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Reflecting Solutions of High Order Elliptic Differential Equations in Two Independent Variables Across Analytic Arcs

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by

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ABSTRACT

Reflecting Solutions of High Order Elliptic Differential Equations in Two Independent Variables Across Analytic Arcs

by

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Consideration is given specifically to sixth order elliptic partial differential equations in two independent real variables x, y such that the coefficients of the highest order terms are real constants. It is assumed that the differential operator has distinct characteristics and that it can be factored as a product of second order operators of the form $\Delta_k + a_k(x_k, y_k)\frac{\partial}{\partial x_k} + b_k(x_k, y_k)\frac{\partial}{\partial y_k} + c_k(x_k, y_k)$, where x_k and y_k are real variables determined from the characteristics, $\Delta_k = \frac{\partial^2}{\partial x_k^{\,2}} + \frac{\partial^2}{\partial y_k^{\,2}}$ and the coefficients a_k, b_k, c_k are, in general, complex-valued analytic functions of their arguments. This class of equations includes those examined by Sloss. By analytically continuing into the complex domain and using the complex characteristic coordinates of the differential equation, it is shown that its solutions, u, may be reflected across analytic arcs on which u satisfies certain analytic boundary conditions. Moreover, a method is given whereby one can determine a region into which the solution is extensible. It is seen that this region of reflection is dependent on the

original domain of definition of the solution, the arc and the coefficients of the highest order terms of the equation and not on any "sufficiently small" quantities; i.e., the reflection is global in nature. The method employed may be applied to similar differential equations of order 2n. Finally, included are some figures illustrating the region of reflection with respect to various arcs for two specific sixth order equations having no lower order terms.

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INTRODUCTION

In this paper, we wish to examine the reflection problem for a class of elliptic partial differential equations that generalizes the type of equations Sloss considered in [7]. Since the geometry of the problem is better illustrated by equations of order greater than four, while equations of order greater than six add nothing new to the understanding of the reflection process presented here, we shall limit our discussion to equations of order six. In particular, we shall consider a sixth order elliptic equation

$$L[u] = \sum_{0 \le i+j \le 6} c_{ij} \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = 0 \tag{0.1}$$

in two independent real variables x, y such that the coefficients of the highest order terms are real constants. We assume that this equation has distinct characteristics and that it can be factored as

$$L[u] = \prod_{k=1}^{3} \left[\Delta_k + a_k \frac{\partial}{\partial x_k} + b_k \frac{\partial}{\partial y_k} + c_k \right] u = 0 \; . \tag{0.2}$$

Here, x_k and y_k are real variables determined from the complex characteristics, $\Delta_k = \frac{\partial^2}{\partial x_k^{\,2}} + \frac{\partial^2}{\partial y_k^{\,2}}$ and the coefficients a_k, b_k, c_k are complex-valued analytic functions of x_k and y_k. We propose to write (0.2) as a system of three second order equations and to assign a general first order analytic boundary condition to each equation of the system. By analytically continuing into the complex domain and using the complex characteristic coordinates of (0.1), we show that solutions, u, of Equation (0.1) may be reflected across analytic arcs on which u satisfies the above-mentioned boundary conditions and, moreover, that this reflection can be carried out in the large. That is to say, we present a method to determine a region into which the solutions can be reflected which depends only on the original domain of definition, the arc, the characteristic coordinates and, possibly, on the choice of certain simply-connected domains.

The problem of reflecting solutions of elliptic partial differential equations in two independent variables by analytic continuation into the complex domain was pioneered by H. Lewy. In a very nice paper, [5], he thoroughly treated the case of a linear second order elliptic equation with analytic coefficients. He showed that if D is a simply-connected domain lying in the half-plane, say $y < 0$, having a segment κ of the x-axis as part of its boundary and such that D contains the portion $y < 0$ of a neighborhood of each point of κ, then u can be analytically continued as a solution into the entire mirror image, $\overline{D}$, of D across κ. Garabedian [3] has examined the problem for second order analytic quasi-linear elliptic equations and indicated how his method may be applied to systems of such equations. In a recent paper, Kraft [4] modified Garabedian's techniques to treat first order elliptic quasi-linear systems in two independent variables. Since both Garabedian and Kraft dealt with nonlinear equations, their results concerning analytic continuation were strictly local and shed no light on the domain of reflection.

As concerns higher order equations and reflection in the large, Sloss [6] has continued solutions of the biharmonic equation across analytic arcs and Brown [1] has investigated the general fourth order elliptic equation with constant coefficients. However, Brown had to restrict himself to convex domains and he reflected only across segments of the x-axis. In [7], Sloss reflected solutions of elliptic equations of order 2n with constant coefficients and with no lower order terms across analytic arcs.

The method of Garabedian and Brown consists of transforming the original elliptic equation in two real variables into a hyperbolic equation in three real variables and then examining certain three dimensional Cauchy problems. This differs from the method of

Lewy and Sloss who utilize the complex characteristic coordinates of the elliptic equation to perform the extension. In this paper, we propose to adapt the techniques of Lewy and Sloss.

Before proceeding, we would like to introduce some notation. Let S be a set in the complex plane. We will consistently use the notation $\overline{S}$ to denote the set $\{\overline{z} : z \in S\}$ where z is a complex number and the bar denotes complex conjugation.

Definition 0.1. Let D be a simply-connected domain in the x, y-plane having an arc κ as part of its boundary. If every point $z_o = x_o + iy_o$ on κ can be joined to every point $z = x + iy$ in D by a rectifiable curve which, except for the end point z_o, lies entirely in D, then D is said to be adjacent to κ.

Finally, unless stated otherwise, we shall deal exclusively with simply-connected domains whose boundaries are closed rectifiable Jordan curves.

CHAPTER I

§1. Continuation into the Complex Domain.

For the most part, we shall be dealing with differential operators of the form

$$L = \Delta + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y) \quad ,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian and a, b, c are, in general, complex-valued analytic functions of the real variables x, y. For this reason, we devote the present chapter to a consideration of the equation

$$(1.1) \qquad L[u] = \Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)\, u = 0 \quad .$$

Our goal is to obtain a suitable representation for the solutions of this equation in terms of analytic functions of a complex variable. We shall use as a guide the presentation given by Vekua in [8], chapter I.

Let $z = x + iy$ (x and y real) and $U(z,\overline{z}) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) = u(x,y)$.

If we formally define the operations

$$(1.2) \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad , \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

then Equation (1.1) can be written as

$$(1.3) \qquad \frac{\partial^2 U}{\partial z \partial \overline{z}} + A(z,\overline{z})\frac{\partial U}{\partial z} + B(z,\overline{z})\frac{\partial U}{\partial \overline{z}} + C(z,\overline{z})\; U = 0 \quad ,$$

where

$$A(z,\zeta) = \frac{1}{4}\left[a\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) + ib\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)\right],$$

(1.4) $$B(z,\zeta) = \frac{1}{4}\left[a\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) - ib\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)\right],$$

$$C(z,\zeta) = \frac{1}{4}\; c\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) .$$

It must be kept in mind that (1.3) is purely a symbolic form of the Equation (1.1). If it happens that $U(z, \bar{z})$ is an analytic function of z and $\bar{z}$ then the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ defined in (1.2) become true derivatives with respect to z and $\bar{z}$; that is,

$$\frac{\partial}{\partial z} U(z,\bar{z}) = \frac{\partial}{\partial z} U(z,\zeta)\Big|_{\zeta=\bar{z}} ,$$

$$\frac{\partial}{\partial \bar{z}} U(z,\bar{z}) = \frac{\partial}{\partial \zeta} U(z,\zeta)\Big|_{\zeta=\bar{z}} ,$$

where ζ is a complex variable independent of z. This prompts us to consider the following equation:

(1.5) $$\mathscr{L}[U] = \frac{\partial^2 U}{\partial z\,\partial\zeta} + A(z,\zeta)\frac{\partial U}{\partial z} + B(z,\zeta)\frac{\partial U}{\partial \zeta} + C(z,\zeta)\,U = 0 .$$

Definition 1.1. Let Ω be a simply-connected domain in the complex plane such that $A(z,\zeta)$, $B(z,\zeta)$ and $C(z,\zeta)$ given in (1.4) are analytic functions of the two independent complex variables z, ζ in the poly-cylindrical region $(\Omega, \bar{\Omega}) = \{(z,\zeta) : z \in \Omega, \zeta \in \bar{\Omega}\}$. Then Ω is said to be a fundamental domain for Equation (1.1) or for Equation (1.5).

We remark that if the coefficients of Equation (1.1) are entire functions of their arguments, then A, B and C of Equation (1.5) will be entire functions of z and ζ and in this case, any simply-connected domain in the complex plane may serve as a fundamental domain for (1.1).

Vekua shows that if Ω is a fundamental domain for Equation (1.1) and if $u(x, y)$ is given as a regular solution of (1.1) in Ω (i.e., u has continuous first and second partial derivatives in Ω), then u is in fact analytic in x and y, and the function $U(z, \zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$ is analytic for $(z, \zeta) \in (\Omega, \overline{\Omega})$ and satisfies Equation (1.5) in that region. On the other hand, if $U(z, \zeta)$ is given to be a solution of (1.5) in $(\Omega, \overline{\Omega})$ (and so necessarily an analytic function of z and ζ), then restriction to the real manifold $\zeta = \overline{z}$ gives $U(z, \overline{z}) = u(x, y)$ as an analytic function of the real variables x, y which satisfies Equation (1.1) in Ω.

Thus, from the preceding discussion, it is evident that Equations (1.1) and (1.5) are in a certain sense equivalent, and that to investigate the solutions of (1.1) we may work with Equation (1.5) and then restrict the results to the real manifold $\zeta = \overline{z}$. We point out that (1.5) has the same form as a real hyperbolic equation. We shall take advantage of this fact to obtain a representation for the solutions of (1.1).

§2. The Riemann Function.

In a manner completely analogous to the real case, the adjoint equation relative to (1.5) is defined to be

$$\mathcal{L}^*[V] = \frac{\partial^2 V}{\partial z \partial \zeta} - \frac{\partial (AV)}{\partial z} - \frac{\partial (BV)}{\partial \zeta} + CV = 0 . \tag{1.6}$$

A straightforward calculation serves to establish the fundamental identity

$$V\mathcal{L}[U] - U\mathcal{L}^*[V] = \frac{\partial}{\partial z}\left(V\frac{\partial U}{\partial \zeta} + AUV\right) + \frac{\partial}{\partial \zeta}\left(-U\frac{\partial V}{\partial z} + BUV\right), \tag{1.7}$$

which is valid for any two functions $U(z, \zeta)$ and $V(z, \zeta)$ analytic in a common region $(z, \zeta) \in (\Omega, \overline{\Omega})$.

Definition 1.2. Let Ω be a fundamental domain for Equation (1.1). The Riemann Function, $R(z, \zeta; t, \tau)$, for (1.1) (or, for Equation (1.5)) is an analytic function of four complex variables z, t in Ω, ζ, τ in $\overline{\Omega}$ which satisfies

$$\mathcal{L}^*[R] = \frac{\partial^2 R}{\partial z \partial \zeta} - \frac{\partial (AR)}{\partial z} - \frac{\partial (BR)}{\partial \zeta} + CR = 0 \tag{1.8}$$

with respect to the variables z and ζ, and the conditions

$$\begin{aligned} R(t, \zeta; t, \tau) &= \exp\left[\int_\tau^\zeta A(t, \eta)\, d\eta\right], \\ R(z, \tau; t, \tau) &= \exp\left[\int_t^z B(\xi, \tau)\, d\xi\right], \end{aligned} \tag{1.9}$$

where t and τ are fixed parameters in Ω and $\overline{\Omega}$, respectively.

The existence and uniqueness of the Riemann function may be established by first writing (1.8) in the form

$$(1.10) \qquad \frac{\partial^2}{\partial z \partial \zeta} \Big[R(z, \zeta; t, \tau) - \int_\tau^\zeta A(z, \eta) R(z, \eta; t, \tau)\, d\eta - \int_t^z B(\xi, \zeta) R(\xi, \zeta; t, \tau) d\xi + \int_t^z d\xi \int_\tau^\zeta C(\xi, \eta) R(\xi, \eta; t, \tau) d\eta \Big] = 0$$

and then integrating with respect to z and ζ to get

$$(1.11) \qquad R(z, \zeta; t, \tau) - \int_\tau^\zeta A(z, \eta) R(z, \eta; t, \tau) d\eta - \int_t^z B(\xi, \zeta) R(\xi, \zeta; t, \tau) d\xi + \int_t^z d\xi \int_\tau^\zeta C(\xi, \eta) R(\xi, \eta; t, \tau) d\eta = 1 ,$$

since (1.9) implies that $R(t, \tau; t, \tau) = 1$. In view of the fact that conditions (1.9) are equivalent to the conditions

$$(1.12) \qquad \begin{aligned} &\frac{\partial}{\partial \zeta} R(t, \zeta; t, \tau) - A(t, \zeta) R(t, \zeta; t, \tau) = 0 , \\ &\frac{\partial}{\partial z} R(z, \tau; t, \tau) - B(z, \tau) R(z, \tau; t, \tau) = 0 , \\ &R(t, \tau; t, \tau) = 1 , \end{aligned}$$

it is not difficult to verify that any solution of Equation (1.11) satisfies

(1.8) and conditions (1.9) and so must be the Riemann function for (1.1). But, the integral equation (1.11) is of Volterra type which, as Vekua shows, has a unique analytic solution for z, t in Ω and ζ, τ in $\overline{\Omega}$. Thus, the existence and uniqueness of the Riemann function is guaranteed.

Just as in the real case, it can be shown that as a function of its last two arguments, t and τ, the Riemann function for (1.5) satisfies Equation (1.5) and the conditions

$$\frac{\partial}{\partial \tau} R(z, \zeta; z, \tau) + A(z, \tau) R(z, \zeta; z, \tau) = 0 ,$$

$$(1.13) \qquad \frac{\partial}{\partial t} R(z, \zeta; t, \zeta) + B(t, \zeta) R(z, \zeta; t, \zeta) = 0 ,$$

$$R(z, \zeta; z, \zeta) = 1 ;$$

i.e., with respect to the variables t and τ, the function $R(z, \zeta; t, \tau)$ is the Riemann function for the adjoint equation (1.6).

In case the coefficients a, b, c of Equation (1.1) are all real valued, the Riemann function, $R(z, \zeta; t, \tau)$, for (1.5) has the additional important property that it assumes real values when $\zeta = \overline{z}$ and $\tau = \overline{t}$ (see [8], § 12).

§ 3. A Representation for Solutions of Equation (1.1).

In order to obtain a representation for the solutions of Equation (1.1) that will lend itself to reflection across analytic arcs κ, it is necessary to examine what happens to the solutions of Equation (1.5) as z and ζ are allowed to approach κ and $\overline{\kappa}$, respectively. Towards this end, we adapt the techniques of Lewy, [5], to establish the following lemma.

Lemma 1.1. Let D be a simply-connected domain adjacent to an arc κ such that $D \cup \kappa$ is contained in a fundamental domain for Equation (1.1). Let $u(x, y)$ be given as a regular solution of (1.1) in D and let $U(z, \zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$. If $u(x, y)$ and its first partial derivatives are continuous in $D \cup \kappa$, then for any pair of fixed points $z_o \in D \cup \kappa$, $\zeta_o \in \overline{D \cup \kappa}$, the functions $U(z, \zeta_o)$ and $U(z_o, \zeta)$ $\left(\text{and therefore } \frac{\partial U(z, \zeta_o)}{\partial z} \text{ and } \frac{\partial U(z_o, \zeta)}{\partial \zeta}\right)$ are analytic for $z \in D$ and $\zeta \in \overline{D}$, respectively. Furthermore, $\frac{\partial U(z, \zeta_o)}{\partial z}$ $\left(\text{and therefore } U(z, \zeta_o)\right)$ is continuous on $D \cup \kappa$ while $\frac{\partial U(z_o, \zeta)}{\partial \zeta}$ $\left(\text{and therefore } U(z_o, \zeta)\right)$ is continuous on $\overline{D \cup \kappa}$.

Proof: Replace V in Equation (1.7) by the Riemann function $R(z, \zeta; t, \tau)$. Keeping in mind that $U(z, \zeta)$ satisfies (1.5) in $(D, \overline{D})$, identity (1.7) becomes

$$\frac{\partial}{\partial z}\left(R\frac{\partial U}{\partial \zeta} + ARU\right) + \frac{\partial}{\partial \zeta}\left(-U\frac{\partial R}{\partial z} + BRU\right) = 0 .$$

Using the fact that $R\frac{\partial U}{\partial \zeta} = \frac{\partial(RU)}{\partial \zeta} - U\frac{\partial R}{\partial \zeta}$, we rewrite this relation as

$$(1.14) \qquad -\frac{\partial^2}{\partial z\,\partial \zeta}\left[R(z, \zeta; t, \tau)\,U(z, \zeta)\right] + \frac{\partial}{\partial z}\left[U\left(\frac{\partial R}{\partial \zeta} - AR\right)\right]$$

$$+ \frac{\partial}{\partial \zeta}\left[U\left(\frac{\partial R}{\partial z} - BR\right)\right] = 0 .$$

Now interchange the pairs (z, ζ) and (t, τ) in (1.14) and set

$$P(t, \tau; z, \zeta) = \frac{\partial}{\partial t}\left[R(t, \tau; z, \zeta)\,U(t, \tau)\right] - U\left(\frac{\partial R}{\partial t} - BR\right) ,$$

$$Q(t, \tau; z, \zeta) = U(t, \tau)\left(\frac{\partial R}{\partial \tau} - AR\right)$$

to obtain

$$(1.15) \qquad \frac{\partial}{\partial t} Q(t, \tau; z, \zeta) - \frac{\partial}{\partial \tau} P(t, \tau; z, \zeta) = 0 ,$$

where (t, τ) varies in $(D, \overline{D})$ and (z, ζ) is arbitrarily fixed in $(D, \overline{D})$.

Pick a point $\overline{\zeta_o}$ on κ and a point z in D and join them by a rectifiable curve, C, lying in D. Let $\overline{C}$ be the conjugate path lying in D having ζ_o and $\overline{z}$ as end points. Fix a point ζ on $\overline{C}$, ζ not equal to ζ_o or $\overline{z}$. Denote the part of $\overline{C}$ joining ζ and $\overline{z}$ by $\overline{C}_\zeta$. Then C_ζ is that part of C joining $\overline{\zeta}$ and z (see Figure 1.1). We form the Cartesian product $S_\zeta^2 = C_\zeta \times \overline{C}_\zeta$ and define

$$d_\zeta = \{(t, \overline{t}) : t \in C_\zeta\} .$$

We shall refer to d_ζ as the diagonal of S_ζ^2 .

