The University of Connecticut

SCHOOL OF ENGINEERING

Storrs, Connecticut 06268

Department of Electrical Engineering

INPUT FREQUENCY REQUIREMENTS FOR *IDENTIFICATION THROUGH LIAPUNOV METHODS*

C. H. Kim
D. P. Lindorff

Technical Report 73-5

July 1973

This work was sponsored by
NASA under Grant NGL07-002-002, and
NSF under Grant GK 37841.

I. Introduction

In this paper a theorem is derived which specifies a sufficient number of input frequencies to guarantee identification of an unknown noise free linear plant. Similar work has been reported by Lion [1], Carroll [2], and Hoberock and Stewart [3], in which various classes of systems were considered. The results in this paper, although essentially in agreement with previous work, are in some respects more general. Since all of the referenced work relates to sufficient conditions for nulling of the parameter error vector, it is to be expected that various conditions on the system to be identified will have been imposed. In this paper it is found that some of these conditions appear to be necessary while others do not. The main contribution is to provide a theorem which considers the effect of unknown parameters in the state equation upon the frequency requirements.

Results are obtained for a linear multi-input system of known structure in which some or all of the coefficients of the state equation are unknown. As in [1],[2], an identification algorithm is assumed by which convergence of the parameter error vector is guaranteed, and requirements on the input frequencies are derived such that convergence to the null is assured. In [1] it was specified that the plant transfer function has a single input, and contains no phase shifts of integral multiples of π radians at the input frequencies. In [2] observability and controllability, but no phase shift requirements were imposed. In [4], conditions for identifiability of the multi-input plant were studied without regard to a specific identification algorithm. The conditions imposed on the plant were rather stringent, however, in that no eigenvalues on the imaginary axis were permitted.

In the present work the identification scheme outlined by Kudva and Narendra [4] is used as a model. However, the theorem is applicable to the model following adaptive control problems and adaptive observers as well.

Simulation results are presented which demonstrate certain interesting aspects of the theorem. It is seen that the sufficient number of input frequencies can be quite small relative to the system order, depending upon the location and number of the unknown coefficients of the state equation. It is further demonstrated that convergence of the parameter error vector to the null may not occur when the number of frequencies fails to satisfy the theorem. Although certain phase shift requirements appear in the theorem, as in [1], it is shown that convergence to the null may in fact occur when these requirements are violated. Finally, it is demonstrated that the assumption of controllability made in the present and referenced theorems is not a necessary condition.

II. Problem Formulation

The mathematical expressions of plant and model are described as follows:

$$\text{PLANT:} \quad \dot{x} = A^{*}x + B^{*}u, \; (A^{*}, B^{*} \text{ constant, unknown}) \qquad (1)$$

$$\text{MODEL:} \quad \dot{x}_m = Ax + C\,[x_m - x] + Bu \qquad (2)$$

where $A, A^{*} \varepsilon R^{nxn}$; $B, B^{*} \varepsilon R^{nxm}$; $x_m, x \varepsilon R^n$; $u \varepsilon R^m$; and C is any stability matrix. It is assumed that all the plant states are measurable and the dimensions of A^{*} and B^{*} are also known. The purpose here is to identify the plant by adjusting the parameters of A and B.

Now defining the tracking error as

$$e \overset{\Delta}{=} x_m - x \qquad (3)$$

the following error equation is obtained:

$$\dot{e} = Ce + \Phi x + \Psi u \tag{4}$$

where the parameter errors are defined by the matrices

$$\Phi = A - A^*$$

$$\Psi = B - B^*$$

Following the procedure shown in [4], it is possible to obtain a Liapunov function which is a positive definite quadratic form in tracking and parameter error space,

$$V = \frac{1}{2}\left(e^T Pe + \sum_{i=1}^{n} \phi_i^T \phi_i + \sum_{i=1}^{m} \psi_i^T \psi_i\right) \tag{5}$$

where

$P = P^T$ is positive definite

$\phi_i \overset{\Delta}{=} i^{th}$ column of Φ

$\psi_i \overset{\Delta}{=} i^{th}$ column of Ψ

The time derivative is in turn given by

$$\dot{V} = -e^T Qe + e^T Pf + \sum_{i=1}^{n} \dot{\phi}_i^T \phi_i + \dot{\psi}_i^T \psi_i \tag{6}$$

where

$-Q \overset{\Delta}{=} (C^T P + PC)/2 \; , \quad Q = Q^T$

$f \overset{\Delta}{=} \Phi x + \psi u$

and Q is any positive definite symetric matrix. If the adaptive law[1] is formulated as

$$\begin{aligned} \dot{\phi}_i^T &= -e^T Px_i \; , \quad i = 1,2,\ldots n \; , \\ \dot{\psi}_i^T &= -e^T Pu_i \; , \quad i = 1,2,\ldots m \; . \end{aligned} \tag{7}$$

---

[1] In this report the problem is formualted as one of identification. However, the results can also be applied to the adaptive control problem as outlined in [5]. The term "adaptive system" will be used to imply either "model reference adaptive control" or "parameter identification".

then (6) becomes

$$\dot{V} = -e^T Qe \tag{9}$$

which is seen to be semi-negative definite in the e, Φ, Ψ space. (5) and (9) show that there is an uncertainty as to the asymptotic behavior of the Φ and Ψ matrices, in the sense that, although Φ, Ψ are assured of attaining constant values, there is no guarantee that they will approach the null.

It is shown in the next section that if a certain class of inputs is applied, then the convergence of Φ, Ψ to the null can be guaranteed.

III. Main Result (Single Input Case)

As a solution to this problem, the following theorem is established.

Theorem:

In a single input adaptive system, the convergence of the Φ, Ψ matrices of (4) to the null is guaranteed if

1) the input is periodic and contains at least q distinct frequencies where $q \overset{\Delta}{=} \text{N.U.I.}[R/2]$ and $R \leq n + 1$,[2,3]

2) there exists at least one x_j in which none of the q distinct frequencies encounters a phase shift of $K\pi/2$ radians, where K is any integer,

3) for all non-zero columns of Φ in (10) the corresponding states are excited and linearly independent in the steady state[4] at the input frequencies.

Proof:

Since Liapunov adaptation quarantees asymptotic stability of the equilibrium in e, then for $t \to \infty$, $e \equiv \dot{e} \equiv 0$. Thus (4) in the steady state becomes

$$\Phi x + \Psi u = 0 \tag{10}$$

[2] N.U.I. $\overset{\Delta}{=}$ NEAREST UPPER INTEGER.

[3] Let R_i = the sum of number of nonzero elements in i^{th} row of Φ, Ψ. Then $R = \text{U.B.}[R_i]$, $i = 1,2,\ldots n$.

[4] By this requirement, the system must also be controllable.

Assuming the i^{th} row of the above equation contains R nonzero elements, and letting R be equal to (n+1), the i^{th} equation in (10) is given by[5]

$$\sum_{j=1}^{n} \phi_{ij} x_j + \psi_i u = 0 \quad , \quad i = 1,2,\ldots n \tag{11}$$

where the input, u, is assumed to be comprised of q sinusoidal signals with distinct frequencies, namely

$$u = \sum_{\ell=1}^{q} u^{\ell} \tag{12}$$

where

$$q \overset{\Delta}{=} \text{N.U.I.}[R/2]$$

$$u^{\ell} \overset{\Delta}{=} \sin \omega_{\ell} t$$

Now applying the superposition property to (11), a set of q equations is obtained,

$$\sum_{j=1}^{n} \phi_{ij} x_j^{\ell} + \psi_i u^{\ell} = 0 \quad , \quad \ell = 1,2,\ldots\ldots q \tag{13}$$

where x^{ℓ} is the state response to u^{ℓ}. Since u^{ℓ} is a periodic function, and the state response, x^{ℓ}, is also periodic in the steady state, all the terms in (13) are periodic. Because of the nature of the periodic input, the time derivative of (13) yields another independent equation, namely

$$\sum_{j=1}^{n} \phi_{ij} \dot{x}_j^{\ell} + \psi_i \dot{u}^{\ell} = 0 \quad , \quad \ell = 1,2,\ldots.q \tag{14}$$

wherein, by Liapunov stability, $\dot{\phi}_{ij} \equiv \dot{\psi}_i \equiv 0$ in the steady state for all j. The 2q (i.e. n+1) equations represented by (13) and (14) can be written in matrix form,

$$[F(t)]\gamma_i = 0 \tag{15}$$

---

[5] The case of R < (n+1), which is a subset of the case of R = n + 1, is not treated explicitly in the proof because of the tedious indexing required. An example is provided to show the case of R < (n+1).

where

$$F(t) = \begin{bmatrix} x_1^1 & x_2^1 & \cdots & & x_n^1 & u^1 \\ \dot{x}_1^1 & \dot{x}_2^1 & \cdots & & \dot{x}_n^1 & \dot{u}^1 \\ \vdots & & & & \vdots & \vdots \\ x_1^k & x_2^k & \cdots & x_j^k \cdots & x_n^k & u^k \\ \dot{x}_1^k & \dot{x}_2^k & \cdots & \dot{x}_j^k \cdots & \dot{x}_n^k & \dot{u}^k \\ \vdots & & & & \vdots & \vdots \\ x_1^{\frac{n+1}{2}} & x_2^{\frac{n+1}{2}} & \cdots & & x_n^{\frac{n+1}{2}} & u^{\frac{n+1}{2}} \\ \dot{x}_1^{\frac{n+1}{2}} & \dot{x}_2^{\frac{n+1}{2}} & \cdots & & \dot{x}_n^{\frac{n+1}{2}} & \dot{u}^{\frac{n+1}{2}} \end{bmatrix}$$

in which

$$x_j^k = \left|K_j^k\ (j\omega_k)\right| \sin\ (\omega_k t + \alpha_j^k)$$

and the vector γ_i is defined by

$$\gamma_i \overset{\Delta}{=} [\phi_{i1}\phi_{i2}\ \cdots\ \phi_{in}\psi_i]^T.$$

Using theorem (5-3) in [6] it is to be proved that $\gamma_i \equiv 0$ for all t or, what is equivalent, that F(t) is a non-singular matrix for all t.

Comment: *Because F(t) is time varying it can be shown by example that linear independence of its rows (columns) need not infer linear independents of its columns (rows).*

From this comment it is seen that linear independence of both rows and columns must be established in order to establish that F(t) is non-singular for all t. Since, as can be readily demonstrated, F(t) in (15) is identical for all i=1,2,...n, once F(t) is proved to be nonsingular it follows that $\gamma_i \equiv 0$ for all i=1,2,...m.

The linear independence of the rows of F(t) will be discussed first. If F(t) consists of analytic functions on the interval $[t_1,t_2]$ then by Theorem (5-3) if the rank of the infinite matrix

$$\left[F(t_0) \vdots F^{(1)}(t_0) \vdots F(t_0)^{(2)} \vdots \cdots \vdots F^{(n-1)}(t_0) \vdots \cdots \right] \qquad (16)$$

is n+1, where $F^{(i)}(t)$ is the i^{th} derivative of F(t), F(t) is an (n+1)x(n+1) matrix, and t_0 is any fixed point in $[t_1,t_2]$, then the rows of F(t) are linearly independent for all $t \in [t_1,t_2]$.

Since F(t) in (15) consists of analytic functions, without loss of generality let $[t_1,t_2]$ be taken as $[0,\infty]$. Now setting $t_o=0$, and selecting the appropriate (n+1) columns[6] from (16), an (n+1)x(n+1) matrix can be formed which has the following factored form:

$$\begin{bmatrix} K_j^1 \sin \alpha_j^1 & & & & & \\ & K_j^1 \cos \alpha_j^1 & & & 0 & \\ & & K_j^2 \sin \alpha_j^2 & & & \\ & & & K_j^2 \cos \alpha_j^2 & & \\ & 0 & & & \ddots & \\ & & & & & K_j^{\frac{n+1}{2}} \cos \alpha_j^{\frac{n+1}{2}} \end{bmatrix} \text{X} \qquad (17)$$

$$\begin{bmatrix} 1 & -\omega_1^2 & \omega_1^4 & \cdots & -\omega_1^{2(n+1)} & (K_h^1 \sin \alpha_h^1 / K_j^1 \sin \alpha_j^1) \\ \omega_1 & -\omega_1^3 & \omega_1^5 & \cdots & \omega_1^{2n-1} & \omega_1 (K_h^1 \cos \alpha_h^1 / K_j^1 \cos \alpha_j^1) \\ 1 & -\omega_2^2 & \omega_2^4 & \cdots & -\omega_2^{2(n-1)} & (K_h^2 \sin \alpha_h^2 / K_j^2 \sin \alpha_j^2) \\ \omega_2 & -\omega_2^3 & \omega_2^5 & \cdots & \omega_2^{2n-1} & \omega_2 (K_h^2 \cos \alpha_h^2 / K_j^2 \cos \alpha_j^2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \omega_{\frac{n+1}{2}} & -\omega_{\frac{n+1}{2}}^3 & \omega_{\frac{n+1}{2}}^5 & \cdots & \omega_{\frac{n+1}{2}}^{2n-1} & \omega_{\frac{n+1}{2}} (K_h^{\frac{n+1}{2}} \cos \alpha_h^{\frac{n+1}{2}} / K_j^{\frac{n+1}{2}} \cos \alpha_j^{\frac{n+1}{2}}) \end{bmatrix}$$

---

[6] To begin with, pick j^{th} column of (16), where $j \leq n$, and then pick every $[j + k(n+1)]^{th}$ column where k=1,2,...n-1. Finally add any column (e.g. h^{th} col. $h \neq j$, $h \leq n+1$) other than j^{th} column within n^{th} column of (16) to form an (n+1)x(n+1) matrix (Note that the last column of the (n+1)x(n+1) matrix is formed by multiplying each element of the finally chosen column by $(K_h^\ell \sin \alpha_h^\ell)/(K_h^\ell \sin \alpha_h^\ell)$ and $(K_h^\ell \cos \alpha_h^\ell)/(K_h^\ell \cos \alpha_h^\ell)$ alternately.)

where

$$j \leq n \;,\quad h \neq j \;\text{ and }\; h \leq n{+}1$$

Theorem (5-3) is satisfied iff both matrices above are nonsingular. The first matrix is non-singular iff there exists at least one non-zero valued state variable, x_j in (15) such that $\alpha_j^k \neq K\pi/2$ for all k.

Due to its structure, only two cases need be considered to be assured that the second matrix is nonsingular. Namely,

1. two adjacent rows containing ω_i will be linearly independent iff neither of the following occurs:
   a. $\cos \alpha_h/\cos \alpha_j \neq \sin \alpha_h/\sin \alpha_j$
   b. neither $K_h = 0$ nor $K_j = 0$. This is equivalent to conditions (2) and (3) of the theorem.
2. Two nonadjacent rows will be linearly independent iff condition (1) of the theorem is satisfied.

Therefore, it is proven that all the rows of F(t) are linearly independent.

