NASA TECHNICAL MEMORANDUM

NASA TM X- 64781

A COMPARISON OF DIGITAL COMPUTER PROGRAMS FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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July 1, 1973

NASA

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1. REPORT NO. TM X-64781	2. GOVERNMENT ACCESSION NO.	3.
4. TITLE AND SUBTITLE A Comparison of Digital Computer Programs for the Numerical Solution of Ordinary Differential Equations		5. July 1, 19[..]
		6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) Hugo L. Ingram		8. PERFORMING ORGANIZATION REPORT #
9. PERFORMING ORGANIZATION NAME AND ADDRESS George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812		10. WORK UNIT NO.
		11. CONTRACT OR GRANT NO.
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D.C. 20546		13. TYPE OF REPORT & PERIOD COVERED Technical Memorandum
		14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Prepared by Aero-Astrodynamics Laboratory
Science and Engineering

16. ABSTRACT

Recently the determination of the best technique for numerically solving systems of ordinary differential equations on a digital computer has received much attention. Phyllis Fox in A Comparative Study of Computer Programs for Integrating Differential Equations; and Hull, Enright, Fellen, and Sedgewick in Comparing Numerical Methods for Ordinary Differential Equations made studies on the computational efficiency of several different numerical integration techniques, but their studies did not include the Runge-Kutta formulas developed by Fehlberg (NASA TR R-287 and R-315). The use of these formulas in conjunction with a stepsize control developed in this report is explained, and one of the formulas is chosen for comparison with other integration techniques. This comparison of one of the best of Fehlberg's formulas with the different numerical techniques described in the aforementioned studies on a variety of test problems clearly shows the superiority of Fehlberg's formula. That is, on each of the test problems, the chosen Fehlberg formula is able to achieve a given accuracy in less computer time than any of the other techniques tested. Also, the computer program for the chosen Fehlberg formula is less complex and easier to use than the computer programs for most of the other techniques. To illustrate the use of the chosen Fehlberg formula, a computer listing of its application to several example problems is included along with a sample of the computer output from these applications.

17. KEY WORDS	18. DISTRIBUTION STATEMENT Unclassified - Unlimited for E. D. Geissler, Director Aero-Astrodynamics Laboratory

19. SECURITY CLASSIF. (of this report)	20. SECURITY CLASSIF. (of this page)	21. NO. OF PAGES	22. PRICE
Unclassified	Unclassified	108	NTIS

MSFC - Form 3292 (Rev December 1972) For sale by National Technical Information Service, Springfield, Virginia 22151

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A COMPARISON OF DIGITAL COMPUTER PROGRAMS FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

When a physical problem is to be simulated with the aid of a digital computer, the result is often a system of ordinary differential equations that must be solved numerically. For this reason an efficient numerical integration algorithm is very desirable to minimize the amount of computer time needed to solve a particular problem.

Many different approaches are available for the numerical solution of system of ordinary differential equations, and an evaluation in this report of some of these approaches was motivated by a desire to compute optimal trajectories as rapidly as possible. In computing optimal trajectories, the system of ordinary differential equations known as the equations of motion and adjoint variables must be integrated many times to satisfy iteratively the boundary conditions. This process consumes much computer time unless the particular numerical integration technique being used is operating efficiently. The Runge-Kutta formulas derived by E. H. Fehlberg in References 1 and 2 and discussed in the next sections of this report meet this requirement of operational efficiency better than any of the other methods tested.

Also described in the section of this report entitled An Explanation of Runge-Kutta Numerical Integration Formulas is a new stepsize control procedure for Fehlberg's numerical integration formula. This new stepsize control procedure is combined with one of the best of Fehlberg's formulas, and the results it achieves for a variety of test problems is compared with the results achieved by other numerical integration techniques for the same test problems. The data obtained from these comparisons is discussed in detail in the sections of this report entitled Fehlberg's Runge-Kutta Formulas With Stepsize Control and A Comparison of Fehlberg's 7-8-13 Formulas With the Numerical Integration Techniques of References 3 and 4.

AN EXPLANATION OF RUNGE-KUTTA NUMERICAL INTEGRATION FORMULAS

Consider the system of differential equations denoted by

$$\dot{x} = f(t, x) \quad , \tag{1}$$

where $\dot{x}$, f, and x are all vectors of dimension n. Note that the symbol $\dot{x}$ denotes dx/dt where t is assumed to be the independent variable for the system. For a given set of initial conditions denoted by $x(t_o) = x_o$ values for $x(t_o + \Delta t)$ can be obtained from Runge-Kutta formulas as follows:

$$x(t_o + \Delta t) = x_o + \Delta t \sum_{k=0}^{m} c_k f_k \quad , \tag{2}$$

where

$$f_o = f(t_o, x_o)$$

$$f_k = f\left(t_o + \alpha_k \Delta t,\ x_o + \Delta t \sum_{\ell=0}^{k-1} \beta_{k\ell} f_\ell\right) \text{ for } k = 1, 2, \ldots, m \quad . \tag{3}$$

The coefficients α_k, c_k, $\beta_{k\ell}$, and the integer m are determined to make the expression for $x(t_o + \Delta t)$ as given in equation (2) equal to a Taylor series expansion for $x(t_o + \Delta t)$ up to a certain order. For this determination of the coefficients to be meaningful, the vector function $f(t, x)$ must be reasonable enough to have a convergent Taylor series expansion in some neighborhood of the initial conditions t_o, x_o.

As an example of how the coefficients α_k, c_k, and $\beta_{k\ell}$ are determined, a second-order Runge-Kutta formula can be derived. To do this, consider a Taylor series expansion of $x(t_o + \Delta t)$ to second order. That is,

$$x\,(t_o + \Delta t) = x_o + \dot{x}_o\,\Delta t + \tfrac{1}{2}\,\ddot{x}_o\,\Delta t^2 \quad . \tag{4}$$

Then note that

$$\dot{x}_o = f\,(t_o, x_o)$$

and

$$\ddot{x}_o = \left[\frac{d}{dt}\,(\dot{x})\right]_{x=x_o} = \left[\left(\frac{\partial f}{\partial t}\right) + \left(\frac{\partial f}{\partial x}\right)\dot{x}\right]_{x=x_o}$$

$$= \left[\left(\frac{\partial f}{\partial t}\right) + \left(\frac{\partial f}{\partial x}\right) f\,(t, x)\right]_{x=x_o}$$

$$= \left(\frac{\partial f}{\partial t}\right)_{x=x_o} + \left(\frac{\partial f}{\partial x}\right)_{x=x_o} f\,(t_o, x_o) \quad .$$

Thus the Taylor series expansion can be rewritten as:

$$x\,(t_o + \Delta t) = x_o + f\left(t_o, x_o\right)\Delta t + \tfrac{1}{2}\left[\left(\frac{\partial f}{\partial t}\right)_0 + \left(\frac{\partial f}{\partial x}\right)_0 f\,(t_o, x_o)\right]\Delta t^2 \tag{5}$$

Now the expression for $x(t_o + \Delta t)$ from the Runge-Kutta approach is given by equation (2). That is,

$$x\,(t_o + \Delta t) = x_o + \Delta t\,[c_o\,f\,(t_o, x_o) + c_1\,f_1 + \ldots + c_m\,f_m] \tag{6}$$

Now equation (3) can be used and f_1 expanded in a multivariable Taylor series to give:

$$f_1 = f\,(t_o + \alpha_1\,\Delta t,\ x_o + \Delta t\,\beta_{10}\,f_o)$$

$$= f\,(t_o,\ x_o) + \left(\frac{\partial f}{\partial t}\right)_0 (\alpha_1\,\Delta t) + \left(\frac{\partial f}{\partial x}\right)_0\ [\Delta t\,\beta_{10}\,f_0] + \ldots \qquad (7)$$

When the above expression is substituted into equation (6), second-order terms will be obtained in Δt. Thus, there is no need to carry the expansion of equation (7) past first order and m in equation (6) is chosen to be one. Thus equation (6) becomes:

$$x\,(t_o + \Delta t) = x_o + \Delta t \left\{ c_o\,f\,(t_o,\ x_o) + c_1\left[f\,(t_o,\ x_o) + \left(\frac{\partial f}{\partial t}\right)_0 (\alpha_1\,\Delta t) + \left(\frac{\partial f}{dx}\right)_0 (\Delta t\,\beta_{10}\,f_0)\right]\right\} \qquad (8)$$

Therefore,

$$x\,(t_o + \Delta t) = x_o + (c_0 + c_1)\,f\,(t_o,\ x_o)\,\Delta t + \left[\left(\frac{\partial f}{\partial t}\right)_0 (c_1\,\alpha_1) + \left(\frac{\partial f}{\partial x}\right)_0 f\,(t_o,\ x_o)\,c_1\,\beta_{10}\right]\Delta t^2 \qquad (9)$$

Now a comparison of equations (5) and (9) shows that:

$$c_0 + c_1 = 1$$

$$c_1\,\alpha_1 = \tfrac{1}{2}$$

$$c_1\,\beta_{10} = \tfrac{1}{2} \qquad . \qquad (10)$$

Any set of coefficients that satisfies equation (10) will give a second-order Runge-Kutta formula that uses only two evaluations of the differential equations. Thus it can be seen that Runge-Kutta formulas are not unique. As an example $c_0 = c_1 = 1/2$, $\alpha_1 = 1$, and $\beta_{10} = 1$ will satisfy equation (10). Equation (10) is called the equations of condition for a second-order Runge-Kutta formula. Also, the number of evaluations of the differential equations required for a particular order is usually called the number of function evaluations needed. With this background information, the Runge-Kutta formulas developed by Fehlberg in References 1 and 2 can now be discussed.

FEHLBERG'S RUNGE-KUTTA FORMULAS WITH STEPSIZE CONTROL

To obtain a useful stepsize control procedure for Runge-Kutta formulas some indication of the truncation error of the series expansion must be determined. Fehlberg's idea is to develop Runge-Kutta formulas of adjacent order that use the same function evaluations, and then the difference in the two formulas is a good approximation to some single term in the Taylor series expansion.

In References 1 and 2, Fehlberg performs the very difficult determination of adjacent Runge-Kutta formulas for orders from one to eight. That is, formulas of order 1 and 2, 2 and 3, 3 and 4, 4 and 5, 5 and 6, 6 and 7, 7 and 8, and 8 and 9 are all developed, and each adjacent pair uses the same function evaluations so that their difference can approximate a corresponding term in the Taylor series expansion for $x(t_o + \Delta t)$. As an example, for $m = 12$ a seventh-order Runge-Kutta formula and an eighth-order Runge-Kutta formula are developed that use the same function evaluations. That is,

$$x\,(t_o + \Delta t) = x_o + \Delta t \sum_{k=0}^{12} c_k f_k \tag{11}$$

$$\hat{x}\,(t_o + \Delta t) = x_o + \Delta t \sum_{k=0}^{12} \hat{c}_k f_k \tag{12}$$

where

$$f_0 = f(t_o, x_o)$$

$$f_k = f\left(t_o + \alpha_k \Delta t,\ x_o + \Delta t \sum_{\ell=0}^{k-1} \beta_{k\ell} f_\ell\right) \text{ for } k = 1, 2, \ldots, 12 \quad . \tag{13}$$

Table 1, which is taken from Reference 1, gives the values for c_k, $\hat{c}_k$, α_k, and $\beta_{k\ell}$. Equation (11) is a Runge-Kutta formula that agrees with the Taylor series expansion to order seven, and equation (12) is a Runge-Kutta formula that agrees with the Taylor series expansion to order eight. Thus, the difference between the two formulas that Fehlberg denotes by TE is a good approximation to the eighth-order term in the Taylor series expansion. This difference is given by the following expression.

$$TE = \frac{41}{840} \left(f_0 + f_{10} - f_{11} - f_{12}\right) \Delta t \quad . \tag{14}$$

Since the expression for TE as given by equation (14) is a good approximation to the eighth-order term in the Taylor series expansion for $x(t_o + \Delta t)$, it can be used to determine Δt. Let $|TE|$ denote the largest component of the vector TE as given in equation (14). Then

$$|TE| \approx a_8 \Delta t^8 \quad , \tag{15}$$

where a_8 is the coefficient of the eighth-order term in the Taylor series expansion for the variable which yields the largest component of TE. Therefore,

$$a_8 \approx \frac{|TE|}{\Delta t^8} \quad . \tag{16}$$

TABLE 1. RK 7(8)

$k \backslash \ell$	α_k	$\beta_{k\ell}$ 0	1	2	3	4	5	6	7	8	9	10	11	c_k	$\hat{c}_k$
0	0	0												$\frac{41}{840}$	0
1	$\frac{2}{27}$	$\frac{2}{27}$												0	0
2	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{12}$											0	0
3	$\frac{1}{6}$	$\frac{1}{24}$	0	$\frac{1}{8}$										0	0
4	$\frac{5}{12}$	$\frac{5}{12}$	0	$-\frac{25}{16}$	$\frac{25}{16}$									0	0
5	$\frac{1}{2}$	$\frac{1}{20}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$								$\frac{34}{105}$	$\frac{34}{105}$
6	$\frac{5}{6}$	$-\frac{25}{108}$	0	0	$\frac{125}{108}$	$-\frac{65}{27}$	$\frac{125}{54}$							$\frac{9}{35}$	$\frac{9}{35}$
7	$\frac{1}{6}$	$\frac{31}{300}$	0	0	0	$\frac{61}{225}$	$-\frac{2}{9}$	$\frac{13}{900}$						$\frac{9}{35}$	$\frac{9}{35}$
8	$\frac{2}{3}$	2	0	0	$-\frac{53}{6}$	$\frac{704}{45}$	$-\frac{107}{9}$	$\frac{67}{90}$	3					$\frac{9}{280}$	$\frac{9}{280}$
9	$\frac{1}{3}$	$-\frac{91}{108}$	0	0	$\frac{23}{108}$	$-\frac{976}{135}$	$\frac{311}{54}$	$-\frac{19}{60}$	$\frac{17}{6}$	$-\frac{1}{12}$				$\frac{9}{280}$	$\frac{9}{280}$
10	1	$\frac{2383}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{301}{82}$	$\frac{2133}{4100}$	$\frac{45}{82}$	$\frac{45}{164}$	$\frac{18}{41}$			$\frac{41}{840}$	0
11	0	$\frac{3}{205}$	0	0	0	0	$-\frac{6}{41}$	$-\frac{3}{205}$	$-\frac{3}{41}$	$\frac{3}{41}$	$\frac{6}{41}$	0			$\frac{41}{840}$
12	1	$-\frac{1777}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{289}{82}$	$\frac{2193}{4100}$	$\frac{51}{82}$	$\frac{33}{164}$	$\frac{12}{41}$	0	1		$\frac{41}{840}$

Thus, for a given value of Δt, equation (14) can be used to compute TE. Then equation (16) can be used to compute a_8. Let ϵ denote the largest acceptable component of TE. Then, if $|TE| \leq \epsilon$, the step taken with Δt is an acceptable step and a Δt for the next step can be computed. If $|TE| > \epsilon$, then the step taken with Δt is not acceptable, and the step must be taken over again with a new Δt that can be computed. Let Δt_n denote the new Δt that is to be computed and Δt_o denote the old Δt. To derive an expression for Δt_n, it is required that

$$\epsilon = a_8 \, \Delta t_n^8 \quad . \tag{17}$$

That is, the new Δt should produce a maximum component of the truncation error of magnitude ϵ. Thus

$$\Delta t_n = \left(\frac{\epsilon}{a_8}\right)^{1/8} \quad . \tag{18}$$

Now the value of a_8 is obtained from equation (16) with the value of $|TE|$ produced by Δt_o. Thus,

$$\Delta t_n = \left[\frac{\epsilon}{\left(\frac{|TE|}{\Delta t_o^8}\right)}\right]^{1/8} = \Delta t_o \left(\frac{\epsilon}{|TE|}\right)^{1/8} \tag{19}$$

Equation (19) gives the computed value of Δt_n that is used for the next integration step if $|TE| \leq \epsilon$ and is used for a repeat step if $|TE| > \epsilon$. In actual practice, only 80 percent of the Δt_n as computed by equation (19) is used for either the next step or a repeat step. Also, in computing $|TE|$ each component is usually divided by the corresponding component from the initial value of x so that a valid comparison can be made in search of the maximum component. Several example computer listings are shown in Appendix A so that the different implementations of the normalization of the truncation error vector can be illustrated.

Note that in the example computer program listings, equation (12) is used to compute $x(t_o + \Delta t)$. This allows more accuracy in the final result to be obtained than if equation (11) were used because equation (12) is an eighth-order formula. In fact, for most problems, ϵ will be a bound for the entire solution time interval because of the conservative stepsize control procedure used. Other less conservative procedures could be implemented if computer speed is of more importance than accuracy of the solutions obtained, but the results in the next sections indicate that this conservative stepsize control gives very satisfactory computer execution times on the example problems. The formula chosen for testing on the example problem is the 7-8 formula which used 13 function evaluations (denoted by Fehlberg's 7-8-13 formula). Fehlberg states in Reference 2 that this formula is probably the best for difficult problems, but for easier problems where values of $x(t)$ are needed at more frequent intervals, a lower order formula might be more efficient. If a lower order formula is used (for example, a 2-3 formula), then equation (19) can still be used to compute Δt, but the power (1/8) is replaced by (1/3).

A COMPARISON OF FEHLBERG'S 7-8-13 FORMULA WITH THE NUMERICAL INTEGRATION TECHNIQUES OF REFERENCES 3 AND 4

From Reference 3, two example problems are selected. These examples are denoted by B1 and F1. The definition of problem B1 is given as follows:

Test Problem B1

$$\dot{x}_1 = x_1^2 x_2 \qquad x_1(t_o) = 1 \qquad t_o = 0$$

$$\dot{x}_2 = -1/x_1 \qquad x_2(t_o) = 1 \qquad t_f = 4$$

The definition of problem F1 is given as follows:

Test Problem F1

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = x_1 + 2\,x_4 - (1-\mu)\frac{(x_1+\mu)}{\left[(x_1+\mu)^2 + x_2^2\right]^{3/2}} - (\mu)\frac{(x_1 - 1 + \mu)}{\left[(x_1 - 1 + \mu)^2 + x_2^2\right]^{3/2}}$$

$$\dot{x}_4 = x_2 - 2\,x_3 - (1-\mu)\frac{x_2}{\left[(x_1+\mu)^2 + x_2^2\right]^{3/2}} - (\mu)\frac{x_2}{\left[(x_1 - 1 + \mu)^2 + x_2^2\right]^{3/2}}$$

$x_1(t_o) = 0.994$ $\quad\quad \mu = 0.012277471$

$x_2(t_o) = 0.$ $\quad\quad t_o = 0$

$x_3(t_o) = 0.$ $\quad\quad t_f = 11.124340337$

$x_4(t_o) = -2.0317326296$

Figure 1 shows the results obtained for test problem B1 using Fehlberg's 7-8-13 formula (denoted by RK713). Figure 2 shows the results obtained for test problem F1 using RK713. In Figure 1, the error shown is calculated as $\epsilon = | e^4 - x(1) |$. Also, note that the percentage error in x(2) is equal to the percentage error in x(1) because of the stepsize control for RK713 that was used for this problem. This can be seen by examining the computer program listing shown in Appendix A and the associated output for this problem.

