)] = E[n_s^2(t)] = \frac{\sigma^2}{2} \quad (6)$$

where

$$\sigma^2 = N_0 B,$$

The input signal $x(t)$ is applied to a soft limiter having an error function transfer characteristic given by

$$y(x) = L \operatorname{erf} \left(\frac{K \sqrt{\pi}}{2L} x \right) \quad (7)$$

where L is the peak limiter output, K is the slope of the transfer characteristic at $x = 0$, and the error function is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (8)$$

The limiter characteristic $y(x)$ is shown in Fig. 2

The output of the soft limiter is passed through an ideal bandpass filter which extracts only the fundamental (first zone) components of $y(t)$, denoted $y_1(t)$; $y_1(t)$ is then mixed with a reference signal of the form

$$u(t) = \sqrt{2} \cos(\omega_0 t + \theta) \quad (9)$$

and passed through an ideal lowpass filter which filters out all double frequency terms. The phase angle θ is a known and fixed parameter and determines the type of demodulation which is being used ($\theta = 0$ for coherent amplitude detection, $\theta = \pi/2$ for phase demodulation). The final system output is denoted $z(t)$.

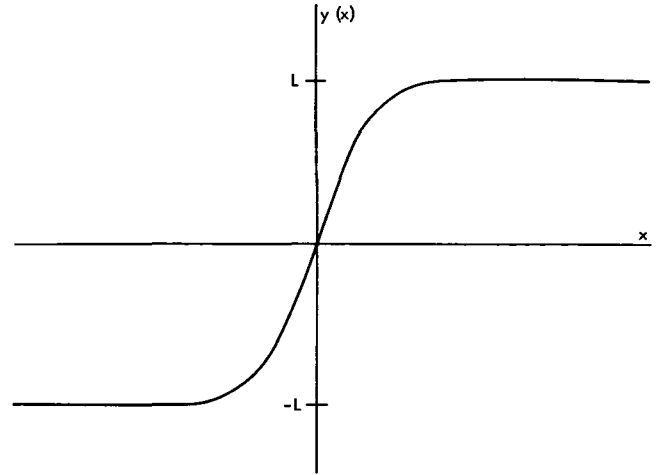


Fig. 2. Error function transfer characteristic

The method for obtaining the output $y_1(t)$ is to apply an input $x(t)$ and compute $y(t)$. Then expand $y(t)$ in a Fourier series, if $x(t)$ is periodic, and retain only those components which are in the first zone. If the input is assumed to be sinusoidal, this method becomes the technique of equivalent linearization (Ref. 12). Thus, if we assume (for the moment) an input of the form

$$x(t) = A_0 \sin(\omega t + \theta_0) \quad (10)$$

then the output $y_1(t)$ becomes

$$y_1(t) = c_1 \sin(\omega t + \theta_0) \quad (11)$$

where

$$c_1 = \frac{L\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} \operatorname{erf} \left[\frac{K \sqrt{\pi}}{2L} A_0 \sin(\omega t + \theta_0) \right] \sin(\omega t + \theta_0) dt \quad (12)$$

Integrating by parts gives

$$c_1 = \frac{KA_0}{\pi} \int_{-\pi}^{\pi} \cos^2 \phi \exp \left(-\frac{K^2 \pi A_0^2}{4L^2} \sin^2 \phi \right) d\phi \quad (13)$$

Now, if we substitute c_1 into Eq (11) and divide by the input (Eq. 10), we find that the equivalent linear gain $N(A_0, \omega)$ of the bandpass soft limiter to a sinusoid of amplitude A_0 and frequency ω is

$$N(A_0, \omega) = \frac{K}{\pi} \int_{-\pi}^{\pi} \cos^2 \phi \exp \left(-\frac{K^2 \pi A_0^2}{4L^2} \sin^2 \phi \right) d\phi \quad (14)$$

Using this result with the input signal given in Eq. (2) we are able to express the first zone output $y_1(t)$ by

$$y_1(t) = N[v(t), \omega_0] v(t) \cos[\omega_0 t + \gamma(t)] \quad (15)$$

Now, if we multiply this output by the coherent reference $u(t)$ and retain only the lowpass components we have

$$\begin{aligned} z(t) &= \frac{\sqrt{2}}{2} N[v(t), \omega_0] v(t) [\cos \gamma(t) \cos \theta + \sin \gamma(t) \sin \theta] \\ &= \frac{K}{\pi} \int_{-\pi}^{\pi} \cos^2 \phi \exp \left(-\{[A + n_c(t)]^2 \right. \\ &\quad \left. + n_s^2(t)\} \frac{K^2 \pi}{2L^2} \sin^2 \phi \right) d\phi \\ &\quad \times \{[A + n_c(t)] \cos \theta + n_s(t) \sin \theta\} \end{aligned} \quad (16)$$

Equation (16) is the mathematical model to be used for the coherent soft limiter. We are interested in evaluating the signal-to-noise ratio performance based on this model. To do this we must first compute the first two moments of z .

III. Mean Value of z

We note that z is a random variable which depends on the random variables n_c and n_s , both of which are gaussian. Thus, if we denote the expected value of z by μ_z we have

$$\mu_z = \frac{1}{\pi \sigma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(n_c, n_s) \exp \left(-\frac{n_c^2 + n_s^2}{\sigma^2} \right) dn_s dn_c \quad (17)$$

where $z(n_c, n_s)$ is given by Eq. (16) with the time arguments suppressed. If we interchange the order of integration, after integrating over n_s we obtain

$$\begin{aligned} \mu_z &= \frac{KL}{\pi^2 \sigma} \int_{-\pi}^{\pi} \left(\frac{2\pi}{2L^2 + K^2 \pi \sigma^2 \sin^2 \phi} \right)^{1/2} \\ &\quad \times \cos^2 \phi \int_{-\infty}^{\infty} \exp \left(-\frac{n_c^2}{\sigma^2} - \frac{K^2 \pi}{2L^2} (A + n_c)^2 \sin^2 \phi \right) \\ &\quad \times [(A + n_c) \cos \theta] dn_c d\phi \end{aligned} \quad (18)$$