We now integrate* the functions $\frac{\partial P}{\partial \tau}$ and $\frac{\partial Q}{\partial t}$ which appear in relation (1.15) over the "triangle" whose sides are d_ζ, $\{(t,\zeta) : t \in C_\zeta\}$ and $\{(z, \tau) : \tau \in \overline{C}_\zeta\}$ (see Figure 1.2) in the following way:

(1.16) $$\int_{\zeta}^{z} dt \int_{\zeta}^{\overline{t}} \frac{\partial P(t,\tau;z,\zeta)}{\partial \tau} d\tau = \int_{\zeta}^{z} [P(t,\overline{t};z,\zeta) - P(t,\zeta;z,\zeta)]\,dt ,$$

(1.17) $$\int_{\zeta}^{z} dt \int_{\zeta}^{\overline{t}} \frac{\partial Q(t,\tau;z,\zeta)}{\partial t} d\tau = \int_{\zeta}^{\overline{z}} d\tau \int_{\overline{\tau}}^{z} \frac{\partial Q(t,\tau;z,\zeta)}{\partial t} dt$$

$$= \int_{\zeta}^{\overline{z}} [Q(z,\tau;z,\zeta) - Q(\overline{\tau},\tau;z,\zeta)]\,d\tau.$$

In view of conditions (1.12) satisfied by the Riemann function, we have

$$\int_{\zeta}^{z} P(t,\zeta;z,\zeta)\,dt = \int_{\zeta}^{z} \frac{\partial}{\partial t} [R(t,\zeta;z,\zeta)\,U(t,\zeta)]\,dt$$

$$= U(z,\zeta) - U(\overline{\zeta},\zeta)R(\overline{\zeta},\zeta;z,\zeta) ,$$

and

$$\int_{\zeta}^{\overline{z}} Q(z,\tau;z,\zeta)\,d\tau = 0 .$$

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* The steps involving integrations over the triangle pictured in Figure 1.2 can be fully justified by parametrizing the curves C and $\overline{C}$ with respect to two real parameters varying over a common interval and then decomposing the complex line integrals in the usual way as a sum of real integrals.

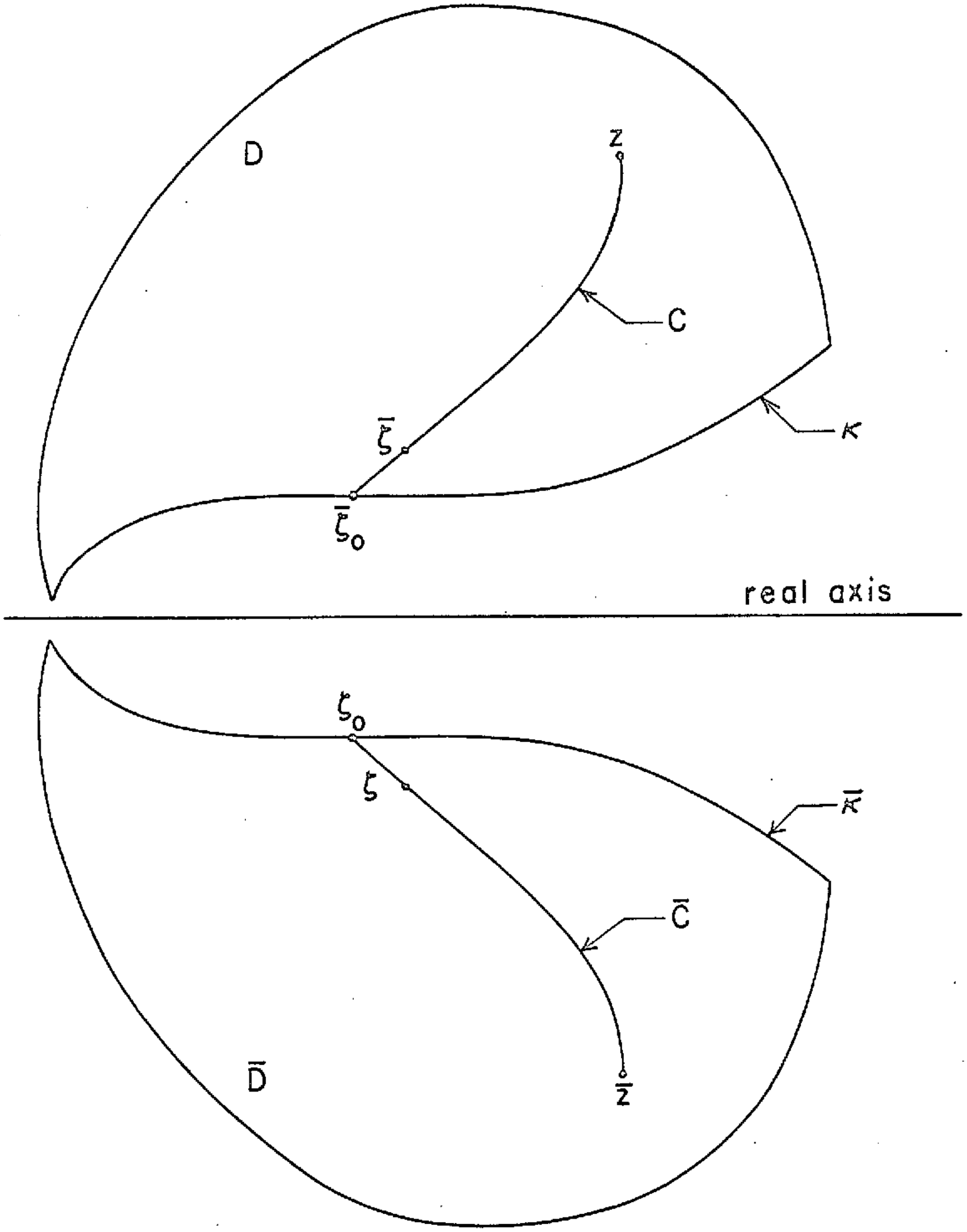
D
z
C
$\bar{\zeta}$
κ
$\bar{\zeta}_0$
real axis
ζ_0
ζ
$\bar{\kappa}$
$\bar{C}$
$\bar{D}$
$\bar{z}$

Figure I.1

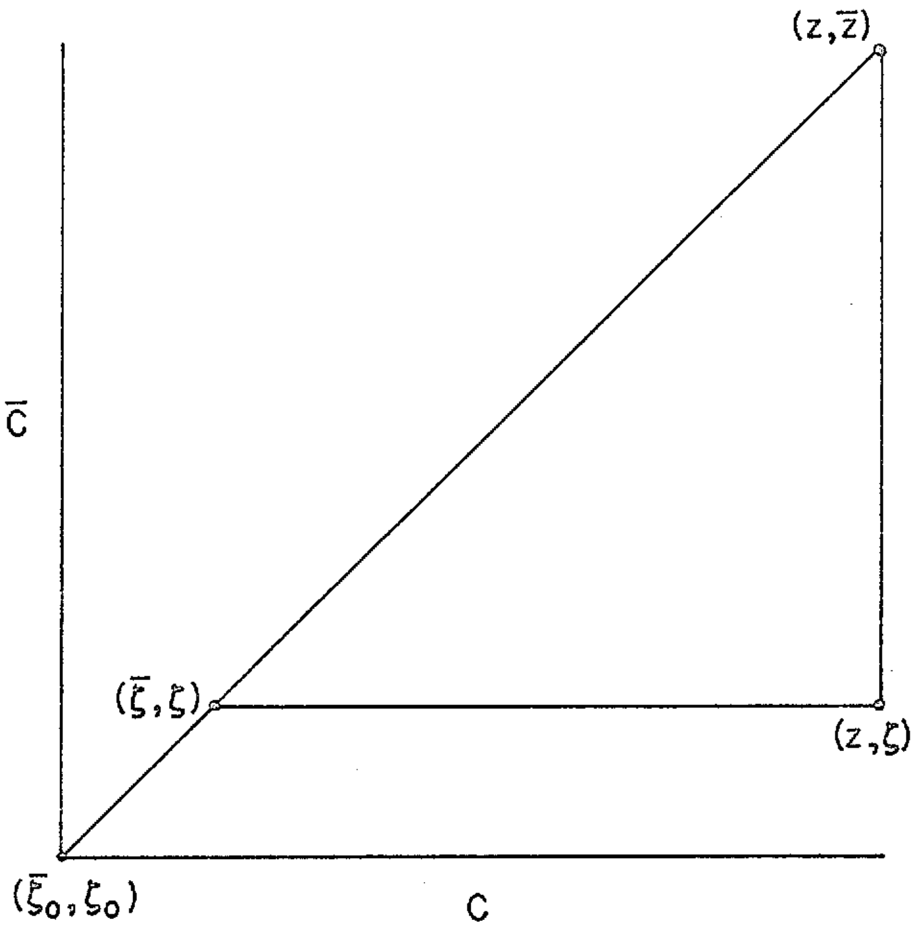
(z,$\overline{z}$)
$\overline{C}$
($\overline{\zeta}$,ζ)
(z,ζ)
($\overline{\zeta}_o$,ζ_o)
C

Figure 1.2

Whence, subtracting (1.16) from (1.17) and taking into account (1.15), we obtain the formula

$$(1.18)\quad U(z,\zeta) = U(\bar{\zeta},\zeta)\,R(\bar{\zeta},\zeta;z,\zeta) + \int_{d_\zeta} P(t,\bar{t};z,\zeta)\,dt + Q(t,\bar{t};z,\zeta)\,d\bar{t}$$

$$= U(\bar{\zeta},\zeta)\,R(\bar{\zeta},\zeta;z,\zeta) + \int_{d_\zeta}\Big\{\frac{\partial}{\partial t}[R(t,\bar{t};z,\zeta)\,U(t,\bar{t})]$$

$$- U\Big(\frac{\partial R}{\partial t} - BR\Big)\Big\}\,dt + \int_{d_\zeta} U(t,\bar{t})\Big(\frac{\partial R}{\partial \bar{t}} - AR\Big)\,d\bar{t}\quad ,$$

where

$$(1.19)\quad \int_{d_\zeta} P(t,\bar{t};z,\zeta)\,dt + Q(t,\bar{t};z,\zeta)\,d\bar{t} = \int_{\bar{\zeta}}^{z} P(t,\bar{t};z,\zeta)\,dt$$

$$+ \int_{\zeta}^{\bar{z}} Q(t,\bar{t};z,\zeta)\,d\bar{t}$$

is a line integral over that part of the diagonal of S_ζ^2 from $(\bar{\zeta},\zeta)$ to $(z,\bar{z})$, and where we have replaced τ by $\bar{t}$. Reverting to the real variables x and y through the transformation

$$x = \frac{t+\bar{t}}{2}\ ,\quad y = \frac{t-\bar{t}}{2i}$$

and writing $dt = dx + i\,dy$, $d\bar{t} = dx - i\,dy$, it is seen that (1.19) is actually a line integral over the path C_ζ in D. Keeping in mind that

$$\frac{\partial}{\partial t} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\ ,\quad \frac{\partial}{\partial \bar{t}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\quad ,$$

it is then easily checked that the condition

$$(1.20) \qquad \frac{\partial\, Q(t,\overline{t};\ z,\zeta)}{dt} - \frac{\partial\, P(t,\overline{t};\ z,\zeta)}{d\overline{t}} = 0$$

is equivalent to the familiar condition that (1.19) be independent of the path. But the truth of (1.20) is immediate from (1.15) when τ is restricted to be $\overline{t}$.

Since $U(t,\overline{t}) = u(x,y)$ as well as its first partial derivatives are continuous on $D \cup \kappa$, we may let the variable ζ approach ζ_o in Formula (1.18) to obtain

$$(1.21) \qquad U(z,\zeta_o) = U(\overline{\zeta}_o,\zeta_o)\, R(\overline{\zeta}_o,\zeta_o;z,\zeta_o) + \int_{d_{\zeta_o}} \left\{ \frac{\partial}{\partial t}\,[R(t,\overline{t};z,\zeta_o)U(t,\overline{t})] - U\left(\frac{\partial R}{\partial t} - BR\right)\right\} dt + \int_{d_{\zeta_o}} U(t,\overline{t}) \left[\frac{\partial}{\partial \overline{t}}\, R(t,\overline{t};z,\zeta_o) - AR\right] d\overline{t}\ ,$$

where the line integral is independent of the path joing ζ_o to z. Formula (1.21) is valid for all z in D and shows that $U(z,\zeta_o)$ is continuous on $D \cup \kappa$. To verify the analyticity of $U(z,\zeta_o)$ on D, it suffices to check the Cauchy-Riemann equations, $\frac{\partial\, U(z,\zeta_o)}{\partial \overline{z}} = 0$. Since the first two terms of (1.21) depend analytically on z, their derivatives with respect to $\overline{z}$ are zero.* Calculating the derivative of the last term we get

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* The line integral $\int_{d_{\zeta_o}} P dt + Q d\overline{t} = \int_{\zeta_o}^{z} P(t,\overline{t};\ z,\zeta_o)dt + \int_{\zeta_o}^{\overline{z}} Q(t,\overline{t};z,\zeta_o)d\overline{t}$ is independent of the path and P is a continuous function of t in D while Q is a continuous function of $\overline{t}$ in $\overline{D}$. Whence, by Morera's theorem, the integral $\int_{\zeta_o}^{z} P(t,\overline{t};z,\zeta_o)dt$ is an analytic function of its upper limit and $P(z,\overline{z};z,\zeta_o)$ is an analytic function of z, while $\int_{\zeta_o}^{\overline{z}} Q(t,\overline{t};z,\zeta_o)d\overline{t}$ is an analytic function of its upper limit.

$$\frac{\partial}{\partial \bar z}\int_{\zeta_o}^{\bar z} U(t,\bar t)\left[\frac{\partial}{\partial \bar t} R(t,\bar t;z,\zeta_o) - A(t,\bar t)R(t,\bar t;z,\zeta_o)\right] d\bar t$$

$$= U(z,\bar z)\left[\frac{\partial}{\partial \zeta} R(z,\zeta;z,\zeta_o) - A(z,\zeta)R(z,\zeta;z,\zeta_o)\right]_{\zeta=\bar z} = 0,$$

where the last equality follows from (1.12). Thus $U(z,\zeta_o)$ is indeed an analytic function of z in D, and therefore so also is $\frac{\partial U(z,\zeta_o)}{\partial z}$. Differentiating the right hand side of (1.21) with respect to z gives

$$\frac{\partial U(z,\zeta_o)}{\partial z} = U(\bar\zeta_o,\zeta_o)\frac{\partial}{\partial z} R(\bar\zeta_o,\zeta_o;z,\zeta_o) + P(z,\bar z;z,\zeta_o)$$

$$+\int_{\bar\zeta_o}^{z}\frac{\partial}{\partial z} P(t,\bar t;z,\zeta_o)\,dt + \int_{\zeta_o}^{\bar z}\frac{\partial}{\partial z} Q(t,\bar t;z,\zeta_o)\,d\bar t,$$

which is a continuous function of z in $D \cup \kappa$.

The desired analyticity and continuity of $U(z_o,\zeta)$ and $\frac{\partial U(z_o,\zeta)}{\partial \zeta}$ may be established by replacing z with z_o in Equation (1.18) and interchanging the limits of integration in (1.19) to get

$$(1.22)\qquad U(z_o,\zeta) = U(\bar\zeta,\zeta)R(\bar\zeta,\zeta;z_o,\zeta) - \int_{z_o}^{\bar\zeta}\left\{\frac{\partial}{\partial t}\left[U(t,\bar t)R(t,\bar t;z_o,\zeta)\right]\right.$$

$$\left. - U\left(\frac{\partial R}{\partial t} - BR\right)\right\} dt - \int_{\bar z_o}^{\zeta} U(t,\bar t)\left[\frac{\partial R}{\partial \bar t} - AR\right] d\bar t .$$

This formula shows that $U(z_o,\zeta)$ is continuous for ζ in $\overline{D \cup \kappa}$. To check the Cauchy-Riemann equations, differentiate (1.22) with respect to $\bar\zeta$. Taking into account Condition (1.12) and the fact that the last term on the right hand side is an analytic function of ζ, this calculation gives $\frac{\partial U(z_o,\zeta)}{\partial\bar{\zeta}} = 0$. Thus, $U(z_o,\zeta)$ as well as $\frac{\partial U(z_o,\zeta)}{\partial\zeta}$ are analytic functions of ζ in $\bar{D}$. Finally, differentiating (1.22) with respect to ζ shows that $\frac{\partial U(z_o,\zeta)}{\partial\zeta}$ is continuous for ζ in $\overline{D \cup \kappa}$.

To finish the proof, we point out that when $\zeta_o \in \bar{D}$ and $z_o \in D$, the analyticity of $U(z,\zeta_o)$ and $U(z_o,\zeta)$ follows immediately from the analyticity of $U(z,\zeta)$ in $(D,\bar{D})$. The expressed continuity of $\frac{\partial U(z,\zeta_o)}{\partial z}$ and $\frac{\partial U(z_o,\zeta)}{\partial\zeta}$ is seen by replacing ζ or z by ζ_o or z_o, respectively, in Formula (1.18) and then performing the necessary differentiation. This completes the proof of the lemma.

An important property of the solutions of Equation (1.5) in case the coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$ of Equation (1.1) are real-valued functions is given by the following theorem.

Theorem 1.1. Let D and κ be given as in the statement of Lemma 1.1. Assume that the coefficients $a(x,y)$ and $b(x,y)$ of Equation (1.1) are real-valued functions. Let $u(x,y)$ be a real-valued regular solution of (1.1) in D such that u and its first partial derivatives are continuous in $D \cup \kappa$. Let $U(z,\zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$. If z_o is a fixed point on κ, then

$$U(z,\bar{z}_o) = \overline{U(z_o,\bar{z})}$$

$$\frac{\partial U(z,\bar{z}_o)}{\partial z} = \overline{\frac{\partial U(z_o,\bar{z})}{\partial\bar{z}}}$$

for z in $D \cup \kappa$. Whence, the functions $\varphi(z)$ and $\varphi^*(\zeta)$ given by

$$\varphi(z) = \frac{\partial U(z, \bar{z}_o)}{\partial z} + B(z, \bar{z}_o) U(z, \bar{z}_o)$$

$$\varphi^*(\zeta) = \frac{\partial U(z_o, \zeta)}{\partial \zeta} + A(z_o, \zeta) U(z_o, \zeta)$$

take complex conjugate values when $\zeta = \bar{z}$.

Proof: We establish the theorem for the case when z_o is in D and then let z_o approach κ .

Fix $z_o = x_o + i\, y_o$ in D. Then u has a Taylor series representation in a neighborhood of the point (x_o, y_o) given by

$$u(x, y) = \sum_{m, n=0}^{\infty} a_{mn} (x - x_o)^m (y - y_o)^n \quad ,$$

where the a_{mn} are real constants dependent upon (x_o, y_o). From this, we obtain

$$U(z, \zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = \sum_{m, n=0}^{\infty} a_{mn} \left(\frac{z+\zeta}{2} - \frac{z_o + \bar{z}_o}{2}\right)^m \left(\frac{z-\zeta}{2i} - \frac{z_o - \bar{z}_o}{2i}\right)^n$$

$$= \sum_{m, n=0}^{\infty} a_{mn} \frac{1}{2^{m+n}} \left(\frac{1}{i}\right)^n \left[(z - z_o) + (\zeta - \bar{z}_o)\right]^m \left[(z - z_o) - (\zeta - \bar{z}_o)\right]^n$$

which is absolutely and uniformly convergent for (z, ζ) in some polycylindrical neighborhood of $(z_o, \bar{z}_o)$. In particular,

$$(1.23) \qquad U(z, \bar{z}_o) = \sum_{m+n=0}^{\infty} a_{mn} \frac{1}{2^{m+n}} \left(\frac{1}{i}\right)^n (z - z_o)^{m+n}$$

and

$$(1.24) \qquad U(z_o, \bar{z}) = \sum_{m+n=0}^{\infty} a_{mn} \frac{1}{2^{m+n}} \left(\frac{-1}{i}\right)^n (\bar{z} - \bar{z}_o)^{m+n} \quad .$$

Whence,

$$\overline{U(z_o, \overline{z})} = \overline{\sum_{m+n=0}^{\infty} a_{mn} \frac{1}{2^{m+n}} \left(\frac{-1}{i}\right)^n (\overline{z} - \overline{z}_o)^{m+n}} \tag{1.25}$$

$$= \sum_{m+n=0}^{\infty} a_{mn} \frac{1}{2^{m+n}} \left(\frac{1}{i}\right)^n (z - z_o)^{m+n}$$

$$= U(z, \overline{z}_o)$$

for all z in some neighborhood for z_o. Since $U(z, \overline{z}_o)$ and $\overline{U(z_o, \overline{z})}$ are analytic for z in D ($\overline{U(z_o, \overline{z})}$ is an antianalytic function of an antianalytic function) and since they agree on a set with a limit point, we may conclude that $U(z, \overline{z}_o) = \overline{U(z_o, \overline{z})}$ throughout all of D. By Lemma 1.1 we can let z_o approach κ to conclude that $U(z, \overline{z}_o) = \overline{U(z_o, \overline{z})}$ for z_o on κ and z in $D \cup \kappa$.

