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On the Establishment and
Evolution of Orbit-Orbit Resonances

A dissertation submitted in partial satisfaction
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by

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ABSTRACT

On the Establishment and
Evolution of Orbit-Orbit Resonances

by

Charles Finney Yoder

In the solar system, there exist several examples of gravitational resonance between two or more satellites or planets in which a special angle variable is observed to librate. Goldreich has suggested that in the case of planetary satellites a tidally induced torque acting on the satellites may have played an essential role in the establishment and subsequent evolution of the observed resonances. This proposal is thoroughly investigated as it applies to the three resonances among pairs of satellites of Saturn and is shown to be a plausible mechanism for their establishment but is less successful, in the Tital-Hyperion case, in providing a reasonable time scale for the damping of the amplitude of libration.

The solution of the problem is reached in three stages. First, a theoretical description of transition is developed for a simple time dependent pendulum plus constant applied torque. The evolution of the system through the various phases (i.e. positive rotation, negative rotatian and libration) is described in terms of the motion of the extremes or "roots" of the momentum variable in

the complex plane. A transition phase is defined and equations of motion of these roots are derived from which a lowest order estimate of the probability for transition from a rotation into libration is obtained.

Second, the two body gravitational interaction is expanded and reduced to a one dimensional time independent Hamiltonian which accurately describes the motion of the resonance variable in the absence of tides - if the satellites' inclinations and eccentricities are relatively small and if the perturbations in the semimajor axes during each phase of its evolution are also small. The effect of the tides is then introduced by redefining the orbital elements in such a way as to recover the Hamiltonian formulation, the important difference being that it is now time dependent.

The theoretical approach outlined for simple pendulum systems is then applied to eccentricity dependent resonances. The dependence of the probability for transition into libration is obtained as a function of the mean eccentricity and the mechanism governing transition in various limits is discussed. The damping of the amplitude of libration as a function of the tidal charge in the orbital parameters (principally semimajor axis) is found via the action integral.

Finally, the theoretical model developed is then applied to the Saturn resonances and found to agree with the recent work of Allan, Greenberg and Sinclair. In addition, a proposal by R. G.

Hipkin that the moon may have been trapped in an orbit-orbit resonance with another planet in the past is examined and found to be untenable.

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1.1 INTRODUCTION

In the solar system there are several instances in which the ratio of the mean motions of a pair of satellites is very nearly a simple fraction. This kind of relationship is called a commensurability. Some examples include the three satellite-satellite commensurabilities of Saturn (Mimas-Tethys, 2:1; Enceladas-Dione, 2:1; Titan-Hyperion, 4:3) and the 3:2 commensurability of Neptune-Pluto. An equivalent statement of a commensurability is the relation

$$j\lambda + j'\lambda' = 0,$$

where j and j' are integers and λ and λ' are the respective mean longitudes of the pair of satellites (or planets). The obvious extension of the above relation to N bodies is

$$\sum_{n=1}^{N} j^n\lambda^n = 0. \tag{1.1.1}$$

The best known example of a commensurability of three bodies involves the JI, JII, and JIII satellites of Jupiter through the relation

$$\lambda_{JI} - 3\lambda_{JII} + 2\lambda_{JIII} = 0.$$

Observed commensurabilities are not restricted to these orbit-orbit types. Another type involves the ratio of the orbital period to the rotational period of either the same or different bodies.

Respective examples of this type are the spin-orbit interaction of Mercury (Goldreich and Peale, 1967) and the possible orbital commensurabilities of artificial satellites with the earth's siderial day (Allan, 1967).

These special relationships would not be nearly as interesting if they did not have a physical basis for their existence. First of all, the commensurability relation is often not the physical variable which best describes the observations. Examining the visual evidence more closely, we find that in many cases there exists a single resonance variable ϕ which appears to librate about either mod(π) or mod(2π), and, for the two-body case, has the form

$$\phi = j\lambda + j'\lambda' + \left\{ \begin{array}{l} \text{linear function of perihelion } (\varpi) \\ \text{and node } (\Omega) \text{ of each body.} \end{array} \right\} \qquad (1.1.2)$$

The mechanism which maintains this commensurability or resonance in every known instance involves a gravitational interaction which is fairly well understood (Hagihara, 1972, pp328-52). A study of the satellite-satellite interaction of the two-body resonance, after the two-body gravitational potential has been expanded in terms of the orbital elements which describe the position of each body, reveals that ϕ is that argument of a cosine function in the expansion which becomes very slowly varying for a nearly commensurate motion. Furthermore, this resonant term in the expansion often acts as a pendulum-like potential, being the dominant factor controlling the very long period motion of both satellites. This suggests that a

one-dimensional Hamiltonian might be derived as an approximation of this very long period behavior, and is the subject of chapter two. If the expansion (and Hamiltonian!) is valid for a range of ϕ which includes both the rotational and librational phases of the resonance variable, then, of course, ϕ may execute either rotations or librations, depending on the parameters of the system. Although the libration of the resonance variable can be explained in terms of the mutual gravitational interactions between the partners, it seems unlikely that a state of libration could have existed since the earliest stages of formation of the solar system. In other words, there should be some mechanism or mechanisms by which the partners evolved into their presently observed state. One interesting fact that Roy and Ovenden (1954) have shown is that the high frequency of commensurabilities in the solar system cannot be assigned to a chance initial arrangement.

There appears to be two basic solutions to the two questions: 1) why so many commensurabilities? and 2) why so many librating resonance variables? One possibility is that a resonant or commensurable configuration is inherently more stable than a slightly off-resonant configuration. As illustration, consider the Trojan asteroids which move in approximately the same orbit as Jupiter, clustered in two groups, 60° ahead and behind Jupiter (Brown and Shook, 1964, pp. 250-88). Their gravitational interaction with Jupiter tends to maintain the one to one commensurability.

Imagine what would happen to an asteroid with a nearly circular orbit which is slightly larger or smaller than Jupiter's. Within a short time span that asteroid would make a close approach to Jupiter. If close enough, the gravitational force of Jupiter could dominate that of the sun, and radically change the orbit of this asteroid, perhaps even removing it from the solar system by changing its orbit from elliptic to hyperbolic, or removing it through a collision. In this example, the lifetime of any asteroid not in resonance with Jupiter but having nearly the same orbit or a crossing orbit, would tend to be very short in comparison with the age of the solar system. Thus time, by Jupiter, endows the Trojans with a divine relationship! A recent proposal by Ovenden (1972), based on this idea of maximum stability, is that the high frequency of commensurabilities is a reflection of the evolution of the solar system towards a "Least Interaction Action" configuration, driven to its present state by <u>purely conservative gravitational forces</u>.

The second possibility is that dissipative effects, which give rise to secular torques on the affected bodies, drive them towards a commensurability with one or more other bodies, and that something, either in the nature of the dissipative mechanism or in its interaction with the gravitational force, leads to transition into a libration of a particular resonance variable. Goldreich (1965) has suggested that the dissipative mechanism operating in satellite systems of the planets is the inelastic tidal response of the planet

to the pull of each satellite. Already a similar theory, when applied to the spin-orbit resonance of Mercury, has led to a satisfactory explanation of evolution and capture into libration (Goldreich and Peale, 1967). In this instance, capture is apparently caused by an asymmetry in the tidal torque acting on the spin of Mercury, as the velocity of the resonance variable $\dot{\phi}$ vanishes and then changes sign. An important difference in the orbit-orbit type of resonance is that the capture mechanism does not appear to depend in any important way on the details of the tidal interaction itself, as it does with the spin-orbit case. Recent numerical studies of Greenberg (1972) and Sinclair (1972) indicate that the tidal mechanism does satisfactorily explain the Saturn resonances and that capture into libration is caused by the tidal torque acting through the gravitational interaction. The existence of a secular torque of any significance acting on the satellites of the major planets has not been documented with corresponding visual evidence of a secular change in their orbital periods. This effect is apparently too small to be measurable at present. Still, an estimate of its magnitude has been inferred from the present size of orbit of the innermost satellite (Goldreich, 1965). The best evidence for tidal friction comes from observations involving the period of our moon (Munk and Macdonald, 1960, p. 198). In fact, the present rate of increase leads to something of a paradox in the age of the earth-moon system compared to the age of the earth, assuming a constant dissipative mechanism. A novel proposal of R. Hipkin (in press) is that the moon may have been trapped in a commensurability with Venus at a fixed radius for a long enough period of time to resolve the time scale paradox. Unfortunately, the proposal does not appear to be feasible because of various factors discussed in section 4.2. Perhaps the solution is that, in the past, the dissipation function has been variable, as indicated by some paleontological evidence (Pannella, MacClintock and Thompson, 1968).

At present, Goldreich's tidal evolution hypothesis must find its support through indirect evidence involving its consistency in explaining present observations and past history. Applied to each satellite-satellite resonance, it should lead to the results 1) that capture into the presently observed resonance is a reasonably probable event, and 2) that the time that this event took place was within the age of the solar system, given a reasonable estimate of the tidal torque. Allan (1969) has already shown for the Mimas-Tethys commensurability, given tidally induced torques acting on each resonance partner, that the evolution of the orbital elements, including the amplitude of libration, could be followed backward in time to determine the initial values of the elements at the time of capture, estimated to be about 2×10^8 years ago. Unfortunately, the approximations that Allan made for this case cannot be used for the remaining two cases. A major problem before us is to carry out a similar analysis of the other satellite-satellite resonances and

follow their evolution back in time.

Attempting to understand how these three satellite-satellite resonances of Saturn evolved, in the context of Goldreich's hypothesis, was certainly one of the major goals of this thesis, although the first problem attempted was to determine the feasibility of Hipkin's lunar resonance hypothesis. Lots of time passed before it was realized that each was governed by a one-dimensional Hamiltonian, although it turns out to be a much poorer approximation in Hipkin's lunar case. More time was spent determining how to best introduce the tidal torque into the Hamiltonian so as to preserve its canonical character. The resulting Hamiltonian is, of course, an explicit function of the time, and it is this explicit dependence which allows the system to evolve. Even more time elapsed before it was realized that no adequate analytical theory existed with which transition for even the simplest pendulum system was thoroughly explained. In the process, the scope of the thesis has broadened considerably and made it difficult to find some point to end the affair and bring it to some conclusion. The exposition of this paper breaks down into three exercises: 1) development of a one-dimensional Hamiltonian from the satellite-satellite interaction, 2) development of transition theory for this pendulum-like Hamiltonian, and 3) applications.

The development of an approximate description of the motion due to the resonance variable is a complex exercise using, for the most part, well-known techniques of celestial mechanics and variational theory. The first step is to reduce the interaction to a one-dimensio[illegible] [illegible]iltonian of the form

$$H(x,\phi,t) = 1/2(x + c(t))^2 + b(x,t)\cos\phi. \qquad (1.1.3)$$

Chapter two takes the specific example of a satellite-satellite gravitational interaction and outlines a procedure for expanding the interaction in terms of the orbital elements. The variational equations of motion of a canonical set of elements are also derived. A method for the elimination of the "short-period" terms in the interaction is sketched, along with a qualitative discussion of the approximations involved in reducing the system to one degree of freedom. The tidal interaction is further discussed and a method is proposed for introducing the tidal interaction into the tide-free Hamiltonian. In addition, a discussion of the other physical situations for which the above Hamiltonian is a good approximation of the motion is given. In chapter three, the analytic behavior of the Hamiltonian $H(x,\phi,t)$ is discussed in detail. First, the similarities to and differences from a simple pendulum are discussed for the tide-free case, and the possible motions of the system for different functional forms of $b(x)$ are found using elementary analytical principles. Then the motion of the time-dependent system is obtained through an investigation of the motion of the "roots" of a polynomial in x. In the time-independent system, a pair of these

roots is exactly the extremes of the motion of x, while in the time-dependent case, they at least bound the motion of x. This method of attack and its associated "picture" have rather wide application and can reduce many difficult problems involving transitions between distinct phases to a tractable form. Capture criteria are specifically developed for two kinds of eccentricity-dependent resonance variables.

The results are then applied in chapter four to the three satellite-satellite resonances of Saturn and to Hipkin's lunar-planetary resonance hypothesis. In the first exercise we find that the existence of a tidally induced torque does successfully explain the capture process. In fact, for two of the three examples discussed, the resonance variable automatically evolves into libration. But the hypothesis is less successful in resolving the evolutionary time scale. The negative results of the second exercise should have a sobering influence on those who might over-estimate the importance of resonance phenomena. Finally, the three-satellite commensurability of Jupiter is briefly discussed, and a probable history of its evolution is given.