Figure 5 of Reference 3 is comparable to Figure 1 of this report. That is, Figure 5 of Reference 3 shows the same type of information for the integration techniques tested in Reference 3, that is shown by Figure 1 of this report for RK713. From Figure 5 of Reference 3, it can be seen that a predictor-corrector technique (denoted by HPCG) uses the fewest function evaluations

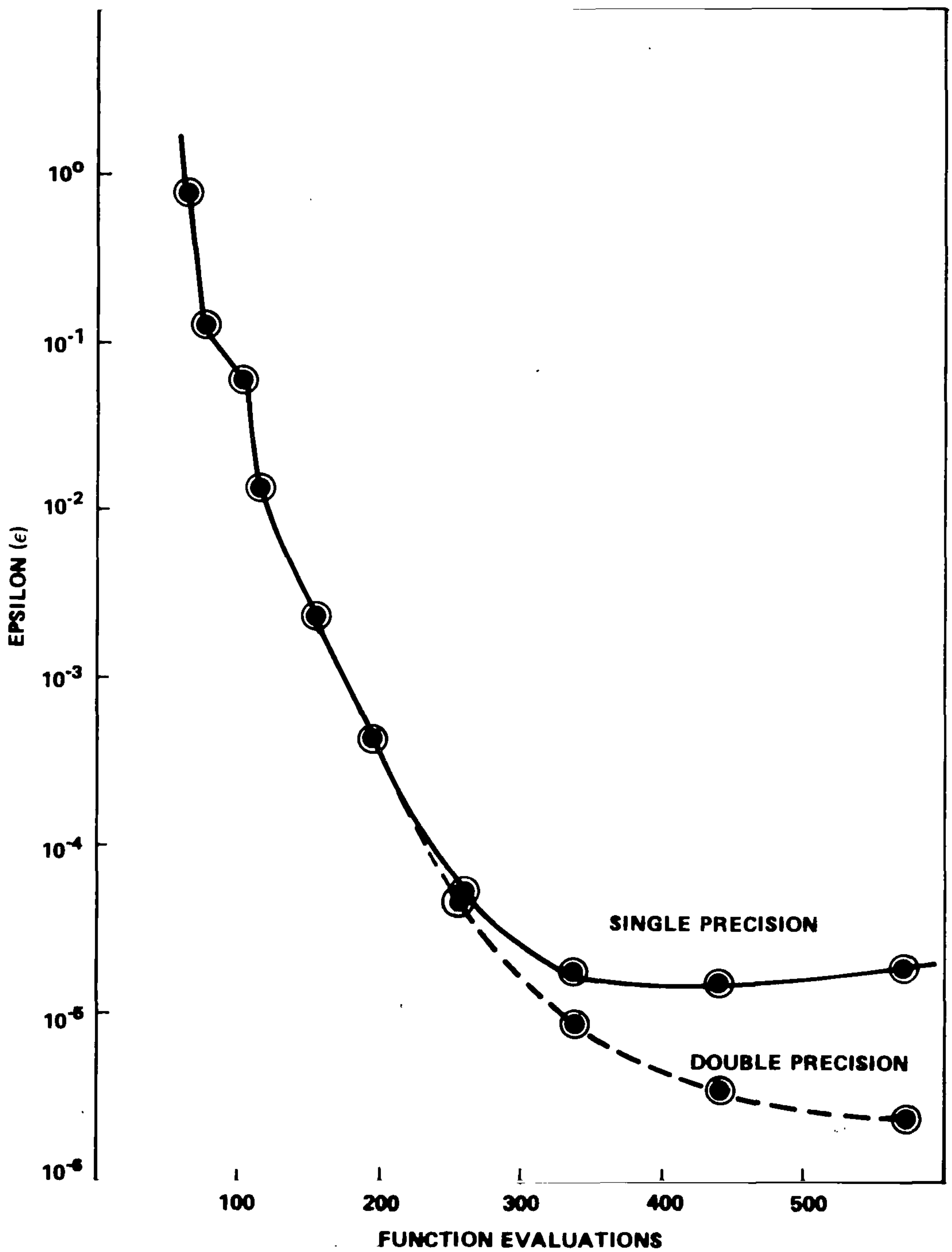


Figure 1. Test problem B1.

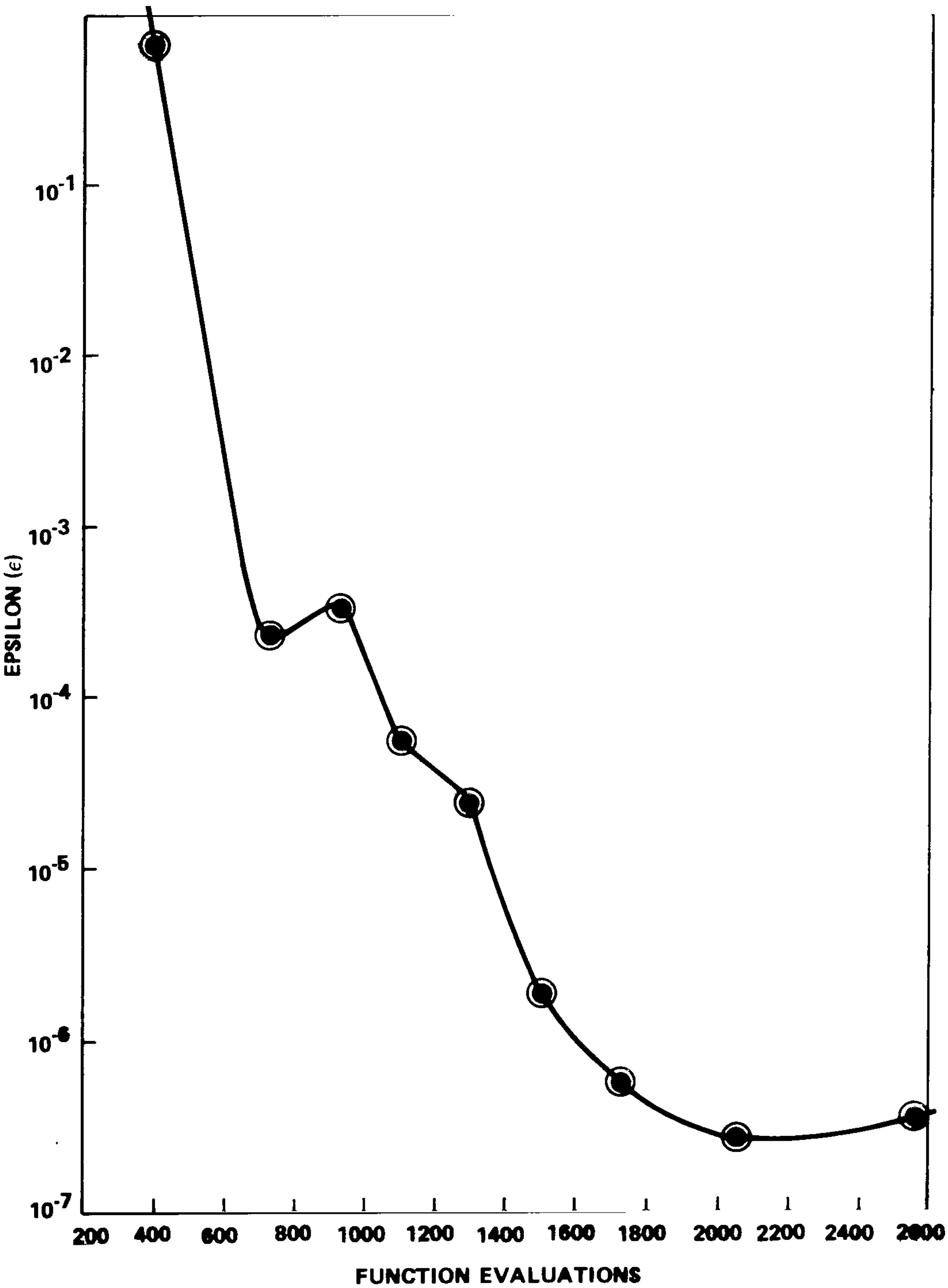


Figure 2. Test problem F1.

(about 100) for an ϵ of 10^{-1}. As seen in Figure 1, RK713 needs about 85 function evaluations for this accuracy. For an accuracy of $\epsilon = 10^{-2}$, RK713 needs only 120 function evaluations, whereas the only two integration techniques tested in Reference 3 that were able to achieve this accuracy (rational extrapolation and extrapolated Runge-Kutta) needed about 300 function evaluations. As seen in Figure 1, RK713 is able to achieve an ϵ of 10^{-4} with about 240 function evaluations. None of the integration techniques tested in Reference 3 was able to achieve this accuracy. The fact that the curve levels off at around 10^{-5} accuracy is due to the precision limits of the computer used and not the difficulty of the problem or any inadequacy in the RK713 technique. The computer runs in this section were made on an XDS-930 computer unless otherwise indicated. This computer is able to carry only 11 or 12 significant figures for each computation. The dotted line in Figure 1 shows the results from the UNIVAC 1108 computer using double precision, which allows 16 to 18 significant figures to be considered in the computations.

When Figure 2 of this report is compared with Figure 6 of Reference 3, the superiority of RK713 is again demonstrated. The error ϵ shown in Figure 2 of this report and Figure 6 of Reference 3 is computed by the formula

$$\epsilon = \left\{ [x_1(t_f) - x_1(t_o)]^2 + [x_2(t_f)]^2 \right\}^{1/2} .$$

RK713 is able to attain an $\epsilon = 10^{-6}$ with 1600 function evaluations. Most of the integration techniques tested in Reference 1 could not achieve an accuracy of even $\epsilon = 10^{-5}$. The only two routines to come close were again the extrapolated Runge-Kutta routine and the rational extrapolation routine, and both required about 2500 function evaluations to get close to the accuracy $\epsilon = 10^{-5}$. Again a sample computer program listing and its output are included in Appendix A for this problem.

From Reference 4, two example problems were also selected for comparison purposes and denoted by the symbols B12 and E22. The definition of problem B12 is given as follows:

Test Problem B12 (this is also problem E of Reference 3)

$$\dot{x}_1 = 2(x_1 - x_1 x_2) \qquad x_1(t_o) = 1 \qquad t_o = 0$$

$$\dot{x}_2 = -x_2(1 - x_1) \qquad x_2(t_o) = 3 \qquad t_f = 20$$

The definition of problem E22 is given as follows:

<u>Test Problem E-22</u>

$$\dot{x}_1 = x_2 \qquad x_1(t_o) = 2 \qquad t_o = 0$$

$$\dot{x}_2 = (1 - x_1^2)\, x_2 - x_1 \qquad x_2(t_o) = 0 \qquad t_f = 20$$

Figures 3 and 4 are plots of the errors achieved by RK713 versus the number of function evaluations needed for problems B12 and E22, respectively. To attempt to compare with the results shown in Reference 4, the error ϵ shown in Figures 3 and 4 is computed as follows:

$$\epsilon = \frac{\max.\ [y_1(t_f) - x_1(t_f),\ y_2(t_f) - x_2(t_f)]}{t_f - t_o}$$

where $[y_1(t_f),\ y_2(t_f)]$ are assumed to be the actual solution of the problems and are obtained by a very accurate integration of the two example sets of differential equations on the UNIVAC 1108 using double precision arithmetic.

From Figure 3 it can be seen that RK713 on problem B12 needs 600 function evaluations to achieve an accuracy of $\epsilon = 10^{-3}$, 915 function evaluations to achieve an accuracy of $\epsilon = 10^{-6}$, and 1615 function evaluations to achieve an accuracy of $\epsilon = 10^{-9}$. From Reference 4, the best method tested (with respect to the actual machine time needed to solve problem B12) is the Bulirsch-Stoer method. The Bulirsch-Stoer method needed 992 function evaluations for $\epsilon = 10^{-3}$, 1873 function evaluations for $\epsilon = 10^{-6}$, and 3128 function evaluations for $\epsilon = 10^{-9}$. Since the computer program DETEST mentioned in Reference 4 was not available, the other statistics given in Reference 4 were not compared. It is hoped that the authors of Reference 4 will be interested enough in RK713 to try it on their DETEST program.

From Figure 4, it can be seen that RK713 on problem E22 needs 665 function evaluations to achieve an accuracy of $\epsilon = 10^{-3}$, 915 function evaluations to achieve an accuracy of $\epsilon = 10^{-6}$, and 1625 function evaluations to achieve an accuracy of $\epsilon = 10^{-9}$. Again, the best method tested in Reference 4 (with respect to the actual machine time needed to solve problem E22) was the Bulirsch-Stoer method. On this problem, the Bulirsch-Stoer method needed 1109 function evaluations for $\epsilon = 10^{-3}$, 2009 function evaluations for $\epsilon = 10^{-6}$,

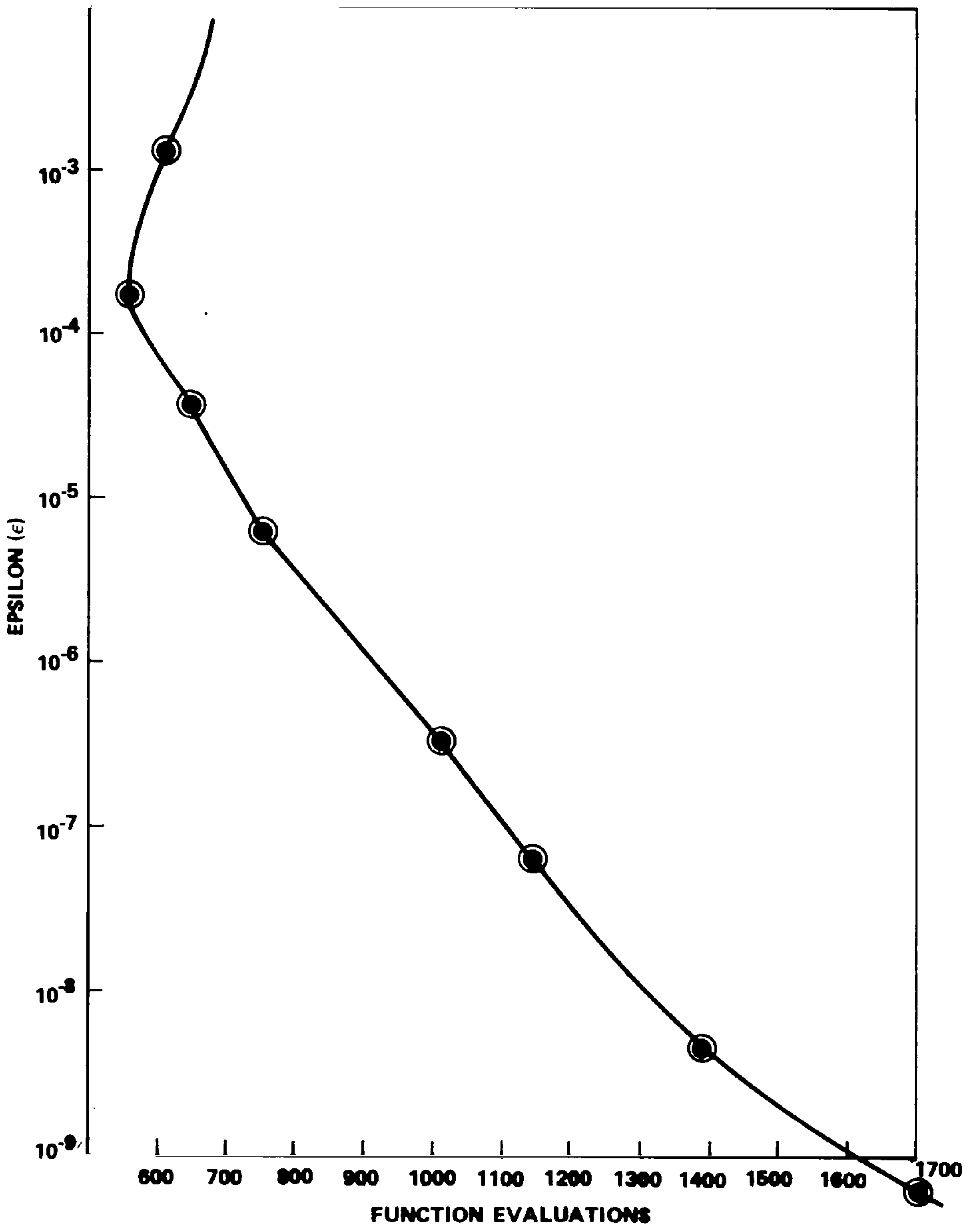


Figure 3. Test problem B12.

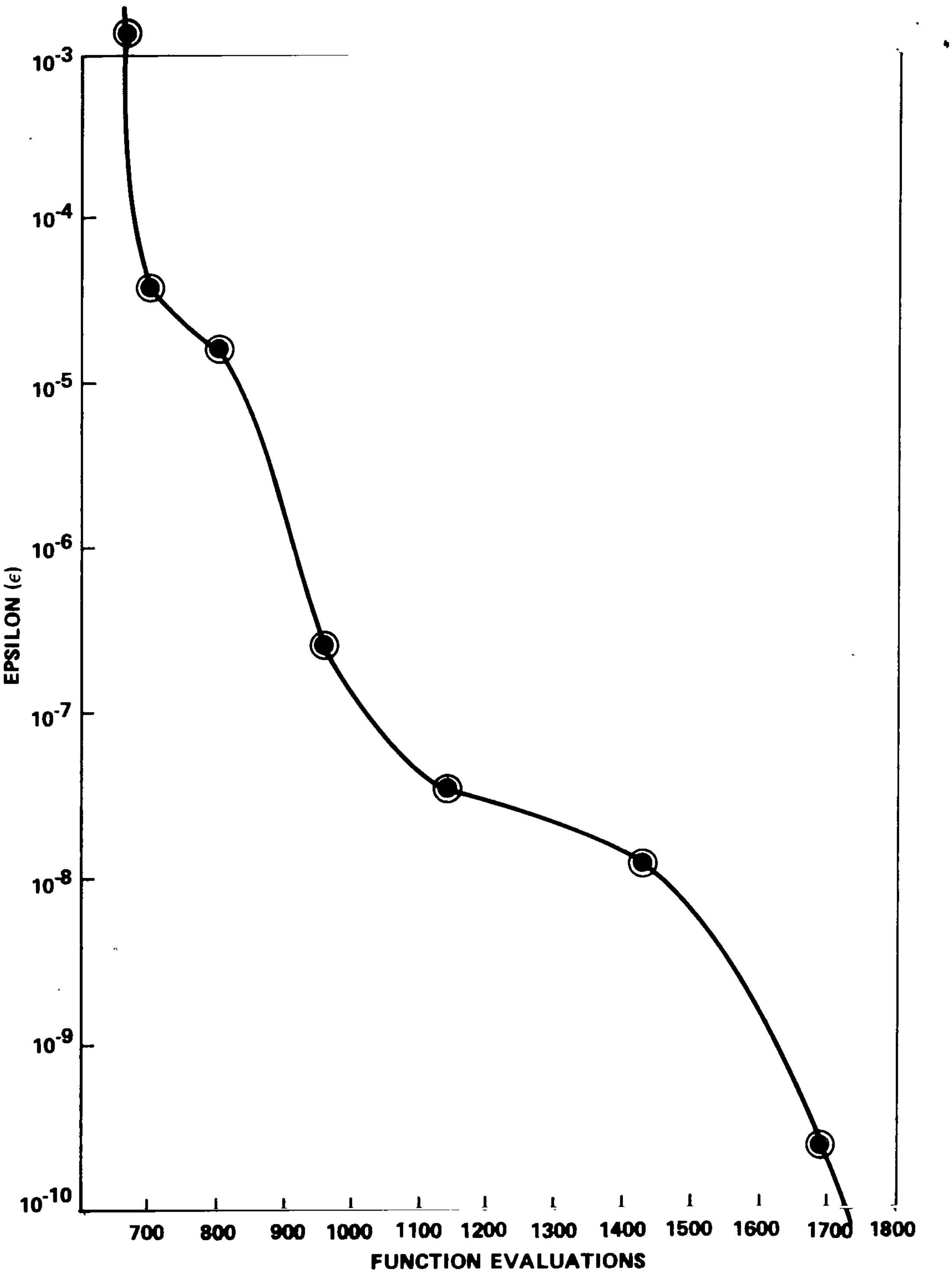


Figure 4. Test problem E22.

and 3588 function evaluations for $\epsilon = 10^{-9}$. Note again that a sample computer program listing and its output for both of the examples from Reference 4 are also contained in Appendix A. For each of the example problems of this section, the last output from each example computer program listing shown in Appendix A is a detailed print so that the actual behavior of each function can be seen. This detailed print was integrated as accurately as possible on the XDS 930 computer and also serves to illustrate the detailed print feature of the computer program.

All of the results of this section indicate that RK713 is superior to any of the other methods tested in References 3 and 4. Since all the errors in this section were absolute errors from the actual solutions of each of the differential equations, a more meaningful comparison is attempted in the next section. That is, RK713 and a rational extrapolation algorithm DIFSYF are compared on the basis of the accuracy each integration algorithm thinks it is achieving. Since actual accurate solutions of most differential equations attempted are not available, this is really the only comparison that can be made for these problems. Another reason for the inclusion of the comparison of the rational extrapolation algorithm with RK713 is that References 3 and 4 both concluded that the rational extrapolation algorithm as developed by Bulirsch-Stoer in Reference 5 was probably the best all-around algorithm they tested.

MORE DETAILED COMPARISONS OF FEHLBERG'S 7-8-13 FORMULA WITH AN EXTRAPOLATION ALGORITHM BASED ON RATIONAL FUNCTIONS

In this section, a comparison of the two integration techniques (Fehlberg's 7-8-13 formula RK713 and the rational function extrapolation algorithm DIFSYF) is made using systems of differential equations describing the following example problems: planar elliptic vacuum orbits with eccentricities ranging from 0.001 to 0.991, a three-dimensional almost circular vacuum orbit, and two reentry trajectories with aerodynamic forces and the adjoint differential equations included.[1]

After preparation of this report was complete the author was informed that a newer version of DIFSYF had been developed by R. Bulirsch. This new

1. The comparisons shown were performed by Dr. E. D. Dickmanns while he was working at Marshall Space Flight Center. They are included in this report with his permission. Dr. Dickmanns is now working in West Germany at the DFVLR - Institut fur Dynamik der Flugsysteme.

version is claimed to be 30 to 40 percent faster than the old version for some accuracy requirements. Appendix B contains a listing of both the old and new versions of DIFSYF. No comparisons were made with the new version of DIFSYF because the results shown in this section and the previous section indicate that a 30- to 40-percent improvement in DIFSYF would not make it superior to RK713 for most accuracy requirements.