After differentiating expressions (1.23) and (1.24) with respect to z and $\overline{z}$, respectively, we may apply the same argument as above to establish that

$$\frac{\partial U(z, \overline{z}_o)}{\partial z} = \overline{\frac{\partial U(z_o, \overline{z})}{\partial \overline{z}}}$$

for z_o on κ and z in $D \cup \kappa$.

To prove the last part of the theorem, let $\tilde{a}(z, \zeta) = a\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$ and $\tilde{b}(z, \zeta) = b\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$. Then $\tilde{a}(z, \overline{z}) = a(x, y)$ and $\tilde{b}(z, \overline{z}) = b(x, y)$ are real analytic functions and so the same argument as above may be used to establish that $\tilde{a}(z, \overline{z}_o) = \overline{\tilde{a}(z_o, \overline{z})}$ and $\tilde{b}(z, \overline{z}_o) = \overline{\tilde{b}(z_o, \overline{z})}$.

Whence, by the definition of $A(z,\zeta)$ and $B(z,\zeta)$ given in (1.4),

$$4\,A(z_o,\overline{z}) = \tilde{a}\,(z_o,\overline{z}) + i\,\tilde{b}\,(z_o,\overline{z})$$

and

$$\begin{aligned} 4\,\overline{A(z_o,\overline{z})} &= \overline{\tilde{a}\,(z_o,\overline{z})} - \overline{i\,\tilde{b}(z_o,\overline{z})} \\ &= \tilde{a}\,(z,\overline{z}_o) - i\,\tilde{b}\,(z,\overline{z}_o) \\ &= 4\,B\,(z,\overline{z}_o)\ . \end{aligned}$$

It is now an easy matter to check that $\varphi(z) = \overline{\varphi^*(\overline{z})}$. This completes the proof of the theorem.

We are now ready to write down a representation for the solutions of Equation (1.1) in terms of analytic functions of a complex variable.

Theorem 1.2. Let D be a simply-connected domain adjacent to an arc κ such that $D \cup \kappa$ is contained in a fundamental domain for Equation (1.1). If $u(x,y)$ is a regular solution of (1.1) in D and if u and its first partial derivatives are continuous in $D \cup \kappa$ and if $U(z,\zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$, then

$$(1.26) \qquad u(x,y) = \alpha_o\,R(z_o,\overline{z}_o;z,\overline{z}) + \int_{z_o}^{z} \varphi(t)\,R(t,\overline{z}_o;z,\overline{z})\,dt + \int_{\overline{z}_o}^{\overline{z}} \varphi^*(\tau)\,R(z_o,\tau;z,\overline{z})\,d\tau\,,$$

where $z = x + iy$ is in D, $z_o = x_o + i y_o$ is a fixed point on κ, $\alpha_o = U(z_o, \overline{z}_o)$, R is the Riemann function for (1.1) and

$$\varphi(z) = \frac{\partial U(z, \overline{z}_o)}{\partial z} + B(z, \overline{z}_o) U(z, \overline{z}_o) ,$$

$$\varphi^*(\zeta) = \frac{\partial U(z_o, \zeta)}{\partial \zeta} + A(z_o, \zeta) U(z_o, \zeta) ,$$

which are analytic functions of z and ζ in D and $\overline{D}$ and continuous in $D \cup \kappa$ and $\overline{D \cup \kappa}$ respectively.

Conversely, if α_o is an arbitrary constant and $\varphi(z)$ and $\varphi^*(\zeta)$ are arbitrary analytic functions in D and $\overline{D}$, continuous in $D \cup \kappa$ and $\overline{D \cup \kappa}$, respectively, then Formula (1.26) represents the regular solutions of Equation (1.1), continuous in $D \cup \kappa$.

Remark: If the coefficients $a(x, y)$, $b(x, y)$ and $c(x, y)$ of Equation (1.1) are real-valued functions and if $u(x, y)$ is a real regular solution of (1.1), then α_o is a real constant, $R(z_o, \overline{z}_o, z, \overline{z})$ is a real-valued function and from Theorem 1.1, $\varphi^*(\zeta) = \overline{\varphi(\overline{\zeta})}$ and $R(t, \overline{z}_o; z, \overline{z}) = \overline{R(z_o, \overline{t}; z, \overline{z})}$. In this case, Formula (1.26) may be written as

$$u(x, y) = \alpha_o R(z_o, \overline{z}_o; z, \overline{z}) + 2 \operatorname{Re}\left\{\int_{z_o}^{z} \varphi(t) R(t, \overline{z}_o; z, \overline{z})\, dt\right\} .$$

Proof of theorem: In identity (1.7), replace V by the Riemann function $R(z, \zeta; t, \tau)$, interchange the ordered pairs (z, ζ) and (t, τ) and use the fact that $U(t, \tau)$ satisfies (1.5) in t and τ to get

$$(1.27) \qquad \frac{\partial^2}{\partial t \partial \tau}\left[U(t,\tau)R(t,\tau;z,\zeta)\right] = \frac{\partial}{\partial t}\left[U(t,\tau)\left(\frac{\partial R}{\partial \tau} - AR\right)\right] + \frac{\partial}{\partial \tau}\left[U(t,\tau)\left(\frac{\partial R}{\partial t} - BR\right)\right] .$$

We wish to integrate the above expression with respect to t from z_o to z and with respect to τ from $\overline{z}_o$ to ζ. Note that because of Lemma 1.1, z_o may be taken to be on κ .

In view of Conditions (1.12), the above integrals may be evaluated as follows:

$$(1.28) \qquad \int_{z_o}^{z} dt \int_{\overline{z}_o}^{\zeta} \frac{\partial^2}{\partial t \partial \tau}\left[U(t,\tau)R(t,\tau;z,\zeta)\right] d\tau$$

$$= \int_{z_o}^{z} \frac{\partial}{\partial t}\left[U(t,\zeta)R(t,\zeta;z,\zeta) - U(t,\overline{z}_o)R(t,\overline{z}_o;z,\zeta)\right] dt$$

$$= U(z,\zeta) - U(z,\overline{z}_o)R(z,\overline{z}_o;z,\zeta) - U(z_o,\zeta)R(z_o,\zeta;z,\zeta)$$

$$+ U(z_o,\overline{z}_o)R(z_o,\overline{z}_o;z,\zeta) ,$$

$$(1.29)\quad \int_{z_o}^{z} dt \int_{\overline{z}_o}^{\zeta} \frac{\partial}{\partial t}\left[U(t,\tau)\left(\frac{\partial R}{\partial \tau} - AR\right)\right] d\tau = \int_{\overline{z}_o}^{\zeta} d\tau \int_{z_o}^{z} \frac{\partial}{\partial t}\left[U(t,\tau)\left(\frac{\partial R}{\partial \tau} - AR\right)\right] dt$$

$$= -\int_{\overline{z}_o}^{\zeta} U(z_o,\tau)\left(\frac{\partial}{\partial \tau} R(z_o,\tau;z,\zeta) - AR\right) d\tau$$

$$= -\int_{\overline{z}_o}^{\zeta} \frac{\partial}{\partial \tau}\left[U(z_o,\tau)R(z_o,\tau;z,\zeta)\right] d\tau$$

$$+\int_{\overline{z}_o}^{\zeta} \left(\frac{\partial U}{\partial \tau} + AU\right) R(z_o,\tau;z,\zeta)\, d\tau$$

$$= -U(z_o,\zeta)\, R(z_o,\zeta;z,\zeta) + U(z_o,\overline{z}_o)\, R(z_o,\overline{z}_o;z,\zeta)$$

$$+\int_{\overline{z}_o}^{\zeta} \left(\frac{\partial U}{\partial \tau} + AU\right) R(z_o,\tau;z,\zeta)\, d\tau,$$

and

$$(1.30)\quad \int_{z_o}^{z} dt \int_{\overline{z}_o}^{\zeta} \frac{\partial}{\partial \tau}\left[U(t,\tau)\left(\frac{\partial R}{\partial t} - BR\right)\right] d\tau$$

$$= -\int_{z_o}^{z} U(t,\overline{z}_o)\left(\frac{\partial}{\partial t} R(t,\overline{z}_o;z,\zeta) - BR\right) dt$$

$$= -\int_{z_o}^{z} \frac{\partial}{\partial t}\left[U(t,\overline{z}_o)\, R(t,\overline{z}_o;z,\zeta)\right] dt$$

$$+\int_{z_o}^{z} \left(\frac{\partial U}{\partial t} + BU\right) R(t,\overline{z}_o;z,\zeta)\, dt$$

$$= -U(z,\overline{z}_o)\, R(z,\overline{z}_o;z,\zeta) + U(z_o,\overline{z}_o)\, R(z_o,\overline{z}_o;z,\zeta)$$

$$+\int_{z_o}^{z} \left(\frac{\partial U}{\partial t} + BU\right) R(t,\overline{z}_o;z,\zeta)\, dt \quad .$$

Combining (1.28), (1.29), and (1.30) according to the identity (1.27), we obtain

$$U(z,\zeta) = U(z_o,\bar{z}_o)\, R(z_o,\bar{z}_o;z,\zeta) + \int_{z_o}^{z} \left[\frac{\partial}{\partial t} U(t,\bar{z}_o) + BU\right] R(t,\bar{z}_o;z,\zeta)\,dt$$

$$+ \int_{\bar{z}_o}^{\zeta} \left[\frac{\partial}{\partial \tau} U(z_o,\tau) + AU\right] R(z_o,\tau;z,\zeta)\, d\tau .$$

Restricting $\zeta = \bar{z}$ in this expression gives Formula (1.26). And, the desired analyticity and continuity of $\varphi(z)$ and $\varphi^*(\zeta)$ follow directly from Lemma 1.1.

The second half of the theorem is easily verified by directly substituting Formula (1.26) into Equation (1.5) with $\zeta = \bar{z}$ and keeping in mind Conditions (1.13). This completes the proof of the theorem.

Consider the nonhomogeneous equation associated with (1.1),

(1.31) $$L[u] = f(x, y),$$

where $f(x, y)$ is an analytic function of x and y in some simply-connected domain of the x, y-plane. The equivalent complex form of this equation is

(1.32) $$\mathcal{L}[U] = F(z,\zeta) \quad ,$$

where $F(z,\zeta) = \frac{1}{4} f\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$. As is well known, the general solution of a linear nonhomogeneous differential equation is obtained by adding a particular solution of the nonhomogeneous equation to the general solution of the homogeneous equation. Taking into account Conditions (1.13), it is easily shown by direct substitution that the function

$$U_p(z,\zeta) = \int_{z_o}^{z} dt \int_{\overline{z}_o}^{\zeta} R(t,\tau;z,\zeta)\, F(t,\tau)\, d\tau \tag{1.33}$$

is a particular solution of (1.32). Restricting $\zeta = \overline{z}$ gives $U_p(z,\overline{z})$ as a particular solution of (1.31). We are thus lead to state the following corollary.

Corollary: Let D and κ be given as in the statement of Theorem 1.2. Assume $F(z,\zeta)$ is analytic in $(D,\overline{D})$ and that it is continuous in each variable separately up to and including the boundary κ, or $\overline{\kappa}$, as the case may be. If $u(x,y)$ is a regular solution of the nonhomogeneous Equation (1.31) in D and if it and its first partial derivatives are continuous on $D \cup \kappa$ and if $U(z,\zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$, then

$$\begin{aligned} u(x,y) = {} & \alpha_o R(z_o,\overline{z}_o;z,\overline{z}) + \int_{z_o}^{z} \varphi(t) R(t,\overline{z}_o;z,\overline{z})\, dt \\ & + \int_{\overline{z}_o}^{\overline{z}} \varphi^*(\tau) R(z_o,\tau;z,\overline{z})\, d\tau \\ & + \int_{z_o}^{z} dt \int_{\overline{z}_o}^{\overline{z}} R(t,\tau;z,\overline{z})\, F(t,\tau)\, d\tau, \end{aligned} \tag{1.34}$$

where $z = x + iy$ is in D, $z_o = x_o + iy_o$ is a fixed point on κ,
$\alpha_o = U(z_o, \overline{z}_o)$, $F(z,\zeta) = \frac{1}{4} f\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$, and

$$\varphi(z) = \frac{\partial U(z, \overline{z}_o)}{\partial z} + B(z, \overline{z}_o)\, U(z, \overline{z}_o),$$

$$\varphi^*(\zeta) = \frac{\partial U(z_o, \zeta)}{\partial \zeta} + A(z_o, \zeta)\, U(z_o, \zeta),$$

which are analytic functions of z and ζ in D and $\overline{D}$ and continuous in $D \cup \kappa$ and $\overline{D \cup \kappa}$, respectively.

Conversely, if α_o is an arbitrary constant and $\varphi(z)$ and $\varphi^*(\zeta)$ are arbitrary analytic functions in D and $\overline{D}$, continuous in $D \cup \kappa$ and $\overline{D \cup \kappa}$, respectively, then (1.34) represents the regular solutions of Equation (1.31), continuous in $D \cup \kappa$.

Proof: If $u(x, y)$ is given as a solution of (1.31), then the function

$$u_h(x, y) = U_h(z, \overline{z}) = U(z, \overline{z}) - U_p(z, \overline{z})$$

satisfies $L[u_h] = 0$. Therefore, from Theorem 1.2,

$$(1.35) \qquad U(z, \overline{z}) = U_h(z, \overline{z}) + U_p(z, \overline{z})$$
$$= U_h(z_o, \overline{z}_o) R(z_o, \overline{z}_o; z, \overline{z}) + \int_{z_o}^{z} \varphi(t) R(t, \overline{z}_o; z, \overline{z})\, dt$$
$$+ \int_{\overline{z}_o}^{\overline{z}} \varphi^*(\tau) R(z_o, \tau; z, \overline{z})\, d\tau + \int_{z_o}^{z} dt \int_{\overline{z}_o}^{\overline{z}} R(t, \tau; z, \overline{z}) F(t, \tau)\, d\tau,$$

where

$$\varphi(z) = \frac{\partial U_h(z,\overline{z}_o)}{\partial z} + B(z,\overline{z}_o)\ U_h(z,\overline{z}_o)$$

and

$$\varphi^*(\zeta) = \frac{\partial U_h(z_o,\zeta)}{\partial \zeta} + A(z_o,\zeta)\ U_h(z_o,\zeta) \quad .$$

But, from (1.33) it is seen that $U_p(z_o, \overline{z}_o) = 0$ and that

$$U_p(z,\overline{z}_o) = 0 \ , \qquad \frac{\partial U_p(z,\overline{z}_o)}{\partial z} = 0 \ ,$$

$$U_p\ (z_o,\zeta) = 0 \ , \qquad \frac{\partial U_p(z_o,\zeta)}{\partial \zeta} = 0 \quad .$$

Therefore, $U_h\ (z_o,\overline{z}_o) = U(z_o,\overline{z}_o)$ and

$$U_h(z,\overline{z}_o) = U(z,\overline{z}_o), \quad \frac{\partial U_h(z,\overline{z}_o)}{\partial z} = \frac{\partial U(z,\overline{z}_o)}{\partial z} \ ,$$

$$U_h\ (z_o,\zeta) = U(z_o,\zeta), \quad \frac{\partial U_h(z_o,\zeta)}{\partial \zeta} = \frac{\partial U(z_o,\zeta)}{\partial \zeta} \quad .$$

Thus, Formula (1.35) is identical to Formula (1.34). The second half of the corollary may be verified by direct substitution. This completes the proof.

CHAPTER II

§ 1. Geometric Reflection Across an Analytic Arc.

If $z = x + iy$ is a point in the complex plane, then reflecting z across the x-axis corresponds to the familiar notion of taking its complex conjugate, $\overline{z} = x - iy$. We would like to extend this notion of reflection to more general types of arcs; in particular, to those analytic arcs treated by Sloss in [6] and [7].

Let κ be an open analytic arc in the xy-plane defined by the relation $F(x, y) = 0$, where F is a real analytic function in some neighborhood of κ and where $F_x^2(x, y) + F_y^2(x, y) \neq 0$ along κ. By κ being open, we mean that it can be considered as a homeomorphic image of an open interval. Make the substitutions $z = x+iy$, $\zeta = x - iy$ to obtain

$$g(z, \zeta) = F\left[\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right]$$

as an analytic function in some polycylindrical neighborhood $|z-z_o| < r$, $|\zeta - z_o| < r$ of $(z_o, \overline{z}_o)$ for every $z_o = x_o + iy_o$ on κ. Moreover,

$$g(z_o, \overline{z}_o) = F\left[\frac{z_o+\overline{z}_o}{2} \quad \frac{z_o-\overline{z}_o}{2i}\right] = F(x_o, y_o) = 0 ,$$

$$\left.\frac{\partial g}{\partial \zeta}\right|_{\substack{z=z_o \\ \zeta=\overline{z}_o}} = \frac{1}{2}\left[F_x(x_o, y_o) + i F_y(x_o, y_o)\right] \neq 0 ,$$

and

$$\left.\frac{\partial g}{\partial z}\right|_{\substack{z = z_o \\ \zeta = \bar{z}_o}} = \frac{1}{2}\left[F_x(x_o, y_o) - i F_y(x_o, y_o)\right] \neq 0 .$$

Therefore, by the implicit function theorem of complex variables, there exists a unique function, $\zeta = G(z)$, defined on a neighborhood, $N(z_o)$, of each point z_o on κ, whose range is a neighborhood of $\bar{z}_o$ and which satisfies the relation $g[z, G(z)] = 0$ for all $z \in N(z_o)$. Also, $G(z)$ is single-valued and analytic in a neighborhood of κ and $G(z_o) = \bar{z}_o$ for all $z_o \in \kappa$. Because $\frac{\partial g}{\partial z}(z_o, \bar{z}_o) \neq 0$ and $\frac{\partial g}{\partial \zeta}(z_o, \bar{z}_o) \neq 0$, $G(z)$ is also one-to-one in some neighborhood of κ.

Definition 2.1. The function $G(z)$ introduced above will be referred to as the reflection function relative to the arc κ.

Definition 2.2. Let D be a simply-connected domain adjacent to an analytic arc κ such that the reflection function, G, is defined, analytic and one-to-one in D. If z is a point in D, we define the point $\hat{z} = \overline{G(z)}$ to be the reflection of z across κ.

Note that $\hat{z} = \overline{G(z)} = z$ for all points z on κ. We let $\hat{D} = \overline{G(D)}$ denote the reflection of D across κ.

To justify the above definition of reflection, we show that if z is not a point of κ, but lies in a sufficiently small neighborhood of κ, then $\hat{z}$ must lie on that side of κ not containing z. Toward this end, let κ be given as the image of an open interval (a, b) under a mapping $h(\xi) = x(\xi) + i\,y(\xi)$ which is analytic and one-to-one. Then h can be extended as a one-to-one analytic function of a complex variable into some sufficiently small simply-connected neighborhood, R, of (a, b) which is symmetric with respect to the real axis, i.e., $R = \overline{R}$. Let R_+ denote that portion of R which lies in the upper half-plane, $y > 0$, and let R_- be that portion which lies in the lower half-plane, $y < 0$. Let $N_+ = h(R_+)$ and $N_- = h(R_-)$. Then N_+ and N_- are domains lying on opposite sides of κ, since $N_+ \cap N_- = \emptyset$ and κ is contained in the boundaries of N_+ and N_-.