The material in chapter three concerning the [illegible] criteria is different enough from other approaches to the proble[illegible] require a lengthy introduction of its own, mainl[illegible] to understand [illegible] nature of the approximations which will be used. First, th[illegible] [illegible]ion of a simple pendulum with a time-dependent restoring f[illegible] is investigated. Attention is focused on the "roots". We discover that they are four in number and are points in the complex plane. One pair of these roots bounds the motion of the momentum variable, and each root can be uniquely labeled. Whether the angle variable ϕ executes positive rotations, librations, or negative rotations is qualitatively determined by the relative position of the roots, along with the specification of these roots which bound the motion. This leads eventually to the precise definition of the transition phase in terms of the mutual motion of the roots and of the momentum variable. To complement this picture, the first-order equations of motion of each root are derived, from which the analytical properties are deduced and a transition integral" is defined. Next, a constant torque term is added to this simple pendulum Hamiltonian and transformed to a new form very like (1.1.3), except that the coefficient b is independent of x. Again the equations of motion of each root are derived, their motion discussed, and a transition integral defined. These simpler systems need to be clearly understood before we apply similar methods to the more complex ca certain amount of repetition is involved, but it is necessar understand the scope of the theoretical approach taken.

1.2 THEORY OF TRANSITION FOR SIMPLE PENDULUM SYSTEMS

The first example we shall examine is that of a simple pendulum governed by the Hamiltonian

$$H(p,\phi) = 1/2\, p^2 + b(t)\cos\phi, \qquad (1.2.1a)$$

where the equations of motion are given by

$$\frac{dp}{dt} = + \frac{\partial H}{\partial \phi} = - b(t)\sin\phi, \qquad b)$$

$$\frac{d\phi}{dt} = - \frac{\partial H}{\partial p} = - p, \qquad c)$$

$$\frac{dH}{dt} = + \frac{\partial H}{\partial t} = \frac{db}{dt}\cos\phi. \qquad d)$$

The sign convention for the equations of motion is the normal convention adopted in celestial mechanics and is used throughout this paper. Ordinarily, changing the sign convention should result in replacing the ordinary Hamiltonian by its negative counterpart to preserve the equations of motion. This should mean that the kinetic energy term ($1/2\, p^2$) should enter in equation (1.2.1a) with a minus sign. The above Hamiltonian is our subject of study simply because of its similarity to the Hamiltonian developed later from the orbit-orbit interaction. In that interaction, the equivalent kinetic energy term is negative-definite.

Initially, say, the pendulum is executing positive rotations. If the coefficient b(t) slowly increases in magnitude with time, then the velocity of the pendulum as it passes over the top will slowly decrease (Best, 1968). Eventually, the pendulum will not have enough energy to pass over the top, and thereafter will librate. An examination of the solution for the momentum variable p will suggest an equivalent picture of the transition. If $b(t)$ were constant, then we could find a solution for p in terms of the time by using the Hamiltonian to eliminate ϕ. The resulting integral solution is:

$$\int_{p_o}^{p} dp \, \frac{\text{sign}(-b\sin\phi)}{\sqrt{R(p)}} = t - t_o, \tag{1.2.2}$$

where

$$R(p) = b^2 - (H - 1/2\, p^2)^2.$$

The function R(p) is a quartic polynomial whose four roots are given by

$$\text{Roots} = \pm\sqrt{2(H + b)}, \quad \pm\sqrt{2(H - b)}. \tag{1.2.3}$$

The motion of p is bounded by a pair of these roots, with p oscillating back and forth between them with increasing time. Inspection of the equation of motion for p reveals that these turning points of p (maxima and minima) occur when

$$\phi = 2n\pi, \quad \text{or} \quad \phi = (2n + 1)\pi,$$

n being an integer. In subsequent discussion, mod(π) and mod(2π) shall designate ϕ equal to $2n\pi$ and $(2n + 1)\pi$, respectively. If the pendulum is executing positive rotations ($\langle\dot{\phi}\rangle > 0$), then the negative

set of roots corresponds to the value of $\dot{\phi}$ at the top and bottom of its swing.

Therefore, the four roots can be completely labeled by determining the value of ϕ (either mod(π) or mod(2π)) and the sign of ($\dot{\phi}$) at that root. Inspection of the equations of motion reveals that the first set corresponds to ϕ = mod(π) and the second set to ϕ = mod(2π), and that they are completely specified by the set of labels $p_{\pi+}$, $p_{\pi-}$, $p_{2\pi+}$, and $p_{2\pi-}$. Factoring R(p) in terms of these roots we have

$$R(p) = 1/4(p_{2\pi+} - p)(p_{\pi+} - p)(p_{\pi-} - p)(p - p_{2\pi-}), \tag{1.2.4a}$$

and

$$p_{\pi\pm} = \pm\sqrt{2(H + b)}, \quad p_{2\pi\pm} = \pm\sqrt{2(H - b)}. \tag{1.2.4b}$$

Physically p is always real, forcing R(p) to be ≥ 0. If all the roots are real, then the motion of p is bound between either $p_{2\pi-}$, $p_{\pi-}$, or $p_{\pi+}$, $p_{2\pi+}$, and corresponds to positive or negative rotation, respectively. The motion is bounded between the two π-roots or two 2π-roots, or, p librates only if the opposite pair is complex. We shall adopt the convention that in the rotation phase (all roots real) the π-roots lie interior to the 2π-roots, or $b < 0$. Diagrammatically, we can represent the three distinct states of the pendulum -- positive rotation, negative rotation, and libration -- by a graph of the relative locations of the roots in the complex plane

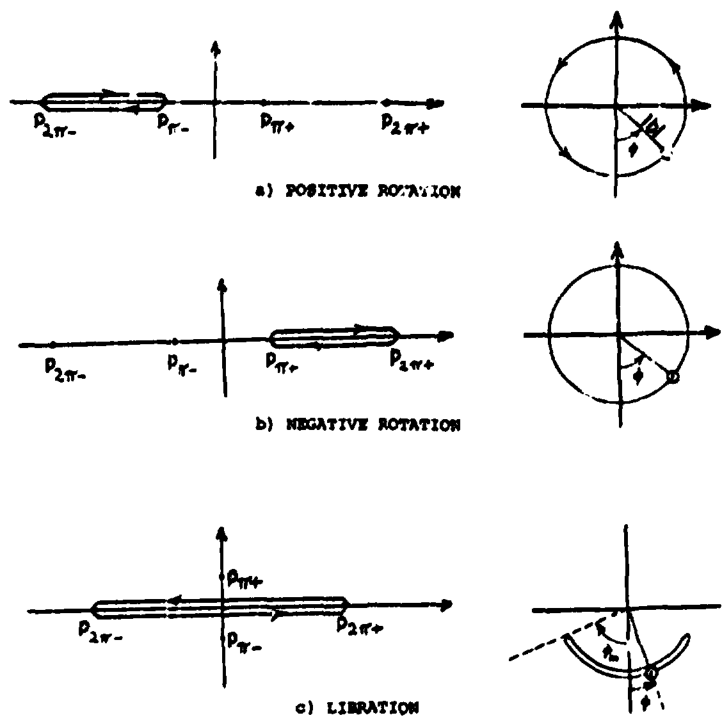


FIGURE 1.2.1 PENDULUM STATES

(see figures 1.2.1a, b, c). Also included in these diagrams are graphs of the function $b(x)$ versus ϕ in polar coordinates. The pendulum librates about ϕ = mod(2π) if the $H < |b|$ and rotates if $H > |b|$. Allowing b to be a function of time does not qualitatively change the integral solution given by (1.2.2) except that both H and b are time-dependent, and therefore the roots are time-dependent. Instead of being fixed, the roots now move in the complex plane. Intuitively, we see that passage from rotation to libration must involve the motion of the π-roots toward the origin and then out along the imaginary axis. This picture of the motion suggests that we look for equations of motion of the roots themselves. Since the roots are only functions of b(t) and $H(x,\phi,t)$, the motion of a given root p_r must satisfy

$$\frac{dp_r}{dt}(b,H) = \frac{db}{dt}\frac{\partial p_r}{\partial b} + \frac{dH}{dt}\frac{\partial p_r}{\partial H} \quad . \tag{1.2.5}$$

The partial derivatives of p with respect to b and H are obtained from the roots themselves (1.2.4b), while the time derivative of H is obtained from its partial time derivative (1.2.1d):

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{db}{dt}\cos\phi . \tag{1.2.6}$$

Thus the equations of motion for the roots are:

$$\frac{dp_{\pi\pm}}{dt} = \frac{1}{p_{\pi\pm}}\frac{db(t)}{dt}(1 + \cos\phi) , \tag{1.2.7a}$$

$$\frac{dp_{2\pi\pm}}{dt} = -\frac{1}{p_{2\pi\pm}}\frac{db(t)}{dt}(1 - \cos\phi). \qquad \text{b)}$$

The qualitative motion of each can be easily determined, given the signs of $\frac{db(t)}{dt}$ and of the root. If $|b(t)|$ increases with time, and since we have adopted the convention that b(t) be negative, then the π-roots move toward the origin and the 2π-roots away from it. Thus the fluctuation in $\dot{\phi}$ caused by the pendulum force grows as transition to libration is approached. One interesting observation is that the π-roots are stationary ($\frac{dp_{\pi\pm}}{dt} = 0$) when $\phi = \text{mod}(\pi)$ or $p = p_\pi$. But p itself is a minimum when it is at the root p_π and is stationary. If the root were not stationary when $p = p_\pi$, then it is a simple exercise to show that p would suffer an infinite acceleration at this point. The analogous situation holds when $p = p_{2\pi}$.

We could use (1.2.7a, b) to determine the first order (in $\frac{db(t)}{dt}$) secular behavior of the roots by integrating them over one revolution and approximating the motion of x and ϕ by replacing b(t) with its mean value over the revolution. In this instance, the action integral represents a simpler method to obtain the secular motions.

The above equations are uniquely useful in the transition phase. This phase will be defined by the conditions that 1) it starts at the instant the π-roots coincide and become imaginary for some initial values (p_i, ϕ_i), 2) it continues as the motion of p is

carried to the opposite 2π-root, and 3) it ends when p returns to the origin (the real part of the p_π root). This motion corresponds roughly to the revolution in which $\dot{\phi}$ goes to zero, reverses sign as the pendulum moves backwards through the bottom and again goes to zero near the top. If initially the pendulum executes positive rotations and $\frac{d|b|}{dt} > 0$, then figure 1.2.2 represents the motions in the complex p plane.

The change in the π-roots during this transition phase can be obtained by integrating the equation of motion. The result is

$$1/2(p_\pi^2(f) - p_\pi^2(i)) = \int_{t_i}^{t_f} dt\frac{db}{dt}(1 + \cos\phi). \qquad (1.2.8)$$

Notice that $p_\pi^2(i) = 0$ from the definition of transition phase. The integrand is negative definite ($\frac{db}{dt} < 0$), implying that $p_\pi^2(f)$ is negative or $p_\pi(f)$ imaginary. The range of the initial value ϕ_i is $+\pi \leq \phi_i \leq 3\pi$. The angle ϕ then increases, passing through the botton position of the pendulum and finally reaching a maximum at $\phi_c = 3\pi - \delta\phi_c$, where $\delta\phi_c$ is a small positive angle of

$$O(|b|^{-3/2}\frac{d|b|}{dt}.$$

$\dot{\phi}$ then reverses sign, and the angle ϕ decreases until it reaches a minimum at $\phi_f = \pi + \delta\phi_f(\delta\phi_f > 0$ and $O(|b|^{-3/2}\frac{d|b|}{dt}))$.