The following sections describe in detail the example problems and the comparisons that were made between RK713 and the old version of DIFSYF.

Systems of Differential Equations Used

Planar Elliptic Orbits. For the range angle from pericenter θ (eccentric anomaly) as independent variable the equations of motion are

$$\frac{du}{d\theta} = -\dot{r}$$

u = horizontal velocity

$$\frac{d\dot{r}}{d\theta} = -\frac{GM}{ur}$$

$\dot{r}$ = radial velocity

r = radial distance from center of gravity

$$\frac{dr}{d\theta} = \frac{\dot{r}r}{u}$$

GM = $3.986032 \cdot 10^{14}$ (metric units)

Earth mass times gravity constant.

Three-Dimensional Vacuum Trajectories. In this case time has been chosen as independent variable. The equations then are

$$\frac{du}{dt} = -\frac{u\dot{r}}{r}$$

$$\frac{d\chi}{dt} = \frac{u}{r} \sin \chi \tan \lambda$$

χ = azimuth from north (positive east)

λ = geocentric latitude

$$\frac{d\dot{r}}{dt} = \frac{u^2}{r} - \frac{GM}{r^2}$$

Λ = longitude

ω_e = rotational speed of the earth

$$\frac{dr}{dt} = \dot{r}$$

$$\frac{d\lambda}{dt} = \frac{u}{r} \cos \chi$$

$$\frac{d\Lambda}{dt} = \frac{u}{r} \frac{\sin \chi}{\cos \lambda} - \omega_e \quad .$$

Reentry Equations for State and Adjoint Variables. These equations were written in a flight path-oriented axis system for a spherical, nonrotating earth and an exponential density-altitude relationship.

$$\dot{v} = -g \sin \gamma - (C_{D_o} + k\, C_L^{\,n}) \frac{S\rho_o}{2m} v^2 e^{-\beta h}$$

$$\dot{\chi} = -\frac{v}{r} \cos \gamma \, \cos \chi \, \tan \Lambda + \frac{C_L S\rho_o}{2m} v e^{-\beta h} \frac{\sin \mu}{\cos \gamma}$$

$$\dot{\theta} = \frac{v}{r} \frac{\cos \gamma}{\cos \Lambda} \cos \chi$$

$$\dot{\Lambda} = \frac{v}{r} \cos \gamma \, \sin \chi$$

$$\dot{h} = v \sin \gamma \qquad\qquad r = R_E + h$$

The differential equations for the adjoint variables are rather lengthy and are not reproduced here, but are shown in Reference 6. The controls C_L and μ are determined from

$$\sin \mu = \frac{\lambda_\chi}{\omega \cos \gamma} \; ; \quad \omega = [(\lambda_\chi / \cos \gamma)^2 + \lambda_\gamma^{\,2}]^{1/2}$$

$$C_L = [-\omega / (v \lambda_v \, k\, n)]^{1/(n-1)} \quad .$$

These are highly nonlinear equations sensitive to small changes in some of the adjoint variables.

Symbols used:

C_{D_o}	Zero-lift drag coefficient (set to 0.04)
C_L	Lift coefficient (control)
$g=\frac{GM}{r^2}$	Local gravitational acceleration
h	Altitude above sea level
k	Factor for lift-dependent drag (0.8 and 1.0)
m	Vehicle mass
n	Power for lift-dependent drag (1.75)
R_E	Earth radius (6371 km)
S	Reference area ($\frac{S}{m}$ = 100 [kg/m^2])
v	Inertial velocity
β	Inverse density scale height (assumed $\beta = 1.45 \cdot 10^{-4}$ [m^{-1}])
γ	Flight path angle
θ	Range angle in initial flight direction
λ_i	Adjoint variable to i
μ	Bank angle (control)
ρ_o	Density for h= 0 (assumed 1.54 [kg/m^2])
χ	Azimuth relative to initial flight direction

Cases Compared

To investigate the stepsize adjustment properties of the integration routines, system 1 has been integrated for eccentricities e from 0.001 (almost circular) to 0.991 (highly elliptic) with the following initial conditions:

$$u(o) = [\frac{GM}{r}(1+e)]^{1/2} \; [m/s]$$

$$\dot{r}(o) = 0$$

$$r(o) = 6\,571\,000 \; [m] \quad .$$

Two complete orbits have been integrated for each set of initial conditions and accuracy requirement.

For the system of differential equations describing three-dimensional motion in vacuum about a rotating earth, an almost circular orbit inclined by 30 deg to the equatorial plane has been chosen:

$$u_o = 7850 \; [m/s]$$

$$\chi_o = 90 \text{ deg}$$

$$\dot{r}_o = 150 \; [m/s]$$

$$r_o = 6\,470\,000 \; [m]$$

$$\lambda_o = 30 \text{ deg}$$

$$\Lambda_o = 0$$

Four complete revolutions have been integrated with the period P given by

$$P = \frac{\Pi \cdot GM}{\sqrt{2 \cdot E^3}} \quad ; \quad E = (u_o^2 + \dot{r}_o^2)\,\tfrac{1}{2} - \frac{GM}{r_o} \quad .$$

With the reentry equations a single three-dimensional skip has been chosen as test case. The initial conditions were

$v_o = 7950$	$\lambda v_o = -0.543197$
$\chi_o = 0$	$\lambda \chi_o = -2054.06$
$\gamma_o = 3$ deg	$\lambda \gamma_o = -46.1287$
$\Lambda_o = 0$	$\lambda \Lambda_o = 538.79$
$h_o = 95\,000$	$\lambda h_o = -0.000703044$

Discussion of Results

For weakly elliptic planar orbits ($e \lesssim 0.25$) DIFSYF requires about 30 to 40 percent more evaluations of the differential equations than RK713. The average stepsize is about five times as large for DIFSYF as for RK713 (Figs. 5 and 6).

For accuracies (tolerance per step) required of 10^{-7}, one complete revolution is integrated in two steps by DIFSYF. As the eccentricity increases the optimal stepsizes for integrating the apse-portions and the flanks of the orbit become more and more different. For these cases, the smaller stepsize of RK713 allows a faster stepsize adjustment, resulting in increasing superiority of RK713 (Figs. 7 and 8). For highly eccentric orbits, RK713 requires less than half the evaluations of the right-hand side of the differential equations. The ratio of maximum to minimum stepsize is about equal for both methods (~ 15 for $e = 0.991$). RK713 seems to be more tolerant against too low accuracy requirements. To obtain approximately periodic orbits, tolerances required for RK713 were $\sim 10^{-3}$ while DIFSYF required 10^{-5} (Fig. 9).

For the almost circular three-dimensional vacuum orbit DIFSYF takes two integration steps per revolution for 10^{-5} tolerance per step compared to 10^{-7} for planar orbits. Here the sinus-like time traces for χ and λ require stepsize adjustment, which causes RK713 to become more superior as the accuracy requirement is increased. The average stepsize for DIFSYF is four times that of RK713. For high accuracies DIFSYF requires twice the number of function evaluations as RK713 (Fig. 10).

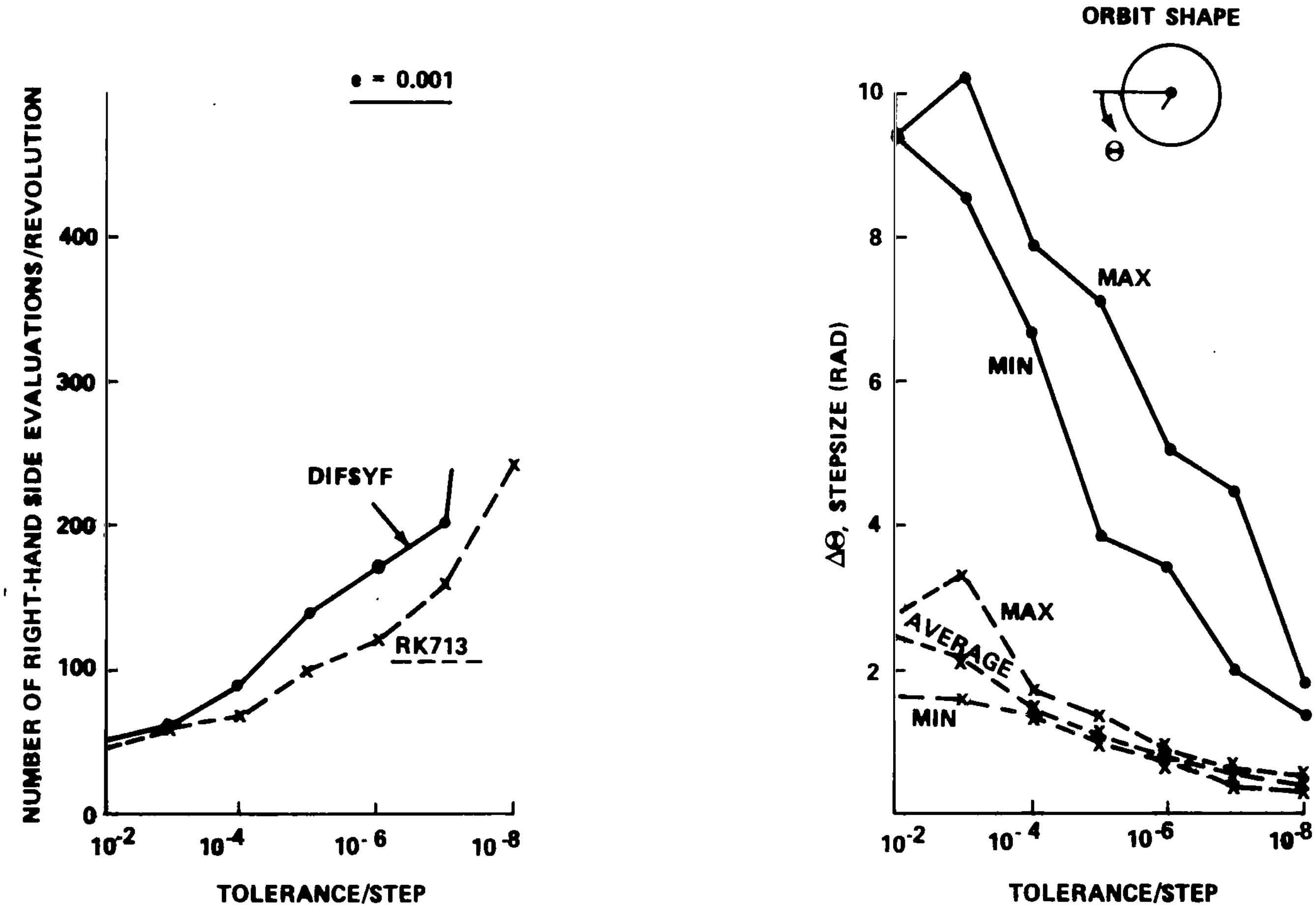


Figure 5. Performance comparison RK713-DIFSYF, planar orbits — e = 0.001, almost circular.

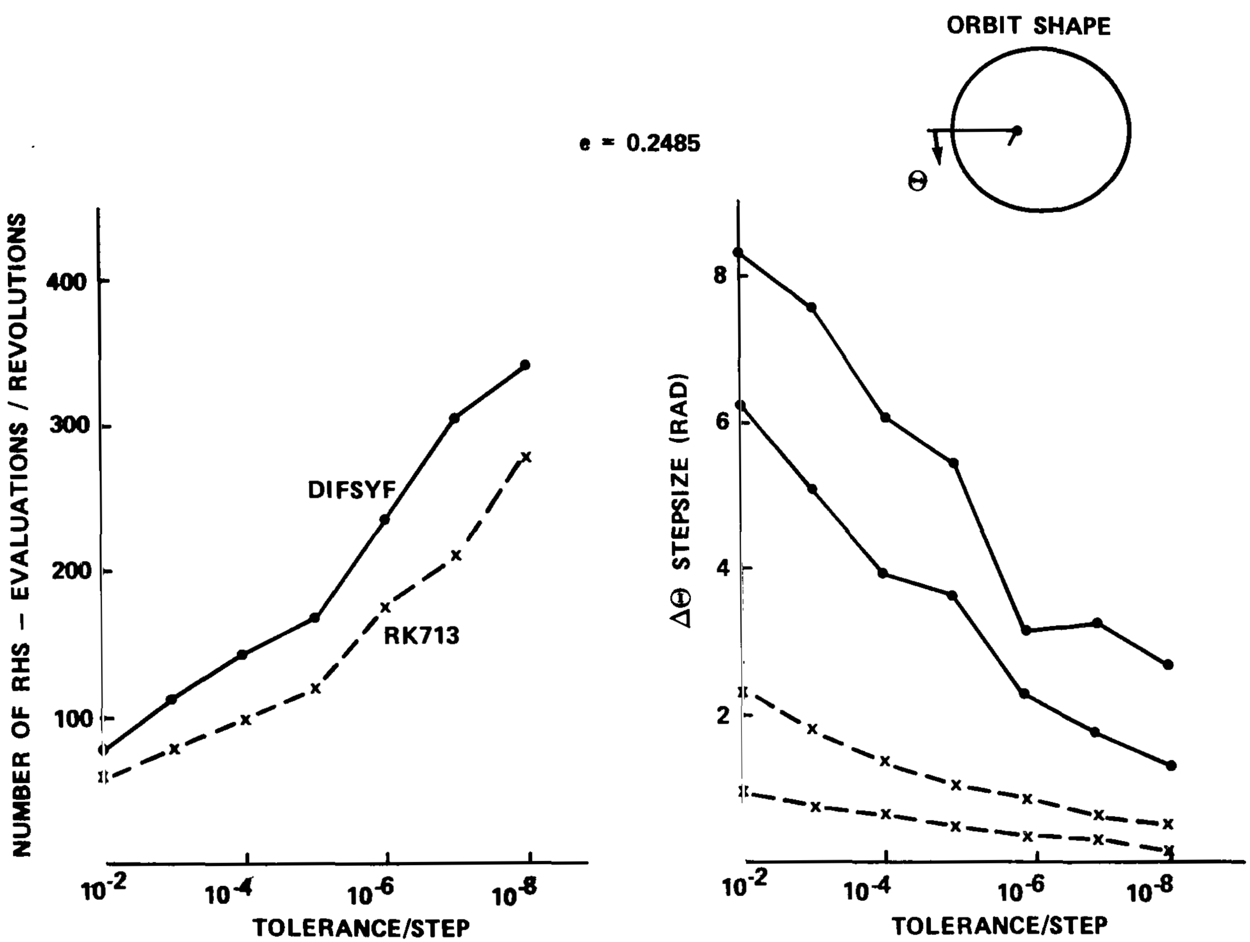


Figure 6. Performance comparison RK713-DIFSYF, planar orbits — e= 0.2485.

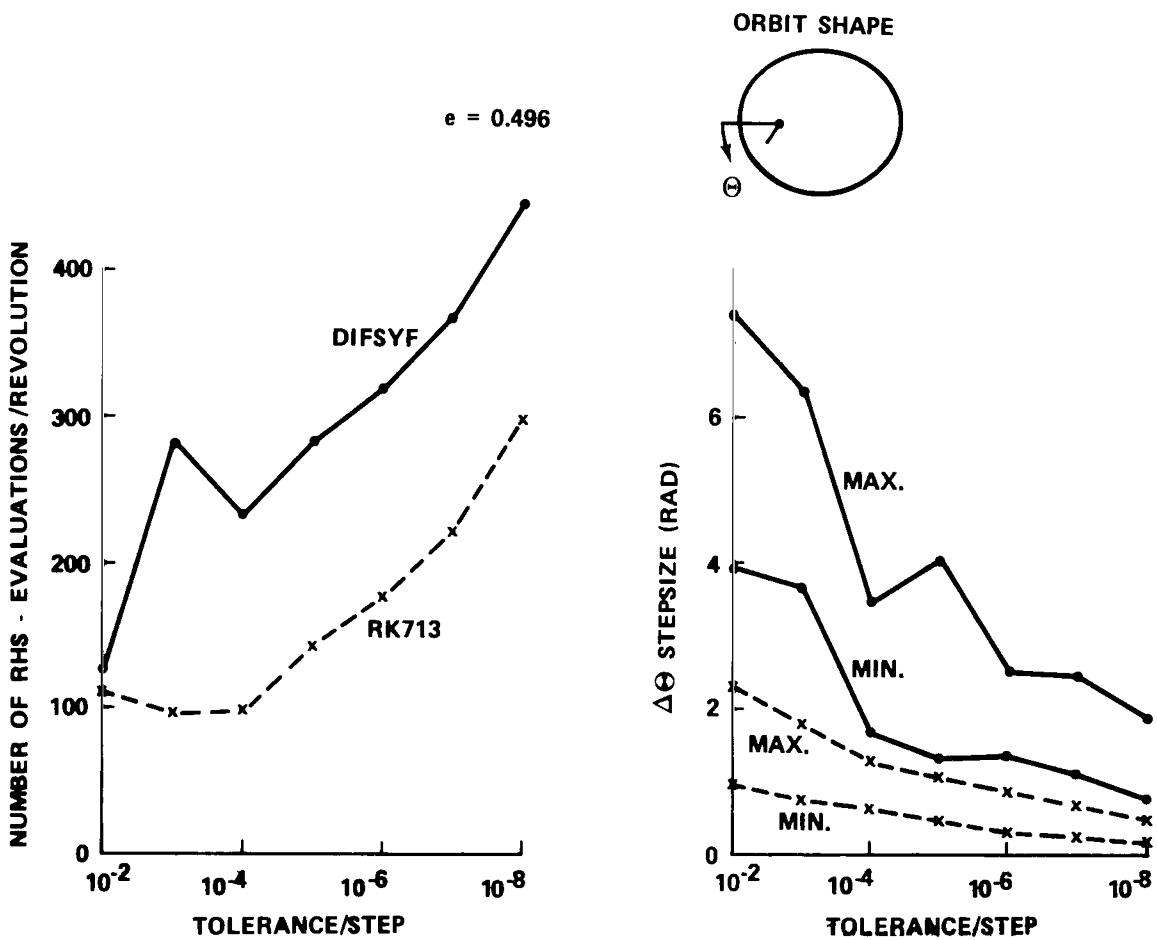


Figure 7. Performance comparison RK713-DIFSYF, planar orbits — e= 0.496.

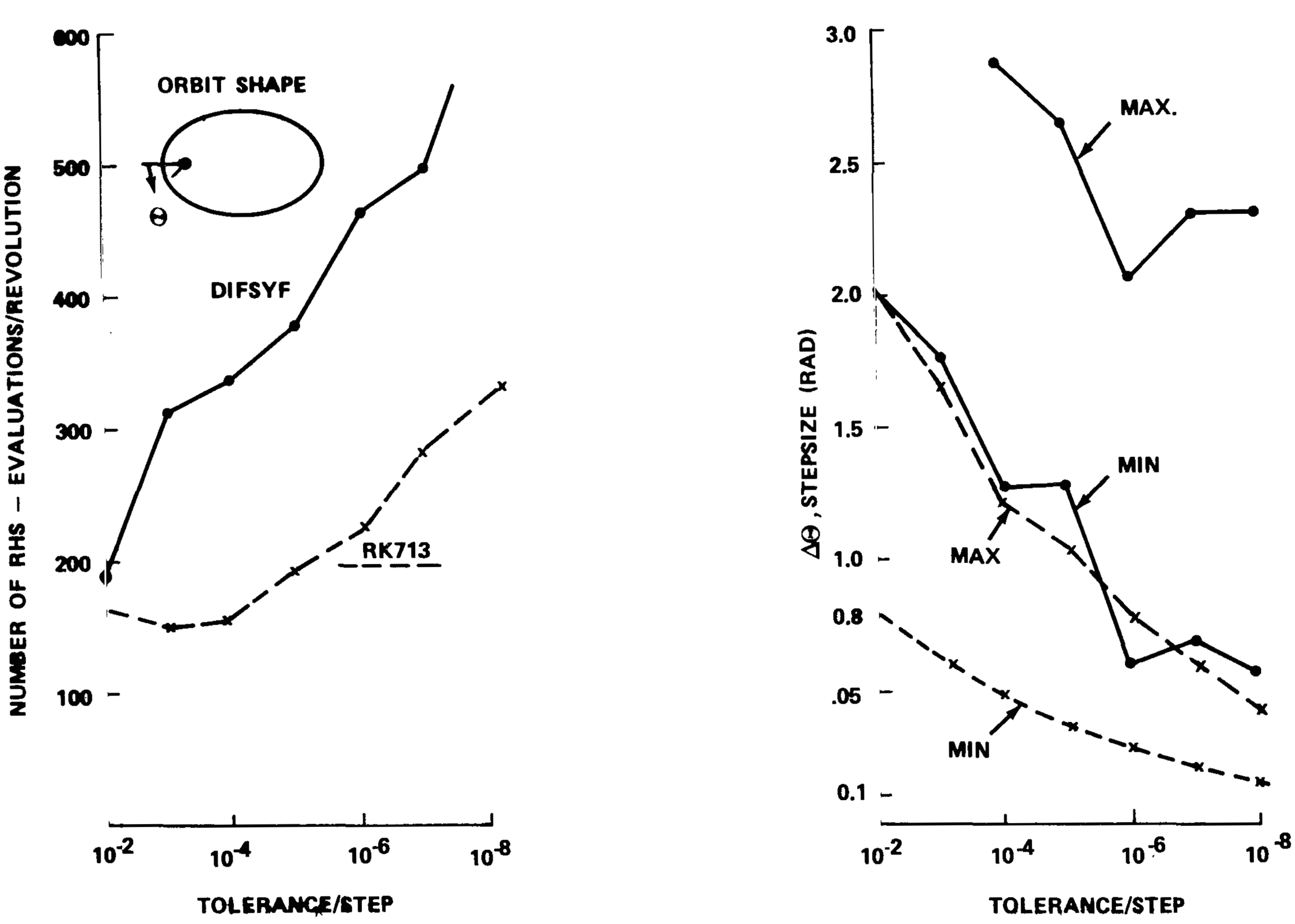


Figure 8. Performance comparison RK713-DIFSYF, planar orbits — e= 0.7435.