Without loss of generality, assume G is one-to-one and analytic in $N_+ \cup \kappa \cup N_-$. We wish to show that if z is in N_-, say, then its reflected image, $\hat{z}$, must lie in N_+. It is clear that the latter will be established if we can show that $\hat{z} = h(\overline{\zeta})$, where ζ is in R_- and $h(\zeta) = z$. However, notice that $\hat{z} = h(\overline{\zeta})$ is equivalent to the expression

$$G[h(\zeta)] = \overline{h(\overline{\zeta})} \quad , \tag{2.1a}$$

since $\hat{z} = \overline{G[h(\zeta)]}$. But $G[h(\zeta)]$ and $\overline{h(\overline{\zeta})}$ are analytic functions of ζ in

R and they agree along (a, b). Indeed, if $\xi \in (a, b)$ then $h(\xi) \in \kappa$ and $\overline{G[h(\xi)]} = h(\xi)$ or $G[h(\xi)] = \overline{h(\xi)} = h(\overline{\xi})$. The last equality follows because ξ is real. Thus, (2.1a) must be valid for all ζ in R and so $\hat{z} = h(\overline{\zeta})$ lies in N_+.

Intuitively speaking, the above shows that points close to κ have reflected images on the opposite side of κ. We will extend this further by assuming throughout the remainder of the paper that whenever D is given as in Defintion 2.2, then $D \cap \hat{D}$ is empty.

Later, it will be useful to know how the reflection function acts on the reflected region $\hat{D}$. Toward this end, consider the function $H(\hat{z}) = \overline{z}$ defined for $\hat{z}$ in $\hat{D} \cup \kappa$. By examining the difference quotient for $H(\hat{z})$ it is easily seen that H has a derivative at every point of $\hat{D}$; in fact, $H'(\hat{z}) = \overline{[G'(z)]}^{-1}$. Thus, H is analytic in $\hat{D}$ and it is continuous on $\hat{D} \cup \kappa$ and agrees with G for $\hat{z}$ on κ. Therefore, H is the analytic continuation of G into $\hat{D}$; i.e., $G(\hat{z}) = H(\hat{z}) = \overline{z}$ for $\hat{z}$ in $\hat{D} \cup \kappa$. As a consequence, we have that $\hat{\hat{z}} = z$ in $D \cup \kappa \cup \hat{D}$. Thus, G(z) is analytic and $G'(z) \neq 0$ in $D \cup \kappa \cup \hat{D}$ and $\overline{G(\hat{D})} = D$.

Of particular interest are the cases where it is possible to obtain an explicit expression for the reflection function. When κ is a segment of the x-axis, we set $F(x, y) = y$ and obtain

$$g(z,\zeta) = \frac{z-\zeta}{2i} .$$

Setting the right hand side equal to zero and solving for ζ, we get that $\zeta = G(z) = z$ and that $\hat{z} = \overline{G(z)} = \bar{z}$. Thus, for the case of the x-axis, the generalized notion of reflection agrees with the usual notion of reflecting a point by taking its complex conjugate.

For the case when κ is a circle or a circular arc of radius r centered at the origin, we set $F(x,y) = x^2 + y^2 - r^2$. Make the complex substitutions to get

$$g(z,\zeta) = \left(\frac{z+\zeta}{2}\right)^2 + \left(\frac{z-\zeta}{2i}\right)^2 - r^2$$

and then solve $g(z,\zeta) = 0$ for ζ to obtain $\zeta = G(z) = r^2 z^{-1}$ and $\hat{z} = \overline{G(z)} = r^2 \bar{z}^{-1}$. We see that the reflected point, $\hat{z}$, obtained by means of the reflection function corresponds to the familiar notion of the inverse point of z relative to the circle of radius r centered at the origin.

As another example, we take the ellipse given by $F(x,y) = b^2x^2 + a^2y^2 - a^2b^2 = 0$ and calculate the reflection function to be

$$G(z) = \frac{(a^2+b^2)z - 2ab\sqrt{z^2+b^2-a^2}}{a^2-b^2} ,$$

where the principle branch of the square root function is used; i.e. $\sqrt{z^2+b^2-a^2} = \sqrt{|z^2+b^2-a^2|}\ \exp[\arg(z^2+b^2-a^2)\cdot 2^{-1}]$.

Finally, for the parabola $F(x, y) = x^2 - y = 0$ we find that

$$G(z) = (i - z) - i\sqrt{4iz + 1} \quad ,$$

where again, the principle branch of the square root is used.

§ 2. Statement of the Problem and Notation.

Consider the elliptic partial differential equation

$$(2.1) \qquad Q[u] = \sum_{i+j=2n} c_{ij} \frac{\partial^{2n}}{\partial x^i \partial y^j} u(x, y) = 0$$

of order 2n where the c_{ij} are real constants. Ellipticity implies that $c_{2n,0} \neq 0$ and so, without loss of generality, we take $c_{2n,0}=1$. Further, the elliptic nature of the operator Q allows it to be decomposed as a product of 2n linear factors as

$$Q = \prod_{k=1}^{n} \left(\frac{\partial}{\partial x} - \alpha_k \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \overline{\alpha}_k \frac{\partial}{\partial y}\right)$$

where $\alpha_k = \beta_k + i\,\delta_k$ and $\overline{\alpha}_k = \beta_k - i\,\delta_k$, with $\delta_k \neq 0$, are the roots of the associated characteristic polynomial

$$P(\xi) = \sum_{k=0}^{2n} c_{2n-k,k}\, \xi^{2n-k} = 0 \quad .$$

Since $\delta_k \neq 0$ and since α_k and $\overline{\alpha}_k$ are both involved in the decomposition of Q, we may choose δ_k to always be positive. Finally, we shall assume that the characteristics of (2.1) are distinct; that is to say, we shall assume that $\alpha_i \neq \alpha_j$ if $i \neq j$.

For each $k = 1, 2, \ldots, n$, introduce the nonsingular coordinate transformation

$$(2.2) \qquad x_k = x, \quad y_k = \delta_k^{-1} (y + \beta_k x)$$

whose inverse is given by

$$(2.3) \qquad x = x_k, \quad y = \delta_k y_k - \beta_k x_k .$$

Under this transformation, observe that

$$\left(\frac{\partial}{\partial x} - \alpha_k \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \bar{\alpha}_k \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} = \Delta_k ,$$

which is just the Laplacian with respect to the coordinates, x_k, y_k. Thus, (2.1) may be written as

$$(2.4) \qquad Q[u] = \left(\prod_{k=1}^{n} \Delta_k\right) u(x, y) = 0 .$$

In [7], Sloss showed that it is possible to analytically continue solutions of Equation (2.4) across analytic arcs by means of reflection provided the solution satisfies certain analytic boundary conditions along the arc. Furthermore, he was able to explicitly describe a region into which the solution could be extended. The region of reflection turned out to be dependent only on the original domain, the analytic arc and the coefficients, c_{ij}, of Equation (2.1).

In this chapter, we would like to examine to what extent Sloss' results carry over to the case when (2.1) has lower order terms present. Because of the reason mentioned in the introduction, we shall limit our examination to sixth order elliptic equations that can be factored as

$$(2.5) \qquad L[u] = \prod_{k=1}^{3} \left[\Delta_k + a_k(x_k, y_k)\frac{\partial}{\partial x_k} + b_k(x_k, y_k)\frac{\partial}{\partial y_k} + c_k(x_k, y_k)\right] u(x, y) = 0 \quad ,$$

where the coefficients a_k, b_k and c_k are complex-valued analytic functions of their arguments, x_k and y_k. Before proceeding with an investigation of this equation, we introduce some transformations and notation that will be used repeatedly throughout what follows.

Let $z = x + iy$ and set $z_k = x_k + i y_k$ where x_k and y_k are given by (2.2), $k = 1, 2, 3$. Solve for z_k and $\overline{z}_k$ in terms of z and $\overline{z}$ to obtain the transformations

$$(2.6) \qquad \begin{aligned} z_k &= \tau_k(z) = A_k z + B_k \overline{z} \\ \overline{z}_k &= \tau_k^*(\overline{z}) = \overline{B}_k z + \overline{A}_k \overline{z} \end{aligned}$$

where the coefficients are uniquely determined as

$$(2.6a) \qquad \begin{aligned} A_k &= \frac{1}{2}\left[(1 + \delta_k^{-1}) + i\,\delta_k^{-1}\beta_k\right] \\ B_k &= \frac{1}{2}\left[(1 - \delta_k^{-1}) + i\,\delta_k^{-1}\beta_k\right] \quad . \end{aligned}$$

Though $\tau_k(z)$ and $\tau_k^*(\bar{z})$ are not analytic functions of their arguments, they are continuous one-to-one transformations having continuous inverses given by

$$(2.7) \qquad \begin{aligned} z &= \tau_k^{-1}(z_k) = \delta_k (\bar{A}_k z_k - B_k \bar{z}_k) \\ \bar{z} &= \tau_k^{*-1}(\bar{z}_k) = \delta_k (-\bar{B}_k z_k + A_k \bar{z}_k) \end{aligned}$$

for all z_k.

We now continue τ_k and τ_k^* into the domain of two independent complex variables. Replace $\bar{z}$ by ζ in (2.6) to get

$$(2.8) \qquad \begin{aligned} z_k &= A_k z + B_k \zeta \\ \zeta_k &= \bar{B}_k z + \bar{A}_k \zeta \end{aligned}$$

as a nonsingular continuous linear transformation between the variables (z, ζ) and (z_k, ζ_k) whose inverse is given by

$$(2.9) \qquad \begin{aligned} z &= \delta_k (\bar{A}_k z_k - B_k \zeta_k) \\ \zeta &= \delta_k (-\bar{B}_k z_k + A_k \zeta_k) . \end{aligned}$$

From (2.8) and (2.9) the relationship between the pairs (z_k, ζ_k) and (z_j, ζ_j), $k \neq j$, can easily be established as a result of straightforward algebraic manipulations to be

$$(2.10)\qquad \begin{aligned} z_k &= T_k(z_j, \zeta_j) = a_{kj}\, z_j + b_{kj}\zeta_j \\ \zeta_k &= T_k^*(\zeta_j, z_j) = \overline{b}_{kj}\, z_j + \overline{a}_{kj}\zeta_j \end{aligned} ,$$

where

$$a_{kj} = \delta_j(A_k\overline{A}_j - B_k\,\overline{B}_j)$$

$$b_{kj} = \delta_j(B_k A_j - A_k\, B_j) \quad .$$

Again, this is a continuous nonsingular linear change of variables and T_k and T_k^* are entire functions of z_j and ζ_j. We remark that when $\zeta_j = \overline{z}_j$, then $T_k(z_j, \overline{z}_j) = \tau_k\, \tau_j^{-1}(z_j) = z_k$ and $T_k^*(\overline{z}_j, z_j) = \tau_k^*\, \tau_j^{*-1}(\overline{z}_j) = \overline{z}_k$. This notation will be used interchangeably.

Finally, let D denote a simply-connected domain in the x, y-plane adjacent to an open analytic arc κ. For each j, let

$$D_j = \left\{ z_j = \tau_j(z) : z \in D \right\} ,$$

$$\kappa_j = \left\{ z_j = \tau_j(z) : z \in \kappa \right\} .$$

Since τ_j is a homeomorphism, D_j will be a simply-connected domain adjacent to κ_j. Moreover, the relations defining transformation (2.3) are analytic and for this reason κ_j will also be an analytic arc. For example, if κ is part of a nondegenerate conic given by

$$F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0 ,$$

then κ_j will also be part of a nondegenerate conic of the same type as κ.

§ 3. Equation (2.5) as a System.

We return now to Equation (2.5) and define operators

$$L_k = \Delta_k + a_k(x_k, y_k)\frac{\partial}{\partial x_k} + b_k(x_k, y_k)\frac{\partial}{\partial y_k}$$

$$+ c_k(x_k, y_k), \quad k = 1, 2, 3$$

where, from (2.3),

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x} - \beta_k \frac{\partial}{\partial y} \;, \quad \frac{\partial}{\partial y_k} = \delta_k \frac{\partial}{\partial y} \quad .$$

Then (2.5) may be written as

(2.11) $$L[u] = L_1 \, L_2 \, L_3 \, u(x, y) = 0 \; .$$

Set

(2.12) $$\begin{aligned} u_3(x, y) &= u(x, y) \;, \\ u_2(x, y) &= L_3[u_3] \;, \\ u_1(x, y) &= L_2[u_2] \; . \end{aligned}$$

Then (2.11) may be written as the system

(2.13) $$\begin{aligned} L_3[u_3] &= u_2(x, y) \\ L_2[u_2] &= u_1(x, y) \\ L_1[u_1] &= 0 \; . \end{aligned}$$

From the coordinate transformation (2.3), we have

$$(2.14)\qquad u_k(x,y) = u_k(x_j, \delta_j y_j - \beta_j x_j)$$

$$= v_{kj}(x_j, y_j)$$

$$= v_{kj}\left(\frac{z_j+\bar{z}_j}{2}, \frac{z_j-\bar{z}_j}{2i}\right)$$

$$= V_{kj}(z_j, \bar{z}_j) .$$

If we now introduce the operations

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right),$$

then system (2.13) takes the form

$$\left[\frac{\partial^2}{\partial z_3 \partial \bar{z}_3} + A_3(z_3,\bar{z}_3)\frac{\partial}{\partial z_3} + B_3(z_3,\bar{z}_3)\frac{\partial}{\partial \bar{z}_3} + C_3(z_3,\bar{z}_3)\right]V_{33}(z_3,\bar{z}_3) = \frac{1}{4}V_{23}(z_3,\bar{z}_3)$$

$$(2.15)\qquad \left[\frac{\partial^2}{\partial z_2 \partial \bar{z}_2} + A_2(z_2,\bar{z}_2)\frac{\partial}{\partial z_2} + B_2(z_2,\bar{z}_2)\frac{\partial}{\partial \bar{z}_2} + C_2(z_2,\bar{z}_2)\right]V_{22}(z_2,\bar{z}_2) = \frac{1}{4}V_{12}(z_2,\bar{z}_2)$$

$$\left[\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + A_1(z_1,\bar{z}_1)\frac{\partial}{\partial z_1} + B_1(z_1,\bar{z}_1)\frac{\partial}{\partial \bar{z}_1} + C_1(z_1,\bar{z}_1)\right]V_{11}(z_1,\bar{z}_1) = 0 ,$$

where

$$A_k(z_k, \overline{z}_k) = \frac{1}{4}\left[a_k\left(\frac{z_k+\overline{z}_k}{2}, \frac{z_k-\overline{z}_k}{2i}\right) + i\, b_k\left(\frac{z_k+\overline{z}_k}{2}, \frac{z_k-\overline{z}_k}{2i}\right)\right] ,$$

$$B_k(z_k, \overline{z}_k) = \frac{1}{4}\left[a_k\left(\frac{z_k+\overline{z}_k}{2}, \frac{z_k-\overline{z}_k}{2i}\right) - i\, b_k\left(\frac{z_k+\overline{z}_k}{2}, \frac{z_k-\overline{z}_k}{2i}\right)\right] ,$$

$$C_k(z_k, \overline{z}_k) = \frac{1}{4} \quad c_k\left(\frac{z_k+\overline{z}_k}{2}, \frac{z_k-\overline{z}_k}{2i}\right) .$$

It is System (2.15) which we shall use to reflect the solutions of Equation (2.5). We point out that $V_{kk}(z_k, \overline{z}_k)$ and $V_{kj}(z_j, \overline{z}_j)$, $j = k+1$, $k = 1, 2$, are equal to the same function $u_k(x, y)$ and that they are related by the expression

$$(2.16) \qquad V_{kj}(z_j, \overline{z}_j) = V_{kk}\,[\tau_k\, \tau_j^{-1}(z_j),\ \tau_k^*\, \tau_j^{*-1}(\overline{z}_j)]$$

$$= V_{kk}\,[T_k(z_j, \overline{z}_j),\ T_k^*(\overline{z}_j, z_j)]$$

$$= V_{kk}(z_k, \overline{z}_k) ,$$

where $z_j = \tau_j\, \tau_k^{-1}(z_k)$. If $u(x, y)$ is given as a solution of (2.5), then $v_{kk}(x_k, y_k) = V_{kk}(z_k, \overline{z}_k)$ becomes a known analytic function of x_k, y_k, and is a solution of the k^{th} equation of (2.15), $k = 1, 2, 3$.

§4. Reflecting solutions of Equation (2.5).

Suppose $f(x_1, y_1) = F(z_1, \overline{z}_1)$ is a known function defined for all z_1 in D_1 such that $F(z_1, \zeta_1)$ is an analytic function of the two complex variables $z_1 \in D_1$, $\zeta_1 \in \overline{D}_1$. Upon performing the coordinate transformation

$$F(z_1, \overline{z}_1) = F[T_1(z_2, \overline{z}_2), \; T_1^*(\overline{z}_2, z_2)] = \widetilde{F}(z_2, \overline{z}_2) \;,$$

where $z_2 = \tau_2 \tau_1^{-1}(z_1) = T_2(z_1, \overline{z}_1)$, we obtain a function $\widetilde{F}(z_2, \overline{z}_2)$, defined for all z_2 in $D_2 = \tau_2 \tau_1^{-1}(D_1)$. However, the function

$$\widetilde{F}(z_2, \zeta_2) = F[T_1(z_2, \zeta_2), \; T_1^*(\zeta_2, z_2)]$$

will not, in general, be analytic for all $z_2 \in D_2$, $\zeta_2 \in \overline{D}_2$ as the following example suggests.

Take D_1 to be the square whose corners are the points (0, 0), (1, 0), (1, 1) and (0, 1) and take $\delta_1 = \delta_2 = 1$, $\beta_1 = 0$, $\beta_2 = 1$. Then

$$z_2 = T_2(z_1, \zeta_1) = (1 + \tfrac{1}{2} i)\, z_1 + \tfrac{1}{2}\, i \zeta_1 \;,$$

$$\zeta_2 = T_2^*(\zeta_1, z_1) = -\tfrac{1}{2}\, i\, z_1 + (1 - \tfrac{1}{2} i)\, \zeta_1 \;,$$

and the inverse of this coordinate transformation is

$$z_1 = T_1(z_2, \zeta_2) = (1 - \tfrac{1}{2} i)\, z_2 - \tfrac{1}{2}\, i\, \zeta_2$$

$$\zeta_1 = T_1^*(\zeta_2, z_2) = \tfrac{1}{2}\, i\, z_2 + (1 + \tfrac{1}{2} i)\, \zeta_2 \;.$$

The set D_2 is just $\left\{ z_2 = (1 + \tfrac{1}{2} i)\, z_1 + \tfrac{1}{2}\, i\, \overline{z}_1 : z_1 \in D_1 \right\}$ and is shown in Figure 2.1.