Now to prove linear independence of the columns of F(t), theorem (5-3) in [5] is applied to the transpose of F(t). Thus, if

$$\rho \left[F^T(0) \,\vdots\, F^{T^{(1)}}(0) \,\vdots\, F^{T^{(2)}} \,\vdots\, \cdots \,\vdots\, F^{T^{(n)}}(0) \,\vdots\, \cdots \right] = n{+}1$$

where $\rho[\cdot]$ is the rank of $[\cdot]$, and

$$F^{T^{(i)}}(0) = \frac{d^i}{dt^i} F^T(t) \Big|_{t=0}$$

then the columns of F(t) are linearly independent. If it can be shown that $\rho[F^T(0)] = n{+}1$, then the proof will be completed.

Noting

$$F^T(0) = \begin{bmatrix} K_1^1 \sin\alpha_1^1 & K_2^1 \sin\alpha_2^1 & \cdots & K_n^1 \sin\alpha_n^1 & 0 \\ \omega_1 K_1^1 \cos\alpha_1^1 & \omega_1 K_2^1 \cos\alpha_2^1 & \cdots & \omega_1 K_n^1 \cos\alpha_n^1 & \omega_1 \\ \vdots & & & & \vdots \\ \omega_{\frac{n+1}{2}} K^{\frac{n+1}{2}} \cos\alpha_1^{\frac{n+1}{2}} & \omega_{\frac{n+1}{2}} K_2^{\frac{n+1}{2}} \cos\alpha_2^{\frac{n+1}{2}} & \cdots & \omega_{\frac{n+1}{2}} K_n^{\frac{n+1}{2}} \cos\alpha_n^{\frac{n+1}{2}} & \omega_{\frac{n+1}{2}} \end{bmatrix}^T$$

and defining

$$F^T(0) \overset{\Delta}{=} [\underline{f}_1 \vdots \underline{f}_2 \vdots \cdots \underline{f}_i \cdots \underline{f}_n \vdots \cdots \underline{u}_n]^T,$$

it is seen from (15) that

$$\underline{f}_i = [x_i^1 \; \dot{x}_i^1 \; x_i^2 \; \dot{x}_i^2 \; \cdots \; \dot{x}_i^{\frac{n+1}{2}}]^T \Big|_{t=0}$$

$$\overset{\Delta}{=} [f_{1i} \; f_{2i} \; \cdots \; f_{ni} \; f_{n+1\; i}]^T$$

and

$$\underline{u} = [0 \; \omega_1 \; 0 \; \omega_2 \; \cdots \; \omega_{\frac{n+1}{2}}]^T$$

The proof will be done by contradiction. Thus assuming that $\rho[F^T(0)] < n+1$, it can be stated that

$$\alpha_1\underline{f}_1 + \alpha_2\underline{f}_2 + \cdots + \alpha_n\underline{f}_n + \alpha_{n+1}\underline{u} = 0 \qquad (18)$$

for $\underline{\alpha} \neq 0$, where $\underline{\alpha} = [\alpha_1\alpha_2 \cdots \alpha_{n+1}]^T$.

If (18) is rewritten as

$$\alpha_1\underline{f}_1 + \alpha_2\underline{f}_2 + \cdots + \alpha_n\underline{f}_n = -\alpha_{n+1}\underline{u}, \qquad (19)$$

then there exists an i^{th} row of (19), where i is an odd integer less than n+1 such that

$$\alpha_1 f_{i1} + \alpha_2 f_{i2} + \cdots + \alpha_n f_{in} = 0 . \qquad (20)$$

However, since the $(i+1)^{th}$ row was obtained by taking the derivative of the i^{th} row, it follows that

$$\alpha_1 \dot{f}_{i+1,1} + \alpha_2 \dot{f}_{i+1,2} + \ldots + \alpha_n \dot{f}_{i+1,n} = \alpha_1 f_{i1} + \alpha_2 f_{i2} + \ldots + \alpha_n f_{in} \tag{21}$$

To satisfy (21), $\alpha_{n+1} = 0$. Hence (18) becomes

$$\alpha_1 \underline{f}_1 + \alpha_2 \underline{f}_2 + \ldots + \alpha_n \underline{f}_n = 0 \quad \text{for } \underline{\alpha} \neq 0. \tag{22}$$

Since all the states are assumed to be linearly independent, (22) shows a contradiction. Therefore, it is proven that all the columns of $F(0)$ are linearly independent and $\rho[F^T(0)] = n+1$. Thus, by the comment in Sec.III it follows that the time varying matrix, $F(t)$, is nonsingular for all $t \in [0, \infty]$. //

Considering a multi-input system having m inputs, (10) becomes

$$\Phi x + \Psi u = 0 \tag{23}$$

where Φ, Ψ, x, u are defined in II, and (11) becomes

$$\sum_{j=1}^{n} \phi_{ij} x_j + \sum_{k=1}^{m} \psi_{ik} u_k = 0 \tag{24}$$

where ϕ_{ij} and ψ_{ik} are the elements of Φ and Ψ matrices respectively. Applying the superposition property to (23),

$$\sum_{j=1}^{n} \phi_{ij} x_j + \psi_{ik} u_k = 0 \quad , \quad \begin{matrix} i=1,2,\ldots n \\ k=1,2,\ldots n \end{matrix} \tag{25}$$

(24) is basically similar to (11), and an extension of the previous theorem to multi-input system follows.

Corollary:

In multi-input adaptive systems, the convergence of Φ, Ψ matrices to the null is guaranteed if

1. each input contains at least q distinct frequencies, where

   $q = \text{N.U.I.}[R/2]$, in which

   $R \overset{\Delta}{=} \text{U.B.}[R_i^k]$

   $R_i^k \overset{\Delta}{=}$ number of nonzero terms in (25), $i=1,2,\ldots n$, $k=1,2,\ldots m$.

2. conditions 2) and 3) of the main theorem are satisfied with respect to each input.[7]

Proof:

When extended to the multi-input case, (11) takes the form of (24). However, due to the multi-input feature, it follows that each input should have a different set of q distinct frequencies so that superposition property can be applied to each term of $\sum_{k=1}^{m} \psi_i ku_k$ in (24) separately, thereby guaranteeing that each $\psi_i \to 0$.

Assuming that each u_k, $k=1,2,\ldots m$, contains the required number of distinct frequencies, u_k may be written as follows:

$$u_k = \sum_{\ell=1}^{q} u_k^\ell \tag{26}$$

Here again, using the superposition property, we have

$$\sum_{j=1}^{n} \phi_{\hat{i}j} x_j^\ell + \psi_{\hat{i}k} u_k = 0 \tag{27}$$

where $\ell=1,2,\ldots,q$,

and the $\hat{i}^{th}$ row is the row of the maximum number of nonzero terms.

(27) is identical to (13) and it has R terms where R is defined in the corollary. Using the same procedure as in the case of the single input system, the convergence of $\psi_{\hat{i}\hat{k}}$ and $\phi_{\hat{i}j}$, $j \in R$, can be proved. Consequently $[\gamma_i] \equiv 0$ for all i.//

[7]The corollary can be extended as in [2] to require only that [A,B] be controllable.

In order to interpret the above results and notations used in the preceding development, an illustrative example will be worked out. However, it is noted that application of the theorem can be made directly upon determination of R.

Example:

Consider a fourth order system with two inputs. Assume from (23) that the Φ, Ψ matrices take the following forms.

$$\Phi = \begin{bmatrix} \phi_{11} & 0 & 0 & 0 \\ \phi_{21} & \phi_{22} & 0 & 0 \\ 0 & 0 & 0 & \phi_{34} \\ \phi_{41} & \phi_{42} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \\ 0 & \Psi_{32} \\ 0 & 0 \end{bmatrix}$$

From (25) and the condition (1) of the corollary, we have

$$R = U.B.[R_1^1, R_1^2, R_2^1, R_2^2, R_3^1, R_3^2, R_4^1, R_4^2]$$
$$= U.B.[2, 1, 3, 3, 1, 2, 1, 1]$$
$$= 3$$

Therefore,

$$q = N.U.I.[R/2]$$
$$= N.U.I.[3/2]$$
$$= 2$$

and by the theorem two frequencies are called for.

Since the second row is identified with the largest value of R, it is used as a starting point. Thus, with two distinct frequencies, ω_1 and ω_2, (27) can be written for the second row of (23) with either u_1 or u_2 equal to zero. Choosing $u_1 = \sin \omega_1 t + \sin \omega_2 t$, $u_2 \equiv 0$,

$$\omega_1:\ \phi_{21}x_1^1 + \phi_{22}x_2^1 + 0\cdot x_3^1 + 0\cdot x_4^1 + \psi_{21}u_1^1 = 0$$
$$\omega_2:\ \phi_{21}x_1^2 + \phi_{22}x_2^2 + 0\cdot x_3^2 + 0\cdot x_4^2 + \psi_{21}u_1^2 = 0 \qquad (28)$$

The time derivative of (28), noting that $\dot{\phi}_{21} = \dot{\phi}_{22} = \dot{\psi}_{21} = 0$ in the steady state, is

$$\phi_{21}\dot{x}_1^1 + \phi_{22}\dot{x}_2^1 + \psi_{21}\dot{u}_1^1 = 0$$
$$\phi_{21}\dot{x}_1^2 + \phi_{22}\dot{x}_2^2 + \psi_{21}\dot{u}_1^2 = 0 \qquad (29)$$

(28) and (29) take the form of (15). Since only three independent equations are needed, it suffices to use (28) and the first equation of (29). Accordingly,

$$\begin{bmatrix} x_1^1 & x_2^1 & u_1^1 \\ \dot{x}_1^1 & \dot{x}_2^1 & \dot{u}_1^1 \\ x_1^2 & x_2^2 & u_1^2 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \\ \psi_{21} \end{bmatrix} = 0 \qquad (30)$$

The first matrix of (30) can be written as

$$F(t) = \begin{bmatrix} K_1^1 \sin(\omega_1 t + \alpha_1^1) & K_2^1 \sin(\omega_1 t + \alpha_2^1) & \sin \omega_1 t \\ \omega_1 K_1^1 \cos(\omega_1 t + \alpha_1^1) & \omega_1 K_2^1 \cos(\omega_1 t + \alpha_2^1) & \omega_1 \cos \omega_1 t \\ K_1^2 \sin(\omega_2 t + \alpha_1^2) & K_2^2 \sin(\omega_2 t + \alpha_2^2) & \sin \omega_2 t \end{bmatrix}$$

corresponding to F(t) in (15).

Now form (16) up to the second derivative of F(t) and set $t = t_0 \equiv 0$. By using the rule described in II, and choosing the appropriate columns we have

$$\begin{bmatrix} K_1^1 \sin \alpha_1^1 & -\omega_1^2 K_1^1 \sin \alpha_1^1 & K_2^1 \sin \alpha_2^1 \\ \omega_1 K_1^1 \cos \alpha_1^1 & -\omega_1^3 K_1^1 \cos \alpha_1^1 & \omega_1 K_2^1 \cos \alpha_2^1 \\ K_1^2 \sin \alpha_1^2 & -\omega_2^2 K_1^2 \sin \alpha_1^2 & K_2^2 \sin \alpha_2^2 \end{bmatrix} \qquad (31)$$

Factoring (31) into two matrices as shown in (17), the following expression is obtained:

$$\begin{bmatrix} K_1^1 \sin\ {}_1^1 & & 0 \\ & K_1^1 \cos \alpha_1^1 & \\ 0 & & K_1^2 \sin \alpha_1^2 \end{bmatrix} \begin{bmatrix} 1 & -\omega_1^2 & (K_2^1 \sin \alpha_2^1)/(K_1^1 \sin \alpha_1^1) \\ \omega_1 & -\omega_1^3 & \omega_1(K_2^1 \cos \alpha_2^1)/(K_1^1 \cos \alpha_1^1) \\ 1 & -\omega_2^2 & (K_2^2 \sin \alpha_2^2)/(K_1^2 \sin \alpha_1^2) \end{bmatrix}$$

It has been shown in II that these two matrices are nonsingular under the conditions described in the corollary. Therefore, from (30)

$$\begin{bmatrix} \phi_{21} \\ \phi_{22} \\ \psi_{21} \end{bmatrix} \equiv 0 \tag{32}$$

Considering the second row of (23) with input u_2 acting (i.e. $u_1 \equiv 0$), and using (32), it follows

$$\psi_{22} u_2 = 0 \tag{33}$$

and

$$\psi_{22} = 0 \tag{34}$$

Now consider the first row of (23) with $u_1 \not\equiv 0$, $u_2 \equiv 0$. Since this row has only two nonzero elements of Φ, only one of the two distinct frequencies used previously need be employed here. Choosing ω_1, we have

$$\omega_1 : \ \phi_{11} x_1^1 + \psi_{11} u_1^1 = 0 \tag{35}$$

Noting that $\dot{\phi}_{11} = \dot{\psi}_{11} = 0$ in the steady state, we have for the time derivative of (35)

$$\phi_n \dot{x}_1^1 + \psi_{11} \dot{u}_1^1 = 0 \tag{36}$$

The matrix form of (35) and (36) become

$$\begin{bmatrix} x_1^1 & u_1^1 \\ \dot{x}_1^1 & \dot{u}_1^1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \psi_{11} \end{bmatrix} = 0 \tag{37}$$

Rewriting the first matrix of (37),

$$F(t) = \begin{bmatrix} K_1^1 \sin(\omega_1 t + \alpha_1^1) & \sin \omega_1 t \\ \omega_1 K_1^1 \cos(\omega_1 t + \alpha_1^1) & \omega_1 \cos \omega_1 t \end{bmatrix}$$

Now form (16), i.e. set $t = t_0 = 0$, and make use of the rule described in II to obtain the following expression:

$$\begin{bmatrix} K_1^1 \sin \alpha_1^1 & 0 \\ \omega_1 K_1^1 \cos \alpha_1^1 & \omega_1 \end{bmatrix} \tag{38}$$

The factored form of (38) becomes

$$\begin{bmatrix} K_1^1 \sin \alpha_1^1 & 0 \\ 0 & K_1^1 \cos \alpha_1^1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \omega_1 & \omega_1 / K_1^1 \cos \alpha_1^1 \end{bmatrix} \tag{39}$$

showing that these two matrices are nonsingular under the required conditions. Therefore, from (37)

$$\begin{bmatrix} \phi_{11} \\ \psi_{11} \end{bmatrix} = 0 \tag{40}$$

A similar exercise can be applied to the other rows with the result that all the rest of the parameter differences converge to zero.

IV. Computer Simulation

Computer simulation was undertaken to test the sufficiency aspect of the theorem. The various cases listed in Table 1 were studied, as discussed below.

1) In [4], an example of a 4th order multivariable system with two inputs and three unknown parameters was simulated in an identification scheme. The input signals consisted of periodic square waves of unit height with fundamental frequencies of ω_1 = 1 rad/sec. and ω_2 = 2 rad/sec for inputs u_1 and u_2 respectively. For this same example, it is found that R = 1. Thus, by the corollary it suffices that each input be excited by a single frequency and that these frequencies be distinct. The inputs chosen were $u_1 = \sin \omega_1 t$ and $u_2 = \sin \omega_2 t$, where ω_1 = 1 rad/sec. and ω_2 = 2 rad/sec. The results shown in Fig. 1-a, b, c for parameters, a^*_{31}, a^*_{33} and b^*_{21} respectively demonstrate that, according to the theorem, one frequency for each input is sufficient.