To simplify our problem, let's choose $\frac{db}{dt}$ to be constant, and demand that it be comparatively small. Also change the integration variable from t to ϕ. The result is

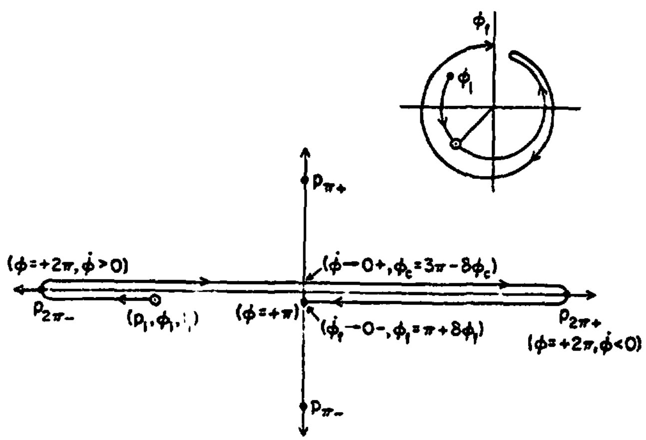


FIGURE 1.2.2 TRANSITION PHASE

This phase will be defined by the conditions that

1) it starts at the instant the π-roots coincide and become imaginary for some initial values $\{p_i, \phi_i\}$;
   Here we choose the minimum possible value of ϕ_i to equal π;
2) it continues as the motion of p is first towards the $p_{2\pi}$-root, where $\phi = 2\pi$ and $\dot\phi(2\pi-) > 0$, and then back towards the point where $\dot\phi$ vanishes at a maximum angle ϕ_c;
3) it continues as $\dot\phi$ reverses sign and the motion of p is carried to the opposite 2π-root where ϕ again equals 2π but $\dot\phi(2\pi+) < 0$;
4) it ends when p returns to the origin (the real part of the p_π-root) where $\dot\phi$ again vanishes at some angle ϕ_f near $\phi = \pi$.

$$1/2p_\pi^2(f) = \frac{db}{dt} \int_{\phi_i}^{\phi_f} \frac{d\phi(1 + \cos\phi)\,\text{sign}(-p)}{(2H(t))^{1/2}(1 + K(t)\cos\phi)^{1/2}}, \qquad (1.2.9)$$

where

$$K(t) = \frac{-b(t)}{H(t)}.$$

It should be emphasized that the above is a path-like integral in which the variation of ϕ in going from ϕ_i to ϕ_f is determined by the transition phase diagram (fig. 1.2.2). During transition K(t) changes from one to a value slightly above it. Since $\frac{dH}{dt} = \frac{-db}{dt}$ at $\phi = \text{mod}(\pi)$, K(t) is very nearly a constant, no matter how long a time the pendulum spends near the top during transition. The contribution to the integral is small for $\phi \cong \pi$ since the integrand vanishes. Except for near the top, the motion of ϕ is fast compared to any change in H(t) or K(t), and the integral is well-bounded.

There is a stationary solution for $p = 0$, $\phi = \text{mod}(\pi)$ or $\text{mod}(2\pi)$, and there exists the singular possibility of a sticking motion in which p slowly approaches the top and "sticks" there. Motions very near this singular event will have very long transition times, implying H(t) could change appreciably as p moves between the complex π-roots. The question is, how near?

It turns out that the set of motions which have a long transition time are restricted to an exponentially small set of $O(|b|^{-3/2} \frac{d|b|}{dt})$ of initial values of ϕ_i. To demonstrate this assertion, let's determine the condition for which the change in H(t) in the time interval $t_a \le t \le t_b$ is of $O(H(t_a))$. In addition, this particular calculation will be restricted to the event where p moves between the complex π-roots while $\phi = 0+$ and reverses sign. The transition time $\Delta t = t_b - t_a$ can be calculated most easily by finding the analogous solution for ϕ in terms of t. We find

$$\int_{\phi_a}^{\phi_c} \frac{d\phi\,\mathrm{sign}(\dot\phi;\dot\phi > 0)}{(1 + K(t)\cos\phi)^{1/2}} + \int_{\phi_c}^{\phi_b} \frac{d\phi\,\mathrm{sign}(\dot\phi;\dot\phi < 0)}{(1 + K(t)\cos\phi)^{1/2}} = \int_{t_a}^{t_b} (2H(t))^{1/2} dt. \tag{1.2.10}$$

The integral over ϕ begins at an angle ϕ_a for which $\dot\phi$ is positive, continues to the angle ϕ_c where $\dot\phi$ vanishes and reverses sign, and ends as ϕ moves back to the angle ϕ_b. The integrand on the left hand side of (1.2.10) is large only for angles very near $\phi = \mathrm{mod}(\pi)$, and its value will tend to be independent of the limits ϕ_a and ϕ_b as long as they are not nearly equal to ϕ_c. Therefore, the calculation can be simplified by choosing $\phi_a = \phi_b$, expanding $\cos\phi$ about $\phi = 3\pi - \delta\phi$ and changing the limits of integration as follows:

$$\phi_c = 3\pi - \delta\phi_c; \quad \phi_a = 3\pi - \delta\phi_a.$$

The small differences, $\delta\phi_c$ and $\delta\phi_a$, are positive and $\delta\phi_c < \delta\phi_a$ since $\phi_c > \phi_a$. Near the top of the pendulum H is very nearly equal to b(t). This means that the function K(t) very nearly equals the value one to $O(b^{-1} \frac{db}{dt})$ during transition, implying that the change in K(t) as the pendulum moves over the top is of $O(|b^{-1} \frac{db}{dt}|^2)$. A first order estimate can be obtained by choosing K(t) to be equal

to its value at $\phi = \phi_c$, and approximating the integrand on the right hand side by

$$(1 + K(t)\cos\phi) \cong -\Delta(\phi_c) + 1/2\ \delta\phi^2, \tag{1.2.11}$$

where $K(t) = 1 + \Delta(t)$. Finally, H(t) can be replaced by b(t) in the integrand of the right hand side of (1.2.10).

For a linearly changing b(t), we can write

$$b(t) = b(t_a)(1 + \frac{(t - t_a)}{\tau}), \tag{1.2.12}$$

and τ is the "slow time" associated with the change in b_o. With these approximations, both sides can be integrated. The result is:

$$2\sqrt{2}\ \ln(\delta\phi + \sqrt{-2\Delta(\phi_c)} + \delta\phi^2)\Big|_{\delta\phi_c}^{\delta\phi_a} - \frac{2\sqrt{2}}{3}\,|b(t_a)|^{1/2}\tau(1 + \frac{t}{\tau})^{3/2}\Big|_0^{t_b - t_a}. \tag{1.2.13}$$

$\delta\phi_a$ can be chosen to be >> $\delta\phi_c$ such that the contribution from the lower limit is relatively small. The angle $\delta\phi_c$ is the angle for which $\dot\phi$ vanishes or for which $(1 + K(\phi_c)\cos\phi) = 0$. Thus $\delta\phi_c$ is directly related to $\Delta(\phi_c)$, and is approximately

$$1/2\ \delta^2\phi_c \approx \Delta(\phi_c). \tag{1.2.14}$$

The transition time $t_b - t_a$ must be of order τ if the change in b(t) is of $O(b(t_a))$. For the sake of calculation, we shall demand that $b(t_a) = 2b(t_a)$ or that $t_b - t_a = \tau$. The natural period, T, of the pendulum is $\frac{2\pi}{\sqrt{|b(t_a)|}}$, which is the period of the pendulum in the small libration limit. Solving for $\delta\phi_c$:

$$\delta\phi_c \cong \sqrt{2\Delta\phi_c} = e^{-\alpha},\ \alpha = \frac{\pi(2^{3/2}-1)}{3}\ \frac{\tau}{T'} \tag{1.2.15}$$

The value of $\Delta(\phi_c)$ is related to $p_\pi^2(t)$ by (1.2.4.6, 1.2.9, 1.2.11)

$$\Delta(\phi_c) = -\frac{p_\pi^2(\phi_c)}{2H(\phi_c)} \cong \frac{p_\pi^2(\phi_c)}{2b(t_c)} \tag{1.2.16}$$

and $p_\pi^2(\phi_c)$, which depends on the initial value ϕ_i, can be approximated by

$$1/2\ p_\pi^2(\phi_c) \cong \frac{dh}{dt}\int_{\phi_i}^{3\pi-\delta\phi_c} d\phi \frac{|\cos(1/2\phi)|}{\sqrt{|b(t)|}},\ +\pi \le \phi_i \le +3\pi. \tag{1.2.17}$$

Observe that (1.2.16) depends on the value of $b(t_c)$, not $b(t_a)$ or $b(t_b)$. The relationship between t_c and t_a can be found from an inspection of (1.2.10), in the light of the approximations so far invoked. We see that each of the terms on the left hand side of 1.2.10 are approximately equal. This implies that the integrals (obtained from the right hand side) evaluated between t_c and t_a and t_b and t_c also equal. Explicitly

$$\int_{t_a}^{t_c}(2H(t))^{1/2}dt \cong \int_{t_c}^{t_b}(2H(t))^{1/2}dt\ .$$

The above integrals are approximated by the right hand side of (1.2.13). Given $t_b - t_a = 2\tau$, the important results are:

$$(t_c - t_a) = \left(\left(\frac{2^{3/2}+1}{2}\right)^{2/3} - 1\right)\tau \approx 1/2\tau$$

$$b(t_c) \cong 3/2\ b(t_a)\ .$$

The next step is to obtain an approximate result for the integral in (1.2.4.7). Clearly, the only values of ϕ_i which correspond to a long transition time are near the value $(3\pi - \delta\phi_c)$. Therefore $\cos\phi/2$ can be expanded about $\phi = 3\pi$. From (1.2.13), the transition time is rougly proportional to $(\ln|\delta\phi(t)+\sqrt{2\Delta + \delta^2\phi(t)}|^{2/3}$, so that the integrand does tend to vanish if $b(t)$ increases indefinitely. Since t change much more slowly than ϕ for ϕ not too near $3\pi - \delta\phi_c$, we can approximate $b(t)$ by its initial value $b(t_a)$. The approximate solution for $p_\pi^2(\phi_c)$ is:

$$p_\pi^2(\phi_c) \cong -\frac{\sqrt{|b(t_a)|}}{2\tau}\,[(3\pi - \phi_i)^2 - (\delta\phi_c)^2]. \tag{1.2.18}$$

By contruction, $3\pi - \phi_i > \delta\phi_c$, implying that $p_\pi(\phi_c)$ is imaginary. $\delta\phi_c$ can be eliminated using (1.2.15). Solving for $(3\pi - \phi_i)$,

$$(3\pi - \phi_i) \simeq \left(\frac{9\alpha}{\sqrt{2}} + 1\right)^{1/2} e^{-\alpha},\quad \alpha = \frac{2\sqrt{2\pi}}{3}\,\frac{\tau}{T}\ . \tag{1.2.19}$$

For large values of α, the set of values of ϕ_i which lead to a long transition time is exponentially small compared to the full range of ϕ_i. The above result agrees qualitatively with Best (1968) although he appears to calculate a quite different parameter not nearly so well related to the initial conditions. Anyway, the important result is that the transition integral is well defined

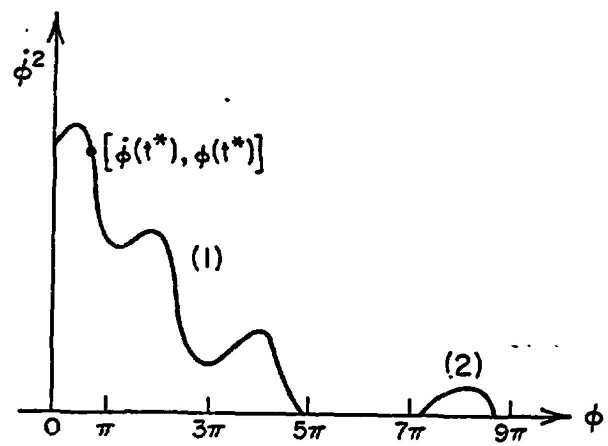


FIGURE 1.2.3a

Diagram of $\dot{\phi}^2$ versus ϕ for equation 1.2.20 where b is constant. (1) A value of H such that ϕ is, at some time t*, a rotating variable. (2) A value of H for which ϕ librates.

except for initial values very close to the sticking motion. But the functional approximation used for the integrand is only good to $O(|b|^{1/2} \frac{d|b|}{dt})$. Therefore, we can effectively ignore those ϕ_i which have a long transition time, if all that is desired is a solution to the integral accurate to first order. Of course, in any real physical system governed by (1.2.1), there are arbitrary fluctuations which would effectively eliminate the possibility of a sticking motion and inhibit transitions which take an exceptionally long time.

A system more nearly related to the problem at hand is that of a pendulum subject to a constant applied torque, $\frac{dc}{dt}$. The Hamiltonian in this case is

$$H(p,\phi,t) = 1/2\, p^2 + b(t)\cos\phi + \frac{dc}{dt}\phi. \qquad (1.2.20)$$

We shall choose $\frac{dc}{dt}$ to be positive such that if the pendulum initially executes positive rotations it will be slowed down by the torque, and $\dot{\phi}$ will eventually *reverse* sign. For the special case where $b(t)$ = const., $H(p,\phi,t)$ is a constant of the motion. Figure 1.2.3a is a graph of $\dot{\phi}^2$ versus ϕ for 1) a value of H such that ϕ is, at some time, a rotating variable, and 2) a value of H for which librates. In the first case, the graph reveals that the path of motion of $\dot{\phi}$ into zero is the same path it follows away from zero. Transition from rotation into libration cannot occur, except for the singular event of a sticking motion. Obviously, if capture is to occur, a non-time-symmetric term must be included in the

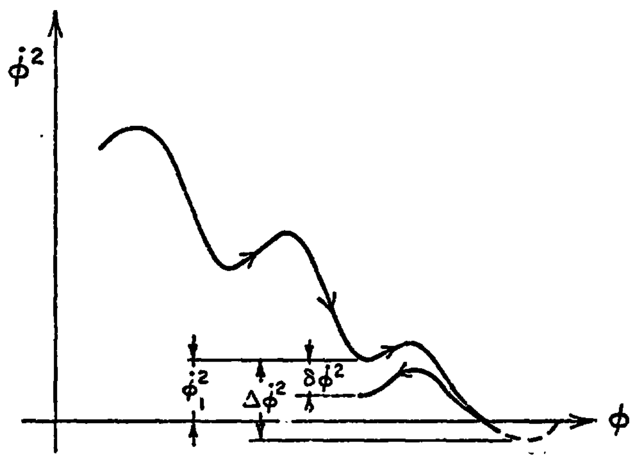


FIGURE 1.2.3b

Diagram of $\dot{\phi}^2$ versus ϕ for a pendulum-like system, subject to a torque, which is asymmetric in time. $\dot{\phi}_i^2$ is the kinetic energy of a pendulum as it goes over the top for the last time, while $\Delta\dot{\phi}_i^2$ is the maximum possible value. $\delta\dot{\phi}^2$ is the kinetic energy after it has reversed direction and again approached the top. Capture occurs if $\delta\dot{\phi}^2 > \dot{\phi}_i^2$.