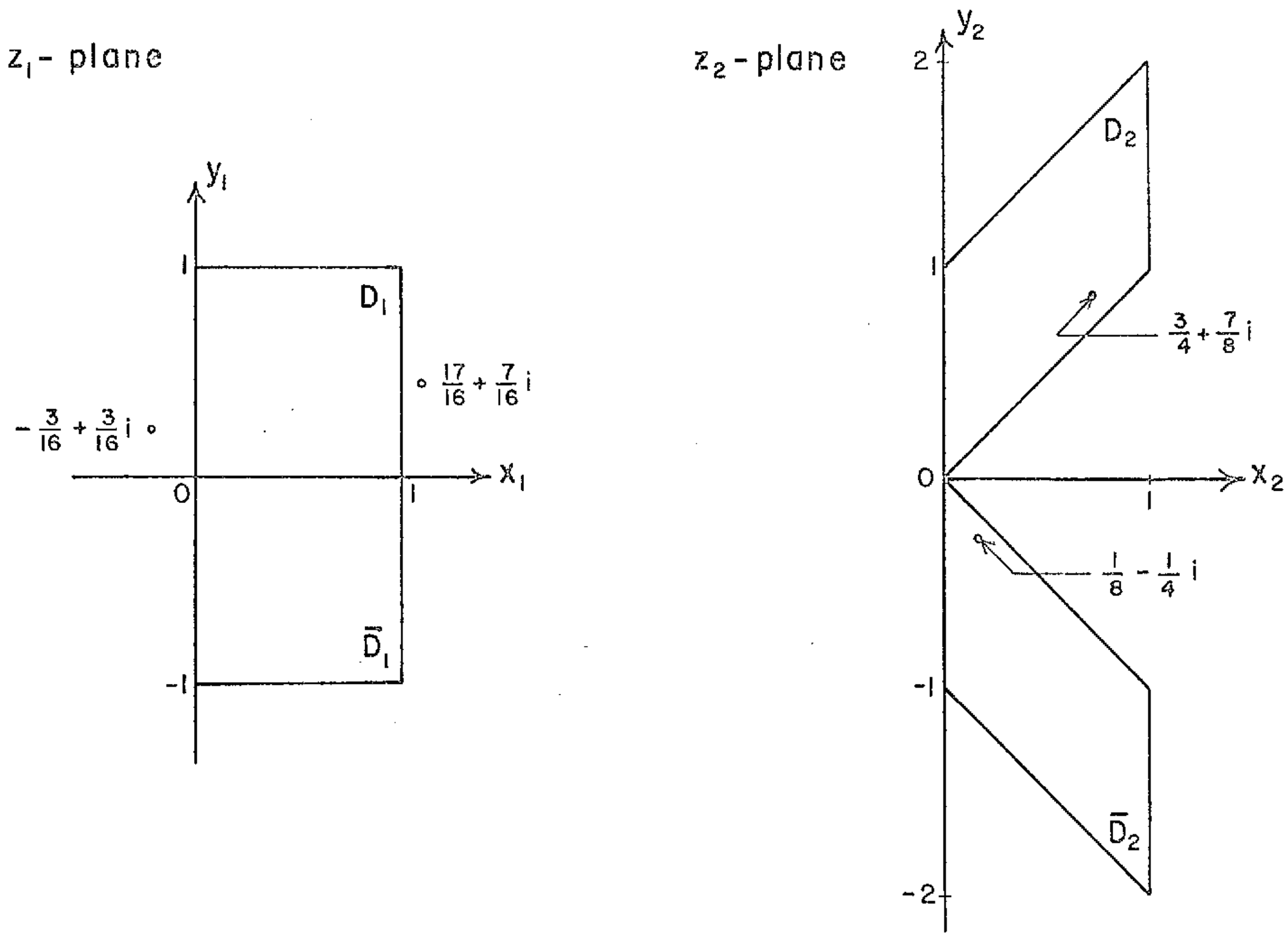
z1- plane
y1
1
D1
17/16 + 7/16 i
-3/16 + 3/16 i
0
1
x1
D̄1
-1
z2- plane
y2
2
D2
1
3/4 + 7/8 i
0
x2
1
1/8 - 1/4 i
-1
D̄2
-2

Figure 2.1

Consider the pair $z_2 = \frac{3}{4} + i\,\frac{7}{8}$, $\zeta_2 = \frac{1}{8} - i\,\frac{1}{4}$ in $D_2 \times \overline{D}_2$. Then

$$z_1 = T_1\left(\frac{3}{4} + i\,\frac{7}{8},\ \frac{1}{8} - i\,\frac{1}{4}\right) = \frac{17}{16} + i\,\frac{7}{16}$$

$$\zeta_1 = T_1^*\left(\frac{1}{8} - i\,\frac{1}{4},\ \frac{3}{4} + i\,\frac{7}{8}\right) = -\frac{3}{16} + i\,\frac{3}{16}$$

and this pair is not in $D_1 \times \overline{D}_1$. Thus $\tilde{F}\left(\frac{3}{4} + i\,\frac{7}{8},\ \frac{1}{8} - i\,\frac{1}{4}\right)$ is not even defined. However, given any point z_2 in D_2 there does exist a neighborhood, N, of z_2 contained in D_2 such that $\tilde{F}(z_2, \zeta_2)$ is defined and is an analytic function of the two complex variables $z_2 \in N$, $\zeta_2 \in \overline{N}$. It is the purpose of the following lemma to establish this fact and to give some idea about the size of N.

We use the notation $N(z,\rho)$ to designate an open disc in the x, y-plane whose boundary is a circle of radius ρ centered at $z = x + iy$.

Lemma 2.1. Let Ω_j be an open connected set in the complex plane and

$$\Omega_k = \left\{ T_k(z_j, \overline{z}_j) = z_k : z_j \in \Omega_j \right\} .$$

Let $F(z_j, \zeta_j)$ be a known analytic function of two complex variables for $(z_j, \zeta_j) \in (\Omega_j, \overline{\Omega}_j)$. Then, for every point z_{ko} in Ω_k there exists an open disc, $N(z_{ko}, r)$, centered at z_{ko} and of radius r contained in Ω_k such that the function $\tilde{F}(z_k, \zeta_k)$ defined by

$$\tilde{F}(z_k, \zeta_k) = F[T_j(z_k, \zeta_k), \; T_j^*(\zeta_k, z_k)] = F(z_j, \zeta_j)$$

is an analytic function of the two complex variables $z_k \in N(z_{ko}, r)$, $\zeta_k \in N(\overline{z}_{ko}, r)$. Moreover, the radius, r, of the largest such disc can be determined explicitly from the choice of the point z_{ko} and from the coefficients a_{jk} and b_{jk} of the transformation $T_j(z_k, \zeta_k)$ to be

$$r(z_{ko}) = \frac{m(z_{ko})}{|a_{jk}| + |b_{jk}|} \quad ,$$

where $m(z_{ko}) = \inf \left\{ |z_j - z_{jo}| \; : z_{jo} = T_j(z_{ko}, \overline{z}_{ko}), \; z_j \in \partial \Omega_j \right\}$.

Proof: Arbitrarily fix a point z_{ko} in Ω_k and let z_{jo} be the point in Ω_j defined by

$$z_{jo} = T_j(z_{ko}, \overline{z}_{ko}) = a_{jk} z_{ko} + b_{jk} \overline{z}_{ko} \quad .$$

Let $N(z_{ko}, r)$ and $N(\overline{z}_{ko}, r)$ be open discs of radius r centered at z_{ko} and $\overline{z}_{ko}$ respectively. We proceed to find an r satisfying the conclusions of the lemma.

Replace z_j by z_{ko} and ζ_j by ζ_k in the coordinate transformation (2.10) and observe that the resulting mappings

$$T_j(z_{ko}, \zeta_k) = a_{jk} z_{ko} + b_{jk} \zeta_k = z_j$$

and

$$T_j^*(\zeta_k, z_{ko}) = \bar{b}_{jk} z_{ko} + \bar{a}_{jk} \zeta_k = \zeta_j$$

are linear with respect to the single complex variable ζ_k and so map circles onto circles. Furthermore,

$$|z_j - z_{jo}| = |T_j(z_{ko}, \zeta_k) - T_j(z_{ko}, \bar{z}_{ko})|$$

$$= |a_{jk} z_{ko} + b_{jk} \zeta_k - a_{jk} z_{ko} - b_{jk} \bar{z}_{ko}|$$

$$= |b_{jk}| \; |\zeta_k - \bar{z}_{ko}| \quad ,$$

and

$$|\zeta_j - \bar{z}_{jo}| = |T_j^* (\zeta_k, z_{ko}) - T_j^* (\bar{z}_{ko}, z_{ko})|$$

$$= |\bar{b}_{jk} z_{ko} + \bar{a}_{jk} \zeta_k - \bar{b}_{jk} z_{ko} - \bar{a}_{jk} \bar{z}_{ko}|$$

$$= |\bar{a}_{jk}| \; |\zeta_k - \bar{z}_{ko}| \quad .$$

From this, it is seen that $T_j(z_{ko}, \zeta_k)$ maps $N(\bar{z}_{ko}, r)$ onto an open disc, $N(z_{jo}, |b_{jk}| r)$, of radius $|b_{jk}| r$ centered at z_{jo} and that $T_j^* (\zeta_k, z_{ko})$ maps $N(\bar{z}_{ko}, r)$ onto an open disc, $N(\bar{z}_{jo}, |a_{jk}| r)$, of radius $|a_{jk}| r$ centered at $\bar{z}_{jo}$. Thus, transformation (2.10) maps the set $\{z_{ko}\} \times N(\bar{z}_{ko}, r)$ onto a subset of $N(z_{jo}, |b_{jk}| r) \times N(\bar{z}_{jo}, |a_{jk}| r)$.

For the next step, arbitrarily pick a point ζ_k' in $N(\bar{z}_{ko}, r)$. Then, transformation (2.10) maps the pair (z_{ko}, ζ_k') onto the pair (z_j', ζ_j') in $N(z_{jo}, |b_{jk}| r) \times N(\bar{z}_{jo}, |a_{jk}| r)$ given by

$$z_j' = T_j(z_{ko}, \zeta_k') = a_{jk} z_{ko} + b_{jk} \zeta_k'$$

$$\zeta_j' = T_j^*(\zeta_k', z_{ko}) = \bar{b}_{jk} z_{ko} + \bar{a}_{jk} \zeta_k' \, .$$

Consider now the set $N(z_{ko}, r) \times \{\zeta_k'\}$ and the linear mappings $T_j(z_k, \zeta_k')$ and $T_j^*(\zeta_k', z_k)$. As functions of z_k, they map circles onto circles and in particular they map $N(z_{ko}, r)$ onto open discs centered at z_j' and ζ_j' respectively (see Figure 2.2). Furthermore,

$$\begin{aligned} |z_j - z_j'| &= |T_j(z_k, \zeta_k') - T_j(z_{ko}, \zeta_k')| \\ &= |a_{jk} z_k + b_{jk} \zeta_k' - a_{jk} z_{ko} - b_{jk} \zeta_k'| \\ &= |a_{jk}| \, |z_k - z_{ko}| \quad , \end{aligned}$$

and

$$\begin{aligned} |\zeta_j - \zeta_j'| &= |T_j^*(\zeta_k', z_k) - T_j^*(\zeta_k', z_{ko})| \\ &= |\bar{b}_{jk} z_k + \bar{a}_{jk} \zeta_k' - \bar{b}_{jk} z_{ko} - \bar{a}_{jk} \zeta_k'| \\ &= |\bar{b}_{jk}| \, |z_k - z_{ko}| \, . \end{aligned}$$

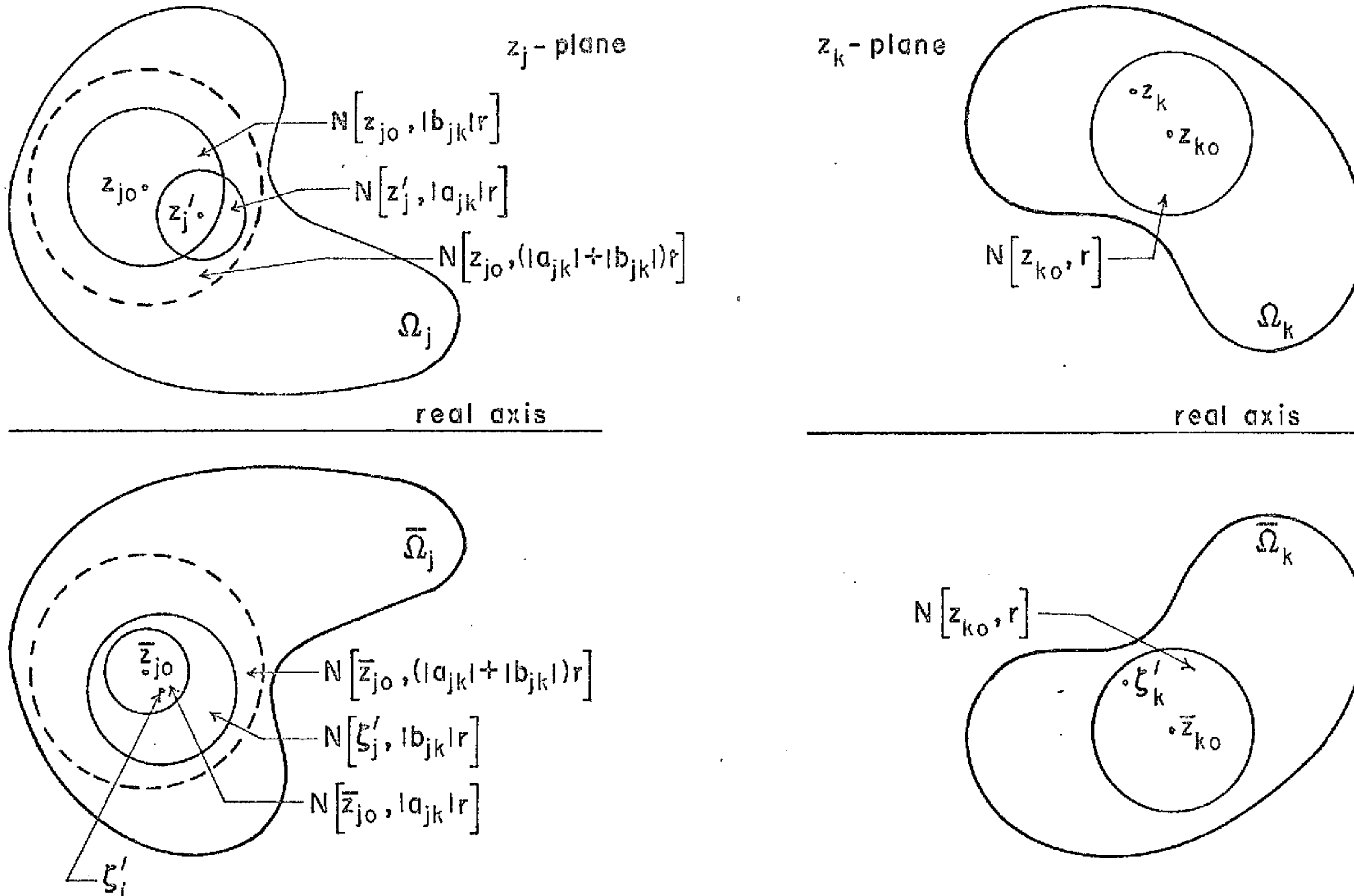
z_j - plane
z_k - plane
N[z_jo, |b_jk| r]
N[z'_j, |a_jk| r]
N[z_jo, (|a_jk| + |b_jk|) r]
N[z_ko, r]
Ω_j
Ω_k
real axis
real axis
Ω̄_j
Ω̄_k
N[z̄_jo, (|a_jk| + |b_jk|) r]
N[ζ'_j, |b_jk| r]
N[z̄_jo, |a_jk| r]
ζ'_j
N[z_ko, r]
ζ'_k
z̄_ko
z_k
z_ko
z_jo
z'_j
z̄_jo

Figure 2.2

Thus, $T_j(z_k, \zeta_k')$ maps $N(z_{ko}, r)$ onto an open disc $N(z_j', |a_{jk}| r)$, of radius $|a_{jk}| r$ centered at z_j' and $T_j^*(\zeta_k', z_k)$ maps $N(z_{ko}, r)$ onto an open disc $N(\zeta_j', |b_{jk}| r)$, of radius $|b_{jk}| r$ centered at ζ_j'. And so, transformation (2.10) maps the set $N(z_{ko}, r) \times \{\zeta_k'\}$ onto a subset of $N(z_j', |a_{jk}| r) \times N(\zeta_j', |b_{jk}| r)$.

Now form the union of all the open discs $N(z_j', |a_{jk}| r)$ whose center, z_j', lies in the disc $N(z_{jo}, |b_{jk}| r)$ to get an open disc centered at z_{jo} and of radius $(|a_{jk}| + |b_{jk}|) r$; i.e.,

$$\bigcup_{z_j' \in N(z_{jo}, |b_{jk}| r)} N(z_j', |a_{jk}| r) = N[z_{jo}, (|a_{jk}| + |b_{jk}|) r].$$

Similarly,

$$\bigcup_{\zeta_j' \in N(\overline{z}_{jo}, |a_{jk}| r)} N(\zeta_j', |b_{jk}| r) = N[\overline{z}_{jo}, (|a_{jk}| + |b_{jk}|) r].$$

From the preceding construction it is seen that transformation (2.10), being a homeomorphism, maps the open set $N(z_{ko}, r) \times N(\overline{z}_{ko}, r)$ onto an open subset of

$$N[z_{jo}, (|a_{jk}| + |b_{jk}|) r] \times N[\overline{z}_{jo}, (|a_{jk}| + |b_{jk}|) r]. \tag{2.17}$$

If we define $m(z_{ko})$ as in the conclusion of the lemma and pick r such that $(|a_{jk}| + |b_{jk}|) r \leq m(z_{ko})$, then (2.17) will be contained in $\Omega_j \times \overline{\Omega}_j$. It is easily seen that $r(z_{ko}) = \dfrac{m(z_{ko})}{|a_{jk}| + |b_{jk}|}$ is the largest such value for r possible and that for r equal to this quantity, $N[z_{ko}, r(z_{ko})]$ is contained in Ω_k .

It is now a simple matter to check that the function $\tilde{F}(z_k, \zeta_k) = F[T_j(z_k, \zeta_k), T_j^*(\zeta_k, z_k)]$ is defined for all $(z_k, \zeta_k) \in N(z_{ko}, r) \times N(\bar{z}_{ko}, r)$, where $r \leq \frac{m(z_{ko})}{|a_{jk}| + |b_{jk}|}$. Furthermore, for each fixed ζ_k in $N(\bar{z}_{ko}, r)$, $\tilde{F}$ is analytic in the single variable z_k in a neighborhood of each point of $N(z_{ko}, r)$ and for each fixed z_k in $N(z_{ko}, r)$, $\tilde{F}$ is analytic in the single variable ζ_k in a neighborhood of each point of $N(\bar{z}_{ko}, r)$. Therefore, by Hartog's theorem on the analyticity of functions of several complex vaiables (see [2] Ch. VII), $\tilde{F}(z_k, \zeta_k)$ is analytic as a function of two complex variables on the domain $N(z_{ko}, r) \times N(\bar{z}_{ko}, r)$, where $r \leq \frac{m(z_{ko})}{|a_{jk}| + |b_{jk}|}$. This completes the proof of the lemma.

In dealing with System (2.15), the approach will be to work with each equation separately, extending first the function V_{11}, then V_{22} and finally V_{33}. To avoid needless repetition, the process we use to reflect solutions of a single second order equation is presented in the following lemma.