2) Using the identification technique developed in [4], a plant of second order with a single input and two unknown parameters were considered. The input frequency was selected such that all the states in the plant should encounter $K\pi/2$ phase shift. The selected frequency was 4 rad./sec. As shown in Fig. 2a and b, the values of the unknown plant parameters were identified showing that the phase shift requirement is not a necessary condition.

The plant and model were as follows:

$$A^* = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix} \qquad b^* = \begin{bmatrix} 0 \\ 16.0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 16 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}$$

where $a_{21}(0) = -25.0$

$a_{22}(0) = -10.00$

$$P = \begin{bmatrix} 75.75 & 10 \\ 10 & 1.75 \end{bmatrix} \qquad Q = \begin{bmatrix} 180 & 0 \\ 0 & 1 \end{bmatrix}$$

3) Using the same system as in 2), the number of unknown parameters was increased from two to three, such that R = 3. Therefore, two distinct frequencies are called for. To test the theorem, only one frequency was provided. Fig. 3a-f show that the model parameters did not converge to the plant parameters, and, furthermore, steady-state values of model parameters depended upon the size of input. Accordingly it is conjectured that the frequency requirement is a necessary condition.

4) It has been conjectured that the controllability requirement is not a necessary condition. To verify the statement above, the following case of an uncontrollable third-order plant with three unknown parameters was simulated.

$$A^* = \begin{bmatrix} 0 & 0 & -12 \\ 1 & 0 & -19 \\ 0 & 1 & -8 \end{bmatrix} \qquad B^* = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 1 & -15 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & -12 \\ a_{21} & 0 & a_{23} \\ 0 & 1 & a_{33} \end{bmatrix}$$

$B = B^*$

where $a_{21}(0) = 3.0$

$a_{23}(0) = -15.0$

$a_{33}(0) = -6.0$

and $\omega_1 = 5.5$ rad/sec

and

$$P = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \qquad Q = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 240 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

As shown in Fig. 4a, b, c, the convergence of the model parameters to the plant parameters is accomplished in spite of the fact that the system is uncontrollable. This example proves the controllability is not a necessary condition for parameter convergence.

5) Due to the fact that the theorem has been developed in state variable rather than transfer function form, in some cases, less frequencies are required than the number of frequencies called for by [1], [2]. The example below illustrates this point, wherein

PLANT:

$$A^* = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \qquad B^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

MODEL:

$$C = \begin{bmatrix} -10 & 0 \\ 0 & -12 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -6 \\ 1 & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $a_{22}(0) = -8.0$, $b_1(0) = 3.0$, $b_2(0) = 5.0$ and

$$P = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \qquad Q = \begin{bmatrix} -100 & 0 \\ 0 & -240 \end{bmatrix}$$

Since R = 2, one frequently is called for. Setting $\omega_1 = 4.5$ rad/sec, simulation shows that the model parameters converged to the plant parameters [See Fig. 5 a,b,c].

In order to apply the theorems in [1] and [2], the plant state equation is first put in form

$$x + \bar{a}_1\dot{x} + \bar{a}_0 x = \bar{b}_0 u + \bar{b}_1\dot{u}$$

where

$$\bar{a}_1 = -a_{22}$$

$$\bar{a}_0 = 6.0$$

$$\bar{b}_0 = -a_{22}b_1 - 6b_2$$

$$\bar{b}_1 = b_1.$$

Since the number of unknown parameters is three, according to [1] and [2], two frequencies are called for.

V. Conclusions

The theorem derived in this paper provides sufficient conditions regarding the frequency content of the inputs to a Liapunov-adaptive identification process, so as to guarantee nullification of the parameter-error vector. As such it complements the several other similar theorems which have appeared in the literature [1],[2],[3]. In contrast to previous work, however, this theorem utilizes knowledge of specific locations of unknown parameters in the A, B matrices, thereby in some instances significantly reducing the frequency requirements.

Because this theorem, as well as others, provide sufficient conditions only, several simulations have been made to test whether these conditions may in fact not be necessary. It is found, notably, that the controllability condition is not necessary, while the number of frequencies required appears to be a necessary condition. As summarized in Table 1, other sufficiency conditions are tested as well.

These results point to the need for deriving a more complete theorem than has yet been reported, in which both necessary and sufficient conditions are established.

Figure	n	m	q	Condition	Result
Fig. 1 a,b,c	4	2	1	Ref. [4] Example	Converged
Fig. 2 a,b	2	1	1	Violation of $K\pi/2$ Phase Shifts	Converged
Fig. 3 a-f	2	1	2	Number of Frequencies less than q	Not Converged
Fig. 4 a,b,c	3	1	1	Uncontrollable plant	Converged
Fig. 5 a,b,c	2	1	1	Number of frequencies less than in Ref. [1],[2]	Converged

TABLE 1

References

[1] Lion, R.M., "Rapid Identification of Linear and Nonlinear Systems", AIAA Journal Vol. 5, No. 10, pp. 1835-1842, (1967).

[2] Carroll, R.L., "The Adaptive Observer", Ph.D. Thesis, Univ. of Conn., Storrs, Conn., (1973).

[3] Hoberock, L.L. and G.W. Stewart, "Input Requirements and Parametric Errors for System Identification under Period Excitation", Trans. ASME, pp. 296-302, (1972).

[4] Kudva, P., and K.S. Narendra, "An Identification Procedure for Linear Multivariable Systems", Tech. Report No. CT-48, Yale University, (1972).

[5] Lindorff, D.P. and R.L. Carroll, "Survey of Adaptive Control Using Liapunov Design", Proc. Int. Conf. on Cybernetics and Soc. (IEEE), pp. 345-352, (1972).

[6] Chen, C.T., Introduction to Linear System Theory, New York: Holt, Rinehart and Winston, Inc., p. 167, (1970).

a_{31} vs. t ($a_{31}^* = 0.2$)

AS31 VERSUS TIME

MINIMUM 3.0489E-02 MAXIMUM 3.3847E-01

TIME	AS31
0.0	1.0020E-01
1.2500E 00	6.8389E-02
2.5000E 00	1.3202E-01
3.7500E 00	3.3026E-01
5.0000E 00	3.2441E-01
6.2500E 00	2.2095E-01
7.5000E 00	2.6023E-01
8.7500E 00	2.6710E-01
1.0000E 01	2.0222E-01
1.1250E 01	1.9880E-01
1.2500E 01	1.8139E-01
1.3750E 01	1.9137E-01
1.5000E 01	2.0825E-01
1.6250E 01	2.0269E-01
1.7500E 01	2.0255E-01
1.8750E 01	1.9473E-01
2.0000E 01	2.0168E-01
2.1250E 01	2.0939E-01
2.2500E 01	2.0463E-01
2.3750E 01	2.0509E-01
2.5000E 01	2.0178E-01
2.6250E 01	2.0532E-01
2.7500E 01	2.0296E-01
2.8750E 01	2.0206E-01
3.0000E 01	2.0187E-01
3.1250E 01	2.0299E-01
3.2500E 01	2.0471E-01
3.3750E 01	1.9948E-01
3.5000E 01	1.9983E-01
3.6250E 01	1.9975E-01
3.7500E 01	2.0141E-01
3.8750E 01	2.0271E-01
4.0000E 01	1.9972E-01
4.1250E 01	2.0037E-01
4.2500E 01	2.0022E-01
4.3750E 01	2.0115E-01
4.5000E 01	2.0159E-01
4.6250E 01	1.9938E-01
4.7500E 01	1.9982E-01
4.8750E 01	1.9985E-01
5.0000E 01	2.0091E-01

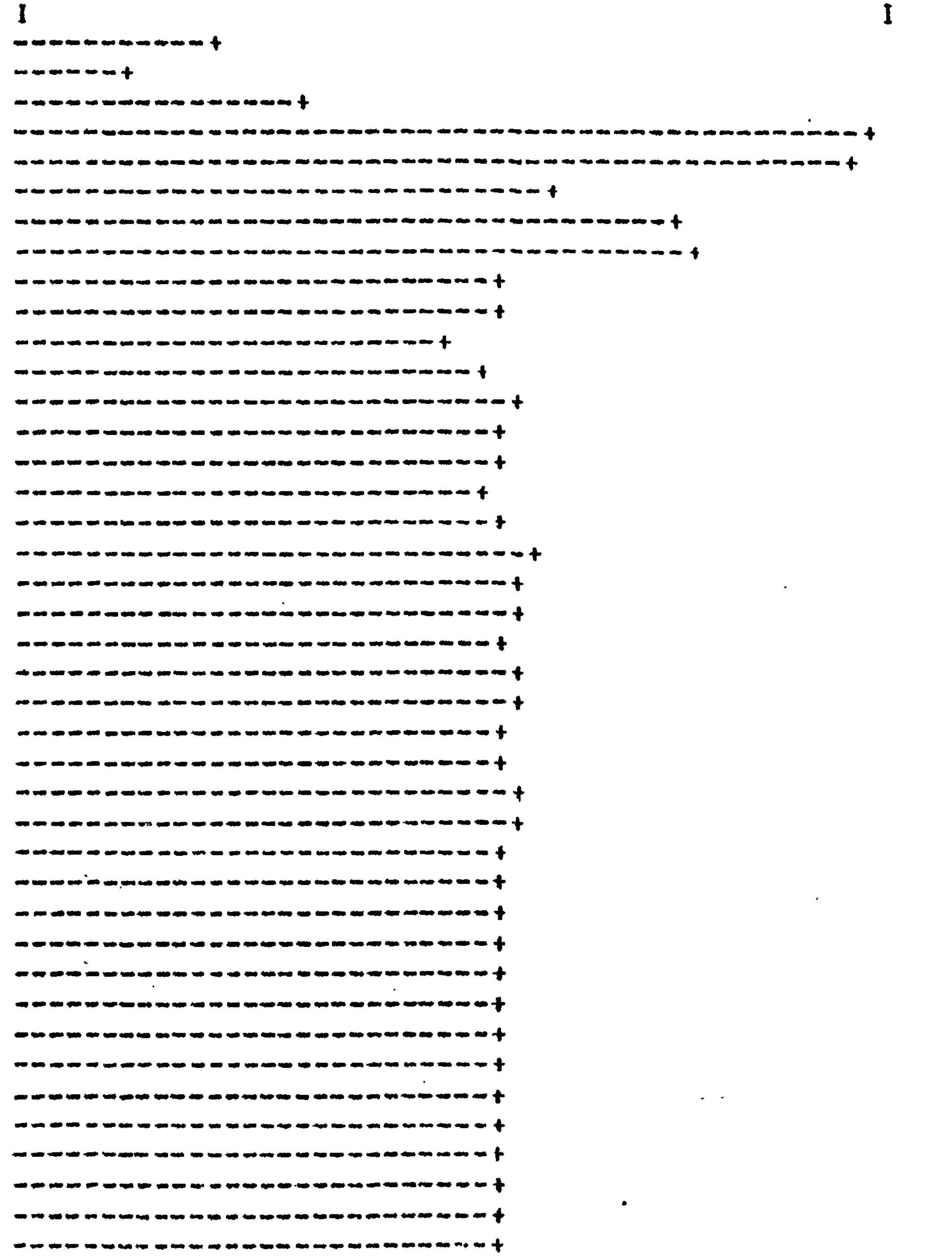


FIG. 1-a

a_{33} vs. t (a_{33}^{*} = -1.95)

AS33 VERSUS TIME

MINIMUM -3.1665E 00 — MAXIMUM -7.7693E-01

TIME	AS33
0.0	-2.3500E 00
1.2500E 00	-2.1330E 00
2.5000E 00	-2.0456E 00
3.7500E 00	-1.8799E 00
5.0000E 00	-2.0219E 00
6.2500E 00	-1.9487E 00
7.5000E 00	-1.9210E 00
8.7500E 00	-1.8893E 00
1.0000E 01	-1.9073E 00
1.1250E 01	-1.9806E 00
1.2500E 01	-1.9476E 00
1.3750E 01	-1.9104E 00
1.5000E 01	-1.9325E 00
1.6250E 01	-1.9630E 00
1.7500E 01	-1.9590E 00
1.8750E 01	-1.9490E 00
2.0000E 01	-1.9319E 00
2.1250E 01	-1.9478E 00
2.2500E 01	-1.9638E 00
2.3750E 01	-1.9488E 00
2.5000E 01	-1.9498E 00
2.6250E 01	-1.9497E 00
2.7500E 01	-1.9484E 00
2.8750E 01	-1.9454E 00
3.0000E 01	-1.9520E 00
3.1250E 01	-1.9502E 00
3.2500E 01	-1.9523E 00
3.3750E 01	-1.9449E 00
3.5000E 01	-1.9504E 00
3.6250E 01	-1.9520E 00
3.7500E 01	-1.9500E 00
3.8750E 01	-1.9499E 00
4.0000E 01	-1.9462E 00
4.1250E 01	-1.9462E 00
4.2500E 01	-1.9515E 00
4.3750E 01	-1.9498E 00
4.5000E 01	-1.9526E 00
4.6250E 01	-1.9486E 00
4.7500E 01	-1.9508E 00
4.8750E 01	-1.9495E 00
5.0000E 01	-1.9497E 00