Now, if we substitute $x = n_c + A$, complete the square in the exponent and integrate, we have

$$\begin{aligned} \mu_z &= \frac{8\sqrt{2}LR^2\beta^3 \cos \theta e^{-R}}{\pi} \\ &\quad \times \int_{-\pi}^{\pi} \frac{\cos^2 \phi \exp \left(\frac{4R^2\beta^2}{4R\beta^2 + \pi \sin^2 \phi} \right)}{(4R\beta^2 + \pi \sin^2 \phi)^2} d\phi \end{aligned} \quad (19)$$

where

$$R = \frac{A^2}{\sigma^2} \quad (\text{input signal-to-noise ratio}) \quad (20)$$

and

$$\beta = \frac{L}{\sqrt{2}KA} \quad (21)$$

The parameter β requires a little more explanation. First, we note that β can be interpreted as the ratio of the peak limiter output to the peak signal output if the limiter acts as a linear amplifier with gain K . More importantly, however, we note that Eq (7) can be expressed as

$$y(x) = L \operatorname{erf} \left(\frac{\sqrt{\pi} x}{\sqrt{2} A \beta} \right) \quad (22)$$

Now, if we let $\beta \rightarrow 0$, then $y(x)$ goes to $\pm L$, depending on the sign of x , or, in other words, the soft limiter approaches a hard limiter. Furthermore, as $\beta \rightarrow \infty$ the asymptotic expansion for the error function yields

$$y(x) \sim \frac{Lx}{\sqrt{2}A\beta} = Kx \quad (23)$$

which is the result for a linear amplifier of gain K . Thus we can interpret β as a measure of "softness" of the soft limiter. Another "softness" parameter which we will find useful is the parameter D defined by

$$D = \frac{4R\beta^2}{\pi} \quad (24)$$

Clearly, for a fixed R , D goes to zero or infinity with β . We shall have more to say about these parameters and their corresponding limiting operations later.

Returning now to Eq. (19) and scaling the variable of integration gives

$$\mu_z = \frac{32\sqrt{2}LR^2\beta^3 \cos \theta e^{-R}}{\pi^3} \int_0^{\pi/2} \frac{\sin^2 x \exp\left(\frac{RD}{D + \cos^2 x}\right)}{(D + \cos^2 x)^2} dx \quad (25)$$

If we substitute the quantity

$$x = \cos^{-1}\left(\frac{\sqrt{D} \sin \phi}{\sqrt{D + \cos^2 \phi}}\right) \quad (26)$$

Eq. (25) becomes

$$\begin{aligned} \mu_z = & \frac{32\sqrt{2}LR^2\beta^3 \cos \theta}{\pi^3 D^{3/2} \sqrt{D+1}} \int_0^{\pi/2} \cos^2 \phi \\ & \times \exp\left(-\frac{R}{D+1} \sin^2 \phi\right) d\phi \end{aligned} \quad (27)$$

Now, by using the identity

$$I_v(x) = \frac{1}{\pi} \int_0^\pi \exp(x \cos \xi) \cos(v\xi) d\xi; \quad v = 0, 1, 2, \dots \quad (28)$$

where $I_v(x)$ is the modified Bessel function of order v , then Eq. (27) becomes

$$\begin{aligned} \mu_z = & \frac{\sqrt{2R}L \cos \theta \exp\left(-\frac{R}{2(1+D)}\right)}{\sqrt{\pi(1+D)}} \\ & \times \left\{ I_0\left[\frac{R}{2(1+D)}\right] + I_1\left[\frac{R}{2(1+D)}\right] \right\} \end{aligned} \quad (29)$$

We note that this result is consistent with earlier results for coherent limiters. For example as $D \rightarrow 0$ the soft coherent limiter degenerates to the hard coherent limiter. From Eq. (29) we have

$$\mu_z|_{D=0} = \frac{\sqrt{2R}L \cos \theta e^{-R/2}}{\sqrt{\pi}} \left[I_0\left(\frac{R}{2}\right) + I_1\left(\frac{R}{2}\right) \right] \quad (30)$$

which is precisely the result determined by Springett and Simon (Ref. 11). For the linear amplifier case ($D \rightarrow \infty$) we note that the I_0 and I_1 Bessel functions reduce to 1 and 0,

respectively. Then, by using the definition of D we have

$$\mu_z|_{D \rightarrow \infty} = \frac{\sqrt{2R}L \cos \theta}{\sqrt{\pi D}} = KA \cos \theta \quad (31)$$

which is, indeed, the answer one expects from an amplifier of gain K followed by a coherent detector.

It is interesting to note that Eq. (29) can be rewritten as

$$\mu_z = \frac{2L\sqrt{2}}{\pi} \alpha\left(\frac{R}{1+D}\right) \cos \theta \quad (32)$$

where

$$\alpha(R) = \frac{\sqrt{\pi R}}{2} e^{-R/2} \left[I_0\left(\frac{R}{2}\right) + I_1\left(\frac{R}{2}\right) \right] \quad (33)$$

The function $\alpha(R)$ is called the hard limiter signal suppression factor (see, for example, Lindsey in Ref. 13). Thus, we see that the mean signal output of a soft limiter can be regarded as the mean signal output of a hard limiter operating at an "effective" input signal-to-noise ratio of $R/(1+D)$. This result was first noticed by Tausworthe (Ref. 9) for the incoherent soft limiter.