Lemma 2.2. Let Ω be a simply-connected domain in the x, y-plane adjacent to an open analytic arc κ such that the reflection function, G, relative to κ is defined, analytic and $G'(z) \neq 0$ on $\Omega \cup \kappa \cup \hat{\Omega}$, where $\hat{\Omega} = \overline{G(\Omega)}$. Let $V(z,\bar{z})$, for $z = x + iy$ in Ω, be given as a regular solution of the equation

$$(2.18) \qquad \frac{\partial^2 V}{\partial z\, \partial \bar{z}} + A(z,\bar{z})\frac{\partial V}{\partial z} + B(z,\bar{z})\frac{\partial V}{\partial \bar{z}} + C(z,\bar{z})V = F(z,\bar{z})$$

which has $\Omega \cup \kappa \cup \hat{\Omega}$ as a fundamental domain. Assume that $V(z,\bar{z})$ is in $C^1(\Omega \cup \kappa)$ and that along κ it satisfies the condition

$$(2.19) \qquad A_o(z,\bar{z})\frac{\partial V}{\partial z} + B_o(z,\bar{z})\frac{\partial V}{\partial \bar{z}} + C_o(z,\bar{z})V + F_o(z,\bar{z}) = 0,$$

where $A_o(z,\zeta)$, $B_o(z,\zeta)$, $C_o(z,\zeta)$, $F_o(z,\zeta)$ as well as $F(z,\zeta)$ are analytic functions for $z \in \Omega \cup \kappa \cup \hat{\Omega}$, $\zeta \in \overline{\Omega \cup \kappa \cup \hat{\Omega}}$ and $A_o(z,\zeta) \neq 0$, $B_o(z,\zeta) \neq 0$. Then $V(z,\bar{z})$ can be extended as a solution of Equation (2.18) across κ into all of $\hat{\Omega}$. Replacing $\bar{z}$ by ζ gives $V(z,\zeta)$ as an analytic function of the two complex variables $z \in \Omega \cup \kappa \cup \hat{\Omega}$, $\zeta \in \overline{\Omega \cup \kappa \cup \hat{\Omega}}$.

Proof: By the corollary to Theorem 1.2, $V(z,\bar{z})$ may be expressed as

$$(2.20) \qquad V(z,\bar{z}) = V(z_o,\bar{z}_o)\, R(z_o,\bar{z}_o; z,\bar{z}) + \int_{z_o}^{z} \varphi(t)R(t,\bar{z}_o;z,\bar{z})\, dt$$

$$+ \int_{\bar{z}_o}^{\bar{z}} \varphi^*(\tau)R(z_o,\tau;z,\bar{z})d\tau + \int_{z_o}^{z} dt \int_{\bar{z}_o}^{\bar{z}} R(t,\tau;z,\bar{z})F(t,\tau)\, d\tau\, ,$$

where z_o is a fixed point on κ and $\varphi(z)$, $\varphi^*(\zeta)$ are uniquely determined from $V(z,\bar{z})$ as analytic functions on Ω, $\bar{\Omega}$ and continuous on

$\Omega \cup \kappa$, $\overline{\Omega \cup \kappa}$, respectively. Our aim is to analytically continue the functions $\varphi(z)$, $\varphi^*(\zeta)$ across κ, $\bar{\kappa}$ into $\hat{\Omega}$, $\bar{\hat{\Omega}}$, respectively. Then, since $F(t, \tau)$ and the Riemann function $R(t, \tau; z, \bar{z})$ are known for $z, t \in \Omega \cup \kappa \cup \hat{\Omega}$ and $\tau \in \overline{\Omega \cup \kappa \cup \hat{\Omega}}$, Formula (2.20) will provide the unique continuation of the given solution, originally known only in Ω, into $\Omega \cup \kappa \cup \hat{\Omega}$.

The extension of $\varphi(z)$ across κ is carried out with the help of Condition (2.19). Keeping in mind that

$$\frac{\partial V(z, \bar{z})}{\partial z} = \frac{\partial V(z, \zeta)}{\partial z}\Big|_{\zeta = \bar{z}}, \quad \frac{\partial V(z, \bar{z})}{\partial \bar{z}} = \frac{\partial V(z, \zeta)}{\partial \zeta}\Big|_{\zeta = \bar{z}}$$

and in view of Property (1.9) of the Riemann function, Formula (2.20) gives rise to the following expressions:

$$(2.21) \qquad \frac{\partial V(z, \bar{z})}{\partial z} = V(z_o, \bar{z}_o)\frac{\partial}{\partial z} R(z_o, \bar{z}_o; z, \bar{z}) + \varphi(z)\exp\Big[\int_{\bar{z}}^{\bar{z}_o} A(z, \eta)\, d\eta\Big]$$
$$+ \int_{z_o}^{z} \varphi(t) \frac{\partial}{\partial z} R(t, \bar{z}_o; z, \bar{z})\, dt$$
$$+ \int_{\bar{z}_o}^{\bar{z}} \varphi^*(\tau) \frac{\partial}{\partial z} R(z_o, \tau; z, \bar{z})\, d\tau$$
$$+ \frac{\partial}{\partial z}\int_{z_o}^{z} dt \int_{\bar{z}_o}^{\bar{z}} R(t, \tau; z, \bar{z}) F(t, \tau)\, d\tau \quad ,$$

and

$$(2.22)\qquad \frac{\partial V(z,\overline{z})}{\partial \overline{z}} = V(z_o,\overline{z}_o)\frac{\partial}{\partial \overline{z}} R(z_o,\overline{z}_o; z,\overline{z}) + \int_{z_o}^{z} \varphi(t)\frac{\partial}{\partial \overline{z}} R(t,\overline{z}_o; z,\overline{z})\,dt + \varphi^*(\overline{z})\exp\left[\int_{z}^{z_o} B(\xi,\overline{z})\,d\xi\right] + \int_{\overline{z}_o}^{\overline{z}} \varphi^*(\tau)\frac{\partial}{\partial \overline{z}} R(z_o,\tau; z,\overline{z})\,d\tau + \frac{\partial}{\partial \overline{z}}\int_{z_o}^{z} dt \int_{\overline{z}_o}^{\overline{z}} R(t,\tau;z,\overline{z})F(t,\tau)\,d\tau\ .$$

For z varying in Ω, substitute Formulas (2.20), (2.21) and (2.22) into the left hand side of (2.19). Without loss of generality, take $A_o(z,\zeta) \equiv 1$. Then, the resulting expression may be written as

$$(2.23)\qquad \exp\left[\int_{\overline{z}}^{\overline{z}_o} A(z,\eta)\,d\eta\right]\left\{\varphi(z) - \int_{z_o}^{z} K(t,\overline{z}_o;z,\overline{z})\varphi(t)\,dt + g(z,\overline{z})\ \varphi^*(\overline{z}) - \int_{\overline{z}_o}^{\overline{z}} K(z_o,\tau;z,\overline{z})\ \varphi^*(\tau)\,d\tau - f(z,\overline{z})\right\}\ ,$$

where we have set

$$- K(t,\tau;z,\zeta) = \exp\left[\int_{\bar{z}_o}^{\zeta} A(z,\eta)d\eta\right]\left\{\frac{\partial}{\partial z} R(t,\tau;z,\zeta)\right.$$

$$\left. + B_o(z,\zeta)\frac{\partial}{\partial\zeta}R(t,\tau;z,\zeta) + C_o(z,\zeta)R(t,\tau;z,\zeta)\right\} ,$$

$$g(z,\zeta) = B_o(z,\zeta)\exp\left[-\int_{z_o}^{z} B(\xi,\zeta)d\xi\right]\exp\left[\int_{\bar{z}_o}^{\zeta} A(z,\eta)d\eta\right] ,$$

$$- f(z,\zeta) = \exp\left[\int_{\bar{z}_o}^{\zeta} A(z,\eta)d\eta\right]\left\{V(z_o,\bar{z}_o)\frac{\partial}{\partial z}R(z_o,\bar{z}_o;z,\zeta)\right.$$

$$+ \frac{\partial}{\partial z}\int_{z_o}^{z} dt\int_{\bar{z}_o}^{\zeta} R(t,\tau;z,\zeta)\, F(t,\tau)d\tau$$

$$+ B_o(z,\zeta)\left[V(z_o,\bar{z}_o)\frac{\partial}{\partial\zeta}R(z_o,\bar{z}_o;z,\zeta)\right.$$

$$\left. + \frac{\partial}{\partial\zeta}\int_{z_o}^{z} dt\int_{\bar{z}_o}^{\zeta} R(t,\tau;z,\zeta)\, F(t,\tau)d\tau\right]$$

$$+ C_o(z,\zeta)\left[V(z_o,\bar{z}_o)R(z_o,\bar{z}_o;z,\zeta)\right.$$

$$\left. + \int_{z_o}^{z} dt\int_{\bar{z}_o}^{\zeta} R(t,\tau;z,\zeta)\, F(t,\tau)d\tau\right]$$

$$\left. + F_o(z,\zeta)\right\} .$$

Note that $K(t,\tau\,;z,\zeta)$, $g(z,\zeta)$ and $f(z,\zeta)$ are analytic functions for z, t in $\Omega \cup \kappa \cup \hat{\Omega}$ and ζ, τ in $\overline{\Omega \cup \kappa \cup \hat{\Omega}}$.

To put Expression (2.23) into a form suitable for reflection, we need to introduce an analytic mapping of a single complex variable which will establish a one-to-one correspondence between the points of $\overline{\Omega}$ and the reflected region $\hat{\Omega}$ and which will map points z on κ onto the corresponding points $\overline{z}$ on $\overline{\kappa}$. The reflection function, G, relative to κ provides this desired connection, since, as seen in § 1 of this chapter, if $\hat{z}$ is a point in $\hat{\Omega}$, then $G(\hat{z}) = \overline{z}$ and $G(z) = \overline{z}$ for all z on κ.

Substitute for $\overline{z}$ in (2.23) to obtain the following expression for the left hand side of (2.19):

$$\exp\left[\int_{G(\hat{z})}^{G(z_o)} A(z,\eta)\, d\eta\right]\Bigg\{\varphi(z) - \int_{z_o}^{z} K[t, G(z_o); z, G(\hat{z})]\varphi(t)\, dt$$

$$+ g[z, G(\hat{z})]\varphi^{*}[G(\hat{z})]$$

$$- \int_{z_o}^{\hat{z}} K[z_o, G(\xi); z, G(\hat{z})]\varphi^{*}[G(\xi)]G'(\xi)d\xi$$

$$- f[z, G(\hat{z})]\Bigg\} .$$

In view of Condition (2.19) and the fact that $\varphi(z)$ and $\varphi^{*}[G(\hat{z})]$ are continuous functions of z in $\Omega \cup \kappa$ and that $\lim_{z\to\kappa} G(\hat{z}) = G(z)$, we can let z approach κ to get the identity

$$(2.24) \qquad \varphi(z) - \int_{z_o}^{z} K[t, G(z_0); z, G(z)]\, \varphi(t)\, dt$$

$$= f[z, G(z)] - g[z, G(z)]\, \varphi^*[G(z)]$$

$$+ \int_{z_o}^{z} K[z_o, G(\xi); z, G(z)]\, \varphi^*[G(\xi)]\, G'(\xi)\, d\xi$$

along κ. Thus, along κ, $\varphi(z)$ satisfies the Volterra integral equation

$$(2.25) \qquad h(z) - \int_{z_o}^{z} K[t, G(z_o); z, G(z)]\, h(t)\, dt$$

$$= f[z, G(z)] - g[z, G(z)]\, \varphi^*[G(z)]$$

$$+ \int_{z_o}^{z} K[z_o, G(\xi); z, G(z)]\, \varphi^*[G(\xi)]\, G'(\xi)\, d\xi$$

in which $h(z)$ plays the role of the unknown.

Because $\varphi^*[G(z)]$ is defined and analytic in $\hat{\Omega}$ and continuous in $\hat{\Omega} \cup \kappa$, the right hand side of Equation (2.25) is actually a known analytic function for z in $\hat{\Omega}$ and is continuous up to and including κ. Similarly, the kernel $K[t, G(z_o); z, G(z)]$ is analytic in $\hat{\Omega}$ and is continuous in $\hat{\Omega} \cup \kappa$. Hence, Equation (2.25) may be solved by the usual method of successive approximations to obtain the unique solution in $\hat{\Omega} \cup \kappa$ given by

$$\psi(z) = \mathcal{R}(z) + \int_{z_o}^{z} \Gamma(t, z)\, \mathcal{R}(t)\, dt \quad , \tag{2.26}$$

where $\mathcal{R}(z)$ denotes the right hand side of (2.25) and $\Gamma(t, z)$ is the so-called resolvent kernel defined by the series

$$\Gamma(t, z) = \sum_{n=1}^{\infty} K^{(n)}(t, z) \quad .$$

By definition,

$$K^{(1)}(t, z) = K[t, G(z_o); z, G(z)]$$

$$K^{(n)}(t, z) = \int_{t}^{z} K^{(1)}(t, \eta)\, K^{(n-1)}(\eta, z) d\eta \,, \quad n = 2, 3, \ldots$$

and it can be shown that the series defining $\Gamma(t, z)$ is absolutely and uniformly convergent on compact subsets of $\hat{\Omega} \cup \kappa$ and that $\Gamma(t, z)$ is an analytic function for z, t in $\hat{\Omega}$ and continuous in $\hat{\Omega} \cup \kappa$ (see [8]). Therefore, $\psi(z)$ is analytic in $\hat{\Omega}$ and is continuous in $\Omega \cup \kappa$. Moreover, $\psi(z)$ satisfies Equation (2.25) along κ. But, as seen earlier, the function $\varphi(z)$ also satisfies (2.25) along κ. Hence, by the uniqueness theorem for Volterra integral equations, the functions $\varphi(z)$ and $\psi(z)$ must assume the same values at points of κ, and so $\psi(z)$ is indeed the unique analytic continuation of $\varphi(z)$ into $\hat{\Omega}$.

If the coefficients $A(z, \bar{z})$, $B(z, \bar{z})$ and $C(z, \bar{z})$ of Equation

(2.18) are real-valued functions, then by Theorem 1.1, $\varphi^*(\zeta)=\overline{\varphi(\overline{\zeta})}$ and the function $\psi^*(\zeta) = \overline{\psi(\overline{\zeta})}$ gives the unique analytic continuation of φ^* across $\overline{\kappa}$ into $\overline{\hat{\Omega}}$. Otherwise, rewrite Expression (2.23) in the form

$$\text{(2.27)} \qquad \exp\left[\int_{z}^{z_o} B(\xi,\overline{z})\, d\xi\right]\left\{\varphi^*(\overline{z}) - \int_{\overline{z}_o}^{\overline{z}} \tilde{K}(z_o,\tau\,;\,z,\overline{z})\,\varphi^*(\tau)\, d\tau + \tilde{g}(z,\overline{z})\,\varphi(z) - \int_{z_o}^{z} \tilde{K}(t,\overline{z}_o\,;\,z,\overline{z})\,\varphi(t)\, dt - \tilde{f}(z,\overline{z})\right\},$$

where $\tilde{K}$, $\tilde{g}$ and $\tilde{f}$ differ from K, g and f only by an exponential factor. Recall that for $\hat{z}$ in $\hat{\Omega}$, $G(\hat{z}) = \overline{z}$; whence, $G(z) = G(\hat{\hat{z}}) = \overline{\hat{z}}$ and $z = G^{-1}(\overline{\hat{z}})$. Substitute for z in (2.27) and then let $\overline{z}$ approach $\overline{\kappa}$ to get that $\varphi^*(\overline{z})$ satisfies a Volterra integral equation analogous to (2.25) along $\overline{\kappa}$. Now proceed as before to obtain the unique analytic continuation of $\varphi^*(\overline{z})$ across $\overline{\kappa}$ into $\hat{\Omega}$.

Finally, in Formula (2.20), replace the functions $\varphi(z)$ and $\varphi^*(\zeta)$ by their extensions defined in $\Omega \cup \kappa \cup \hat{\Omega}$ and $\overline{\Omega \cup \kappa \cup \hat{\Omega}}$, respectively, to obtain $V(z, \bar{z})$ as the unique solution of Equation (2.18) in $\Omega \cup \kappa \cup \hat{\Omega}$ which is identical to the original solution throughout Ω. And, replacing $\bar{z}$ by ζ in Formula (2.20) gives $V(z, \zeta)$ as an analytic function of two complex variables in $(\Omega \cup \kappa \cup \hat{\Omega}, \overline{\Omega \cup \kappa \cup \hat{\Omega}})$. This completes the proof.

Remark: Lemma 2.2 is also valid if the condition $A_o(z, \zeta) \neq 0$, $B_o(z, \zeta) \neq 0$ is replaced by the condition $A_o(z, \zeta) = B_o(z, \zeta) = 0$ and $C_o(z, \zeta) = 1$.

In preparation for the next theorem, let $a_{k,o}(x, y)$, $b_{k,o}(x, y)$, $c_{k,o}(x, y)$ and $f_{k,o}(x, y)$ be analytic functions in some region of the real x, y-plane. Apply the change of coordinates (2.3) and then replace x_k and y_k by the expressions $\frac{z_k+\zeta_k}{2}$ and $\frac{z_k-\zeta_k}{2i}$, respectively, to obtain the four new functions

$$A_{k,o}(z_k, \zeta_k) = a_{k,o}\left(\frac{z_k+\zeta_k}{2}, \delta_k \frac{z_k-\zeta_k}{2i} - \beta_k \frac{z_k+\zeta_k}{2}\right) + i\, b_{k,o}$$

$$B_{k,o}(z_k, \zeta_k) = a_{k,o}\left(\frac{z_k+\zeta_k}{2}, \delta_k \frac{z_k-\zeta_k}{2i} - \beta_k \frac{z_k+\zeta_k}{2}\right) - i\, b_{k,o}$$

$$(2.28) \qquad C_{k,o}(z_k, \zeta_k) = c_{k,o}\left(\frac{z_k+\zeta_k}{2}, \delta_k \frac{z_k-\zeta_k}{2i} - \beta_k \frac{z_k+\zeta_k}{2}\right)$$

$$F_{k,o}(z_k, \zeta_k) = f_{k,o}\left(\frac{z_k+\zeta_k}{2}, \delta_k \frac{z_k-\zeta_k}{2i} - \beta_k \frac{z_k+\zeta_k}{2}\right) .$$

We are now ready to present the main reflection theorem.

Theorem 2.1 . Let D be a simply-connected domain in the x, y-plane adjacent to an open analytic arc κ such that the reflection function, G_k, relative to the arc $\kappa_k = \{ z_k' = \tau_k(z) : z \in \kappa \}$ ($\tau_k(z)$ given by (2.6)) is defined, analytic and $G_k'(z_k) \neq 0$ on $D_k \cup \kappa_k \cup \hat{D}_k$, k = 1, 2, 3. Let u(x, y) be in $C^6(D) \cap C^5(D \cup \kappa)$ and satisfy

$$(2.29) \qquad \prod_{k=1}^{3} [\Delta_k + a_k(x_k, y_k) \frac{\partial}{\partial x_k} + b_k(x_k, y_k) \frac{\partial}{\partial y_k} + c_k(x_k, y_k)] \; u(x, y) = 0$$

in D, where the coefficients a_k, b_k and c_k are nonzero analytic complex-valued functions of their arguments and where $D_k \cup \kappa_k \cup \hat{D}_k$ is a fundamental domain for the operator

$$L_k = \Delta_k + a_k(x_k, y_k)\frac{\partial}{\partial x_k} + b_k(x_k, y_k)\frac{\partial}{\partial y_k} + c(x_k, y_k) \; ,$$

k = 1, 2, 3. Along κ, let u(x, y) satisfy the conditions

$$(2.30) \qquad a_{k,o}(x, y) \frac{\partial u_k}{\partial x} + \left[\delta_k \, b_{k,o}(x, y) - \beta_k \, a_{k,o}(x, y)\right] \frac{\partial u_k}{\partial y} + c_{k,o}(x, y)\, u_k + f_{k,o}(x, y) = 0 \; ,$$

k = 1, 2, 3, where $u_3 = u$, $u_2 = L_3[u_3]$, $u_1 = L_2[u_2]$ and β_k and δ_k are the same numbers that appear in the coordinate transformation (2.2). The functions $a_{k,o}$, $b_{k,o}$, $c_{k,o}$ and $f_{k,o}$ are assumed to be analytic in x and y and to be such, that the functions $A_{k,o}(z_k, \zeta_k)$,

$B_{k,o}(z_k, \zeta_k)$, $C_{k,o}(z_k, \zeta_k)$ and $F_{k,o}(z_k, \zeta_k)$ given in (2.28) are analytic for z_k in $D_k \cup \kappa_k \cup \hat{D}_k$, ζ_k in $\overline{D_k \cup \kappa_k \cup \hat{D}_k}$, and $A_{k,o}(z, \zeta_k) \neq 0$, $B_{k,o}(z_k, \zeta_k) \neq 0$ for $k = 1, 2, 3$. Then, $u(x, y)$ can be analytically continued as a solution of (2.29) across κ into a simply-connected domain, $\hat{R}$, adjacent to κ. The region $\hat{R}$ can be explicitly determined and is seen to depend only on the coefficients of the highest order terms of (2.29) as given in (2.1), on the original domain D, on the arc κ and, possibly, on the choice of a finite number of simply-connected domains. Thus, the extension is global in nature.