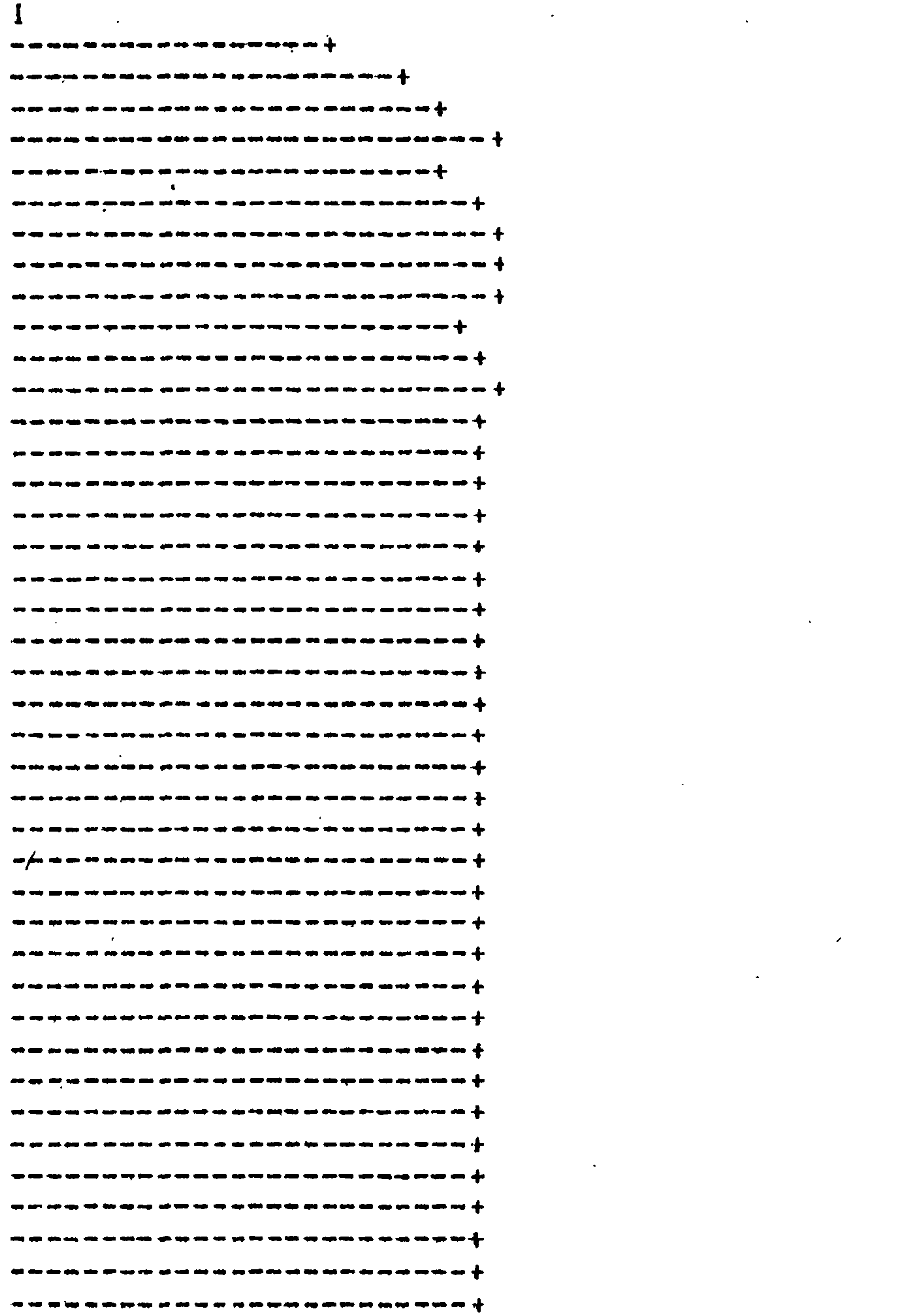


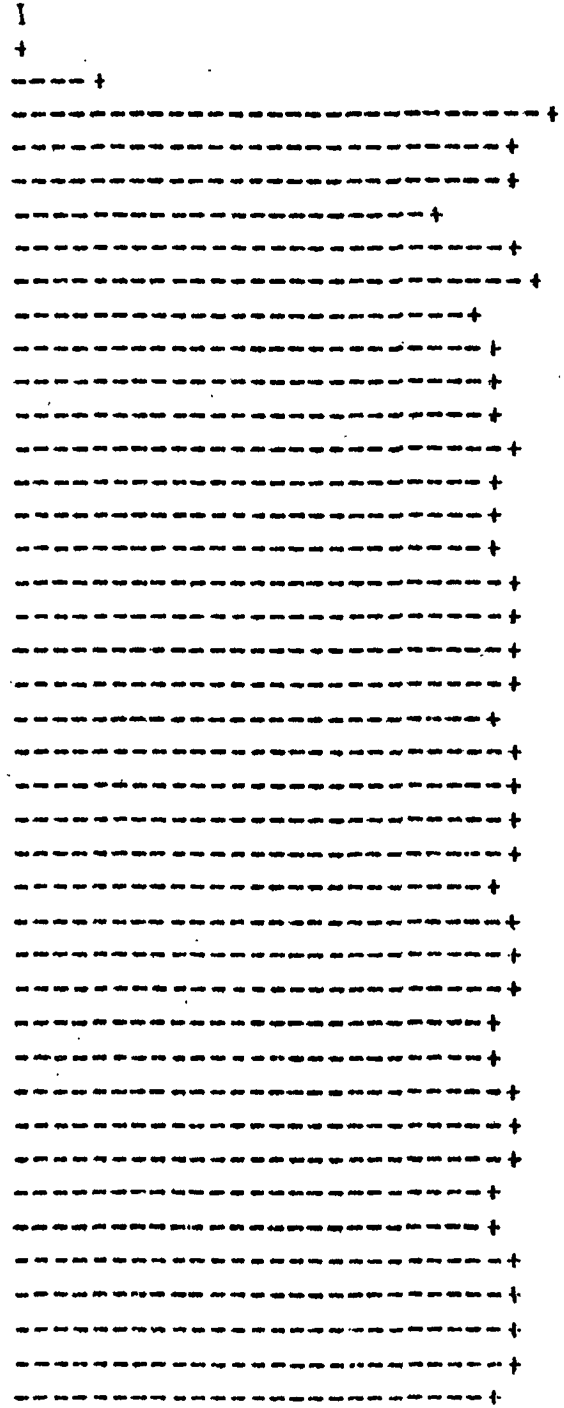
FIG. 1-b

b_{21} vs. t ($b_{21}^{*} = 5.0$)

Note: See the tabulated values instead of graph

MINIMUM 3.5400E 00   BS21 VERSUS TIME   MAXIMUM 6.4513E 00

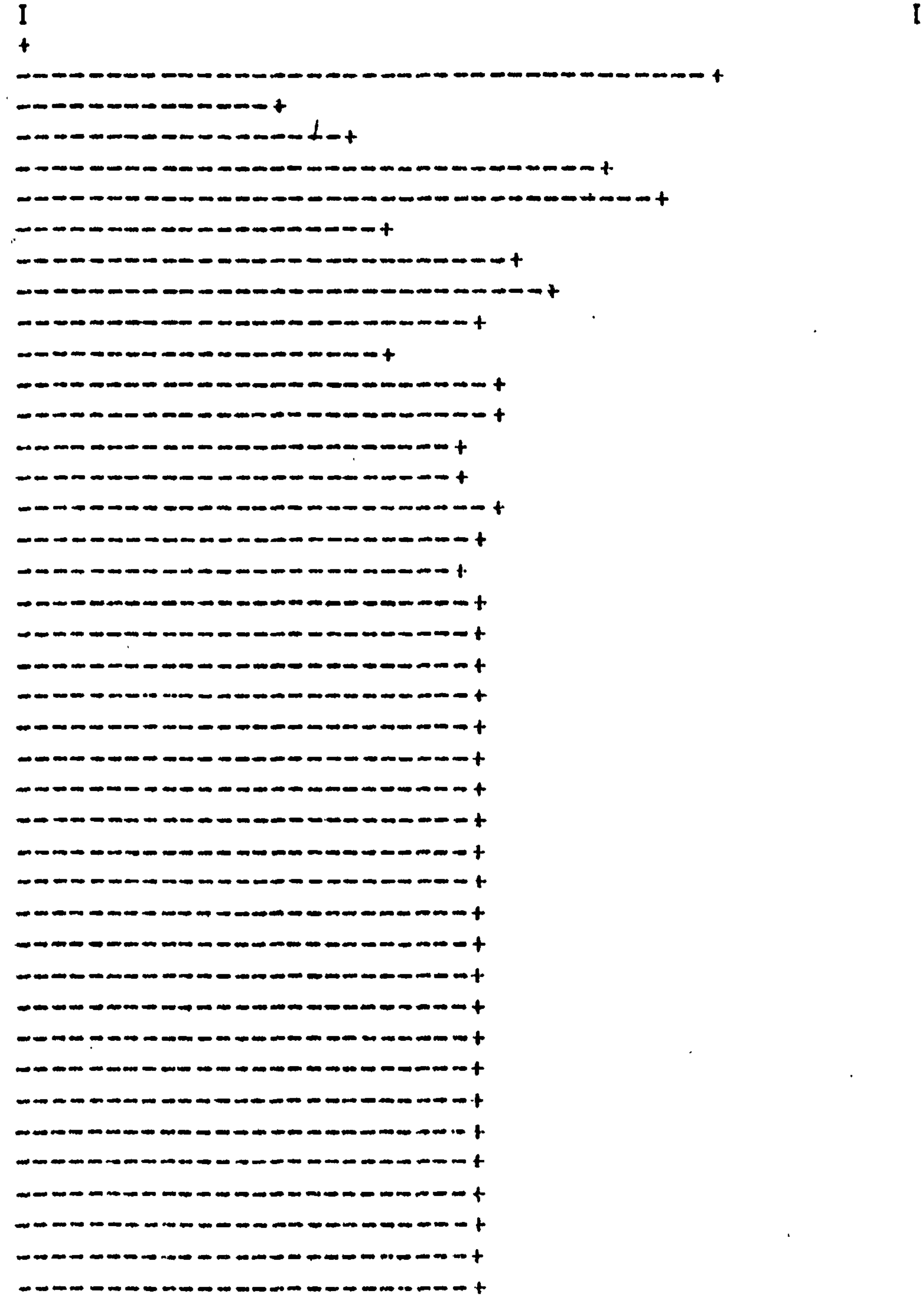
TIME	BS21
0.0	3.5400E 00
1.2500E 00	3.8271E 00
2.5000E 00	5.1685E 00
3.7500E 00	5.0253E 00
5.0000E 00	5.0372E 00
6.2500E 00	4.7900E 00
7.5000E 00	5.0034E 00
8.7500E 00	5.0898E 00
1.0000E 01	4.9305E 00
1.1250E 01	4.9457E 00
1.2500E 01	4.9894E 00
1.3750E 01	4.9956E 00
1.5000E 01	5.0317E 00
1.6250E 01	4.9744E 00
1.7500E 01	4.9846E 00
1.8750E 01	4.9845E 00
2.0000E 01	4.9999E 00
2.1250E 01	5.0019E 00
2.2500E 01	5.0055E 00
2.3750E 01	5.0076E 00
2.5000E 01	4.9810E 00
2.6250E 01	5.0014E 00
2.7500E 01	5.0039E 00
2.8750E 01	5.0131E 00
3.0000E 01	4.9967E 00
3.1250E 01	4.9895E 00
3.2500E 01	5.0013E 00
3.3750E 01	5.0092E 00
3.5000E 01	5.0096E 00
3.6250E 01	4.9940E 00
3.7500E 01	4.9929E 00
3.8750E 01	5.0010E 00
4.0000E 01	5.0064E 00
4.1250E 01	5.0070E 00
4.2500E 01	4.9956E 00
4.3750E 01	4.9913E 00
4.5000E 01	4.9995E 00
4.6250E 01	5.0009E 00
4.7500E 01	5.0052E 00
4.8750E 01	5.0012E 00
5.0000E 01	4.9905E 00



a_{21} vs. t (a_{21}^{*} = -16.0)

MINIMUM -2.5156E 01 — AS21 VERSUS TIME, R = 1.0000E 00 — MAXIMUM -7.1063E 00

TIME	AS21
0.0	-2.5000E 01
1.2500E 00	-1.1187E 01
2.5000E 00	-1.9867E 01
3.7500E 00	-1.8658E 01
5.0000E 00	-1.3600E 01
6.2500E 00	-1.2197E 01
7.5000E 00	-1.7786E 01
8.7500E 00	-1.5216E 01
1.0000E 01	-1.4562E 01
1.1250E 01	-1.5861E 01
1.2500E 01	-1.7604E 01
1.3750E 01	-1.5731E 01
1.5000E 01	-1.5608E 01
1.6250E 01	-1.6482E 01
1.7500E 01	-1.6275E 01
1.8750E 01	-1.5745E 01
2.0000E 01	-1.5967E 01
2.1250E 01	-1.6463E 01
2.2500E 01	-1.5955E 01
2.3750E 01	-1.5860E 01
2.5000E 01	-1.5959E 01
2.6250E 01	-1.6100E 01
2.7500E 01	-1.5806E 01
2.8750E 01	-1.5925E 01
3.0000E 01	-1.6062E 01
3.1250E 01	-1.6045E 01
3.2500E 01	-1.5933E 01
3.3750E 01	-1.6027E 01
3.5000E 01	-1.6041E 01
3.6250E 01	-1.5950E 01
3.7500E 01	-1.5985E 01
3.8750E 01	-1.6035E 01
4.0000E 01	-1.6018E 01
4.1250E 01	-1.5978E 01
4.2500E 01	-1.6021E 01
4.3750E 01	-1.6013E 01
4.5000E 01	-1.5992E 01
4.6250E 01	-1.5985E 01
4.7500E 01	-1.6007E 01
4.8750E 01	-1.5997E 01
5.0000E 01	-1.5994E 01



a_{22} vs. t ($a_{22}^* = -8.0$)

MINIMUM -1.6307E 01 | AS22 VERSUS TIME R = 1.0000E 00 | MAXIMUM 8.4350E-01

TIME	AS22
0.0	-1.0000E 01
1.2500E 00	-1.1715E 01
2.5000E 00	-4.5161E 00
3.7500E 00	-2.8383E 00
5.0000E 00	-1.1065E 01
6.2500E 00	-7.0284E 00
7.5000E 00	-6.1094E 00
8.7500E 00	-1.0689E 01
1.0000E 01	-8.4480E 00
1.1250E 01	-9.2735E 00
1.2500E 01	-7.8680E 00
1.3750E 01	-6.7817E 00
1.5000E 01	-7.4640E 00
1.6250E 01	-8.8849E 00
1.7500E 01	-7.3776E 00
1.8750E 01	-8.4255E 00
2.0000E 01	-8.1653E 00
2.1250E 01	-8.0361E 00
2.2500E 01	-8.0025E 00
2.3750E 01	-7.8098E 00
2.5000E 01	-7.7030E 00
2.6250E 01	-8.2375E 00
2.7500E 01	-8.0390E 00
2.8750E 01	-7.8317E 00
3.0000E 01	-8.1312E 00
3.1250E 01	-8.0339E 00
3.2500E 01	-8.0054E 00
3.3750E 01	-7.9821E 00
3.5000E 01	-7.9805E 00
3.6250E 01	-7.9964E 00
3.7500E 01	-8.0470E 00
3.8750E 01	-7.9767E 00
4.0000E 01	-8.0184E 00
4.1250E 01	-8.0185E 00
4.2500E 01	-7.9849E 00
4.3750E 01	-7.9891E 00
4.5000E 01	-7.9893E 00
4.6250E 01	-7.9937E 00
4.7500E 01	-8.0092E 00
4.8750E 01	-8.0043E 00
5.0000E 01	-7.9945E 00