IV. Second Moment of z

Recall from Eq. (16) the expression for $z(n_c, n_s)$. If we expand the trigonometric functions and use the definition of the modified Bessel functions (Eq. 28), we obtain directly

$$\begin{aligned} z(n_c, n_s) = & K \exp\left(-\frac{K^2\pi}{4L^2} [(A + n_c)^2 + n_s^2]\right) \\ & \times \left\{ I_0\left\{\frac{K^2\pi}{4L^2} [(A + n_c)^2 + n_s^2]\right\} \right. \\ & \left. + I_1\left\{\frac{K^2\pi}{4L^2} [(A + n_c)^2 + n_s^2]\right\} \right\} \\ & \times [(A + n_c) \cos \theta + n_s \sin \theta] \end{aligned} \quad (34)$$

Thus, the second moment of z is given by

$$\begin{aligned} E(z^2) = & \frac{1}{\pi\sigma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [z(x - A, n_s)]^2 \\ & \times \exp\left(-\frac{(x - A)^2}{\sigma^2} - \frac{n_s^2}{\sigma^2}\right) dn_s dx \end{aligned} \quad (35)$$

where we have made the substitution $x = n_c + A$. If we change to polar coordinates, Eq. (35) becomes

$$\begin{aligned}
 E(z^2) &= \frac{K^2 \exp\left(-\frac{A^2}{\sigma^2}\right)}{\pi \sigma^2} \int_0^\infty r^3 \exp\left[-\left(\frac{K^2 \pi \sigma^2 + 2L^2}{2L^2 \sigma^2}\right) r^2\right] \\
 &\quad \times \left[I_0\left(\frac{K^2 \pi r^2}{4L^2}\right) + I_1\left(\frac{K^2 \pi r^2}{4L^2}\right) \right]^2 \\
 &\quad \times \int_0^{2\pi} (\cos^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta \\
 &\quad + 2 \sin \phi \sin \theta \cos \phi \cos \theta) \exp\left(\frac{2rA}{\sigma^2} \cos \phi\right) d\phi dr
 \end{aligned} \tag{36}$$

The last term in the integrand of the ϕ integral integrates to zero, since we can shift ϕ to produce a symmetric integral over an odd function. Then if we use the definitions of R and D we have

$$\begin{aligned}
 E(z^2) &= \frac{2L^2 R^2 e^{-R}}{\pi^2 D} \int_0^\infty y^3 \exp\left[-R\left(\frac{1+D}{D}\right) y^2\right] \\
 &\quad \times \left[I_0\left(\frac{Ry^2}{2D}\right) + I_1\left(\frac{Ry^2}{2D}\right) \right]^2 \\
 &\quad \times \int_0^{2\pi} [\cos^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta] \\
 &\quad \times \exp(2Ry \cos \phi) d\phi dy
 \end{aligned} \tag{37}$$

Now note from Eq. (28)

$$\int_0^{2\pi} \cos^2 \phi \exp(2Ry \cos \phi) d\phi = \pi [I_0(2Ry) + I_2(2Ry)] \tag{38}$$

and

$$\int_0^{2\pi} \sin^2 \phi \exp(2Ry \cos \phi) d\phi = \pi [I_0(2Ry) - I_2(2Ry)] \tag{39}$$

Substituting these expressions into Eq. (37) and using the recurrence relations for modified Bessel functions yields the result

$$\begin{aligned}
 E(z^2) &= \frac{2L^2 R^2 e^{-R}}{\pi D} \int_0^\infty y^2 \exp\left[-R\left(\frac{1+D}{D}\right) y^2\right] \\
 &\quad \times \left[I_0\left(\frac{Ry^2}{2D}\right) + I_1\left(\frac{Ry^2}{2D}\right) \right]^2 \\
 &\quad \times \left[2y I_0(2Ry) \cos^2 \theta - \frac{1}{R} I_1(2Ry) \cos(2\theta) \right] dy
 \end{aligned} \tag{40}$$

The author has tried repeatedly to solve this integral but with no success. Hence, we must resort to numerical evaluation. However, the infinite limit in Eq. (40) makes numerical integration somewhat cumbersome. By a series of substitutions one can show that Eq. (40) can be expressed as

$$\begin{aligned}
 E(z^2) &= \frac{4L^2 D e^{-R}}{\pi^2} \int_0^1 \frac{\sqrt{1-y^2} \sqrt{g(y)}}{\sqrt{f(y)}^5} \\
 &\quad \times \left\{ [f(y) + 2RD \cos^2 \theta] I_0\left[\frac{RDg(y)}{2f(y)}\right] \right. \\
 &\quad \left. + [2RD \cos^2 \theta - f(y) \cos(2\theta)] I_1\left[\frac{RDg(y)}{2f(y)}\right] \right\} \\
 &\quad \times \exp\left(-\frac{RD}{2f(y)} [g(y) - 2]\right) dy
 \end{aligned} \tag{40a}$$

where

$$f(y) = y^2 + D \tag{40b}$$

and

$$g(y) = [f(y) + 1]^{-1} \tag{40c}$$

Equation (40a), although functionally more complex, can be numerically evaluated with little effort.

To gain some confidence in the validity of Eq. (40) let us examine its limiting behavior. For the hard limiter ($D \rightarrow 0$) we can use the asymptotic expansion for modified Bessel functions

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{41}$$

to obtain

$$E(z^2)|_{D \rightarrow 0} = \frac{8RL^2 e^{-R}}{\pi^2} \left\{ 2 \int_0^\infty y \exp(-Ry^2) I_0(2Ry) dy \cos^2 \theta - \frac{1}{R} \int_0^\infty \exp(-Ry^2) I_1(2Ry) dy \cos(2\theta) \right\} \quad (42)$$

Now, from Gradshteyn and Ryzhik (Ref. 14) we have the identities

$$\int_0^\infty y^{\nu+1} \exp(-\alpha y^2) I_\nu(\beta y) dy = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left\{\frac{\beta^2}{4\alpha}\right\} \quad \begin{matrix} \operatorname{Re}(\alpha) > 0 \\ \operatorname{Re}(\nu) > -1 \end{matrix} \quad (43)$$

$$\int_0^\infty y^{\nu-1} \exp(-\alpha y^2) I_\nu(\beta y) dy = (-1)^\nu 2^{\nu-1} \beta^{-\nu} \gamma\left(\nu, -\frac{\beta^2}{4\alpha}\right), \quad \begin{matrix} \operatorname{Re}(\alpha) > 0 \\ \operatorname{Re}(\nu) > 0 \end{matrix} \quad (44)$$

and

$$\gamma(1+n, x) = n! \left[1 - e^{-x} \left(\sum_{m=0}^n \frac{x^m}{m!} \right) \right], \quad n=0, 1, 2, \dots \quad (45)$$

where $\gamma(a, x)$ is the incomplete gamma function.