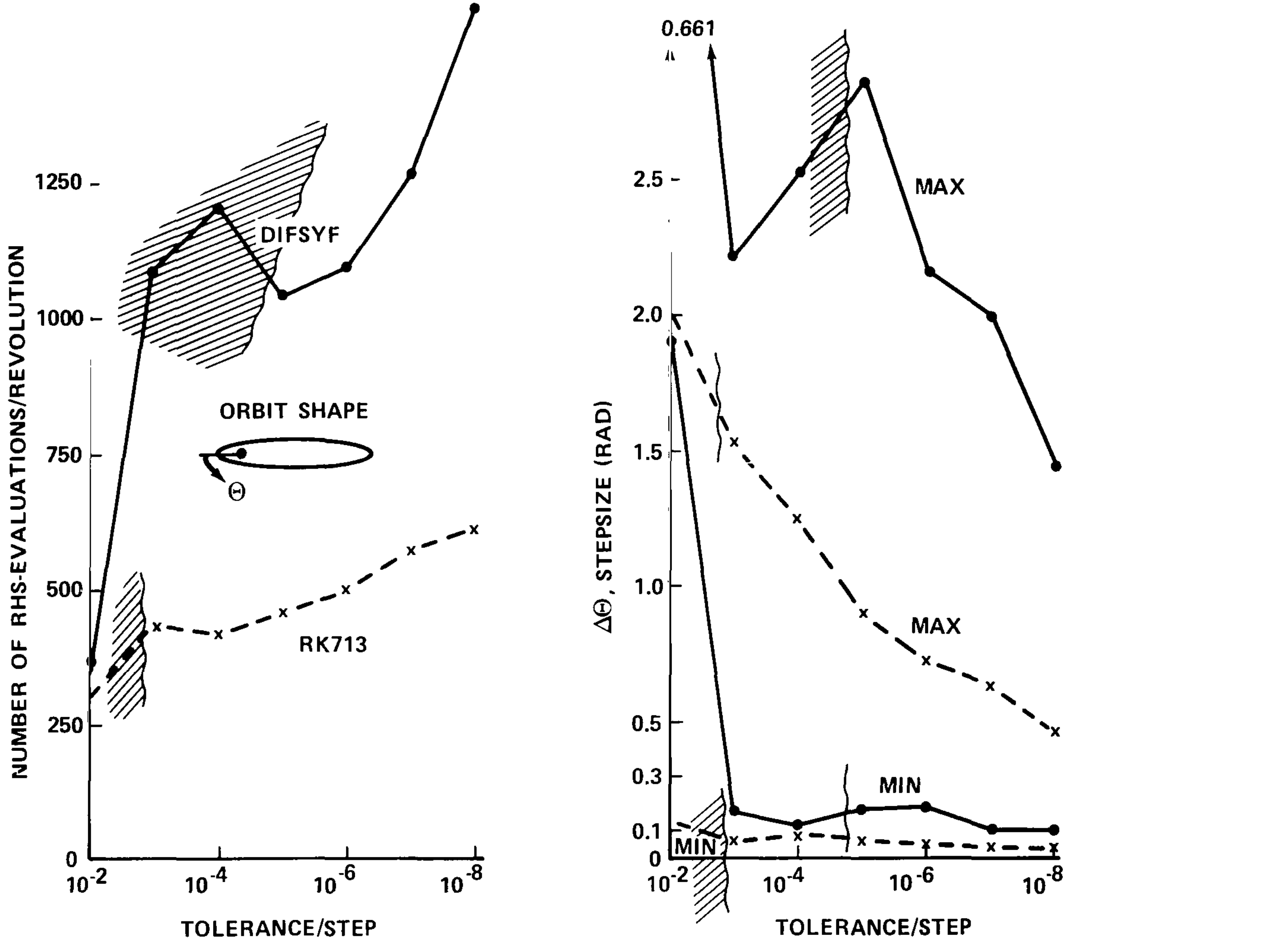


Figure 9. Performance comparison RK713-DIFSYF, planar orbits — e= 0.991; highly elliptic.

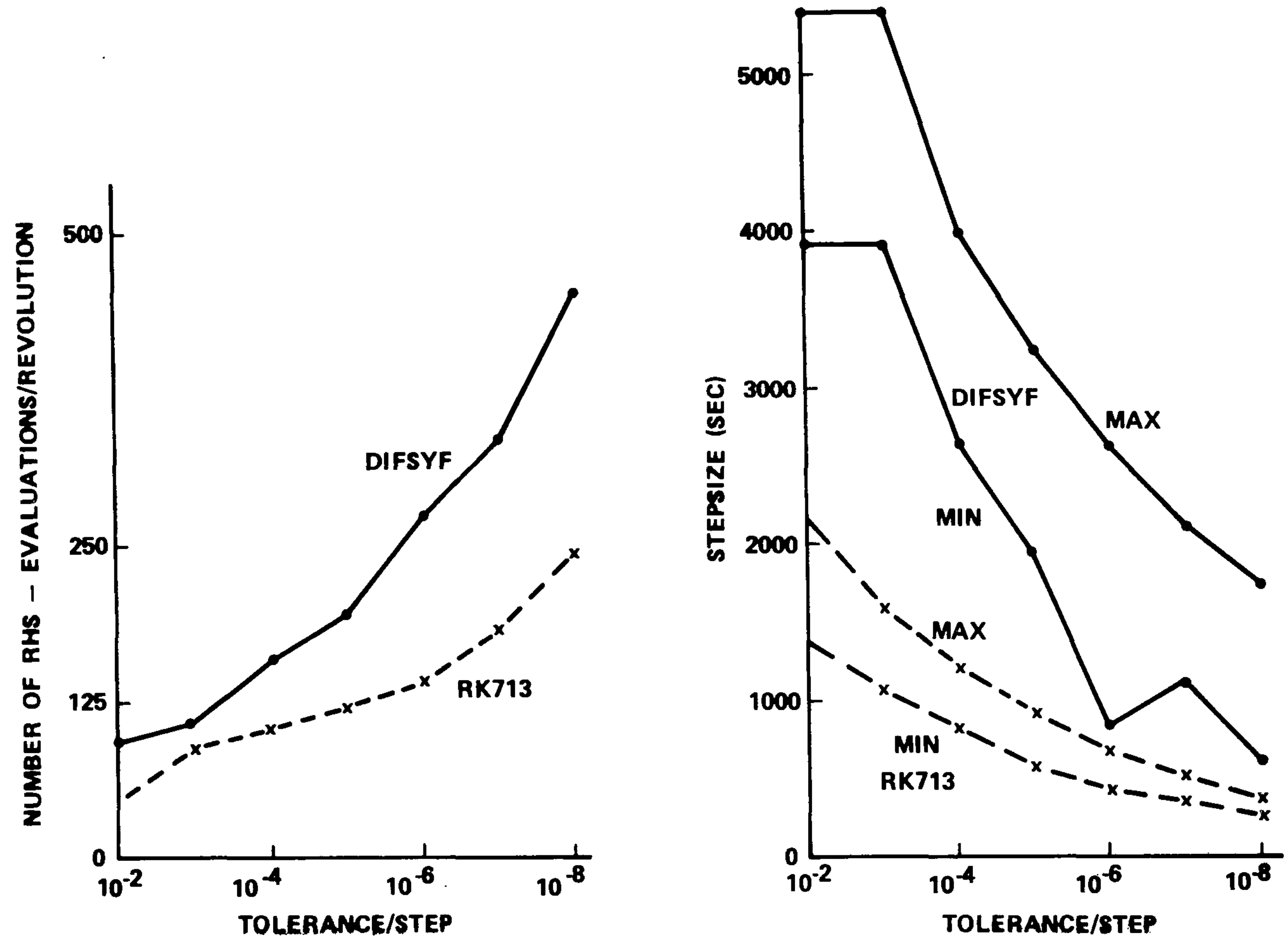


Figure 10. Three-dimensional vacuum orbit (almost circular).

The picture is very similar for the integration of the reentry equations. These test runs were made on a UNIVAC 1108 in double precision for accuracies per step of 10^{-14} to 10^{-2}. For a three-dimensional skip (45-deg plane change) the results are shown in Figure 11. In the usually taken accuracy range 10^{-5} to 10^{-10}, RK713 is slightly superior to DIFSYF. For high accuracy requirements, this superiority becomes more pronounced. It again seems to be a question of stepsize adjustment.

To test the properties of the integration routines with a very demanding trajectory, k has been changed from 1.0 to 0.8, everything else remaining constant.

The resulting trajectory is a highly oscillating spiral dive with the flight path angle varying from -2 to -55 deg and normal accelerations of 150 g. The results are shown in Figure 12. Again RK713 is superior and takes about one-fourth the stepsize of DIFSYF. It also shows a more benevolent behavior against low accuracy requirements for this numerically unstable system of differential equations. DIFSYF breaks down for accuracies lower than 10^{-9} while RK713 yields good results up to 10^{-4}. The breakdown[2] seems to be related to the minimum stepsize used (lower part of Figure 12). Also, the conservative stepsize control for RK713 discussed in the section of this report entitled Fehlberg's Runge-Kutta Formulas With Stepsize Control contributes to the more stable behavior of RK713 for low accuracy specifications.

Check computations with simple precision (eight digits) for tolerances greater than 10^{-7} were made with RK713. Supposedly because of roundoff errors the number of function evaluations was slightly higher, while the minimum stepsize was slightly smaller. Breakdown occurred at the same tolerance requirement of 10^{-3}.

2. Breakdown means that when evaluating the right-hand side of the differential equations numbers larger than allowed for in the standard library functions occurred.

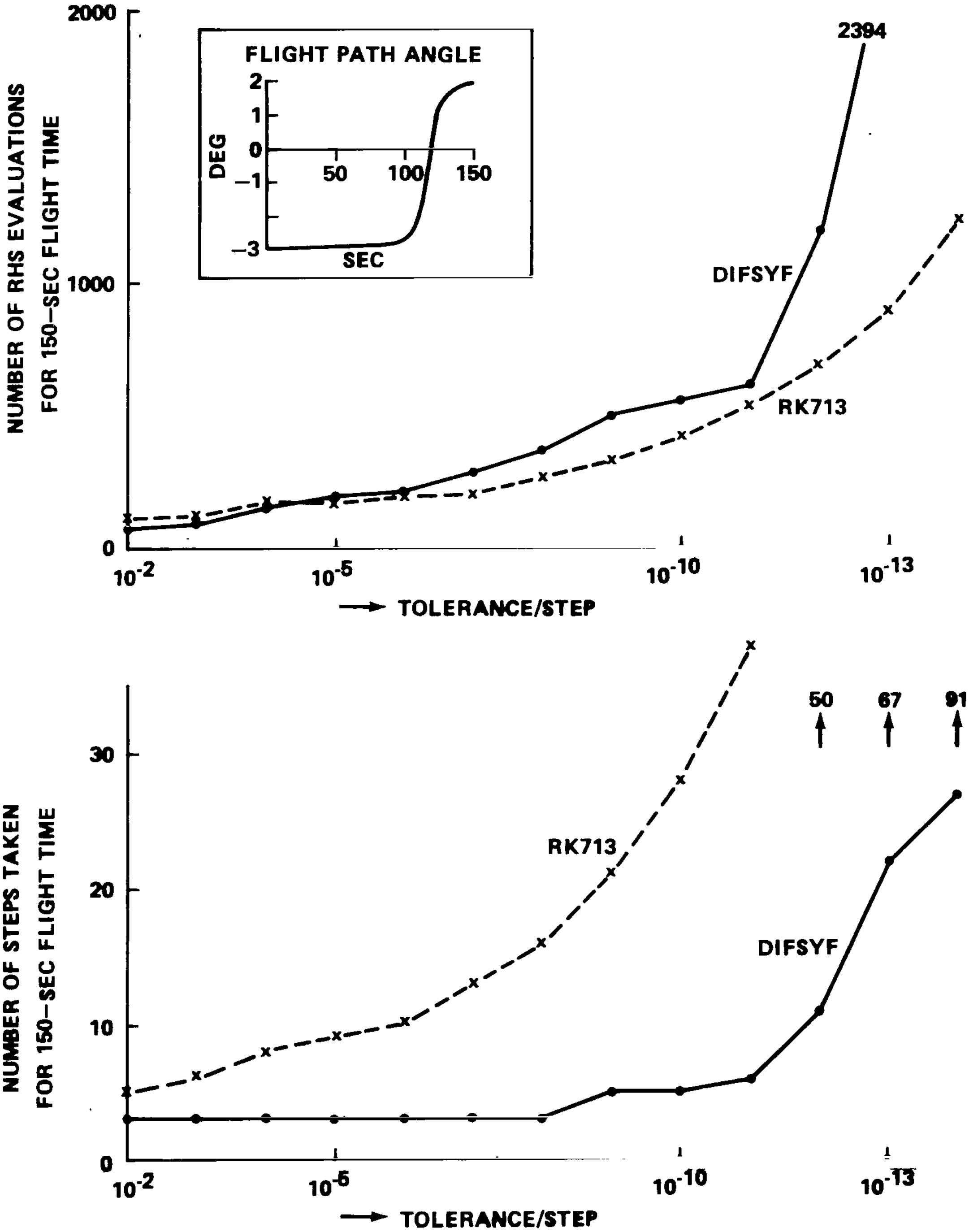
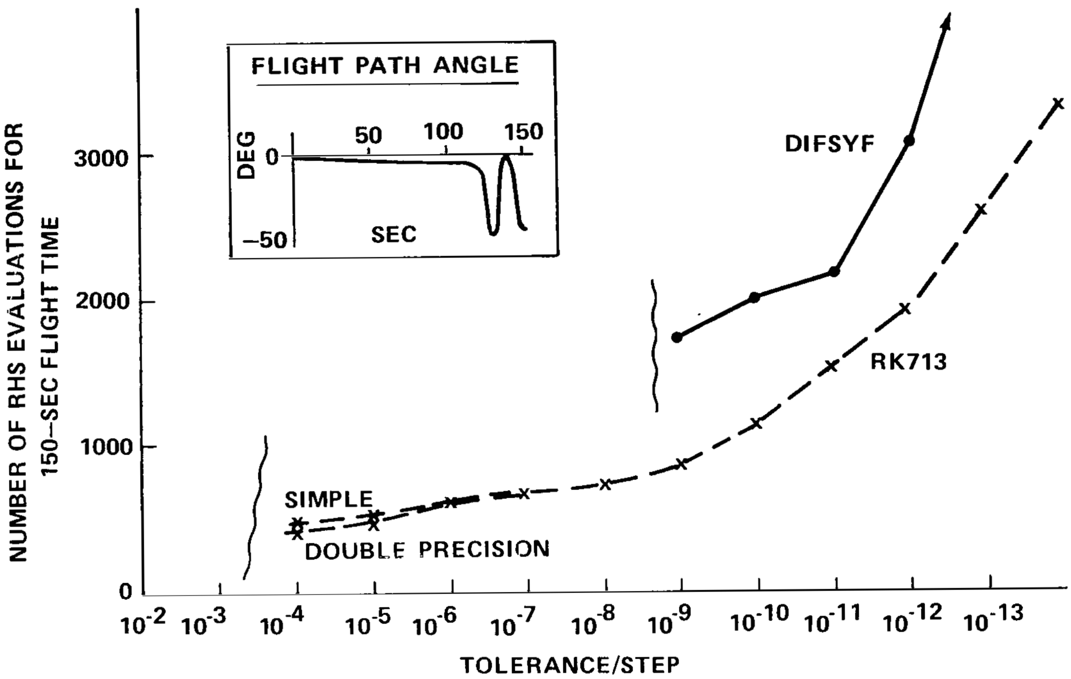


Figure 11. Comparison RK713-DIFSYF, reentry equations — three-dimensional skip trajectory.



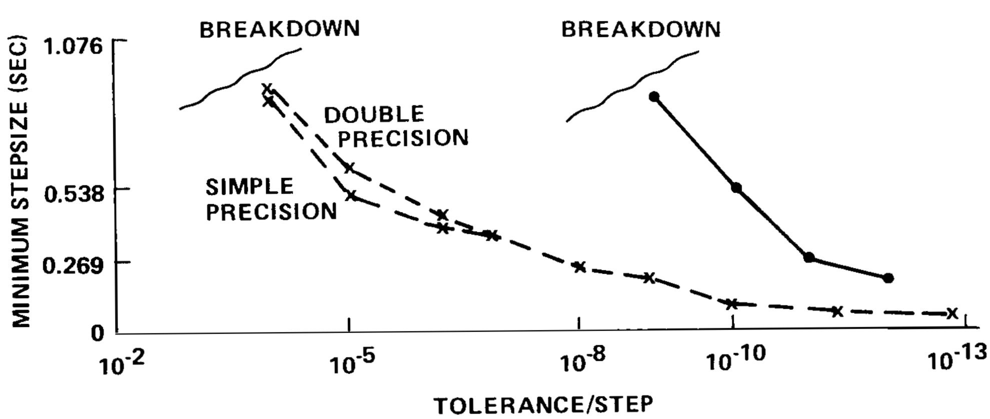


Figure 12. Comparison RK713-DIFSYF, reentry equations — oscillatory spiral dive [highly demanding trajectory (K= 0.8)].

CONCLUSIONS

The results of this report indicate that Fehlberg's 7-8-13 (RK713) formula with the stepsize control developed in this report is superior to all of the other techniques with which it was compared. That is, it was able to solve all the example problems as fast or faster than any of the other techniques for the entire range of integration accuracies tested. For higher accuracies the superiority of RK713 is particularly evident and in fact RK713 was able to achieve better accuracies on all of the problems than any of the other techniques with which it was compared. Since the rapid solution of systems of differential equations is extremely important to the author of this report, he would appreciate being informed by any of the readers of more efficient techniques for solving these problems.

APPENDIX A

SAMPLE COMPUTER PROGRAM LISTINGS AND SAMPLE OUTPUT USING THE FEHLBERG 7-8-13 FORMULA

The following list of symbols is given in order to facilitate the use of the programs:

DTGI	An estimate of the initial t to be used by the program. After this DTGI is used once, the program is able to compute its own Δt so that the value of DTGI used has practically no effect on the number of function evaluations needed for a solution of a particular problem.
DTPI	The delta print times for the full print option. That is, if FPTI < TF, then intermediate prints will occur at intervals of DTPI after the print at FPTI.
FPTI	The control for the full-print option of the program. If FPTI $\geq$ TF, then only the initial and final time are printed. If FPTI < TF, then FPTI is the time of the first intermediate print.
KT	An input control number that allows the program to print every KT integration steps if FPTI and DTPI are both larger than TF. If FPTI and DTPI are less than TF, then KT must be a large number.
TF	The final time.
TI	The initial time.