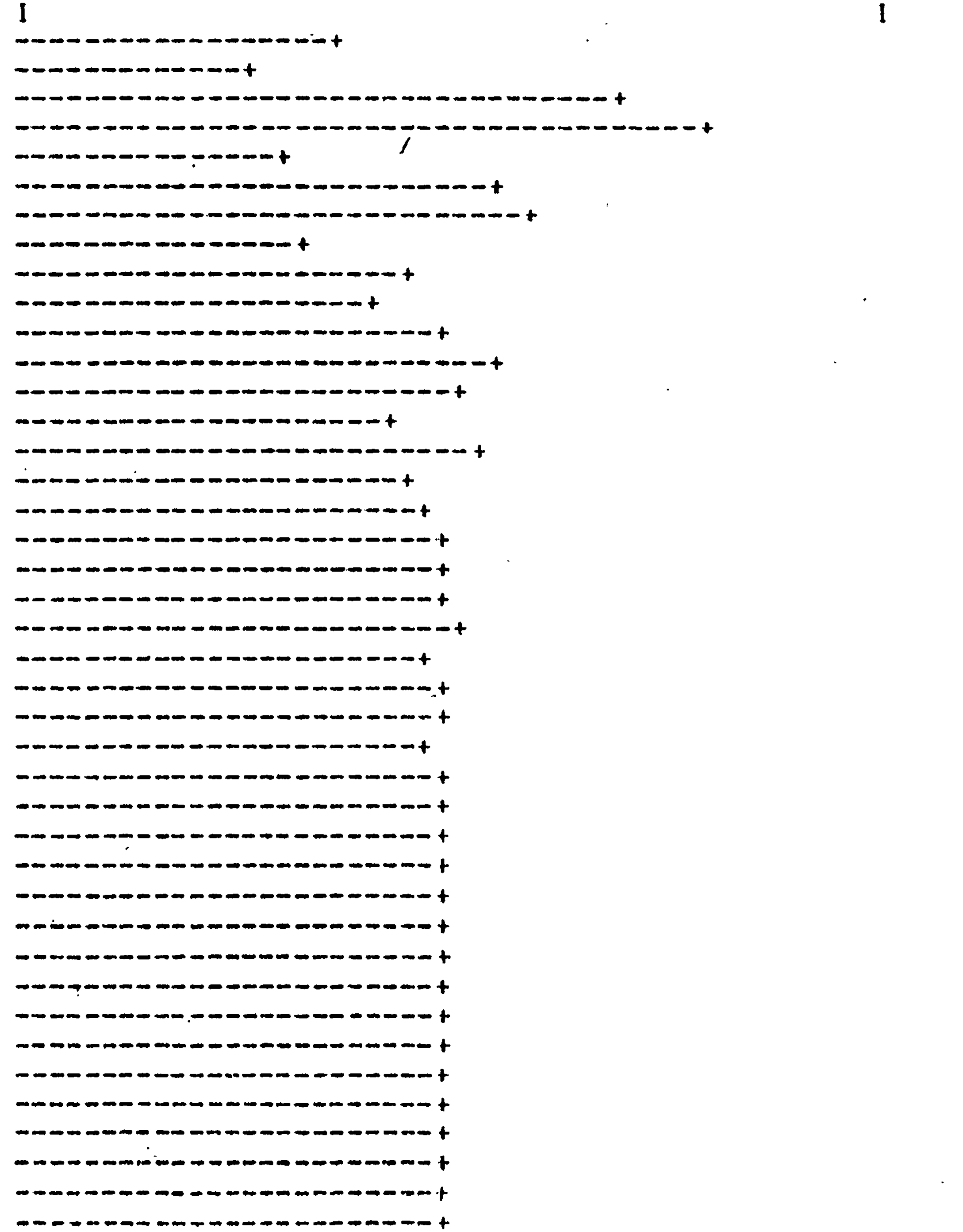


FIG. 2-b

a_{21} vs. t ($a_{21}^* = -16.0$)

Note: Input Amplitude = 1.0

MINIMUM -2.5022E 01 AS21 VERSUS TIME R = 1.0000E 00 MAXIMUM -8.7608E 00

TIME	AS21
0.0	-2.5000E 01
1.2500E 00	-1.8048E 01
2.5000E 00	-2.0945E 01
3.7500E 00	-1.8220E 01
5.0000E 00	-1.4939E 01
6.2500E 00	-1.5785E 01
7.5000E 00	-1.8346E 01
8.7500E 00	-1.9040E 01
1.0000E 01	-1.7286E 01
1.1250E 01	-1.6373E 01
1.2500E 01	-1.6419E 01
1.3750E 01	-1.7477E 01
1.5000E 01	-1.8002E 01
1.6250E 01	-1.7354E 01
1.7500E 01	-1.6603E 01
1.8750E 01	-1.6512E 01
2.0000E 01	-1.7143E 01
2.1250E 01	-1.7492E 01
2.2500E 01	-1.7223E 01
2.3750E 01	-1.6790E 01
2.5000E 01	-1.6730E 01
2.6250E 01	-1.6960E 01
2.7500E 01	-1.7217E 01
2.8750E 01	-1.7129E 01
3.0000E 01	-1.6897E 01
3.1250E 01	-1.6862E 01
3.2500E 01	-1.6943E 01
3.3750E 01	-1.7105E 01
3.5000E 01	-1.7080E 01
3.6250E 01	-1.6965E 01
3.7500E 01	-1.6934E 01
3.8750E 01	-1.6968E 01
4.0000E 01	-1.7061E 01
4.1250E 01	-1.7063E 01
4.2500E 01	-1.7004E 01
4.3750E 01	-1.6972E 01
4.5000E 01	-1.6990E 01
4.6250E 01	-1.7038E 01
4.7500E 01	-1.7049E 01
4.8750E 01	-1.7019E 01
5.0000E 01	-1.6994E 01

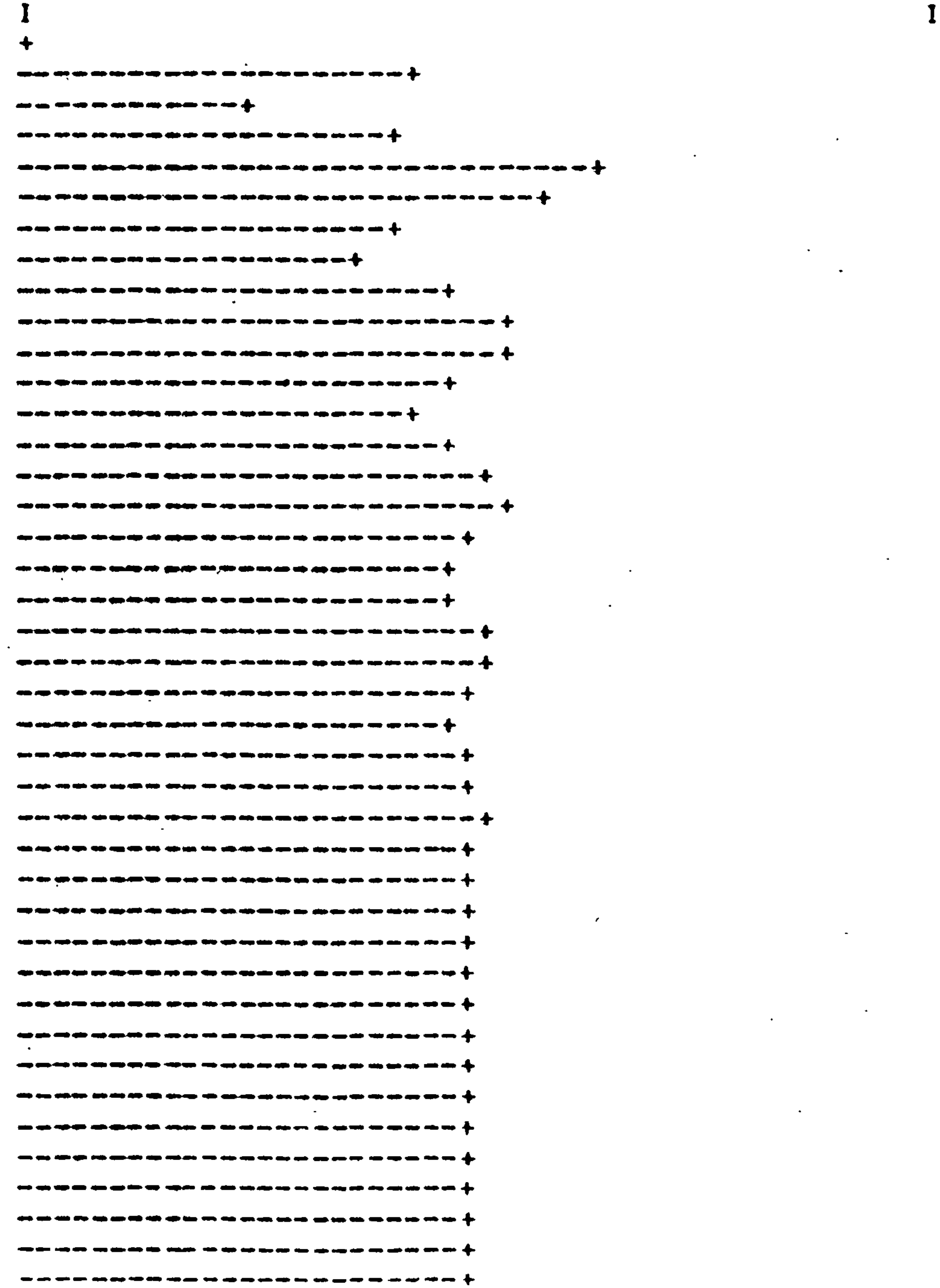


Fig. 3-a

a_{21} vs. t (a_{21}^{*} = -16.0)

Note: Input Amplitude = 5.0

MINIMUM -2.5022E 01   AS21 VERSUS TIME   R = 5.0000E 00   MAXIMUM -8.7608E 00

TIME	AS21
0.0	-2.5000E 01
1.2500E 00	-1.6173E 01
2.5000E 00	-1.0566E 01
3.7500E 00	-1.0160E 01
5.0000E 00	-1.3666E 01
6.2500E 00	-1.3251E 01
7.5000E 00	-1.1348E 01
8.7500E 00	-1.1922E 01
1.0000E 01	-1.2634E 01
1.1250E 01	-1.1913E 01
1.2500E 01	-1.1895E 01
1.3750E 01	-1.2347E 01
1.5000E 01	-1.2283E 01
1.6250E 01	-1.2051E 01
1.7500E 01	-1.2247E 01
1.8750E 01	-1.2227E 01
2.0000E 01	-1.2123E 01
2.1250E 01	-1.2130E 01
2.2500E 01	-1.2190E 01
2.3750E 01	-1.2163E 01
2.5000E 01	-1.2149E 01
2.6250E 01	-1.2175E 01
2.7500E 01	-1.2174E 01
2.8750E 01	-1.2159E 01
3.0000E 01	-1.2163E 01
3.1250E 01	-1.2170E 01
3.2500E 01	-1.2164E 01
3.3750E 01	-1.2165E 01
3.5000E 01	-1.2169E 01
3.6250E 01	-1.2169E 01
3.7500E 01	-1.2167E 01
3.8750E 01	-1.2169E 01
4.0000E 01	-1.2169E 01
4.1250E 01	-1.2169E 01
4.2500E 01	-1.2169E 01
4.3750E 01	-1.2170E 01
4.5000E 01	-1.2170E 01
4.6250E 01	-1.2170E 01
4.7500E 01	-1.2171E 01
4.8750E 01	-1.2171E 01
5.0000E 01	-1.2171E 01

Fig. 3-b

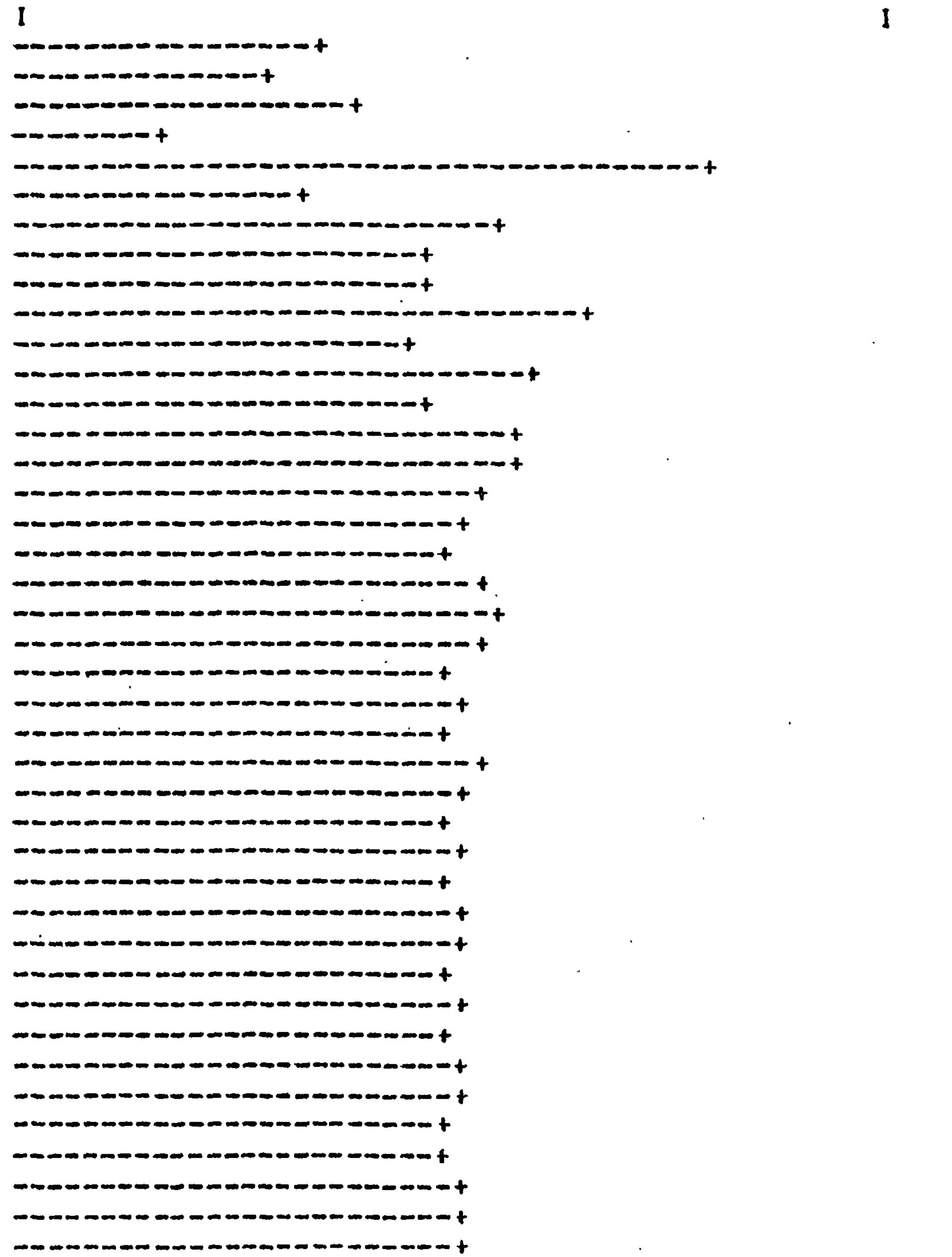
a_{22} vs. t ($a_{22}^{*} = -8.0$)