Hamiltonian. Figure 1.2.3b shows how such a term breaks the time symmetry. From the previous example we can deduce that if $\frac{dc}{dt}$ is constant, $|b(t)|$ must be an increasing function of the time for capture to occur (also see Sinclair, 1972). Also, we except that the criteria for capture will depend on the torque, the function b(t), its derivative, and on the initial conditions. Incidently, in spin-orbit coupling, the time symmetry of (1.2.20) is broken by a velocity dependent torque (see Goldreich and Peale, 1966, and 3.1.10-17).

The above Hamiltonian lacks the simplicity necessary to express ϕ as a function of p. Fortunately it can be transformed to a new Hamiltonian H(x,ϕ,t) which has the requisite simplicity, defined by (cf. 2.9.12):

$$H(x,\phi,t) = 1/2(x + c(t))^2 + b(t)\cos\phi, \qquad (1.2.21a)$$

where

$$x + c(t) = p, \quad c(t) = \int_{t_o}^{t} \frac{dc}{dt}\,dt,$$

and

$$\frac{dx}{dt} = \frac{\partial H}{\partial \phi} = -b(t)\sin\phi, \qquad b)$$

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial x} = -(x + c), \qquad c)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{db}{dt}\cos\phi + \frac{dc}{dt}(x + c). \qquad d)$$

The value of c(t) can be chosen such that $\dot{\phi} = -c$ in the absence of the pendulum force. The variable x then represents a fluctuation in $\dot{\phi}$ caused by the pendulum force.

For c(t) and b(t) constant, the turning points in the motion of x occur at $\phi = \mathrm{mod}(\pi)$, $\mathrm{mod}(2\pi)$, and we can solve for x as a function of t, as was done for the simple pendulum:

$$\int_{x_o}^{x} \frac{dx\ \mathrm{sign}(-b \sin\phi)}{\sqrt{R(x)}} = t - t_o, \qquad (1.2.22)$$

$$R(x) = b^2 - (H - 1/2(x + c)^2)^2.$$

Again, the quartic polynomial R(x) can be factored in terms of its four roots, and the four roots uniquely labeled by the values of ϕ and $\mathrm{sign}(-\dot{\phi})$ for x equal to that root. Allowing c and b to be time dependent does not change this situation in the rotation phase, since the maximum and minimum of x still occur at mod(π) or mod(2π). These roots are

$$x_{\pi\pm} = -c(t) \pm\sqrt{2(H + b(t))} \qquad (1.2.23a)$$

$$x_{2\pi\pm} = -c(t) \pm\sqrt{2(H - b(t))}. \qquad b)$$

The equations of motion for each of the roots are obtained in a manner analogous to that used earlier for the simple pendulum, except that the roots are dependent on three variables: c(t), b(t), and $H(x,\phi,t)$. The equations are

$$\frac{dx}{dt}_{\pi\pm} = \frac{dc}{dt}\left(\frac{x - x_{\pi\pm}}{x_{\pi\pm} + c(t)}\right) + \frac{db}{dt}\left(\frac{1 + \cos\phi}{x_{\pi\pm} + c}\right), \qquad (1.2.24a)$$

$$\frac{dx}{dt}_{2\pi\pm} = \frac{dc}{dt}\left(\frac{x - x_{2\pi\pm}}{x_{2\pi\pm} + c(t)}\right) - \frac{dh}{dt}\left(\frac{1 - \cos\phi}{x_{2\pi\pm} + c}\right). \qquad b)$$

The denominators are equal to the value of $(-\dot{\phi})$ evaluated at x = root. The equations differ from those derived for the simple pendulum in the first term. Like those equations, the motion of the roots is stationary whenever $x = x_{\pi\pm}$ or $x = x_{2\pi\pm}$ and $\phi = \pi$ or $\phi = 2\pi$, respectively. As before, the π-roots lie interior to the 2π-roots. Unlike those equations, however, the roots are not symmetric about the origin but about (-c(t)). Also, the motion of each pair of π and 2π roots is equal in magnitude and opposite in direction about this moving point (-c(t)) (see 1.2.23).

Let's choose x to move between $x_{\pi-}$ and $x_{2\pi-}$ (positive rotation). If we ignore the second term in each equation, then $x_{\pi-}$ moves toward the right and $x_{2\pi-}$ towards the left, implying that the fluctuation, δx (Def: $\delta x = x_{max} - x_{min}$), grows as the system approaches transition. The other pair of roots, besides separating, has a secular motion towards the left of $O(\frac{dc}{dt})$ (see figure 1.2.4).

The transition phase begins, as before, when the two π-roots coincide at time t_i and thereafter become complex. The equations of motion for the π-roots could be separated into their real and imaginary parts, but this procedure can be circumvented here by observing that $x_{\pi\pm} + c$ goes to zero as $H \Rightarrow -b(t)$ and then becomes

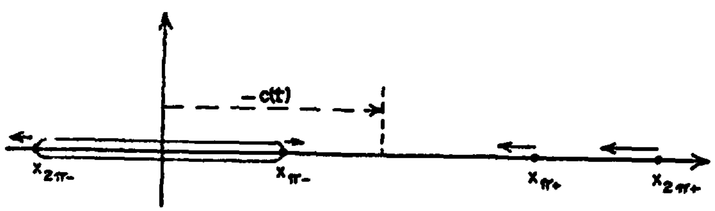


FIGURE 1.2.4 POSITIVE ROTATION PHASE

The arrows indicate the relative velocity of each root.

imaginary (see 1.2.23a). Thus the real and imaginary parts obey the equations:

$$x = \mathrm{Re}\, x + i\, \mathrm{Im}\, x, \tag{1.2.25a}$$

$$\mathrm{Re}\, x_{\pi\pm} = -c(t), \qquad \text{b)}$$

$$1/2 \frac{d}{dt} \mathrm{Im}^2 x_{\pi\pm} = \frac{-dc}{dt}(x - \mathrm{Re}\, x_{\pi\pm}) \frac{db}{dt}(1 + \cos\phi), \qquad \text{c)}$$

We should note that if x is complex, then $\mathrm{Im}\, x$ is real and $\mathrm{Im}^2 x$ is positive definite.

The related integral which determines the value of $\mathrm{Im}\, x_\pi$ at time t_f when x makes the second coincidence with $\mathrm{Re}\, x_\pi$ is

$$1/2\ \mathrm{Im}^2 x_\pi(f) = -\int_{t_i}^{t_f} dt \frac{da}{dt}(x - \mathrm{Re}\, x_\pi) - \int_{t_i}^{t_f} dt \frac{db}{dt}(1 + \cos\phi). \tag{1.2.26}$$

From the previous example we expect that if $\mathrm{Im}^2 x_\pi(f)$ is positive definite, then the π-roots are still imaginary at time t_f and $\dot\phi$ reverses sign, implying transition into libration has occurred. But if $\mathrm{Im}^2 x_\pi(f)$ is negative, it implies that the π-roots returned to the real axis before x reached $\mathrm{Re}\, x$ from the right, and the pendulum executes negative rotations. Therefore $\mathrm{Im}\, x(f) = 0$ corresponds to the sticking motion ($\dot\phi \Rightarrow 0-$) which separates the transition into the libration phase from the transition into the negative rotation phase. Note that this occurs after the first sign reversal of $\dot\phi$ in which $\dot{\phi} \Rightarrow 0+$. This condition (that Im $x(f)$ vanish) is not completely accurate as shall presently be demonstrated. The important point to make here is that the above relation can still be used to find to lowest order the critical initial angle ϕ_{ic} which leads to this sticking motion.

The description of the transition phase is more complex than that defined earlier (figure 1.2.2). The important question to resolve is the relative motion of x with respect to Re x_π for the period of time that the π-roots are complex. We should keep in mind that the major goal is to define the appropriate "transition integral" which can be approximated to first order in the small parameters. There are two small parameters in this system of pendulum plus constant torque: 1) the first parameter is the ratio of the constant torque to the maximum pendulum torque and is small if

$$|b^{-1} \frac{dc}{dt}| << 1; \tag{1.2.27a}$$

2) the second parameter is the ratio of the relative change in b of $O(b^{-1/2} \frac{db}{dt})$ to the initial value $b(t_i)$, and is small if

$$|b^{-3/2} \frac{db}{dt}| << 1. \tag{b}$$

Since the equation of motion of Im x_π^2 is already first order in these small parameters, we expect that the motion of x and ϕ can be replaced by their zero-order motion in calculating the transition integral. Equivalently, the transition phase can be replaced by its lowest order approximation in defining the appropriate integral.

The meaning of this statement shall become clearer as we proceed.

The relative motion of x during transition can be discovered from the simpler case where b(t) is constant. First of all, the explicit time dependence of $H(x,\phi,t)$ can be derived and is (from 1.2.21c,d):

$$H(x,\phi,t) = \text{const.} - \frac{dc}{dt}\,\phi. \tag{1.2.28}$$

The constant in this equation can be chose such that the argument of the radical in (1.2.23a) for the π-roots vanishes when $\phi = \phi_i$. The resulting equation for the π-roots is:

$$x_{\pi\pm} = -a \pm \sqrt{(+2\frac{dc}{dt})(\phi_i - \phi)}, \tag{1.2.29}$$

and the π-roots become complex for $\phi > \phi_i$. The minimum initial angle is mod(π) and shall be chosen for this discussion to equal π. The initial angular velocity $\dot{\phi}_i$ must be ≥ 0; otherwise $\dot{\phi}$ would have previously vanished and reversed sign. Also, if $\dot{\phi}$ vanishes at the moment the π-roots coincide, then ϕ_i must be equal to π. Unless $\phi_i = \pi$, the angle ϕ must increase for $t > t_i$ until ϕ reaches a maximum ϕ_c, at which time $\dot{\phi}$ vanishes, and ϕ thereafter decreases. Eventually ϕ returns to the value ϕ_i at a later time t_j and the π-roots are thereafter real. From (fig. 1.2.3a), we find that when ϕ returns to the value ϕ_i, $\dot{\phi}^2$ also returns to its initial value $\dot{\phi}_i^2$.

In the complex x-plane, $\dot{\phi}$ vanishes when $x = $ Re x_π, and reverses sign as x moves to the right of Re x. After x moves to the right of Re x_π, ϕ decreases. We see that once ϕ returns to the value ϕ_i, x is still to the right of re x_π since $\dot{\phi} < 0$, and the motion of x is trapped between $\pi+$ and $2\pi+$ and has entered the negative rotation phase. This sequence neglects the possibility of a sticking motion. Exactly how this motion of x would appear in the complex plane depends on the relative magnitude and direction of the pendulum torque at $\phi = \phi_i$ as compared to the constant applied torque. The relative motion of x with respect to Re x_π for the time interval $t_i \leq t \leq t_j$, when x_π is complex, is found from the equation of motion of e :h variable, x and Re x_π:

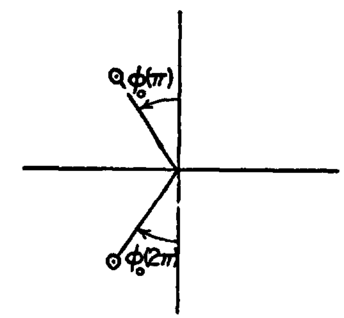
$$\frac{dx}{dt} = -b \sin\phi, \qquad \frac{d \text{ Re } x_\pi}{dt} = -\frac{dc}{dt} . \tag{1.2.30}$$

The two velocities (and the two torques!) are equal for angles $\phi_o(-\pi)$ and $\phi_o(-2\pi)$ given by the relation

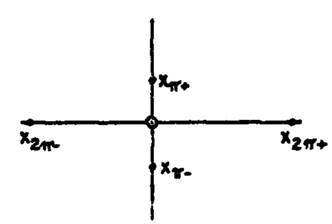
$$\sin\phi_o = b^{-1} \frac{dc}{dt} . \tag{1.2.31}$$

Clearly $|b^{-1} \frac{dc}{dt}| \leq 1$ for these angles to exist. Furthermore, the two stationary solutions (ϕ = constant) of the system correspond to these two angles. Figure 1.2.5a is a "physical" picture of these two stationary solutions, while figures 1.2.5b,c are their equivalent representations in the complex x-plane. Incidently, the distance of each of the roots from re $X_{\pi\pm}$ in figures 1.2.5b,c is obtained from (1.2.23), and

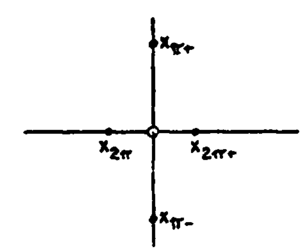
$$H_o = b\cos\phi_o .$$



a) Diagram of stationary positions of "physical" pendulum.



b) Equivalent picture of unstable stationary solution in complex x-plane. $\phi_o(\pi)$ is near π such that $\cos\phi_o \leq -1$.



c) Equivalent picture of stable stationary solution. $\phi_o(2\pi)$ is -2π such that $\cos\phi_o \leq +1$.