Using these substitutions we obtain

$$\begin{aligned} E(z^2)|_{D \rightarrow 0} &= \frac{8L^2 e^{-R}}{\pi^2} \left[e^R \cos^2 \theta + \frac{\gamma(1, -R)}{2R} \right] \\ &= \frac{8L^2}{\pi^2} \left[\cos^2 \theta - \frac{(1 - e^{-R})}{2R} \cos(2\theta) \right] \end{aligned} \quad (46)$$

Equation (46) is recognized as the result obtained by Springett and Simon (Ref. 11) for the coherent hard limiter.

For the linear amplifier result we note that Eq. (40) reduces to

$$E(z^2)|_{D \rightarrow \infty} = \frac{2R^2 L^2 e^{-R}}{\pi D} \left\{ 2 \int_0^\infty y^3 \exp(-Ry^2) I_0(2Ry) dy \cos^2 \theta - \frac{1}{R} \int_0^\infty y^2 \exp(-Ry^2) I_1(2Ry) dy \cos(2\theta) \right\} \quad (47)$$

Again, from Gradshteyn and Ryzhik, we have the identities

$$\begin{aligned} \int_0^\infty x^\mu \exp(-\alpha x^2) I_\nu(\beta x) dx &= \frac{\beta^\nu \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\right) {}_1F_1\left(\frac{\nu + \mu + 1}{2}; \nu + 1; \frac{\beta^2}{4\alpha}\right)}{2^{\nu+1} \alpha^{1/2} (\mu + \nu + 1) \Gamma(\nu + 1)} \\ \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\mu + \nu) > -1 \end{aligned} \quad (48)$$

and

$${}_1F_1(2; 1, x) = {}_1F_1(1; 1, x) + x \times {}_1F_1(2, 2; x) = e^x + xe^x \quad (49)$$

where ${}_1F_1(a, c; x)$ is the confluent hypergeometric function. With these results and the identity of Eq. (43) we see that Eq. (47) becomes

$$E(z^2)|_{D \rightarrow \infty} = \frac{2L^2}{\pi D} \left[(1 + R) \cos^2 \theta - \frac{1}{2} \cos(2\theta) \right] \quad (50)$$

Now, note from the definitions of D , β , and R that D can be expressed as

$$D = \frac{2L^2}{\pi K^2 \sigma^2} \quad (51)$$

If we fix K and σ^2 then $D \rightarrow \infty$ implies $L \rightarrow \infty$. Thus, we have

$$\begin{aligned} E(z^2)|_{D \rightarrow \infty} &= K^2 A^2 \left[\cos^2 \theta + \frac{1}{2R} \right] \\ &= K^2 A^2 \cos^2 \theta + \frac{K^2 \sigma^2}{2} \end{aligned} \quad (52)$$

Therefore, we see that as $D \rightarrow \infty$ the second moment of z reduces to the second moment of the amplified signal and noise terms after projection onto the coherent reference axis, which is exactly as expected.

V. Signal-to-Noise Power Ratios

Armed with the first two moments of z we are now in a position to consider signal-to-noise power ratios. The output signal-to-noise ratio when the detection angle θ is zero is given by

$$(SNR)_o|_{\theta=0} = \frac{\mu_z^2}{E(z^2) - \mu_z^2} \quad (53)$$

and if one is interested in the ratio of the output signal-to-noise ratio to that of the input we have

$$\frac{(SNR)_o}{2(SNR)_i} \Big|_{\theta=0} = \frac{1}{2} \times \frac{\mu_z^2}{R[E(z^2) - \mu_z^2]} \quad (54)$$

where the factor of $1/2$ is needed to account for the band-pass to lowpass transformation. Note, however, that Eq. (53) becomes meaningless when the detection angle is $\pi/2$; i.e., the limiter is followed by a phase detector. Springett and Simon (Ref. 11) handled this case by redefining the mean signal output by

$$\mu_z\left(\frac{\pi}{2}\right) = E\left\{\frac{dz(\theta)}{d\theta}\right\}_{\theta=\pi/2} = -\mu_z|_{\theta=0}; \quad \text{if } \theta = \frac{\pi}{2} \quad (55)$$

and then using $\mu_z(\pi/2)$ instead of μ_z in the numerators of Eqs. (53) and (54). Yet, even if we adopt this additional definition, it is still not immediately clear how we would handle cases involving other detection angles.

We can alleviate all of the problems associated with output signal definitions for various detection angles if we define the ratio of output-to-input signal-to-noise ratios directly. To this extent we shall define the ratio of signal-to-noise ratios for any value of θ by

$$\frac{(SNR)_o}{2(SNR)_i} \triangleq \frac{\mu_z^2}{R'[E(z^2) - \mu_z^2]} \quad (56)$$

where

$$\frac{\mu_z^2}{E(z^2) - \mu_z^2}$$

is the "actual" output signal-to-noise ratio and R' is the signal-to-noise ratio at the input after projection onto the coherent reference axis. (Equation 56 can also be interpreted as the ratio of the output SNR using the limiter to the corresponding output SNR with the limiter removed.)

Clearly, we have that

$$R' = \frac{A^2 \cos^2 \theta}{\frac{\sigma^2}{2}} = 2R \cos^2 \theta \quad (57)$$

so that

$$\frac{(SNR)_o}{2(SNR)_i} = \frac{\mu_z^2|_{\theta=0}}{2R[E(z^2) - \mu_z^2]}; \quad \text{for any } \theta \quad (58)$$

Note that when $\theta = 0$ we obtain Eq. (54) directly. Furthermore, if $\theta = \pi/2$, Eq. (58) reduces to the right-hand side of Eq. (54) with the numerator replaced by Eq. (55).