Note: Input Amplitude = 1.0

AS22 VERSUS TIME
R = 1.0000E 00

MINIMUM -1.5824E 01 — MAXIMUM 3.7997E-01

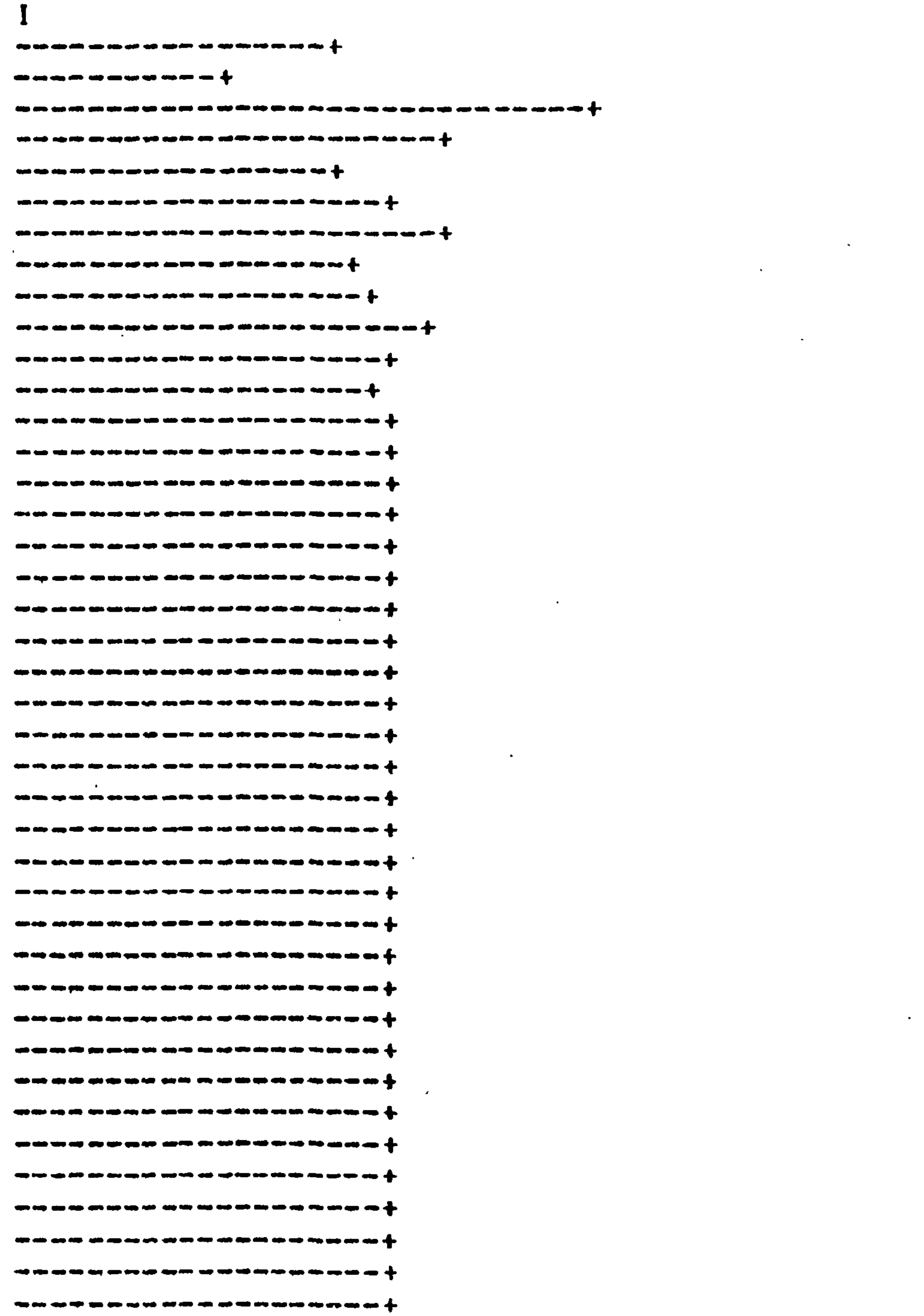
TIME	AS22
0.0	-1.0000E 01
1.2500E 00	-1.1215E 01
2.5000E 00	-9.5574E 00
3.7500E 00	-1.3037E 01
5.0000E 00	-3.0747E 00
6.2500E 00	-1.0412E 01
7.5000E 00	-6.9134E 00
8.7500E 00	-8.2857E 00
1.0000E 01	-8.3509E 00
1.1250E 01	-5.1850E 00
1.2500E 01	-8.6437E 00
1.3750E 01	-6.3777E 00
1.5000E 01	-8.2127E 00
1.6250E 01	-6.5453E 00
1.7500E 01	-6.4462E 00
1.8750E 01	-7.1744E 00
2.0000E 01	-7.6219E 00
2.1250E 01	-7.8284E 00
2.2500E 01	-7.3506E 00
2.3750E 01	-7.0509E 00
2.5000E 01	-7.0851E 00
2.6250E 01	-7.9891E 00
2.7500E 01	-7.5620E 00
2.8750E 01	-7.9786E 00
3.0000E 01	-7.3107E 00
3.1250E 01	-7.4523E 00
3.2500E 01	-7.8973E 00
3.3750E 01	-7.5223E 00
3.5000E 01	-7.9403E 00
3.6250E 01	-7.4754E 00
3.7500E 01	-7.6855E 00
3.8750E 01	-7.7843E 00
4.0000E 01	-7.6322E 00
4.1250E 01	-7.7885E 00
4.2500E 01	-7.6037E 00
4.3750E 01	-7.7131E 00
4.5000E 01	-7.7330E 00
4.6250E 01	-7.7305E 00
4.7500E 01	-7.7208E 00
4.8750E 01	-7.6901E 00
5.0000E 01	-7.6820E 00



a_{22} vs. t (a_{22}^{*} = -8.0)

Note: Input Amplitude = 5.0

MINIMUM -1.5824E 01 — AS22 VERSUS TIME R = 5.0000E 00 — MAXIMUM 3.7997E-01



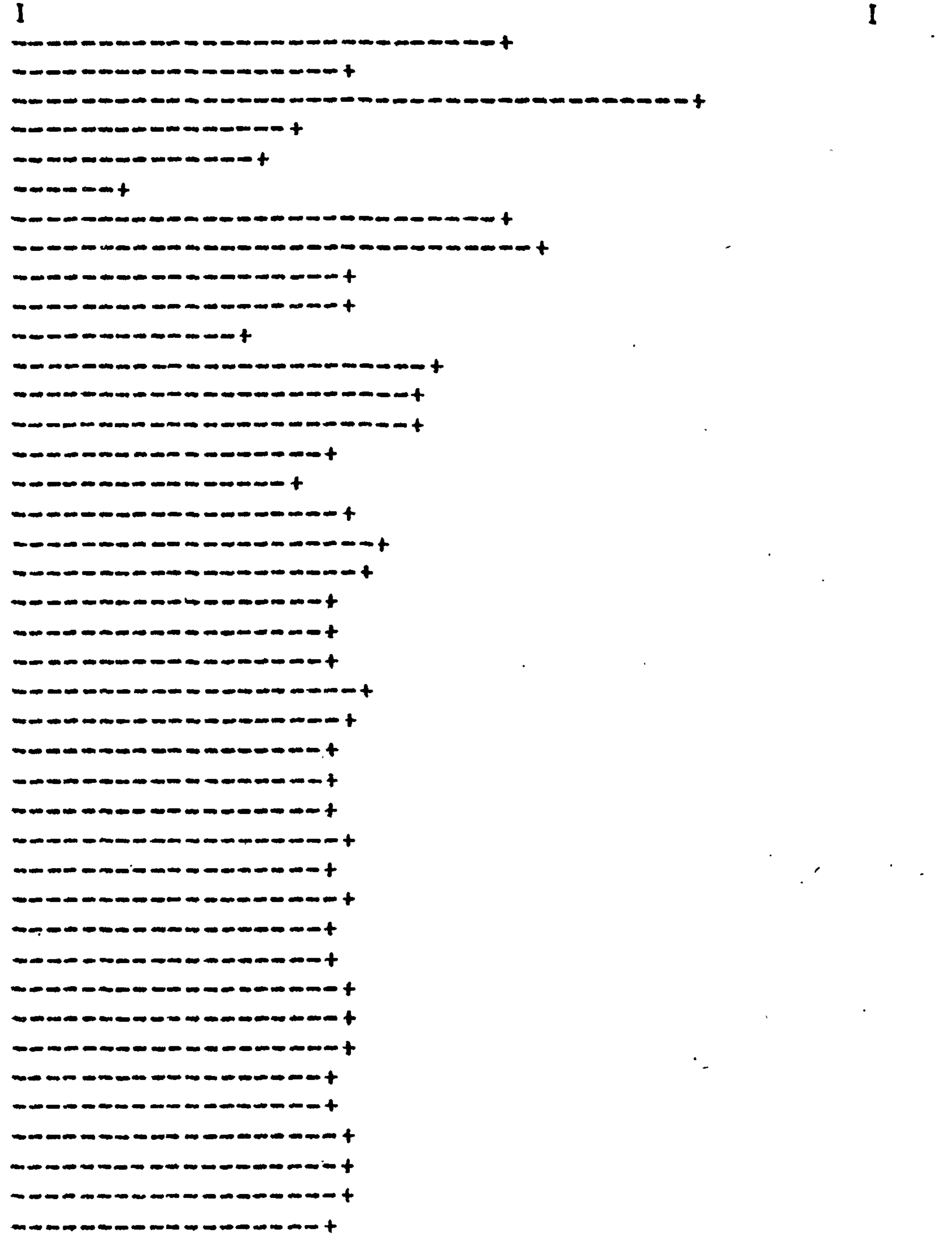
TIME	AS22
0.0	-1.0000E 01
1.2500E 00	-1.1960E 01
2.5000E 00	-5.6558E 00
3.7500E 00	-8.1414E 00
5.0000E 00	-1.0180E 01
6.2500E 00	-9.3085E 00
7.5000E 00	-8.3680E 00
8.7500E 00	-9.8961E 00
1.0000E 01	-9.4584E 00
1.1250E 01	-8.5640E 00
1.2500E 01	-9.0836E 00
1.3750E 01	-9.4096E 00
1.5000E 01	-9.0980E 00
1.6250E 01	-9.0943E 00
1.7500E 01	-9.2550E 00
1.8750E 01	-9.1985E 00
2.0000E 01	-9.0993E 00
2.1250E 01	-9.1703E 00
2.2500E 01	-9.1871E 00
2.3750E 01	-9.1409E 00
2.5000E 01	-9.1605E 00
2.6250E 01	-9.1801E 00
2.7500E 01	-9.1706E 00
2.8750E 01	-9.1649E 00
3.0000E 01	-9.1724E 00
3.1250E 01	-9.1708E 00
3.2500E 01	-9.1650E 00
3.3750E 01	-9.1674E 00
3.5000E 01	-9.1688E 00
3.6250E 01	-9.1676E 00
3.7500E 01	-9.1673E 00
3.8750E 01	-9.1683E 00
4.0000E 01	-9.1681E 00
4.1250E 01	-9.1677E 00
4.2500E 01	-9.1671E 00
4.3750E 01	-9.1675E 00
4.5000E 01	-9.1665E 00
4.6250E 01	-9.1671E 00
4.7500E 01	-9.1671E 00
4.8750E 01	-9.1667E 00
5.0000E 01	-9.1669E 00

b_{21} vs. t (b_{21}^{*} = 16.0)

Note: Input Amplitude = 1.0

MINIMUM 1.0204E 01 — BS21 VERSUS TIME R = 1.0000E 00 — MAXIMUM 2.3778E 01

TIME	BS21
0.0	1.8000E 01
1.2500E 00	1.5509E 01
2.5000E 00	2.0846E 01
3.7500E 00	1.4636E 01
5.0000E 00	1.4234E 01
6.2500E 00	1.1954E 01
7.5000E 00	1.7905E 01
8.7500E 00	1.8359E 01
1.0000E 01	1.5448E 01
1.1250E 01	1.5535E 01
1.2500E 01	1.3881E 01
1.3750E 01	1.6751E 01
1.5000E 01	1.6697E 01
1.6250E 01	1.6468E 01
1.7500E 01	1.5287E 01
1.8750E 01	1.4774E 01
2.0000E 01	1.5589E 01
2.1250E 01	1.6058E 01
2.2500E 01	1.5858E 01
2.3750E 01	1.5298E 01
2.5000E 01	1.5184E 01
2.6250E 01	1.5111E 01
2.7500E 01	1.5747E 01
2.8750E 01	1.5396E 01
3.0000E 01	1.5351E 01
3.1250E 01	1.5224E 01
3.2500E 01	1.5137E 01
3.3750E 01	1.5591E 01
3.5000E 01	1.5341E 01
3.6250E 01	1.5386E 01
3.7500E 01	1.5232E 01
3.8750E 01	1.5240E 01
4.0000E 01	1.5470E 01
4.1250E 01	1.5396E 01
4.2500E 01	1.5393E 01
4.3750E 01	1.5288E 01
4.5000E 01	1.5309E 01
4.6250E 01	1.5391E 01
4.7500E 01	1.5415E 01
4.8750E 01	1.5383E 01
5.0000E 01	1.5347E 01

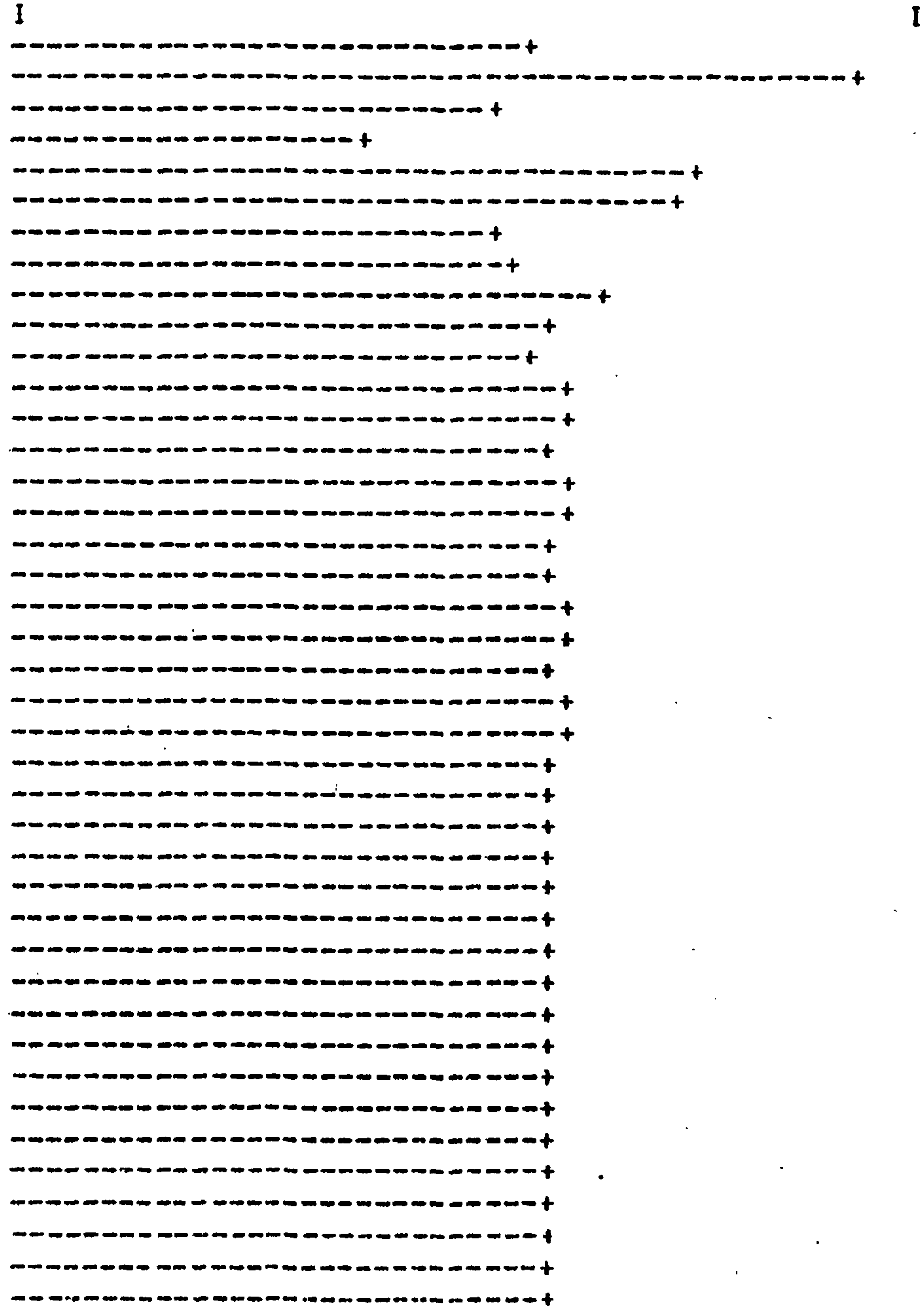


b_{21} vs. t (b_{21}^{*} = 16.0)