FIGURE 1.2.5

Location of stationary solutions of simple pendulum plus torque.

Furthermore, from (1.2.30-31), the relative motion of x is towards $x_{2\pi-}$ if

$$\frac{dx(t_i)}{dt} < \frac{d \operatorname{Re} x_{\pi\pm}}{dt} ,$$

and towards $\operatorname{Re} x_\pi$ otherwise.

Careful consideration of these facts reveals that there are two qualitatively distinct "transition phases" involving the motion of the π-roots and the variable x. Figure 1.2.6a shows the relative motion of x during transition for the case where $\phi_i > \phi_o$, along with the equivalent picture for the real pendulum. This diagram is very similar to the transition phase of a simple time-dependent pendulum without an applied torque (figure 1.2.2). On the other hand, if ϕ_i lies in the range $\pi \le \phi_i \le \phi_o$, then figure 1.2.6b is a picture of the motion during transition. The next question is, which of these diagrams is important?

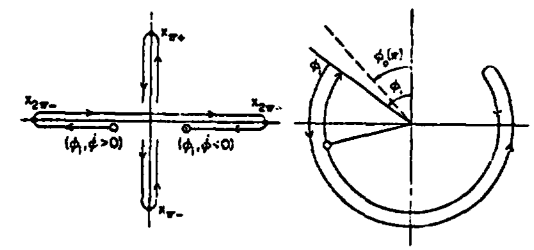
From (1.2.29), $\operatorname{Im}^2 x_\pi(\phi)$ is most positive when ϕ reaches its maximum value at $\phi = \phi_c$ and $\dot\phi$ vanishes. Since $H = b\cos\phi_c$ when $\dot\phi$ vanishes, $\operatorname{Im}^2 x_\pi(\phi_c)$ is also given by

$$\operatorname{Im}^2 x_\pi(\phi_c) = 2|b|(\cos\phi_c + 1).$$

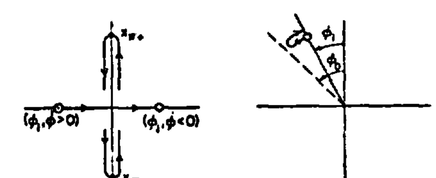
Thus ϕ_i and ϕ_c are related by (1.2.29):

$$\frac{da}{dt}(\phi_c - \phi_i) = |b|(\cos\phi_c + 1). \tag{1.2.32}$$

The sticking motion $\dot\phi \Rightarrow 0+$ must correspond to a motion described



a) Here the initial angle ϕ_i lies in the range $\phi_o \le \phi_i \le 3\pi$. The diagram on the left is a description of transition in the complex x-plane, while the diagram on the right is the equivalent description of the motion of a physical pendulum.



b) In these diagrams, ϕ_i lies in the range $\pi \le \phi_i \le \phi_o(\pi)$.

FIGURE 1.2.6

TRANSITION PHASE FOR PENDULUM PLUS TORQUE WHERE b = CONSTANT by figure 1.2.6b, in which the initial angle ϕ_i (sticking) lies in the range $\pi \le \phi_i \le \phi_o(\pi)$. Otherwise, the pendulum has not just passed over the top prior to the coincidence of the π-roots. Recall that $\phi_o(\pi)$ is the unstable equilibrium position. Since the right hand side of (1.2.32) is a maximum for $\phi_c = \phi_o(\pi)$, it follows that the maximum amount of the π-roots can move off the real axis for the set of transitions defined by figure 1 2.6b is given by the sticking motion. Since $\phi_o - b^{-1}\frac{dc}{dt}$, $Im^2x(st.)$ is of $O([b^{-1}\frac{dc}{dt}]^2)$ and is effectively second order in the small parameter associated with $\frac{dc}{dt}$. But the maximum possible value of $(\quad - \phi_i)$ is 2π, which corresponds to a motion given by figure 1.2.6a. The maximum of Im^2x is therefore of $O(\frac{dc}{dt})$.

This suggests that the following approximations be employed to find the first order motion (in $b^{-1}\frac{dc}{dt}$) of the π-roots: 1) neglect transitions involving figure 1.2.6b for which ϕ_i is in range $\pi \le \phi_i \le \phi_o$, since this set of motions are of second order; 2) for the case $\frac{dh}{dt} \ne 0$, approximate the sticking motion where $\dot{\phi} \Rightarrow 0-$ (and $x = Re\, x_\pi$ by the condition that $Im\, x = 0$ when $x = Re\, x_x$; 3) neglect any effect connected with exceptionally long transition times. With these approximations, the description of the transition phase for the system of pendulum plus torque reduces to that for the simple pendulum (see figure 1.2.2).

The next step is to find the first order approximation to each integral in (1.2.26). For the first integrand, $x + a$ equals $(-\dot{\phi})$.

If $\frac{dc}{dt}$ = constant, then

$$-\int_{t_i}^{t_f} dt\frac{dc}{dt}(x - Re\, x_\pi) = +\frac{dc}{dt}(\phi_f - \phi_i). \qquad (1.2.33)$$

ϕ_i lies in the range $\pi \le \phi_i \le 3\pi$ while ϕ_f equals 3π to lowest order. Also, the contribution to $Im^2x(f)$ from the first integral tends to be negative definite.

The lowest order approximation to the second integral is identical to that found for the simple pendulum (1.2.17) and is

$$-|b|^{-1/2}\frac{db}{dt}\int_{\{\phi_i \le \phi \le 3\pi;\ 3\pi \ge \phi \ge \pi\}} |d\phi \cos \phi/2| =$$

$$-2|b|^{-1/2}\frac{db}{dt}(3 + \sin\phi_i/2). \qquad (1.2.34)$$

Thus the critical initial angle ϕ_{ic} which separates transition into libration from escape into negative rotation satisfies the relation

$$\frac{dc}{dt}(3\pi - \phi_{ic}) - 2| \qquad + \sin\phi_{ic}/2) \quad 0 \qquad (1.2.35)$$

to first order. Capture into libration occurs for ϕ_i in the range $\pi \le \phi_i \le \phi_{ic}$, since the first integral proportional to $\frac{dc}{dt}$ is smaller than the second integral proportional to $\frac{db}{dt}$ for ϕ_i in this range. On the other hand, if ϕ_i lies in the range $\phi_{ic} \le \phi_i \le 3\pi$, then the pendulum has escaped into the negative rotation phase. If ϕ_{ic}, as determined by (1.2.35), is greater than 3π, the implication is that the pendulum will inevitably enter the libration phase, independent of the initial conditions.

Often it happens that the a ue ϕ_i is unknown. In such an instance, a more valuable tool would be a function which describes the probability that capture into libration will occur, given probability densities for the initial angle ϕ_i of the system. The most physically reasonable assignment of prob ility is the following: If we measure the values of ϕ and $\dot{\phi}$ far from transition ($t \to -\infty$), then for a fixed value of $\dot{\phi}(-\infty)$ the angle $\phi(-\infty)$ would be equally distributed in the range $\phi^* \le \phi(-\infty) \le \phi^* + 2\pi$, where ϕ^* is arbitrary. Unfortunately, it is not clear how this statement translates in defining the probability associated with a given value of ϕ_i at transition. For the special case b = constant, the translation is that ϕ_i is equally distributed in range $\pi \le \phi_i \le 3\pi$.

Another parameter which can be assumed to be equally distributed in some closed range for this case is the value of $\dot{\phi}^2$ as the pendulum moves over the top for the last time. Both Goldreich and Peale (1966) and Sinclair (1972) adopt this definition of probability in their respective studies in which the time symmetry is broken, in the former case by a velocity-dependent torque, and in the latter case by a time-dependent coefficient b(t). This second case exactly corresponds to the example being discussed. Since it is always desirable to make contact with others' results, this definition of probability density will be adopted here.

From figure 1.2.3b, the value of $\dot{\phi}^2$ during the last passage of

the pendulum over the top before $\dot{\phi}$ reverses sign is $\dot{\phi}_1^2$ and is equally distributed between 0 and $\Delta\dot{\phi}^2$, where $\Delta\dot{\phi}^2$ is the maximum possible value of $\dot{\phi}_1^2$. In order to define $\Delta\dot{\phi}^2$, we had to analytically continue the graph of $\dot{\phi}^2$ versus ϕ to negative values of $\dot{\phi}^2$ (dashed line). Thus $\Delta^2\dot{\phi}$ is the difference in $\dot{\phi}^2$ between successive minima. The value $\delta^2\dot{\phi}$ is the decrease in the kinetic energy over one revolution as measured at the top, caused by the term which breaks the time-symmetry. Capture just occurs if

$$\dot{\phi}_1^2 = \dot{\phi}_1^2(\phi_{ic}) = \delta\dot{\phi}^2.$$

Therefore, the probability of capture (P_c) is

$$P_c = \frac{\dot{\phi}_1^2(\phi_{ic})}{\Delta\dot{\phi}^2} = \frac{\delta\dot{\phi}^2}{\Delta\dot{\phi}^2} \quad . \qquad (1.2.36)$$

The next step is to relate $\mathrm{Im}^2 x_\pi$, evaluated between appropriate limits, to each of the quantities appearing in (1.2.36). In order to accomplish this, the meaning of negative values of $\dot{\phi}^2$, implied by the continuation of the graph below the ϕ-axis, must be explained. The quantity

$$\dot{\phi}_2^2 = \dot{\phi}_1^2 - \Delta\dot{\phi}^2$$

is a measure of the decrease in the kinetic energy below the value zero as measured from near the top. But $\dot{\phi}_2^2$, to lowest order, equals $\varphi''(\pi)$, which equals:

$$\dot{\phi}^2(\pi \quad (x_1 + c)^2.$$

At coincidence of the x_π roots, $\dot{\phi}(\tau)$ vanishes and is thereafter imaginary. Therefore

$$\dot{;}^2 \quad \cdot^2 \cdot \Delta\dot{\phi}^2 = -\mathrm{Im} \quad \Big|_{\phi_1}^{3\pi} \,. \tag{1.2.37}$$

$\Delta\dot{\phi}^2$ c... ned by letting $\dot{\phi}_1^2$ vanish. Thus, the maximum change in Im ... and is obtained by evaluating $\mathrm{Im}^2 x_\pi$ between the limits ... $= 3\pi$ (where $\dot{\phi} \to 0+$):

$$1/2 \qquad \mathrm{Im}^2 x_\pi \Big|_\pi^{3\pi} = +2\pi\frac{dc}{dt} - 4|b|^{-1/2}\frac{db}{dt}\,. \tag{1.2.38}$$

$\dot{\phi}_1^2(\phi_{1c})$ equals the maximum decrease in $\mathrm{Im}^2 x_\pi$ minus the decrease $\mathrm{Im}^2 x_\pi(\phi_{1c})$, or

$$1/2\ \dot{\phi}_1^2(\phi_{1c}) = 1/2\ \mathrm{Im}^2 x_\pi \Big|_\pi^{3\pi} - 1/2\ \mathrm{Im}^2 x_\pi \Big|_{\phi_{1c}}^{3\pi} \tag{1.2.39}$$

$$= \frac{dc}{dt}(3\pi - \phi_{1c}) - 2|b|^{-1/2}\frac{db}{dt}\left(-1 + \sin\left(\frac{\phi_{1c}}{2}\right)\right).$$

Using (1.2.35) to eliminate the explicit dependence on ϕ_{1c}, the probability is

$$P_c = \frac{2}{1 - \pi/2\left(|b|^{-1}\frac{dc}{dt}\right)\left(|b|^{-3/2}\frac{db}{dt}\right)^{-1}}. \tag{1.2.40}$$

P_c is zero if $\frac{db}{dt} = 0$, while P_c approaches its maximum value of unity when the ratio of the small parameters of the system is of O(1). Also, it should be pointed out that the above formula does not apply to the special case $\frac{dc}{dt} = 0$ since (1.2.35) is then invalid. In

section 3.1 we shall find that the estimate of capture probability for possible resonance associated with the Mimas-Tethys commensurability agree with the numerical calculations of Sinclair.