We see from the expressions for the first two moments of z that the ratio of SNRs depends on R , θ , and the softness parameter D . In other words

$$\frac{(SNR)_o}{2(SNR)_i} = f_1(R, \theta, D) \quad (59)$$

However, from Eq (24) we know that D can be written in terms of the second softness parameter β ; to that

$$\frac{(SNR)_o}{2(SNR)_i} = f_2(R, \theta, \beta) \quad (60)$$

Thus, for a fixed value of θ , the ratio $(SNR)_o/2(SNR)_i$ will be a family of functions of R in terms of the parameters β or D . Let us first consider the parameter β

A. SNRs and the Parameter β

Figures 3-5 illustrate the behavior of $(SNR)_o/2(SNR)_i$ for detection angles 0, $\pi/4$, and $\pi/2$, respectively, for several values of β . We note first that for $\beta = 0$ (hard limiter) we obtain (Figs. 3 and 5), precisely the results reported by Springett and Simon (Ref. 11). Furthermore, we see from Fig. 4 with $\beta = 0$ that our result is the same as Davenport's incoherent limiter result (Ref. 1). The reason for this is that for a detection angle of $\pi/4$ the coherent detector extracts equal percentages of signal power and noise power, and hence neither is favored by the detector. Finally, we observe that for small input signal-to-noise ratios all of the curves in Figs. 3 to 5 approach Davenport's lower asymptotic limit of $\pi/4$.

The interpretation of these results is as follows: Recall the definition of β as given in Eq. (21). If we assume that the physical limiter parameters L and K are fixed, then fixing β at some constant value establishes the rms signal

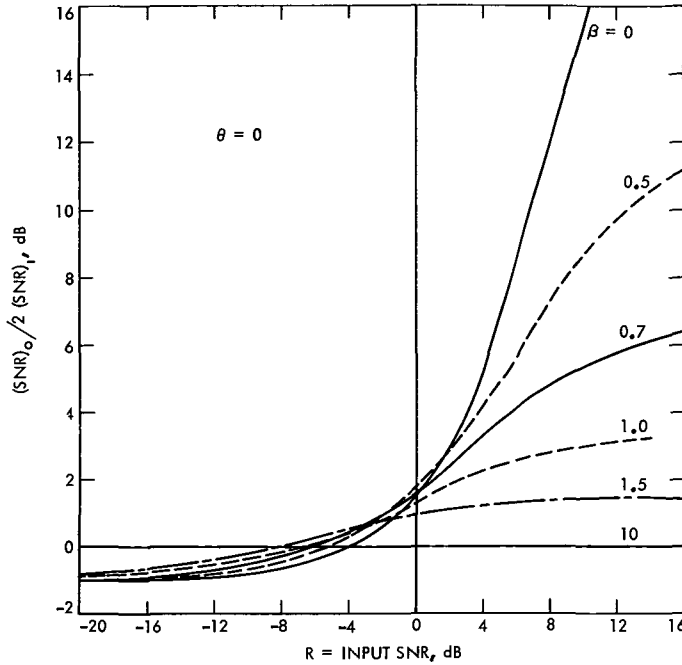


Fig. 3. Ratio of signal-to-noise ratios, parameter β , $\theta = 0$

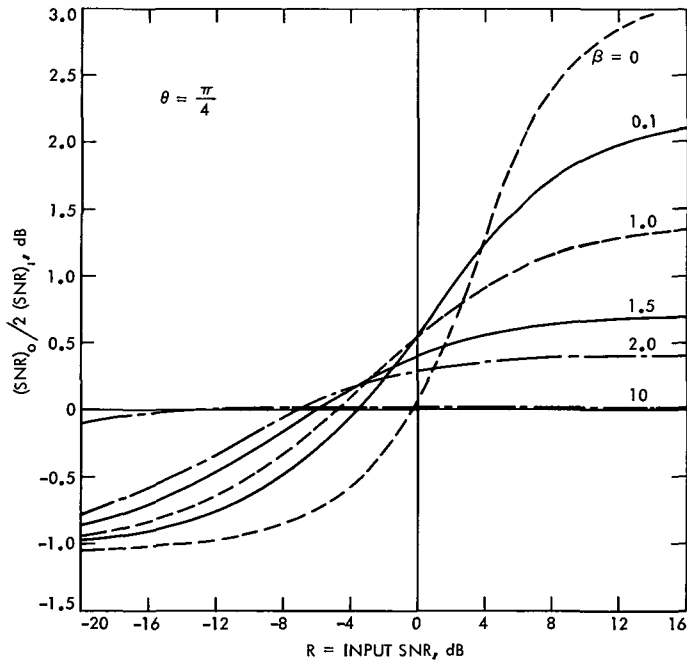


Fig. 4. Ratio of signal-to-noise ratios, parameter β , $\theta = \pi/4$

level at some fixed value. This is precisely what happens if we precede the limiter by an automatic gain control. Hence, we know that a variation in the input SNR corresponds to a variation of the noise level only. Thus, allowing the input SNR to approach zero implies an unbounded increase in the input noise level, and hence the soft limiter

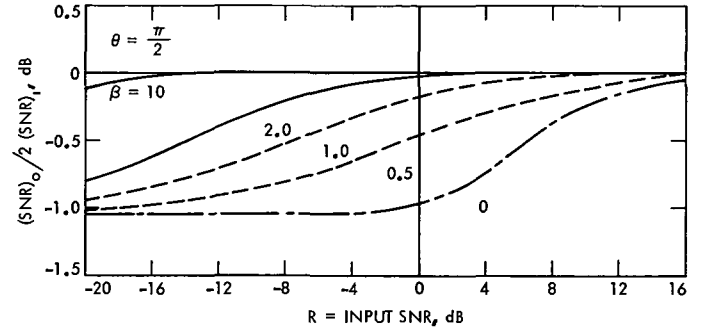


Fig. 5. Ratio of signal-to-noise ratios, parameter β , $\theta = \pi/2$

will always approach the hard limiter, regardless of the value of β . Conversely, an increasing input SNR corresponds to a decrease in the input noise level, and hence the limiter tends to "soften." This softening continues until the input noise level is insignificant relative to the projected signal level. Thus, the softening will be terminated at some input SNR (which depends on β) for all detection angles other than $\pi/2$. For $\theta = \pi/2$ we note that the projected signal level is zero, and the softening continues indefinitely.

B. SNRs and the Parameter D

We have already seen the utility of the parameter D in describing the characteristics of a soft limiter. Another place where this quantity appears is in the paper by Baum (Ref. 2), in which he found that the output autocorrelation function $\psi(\tau)$ of an error function (incoherent) limiter when driven by gaussian noise only was given by

$$\psi(\tau) = \frac{2L^2}{\pi} \sin^{-1} \left[\frac{\rho(\tau)}{1 + D} \right] \quad (61)$$

where $\rho(\tau)$ is the autocorrelation function of the input noise process. We will see now that D has a profound effect with regard to signal-to-noise ratios.