Note: Input Amplitude = 5.0

MINIMUM 1.0204E 01 — BS21 VERSUS TIME, R = 5.0000E 00 — MAXIMUM 2.3778E 01

TIME	BS21
0.0	1.8000E 01
1.2500E 00	2.2742E 01
2.5000E 00	1.7480E 01
3.7500E 00	1.5576E 01
5.0000E 00	2.0309E 01
6.2500E 00	2.0062E 01
7.5000E 00	1.7409E 01
8.7500E 00	1.7587E 01
1.0000E 01	1.8973E 01
1.1250E 01	1.8237E 01
1.2500E 01	1.7947E 01
1.3750E 01	1.8524E 01
1.5000E 01	1.8574E 01
1.6250E 01	1.8194E 01
1.7500E 01	1.8435E 01
1.8750E 01	1.8429E 01
2.0000E 01	1.8307E 01
2.1250E 01	1.8282E 01
2.2500E 01	1.8373E 01
2.3750E 01	1.8350E 01
2.5000E 01	1.8317E 01
2.6250E 01	1.8349E 01
2.7500E 01	1.8350E 01
2.8750E 01	1.8329E 01
3.0000E 01	1.8330E 01
3.1250E 01	1.8342E 01
3.2500E 01	1.8334E 01
3.3750E 01	1.8333E 01
3.5000E 01	1.8338E 01
3.6250E 01	1.8337E 01
3.7500E 01	1.8335E 01
3.8750E 01	1.8336E 01
4.0000E 01	1.8336E 01
4.1250E 01	1.8334F 01
4.2500E 01	1.8335E 01
4.3750E 01	1.8335E 01
4.5000E 01	1.8335E 01
4.6250E 01	1.8334E 01
4.7500E 01	1.8334E 01
4.8750E 01	1.8334F 01
5.0000E 01	1.8333E 01

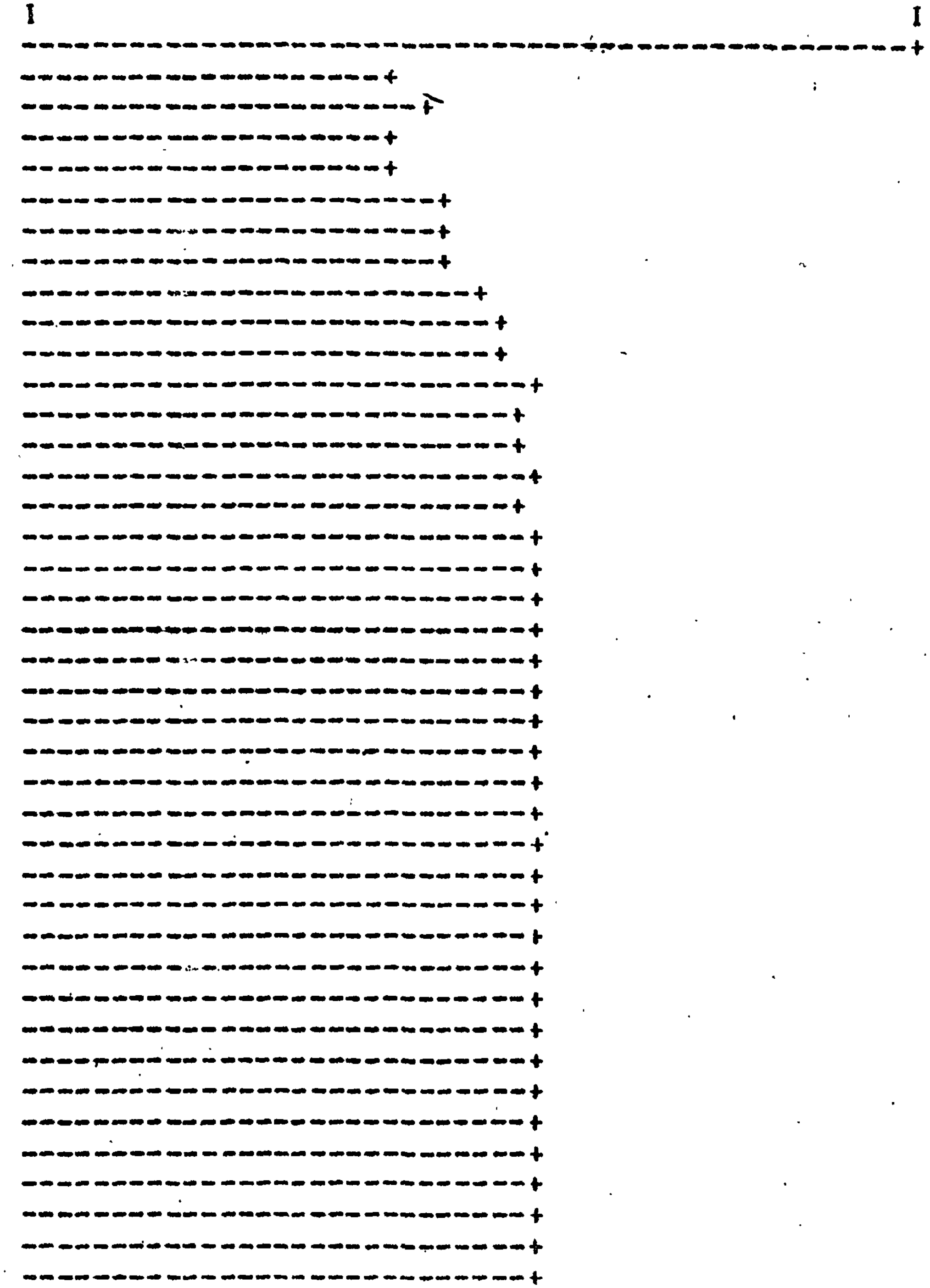


a_{21} vs. t ($a_{21}^* = 1.0$)

AS21 VERSUS TIME

MINIMUM -1.7057E 00 MAXIMUM 3.0000E 00

TIME	AS21
0.0	3.0000E 00
7.5000E-01	2.3390E-01
1.5000E 00	3.9098E-01
2.2500E 00	2.2244E-01
3.0000E 00	2.2177E-01
3.7500E 00	4.7535E-01
4.5000E 00	5.2863E-01
5.2500E 00	5.3865E-01
6.0000E 00	7.0595E-01
6.7500E 00	7.8944E-01
7.5000E 00	7.4951E-01
8.2500E 00	9.4720E-01
9.0000E 00	8.7241E-01
9.7500E 00	8.6914E-01
1.0500E 01	9.6772E-01
1.1250E 01	9.1544E-01
1.2000E 01	9.3229E-01
1.2750E 01	9.5222E-01
1.3500E 01	9.7124E-01
1.4250E 01	9.6070E-01
1.5000E 01	9.6118E-01
1.5750E 01	9.7513E-01
1.6500E 01	9.7540E-01
1.7250E 01	9.7580E-01
1.8000E 01	9.8323E-01
1.8750E 01	9.8815E-01
1.9500E 01	9.8610E-01
2.0250E 01	9.9631E-01
2.1000E 01	9.9297E-01
2.1750E 01	9.9207E-01
2.2500E 01	9.9829E-01
2.3250E 01	9.9441E-01
2.4000E 01	9.9530E-01
2.4750E 01	9.9687E-01
2.5500E 01	9.9757E-01
2.6250E 01	9.9674E-01
2.7000E 01	9.9662E-01
2.7750E 01	9.9778E-01
2.8500E 01	9.9734E-01
2.9250E 01	9.9715E-01
3.0000E 01	9.9770E-01



a_{23} vs. t ($a_{23}^* = 19.0$)

AS23 VERSUS TIME

MINIMUM -1.8997E 01 — MAXIMUM -1.5000E 01

TIME	AS23
0.0	-1.5000E 01
7.5000E-01	-1.7120E 01
1.5000E 00	-1.7316E 01
2.2500E 00	-1.7676E 01
3.0000E 00	-1.7998E 01
3.7500E 00	-1.8142E 01
4.5000E 00	-1.8272E 01
5.2500E 00	-1.8423E 01
6.0000E 00	-1.8516E 01
6.7500E 00	-1.8558E 01
7.5000E 00	-1.8661E 01
8.2500E 00	-1.8677E 01
9.0000E 00	-1.8755E 01
9.7500E 00	-1.8800E 01
1.0500E 01	-1.8818E 01
1.1250E 01	-1.8868E 01
1.2000E 01	-1.8882E 01
1.2750E 01	-1.8906E 01
1.3500E 01	-1.8915E 01
1.4250E 01	-1.8933E 01
1.5000E 01	-1.8949E 01
1.5750E 01	-1.8955E 01
1.6500E 01	-1.8962E 01
1.7250E 01	-1.8970E 01
1.8000E 01	-1.8975E 01
1.8750E 01	-1.8976E 01
1.9500E 01	-1.8981E 01
2.0250E 01	-1.8982E 01
2.1000E 01	-1.8986E 01
2.1750E 01	-1.8988E 01
2.2500E 01	-1.8989E 01
2.3250E 01	-1.8991E 01
2.4000E 01	-1.8992E 01
2.4750E 01	-1.8993E 01
2.5500E 01	-1.8993E 01
2.6250E 01	-1.8994E 01
2.7000E 01	-1.8995E 01
2.7750E 01	-1.8995E 01
2.8500E 01	-1.8996E 01
2.9250E 01	-1.8996E 01
3.0000E 01	-1.8997E 01

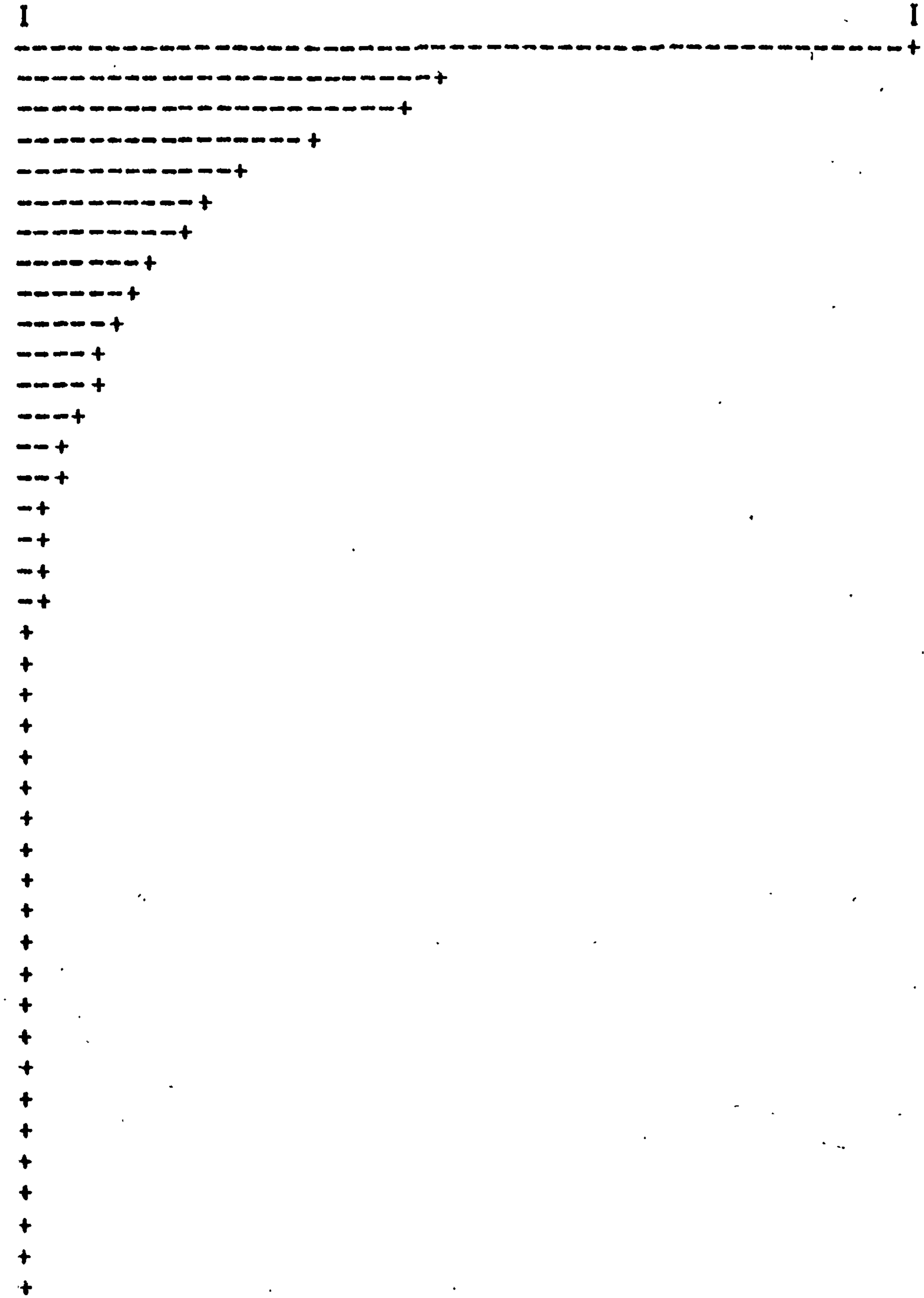


FIG. 4-b

a_{33} vs. t ($a_{33}^* = -8.0$)