Before concluding this discussion, there is one more interesting feature we shall investigate, related to transition from the positive rotation into the negative rotation phase, and it is described by the following: If we look at the system of pendulum plus torque far from transition, at approximately equal time intervals before and after transition, we observe that there is a secular change in the mean value of x, or equivalently, in the mean value of $\dot{\phi}$. That is, if we measure the mean value of $\dot{\phi}$ in the positive rotation phase far from transition and find that it is equal to, say, $(-c(t_a))$ at time t_a (where $t_a \Rightarrow -\infty$), then the result of performing a similar measurement in the negative rotation phase at time t_b (where $t_b \Rightarrow +\infty$) shall be:

$$\langle\dot{\phi}\rangle_{\text{neg.rot.}} = -\Delta x - c(t_b),$$

where Δx is this secular change. The value of Δx can be determined using 1) the action integral and 2) the condition $\langle\dot{\phi}(t_a)\rangle_{\text{pos.rot.}} = -c(t_a)$. The time average of $\dot{\phi}$ at time t_a is

$$\langle\dot{\phi}(t_a)\rangle_{\text{pos.rot.}} = \frac{1}{T(t_a)}\oint(x + c)\,dt = -\langle x\rangle_{\text{pos.rot.}} - c(t_b). \tag{1.2.41}$$

Applying the second condition, we find

$$\langle x(t_a)\rangle_{pos.rot.} = 0. \qquad (1.2.42)$$

As the mean angular velocity decreases and the system approaches transition, $\langle x\rangle$ tends to increase and move toward the x_π-root. This is a consequence of the fact that the system tends to spend more time near the top or π position of the pendulum motion. At transition, the mean value of x is $\sim$ Re x_π.

The next step is to evaluate the action integral J in the positive rotation phase far from transition:

$$J_{pos.rot.} = \oint_{pos.rot.} x d\phi = \oint_{pos.rot.} x\dot{\phi}dt. \qquad (1.2.43)$$

Since the flucturations in $\dot{\phi}$ tend to vanish the further the system is from transition, $\dot{\phi}$ is approximately constant, and $J_{pos.rot.}$ vanishes since it is approximately proportional to $\langle x\rangle$.

The action integral is an adiabatic constant (Born, 1925) in each phase, as long as the instantaneous frequency of the pendulum is large compared to the slow changes in the system induced by the torque $\frac{dc}{dt}$ and the time-dependent coefficient $b(t)$. In the examples being discussed, the restriction is violated only when the π-roots are so close that the instantaneous frequency is very small. Recall that the instantaneous frequency for the first example tended to blow up logarithmically as a function of the π-root separation. In addition, we have found that the roots do not move very much during transition except for the situation where the transition time is exceptionally long. Furthermore, the instantaneous frequency

rapidly increases for small changes in the relative separation of the π-roots. Therefore J can be calculated at transition to determine the secular change in the time-dependent functions. At time t_i, $H_i = b(t_i)$ and J is:

$$J_{pos.rot.} = 0 = \int_0^{2\pi} x d\phi = \int_0^{2\pi}(-\dot{\phi})d\phi - 2\pi c(t_i)$$

$$\doteq -4|b(t_i)|^{1/2}\int_0^{2\pi}|\cos\phi/2| - 2\pi c(t_i), \qquad (1.2.44)$$

or $\quad J_{pos.rot.} \doteq -8|b(t_i)|^{1/2} - 2\pi c(t_i) \doteq 0.$

After the pendulum has made the transition into negative rotation at time t_f, J can again be calculated.

$$J_{neg.rot.} = \int_0^{2\pi} x d\phi = \int_0^{2\pi}(-\dot{\phi})d\phi - 2\pi c(t_f). \qquad (1.2.45)$$

Unless $t_f - t_i$ is exceptionally large, $c(t_i) \doteq c(t_f)$, and $b(t_i) \doteq b(t_f) \doteq H(t_f)$. The result is

$$J_{neg.rot.} = +8|b(t_i)|^{1/2} - 2\pi c(t_i), \qquad (1.2.46)$$

and $J_{neg.rot.}$ is nonzero. Far from transition, the mean value of x tends to Δx in the negative rotation phase, and the value of $J_{neg.rot.}$ corresponds to $2\pi\Delta x$. Thus

$$\Delta x \doteq \frac{8}{\pi}|b(t_i)|^{1/2}. \qquad (1.2.47)$$

For a simple pendulum, $(-\Delta x)$ corresponds to the delay in the evolution of $\langle\dot{\phi}\rangle$ by the pendulum potential due to the constant applied torque. In the orbit-orbit interaction, the variable x is related to fluctuations in the orbital elements a,e, and I (see section 2.1). This means that there is a secular change in the orbital elements associated with passage through resonance, not entir ly connected to the tidal interaction.

The purpose of this investigation has been to develop a description of transition along with some analytical tools which shall prove useful when applied to the Hamiltonian governing the orbit-orbit interaction. Before we proceed to discuss transition for the orbit-orbit case, the Hamiltonian which approximates this interaction will be derived.

2.' NEWTONIAN THEORY FOR PLANETARY SYSTEMS

The development of a one-dimensional Hamiltonian consumes more pages than anyone in his right mind would want to read. This exercise is, for the most part, a rehashing of old material to make it suit our own purposes, and breaks down into four stages. First is the formulation of the many-body planetary problem in terms of a disturbing function, acting on each planet, which is distinct from the more common potential function. In the planetary problem, the principal interaction of the planets is wi n the sun, and is the major factor in determining their orbits. The planet-planet (or satellite-satellite) interaction can be developed as a perturbation on the two-body planet-sun or satellite-planet orbit. Instead of considering perturbations on the coordinates, it is more useful to find the perturbations of the two-body "constants" of the motion.

The second stage is the expansion of the disturbing function in terms of the Keplerian elements of the two-body orbit. It is amazing that the resulting expansion is of any use since it is so complex, but useful approximations can be more readily invoked with the disturbing function in this form. The terms in the expansion are classified, and the relative importance of each class is discussed. We determine the restrictions which must be imposed on these classes such that a single hypothetical resonance variable will dominate the long-term behavior of the system. Also, a procedure is outlined for the analytic elimination of the short-period terms order by order in powers of a "small" expansion parameter. This serves two purposes. First, it explicitly shows that the effect of such terms on the resonance is of second order. We see that these terms have a minor influence on the two-body resonances discussed later in (4.1); on the other hand, in the lunar resonance problem discussed in (4.?) the second-order mixing of short-period terms is important, because the sun can be a substantial indirect participant in the moon-earth-planet resonance. Second, we see that the terms in the expansion which look like a sum of pendulum-like potentials, all with different angles, is not entirely an accident of the expansion procedure -- a libration of one of these angles is implicitly possible.

Once satisfied that the original interaction can be reduced to one which involves only one angle variable, we then show that the system of equations can be reduced, in most cases, to a single canonical set $\{x,\phi\}$. The method used is similar to the first step in Delauney's solution to the sun-moon interaction (Brown 1960, p. 140). The Hamiltonian derived is a constant of the motion in the absence of any dissipative tidal interactions.

Our last act is to interject the tides into the tide-free Hamiltonian just developed. Essentially this is accomplished by subtracting out the secular effect on the canonical variables. Finally, we summarize the results in (2.10). Also, the relation between our Hamiltonian and the second-order equation of motion

derived independently by Allan and Sinclair is discussed, to instill confidence that no serious flaws exist between beginning and end. The beginning, of course, is Newton's laws of gravitation.

The gravitational force between two bodies depends on their mass, shape, and the distance between them. Most celestial bodies approach sphericity, allowing the gravitational force to be approximated by that between two point masses. Given a set of n interacting point masses, the forces acting on the i'th body are additive and individually derivable from a potential:

$$\vec{F} = m_i \frac{d^2\vec{r}_i}{dt^2} = -m_i \vec{\nabla}_i V^i , \tag{2.1.1}$$

where $\vec{\nabla}_i$ operates on the coordinates of m_i and the potential V^i is

$$V^i = -\sum_{j \neq i}^{n} \frac{\mu_j}{r_{ij}} , \quad \mu_j = G_m . \tag{2.1.2}$$

For two bodies the path of motion each describes is a conic section, either an ellipse, a hyperbola or a parabola. If the system is bound, the shape and size of the ellipse are specified by its eccentricity e and semimajor axis a. Its orientation with respect to a reference frame is given by the Euler angles Ω, I, ω (cf. f. 2.1.1). These symbols and their definitions are peculiar to astronomy. Ω is the "longitude of the ascending node," ω is the "argument of pericenter," whereas I is the "inclination". Although not a physical angle, another frequently used "broken angle" is $\tilde{\omega}$, defined by

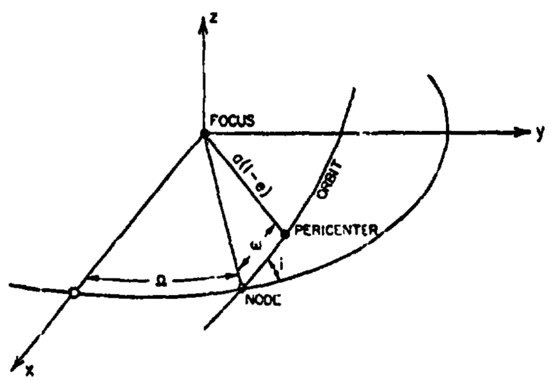


FIGURE 2.1.1. ORBITAL ORIENTATION

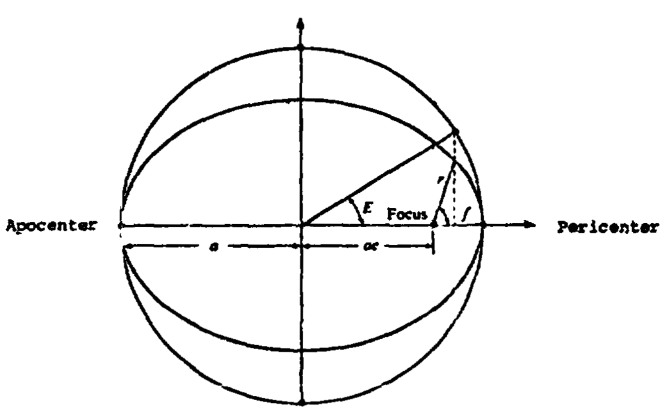


FIGURE 2.1.2. ELLIPTIC VARIABLES

$$\tilde{\omega} = \omega + \Omega, \qquad (2.1.3)$$

and called the "*longitude of pericenter*".

The position of a body on this ellipse, with respect to the pericenter, can be specified by its true anomaly f, eccentric anomaly E (fig. 2.1.2), or mean anomaly M. The angles E and f are related to the distance r by

$$r = a(1 - e \cdot \cos E) \qquad (2.1.4a)$$

$$r = \frac{a(1 - e^2)}{1 + e \cdot \cos f} \qquad b)$$

The mean anomaly is defined by

$$M = n(t - \tau) = E - e \cdot \cos E. \qquad (2.1.5)$$

In most analytical work, M is the most useful anomaly, since it is a linear function of the time in the absence of perturbations. The element τ is the time of perihelion passage and is the sixth constant which fully specifies the two-body system. Instead of τ, another choice for the sixth constant is , the epoch, which is defined by

$$n\tau = \varepsilon - \tilde{\omega} \qquad (2.1.6)$$

The constant n is the "mean motion" and is related to the semimajor axis by

$$\mu_o = G(m_o + m) = n^2 a^3, \qquad (2.1.7)$$

which is recognized as Kepler's Third Law.

The general problem of three interacting bodies is still unsolved.

In a planetary system, one body is much more massive than any of the others, predominantly determining the path of motion of all the other bodies. This fact suggests that a zero-order solution for each secondary mass would be an ellipse whose focus is at the center of mass of the two-body system (primary and secondary), the mutual interactions between the secondaries being ignored. The effect of the secondary interactions can be developed as perturbations on the zero-order ellipse, in which the six constants just described $(a,e,I,\varepsilon,\Omega,\omega)$ become variables. The method is known as "the variation of arbitrary constants" (Brouwer and Clemence, 1961a, pp. 273-307). The six differential equations of the elements are first order in time, compared to the three equations of the coordinates, which are second order.