In Figs. 6 to 8 the behavior of $(SNR)_o/2 (SNR)_i$, is again illustrated for detection angles 0 , $\pi/4$, and $\pi/2$, respectively, except this time for constant values of D . We note first that the hard limiter result ($D \rightarrow 0$) still agrees with the previous results. However, we see that there are two significant differences between this set of curves and the previous set. Note that as the input SNR decreases, the curves no longer all approach Davenport's lower limit of $\pi/4$. Instead they approach a continuum of values from $\pi/4$ to 1. Secondly, we see that for increasing input SNRs all of the curves have a tendency to converge to their corresponding hard limiter result. These characteristics can

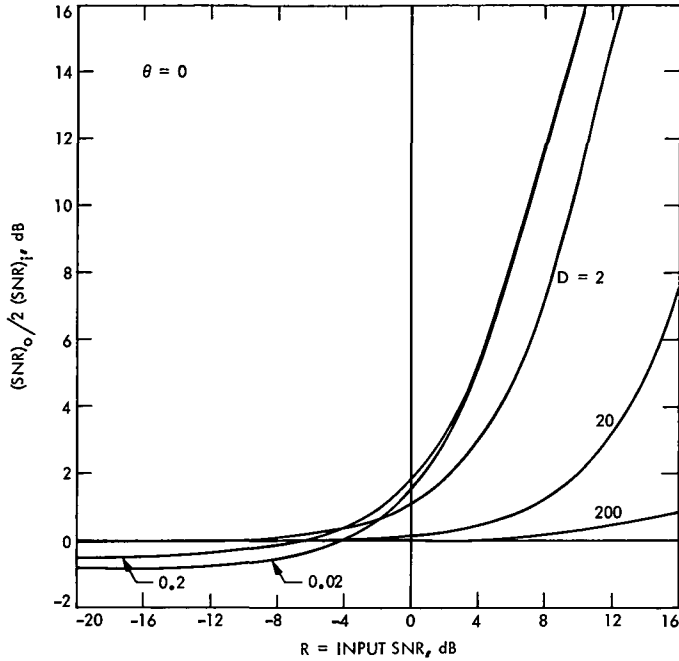


Fig. 6. Ratio of signal-to-noise ratios, parameter D , $\theta = 0$

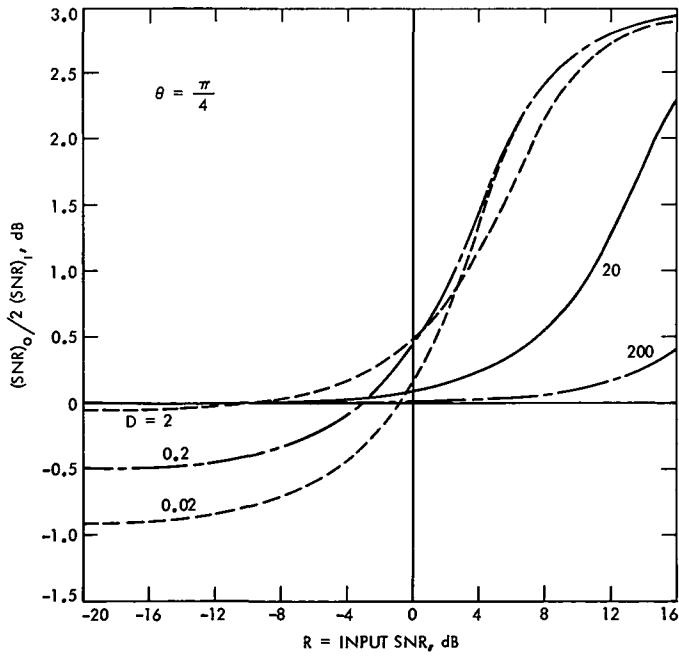


Fig. 7. Ratio of signal-to-noise ratios, parameter D , $\theta = \pi/4$

be easily explained by a closer examination of D . We note that

$$D = \frac{4R\beta^2}{\pi} = \frac{2L^2}{\pi K^2 \sigma^2} \quad (62)$$

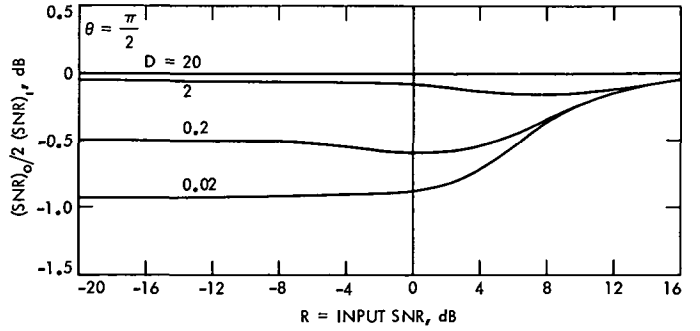


Fig. 8. Ratio of signal-to-noise ratios, parameter D , $\theta = \pi/2$

Thus, D can be considered (except for the factor $2/\pi$) as the ratio of the output variance of a hard limiter with limit L to the variance of the input noise multiplied by K . An alternate interpretation, due to Baum (Ref. 2), can be obtained by noting that the error function characteristic can be rewritten as

$$y(x) = L \operatorname{erf}\left(\frac{K\sqrt{\pi}}{2L}x\right) = K \int_0^x \exp\left(-\frac{\mu^2}{2\sigma_0^2}\right) d\mu \quad (63)$$

where σ_0^2 is the "variance" of the error function characteristic. In this case we have

$$D = \frac{\sigma_0^2}{\sigma^2} \quad (64)$$

so that D is the ratio of the variance of the error function characteristic to the input variance.

No matter which interpretation is used for D , it is immediately clear that, for fixed values of L and K , if we establish a value for D we have, in turn, fixed the input noise variance. Thus, variations in the input signal-to-noise ratio correspond to variations of the signal level only. This means that as the input SNR increases, the input signal will drive the limiter into saturation, causing the limiter to "harden." Conversely, as the input SNR decreases the limiter will continue to "soften" until the point at which the signal level becomes insignificant relative to the input noise level. From this point on, no further softening will occur, so that the asymptotic softness of the limiter will depend on D .