TABLE A-1. TEST PROBLEM B1

```
ΔASSIGN S=MT0,SI CR,BØ MT1,LØ LP.
ΔREWIND MT1.
ΔFØRTRAN BØ,LØ.
*             DIMENSIØN X[2]
*             DIMENSIØN ALPH[13],BETA[13,12],CH[13]
*             CØMMØN ALPH,BETA,CH
*          20 READ 11,    TI,DTGI,TØL
*             READ 11,X[1],X[2]
*             READ 11,TF,FPT1,DTP1
*             READ 2025,KT
*       2025  FØRMAT[I4]
*       C     CØNSTANTS FØR INTEGRATIØN SUBRØUTINE
*             DØ 60 I=1,13
*             DØ 50 J=1,12
*          50 BETA[I,J]=0.
*             ALPH[I]=0.
*          60 CH[I]=0.
*             CH[6]=34./105.
*             CH[7]=9./35.
*             CH[8]=CH[7]
*             CH[9]=9./280.
*             CH[10]=CH[9]
*             CH[12]=41./840.
*             CH[13]=CH[12]
*             ALPH[2]=2./27.
*             ALPH[3]=1./9.
*             ALPH[4]=1./6.
*             ALPH[5]=5./12.
*             ALPH[6]=.5
*             ALPH[7]=5./6.
*             ALPH[8]=1./6.
*             ALPH[9]=2./3.
*             ALPH[10]=1./3.
*             ALPH[11]=1.
*             ALPH[13]=1.
*             BETA[2,1]=2./27.
*             BETA[3,1]=1./36.
*             BETA[4,1]=1./24.
*             BETA[5,1]=5./12.
*             BETA[6,1]=.05
*             BETA[7,1]=-25./108.
*             BETA[8,1]=31./300.
*             BETA[9,1]=2.
*             BETA[10,1]=-91./108.
*             BETA[11,1]=2383./4100.
*             BETA[12,1]=3./205.
*             BETA[13,1]=-1777./4100.
*             BETA[3,2]=1./12.
*             BETA[4,3]=1./8.
*             BETA[5,3]=-25./16.
*             BETA[5,4]=-BETA[5,3]
*             BETA[6,4]=.25
*             BETA[7,4]=125./108.
```

```
      BETA[9,4]=-53./6.
      BETA[10,4]=23./108.
      BETA[11,4]=-341./164.
      BETA[13,4]=BETA[11,4]
      BETA[6,5]=.2
      BETA[7,5]=-65./27.
      BETA[8,5]=61./225.
      BETA[9,5]=704./45.
      BETA[10,5]=-976./135.
      BETA[11,5]=4496./1025.
      BETA[13,5]=BETA[11,5]
      BETA[7,6]=125./54.
      BETA[8,6]=-2./9.
      BETA[9,6]=-107./9.
      BETA[10,6]=311./54.
      BETA[11,6]=-301./82.
      BETA[12,6]=-6./41.
      BETA[13,6]=-289./82.
      BETA[8,7]=13./900.
      BETA[9,7]=67./90.
      BETA[10,7]=-19./60.
      BETA[11,7]=2133./4100.
      BETA[12,7]=-3./205.
      BETA[13,7]=2193./4100.
      BETA[9,8]=3.
      BETA[10,8]=17./6.
      BETA[11,8]=45./82.
      BETA[12,8]=-3./41.
      BETA[13,8]=51./82.
      BETA[10,9]=-1./12.
      BETA[11,9]=45./164.
      BETA[12,9]=3./41.
      BETA[13,9]=33./164.
      BETA[11,10]=18./41.
      BETA[12,10]=6./41.
      BETA[13,10]=12./41.
      BETA[13,12]=1.
      NS=0
      NR=0
      NST=0
      NRT=0
      PRINT 800
  800 FORMAT [1H1,/,51X,14HINITIAL VALUES]
      CALL PRINT [TI,X,   FPT1 ,DTP1,DTGI,TOL]
      T=TI
      STEP=FPT1
      DTG=FPT1-T
      IF[ABS[DTG]-ABS[DTGI]]6,6,7
    7 DTG=DTGI
    6 NSF=0
      NRF=0
      IF[ABS[TF-TI]-ABS[FPT1-TI]]112,121,121
  112 STEP=TF
      DTG=DTGI
```

```
 121     CALL INTEGR[T,STEP,DTG,TØL,X, 2,KT ,NSF,NRF]
         IF[TF-STEP]161,151,161
 161     T=STEP
         STEP=T+DTP1
         IF[ABS[DTG]-ABS[DTP1]]8,8,9
 9       DTG=DTP1
 8       IF[ABS[TF-T]-ABS[DTP1]]132,143,143
 132     STEP=TF
         IF[ABS[DTG]-ABS[TF-T]]143,143,2024
 2024    DTG=TF-T
 143     PRINT 190,NSF,NRF
   190   FØRMAT[1H ,////,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HGØØD STEPS T
        1AKEN ,  ,1X,I4,1X,15HBAD STEPS TAKEN]
         CALL PRINT [T ,X,    FPT1,DTP1,DTGI,TØL]
         NS=NS+NSF
         NR=NR+NRF
         GØ TØ 121
 151     NS=NS+NSF
         NR=NR+NRF
         NST=NST+NS
         NRT=NRT+NR
         PRINT 840,NST,NRT
 840     FØRMAT[1H0,50X,14HFINAL VALUES  ,I4,16HGØØD STEPS TAKEN,I4,
        115HBAD STEPS TAKEN]
         CALL PRINT[TF,X,    FPT1 ,DTP1,DTGI,TØL]
         GØ TØ 20
  11     FØRMAT[3E18.11]
         END

CØMMØN ALLØCATIØN

  77746 ALPH        77256 BETA        77224 CH

PRØGRAM ALLØCATIØN

  00011 X           00015 KT          00016 I           00017 J
  00020 NS          00021 NR          00022 NST         00023 NRT
  00024 NSF         00025 NRF         00026 TI          00030 DTGI
  00032 TØL         00034 TF          00036 FPT1        00040 DTP1
  00042 T           00044 STEP        00046 DTG

SUBPRØGRAMS REQUIRED

  PRINT       ABS         INTEGR
THE END
```

TABLE A-1. (Continued)

```
          SUBROUTINE INTEGR[TI,T,DTS,TOL    ,X,N,KT,NS,NR]
          DIMENSION            X[ 2 ],XE[2 ]
          NT=C
          NS=C
          NR=C
          DTG=DTS
          TO=TI
    20    XE[1]=X[1]
          XE[2]=X[2]
          STEP=TO+DTG
          CALL RK713[TO,STEP,DTG,TOL   ,X, 2 ,     MS,MR,XE,2 ]
          TO=STEP
          NS=MS+NS
          NR=NR+MR
          DTS=DTG
          NT=NT+MS+MR
          IF[STEP-T]240,230,240
   240    IF[ABS[DTG]-ABS[T- STEP]]210,210,260
   260    DTG=T-STEP
   210    IF[NT-KT]20,220,220
   220    T=STEP
   230    RETURN
          END
```

```
PROGRAM ALLOCATION

  DUMMY X          00030 XE         00034 INTEGR     00035 NT
  DUMMY NS         DUMMY NR         00036 MS         00037 MR
  DUMMY KT         00040 DTG        DUMMY DTS        00042 TO
  DUMMY TI         00044 STEP       DUMMY TOL        DUMMY T

SUBPROGRAMS REQUIRED

  RK713      ABS
THE END
```

TABLE A-1. (Continued)

```
      SUBROUTINE RK713 [TI,TF,DT ,TOL    ,X,N,    NS,NR,XE,M]
C     SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
C     TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
C     NS IS THE NUMBER OF SUCCESFULL STEPS TAKEN
C     NR IS THE NUMBER OF REJECTED STEPS TAKEN
C     N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
C     KT IS MAX NUMBER OF ITERATIONS
C     ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
C     SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
C     F[0] IN FEHLBERGS REPORT IS IN F[1,J]
C     F[I] IS IN F[I+1,J]
C     FEHLBERGS REPORT REFERENCED IS NASA TR R-287
C     PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
C     DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
C     NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
      DIMENSION F[13, 2 ],XDUM[ 2 ],TE[ 2 ]          ,ALPH[13],
     1BETA[13,12],X[ 2 ],CH[13],XE[2 ]
      COMMON ALPH,BETA,CH
      T=TI
      NS=0
      NR=0
   20 CALL DEQ [X,T,TE]
      DO 30 I=1,N
   30 F[1,I]=TE[I]
      DO 70 K=2,13
      DO 40 I=1,N
   40 XDUM[I]=X[I]
      NN=K-1
      DO 50 I=1,N
      DO 50 J=1,NN
   50 XDUM[I]=XDUM[I]+DT*BETA[K,J]*F[J,I]
      TDUM=T+ALPH[K]*DT
      CALL DEQ [XDUM,TDUM,TE]
      DO 60 I=1,N
   60 F[K,I]=TE[I]
   70 CONTINUE
      ER=0.
C  M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
C  THE ERROR CONTROL LOOP
C  XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
C  THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
      DO 120 I=1,M
140   TE[I]=DT*[F[1,I]+F[11,I]-F[12,I]-F[13,I]]*41./840./XE[I]
      IF [ABS[TE[I]]-ER] 120,120,130
  130 ER=ABS[TE[I]]
  120 CONTINUE
      DT1=DT
      AK=.8
      IF[ER]141,142,141
 142  DT=10.*DT1
      GO TO 150
  141 DT=[SQRT[SQRT[SQRT[TOL/ER]]]]
      DT=AK*DT*DT1
```

TABLE A-1. (Continued)

```
          IF (ER    -TØL) 150,150,180
      150 TF=T+DT1
          DØ 90 I=1,N
          DØ 90 L=1,13
       90 X(I)=X(I)+DT1*CH(L)*F(L,I)
          NS=NS+1
          GØ TØ 230
  180     NR=NR+1
          TF=T
      230 RETURN
          END
```

```
CØMMØN ALLØCATIØN

  77746 ALPH        77256 BETA        77224 CH

PRØGRAM ALLØCATIØN

  00037 F           00123 XDUM        00127 TE          DUMMY X
  DUMMY XE          DUMMY NS          DUMMY NR          00133 I
  DUMMY N           00134 K           00135 NN          00136 J
  DUMMY M           00137 L           00140 RK713       00142 T
  DUMMY TI          DUMMY DT          00144 TDUM        00146 ER
  00150 DT1         00152 AK          DUMMY TØL         DUMMY TF

SUBPRØGRAMS REQUIRED

  DEQ         ABS         SQRT
THE END
```

```
          SUBRØUTINE PRINT (T,X,    FPT,DTP,DTG,TØL)
          DIMENSION X(2)
          PRINT 1,T,FPT,DTP,DTG,TØL
         1,X(1),X(2)
    1     FØRMAT(1H0,5HT     =E18.11,2X,5HFPT  =E18.11,2X,5HDTP  =E18.11,2X,
         15HDTG  =E18.11,2X,5HTØL  =E18.11
         2,/,6H X(1)=E18.11,2X,5HX(2)=E18.11)
          RETURN
          END
```

```
PRØGRAM ALLØCATIØN

  DUMMY X           00014 PRINT       DUMMY T           DUMMY FPT
  DUMMY DTP         DUMMY DTG         DUMMY TØL
THE END
```

TABLE A-1. (Continued)

```
*      SUBROUTINE DEQ[X,T,DX]
*      DIMENSION X[2],      DX[2]
*      DX[1]=X[1]**2*X[2]
*      DX[2]=-1./X[1]
*      RETURN
*      END

PROGRAM ALLOCATION

  DUMMY X           DUMMY DX          00006 DEQ
THE END
```

```
▲ASSIGN BI=MT1.
▲REWIND MT1.
▲FORTLOAD BIU.
```

TABLE A-1. (Continued)

```
NAME      ENTRY ØRIGIN  LAST SIZE/10 CØMMØN  BASE

•PRØGRAM  03507  03477  10636   2656  17063
```

```
                                                       INITIAL VALUES
 T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TØL =  .10000000000E 00
 X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                                       FINAL VALUES     4GØØD STEPS TAKEN   0BAD STEPS TAKEN
 T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TØL =  .10000000000E 00
 X(1)=  .65333167817E 02  X(2)=  .21196337833E-01
```

TABLE A-1. (Continued)

```
                                                       INITIAL VALUES

T     =   .00000000000E 00   FPT =   .50000000000E 10   DTP =   .50000000000E 00   DTG =   .10000000000E 00   TØL =   .10000000000E-01
X(1)=   .10000000000E 01   X(2)=   .10000000000E 01

                                                       FINAL VALUES      5GØØD STEPS TAKEN    0BAD STEPS TAKEN

T     =   .40000000000E 01   FPT =   .50000000000E 10   DTP =   .50000000000E 00   DTG =   .10000000000E 00   TØL =   .10000000000E-01
X(1)=   .55368223122E 02   X(2)=   .18555934117E-01
```

```
                                                       INITIAL VALUES

T     =   .00000000000E 00   FPT =   .50000000000E 10   DTP =   .50000000000E 00   DTG =   .10000000000E 00   TØL =   .10000000000E-02
X(1)=   .10000000000E 01   X(2)=   .10000000000E 01

                                                       FINAL VALUES      6GØØD STEPS TAKEN    0BAD STEPS TAKEN

T     =   .40000000000E 01   FPT =   .50000000000E 10   DTP =   .50000000000E 00   DTG =   .10000000000E 00   TØL =   .10000000000E-02
X(1)=   .54473429571E 02   X(2)=   .18272071378E-01
```

TABLE A-1. (Continued)

```
                                          INITIAL VALUES

T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TØL =  .10000000000E-03
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                          FINAL VALUES      8GØØD STEPS TAKEN   0BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TØL =  .10000000000E-03
X(1)=  .54540130439E 02  X(2)=  .18295966039E-01
```

```
                                          INITIAL VALUES

T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TØL =  .10000000000E-04
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                          FINAL VALUES      9GØØD STEPS TAKEN   0BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TØL =  .10000000000E-04
X(1)=  .54585324473E 02  X(2)=  .18311318438E-01
```

TABLE A-1. (Continued)

```
                                           INITIAL VALUES

T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-05
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                           FINAL VALUES    12GOOD STEPS TAKEN   0BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-05
X(1)=  .54595981273E 02  X(2)=  .18314910017E-01
```

```
                                           INITIAL VALUES

T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-06
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                           FINAL VALUES    15GOOD STEPS TAKEN   0BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-06
X(1)=  .54597829417E 02  X(2)=  .18315531242E-01
```

TABLE A-1. (Continued)

```
                                                         INITIAL VALUES

T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-07
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                                         FINAL VALUES     20GOOD STEPS TAKEN   0BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-07
X(1)=  .54598099498E 02  X(2)=  .18315621928E-01
```

```
                                                         INITIAL VALUES

T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-08
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01

                                                         FINAL VALUES     26GOOD STEPS TAKEN   0BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-08
X(1)=  .54598133849E 02  X(2)=  .18315633457E-01
```

TABLE A-1. (Continued)

```
                                           INITIAL VALUES
T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-09
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01
                                           FINAL VALUES    34GOOD STEPS TAKEN   0BAD STEPS TAKEN
T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-09
X(1)=  .54598135326E 02  X(2)=  .18315633952E-01
```

```
                                           INITIAL VALUES
T    =  .00000000000E 00  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .10000000000E 01  X(2)=  .10000000000E 01
                                           FINAL VALUES    44GOOD STEPS TAKEN   0BAD STEPS TAKEN
T    =  .40000000000E 01  FPT =  .50000000000E 10  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .54598133004E 02  X(2)=  .18315633170E-01
```

TABLE A-1. (Continued)

```
                                                    INITIAL VALUES

T    =  .00000000000E 00   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .10000000000E 01   X(2)=  .10000000000E 01


                                                 INTERMEDIATE VALUES       6 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .50000000000E 00   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .16487212704E 01   X(2)=  .60653065960E 00


                                                 INTERMEDIATE VALUES       7 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .10000000000E 01   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .27182818265E 01   X(2)=  .36787944091E 00


                                                 INTERMEDIATE VALUES      10 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .15000000000E 01   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .44816890600E 01   X(2)=  .22313015963E 00


                                                 INTERMEDIATE VALUES       9 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .20000000000E 01   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .73890560505E 01   X(2)=  .13533528234E 00


                                                 INTERMEDIATE VALUES       8 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .25000000000E 01   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .12182493740E 02   X(2)=  .82084997121E-01


                                                 INTERMEDIATE VALUES       7 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .30000000000E 01   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .20085535926E 02   X(2)=  .49787065882E-01


                                                 INTERMEDIATE VALUES       6 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .35000000000E 01   FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .33115447475E 02   X(2)=  .30197379324E-01
```

TABLE A-1. (Concluded)

```
                                                FINAL VALUES    59GOOD STEPS TAKEN   0BAD STEPS TAKEN
T    =  .40000000000E 01  FPT =  .50000000000E 00  DTP =  .50000000000E 00  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .54598129917E 02  X(2)=  .18315632132E-01
```

TABLE A-2. TEST PROBLEM F1

```
ΔASSIGN S=MTO,SI CR,B0 MT1,L0 LP.
ΔREWIND MT1.
ΔFORTRAN B0,L0.
*                DIMENSION X[4]
*                DIMENSION ALPH[13],BETA[13,12],CH[13]
*                COMMON GMM
*                COMMON ALPH,BETA,CH
*           20   READ 11,    TI,DTGI
*                READ 11,[X[I],I=1,4]
*                READ 11,TF,FPT1,DTP1
*                READ 11,GMM,TOL
*                READ 2025,KT
*        2025    FORMAT[I4]
*        C       CONSTANTS FOR INTEGRATION SUBROUTINE
*                DO 60 I=1,13
*                DO 50 J=1,12
*           50   BETA[I,J]=0.
*                ALPH[I]=0.
*           60   CH[I]=0.
*                CH[6]=34./105.
*                CH[7]=9./35.
*                CH[8]=CH[7]
*                CH[9]=9./280.
*                CH[10]=CH[9]
*                CH[12]=41./840.
*                CH[13]=CH[12]
*                ALPH[2]=2./27.
*                ALPH[3]=1./9.
*                ALPH[4]=1./6.
*                ALPH[5]=5./12.
*                ALPH[6]=.5
*                ALPH[7]=5./6.
*                ALPH[8]=1./6.
*                ALPH[9]=2./3.
*                ALPH[10]=1./3.
*                ALPH[11]=1.
*                ALPH[13]=1.
*                BETA[2,1]=2./27.
*                BETA[3,1]=1./36.
*                BETA[4,1]=1./24.
*                BETA[5,1]=5./12.
*                BETA[6,1]=.05
*                BETA[7,1]=-25./108.
*                BETA[8,1]=31./300.
*                BETA[9,1]=2.
*                BETA[10,1]=-91./108.
*                BETA[11,1]=2383./4100.
*                BETA[12,1]=3./205.
*                BETA[13,1]=-1777./4100.
*                BETA[3,2]=1./12.
*                BETA[4,3]=1./8.
*                BETA[5,3]=-25./16.
*                BETA[5,4]=-BETA[5,3]
```

TABLE A-2. (Continued)

```
      BETA[6,4]=.25
      BETA[7,4]=125./108.
      BETA[9,4]=-53./6.
      BETA[10,4]=23./108.
      BETA[11,4]=-341./164.
      BETA[13,4]=BETA[11,4]
      BETA[6,5]=.2
      BETA[7,5]=-65./27.
      BETA[8,5]=61./225.
      BETA[9,5]=704./45.
      BETA[10,5]=-976./135.
      BETA[11,5]=4496./1025.
      BETA[13,5]=BETA[11,5]
      BETA[7,6]=125./54.
      BETA[8,6]=-2./9.
      BETA[9,6]=-107./9.
      BETA[10,6]=311./54.
      BETA[11,6]=-301./82.
      BETA[12,6]=-6./41.
      BETA[13,6]=-289./82.
      BETA[8,7]=13./900.
      BETA[9,7]=67./90.
      BETA[10,7]=-19./60.
      BETA[11,7]=2133./4100.
      BETA[12,7]=-3./205.
      BETA[13,7]=2193./4100.
      BETA[9,8]=3.
      BETA[10,8]=17./6.
      BETA[11,8]=45./82.
      BETA[12,8]=-3./41.
      BETA[13,8]=51./82.
      BETA[10,9]=-1./12.
      BETA[11,9]=45./164.
      BETA[12,9]=3./41.
      BETA[13,9]=33./164.
      BETA[11,10]=18./41.
      BETA[12,10]=6./41.
      BETA[13,10]=12./41.
      BETA[13,12]=1.
      NS=0
      NR=0
      NST=0
      NRT=0
      PRINT 800
  800 FORMAT [1H1,/,51X,14HINITIAL VALUES]
      CALL PRINT [TI,X,   FPT1 ,DTP1,DTGI,TOL]
      T=TI
      STEP=FPT1
      DTG=FPT1-T
      IF[ABS[DTG]-ABS[DTGI]]6,6,7
    7 DTG=DTGI
    6 NSF=0
      NRF=0
      IF[ABS[TF-TI]-ABS[FPT1-TI]]112,121,121
```

```
      112   STEP=TF
            DTG=DTGI
      121   CALL INTEGR(T,STEP,DTG,TØL,X, 4,KT ,NSF,NRF)
            IF(TF-STEP)161,151,161
      161   T=STEP
            STEP=T+DTP1
            IF(ABS(DTG)-ABS(DTP1))8,8,9
      9     DTG=DTP1
      8     IF(ABS(TF-T)-ABS(DTP1))132,143,143
      132   STEP=TF
            IF(ABS(DTG)-ABS(TF-T))143,143,2024
      2024  DTG=TF-T
      143   PRINT 190,NSF,NRF
        190 FØRMAT(1H ,////,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HGØØD STEPS T
           1AKEN ,  ,1X,I4,1X,15HBAD STEPS TAKEN)
            CALL PRINT (T ,X,    FPT1,DTP1,DTGI,TØL)
            NS=NS+NSF
            NR=NR+NRF
            GØ TØ 121
      151   NS=NS+NSF
            NR=NR+NRF
            NST=NST+NS
            NRT=NRT+NR
            PRINT 840,NST,NRT
      840   FØRMAT(1H0,50X,14HFINAL VALUES   ,I4,16HGØØD STEPS TAKEN,I4,
           115HBAD STEPS TAKEN)
            CALL PRINT(TF,X,    FPT1 ,DTP1,DTGI,TØL)
            GØ TØ 20
       11   FØRMAT(3E18.11)
            END

CØMMØN ALLØCATIØN

  77776 GMM        77744 ALPH       77254 BETA       77222 CH

PRØGRAM ALLØCATIØN

  00011 X          00021 I          00022 KT         00023 J
  00024 NS         00025 NR         00026 NST        00027 NRT
  00030 NSF        00031 NRF        00032 TI         00034 DTGI
  00036 TF         00040 FPT1       00042 DTP1       00044 TØL
  00046 T          00050 STEP       00052 DTG

SUBPRØGRAMS REQUIRED

  PRINT      ABS        INTEGR
THE END
```