The first step towards a solution is to expand the potential function (2.1.1) in terms of the orbital elements of the i'th disturbed body and the other n-2 disturbing bodies. To accomplish this goal, it is convenient to choose the primary mass as the coordinate origin and subtract the motion of the primary (m_o) caused by the disturbing mass. The result of such an operation is the equation of motion of the relative position vector $\vec{r}_i$:

$$\frac{d^2\vec{r}_i}{dt^2} + \vec{\nabla} V_o^i = + \vec{\nabla} R^i, \qquad (2.1.8a)$$

where it is understood that $\vec{r}_i = \vec{r}_{io}$. The two functions V_o^i and R^i are:

$$V_o^i = - \frac{(m_o + m_i)G}{r_i} \qquad b)$$

$$R^i = \sum_{j\neq i}^{n} R_j^i = \mu_i \sum_{j\neq i}^{n} \left(\frac{1}{r_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{|r_j^3|}\right). \qquad c)$$

R_j^i is known as the disturbing function of the i'th body due to the action of the j'th body, and has the opposite sign from that commonly assigned a potential function. Its parts are called the Direct and Indirect terms respectively. In the absence of any disturbing body except the primary, $R^i = 0$ and (2.1.8a) describes the motion of the i'th mass with respect to the center of mass of the two-body system.

2.2 ANALYTICAL DEVELOPMENT OF THE DISTURBING FUNCTION

The disturbing function acting on any given secondary is the sum of the individual disturbing functions due to other secondaries. These may be other satellites, planets, or even the sun itself if the given secondary be a "moon" of a planet. Our primary concern is to understand the two-body satellite-satellite resonances of Saturn in which the important perturbations involve a single two-body interaction between the partners of the resonance. The critical development involves the expansion of $\Delta^{-1} = |\vec{r}_1 - \vec{r}_2|^{-1}$. The inverse separation, Δ^{-1} in terms of r_1, r_2, and , the angle between $\vec{r}_1$ and $\vec{r}_2$, is

$$\Delta^{-1} = \sum_{\ell=0}^{\infty} \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell} P_\ell(\cos\Theta), \qquad (2.2.1)$$

(Jackson, 1962, p. 62), where $r_>$ is the greater and $r_<$ is the lesser of r_1 and r_2. If the orbits are coplanar, Θ is the difference in true logitudes of $\vec{r}_1$ and $\vec{r}_2$

$$\Theta = L_1 - L_2 \qquad (2.2.2a)$$

where

$$L \equiv f + \omega + \Omega = f + \varpi. \qquad b)$$

If the orbits are also circular, then Θ reduces to $\Theta = \lambda_1 - \lambda_2$, where λ is called the "mean longitude" and is defined by

$$\lambda \equiv M + \varpi. \qquad (2.2.3)$$

In general, orbits are not coplanar and the relationship of Θ to the Euler angles is much more complicated.

The next step is to perform the spherical harmonic expansion of the Legendre polynomical to relate the vectors $\vec{r}_1$ and $\vec{r}_2$ to a common reference frame (ibid.) through their spherical coordinates (r,θ,ϕ) (fig. 2.2.1):

$$P_\ell(\cos\quad) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta_1,\phi_1)Y_{\ell m}(\theta_2,\phi_2) \qquad (2.2.4a)$$

where

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P^m_\ell(\cos\theta)e^{im\phi} \qquad b)$$

and the associated Legendre function is defined by

$$P^m_\ell(x) = \frac{(-)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell. \qquad c)$$

where the * superscript indicates the complex conjugate operation. The spherical harmonics can be related to the Euler angles $(\Omega, I, f+\omega)$ through an explicit expansion in which the trigonometric relations between (θ,ϕ) and $(\Omega, I, f+\omega)$ are utilized (Kaula, 1966, pp. 30-35). Another approach is to use the group properties of the spherical harmonics under rotations (Iszak, 1964). The results are:

$$Y_{\ell m}(\theta,\phi) = \left(\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}\right)^{1/2}$$

$$\sum_{p=o}^{\ell} F_{\ell mp}(I) e^{i[(\ell - 2p)(f + \omega) + m\Omega]}, \qquad (2.2.5a)$$

where

$$F_{\ell mp}(I) = i^{\ell - m} \frac{(\ell + m)!}{2^{\ell} p! (\ell - p)!} \sum_{k} (-)^k \binom{2\ell - 2p}{k}$$

$$\binom{2p}{\ell - m - k} \gamma^{2\ell - \nu} \sigma^{\nu}, \qquad b)$$

$$\gamma = \cos\frac{I}{2}, \quad \sigma = \sin\frac{I}{2}, \quad \nu = m - \ell + 2p + 2k; \quad i = \sqrt{-1},$$

and k is summed over all non-negative factorials (cf. Allan, 1967). The inclination function $F_{\ell mp}$ has an important symmetry property which relates the coefficients of angles that differ only in sign:

$$F_{\ell mp}(I) = (-)^{\ell - m} \frac{(\ell - m)!}{(\ell + m)!} F_{\ell,m,\ell - p}(I). \qquad (2.2.6)$$

Therefore, the coefficients of the angle $\{(\ell - 2p)(f + \omega) + m\Omega\}$ and its negative are identical except for sign. Since (2.2.4a) involves products products of spherical harmonics with the same ℓ and m, the sign of the product of coefficients will be the same for angles which differ only in sign. The basic properties of $F_{\ell mp}(I)$ for small I are

$$F_{\ell mp}(I) \sim O(I^{|\ell - m - 2p|}), \qquad (2.2.7a)$$

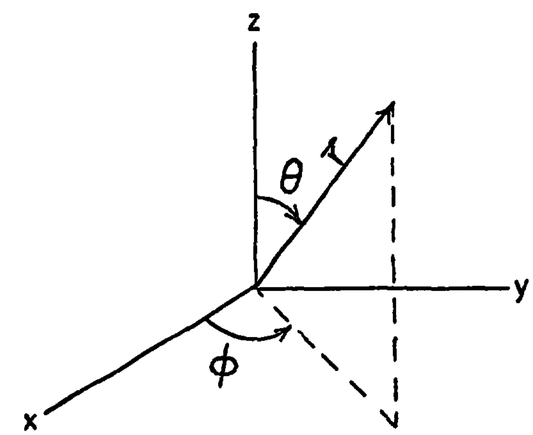


FIGURE 2.2.1

SPHERICAL COORDINATES OF VECTOR $\hat{r}$

$$F_{\ell mp}(0) = \delta_{\ell - 2p, m} P_\ell^m(0) = \delta_{\ell - 2p, m} \cos(p\pi)$$

$$\frac{(\ell + m)!}{2^\ell p!(\ell - p)!} \qquad \text{b)}$$

The final expansion relates the true anomaly f and the distance r to the mean anomaly M. From equations (2.2.1), (2.2.4a), and (2.2.5a) we see that the function which must be expanded is $\{(\frac{r}{a})^s \exp i\,(tf)\}$. Hansen's coefficients are defined by the relation:

$$\left(\frac{r}{a}\right)^s \exp i\,(tf) = \sum_{q=-\infty}^{\infty} X_{s,t}^q(e) \exp i\,(q + t)M. \qquad (2.2.8)$$

The expansion of Hansen's coefficients in a power series in e is fairly complicated and can be found in Plummer (1960, p. 44). Table (4.2.1) gives a few typical values to lowest order in e, while more extensive tabulations are published by Cayley (1961). The basic properties of $X_{s,t}^q(e)$ are

$$X_{s,t}^q(e) = X_{s,-t}^{-q}(e) \qquad (2.2.9\text{a})$$

$$X_{s,t}^q(e) \sim O(e^{||t| - |q||}) \qquad \text{b)}$$

$$X_{s,t}^q(0) = \delta_{tq}. \qquad \text{c)}$$

Using these expansions (2.2.1, 2.2.4, 2.2.5a, 2.2.8) we find the direct part of the disturbing function R_2^1 (2.1.8) to be

$$R_{2D}^1 = \frac{\mu_2}{\Delta} = \mu_2 \sum_{\ell=0}^{\infty} \frac{1}{a_>} \left(\frac{a_<}{a_>}\right)^\ell \sum_m \frac{(\ell-m)!}{(\ell+m)!} \sum_{p_1 p_2} F_{\ell m p_1}(I_1) F^*_{\ell m p_2}(I_2) \cdot$$

$$\sum_{q_1 q_2} X_{\ell, \ell-2p_1}^{\ell+q_1-2p_1}(e_1)\, X_{-(\ell+1), \ell-2p_2}^{\ell+q_2-2p_2}(e_2)\, e^{i\phi_{\ell m p_1 p_2 q_1 q_2}}. \qquad (2.2.10)$$

$$\phi_{\ell m p_1 p_2 q_1 q_2} = (\ell-2p_1)(M_1+\omega_1) - (\ell-2p_2)(M_2+\omega_2) + m(\Omega_1-\Omega_2)$$
$$+ q_1 M_1 - q_2 M_2.$$

This expansion can be reduced to a cosine series from the symmetry relations (2.2.7, 2.2.9).

The expansion of the indirect part of R is, from (2.1.8),

$$R_{21}^1 \equiv -\mu_2 \frac{r_1}{r_2^2} \cos\Theta \qquad (2.1.11\text{a})$$

$$= -\mu_2 \frac{a_1}{a_2^2} \sum_{m,p,q} \frac{(1-m)!}{(1+m)!} F_{1mp_1}(I_1) F_{1mp_2}(I_2)$$

$$X_{1,\ell-2p_1}^{1+q_1-2p_1}(e_1)\, X_{-2,\ell-2p_2}^{1+q_2-2p_2}(e_2)$$

$$e^{i\phi_{1mp_1p_2q_1q_2}},$$

and the sums are restricted to the terms where

$$\ell = 1; \quad m = \{-1, 0, 1\}; \quad p_1, p_2 = \{0, 1\} \qquad (2.2.11\text{b})$$

The qualitative dependence of R_2^1 for small e and I can be deduced from (2.1.6) and (2.1.8). Formally, R_2^1 is

$$R_2^1 = \sum_{\phi} \frac{\mu_2}{a_>} C I_1^{|\ell - m - 2p_1|} I_2^{|\ell - m - 2p_2|} e_1^{|q_1|} e_2^{|q_2|} \cos\phi_{\ell m p_1 p_2 q_1 q_2} \qquad (2.2.12)$$

where C is a series in I^2, e^2 and $\alpha = \frac{a_<}{a_>}$, and is formally presented by:

$$C = \sum_{\sigma = \ell_{mn}}^{\infty} \sum_{\substack{\mu_1 = 0 \\ \mu_2 = 0}}^{\infty} \sum_{\substack{\nu_1 = 0 \\ \nu_2 = 0}}^{\infty} C_{\sigma\mu_{1,2}\nu_{1,2}} \alpha^{\sigma} I_1^{2\mu_1} I_2^{2\mu_2} e_1^{2\nu_1} e_2^{2\nu_2}. \qquad (2.2.13)$$

The restriction on the sums in (2.2.13) are that $\ell - 2p$, m and q are constant (i.e. $\phi_{\ell m p_1 p_2 q_1 q_2}$ is fixed). The leading term is of order $\alpha^{\ell_{min}}$, where ℓ_{min} is the smallest value of ℓ consistent with the argument of the cosine and the range of values for ℓ and p. Terms in C which contain e or I are at least of order e^2 or I^2 smaller than the leading terms which only contain factors of α. The important point, which will be demonstrated later (2.6.10,11), is that any variation in C (due to a variation of the orbital parameters) is of $O(e^2$ or $I^2)$ smaller than the corresponding variation of the leading factors. This implies that the variation of C with respect to x can be neglec[illegible]. [illegible]ly,

the secular term of the disturbing function is a polynomial series in powers of α, I^2 and e^2, and this term has no leading factors of e or I.

Obviously, the expansion of R_2^1 is a very complicated function of the orbital parameters, and would not be very useful if it were not the case that the perturbations produced by the disturbing body are small, and, furthermore, that the major portion of the variation of the factor multiplying the cosine function is determined by the leading factors of e and I, as demonstrated in (2.6.1.2).