It would appear that these latest characteristics could be exploited to enhance the process of coherent detection. When using such detectors it is necessary to limit the input signal so that the detector (mixer) is not driven past

its allowable input range. This limiting usually is implemented as a hard bandpass limiter. Yet we know that hard limiters produce a degradation of about 1 dB in the output SNR at low input SNRs. Notice, however, from Figs. 6 to 8 that for D larger than about 1.0 this low SNR degradation is all but eliminated. Let us, for example, consider the case where $\theta = 0$. Figure 9 shows the SNR curves for $D = 0.5, 1.0$, and 1.5 . From this we see that for D in the vicinity of 1.0 (that is, when the noise alone is on the verge of driving the limiter into saturation), the low input SNR degradation is effectively eliminated with only a slight change in the large SNR enhancement. Note, also, that the point at which enhancement $((SNR)_o \geq 2 (SNR)_i)$ begins is at an input SNR of -8 dB as compared to -4 dB for the hard limiter case. This suggests that properly adjusted soft limiters might prove useful in the area of threshold extension.

VI. Some Comments on Signal-to-Noise Spectral Densities

Thus far we have considered only systems where the coherent detector is followed by a lowpass filter having a bandwidth equal to the output zeroth zone spectral width. In most applications, however, the detector is followed by a lowpass filter with a bandwidth much smaller than the zonal bandwidth. For cases such as these we must determine the detector output noise spectral density at zero frequency.

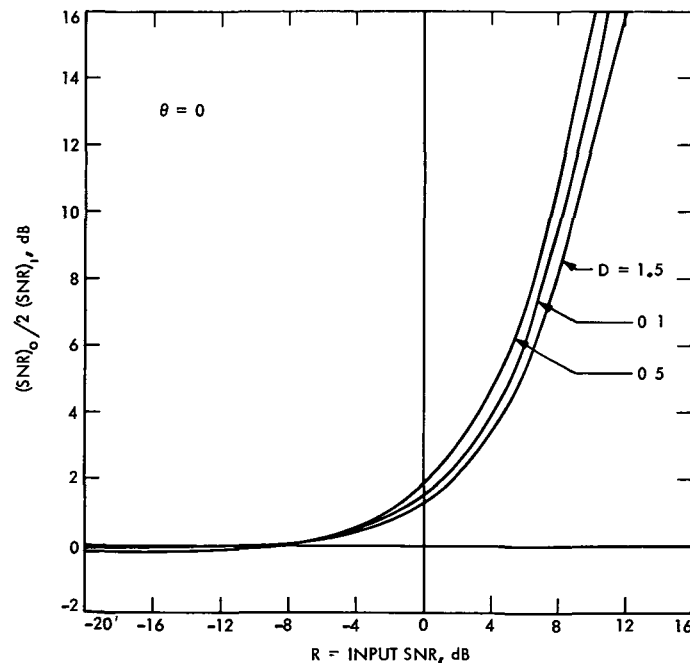


Fig. 9. Ratio of signal-to-noise ratios, $D = 0.5, 1.0, 1.5$, $\theta = 0$

The usual technique for obtaining the output noise density is to use Rice's characteristic function method (Ref. 15). This method entails computing the output autocorrelation function $R_z(\tau)$ given by

$$R_z(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{1}{4\pi^2} \int_{-\infty}^{\infty} f(j\mu, t) \exp\left(-\frac{\sigma^2 \mu^2}{2}\right) \times \int_{-\infty}^{\infty} f(j\nu, t + \tau) \exp\left(-\frac{\sigma^2 \nu^2}{2}\right) \exp(-\sigma^2 \rho(\tau) \mu \nu) \times \exp[j\mu x_s(t) + j\nu x_s(t + \tau)] du dv dt \quad (65)$$

where $x_s(t)$ is the noise-free input signal at time t , $\rho(\tau)$ is the autocorrelation function of the input narrowband noise process, and $f(j\mu, t)$ is the Fourier transform of the coherent soft limiter characteristic given by

$$f(j\mu, t) = L \sqrt{2} \cos(\omega_0 t + \theta) \int_{-\infty}^{\infty} \operatorname{erf}\left(\frac{K \sqrt{\pi}}{2L} x\right) e^{-j\mu x} dx = \frac{2L \sqrt{2} \exp\left(-\frac{L^2 \mu^2}{K^2 \pi}\right)}{j\mu} \cos(\omega_0 t + \theta) \quad (66)$$

By expanding all of the quantities involving trigonometric expressions, including $\rho(\tau)$ since the input noise is narrowband, and retaining only those components in the vicinity of $\tau = 0$, the output zeroth zone autocorrelation function is obtained. Unfortunately, the resulting expression involves a doubly infinite series of confluent hypergeometric functions and does not appear to have a more manageable reduced form. The result (for those interested in pursuing the topic further) is given by

$$R_z(\tau)|_{\text{zone 0}} = \mu_z^2 + \sum_{n=0}^{\infty} [\rho_L(\tau)]^{2n+1} \left\{ \frac{g_{1-2m, 1-2m}^2}{n! (n+1)!} + \frac{(1 + 2 \cos^2 \theta) f_{3-2m, 2-2m}^2}{[(n+1)!]^2} \rho_L(\tau) + \sum_{m=0}^n \left[\frac{g_{1,1}^2 + g_{3,3}^2 - 2g_{1,1} g_{3,3} \cos(2\theta)}{(n-m)! (n-m+1)!} + \frac{f_{3,2}^2 + f_{5,4}^2 - 2f_{3,2} f_{5,4} \cos(2\theta)}{(n-m)! (n-m+2)!} \rho_L(\tau) \right] \right\} \quad (67)$$

where

$$g_{k,l} \triangleq \frac{R^{(2m+k-1)/2} \Gamma\left(\frac{2n+2m+k}{2}\right) {}_1F_1\left(\frac{2n+2m+k}{2}; 2m+l, \frac{-R}{1+D}\right)}{(2m+k-1)!(1+D)^{(2n+2m+k)/2}} \quad (68)$$

$$f_{k,l} = \frac{(2m+k-1)}{\sqrt{R}} g_{k,l} \quad (69)$$

and $\rho_L(\tau)$ is the equivalent lowpass autocorrelation function of the input narrowband noise. The output noise spectral density at zero frequency is then given by

$$S_z(0) = \int_{-\infty}^{\infty} [R_z(\tau)|_{\text{zone 0}} - \mu_z^2] d\tau \quad (70)$$

which is an even more complicated expression.