TABLE A-2. (Continued)

```
            SUBROUTINE INTEGR[TI,T,DTS,TOL   ,X,N,KT,NS,NR]
            DIMENSION          X[ 4 ],XE[4 ]
            NT=0
            NS=0
            NR=0
            DTG=DTS
            TO=TI
      20    R2=X[1]**2+X[2]**2
            V2=X[3]**2+X[4]**2
            R=SQRT[R2]
            V=SQRT[V2]
            DO 1 I=1,2
            XE[I]=R
      1     XE[I+2]=V
            STEP=TO+DTG
            CALL RK713[TO,STEP,DTG,TOL   ,X, 4 ,     MS,MR,XE,4 ]
            TO=STEP
            NS=MS+NS
            NR=NR+MR
            DTS=DTG
            NT=NT+MS+MR
            IF[STEP-T]240,230,240
      240   IF[ABS[DTG]-ABS[T- STEP]]210,210,260
      260   DTG=T-STEP
      210   IF[NT-KT]20,220,220
      220   T=STEP
      230   RETURN
            END

PROGRAM ALLOCATION

  DUMMY X         00031 XE        00041 INTEGR      00042 NT
  DUMMY NS        DUMMY NR        00043 I           00044 MS
  00045 MR        DUMMY KT        00046 DTG         DUMMY DTS
  00050 TO        DUMMY TI        00052 R2          00054 V2
  00056 R         00060 V         00062 STEP        DUMMY TOL
  DUMMY T

SUBPROGRAMS REQUIRED

  SQRT      RK713      ABS
THE END
```

TABLE A-2. (Continued)

```
      SUBROUTINE RK713 [TI,TF,DT ,TOL    ,X,N,    NS,NR,XE,M]
C     SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
C     TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
C     NS IS THE NUMBER OF SUCCESFULL STEPS TAKEN
C     NR IS THE NUMBER OF REJECTED STEPS TAKEN
C     N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
C     KT IS MAX NUMBER OF ITERATIONS
C     ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
C     SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
C     F[0] IN FEHLBERGS REPORT IS IN F[1,J]
C     F[I] IS IN F[I+1,J]
C     FEHLBERGS REPORT REFERENCED IS NASA TR R-287
C     PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
C     DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
C     NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
      DIMENSION F[13, 4 ],XDUM[ 4 ],TE[ 4 ]         ,ALPH[13],
     1BETA[13,12],X[ 4 ],CH[13],XE[4 ]
      COMMON GMM
      COMMON ALPH,BETA,CH
      T=TI
      NS=0
      NR=0
   20 CALL DEQ [X,T,TE]
      DO 30 I=1,N
   30 F[1,I]=TE[I]
      DO 70 K=2,13
      DO 40 I=1,N
   40 XDUM[I]=X[I]
      NN=K-1
      DO 50 I=1,N
      DO 50 J=1,NN
   50 XDUM[I]=XDUM[I]+DT*BETA[K,J]*F[J,I]
      TDUM=T+ALPH[K]*DT
      CALL DEQ [XDUM,TDUM,TE]
      DO 60 I=1,N
   60 F[K,I]=TE[I]
   70 CONTINUE
      ER=0.
C  M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
C  THE ERROR CONTROL LOOP
C  XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
C  THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
      DO 120 I=1,M
 140  TE[I]=DT*[F[1,I]+F[11,I]-F[12,I]-F[13,I]]*41./840./XE[I]
      IF [ABS[TE[I]]-ER] 120,120,130
  130 ER=ABS[TE[I]]
  120 CONTINUE
      DT1=DT
      AK=.8
      IF[ER]141,142,141
 142  DT=10.*DT1
      GO TO 150
  141 DT=[SQRT[SQRT[SQRT[TOL/ER]]]]
```

```
            DT=AK*DT*DT1
            IF [ER   -TØL] 150,150,180
       150  TF=T+DT1
            DØ 90 I=1,N
            DØ 90 L=1,13
        90  X[I]=X[I]+DT1*CH[L]*F[L,I]
            NS=NS+1
            GØ TØ 230
   180      NR=NR+1
            TF=T
       230  RETURN
            END
```

```
CØMMØN ALLØCATIØN

  77776 GMM        77744 ALPH        77254 BETA        77222 CH

PRØGRAM ALLØCATIØN

  00037 F          00207 XDUM        00217 TE          DUMMY X
  DUMMY XE         DUMMY NS          DUMMY NR          00227 I
  DUMMY N          00230 K           00231 NN          00232 J
  DUMMY M          00233 L           00234 RK713       00236 T
  DUMMY TI         DUMMY DT          00240 TDUM        00242 ER
  00244 DT1        00246 AK          DUMMY TØL         DUMMY TF

SUBPRØGRAMS REQUIRED

  DEQ         ABS         SQRT
THE END
```

```
             SUBRØUTINE PRINT [T,X,    FPT,DTP,DTG,TØL]
             DIMENSIØN X[4]
             CØMMØN GMM
             PRINT 1,T,FPT,DTP,DTG,TØL
            1,[X[I],I=1,4],GMM
     1       FØRMAT[1H0,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
            15HDTG =E18.11,2X,5HTØL =E18.11
            2,//,6H X[1]=E18.11,2X,5HX[2]=E18.11,2X,5HX[3]=E18.11,2X,5HX[4]=E18.
            311,2X,5HGMM =E18.11]
             RETURN
             END
```

```
CØMMØN ALLØCATIØN

  77776 GMM

PRØGRAM ALLØCATIØN

  DUMMY X          00014 I           00015 PRINT       DUMMY T
  DUMMY FPT        DUMMY DTP         DUMMY DTG         DUMMY TØL

THE END
```

TABLE A-2. (Continued)

```
      SUBRØUTINE DEQ[X,T,DX]
      DIMENSIØN X[4],         DX[4]
      CØMMØN GMM
      C1=GMM-1.
      C2=X[1]+GMM
      C3=C2-1.
      R12=C2**2+X[2]**2
      R22=C3**2+X[2]**2
      R1=SQRT[R12]
      R2=SQRT[R22]
      DEN1=C1/R1/R12
      DEN2=GMM/R2/R22
      DX[1]=X[3]
      DX[2]=X[4]
      DX[3]=X[1]+2.*X[4]+DEN1*C2-DEN2*C3
      DX[4]=X[2]-2.*X[3]+[DEN1-DEN2]*X[2]
      RETURN
      END

CØMMØN ALLØCATIØN

  77776 GMM

PRØGRAM ALLØCATIØN

  DUMMY X          DUMMY DX         00013 DEQ          00015 C1
  00017 C2         00021 C3         00023 R12          00025 R22
  00027 R1         00031 R2         00033 DEN1         00035 DEN2

SUBPRØGRAMS REQUIRED

  SQRT
THE END
```

```
▲ASSIGN BI=MT1.
▲REWIND MT1.
▲FØRTLØAD BIU.
```

TABLE A-2. (Continued)

```
NAME      ENTRY ORIGIN  LAST SIZE/10 COMMON  BASE

*PROGRAM  03507  03477  11237    2913  17061
```

```
                                              INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E 00
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01

                                              FINAL VALUES    16GOOD STEPS TAKEN  12BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E 00
X(1)=  .14834414560E 02  X(2)= -.10497957781E 03  X(3)= -.10353360993E 03  X(4)= -.26034096011E 02  GMM =  .12277471000E-01
```

TABLE A-2. (Continued)

```
                                          INITIAL VALUES

T    = .00000000000E 00  FPT = .10000000000E 11  DTP = .10000000000E 01  DTG = .10000000000E 00  TØL = .10000000000E-01
X(1) = .99400000000E 00  X(2)= .00000000000E 00  X(3)= .00000000000E 00  X(4)= -.20317326296E 01  GMM = .12277471000E-01

                                          FINAL VALUES    21GOOD STEPS TAKEN   9BAD STEPS TAKEN

T    = .11124340337E 02  FPT = .10000000000E 11  DTP = .10000000000E 01  DTG = .10000000000E 00  TØL = .10000000000E-01
X(1) = .46094020203E 00  X(2)= -.52836197765E 00  X(3)= .63686511117E 00  X(4)= -.36212622822E-01  GMM = .12277471000E-01
```

```
                                          INITIAL VALUES

T    = .00000000000E 00  FPT = .10000000000E 11  DTP = .10000000000E 01  DTG = .10000000000E 00  TØL = .10000000000E-02
X(1) = .99400000000E 00  X(2)= .00000000000E 00  X(3)= .00000000000E 00  X(4)= -.20317326296E 01  GMM = .12277471000E-01

                                          FINAL VALUES    38GOOD STEPS TAKEN  18BAD STEPS TAKEN

T    = .11124340337E 02  FPT = .10000000000E 11  DTP = .10000000000E 01  DTG = .10000000000E 00  TØL = .10000000000E-02
X(1) = .99394287931E 00  X(2)= .20675093999E-03  X(3)= .30556134955E-01  X(4)= -.20396719553E 01  GMM = .12277471000E-01
```

TABLE A-2. (Continued)

```
                                   INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-03
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01

                                   FINAL VALUES    49GOOD STEPS TAKEN  23BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-03
X(1)=  .99395953273E 00  X(2)=  .31208898899E-03  X(3)=  .47930058035E-01  X(4)= -.20362226800E 01  GMM =  .12277471000E-01
```

```
                                   INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-04
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01

                                   FINAL VALUES    59GOOD STEPS TAKEN  26BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-04
X(1)=  .99398586886E 00  X(2)= -.52725004310E-04  X(3)= -.85866794634E-02  X(4)= -.20338573144E 01  GMM =  .12277471000E-01
```

TABLE A-2. (Continued)

```
                                        INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-05
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01

                                        FINAL VALUES    71GOOD STEPS TAKEN  29BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-05
X(1)=  .99399278977E 00  X(2)= -.23143635052E-04  X(3)= -.37964340325E-02  X(4)= -.20328307961E 01  GMM =  .12277471000E-01
```

```
                                        INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-06
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01

                                        FINAL VALUES    88GOOD STEPS TAKEN  28BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-06
X(1)=  .99399965197E 00  X(2)= -.18169492185E-05  X(3)= -.29115260859E-03  X(4)= -.20317861661E 01  GMM =  .12277471000E-01
```

TABLE A-2. (Continued)

```
                                                   INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-07
X[1]=  .99400000000E 00  X[2]=  .00000000000E 00  X[3]=  .00000000000E 00  X[4]= -.20317326296E 01  GMM =  .12277471000E-01

                                                   FINAL VALUES   110GOOD STEPS TAKEN  23BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-07
X[1]=  .99399987755E 00  X[2]= -.52863154165E-06  X[3]= -.85480979408E-04  X[4]= -.20317514150E 01  GMM =  .12277471000E-01
```

```
                                                   INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-08
X[1]=  .99400000000E 00  X[2]=  .00000000000E 00  X[3]=  .00000000000E 00  X[4]= -.20317326296E 01  GMM =  .12277471000E-01

                                                   FINAL VALUES   142GOOD STEPS TAKEN  17BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-08
X[1]=  .99399992998E 00  X[2]= -.25967018380E-06  X[3]= -.42320802805E-04  X[4]= -.20317433546E 01  GMM =  .12277472000E-01
```

TABLE A-2. (Continued)

```
                                        INITIAL VALUES
T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-09
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01
                                        FINAL VALUES   187GØØD STEPS TAKEN  11BAD STEPS TAKEN
T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-09
X(1)=  .99399990570E 00  X(2)= -.33857788284E-06  X(3)= -.55281726418E-04  X(4)= -.20317470697E 01  GMM =  .12277471000E-01
```

```
                                        INITIAL VALUES
T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01
                                        FINAL VALUES   284GØØD STEPS TAKEN   6BAD STEPS TAKEN
T    =  .11124340337E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .99399975458E 00  X(2)= -.89878453735E-06  X(3)= -.14661985150E-03  X(4)= -.20317701969E 01  GMM =  .12277471000E-01
```

TABLE A-2. (Continued)

```
                                        INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .99400000000E 00  X(2)=  .00000000000E 00  X(3)=  .00000000000E 00  X(4)= -.20317326296E 01  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES     91 GØØD STEPS TAKEN ,     3 BAD STEPS TAKEN

T    =  .10000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .51306964334E-01  X(2)=  .33261806494E 00  X(3)= -.17725293480E 01  X(4)=  .30780989933E 00  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES     25 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .20000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)= -.93131060560E 00  X(2)=  .24547656687E 00  X(3)= -.43129158899E 00  X(4)=  .31006193716E 00  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES     10 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .30000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)= -.97614412344E 00  X(2)=  .77776335194E 00  X(3)=  .33149953230E 00  X(4)=  .56540311203E 00  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES      8 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)= -.44270603051E 00  X(2)=  .10758527125E 01  X(3)=  .57852776536E 00  X(4)= -.49441502508E-01  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES     11 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .50000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)= -.18006366263E 00  X(2)=  .66950196427E 00  X(3)= -.21786104633E 00  X(4)= -.76073847667E 00  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES     31 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .60000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)= -.21621583708E 00  X(2)= -.56708820368E 00  X(3)=  .36135607595E 00  X(4)= -.89406724136E 00  GMM =  .12277471000E-01


                                     INTERMEDIATE VALUES     14 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .70000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)= -.37318694314E 00  X(2)= -.10638047999E 01  X(3)= -.53675403927E 00  X(4)= -.14407343734E 00  GMM =  .12277471000E-01
```

TABLE A-2. (Concluded)

```
                                   INTERMEDIATE VALUES      9 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .80000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.93001977239E 00  X(2)= -.84552159396E 00  X(3)= -.40885566983E 00  X(4)=  .52215234765E 00  GMM =  .12277471000E-01

                                   INTERMEDIATE VALUES     10 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .90000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.97913676968E 00  X(2)= -.29005836245E 00  X(3)=  .33768585314E 00  X(4)=  .40414413046E 00  GMM =  .12277471000E-01

                                   INTERMEDIATE VALUES     19 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .10000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.16592064157E 00  X(2)= -.33677902506E 00  X(3)=  .16497137116E 01  X(4)= -.20392918830E 00  GMM =  .12277471000E-01

                                   INTERMEDIATE VALUES     31 GOOD STEPS TAKEN ,      1 BAD STEPS TAKEN

T    =  .11000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .89885377499E 00  X(2)=  .51450563383E-01  X(3)=  .68133864168E 00  X(4)= -.12386168192E 00  GMM =  .12277471000E-01
                                       FINAL VALUES   303GOOD STEPS TAKEN   6BAD STEPS TAKEN

T    =  .11124340337E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .99399975514E 00  X(2)= -.89776535297E-06  X(3)= -.14644710143E-03  X(4)= -.20317701066E 01  GMM =  .12277471000E-01
```

TABLE A-3. TEST PROBLEM B12

```
▲ASSIGN S=MT0,SI CR,B0 MT1,L0 LP.
▲REWIND MT1.
▲FORTRAN B0,L0.
            DIMENSION X[2]
            DIMENSION ALPH[13],BETA[13,12],CH[13]
            COMMON ALPH,BETA,CH
       20   READ 11,    TI,DTGI,TOL
            READ 11,X[1],X[2]
            READ 11,TF,FPT1,DTP1
            READ 2025,KT
     2025   FORMAT[I4]
C           CONSTANTS FOR INTEGRATION SUBROUTINE
            DO 60 I=1,13
            DO 50 J=1,12
       50   BETA[I,J]=0.
            ALPH[I]=0.
       60   CH[I]=0.
            CH[6]=34./105.
            CH[7]=9./35.
            CH[8]=CH[7]
            CH[9]=9./280.
            CH[10]=CH[9]
            CH[12]=41./840.
            CH[13]=CH[12]
            ALPH[2]=2./27.
            ALPH[3]=1./9.
            ALPH[4]=1./6.
            ALPH[5]=5./12.
            ALPH[6]=.5
            ALPH[7]=5./6.
            ALPH[8]=1./6.
            ALPH[9]=2./3.
            ALPH[10]=1./3.
            ALPH[11]=1.
            ALPH[13]=1.
            BETA[2,1]=2./27.
            BETA[3,1]=1./36.
            BETA[4,1]=1./24.
            BETA[5,1]=5./12.
            BETA[6,1]=.05
            BETA[7,1]=-25./108.
            BETA[8,1]=31./300.
            BETA[9,1]=2.
            BETA[10,1]=-91./108.
            BETA[11,1]=2383./4100.
            BETA[12,1]=3./205.
            BETA[13,1]=-1777./4100.
            BETA[3,2]=1./12.
            BETA[4,3]=1./8.
            BETA[5,3]=-25./16.
            BETA[5,4]=-BETA[5,3]
            BETA[6,4]=.25
            BETA[7,4]=125./108.
```

TABLE A-3. (Continued)

```
              BETA[9,4]=-53./6.
              BETA[10,4]=23./108.
              BETA[11,4]=-341./164.
              BETA[13,4]=BETA[11,4]
              BETA[6,5]=.2
              BETA[7,5]=-65./27.
              BETA[8,5]=61./225.
              BETA[9,5]=704./45.
              BETA[10,5]=-976./135.
              BETA[11,5]=4496./1025.
              BETA[13,5]=BETA[11,5]
              BETA[7,6]=125./54.
              BETA[8,6]=-2./9.
              BETA[9,6]=-107./9.
              BETA[10,6]=311./54.
              BETA[11,6]=-301./82.
              BETA[12,6]=-6./41.
              BETA[13,6]=-289./82.
              BETA[8,7]=13./900.
              BETA[9,7]=67./90.
              BETA[10,7]=-19./60.
              BETA[11,7]=2133./4100.
              BETA[12,7]=-3./205.
              BETA[13,7]=2193./4100.
              BETA[9,8]=3.
              BETA[10,8]=17./6.
              BETA[11,8]=45./82.
              BETA[12,8]=-3./41.
              BETA[13,8]=51./82.
              BETA[10,9]=-1./12.
              BETA[11,9]=45./164.
              BETA[12,9]=3./41.
              BETA[13,9]=33./164.
              BETA[11,10]=18./41.
              BETA[12,10]=6./41.
              BETA[13,10]=12./41.
              BETA[13,12]=1.
              NS=0
              NR=0
              NST=0
              NRT=0
              PRINT 800
        800   FORMAT [1H1,/,51X,14HINITIAL VALUES]
              CALL PRINT [TI,X,    FPT1 ,DTP1,DTGI,TOL]
              T=TI
              STEP=FPT1
              DTG=FPT1-T
              IF[ABS[DTG]-ABS[DTGI]]6,6,7
     7        DTG=DTGI
     6        NSF=0
              NRF=0
              IF[ABS[TF-TI]-ABS[FPT1-TI]]112,121,121
     112      STEP=TF
              DTG=DTGI
```

```
 121    CALL  INTEGR[T,STEP,DTG,TOL,X, 2,KT ,NSF,NRF]
        IF[TF-STEP]161,151,161
 161    T=STEP
        STEP=T+DTP1
        IF[ABS[DTG]-ABS[DTP1]]8,8,9
 9      DTG=DTP1
 8      IF[ABS[TF-T]-ABS[DTP1]]132,143,143
 132    STEP=TF
        IF[ABS[DTG]-ABS[TF-T]]143,143,2024
 2024   DTG=TF-T
 143    PRINT 190,NSF,NRF
   190 FORMAT[1H ,////,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HGOOD STEPS T
      1AKEN , ,1X,I4,1X,15HBAD STEPS TAKEN]
        CALL PRINT [T ,X,    FPT1,DTP1,DTGI,TOL]
        NS=NS+NSF
        NR=NR+NRF
        GO TO 121
 151    NS=NS+NSF
        NR=NR+NRF
        NST=NST+NS
        NRT=NRT+NR
        PRINT 840,NST,NRT
 840    FORMAT[1HO,50X,14HFINAL VALUES  ,I4,16HGOOD STEPS TAKEN,I4,
       115HBAD STEPS TAKEN]
        CALL PRINT[TF,X,   FPT1 ,DTP1,DTGI,TOL]
        GO TO 20
  11    FORMAT[3E18.11]
        END

COMMON ALLOCATION

  77746 ALPH        77256 BETA        77224 CH

PROGRAM ALLOCATION

  00011 X           00015 KT          00016 I           00017 J
  00020 NS          00021 NR          00022 NST         00023 NRT
  00024 NSF         00025 NRF         00026 TI          00030 DTGI
  00032 TOL         00034 TF          00036 FPT1        00040 DTP1
  00042 T           00044 STEP        00046 DTG

SUBPROGRAMS REQUIRED

  PRINT       ABS         INTEGR
THE END
```