Remark: Theorem 2.1 remains valid if the condition $A_{k,o} \neq 0$, $B_{k,o} \neq 0$ is replaced by the condition $A_{k,o} = B_{k,o} = 0$, $C_{k,o} = 1$ for any k .

Proof of theorem: As seen in §3 of this chapter, Equation (2.29) may be written as the system

$$(2.31a)\quad \left[\frac{\partial^2}{\partial z_3 \partial \bar{z}_3} + A_3(z_3, \bar{z}_3)\frac{\partial}{\partial z_3} + B_3(z_3, \bar{z}_3)\frac{\partial}{\partial \bar{z}_3} + C_3(z_3, \bar{z}_3)\right] V_{33}(z_3, \bar{z}_3) = \frac{1}{4} V_{23}(z_3, \bar{z}_3),\ z_3 \in D_3,$$

$$(2.31b)\quad \left[\frac{\partial^2}{\partial z_2 \partial \bar{z}_2} + A_2(z_2, \bar{z}_2)\frac{\partial}{\partial z_2} + B_2(z_2, \bar{z}_2)\frac{\partial}{\partial \bar{z}_2} + C_2(z_2, \bar{z}_2)\right] V_{22}(z_2, \bar{z}_2) = \frac{1}{4} V_{12}(z_2, \bar{z}_2),\ z_2 \in D_2,$$

$$(2.31c)\quad \left[\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + A_1(z_1,\bar{z}_1)\frac{\partial}{\partial z_1} + B_1(z_1,\bar{z}_1)\frac{\partial}{\partial \bar{z}_1} + C_1(z_1,\bar{z}_1)\right] V_{11}(z_1,\bar{z}_1) = 0,\ z_1 \in D_1 \ .$$

To rewrite Conditions (2.30), rearrange the first two terms as

$$a_{k,o}(x,y)\left(\frac{\partial u_k}{\partial x} - \beta_k \frac{\partial u_k}{\partial y}\right) + \delta_k b_{k,o}(x,y)\frac{\partial u_k}{\partial y} \ ,$$

and then refer to (2.14) to see that

$$\frac{\partial v_{kk}(x_k,y_k)}{\partial x_k} = \frac{\partial u_k(x,y)}{\partial x} - \beta_k \frac{\partial u_k(x,y)}{\partial y} \ ,$$

$$\frac{\partial v_{kk}(x_k,y_k)}{\partial y_k} = \delta_k \frac{\partial u_k(x,y)}{\partial y} \ .$$

Thus, with respect to the variables x_k and y_k, Conditions (2.30) become

$$a_{k,o}(x_k, \delta_k y_k - \beta_k x_k)\frac{\partial v_{kk}}{\partial x_k} + b_{k,o}\frac{\partial v_{kk}}{\partial y_k} + c_{k,o} v_{kk} + f_{k,o} = 0 \ .$$

Introducing the transformation

$$x_k = \frac{z_k + \bar{z}_k}{2}, \quad y_k = \frac{z_k - \bar{z}_k}{2i}$$

into this last expression, we see that when z_k is on κ_k, $V_{kk}(z_k,\bar{z}_k)$ must satisfy the condition

$$(2.32)\quad A_{k,o}(z_k,\bar{z}_k)\frac{\partial V_{kk}}{\partial z_k} + B_{k,o}(z_k,\bar{z}_k)\frac{\partial V_{kk}}{\partial \bar{z}_k} + C_{k,o}(z_k,\bar{z}_k)V_{kk} + F_{k,o}(z_k,\bar{z}_k) = 0\,,$$

$k = 1, 2, 3.$

We begin the reflection process with $V_{11}(z_1, \bar{z}_1)$, a known function satisfying Equation (2.31c) for z_1 in D_1 and Condition (2.32) along κ_1. Apply Lemma 2.2 to extend $V_{11}(z_1, \bar{z}_1)$ as a solution of (2.31c) into the entire reflected domain $\hat{D}_1$ adjacent to κ_1. Refering to (2.14) and (2.7), transform $V_{11}(z_1, \bar{z}_1)$ back to the x, y coordinates to obtain $u_1(x, y)$ as defined and satisfying $L_1[u_1] = 0$ in $D \cup \kappa \cup \tau_1^{-1}(\hat{D}_1)$, where $\tau_1^{-1}(\hat{D}_1)$ is a simply-connected domain adjacent to κ and disjoint from D.

Next, we consider the function $V_{22}(z_2, \bar{z}_2)$, which is known and satisfies (2.31b) in D_2 and Condition (2.32) along κ_2. The function $V_{12}(z_2, \bar{z}_2) = V_{11}[T_1(z_2, \bar{z}_2), T_1^*(\bar{z}_2, z_2)]$ appearing in the right hand side of (2.31b), which is originally known only for z_2 in D_2, is now defined for z_2 in $\tau_2\tau_1^{-1}(D_1 \cup \kappa_1 \cup \hat{D}_1) = D_2 \cup \kappa_2 \cup \tau_2\tau_1^{-1}(\hat{D}_1)$, where $\tau_2\tau_1^{-1}(\hat{D}_1)$ is a simply-connected domain adjacent to κ_2 and disjoint from D_2. Since $V_{11}(z_1, \zeta_1)$ is analytic for $(z_1, \zeta_1) \in (D_1 \cup \kappa_1 \cup \hat{D}_1, \overline{D_1 \cup \kappa_1 \cup \hat{D}_1})$, then, by Lemma 2.1, for each z_{20} on κ_2 there is an open disc,

$N[z_{20}, r(z_{20})]$, of maximum radius $r(z_{20})$ and centered at z_{20}, contained in $D_2 \cup \kappa_2 \cup \tau_2\tau_1^{-1}(\hat{D}_1)$ such that $V_{12}(z_2, \zeta_2) = V_{11}[T_1(z_2, \zeta_2), T_1^*(\zeta_2, z_2)]$ is analytic for $z_2 \in N[z_{20}, r(z_{20})]$, $\zeta_2 \in N[\bar{z}_{20}, r(z_{20})]$. For simplicity, we limit $r(z_{20})$ so that the boundary of $N[z_{20}, r(z_{20})]$ meets κ_2 in at most two points. Restricting our attention to the domain $N[z_{20}, r(z_{20})]$, we use Lemma 2.2 to extend $V_{22}(z_2, \bar{z}_2)$ as a solution of (2.31b) into that simply-connected component, $\hat{R}(z_{20})$, of $\left\{\widehat{N[z_{20}, r(z_{20})] \cap D_2}\right\} \cap N[z_{20}, r(z_{20})]$ which is adjacent to that portion of κ_2 contained in $N[z_{20}, r(z_{20})]$ (see Figure 2.3).

We proceed to show that if $\hat{R}(z_{20}) \cap \hat{R}(z'_{20})$ is nonempty and simply-connected for two distinct points z_{20} and z'_{20} on κ_2, then the extensions of $V_{22}(z_2, \bar{z}_2) = v_{22}(x_2, y_2)$ into $\hat{R}(z_{20})$ and $\hat{R}(z'_{20})$ agree on $\hat{R}(z_{20}) \cap \hat{R}(z'_{20})$. First, note that if $\hat{R}(z_{20}) \cap \hat{R}(z'_{20})$ is nonempty, then the assumption that $G'_2(z_2) \neq 0$ on $D_2 \cup \kappa_2 \cup \hat{D}_2$ implies that $N[z_{20}, r(z_{20})] \cap N[z'_{20}, r(z'_{20})] \cap D_2$ is nonempty. Indeed, let η be a point in $\hat{R}(z_{20}) \cap \hat{R}(z'_{20})$. Then, the fact that

$$\eta \in \hat{R}(z_{20}) \subset \left\{\widehat{N[z_{20}, r(z_{20})] \cap D_2}\right\} \cap N[z_{20}, r(z_{20})]$$

and

$$\eta \in \hat{R}(z'_{20}) \subset \left\{\widehat{N[z'_{20}, r(z'_{20})] \cap D_2}\right\} \cap N[z'_{20}, r(z'_{20})]$$

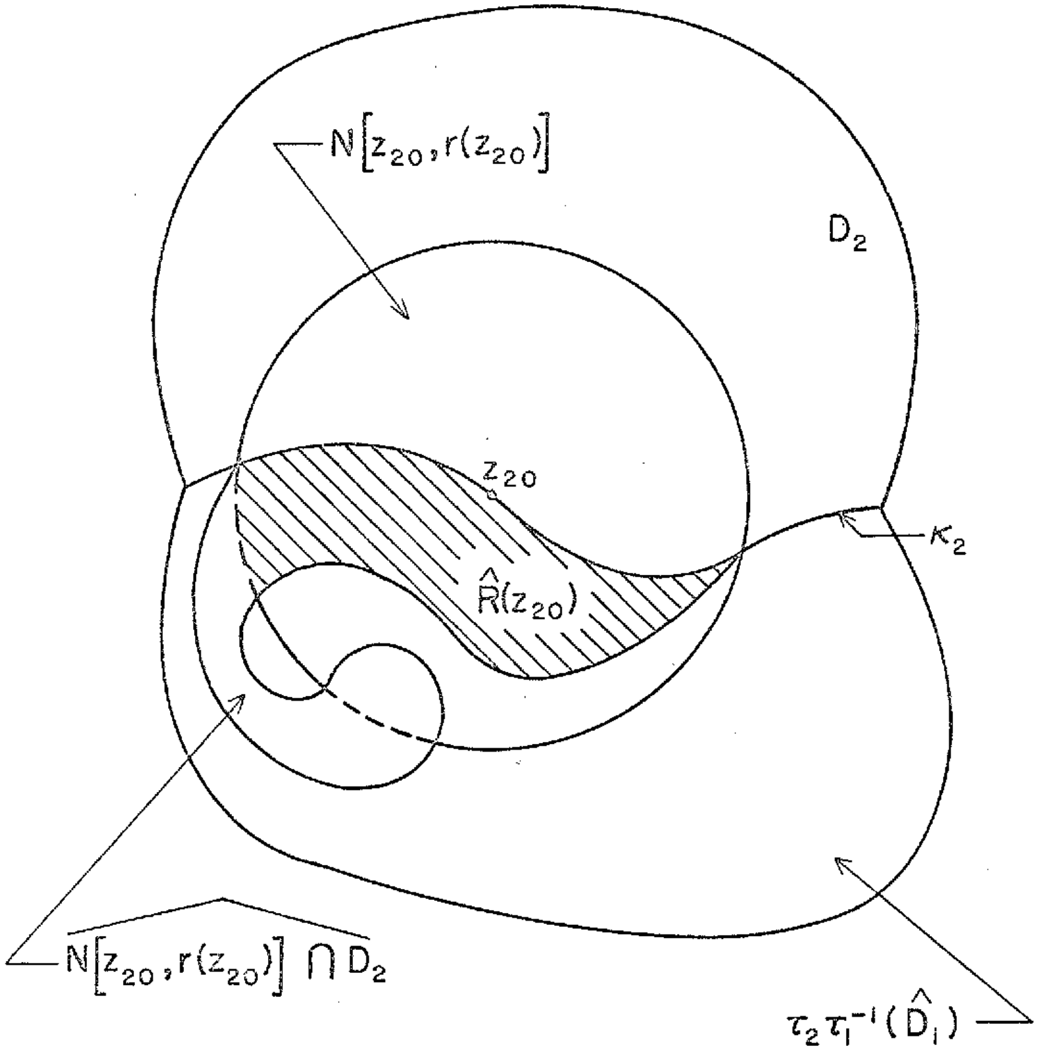

Figure 2.3

implies that there exist points ξ and ξ' in $N[z_{20}, r(z_{20})] \cap D_2$ and $N[z'_{20}, r(z'_{20})] \cap D_2$, respectively, such that $\overline{G_2(\xi)} = \hat{\xi} = \eta = \hat{\xi}' = \overline{G_2(\xi')}$. But by assumption, G_2 is one-to-one on $D_2 \cup \kappa \cup \hat{D}_2$. Whence

$$\xi = \xi' \in N[z_{20}, r(z_{20})] \cap N[z'_{20}, r(z'_{20})] \cap D_2$$

(see Figure 2.4).

Also, note that the set

$$\left| N[z_{20}, r(z_{20})] \cap N[z'_{20}, r(z'_{20})] \cap (D_2 \cup \kappa_2) \right| \cup \left| \hat{R}(z_{20}) \cap \hat{R}(z'_{20}) \right|$$

is a simply-connected domain.

Now let $v_{z_{20}}(x_2, y_2)$ and $v_{z'_{20}}(x_2, y_2)$ be the extensions of $V_{22}(z_2, \overline{z}_2) = v_2(x_2, y_2)$ in the domain

$$\left| N[z_{20}, r(z_{20})] \cap (D_2 \cup \kappa_2) \right| \cup \hat{R}(z_{20})$$

and in the domain

$$\left| N[z'_{20}, r(z'_{20})] \cap (D_2 \cup \kappa_2) \right| \cup \hat{R}(z'_{20}) \quad ,$$

respectively. Then $v_{z_{20}}(x_2, y_2)$ and $v_{z'_{20}}(x_2, y_2)$ are analytic functions of x_2 and y_2 which agree on the open set

$$N[z_{20}, r(z_{20})] \cap N[z'_{20}, r(z'_{20})] \cap D_2 \quad ,$$

and so they must also agree on $\hat{R}(z_{20}) \cap \hat{R}(z'_{20})$.

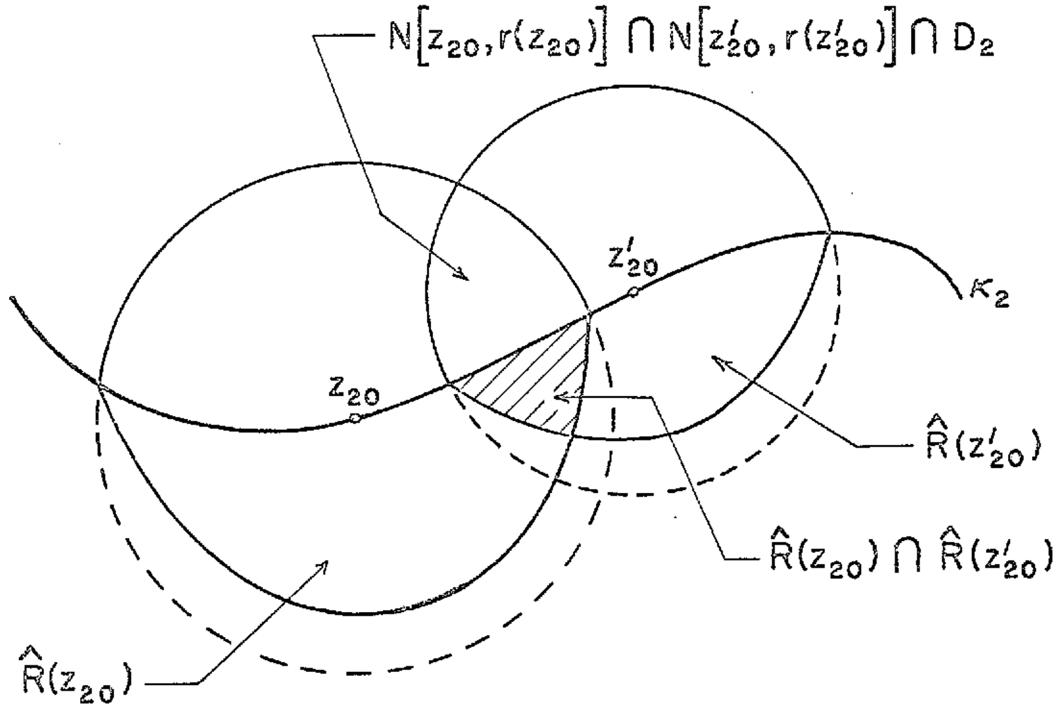

Figure 2.4

We form the set $\bigcup_{z_{20} \in \kappa_2} \hat{R}(z_{20})$, which is adjacent to κ_2. If it is simply-connected, then we set $\hat{R}_2$ equal to it. Otherwise, we let $\hat{R}_2$ be a simply-connected subdomain of $\bigcup_{z_{20} \in \kappa_2} \hat{R}(z_{20})$ adjacent to κ_2. Now use the preceding argument to analytically continue $V_{22}(z_2, \bar{z}_2) = v_{22}(x_2, y_2)$ throughout $\hat{R}_2$. Then $V_{22}(z_2, \bar{z}_2)$ becomes defined throughout $D_2 \cup \kappa_2 \cup \hat{R}_2$ as a solution of Equation (2.31b) and $V_{22}(z_2, \zeta_2)$ is an analytic function of two complex variables whenever $z_{20} \in \kappa_2$ and

$$z_2 \in \left\{N[z_{20}, r(z_{20})] \cap (D_2 \cup \kappa_2)\right\} \cup \left\{\hat{R}(z_{20}) \cap \hat{R}_2\right\},$$

$$\zeta_2 \in \overline{\left\{N[z_{20}, r(z_{20})] \cap (D_2 \cup \kappa_2)\right\} \cup \left\{\hat{R}(z_{20}) \cap \hat{R}_2\right\}}.$$

Again, referring to (2.14) and (2.7), we can transform $V_{22}(z_2, \bar{z}_2)$ back to the x, y coordinates to obtain $u_2(x, y)$ as defined and satisfying $L_2[u_2] = u_1(x, y)$ in $D \cup \kappa \cup \tau_2^{-1}(\hat{R}_2)$. Note, since $\hat{R}_2 \subset \tau_2 \tau_1^{-1}(\hat{D}_1)$, then $\tau_2^{-1}(\hat{R}_2) \subset \tau^{-1}(\hat{D}_1)$.

Finally, repeat the same argument for $V_{33}(z_3, \bar{z}_3)$ as we did for the function $V_{22}(z_2, \bar{z}_2)$ to obtain a simply-connected domain $\hat{R}_3$ adjacent to κ_3 into which $V_{33}(z_3, \bar{z}_3) = v_{33}(x_3, y_3)$ can be extended as a soltuion of (2.31a). We remark that when applying Lemma 2.1, the set Ω_j becomes $\left\{N[z_{20}, r(z_{20})] \cap (D_2 \cup \kappa_2)\right\}$

$\cup \left\{ \hat{R}(z_{20}) \cap \hat{R}_{20} \right\}$ and the image of this latter set under the mapping $\tau_3 \tau_2^{-1}$ becomes Ω_k. Again, perform a change of variables from the x_3, y_3 coordinates to the original x, y coordinates to get $u_3(x, y)$ as being defined and satisfying $L_3[u_3] = u_2(x, y)$ in $D \cup \kappa \cup \tau_3^{-1}(\hat{R}_3)$. As before, $\tau_3^{-1}(\hat{R}_3) \subset \tau_2^{-1}(\hat{R}_2) \subset \tau_1^{-1}(\hat{D}_1)$. Thus, keeping in mind that $u_3(x, y) = u(x, y)$, it is seen that u(x, y) satisfies Equation (2.29) in $D \cup \kappa \cup \hat{R}$ and agrees with the original solution in $D \cup \kappa$, where $\hat{R} = \tau_3^{-1}(\hat{R}_3)$ is a simply-connected domain adjacent to κ and disjoint from D. This completes the proof.