The evaluation of Eq (70) appears to be an insurmountable task. However, if we restrict our attention to the ratio of output-to-input signal-to-noise spectral densities we can approximate this ratio quite well by using the ratio of signal-to-noise power ratios. To see this, let us define $1/\Gamma$ to be the ratio of signal-to-noise density ratios. Then it is clear that

$$\frac{1}{\Gamma} \triangleq \frac{(S/N_o)_o}{(S/N_o)_i} = \frac{(SNR)_o}{2(SNR)_i} \times \frac{B_o}{B_i} \quad (71)$$

where B_o is the coherent detector zeroth zone output bandwidth. In the paper by Springett and Simon it is shown that this bandwidth ratio for the hard limiter (when noise only is present) is bounded by

$$1 < \frac{B_o}{B_i} < \frac{4}{\pi} \quad (72)$$

and that B_o/B_i goes to unity as the input SNR increases. But we know that for a linear amplifier, which is the opposite extreme case of a soft limiter,

$$\frac{B_o}{B_i} \equiv 1 \quad (73)$$

for all input SNRs. From this we see that the bandwidth ratio for the soft limiter is "squeezed" between two curves: one which is identically 1 and the other which is approximately 1 at low input SNRs and asymptotically equal to 1 at high SNR. Thus we have for the soft limiter

$$\frac{1}{\Gamma} \approx \frac{(SNR)_o}{2(SNR)_i} \quad (74)$$

for all values of input SNR.

As a final comment it would appear that an even better approximation to $1/\Gamma$ might be obtained if we assume that the soft limiter, and consequently B_o/B_i , can be represented as a convex combination of their two extreme cases (i.e., hard limiter and linear amplifier). Thus, if $0 \leq \xi \leq 1$ then we have

$$\left(\frac{B_o}{B_i}\right)_{\text{soft limiter}} = \xi \left(\frac{B_o}{B_i}\right)_{\text{hard limiter}} + (1 - \xi) \times 1 \quad (75)$$

where we recall that $B_o/B_i \equiv 1$ for a linear amplifier. Based on our previous results, the most logical choice for the convex parameter ξ is

$$\xi = \frac{1}{1+D} \quad (76)$$

so that

$$\left(\frac{B_o}{B_i}\right)_{\text{soft limiter}} = \frac{1}{1+D} \left(\frac{B_o}{B_i}\right)_{\text{hard limiter}} + \frac{D}{1+D} \quad (77)$$

An experimentally determined expression for the bandwidth ratio of the hard limiter is given in the Springett and Simon paper (Ref. 11).

References

1. Davenport, W. B., Jr., "Signal to Noise Ratios in Bandpass Limiters," *J. Appl. Phys.*, Vol. 24, No. 6, pp. 720-727, June 1953.
2. Baum, R. F., "The Correlation Function of Smoothly Limited Gaussian Noise," *IRE Trans. Inform. Theory*, Vol. IT-3, pp. 193-197, September 1957.
3. Galejs, J., "Signal to Noise Ratios in Smooth Limiters," *IRE Trans. Inform. Theory*, Vol. IT-5, pp. 79-85, June 1959.
4. Blachman, N. M., "The Correlation Function of Gaussian Noise After Error Function Limiting," *J. Electron. Control*, Vol. 16, pp. 509-511, 1964.
5. Van Vleck, J. H., and Middleton, D., "The Spectrum of Clipped Noise," *Proc. IEEE*, Vol. 54, No. 1, pp. 2-19, January 1966.
6. Rauch, S., and Fukada, M., "Output Correlation Function of a Biased Full Wave Limiter," *Int. J. Control*, Vol. 4, No. 3, pp. 241-245, 1966.
7. Rauch, S., and Fukada, M., "Output Signal to Noise Ratio of an Error Function Limiter," *Int. J. Control*, Vol. 5, No. 1, pp. 47-54, 1967.
8. Lee, J. S., "Signal to Crosstalk Power Ratio in Smoothly Limited Multichannel FDM Signals," *IEEE Trans. Comm. Tech.*, Vol. COM-16, No. 1, pp. 63-67, February 1968.
9. Tausworthe, R. C., "Analysis of Narrowband Signals Through the Bandpass Soft Limiter," in *Supporting Research and Advanced Development*, Space Programs Summary 37-53, Vol. III, Jet Propulsion Laboratory, Pasadena, Calif., pp. 209-214, October 31, 1968.
10. Tausworthe, R. C., "Limiters in Phase Locked Loops: A Correction to Previous Theory," in *Supporting Research and Advanced Development*, Space Programs Summary 37-54, Vol. III, Jet Propulsion Laboratory, Pasadena, Calif., pp. 201-204, December 31, 1968.
11. Springgett, J. C., and Simon, M. K., "An Analysis of the Phase Coherent-Incoherent Output of the Bandpass Limiter," *IEEE Trans. Commun. Tech.*, Vol. COM-19, No. 1, pp. 42-49, February 1971.
12. Hsu, J. C., and Meyer, A. U., *Modern Control Principles and Applications*, McGraw-Hill Book Co. Inc., New York, 1968.
13. Lindsey, W. C., "Performance of Phase-Coherent Receivers Preceded by Bandpass Limiters," *IEEE Trans. Commun. Tech.*, Vol. COM-16, No. 2, pp. 245-251, April 1968.
14. Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series, and Products*, Translated by A. Jeffrey. Academic Press, New York, 1965.
15. Rice, S. O., "Mathematical Analysis of Random Noise," in *Selected Papers on Noise and Stochastic Processes*, N. Wax, Ed. Dover Publications, Inc., New York, 1954.