TABLE A-3. (Continued)

```
*           SUBRØUTINE INTEGR[TI,T,DTS,TØL    ,X,N,KT,NS,NR]
*           DIMENSIØN          X[ 2 ],XE[2 ]
*           NT=0
*           NS=0
*           NR=0
*           DTG=DTS
*           TO=TI
*      20   XE[1]=X[1]
*           XE[2]=X[2]
*           STEP=TO+DTG
*           CALL RK713[TO,STEP,DTG,TØL  ,X, 2 ,    MS,MR,XE,2 ]
*           TO=STEP
*           NS=MS+NS
*           NR=NR+MR
*           DTS=DTG
*           NT=NT+MS+MR
*           IF[STEP-T]240,230,240
*      240  IF[ABS[DTG]-ABS[T- STEP]]210,210,260
*      260  DTG=T-STEP
*      210  IF[NT-KT]20,220,220
*      220  T=STEP
*      230  RETURN
*           END

PRØGRAM ALLØCATIØN

  DUMMY X         00030 XE        00034 INTEGR     00035 NT
  DUMMY NS        DUMMY NR        00036 MS         00037 MR
  DUMMY KT        00040 DTG       DUMMY DTS        00042 TO
  DUMMY TI        00044 STEP      DUMMY TØL        DUMMY T

SUBPRØGRAMS REQUIRED

  RK713       ABS
THE END
```

TABLE A-3. (Continued)

```
      SUBRØUTINE RK713 (TI,TF,DT ,TØL    ,X,N,    NS,NR,XE,M)
C     SEVENTH ØRDER RUNGE-KUTTA INTEGRATIØN WITH STEPSIZE CØNTRØL
C     TF CAN BE GREATER THAN TI ØR LESS THAN TI AND RK713 WILL WØRK
C     NS IS THE NUMBER ØF SUCCESFULL STEPS TAKEN
C     NR IS THE NUMBER ØF REJECTED STEPS TAKEN
C     N IS THE NUMBER ØF DIFFERENTIAL EQUATIØNS
C     KT IS MAX NUMBER ØF ITERATIØNS
C     ARRAY F STØRES THE 13 EVALUATIØNS ØF THE DIFFERENTIAL EQUATIØNS
C     SUBSCRIPTS FØR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
C     F(Ø) IN FEHLBERGS REPØRT IS IN F(1,J)
C     F(I) IS IN F(I+1,J)
C     FEHLBERGS REPØRT REFERENCED IS NASA TR R-287
C     PARAMETERS FØR DEQ SUBRØUTINE MUST BE STØRED IN CØMMØN
C     DIMENSIØNS MUST AGREE WITH NUMBER ØF DIFFERENTIAL EQUATIØNS AND
C     NUMBER ØF CØNSTANTS IN THE PARTICULAR FEHLBERG FØRMULA USED
      DIMENSIØN F(13, 2 ),XDUM( 2 ),TE( 2 )         ,ALPH(13),
     1BETA(13,12),X( 2 ),CH(13),XE(2 )
      CØMMØN ALPH,BETA,CH
      T=TI
      NS=0
      NR=0
   20 CALL DEQ (X,T,TE)
      DØ 30 I=1,N
   30 F(1,I)=TE(I)
      DØ 70 K=2,13
      DØ 40 I=1,N
   40 XDUM(I)=X(I)
      NN=K-1
      DØ 50 I=1,N
      DØ 50 J=1,NN
   50 XDUM(I)=XDUM(I)+DT*BETA(K,J)*F(J,I)
      TDUM=T+ALPH(K)*DT
      CALL DEQ (XDUM,TDUM,TE)
      DØ 60 I=1,N
   60 F(K,I)=TE(I)
   70 CØNTINUE
      ER=0.
C  M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER ØF VARIABLES USED IN
C  THE ERRØR CØNTRØL LØØP
C  XE IS AN INPUT VECTØR WITH DIMENSIØN M WHICH IS USED TØ NØRMALIZE
C  THE TRUNCATIØN ERRØR CØMPUTATIØNS IN THE ERRØR CØNTRØL LØØP
      DØ 120 I=1,M
  140 TE(I)=DT*(F(1,I)+F(11,I)-F(12,I)-F(13,I))*41./840./XE(I)
      IF (ABS(TE(I))-ER) 120,120,130
  130 ER=ABS(TE(I))
  120 CØNTINUE
      DT1=DT
      AK=.8
      IF(ER)141,142,141
  142 DT=10.*DT1
      GØ TØ 150
  141 DT=(SQRT(SQRT(SQRT(TØL/ER))))
      DT=AK*DT*DT1
```

TABLE A-3. (Continued)

```
               IF [ER    -TØL] 150,150,180
           150 TF=T+DT1
               DØ 90 I=1,N
               DØ 90 L=1,13
            90 X[I]=X[I]+DT1*CH[L]*F[L,I]
               NS=NS+1
               GØ TØ 230
       180     NR=NR+1
               TF=T
           230 RETURN
               END

CØMMØN ALLØCATIØN

  77746 ALPH        77256 BETA        77224 CH

PRØGRAM ALLØCATIØN

  00037 F           00123 XDUM        00127 TE          DUMMY X
  DUMMY XE          DUMMY NS          DUMMY NR          00133 I
  DUMMY N           00134 K           00135 NN          00136 J
  DUMMY M           00137 L           00140 RK713       00142 T
  DUMMY TI          DUMMY DT          00144 TDUM        00146 ER
  00150 DT1         00152 AK          DUMMY TØL         DUMMY TF

SUBPRØGRAMS REQUIRED

  DEG         ABS         SQRT
THE END
```

```
               SUBRØUTINE PRINT [T,X,    FPT,DTP,DTG,TØL]
               DIMENSION X[2]
               PRINT 1,T,FPT,DTP,DTG,TØL
              1,X[1],X[2]
       1       FØRMAT[1HO,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
              15HDTG =E18.11,2X,5HTØL =E18.11
              2,//,6H X[1]=E18.11,2X,5HX[2]=E18.11]
               RETURN
               END

PRØGRAM ALLØCATIØN

  DUMMY X           00014 PRINT       DUMMY T           DUMMY FPT
  DUMMY DTP         DUMMY DTG         DUMMY TØL
THE END
```

TABLE A-3. (Continued)

```
*          SUBRØUTINE DEQ[X,T,DX]
*          DIMENSIØN X[2],      DX[2]
*          DX[1]=2.*[1.-X[2]]*X[1]
*          DX[2]=-X[2]*[1.-X[1]]
*          RETURN
*          END

PRØGRAM ALLØCATIØN

  DUMMY X            DUMMY DX          00006 DEQ
THE END
```

```
ΔASSIGN BI=MT1.
ΔREWIND MT1.
ΔFØRTLØAD BIU.
```

TABLE A-3. (Continued)

```
NAME       ENTRY ØRIGIN   LAST SIZE/10 CØMMØN   BASE

*PRØGRAM   03507  03477   10024     2262  17063
```

```
                                                        INITIAL VALUES

T     =  .00000000000E 00   FPT =   .10000000000E 11   DTP =   .10000000000E 01   DTG =   .10000000000E 00   TØL =   .10000000000E 00
X(1)=   .10000000000E 01   X(2)=   .30000000000E 01

                                                        FINAL VALUES     41GØØD STEPS TAKEN   12BAD STEPS TAKEN

T     =  .20000000000E 02   FPT =   .10000000000E 11   DTP =   .10000000000E 01   DTG =   .10000000000E 00   TØL =   .10000000000E 00
X(1)=   .11832482416E 01   X(2)=   .22060344093E 00
```

TABLE A-3. (Continued)

```
                                              INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TØL =  .10000000000E-01
X(1)=   .10000000000E 01   X(2)=  .30000000000E 01

                                              FINAL VALUES     32GØØD STEPS TAKEN   15BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TØL =  .10000000000E-01
X(1)=   .65544148974E 00   X(2)=  .18534633031E 00
```

```
                                              INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TØL =  .10000000000E-02
X(1)=   .10000000000E 01   X(2)=  .30000000000E 01

                                              FINAL VALUES     32GØØD STEPS TAKEN   11BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TØL =  .10000000000E-02
X(1)=   .67265193093E 00   X(2)=  .18597458435E 00
```

TABLE A-3. (Continued)

```
                                                  INITIAL VALUES

T     =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-03
X(1)=  .10000000000E 01   X(2)=  .30000000000E 01

                                                  FINAL VALUES    38GOOD STEPS TAKEN  12BAD STEPS TAKEN

T     =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-03
X(1)=  .67540936703E 00   X(2)=  .18604354748E 00
```

```
                                                  INITIAL VALUES

T     =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-04
X(1)=  .10000000000E 01   X(2)=  .30000000000E 01

                                                  FINAL VALUES    47GOOD STEPS TAKEN  11BAD STEPS TAKEN

T     =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-04
X(1)=  .67606171007E 00   X(2)=  .18607941743E 00
```

TABLE A-3. (Continued)

```
                                  INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-05
X(1)=   .10000000000E 01   X(2)=  .30000000000E 01

                                  FINAL VALUES     61GOOD STEPS TAKEN   17BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-05
X(1)=   .67618129079E 00   X(2)=  .18608146993E 00
```

```
                                  INITIAL V.LUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-06
X(1)=   .10000000000E 01   X(2)=  .30000000000E 01

                                  FINAL VALUES     75GOOD STEPS TAKEN   13BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-06
X(1)=   .67618638203E 00   X(2)=  .18608158017E 00
```

TABLE A-3. (Continued)

```
                                                   INITIAL VALUES

T    =  .00000000000E 00   FPT =   .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-07
X(1)=  .10000000000E 01   X(2)=   .30000000000E 01

                                                   FINAL VALUES     96GOOD STEPS TAKEN  11BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =   .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-07
X(1)=  .67618750253E 00   X(2)=   .18608160742E 00
```

```
                                                   INITIAL VALUES

T    =  .00000000000E 00   FPT =   .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-08
X(1)=  .10000000000E 01   X(2)=   .30000000000E 01

                                                   FINAL VALUES    124GOOD STEPS TAKEN   7BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =   .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-08
X(1)=  .67618758705E 00   X(2)=   .18608160911E 00
```

TABLE A-3. (Continued)

```
                                                INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-09
X(1)=  .10000000000E 01   X(2)=  .30000000000E 01

                                                FINAL VALUES   162GOOD STEPS TAKEN   6BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-09
X(1)=  .67618760770E 00   X(2)=  .18608160965E 00
```

```
                                                INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-10
X(1)=  .10000000000E 01   X(2)=  .30000000000E 01

                                                FINAL VALUES   217GOOD STEPS TAKEN   15BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-10
X(1)=  .67618761279E 00   X(2)=  .18608160961E 00
```

TABLE A-3. (Continued)

```
                                        INITIAL VALUES
T    =  .00000000000E 00  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .10000000000E 01  X(2)=  .30000000000E 01


                                        INTERMEDIATE VALUES     14 GØØD STEPS TAKEN ,     2 BAD STEPS TAKEN

T    =  .10000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .77344016154E-01  X(2)=  .14644481571E 01


                                        INTERMEDIATE VALUES     10 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .20000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .84977753179E-01  X(2)=  .57795270690E 00


                                        INTERMEDIATE VALUES      9 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .30000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .29089135173E 00  X(2)=  .24925317274E 00


                                        INTERMEDIATE VALUES      9 GØØD STEPS TAKEN ,     1 BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .14466020925E 01  X(2)=  .18721896503E 00


                                        INTERMEDIATE VALUES     14 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .50000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .40514470648E 01  X(2)=  .14394903986E 01


                                        INTERMEDIATE VALUES     19 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .60000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .17561472735E 00  X(2)=  .22585894715E 01


                                        INTERMEDIATE VALUES     12 GØØD STEPS TAKEN ,     1 BAD STEPS TAKEN

T    =  .70000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .65310426683E-01  X(2)=  .90879526320E 00
```

TABLE A-3. (Continued)

```
                                        INTERMEDIATE VALUES      8 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .80000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .14722682008E 00  X(2)=  .36671583527E 00

                                        INTERMEDIATE VALUES      8 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .90000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .65059555881E 00  X(2)=  .18757387506E 00

                                        INTERMEDIATE VALUES     11 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .10000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .31443368010E 01  X(2)=  .34881916525E 00

                                        INTERMEDIATE VALUES     20 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .11000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .90951832589E 00  X(2)=  .29967302638E 01

                                        INTERMEDIATE VALUES     14 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .12000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .75715455036E-01  X(2)=  .14327141371E 01

                                        INTERMEDIATE VALUES      9 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .13000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .86722147974E-01  X(2)=  .56555380118E 00

                                        INTERMEDIATE VALUES      7 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .14000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
X(1)=  .30146933561E 00  X(2)=  .24512579198E 00

                                        INTERMEDIATE VALUES      9 GØØD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .15000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TØL =  .10000000000E-10
```

TABLE A-3. (Concluded)

```
X(1)=  .15034034838E 01  X(2)=  .18933981458E 00

                                        INTERMEDIATE VALUES     16 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .16000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .39579039745E 01  X(2)=  .15458979640E 01

                                        INTERMEDIATE VALUES     20 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .17000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .16560785758E 00  X(2)=  .22145804662E 01

                                        INTERMEDIATE VALUES     13 GOOD STEPS TAKEN ,      1 BAD STEPS TAKEN

T    =  .18000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .65624788633E-01  X(2)=  .88886887967E 00

                                        INTERMEDIATE VALUES      8 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .19000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .15174413747E 00  X(2)=  .35939068967E 00
                                          FINAL VALUES   239GOOD STEPS TAKEN   5BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .67618761288E 00  X(2)=  .18608160960E 00
```

TABLE A-4. TEST PROBLEM E22

```
▲ASSIGN S=MTO,SI CR,BØ MT1,LØ LP.
▲REWIND MT1.
▲FØRTRAN BØ,LØ.
*              DIMENSIØN X(2)
*              DIMENSIØN ALPH(13),BETA(13,12),CH(13)
*              CØMMØN ALPH,BETA,CH
*           20 READ 11,    TI,DTGI,TØL
*              READ 11,X(1),X(2)
*              READ 11,TF,FPT1,DTP1
*              READ 2025,KT
*     2025     FØRMAT(I4)
*     C        CØNSTANTS FØR INTEGRATIØN SUBRØUTINE
*              DØ 60 I=1,13
*              DØ 50 J=1,12
*           50 BETA(I,J)=0.
*              ALPH(I)=0.
*           60 CH(I)=0.
*              CH(6)=34./105.
*              CH(7)=9./35.
*              CH(8)=CH(7)
*              CH(9)=9./280.
*              CH(10)=CH(9)
*              CH(12)=41./840.
*              CH(13)=CH(12)
*              ALPH(2)=2./27.
*              ALPH(3)=1./9.
*              ALPH(4)=1./6.
*              ALPH(5)=5./12.
*              ALPH(6)=.5
*              ALPH(7)=5./6.
*              ALPH(8)=1./6.
*              ALPH(9)=2./3.
*              ALPH(10)=1./3.
*              ALPH(11)=1.
*              ALPH(13)=1.
*              BETA(2,1)=2./27.
*              BETA(3,1)=1./36.
*              BETA(4,1)=1./24.
*              BETA(5,1)=5./12.
*              BETA(6,1)=.05
*              BETA(7,1)=-25./108.
*              BETA(8,1)=31./300.
*              BETA(9,1)=2.
*              BETA(10,1)=-91./108.
*              BETA(11,1)=2383./4100.
*              BETA(12,1)=3./205.
*              BETA(13,1)=-1777./4100.
*              BETA(3,2)=1./12.
*              BETA(4,3)=1./8.
*              BETA(5,3)=-25./16.
*              BETA(5,4)=-BETA(5,3)
*              BETA(6,4)=.25
*              BETA(7,4)=125./108.
```

TABLE A-4. (Continued)

```
      BETA[9,4]=-53./6.
      BETA[10,4]=23./108.
      BETA[11,4]=-341./164.
      BETA[13,4]=BETA[11,4]
      BETA[6,5]=.2
      BETA[7,5]=-65./27.
      BETA[8,5]=61./225.
      BETA[9,5]=704./45.
      BETA[10,5]=-976./135.
      BETA[11,5]=4496./1025.
      BETA[13,5]=BETA[11,5]
      BETA[7,6]=125./54.
      BETA[8,6]=-2./9.
      BETA[9,6]=-107./9.
      BETA[10,6]=311./54.
      BETA[11,6]=-301./82.
      BETA[12,6]=-6./41.
      BETA[13,6]=-289./82.
      BETA[8,7]=13./900.
      BETA[9,7]=67./90.
      BETA[10,7]=-19./60.
      BETA[11,7]=2133./4100.
      BETA[12,7]=-3./205.
      BETA[13,7]=2193./4100.
      BETA[9,8]=3.
      BETA[10,8]=17./6.
      BETA[11,8]=45./82.
      BETA[12,8]=-3./41.
      BETA[13,8]=51./82.
      BETA[10,9]=-1./12.
      BETA[11,9]=45./164.
      BETA[12,9]=3./41.
      BETA[13,9]=33./164.
      BETA[11,10]=18./41.
      BETA[12,10]=6./41.
      BETA[13,10]=12./41.
      BETA[13,12]=1.
      NS=0
      NR=0
      NST=0
      NRT=0
      PRINT 800
  800 FØRMAT [1H1,/,51X,14HINITIAL VALUES]
      CALL PRINT [TI,X,    FPT1 ,DTP1,DTGI,TØL]
      T=TI
      STFP=FPT1
      DTG=FPT1-T
      IF[ABS[DTG]-ABS[DTGI]]6,6,7
7     DTG=DTGI
6     NSF=0
      NRF=0
      IF[ABS[TF-TI]-ABS[FPT1-TI]]112,121,121
112   STEP=TF
      DTG=DTGI
```

TABLE A-4. (Continued)

```
121     CALL INTEGR[T,STEP,DTG,TØL,X, 2,KT ,NSF,NRF]
        IF[TF-STEP]161,151,161
161     T=STEP
        STEP=T+DTP1
        IF[ABS[DTG]-ABS[DTP1]]8,8,9
9       DTG=DTP1
8       IF[ABS[TF-T]-ABS[DTP1]]132,143,143
132     STEP=TF
        IF[ABS[DTG]-ABS[TF-T]]143,143,2024
2024    DTG=TF-T
143     PRINT 190,NSF,NRF
  190   FØRMAT[1H ,///,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HGØØD STEPS T
       1AKEN , ,1X,I4,1X,15HBAD STEPS TAKEN]
        CALL PRINT [T ,X,    FPT1,DTP1,DTGI,TØL]
        NS=NS+NSF
        NR=NR+NRF
        GØ TØ 121
151     NS=NS+NSF
        NR=NR+NRF
        NST=NST+NS
        NRT=NRT+NR
        PRINT 840,NST,NRT
840     FØRMAT[1HO,50X,14HFINAL VALUES  ,I4,16HGØØD STEPS TAKEN,I4,
       115HBAD STEPS TAKEN]
        CALL PRINT[TF,X,   FPT1 ,DTP1,DTGI,TØL]
        GØ TØ 20
 11     FØRMAT[3E18.11]
        END

CØMMØN ALLØCATIØN

  77746 ALPH      77256 BETA      77224 CH

PRØGRAM ALLØCATIØN

  00011 X         00015 KT        00016 I         00017 J
  00020 NS        00021 NR        00022 NST       00023 NRT
  00024 NSF       00025 NRF       00026 TI        00030 DTGI
  00032 TØL       00034 TF        00036 FPT1      00040 DTP1
  00042 T         00044 STEP      00046 DTG

SUBPRØGRAMS REQUIRED

  PRINT     ABS       INTEGR
THE END
```