§ 5. Concluding Remarks and Illustrations.

Conceptually, it is not difficult to see that the process involved in the proof of Theorem 2.1 may be continued to include equations of order higher than six. Thus, with the obvious modifications in the hypotheses, an inductive argument may be adapted to establish the conclusions of the reflection theorem for elliptic equations of the form

$$\left(\prod_{k=1}^{n} L_k \right) u(x, y) = 0 ,$$

where the operators L_k are defined in § 3 of this chapter.

We would like now to point out a couple of differences between the cases treated here and by Sloss in [7]. First, unlike the method used in [7], the reflection process of Theorem 2.1 does not depend on the characteristics of the differential equation being distinct. That is, it is not necessary that $\alpha_j \neq \alpha_k$ if $j \neq k$ (see §2).

The second and most outstanding difference between the two cases lies in the Riemann functions used to represent solutions of second order elliptic equations with analytic coefficients (see Theorem 1.2). If the given differential equation has no lower order terms and is of the type considered by Sloss, then it can be decomposed as a system of second order equations whose Riemann functions are all identically equal to unity. On the other hand, for equations of the type considered in this paper, where lower order terms are present, the Riemann functions depend nontrivially on four independent complex variables. The introduction of these independent complex variables complicates the geometry of the problem considerably by requiring computations to always be performed in certain fundamental domains, namely those which are determined by Lemma 2.1.

When the differential equation has no lower order terms present, the computations necessary for reflection involve analytic functions of a single complex variable. In this case, there is a method whereby a relatively simple and explicit formula may be derived for the domain of reflection. In fact, let D be a simply-connected domain adjacent to an open analytic arc κ such that the reflection function, G, relative to κ is defined, analytic and $G'(z) \neq 0$, $G'(z) \neq -A_j/B_j$ for z in $D \cup \kappa \cup \hat{D}$. Here, A_j and B_j are the constants given in (2.6a). Let $\tau_j(z)$ be given as in (2.6) and let

$$\sigma_j(z) = A_j z + B_j G(z) \quad .$$

The function $\sigma_j(z)$ is analytic in $D \cup \kappa \cup \hat{D}$ and $\sigma'(z) \neq 0$ there, and so it has an inverse. If u(x, y) is given in D to be a solution of the differential equation

$$\Delta_1 \Delta_2 \ldots \Delta_n u(x, y) = 0 \quad ,$$

where u(x, y) satisfies certain analytic boundary conditions along κ, then Sloss has shown in [7] that it is possible to continue u as a solution of the differential equation into the region

$$R_o = \bigcap_{j=1}^{n} \tau_j^{-1} \left\{ \sigma_j(\hat{R}) \cap \tau_j(\hat{D}) \right\} \subset \hat{D}$$

where $R = R_1 \cap R_2 \cap \ldots \cap R_n$, $R_j = \sigma_j^{-1} \left\{ \sigma_j(D) \cap \tau_j(D) \right\}$ and $\hat{R} = \overline{G(R)}$.

The following four figures were obtained with the invaluable aid of the U. C. S. B. on-line system. They were photographed from the display scope and then drawn from an enlarged print. The first three illustrate the region of reflection for solutions of the specific equation

$$\frac{\partial^6 u}{\partial x^6} + 2\frac{\partial^6 u}{\partial x^5 \partial y} + 3\frac{\partial^6 u}{\partial x^4 \partial y^2} + 6\frac{\partial^6 u}{\partial x^3 \partial y^3} + 6\frac{\partial^6 u}{\partial x^2 \partial y^4}$$

$$- 8\frac{\partial^6 u}{\partial x \partial y^5} + 40\frac{\partial^6 u}{\partial y^6} = 0$$

for various choices of the arc. Here, $\alpha_1 = 1 + i$, $\alpha_2 = 2i$ and $\alpha_3 = -2 + i$. The last illustration is for the differential equation having $\alpha_1 = 1 + 1.5i$, $\alpha_2 = i$ and $\alpha_3 = .5 + .5i$. Each figure shows the original domain D, the arc κ, the reflected domain $\hat{D}$, and the region of reflection, R_o, which is the shaded portion of the drawing. In Figure 2.5, κ is the interval $(-1, 1)$ of the x-axis and D is the region between the curve whose equation is $y^2 = x^2 + .5$ and κ. Figure 2.6 shows reflection across a portion of an ellipse. The domain D is the intersection of the open circular disc centered at $(0, -.7)$ of radius one with the inside of the ellipse $\cos\theta + i\frac{1}{2}\sin\theta$, $-\pi \le \theta \le \pi$. In Figure 2.7, κ is part of a parabola whose equation is $y = x^2$ for $.1 \le x \le \sqrt{.7}$. Finally, in Figure 2.8, D is the annular domain bounded by the circle of radius .7 centered at the origin and κ, which is the unit circle. In this case, since D is not simply-connected, we must restrict our attention to the single valued solutions of the differential equation. We emphasize that in these illustrations $\hat{D}$ contains R_o and is adjacent to κ .

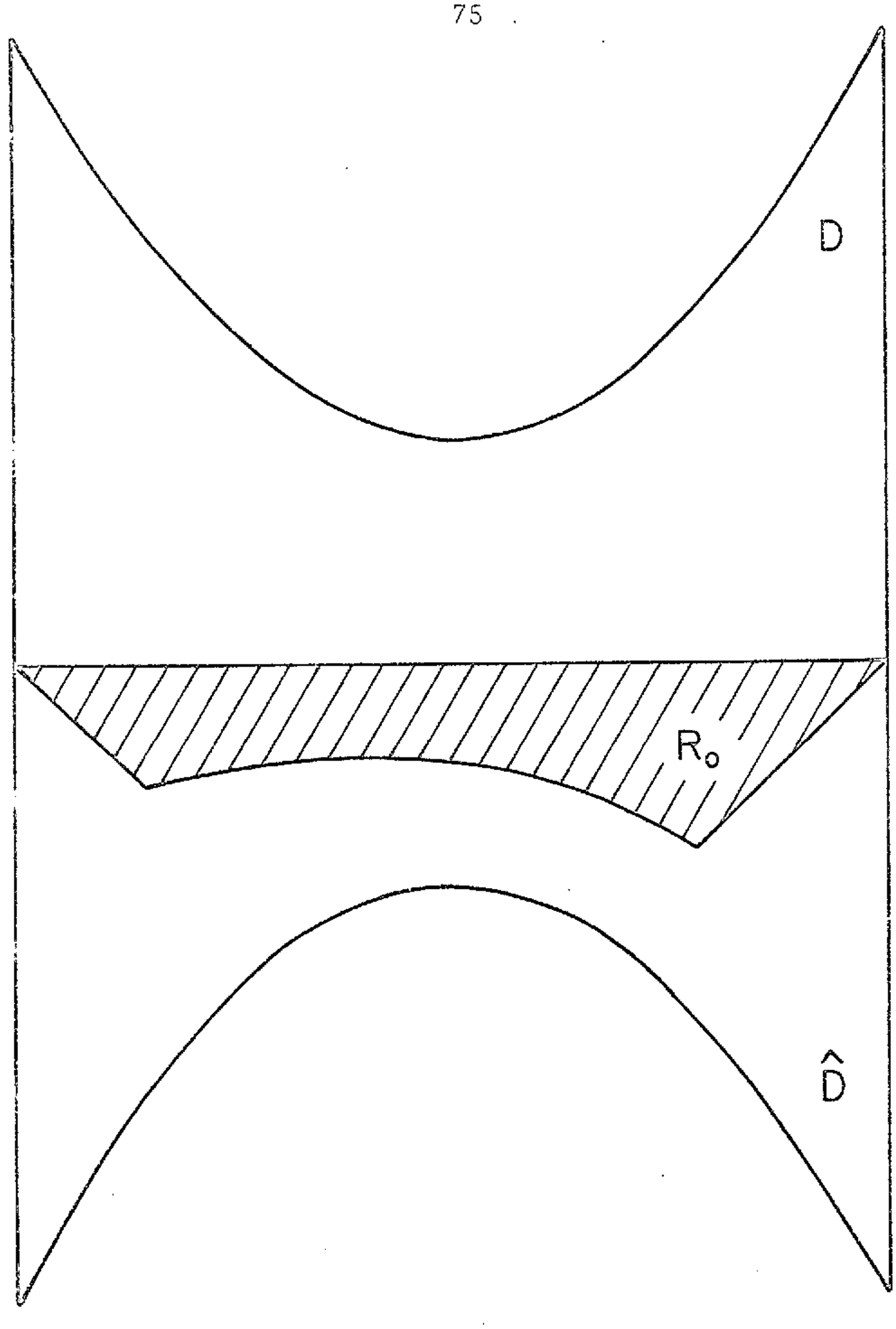

Figure 2.5

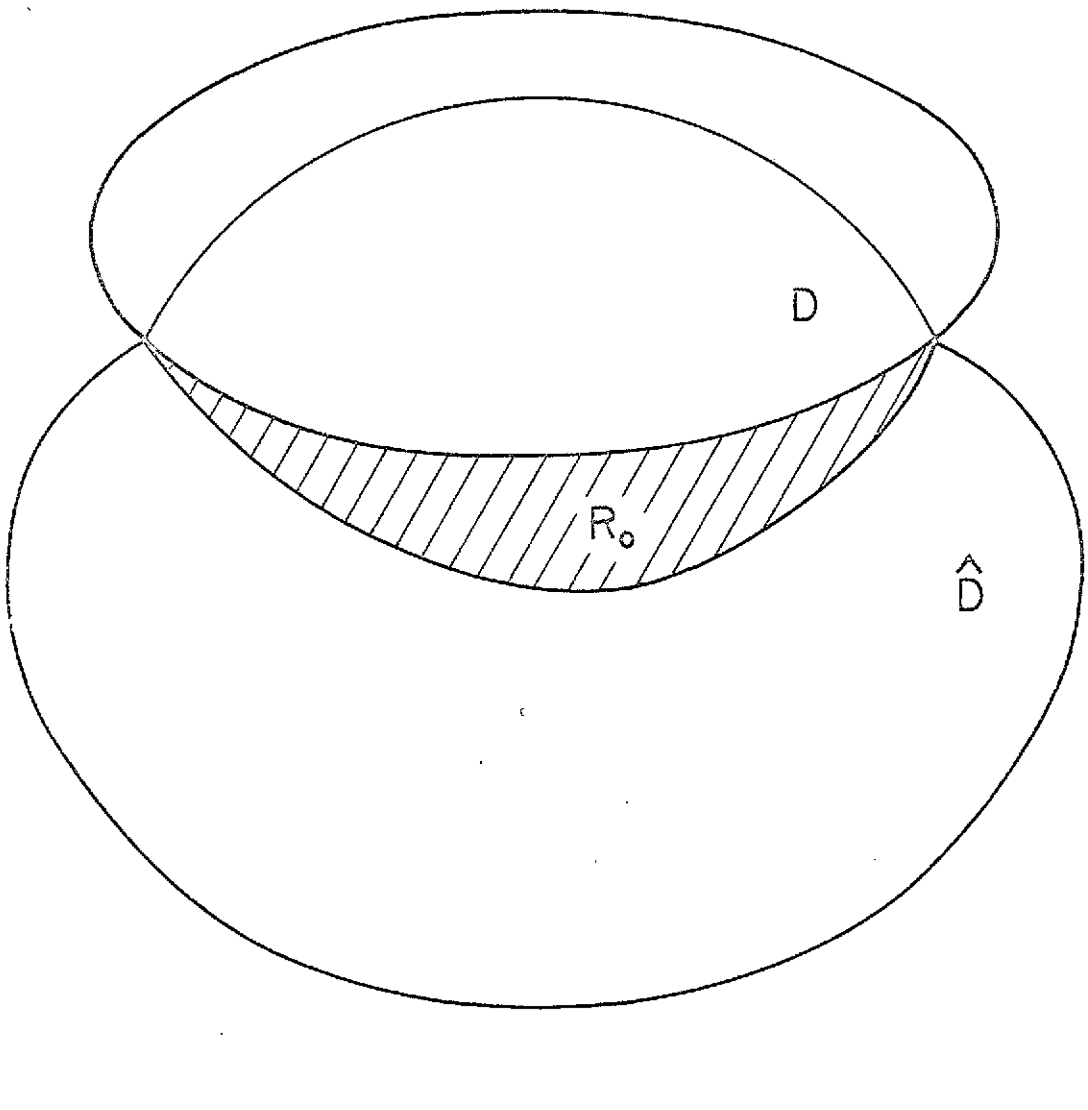

Figure 2.6

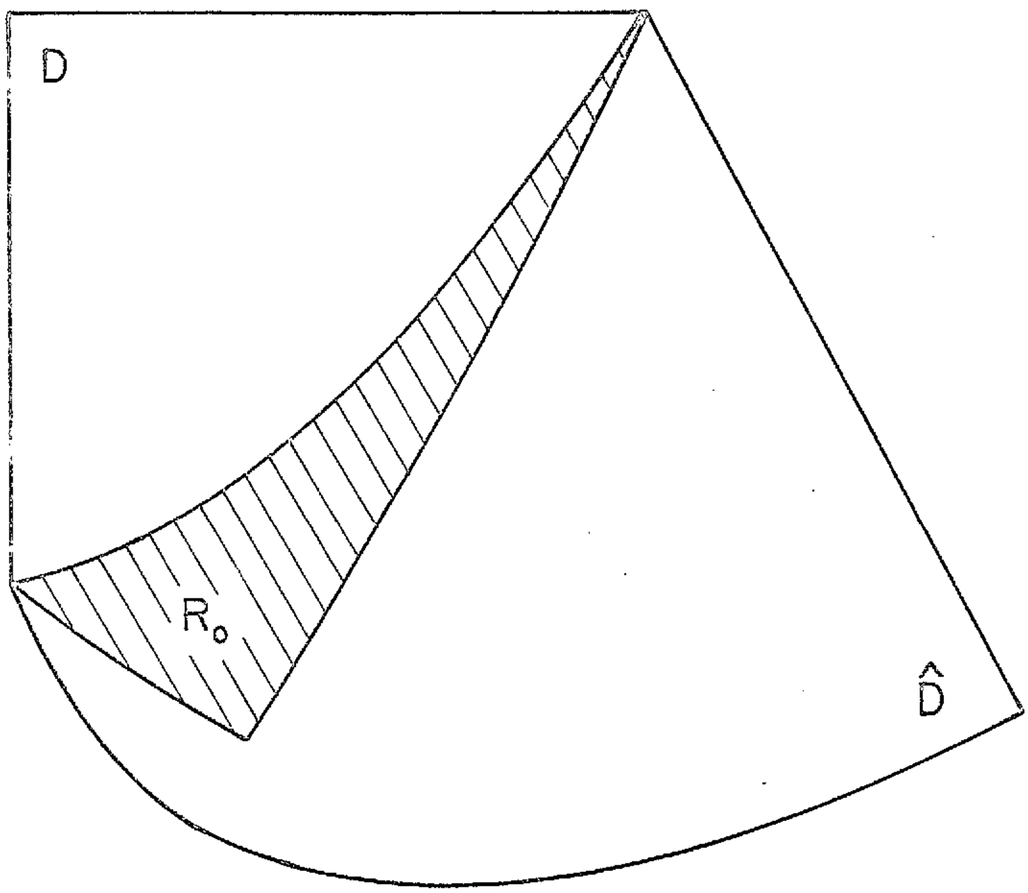

Figure 2.7

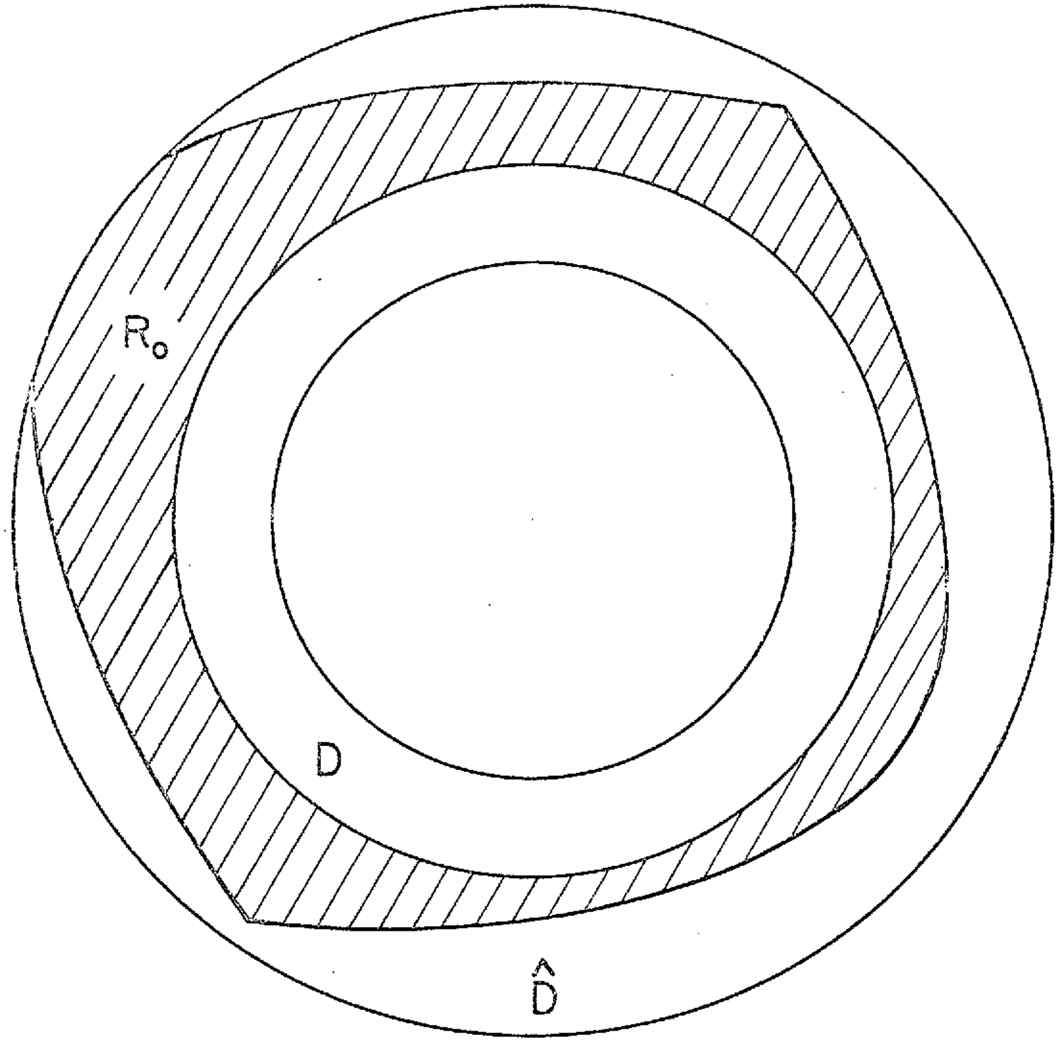

Figure 2.8

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