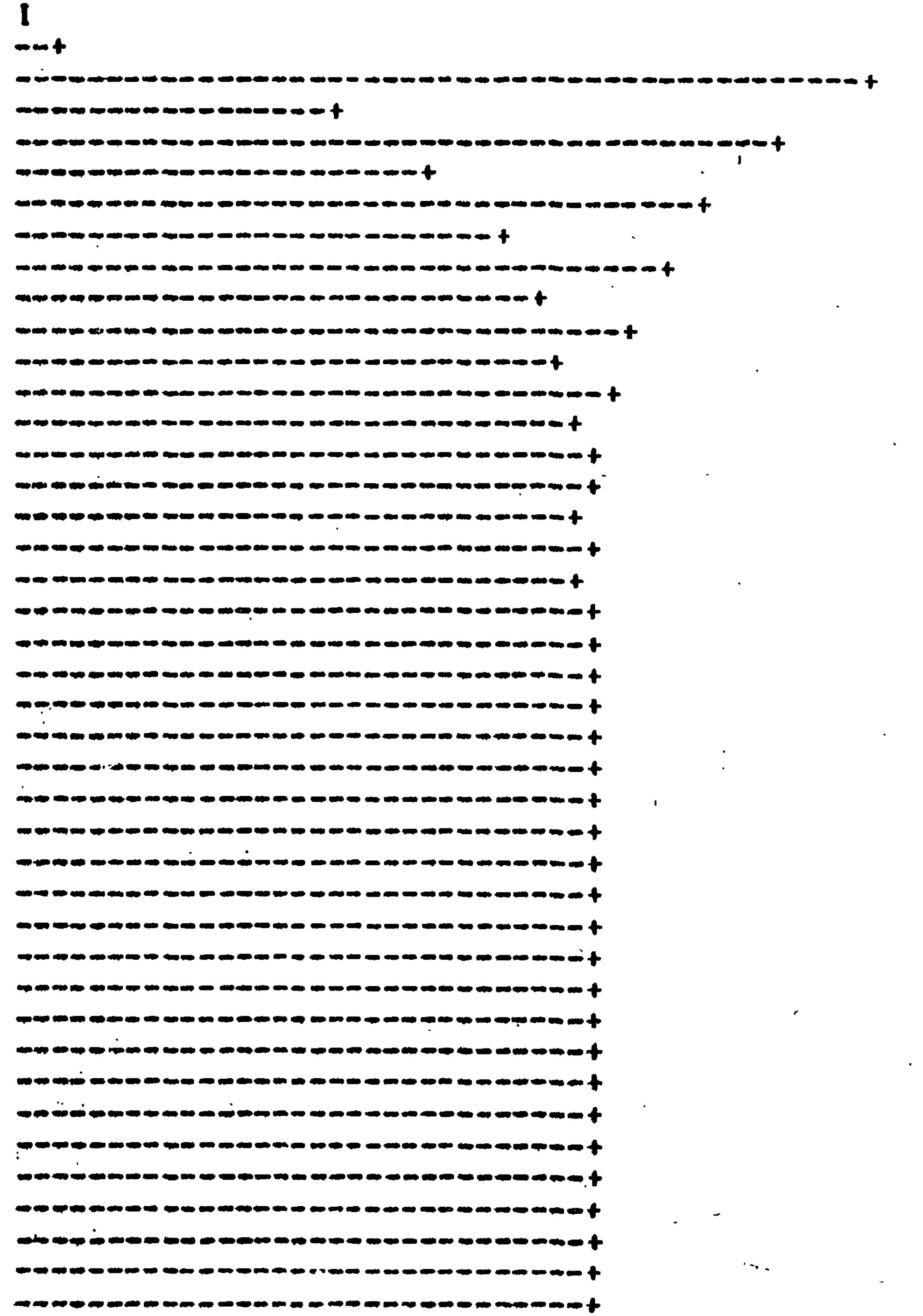

```
                                 MINIMUM                   AS33   VERSUS TIME              MAXIMUM
                               -8.5331E 00                                               -6.0000E 00
  TIME            AS33             I                                                          I
0.0            -6.0000E 00     ---------------------------------------------------------------+
7.5000E-01     -7.9872E 00     -----------+
1.5000E 00     -8.0000E 00     -----------+
2.2500E 00     -8.0000E 00     -----------+
3.0000E 00     -8.0000E 00     -----------+
3.7500E 00     -8.0000E 00     -----------+
4.5000E 00     -8.0000E 00     -----------+
5.2500E 00     -8.0000E 00     -----------+
6.0000E 00     -8.0000E 00     -----------+
6.7500E 00     -8.0000E 00     -----------+
7.5000E 00     -8.0000E 00     -----------+
8.2500E 00     -8.0000E 00     -----------+
9.0000E 00     -8.0000E 00     -----------+
9.7500E 00     -8.0000E 00     -----------+
1.0500E 01     -8.0000E 00     -----------+
1.1250E 01     -8.0000E 00     -----------+
1.2000E 01     -8.0000E 00     -----------+
1.2750E 01     -8.0000E 00     -----------+
1.3500E 01     -8.0000E 00     -----------+
1.4250E 01     -8.0000E 00     -----------+
1.5000E 01     -8.0000E 00     -----------+
1.5750E 01     -8.0000E 00     -----------+
1.6500E 01     -8.0000E 00     -----------+
1.7250E 01     -8.0000E 00     -----------+
1.8000E 01     -8.0000E 00     -----------+
1.8750E 01     -8.0000E 00     -----------+
1.9500E 01     -8.0000E 00     -----------+
2.0250E 01     -8.0000E 00     -----------+
2.1000E 01     -8.0000E 00     -----------+
2.1750E 01     -8.0000E 00     -----------+
2.2500E 01     -8.0000E 00     -----------+
2.3250E 01     -8.0000E 00     -----------+
2.4000E 01     -8.0000E 00     -----------+
2.4750E 01     -8.0000E 00     -----------+
2.5500E 01     -8.0000E 00     -----------+
2.6250E 01     -8.0000E 00     -----------+
2.7000E 01     -8.0000E 00     -----------+
2.7750E 01     -8.0000E 00     -----------+
2.8500E 01     -8.0000E 00     -----------+
2.9250E 01     -8.0000E 00     -----------+
3.0000E 01     -8.0000E 00     -----------+
```

FIG. [illegible]

a_{22} vs. t $(a_{22}^* = -5.0)$

MINIMUM AS22 VERSUS TIME MAXIMUM

-8.2405E 00 -3.0466E 00



TIME	AS22
0.0	-8.0000E 00
7.5000E-01	-3.4455E 00
1.5000E 00	-6.3823E 00
2.2500E 00	-3.8916E 00
3.0000E 00	-5.8620E 00
3.7500E 00	-4.3311E 00
4.5000E 00	-5.5168E 00
5.2500E 00	-4.6009E 00
6.0000E 00	-5.3076E 00
6.7500E 00	-4.7641E 00
7.5000E 00	-5.1793E 00
8.2500E 00	-4.8668E 00
9.0000E 00	-5.0925E 00
9.7500E 00	-4.9532E 00
1.0500E 01	-4.9806E 00
1.1250E 01	-5.0469E 00
1.2000E 01	-4.9644E 00
1.2750E 01	-5.0274E 00
1.3500E 01	-4.9786E 00
1.4250E 01	-5.0165E 00
1.5000E 01	-4.9872E 00
1.5750E 01	-5.0099E 00
1.6500E 01	-4.9924E 00
1.7250E 01	-5.0058E 00
1.8000E 01	-4.9956E 00
1.8750E 01	-5.0032E 00
1.9500E 01	-4.9980E 00
2.0250E 01	-5.0004E 00
2.1000E 01	-5.0014E 00
2.1750E 01	-4.9989E 00
2.2500E 01	-5.0009E 00
2.3250E 01	-4.9994E 00
2.4000E 01	-5.0005E 00
2.4750E 01	-4.9996E 00
2.5500E 01	-5.0003E 00
2.6250E 01	-4.9998E 00
2.7000E 01	-5.0002E 00
2.7750E 01	-4.9999E 00
2.8500E 01	-5.0001E 00
2.9250E 01	-4.9999E 00
3.0000E 01	-5.0000E 00

FIG. 5-a

$b_1(t)$ vs. t   ($b_1^* = 1.0$)

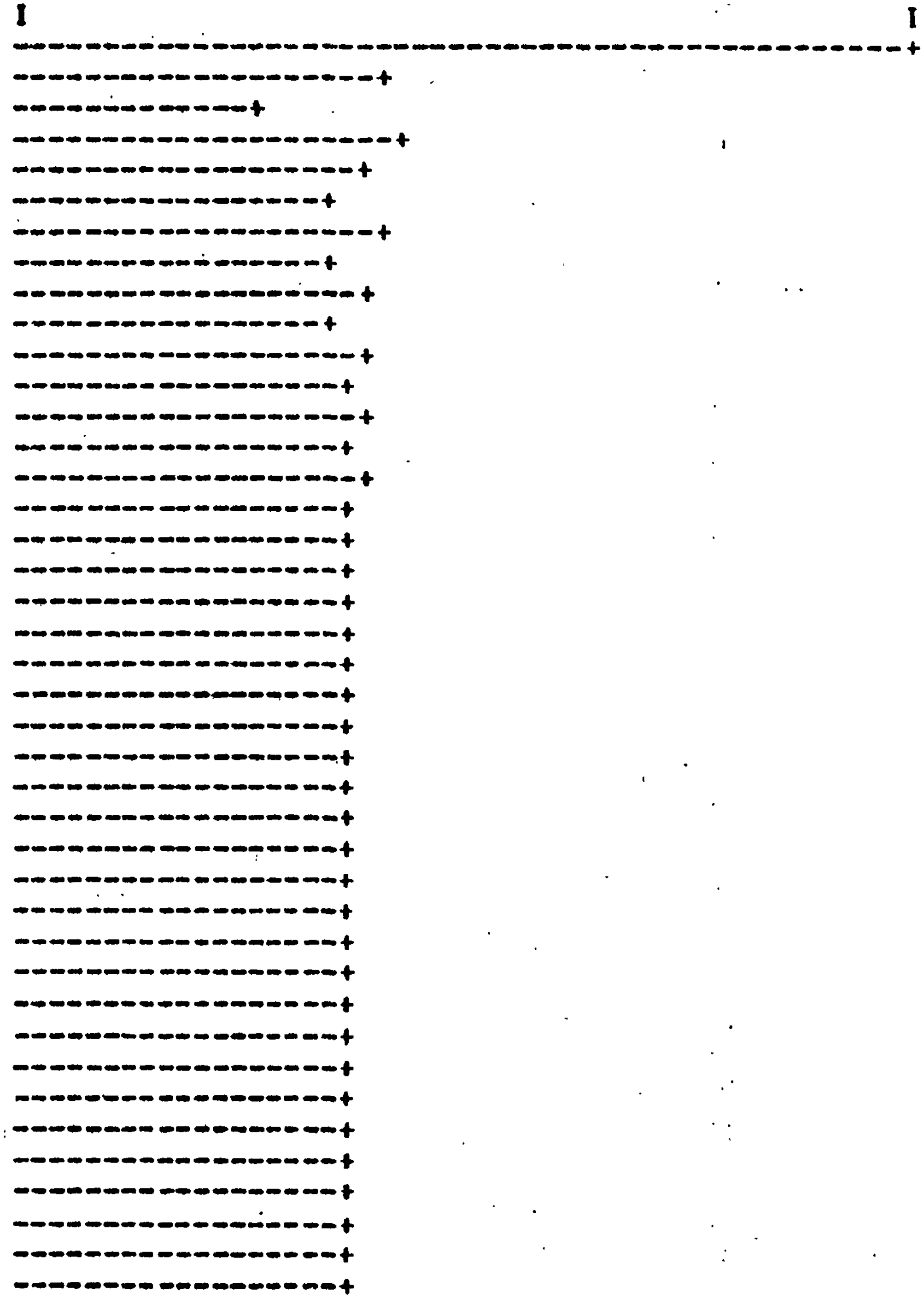
```
                               MINIMUM                BS1    VERSUS TIME                MAXIMUM
                             -3.5963E-01                                              3.0000E 00
 TIME              BS1       I                                                             I
0.0             3.0000E 00   ----------------------------------------------------------------+
7.5000E-01      1.0088E 00   -------------------------+
1.5000E 00      9.9996E-01   -------------------------+
2.2500E 00      1.0000E 00   -------------------------+
3.0000E 00      1.0000E 00   -------------------------+
3.7500E 00      1.0000E 00   -------------------------+
4.5000E 00      1.0000E 00   -------------------------+
5.2500E 00      1.0000E 00   -------------------------+
6.0000E 00      1.0000E 00   -------------------------+
6.7500E 00      1.0000E 00   -------------------------+
7.5000E 00      1.0000E 00   -------------------------+
8.2500E 00      1.0000E 00   -------------------------+
9.0000E 00      1.0000E 00   -------------------------+
9.7500E 00      1.0000E 00   -------------------------+
1.0500E 01      1.0000E 00   -------------------------+
1.1250E 01      1.0000E 00   -------------------------+
1.2000E 01      1.0000E 00   -------------------------+
1.2750E 01      1.0000E 00   -------------------------+
1.3500E 01      1.0000E 00   -------------------------+
1.4250E 01      1.0000E 00   -------------------------+
1.5000E 01      1.0000E 00   -------------------------+
1.5750E 01      1.0000E 00   -------------------------+
1.6500E 01      1.0000E 00   -------------------------+
1.7250E 01      1.0000E 00   -------------------------+
1.8000E 01      1.0000E 00   -------------------------+
1.8750E 01      1.0000E 00   -------------------------+
1.9500E 01      1.0000E 00   -------------------------+
2.0250E 01      1.0000E 00   -------------------------+
2.1000E 01      1.0000E 00   -------------------------+
2.1750E 01      1.0000E 00   -------------------------+
2.2500E 01      1.0000E 00   -------------------------+
2.3250E 01      1.0000E 00   -------------------------+
2.4000E 01      1.0000E 00   -------------------------+
2.4750E 01      1.0000E 00   -------------------------+
2.5500E 01      1.0000E 00   -------------------------+
2.6250E 01      1.0000E 00   -------------------------+
2.7000E 01      1.0000E 00   -------------------------+
2.7750E 01      1.0000E 00   -------------------------+
2.8500E 01      1.0000E 00   -------------------------+
2.9250E 01      1.0000E 00   -------------------------+
3.0000E 01      1.0000E 00   -------------------------+
```

FIG. 5-b

b_2 vs. t ($b_2^* = 2.0$)

BS2 VERSUS TIME

MINIMUM 1.8100E-01 — MAXIMUM 5.0000E 00



TIME	BS2
0.0	5.0000E 00
7.5000E-01	2.1195E 00
1.5000E 00	1.4504E 00
2.2500E 00	2.2060E 00
3.0000E 00	2.1073E 00
3.7500E 00	1.8901E 00
4.5000E 00	2.1181E 00
5.2500E 00	1.8888E 00
6.0000E 00	2.1015E 00
6.7500E 00	1.9082E 00
7.5000E 00	2.0838E 00
8.2500E 00	1.9214E 00
9.0000E 00	2.0775E 00
9.7500E 00	1.9200E 00
1.0500E 01	2.0541E 00
1.1250E 01	1.9979E 00
1.2000E 01	1.9936E 00
1.2750E 01	2.0041E 00
1.3500E 01	1.9975E 00
1.4250E 01	2.0033E 00
1.5000E 01	1.9967E 00
1.5750E 01	2.0030E 00
1.6500E 01	1.9973E 00
1.7250E 01	2.0025E 00
1.8000E 01	1.9977E 00
1.8750E 01	2.0022E 00
1.9500E 01	1.9977E 00
2.0250E 01	2.0021E 00
2.1000E 01	1.9996E 00
2.1750E 01	2.0004E 00
2.2500E 01	1.9999E 00
2.3250E 01	1.9999E 00
2.4000E 01	2.0001E 00
2.4750E 01	1.9999E 00
2.5500E 01	2.0001E 00
2.6250E 01	1.9999E 00
2.7000E 01	2.0001E 00
2.7750E 01	1.9999E 00
2.8500E 01	2.0000E 00
2.9250E 01	1.9999E 00
3.0000E 01	2.0000E 00

FIG. 5-c