TABLE A-4. (Continued)

```
*         SUBRØUTINE INTEGR[TI,T,DTS,TØL    ,X,N,KT,NS,NR]
*         DIMENSIØN          X[ 2 ],XE[2 ]
*         NT=0
*         NS=0
*         NR=0
*         DTG=DTS
*         TO=TI
*   20    XE[1]=1.
*         XE[2]=1.
*         STEP=TO+DTG
*         CALL RK713[TO,STEP,DTG,TØL  ,X, 2 ,    MS,MR,XE,2 ]
*         TO=STEP
*         NS=MS+NS
*         NR=NR+MR
*         DTS=DTG
*         NT=NT+MS+MR
*         IF[STEP-T]240,230,240
*   240   IF[ABS[DTG]-ABS[T- STEP]]210,210,260
*   260   DTG=T-STEP
*   210   IF[NT-KT]20,220,220
*   220   T=STEP
*   230   RETURN
*         END

PRØGRAM ALLØCATIØN

  DUMMY X        00030 XE        00034 INTEGR     00035 NT
  DUMMY NS       DUMMY NR        00036 MS         00037 MR
  DUMMY KT       00040 DTG       DUMMY DTS        00042 TO
  DUMMY TI       00044 STEP      DUMMY TØL        DUMMY T

SUBPRØGRAMS REQUIRED

  RK713     ABS
THE END
```

TABLE A-4. (Continued)

```
      SUBROUTINE RK713 (TI,TF,DT ,TOL    ,X,N,    NS,NR,XE,M)
C     SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
C     TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
C     NS IS THE NUMBER OF SUCCESFULL STEPS TAKEN
C     NR IS THE NUMBER OF REJECTED STEPS TAKEN
C     N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
C     KT IS MAX NUMBER OF ITERATIONS
C     ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
C     SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
C     F(0) IN FEHLBERGS REPORT IS IN F(1,J)
C     F(I) IS IN F(I+1,J)
C     FEHLBERGS REPORT REFERENCED IS NASA TR R-287
C     PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
C     DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
C     NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
      DIMENSION F(13, 2 ),XDUM( 2 ),TE( 2 )         ,ALPH(13),
     1BETA(13,12),X( 2 ),CH(13),XE(2 )
      COMMON ALPH,BETA,CH
      T=TI
      NS=0
      NR=0
   20 CALL DEQ (X,T,TE)
      DO 30 I=1,N
   30 F(1,I)=TE(I)
      DO 70 K=2,13
      DO 40 I=1,N
   40 XDUM(I)=X(I)
      NN=K-1
      DO 50 I=1,N
      DO 50 J=1,NN
   50 XDUM(I)=XDUM(I)+DT*BETA(K,J)*F(J,I)
      TDUM=T+ALPH(K)*DT
      CALL DEQ (XDUM,TDUM,TE)
      DO 60 I=1,N
   60 F(K,I)=TE(I)
   70 CONTINUE
      ER=0.
C  M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
C  THE ERROR CONTROL LOOP
C  XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
C  THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
      DO 120 I=1,M
  140 TE(I)=DT*(F(1,I)+F(11,I)-F(12,I)-F(13,I))*41./840./XE(I)
      IF (ABS(TE(I))-ER) 120,120,130
  130 ER=ABS(TE(I))
  120 CONTINUE
      DT1=DT
      AK=.8
      IF(ER)141,142,141
  142 DT=10.*DT1
      GO TO 150
  141 DT=(SQRT(SQRT(SQRT(TOL/ER))))
      DT=AK*DT*DT1
```

```
          IF (ER    -TOL) 150,150,180
    150 TF=T+DT1
          DO 90 I=1,N
          DO 90 L=1,13
     90 X(I)=X(I)+DT1*CH(L)*F(L,I)
          NS=NS+1
          GO TO 230
  180     NR=NR+1
          TF=T
    230 RETURN
          END
```

COMMON ALLOCATION

77746 ALPH	77256 BETA	77224 CH

PROGRAM ALLOCATION

00037 F	00123 XDUM	00127 TE	DUMMY X
DUMMY XE	DUMMY NS	DUMMY NR	00133 I
DUMMY N	00134 K	00135 NN	00136 J
DUMMY M	00137 L	00140 RK713	00142 T
DUMMY TI	DUMMY DT	00144 TDUM	00146 ER
00150 DT1	00152 AK	DUMMY TOL	DUMMY TF

SUBPROGRAMS REQUIRED

DEQ	ABS	SQRT

THE END

```
          SUBROUTINE PRINT (T,X,    FPT,DTP,DTG,TOL)
          DIMENSION X(2)
          PRINT 1,T,FPT,DTP,DTG,TOL
         1,X(1),X(2)
     1    FORMAT(1H0,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
         15HDTG =E18.11,2X,5HTOL =E18.11
         2,/,6H X(1)=E18.11,2X,5HX(2)=E18.11)
          RETURN
          END
```

PROGRAM ALLOCATION

DUMMY X	00014 PRINT	DUMMY T	DUMMY FPT
DUMMY DTP	DUMMY DTG	DUMMY TOL	

THE END

TABLE A-4. (Continued)

```
*       SUBROUTINE DEQ[X,T,DX]
*       DIMENSION X[2],      DX[2]
*       DX[1]=X[2]
*       DX[2]=[1.-X[1]**2]*X[2]-X[1]
*       RETURN
*       END

PROGRAM ALLOCATION

  DUMMY X          DUMMY DX          00006 DEQ
THE END
```

```
▲ASSIGN BI=MT1.
▲REWIND MT1.
▲FORTLOAD BIU.
```

TABLE A-4. (Continued)

NAME	ENTRY	ORIGIN	LAST	SIZE/10	COMMON	BASE
*PROGRAM	03507	03477	10637		2657	17063

```
                                                    INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E 00
X(1)=  .20000000000E 01  X(2)=  .00000000000E 00

                                                    FINAL VALUES    35GOOD STEPS TAKEN  13BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E 00
X(1)=  .19735582007E 01  X(2)=  .41827137254E 00
```

TABLE A-4. (Continued)

```
                                         INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-01
X(1)=  .20000000000E 01  X(2)=  .00000000000E 00

                                         FINAL VALUES    36GOOD STEPS TAKEN  15BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-01
X(1)=  .20069643660E 01  X(2)= -.71845049017E-01
```

```
                                         INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-02
X(1)=  .20000000000E 01  X(2)=  .00000000000E 00

                                         FINAL VALUES    37GOOD STEPS TAKEN  17BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-02
X(1)=  .20081874022E 01  X(2)= -.41737664121E-01
```

TABLE A-4. (Continued)

```
                                                     INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-03
X(1)=  .20000000000E 01  X(2)=  .00000000000E 00

                                                     FINAL VALUES    46GOOD STEPS TAKEN  16BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-03
X(1)=  .20081644290E 01  X(2)= -.42197190845E-01
```

```
                                                     INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-04
X(1)=  .20000000000E 01  X(2)=  .00000000000E 00

                                                     FINAL VALUES    56GOOD STEPS TAKEN  18BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 11  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-04
X(1)=  .20081537041E 01  X(2)= -.42504030549E-01
```

TABLE A-4. (Continued)

```
                                              INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-05
X(1)=  .20000000000E 01   X(2)=  .00000000000E 00

                                              FINAL VALUES     70GOOD STEPS TAKEN  18BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-05
X(1)=  .20081500503E 01   X(2)= -.42508225085E-01
```

```
                                              INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-06
X(1)=  .20000000000E 01   X(2)=  .00000000000E 00

                                              FINAL VALUES     89GOOD STEPS TAKEN  21BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-06
X(1)=  .20081497884E 01   X(2)= -.42508639103E-01
```

TABLE A-4. (Continued)

```
                                        INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-07
X(1)=  .20000000000E 01   X(2)=  .00000000000E 00

                                        FINAL VALUES   112GOOD STEPS TAKEN  18BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-07
X(1)=  .20081497650E 01   X(2)= -.42508870414E-01
```

```
                                        INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-08
X(1)=  .20000000000E 01   X(2)=  .00000000000E 00

                                        FINAL VALUES   144GOOD STEPS TAKEN  16BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-08
X(1)=  .20081497620E 01   X(2)= -.42508880677E-01
```

TABLE A-4. (Continued)

```
INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-09
X(1)=  .20000000000E 01   X(2)=  .00000000000E 00

FINAL VALUES   186GOOD STEPS TAKEN  12BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-09
X(1)=  .20081497617E 01   X(2)= -.42508884626E-01
```

```
INITIAL VALUES

T    =  .00000000000E 00   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-10
X(1)=  .20000000000E 01   X(2)=  .00000000000E 00

FINAL VALUES   242GOOD STEPS TAKEN   4BAD STEPS TAKEN

T    =  .20000000000E 02   FPT =  .10000000000E 11   DTP =  .10000000000E 01   DTG =  .10000000000E 00   TOL =  .10000000000E-10
X(1)=  .20081497615E 01   X(2)= -.42508886517E-01
```

TABLE A-4. (Continued)

```
                                          INITIAL VALUES

T    =  .00000000000E 00  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .20000000000E 01  X[2]=  .00000000000E 00


                                        INTERMEDIATE VALUES     13 GOOD STEPS TAKEN ,     1 BAD STEPS TAKEN

T    =  .10000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .15081442366E 01  X[2]= -.78021807484E 00


                                        INTERMEDIATE VALUES     13 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .20000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .32331666552E 00  X[2]= -.18329745696E 01


                                        INTERMEDIATE VALUES     16 GOOD STEPS TAKEN ,     1 BAD STEPS TAKEN

T    =  .30000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]= -.18660739124E 01  X[2]= -.10210603358E 01


                                        INTERMEDIATE VALUES     14 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .40000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]= -.17417683244E 01  X[2]=  .62466616401E 00


                                        INTERMEDIATE VALUES      9 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .50000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]= -.83707745035E 00  X[2]=  .13070889378E 01


                                        INTERMEDIATE VALUES     15 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .60000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .12790420293E 01  X[2]=  .24378144489E 01


                                        INTERMEDIATE VALUES     19 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .70000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .19201524166E 01  X[2]= -.43583853332E 00
```

TABLE A-4. (Continued)

```
                                INTERMEDIATE VALUES     10 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .80000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .12132324410E 01  X(2)= -.98781392226E 00


                                INTERMEDIATE VALUES     11 GOOD STEPS TAKEN ,     1 BAD STEPS TAKEN

T    =  .90000000000E 01  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.41291605205E 00  X(2)= -.25269034480E 01


                                INTERMEDIATE VALUES     17 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .10000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.20083407830E 01  X(2)=  .32907070692E-01


                                INTERMEDIATE VALUES     13 GOOD STEPS TAKEN ,     1 BAD STEPS TAKEN

T    =  .11000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.15049739797E 01  X(2)=  .78444442429E 00


                                INTERMEDIATE VALUES     13 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .12000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)= -.31376909536E 00  X(2)=  .18440269613E 01


                                INTERMEDIATE VALUES     17 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .13000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .18717708928E 01  X(2)=  .99758656955E 00


                                INTERMEDIATE VALUES     20 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .14000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X(1)=  .17386417030E 01  X(2)= -.62715364635E 00


                                INTERMEDIATE VALUES      8 GOOD STEPS TAKEN ,     0 BAD STEPS TAKEN

T    =  .15000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
```

TABLE A-4. (Concluded)

```
X[1]=  .83043742862E 00   X[2]= -.13133658842E 01


                                             INTERMEDIATE VALUES      14 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .16000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]= -.12913899229E 01   X[2]= -.24232638004E 01


                                             INTERMEDIATE VALUES      17 GOOD STEPS TAKEN ,      1 BAD STEPS TAKEN

T    =  .17000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]= -.19179362020E 01   X[2]=  .43962156405E 00


                                             INTERMEDIATE VALUES      10 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .18000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]= -.12082152063E 01   X[2]=  .99161534742E 00


                                             INTERMEDIATE VALUES      10 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN

T    =  .19000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .42574876675E 00   X[2]=  .25353542246E 01
                                               FINAL VALUES    276GOOD STEPS TAKEN    6BAD STEPS TAKEN

T    =  .20000000000E 02  FPT =  .10000000000E 01  DTP =  .10000000000E 01  DTG =  .10000000000E 00  TOL =  .10000000000E-10
X[1]=  .20081497614E 01   X[2]= -.42508886890E-01
```

APPENDIX B

A LISTING OF THE RATIONAL FUNCTION EXTRAPOLATION ALGORITHM

A listing of both the old and new versions of DIFSYF is included here. These subroutines are versions of the rational function extrapolation technique as developed by Bulirsch and Stoer [5]. The listings are included here so that they may be compared with other versions that exist.

Old Version of DIFSYF

```
      SUBROUTINE DIFSYF (N,F,EFS,H,X,Y)
      DIMENSION Y(17), DTT(17), D(6), YA(17), YL(17), YM(17),
     DY(17), D1(17), DT(17,6), YG(8,17), YH(8,17), S(17)
      INTEGER R,SR
      LOGICAL KONV,BO,BH,FIN
      EP=ABS(EPS)
      N1=N
      HH=H
      IF (EP.LT.5.E-8) EP=5.E-8
      CALL F (X,Y,DZ)
      BH=.FALSE.
      FIN=.FALSE.
      DO 1 I=1,N1
      S(I)=0.
    1 YA(I)=Y(I)
    2 A=HH+X
      FC=1.5
      BO=.FALSE.
      M=1
      R=2
      SR=3
      JJ=0
      DO 23 J=1,10
      IF (BO) GO TO 3
      D(1)=2.25
      D(3)=9.
      D(5)=36.
      GO TO 4
    3 D(1)=1.7777777778
      D(3)=7.1111111111
      D(5)=28.4444444444
    4 KONV=J.GT.5
      IF (J.LE.7) GO TO 5
      L=6
      D(6)=64.
      FC=.6*FC
      GO TO 7
    5 L=J-1
      IF (J-1) 7,7,6
    6 D(L)=M*M
    7 M=M+M
      G=HH/FLOAT(M)
```

```
      B=G+G
      IF (BH.AND.J.LT.9) GO TO 14
      KK=(M-2)/2
      M=M-1
      DO 8 I=1,N1
      YL(I)=YA(I)
    8 YM(I)=G*DZ(I)+YA(I)
      DO 13 K=1,M
      CALL F (X+FLOAT(K)*G,YM,DY)
      IF (DY(1).GT.1.E38) GO TO 25
      DO 10 I=1,N1
      U=B*DY(I)+YL(I)
      YL(I)=YM(I)
      YM(I)=U
      U=ABS(U)
      IF (U-S(I)) 10,10,9
    9 S(I)=U
   10 CONTINUE
      IF (K.EQ.KK.AND.K.NE.2) GO TO 11
      GO TO 13
   11 JJ=1+JJ
      DO 12 I=1,N1
      YH(JJ,I)=YM(I)
   12 YG(JJ,I)=YL(I)
   13 CONTINUE
      GO TO 16
   14 DO 15 I=1,N1
      YM(I)=YH(J,I)
   15 YL(I)=YG(J,I)
   16 CALL F (A,YM,DY)
      IF (DY(1).GT.1.E38) GO TO 25
      DO 22 I=1,N1
      V=DTT(I)
      DTT(I)=(YM(I)+YL(I)+G*DY(I))*.5
      C=DTT(I)
      TA=C
      IF (L.LT.1) GO TO 20
      DO 19 K=1,L
      B1=D(K)*V
      B=B1-C
      U=V
```

```
      IF (ABS(B)-1.E-10) 18,18,17
17    B=(C-V)/B
      U=C*B
      C=B1*B
18    V=DT(I,K)
      DT(I,K)=U
19    TA=U+TA
20    IF (ABS(Y(I)-TA).GT.EP*S(I)) GO TO 21
      GO TO 22
21    KONV=.FALSE.
22    Y(I)=TA
      IF (KONV) GO TO 24
      D(2)=4.
      D(4)=16.
      BO=.NOT.BO
      M=R
      R=SR
23    SR=M+M
      BH=.NOT.BH
      FIN=.TRUE.
      HH=HH*.5
      GO TO 2
24    H=FC*HH
      X=A
      RETURN
25    H=0
      DO 26 I=1,N1
26    Y(I)=YA(I)
      END
```

END OF COMPILATION: NO DIAGNOSTICS.

New Version of DIFSYF

```
PROTOKOLL
      SUBROUTINE   DIFSYS(N,F,EPS,H,X,Y)
      EXTERNAL  F
      DIMENSION Y(17),YA(21),YL(21),YM(21),DY(21),
     1DZ(21),DT(21,7),D(7),S(21),EP(4)
      LOGICAL KONV,BO,GR,KL
      DATA EP/0.4E-1,0.16E-2,0.64E-4,0.256E-5/
      JTI=0
      FY=1.
      ETA=ABS(EPS)
      IF(ETA.LT.1.E-9) ETA=1.E-9
      DO 100 I=1,N
  100 YA(I)=Y(I)
      CALL  F(X,Y,DZ)
   10 XN=X+H
      BO=.FALSE.
      DO 110 I=1,N
  110 S(I)=0.
      M=1
      JR=2
      JS=3
      DO 260 J=1,10
      IF(.NOT.BO) GO TO 200
      D(2)=1.7777777778
      D(4)=7.1111111111
      D(6)=28.444444444
      GO TO 201
  200 D(2)=2.25
      D(4)=9.
      D(6)=36.
  201 IF(J.LE.7) GO TO 202
      L=7
      D(7)=64.
      GO TO 203
  202 L=J
      D(L)=M*M
  203 KONV=L.GT.3
      M=M+M
      G=H/FLOAT(M)
      B=G+G
      DO 210 I=1,N
      YL(I)=YA(I)
  210 YM(I)=YA(I)+G*DZ(I)
      M=M-1
      DO 220 K=1,M
      CALL F(X+FLOAT(K)*G,YM,DY)
      DO 220 I=1,N
      U=YL(I)+B*DY(I)
      YL(I)=YM(I)
      YM(I)=U
      U=ABS(U)
  220 IF(U.GT.S(I)) S(I)=U
      CALL  F(XN,YM,DY)
      KL=L.LT.2
      GR=L.GT.5
      FS=0.
      DO 233 I=1,N
      V=DT(I,1)
      C=(YM(I)+YL(I)+G*DY(I))*0.5
      DT(I,1)=C
      TA=C
      IF(KL) GOTO 233
```

```
      DO 231 K=2,L
      B1=D(K)*V
      B=B1-C
      W=C-V
      U=V
      IF(B.EQ.0.  ) GOTO 230
      B=W/B
      U=C*B
      C=B1*B
  230 V=DT(I,K)
      DT(I,K)=U
  231 TA=U+TA
      IF(.NOT.KONV) GOTO 232
      IF(ABS(Y(I)-TA).GT.S(I)*ETA) KONV=.FALSE.
 232  IF(GR.OR.S(I).EQ.0.) GO TO 233
      FV=ABS(W)/S(I)
      IF(FS.LT.FV) FS=FV
  233 Y(I)=TA
      IF(FS.EQ.0.) GOTO 250
      FA=FY
      K=L-1
      FY=(EP(K)/FS)**(1./FLOAT(L+K))
      IF(L.EQ.2) GOTO 240
      IF(FY.LT.0.7*FA) GO TO 250
  240 IF(FY.GT.0.7) GOTO 250
      H=H*FY
      JTI=JTI+1
      IF(JTI.GT.  5) GO TO 30
      GO TO 10
  250 IF(KONV) GO TO 20
      D(3)=4.
      D(5)=16.
      BO=.NOT.BO
      M=JR
      JR=JS
  260 JS=M+M
      H=H*0.5
      GO TO 10
 20   X=X
      H=H*FY
      RETURN
  30  H=0.
      DO 300 I=1,N
  300 Y(I)=YA(I)
      RETURN
      END
```

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APPROVAL

TM X-64781

A COMPARISON OF DIGITAL COMPUTER PROGRAMS FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

By Hugo L. Ingram

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

for E. D. GEISSLER
Director, Aero-Astrodynamics Laboratory

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