The expansion as outlined does have one defect. For the satellite-satellite interaction (or planet-planet) the ratio $(\frac{a_<}{a_>})$ is not much different from one. Instead of expanding Δ^{-1} using (2.2.1), we can expand it directly in a cosine series in Θ. The result of this operation is

$$r_> \Delta^{-1} = 1/2 \sum_{j=-\infty}^{+\infty} b_{1/2}^{j}(\alpha)\cos(j\Theta), \quad \alpha = \frac{r_<}{r_>}, \qquad (2.2.14a)$$

and the Laplace coefficient $b_{1/2}^1(\alpha)$ is obtained from the integral

$$b_{1/2}^{j}(\alpha) = k_j \frac{r_>}{\pi} \int \frac{\cos(j\Theta)\,d\Theta}{\sqrt{r_>^2 + r_<^2 - 2r_>r_>\cos\Theta}}; \; K_j = 1 - 1/2\delta_{j,o} \, . \qquad b)$$

The drawback with this approach is that Θ is a complicated function of the Euler angles, and α is a function of the mean anomalies. Subsequent expansions which reduce the disturbing functions to a function of the elements $\{a, e, I; \lambda, \tilde{\omega}, \Omega\}$ comprise a very tedious exercise, as anyone familiar with the topic well knows (cf. Plummer, 1960, pp. 133-48). *The resulting expression is formally* identical to the expansion given by (2.2.12,13) in that the leading factors of e and I in any given term can be factored out and the remaining sum of terms can be lumped together as the coefficient C. The only difference would be that C is not a power series in α, I^2 and e^2, but a series in which the sum over α is expressed as functions of Laplace coefficients.

The first approach outlined is usually reserved for the expansion of the non-spherical geopotential acting on an artificial satellite (cf. Allan, 1967). The ratio $\frac{h}{a}$ enters in the expansion where R is the *earth's mean radius* and a is the semimajor axis of the satellite, but the *problem of convergence in powers of* $\frac{R}{a}$ *is* avoided because of the functional dependence on the appropriate multipole moment in each term of the disturbing function. Rapid convergence of the expansion is obtained, because the multipole moments decrease rapidly with the order of the moment.

A mixture of the two procedures can be applied to the expansion of the disturbing function of a planet acting on the moon, with a definite reduction in the amount of labor usually required (Brown, 1960, p 252). Higher order Laplace coefficients appear in the expansion and are defined by

$$(a_{>}\Delta^{-1})^{2s} = 1/2 \sum_{j=-\infty}^{+\infty} b_s^j(\alpha) \exp i\,(j\,\Theta\,). \qquad (2.2.15a)$$

The series expansion of $b_s^j(\alpha)$, which can be used to numerically evaluate these coefficients, is given by

$$b_s^j(\alpha) = 2\alpha^j \frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)} F(s, s+j, j+1; \alpha^2), \qquad b)$$

where the function $F(s, s+j, j+1; \alpha^2)$ is the hypergeometric series

$$F(s, s+j, j+1; \alpha^2) = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)\Gamma(j+1+n)}{\Gamma(s+j+n)} \alpha^{2n}$$

(cf. Plummer, 1960, p. 158).

The important point to emphasize is that the formal expansions of the satellite-satellite interaction, the lunar-planetary interaction, and the interaction of an artificial satellite with a non-spherical geopotential lead to similar terms and that the behavior of these systems near or in a *"resonance"* where one of these terms dominates the disturbing function is *essentially the same*.

The next series of operations involves the introduction of a Hamiltonian whose variables are functions of the orbital elements rather than of the coordinates and momenta. Eventually this Hamiltonian is reduced to a single degree of freedom and a constant of the motion for the tide-free case. To aid this process, let's discuss the various types of terms in R_2^1 and the different types of resonance variables which can occur.

2.3 *CLASSIFICATION SCHEMES*

The art of classification is to impose an order on a hodgepodge of material, be it physical objects or ideas or terms of a disturbing function, using criteria which suggest a meaningful way of thinking about that material. The first scheme presented classifies these terms using the relative period corresponding to the cosine argument as the principal criterion. Its purpose is to suggest which kinds of terms can be handled using standard perturbation techniques and which cannot. Those terms which cannot be removed may be ignored if they satisfy the basic criterion that their corresponding coefficient of the cosine is much smaller than the coefficient of the given "resonant" term. Otherwise the system cannot be approximated by a Hamiltonian having one degree of freedom. Incidently, a "resonance term" is a term in the expansion of R whose cosine argument is nearly constant due to special values of the orbital angular parameters. This term is distinct from "secular" terms whose cosine arguments are identically zero. We shall define a "resonance variable" as the argument of such a resonant term.

The Hamiltonian contains an infinite number of terms which may or may not affect the resonance. A useful classification in treating the "non-resonant" terms is the following:

a) Short Period Terms. These are terms which explicitly contain λ and have periods on the order of (or less than) the period of disturbed or disturbing bodies.

b) Long Period Terms. Those terms with arguments which do not contain λ, but do contain $\tilde{\omega}$ or Ω, are in this category. The largest terms in this class have two powers of e or I in the coefficient of the cosine.

c) Secular Terms. The secular terms are those for which $\phi \equiv 0$. Only even powers of the e and I occur as factors. If the Hamiltonian is reduced to terms of this class, then the action variables (defined later) become constants and the angle variables are linear functions of the time.

There are still terms whose cosine argument is not some multiple of the resonant angle and which do not belong to any of the classes just defined but have long periods. They are of two types:

a) Those terms which "almost" satisfy the commensurability condition $j_1n_1 + j_2n_2 \sim 0$, but which have different j_1, j_2 than the resonance variable. If the integer pair is much larger than the resonant pair, then its coefficient will be much smaller than the resonant coefficient, and for this reason is ignored. If this is not the case, these terms must be directly compared with the term (or terms) containing the resonance variable.

b) If an angle differs from the resonant angle by a function of the slow angle variables $\tilde{\omega}$ or Ω, then the period of that angle will be of the order of the slow variables near resonance. They can be neglected if either the coefficient of such terms is much smaller than the resonant term or if the period of the slow variable is relatively fast compared to the libration period of the resonance variable.

There are several types of resonance angles which shall be discussed in detail. Below is a list of these types, along with a list of specific examples.

a) Purely Synodic: $\ell - 2p_{1,2} = m$; $q_{1,2} = 0$; $\phi = m(\lambda_1 - \lambda_2)$. The Trojan asteroids librate about the Lagrangian triangular points of Jupiter with a resonance angle of the above type (Brown and Shook, 1964, ch. 9). Unfortunately, the perturbation expansion of the type developed in (2.1) cannot be used for the Trojan resonance problem.

b) Simple e Type: $\ell - 2p_{1,2} = m$; $q_2 = 0$; $\phi = m(\lambda_1 - \lambda_2) + q_1 M_1$. Most known resonances fall into this class. The Enceladas-Dione ($\lambda_{En} - 2\lambda_{Di} + \tilde{\omega}_{Di}$), The Titan-Hyperion ($4\lambda_{Hy} - 3\lambda_{Ti} - \tilde{\omega}_{Hy}$), and the Neptune-Pluto ($3\lambda_p - 2\lambda_N - \tilde{\omega}_P$) resonances are well-known examples. The leading terms in these types of resonances contain a factor of $e^{|q|}$. Actually, the class of observed e-types is restricted to the $|q| = 1$ case.

c) Simple I type: $q_1 = q_2 = 0$; $\ell - 2p_2 = m$; $\phi = (\ell - 2p_1)(f_1 + \omega_1) + m\Omega_1 - m\lambda_2$. There are no known examples of this type.

d) Mixed I type: $q_1 = q_2 = 0$; $\phi = \{(\ell - 2p_1)(f_1 + \omega_1) - (\ell - 2p_2)(f_2 + \omega_2) + m(\Omega_1 - \Omega_2)\}$. The only example is the Mimas-Tethys resonance ($2\lambda_{Mi} - 4\lambda_{Te} + \Omega_{Mi} + \Omega_{Te}$).

Other types are the mixed e-type and mixed type whose definitions are obvious. No naturally occurring examples belong to either type.

2.4 HAMILTON'S EQUATIONS

The Hamiltonian which describes the motion of a disturbed body can be written as

$$H = H_o + R; \quad H_o = -1/2\ \vec{v}\cdot\vec{v} + \frac{\mu_o + \mu_i}{r} = \frac{\mu_o + \mu_i}{2a}, \qquad (2.4.1)$$

where H_o is the Hamiltonian of the unperturbed two-body system of primary and secondary. (Note again that H_o is the negative of its usual counterpart in ordinary mechanics. Of course, the sign has no significance except at tradition.) The canonical elements of H are the conjugate momenta and position coordinates. H is not a constant of the motion since the coordinates of the disturbing body are contained explicitly in R. Since H_o supposedly dominates R, the motion of the disturbed body can be described in terms of the variation of an instantaneous ellipse which determines its position and velocity. The simplest method for deriving a canonical set of conjugate action and angle variables is a method involving the Hamilton-Jacobi equation (cf. Appendix A).

Several sets of canonical variables have been derived (Hagihara, 1970, pp. 526-555). The set chosen for this discussion is known as the modified Delauney variables $\{L, \Gamma, Z, [illegible], \tilde{\omega}, \Omega\}$. The angle variables have already been defined. The action variables are related to the Keplerian elements by:

$$L = \sqrt{\mu_o a}$$

$$\Gamma = L(\sqrt{1 - e^2} - 1) \qquad (2.4.2)$$

$$Z = L\sqrt{1 - e^2}\ (\cos I - 1)$$

and we shall adopt the convention that μ_o replaces $\mu_o + \mu_i$ in (2.4.1). The H-J equations in these variables are

$$\frac{dL}{dt} = \frac{\partial H}{\partial \lambda} \qquad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial L}$$

$$\frac{d\Gamma}{dt} = \frac{\partial H}{\partial \tilde{\omega}} \qquad \frac{d\tilde{\omega}}{dt} = -\frac{\partial H}{\partial \Gamma} \qquad (2.4.3)$$

$$\frac{dZ}{dt} = \frac{\partial H}{\partial \Omega} \qquad \frac{d\Omega}{dt} = -\frac{\partial H}{\partial Z}\ .$$

The above set has the advantage that Z is an approximate constant of the motion for [illegible] -type resonance, and Γ is a constant for an I-type. Also, if [illegible] and [illegible] small, then

$$\Gamma \simeq -[illegible]2e^2 L, \quad Z \simeq -1/2I^2L \qquad (2.4.4)$$

or both Γ and Z are very small quantities compared to L.

If $R \equiv 0$, then all of the variables except λ are constants of motion. From (2.2.3) and (2.4.1-3), the equation for $\frac{d\lambda}{dt}$ is

$$\frac{d\lambda}{dt} = -\frac{\partial H_o}{\partial L} = n = \text{constant} \qquad (2.4.5)$$

and λ is a linear function of time.

The disturbing function contains many terms other than the resonant term. It is important that their effect on the resonance and the meaning of the approximation where they are neglected be understood. Sometimes such an approximation for certain terms in R is unjustified, and a procedure is outlined which determines their effect on the resonance.

In the lunar-planetary resonance problem, the sun is an important participant through a coupling or mixing of short period terms in H. Therefore, a method is discussed which successively eliminates those terms order by order in terms of a small expansion parameter. The expansion parameter (or parameters) in the satellite-satellite case is the mass of the disturbing body μ' (really the ratio $\frac{\mu'}{\mu_0}$). The mass ratio ranges from 10^{-3} to 10^{-8} for satellites of the major planets. In the lunar-solar problem, the appropriate parameter is the ratio of the mean motions of the moon to the earth, $\frac{r_2}{r_1}$, which is of order 10^{-1}. Thus the rate of convergence of a conventional perturbation expansion in the lunar theory is comparatively slow.

2.5 ELIMINATION OF THE SHORT PERIOD TERMS

The expansion of the disturbing function for two satellites, developed in section 2.2, contains an infinity of terms which were classified according to various schemes in section 2.3. If the system is near a commensurability, the usual assumption is that the secular term and the given resonance term are relatively more important than any of the others which occur in R. When restricted to this set of terms, the Hamiltonian can be reduced to a form very like that of a pendulum (2.7.8). This restricted Hamiltonian allows the angle variable to librate and the system to exhibit the property of resonance. One might question whether this resonance phenomenon really exists. Perhaps it is just a property of (1) the particular expansion which conveniently expresses R as the sum of pendulum-like potentials and (2) the given assumption that one of them dominates the motion over a long time scale. We shall try to allay such doubts. Also, the expanded disturbing function is non-linear in the various orbital elements. This non-linearity in R should lead to a "coupling" of terms not directly involved in the resonance. This indirect effect may have the same frequency as the resonant term and thus change its effective potential. Therefore, the principal subject discussed in this section will be this coupling effect. To explicitly display the coupling, a procedure similar to Brown's method will be outlined, by which the short period terms in R can be eliminated from the Hamiltonian order by order in terms of a "small" expansion parameter (Brown and Shook, 1964, ch. 6). The coupling will be shown to be smaller by a factor of an expansion parameter, although the parameter may be different from that which occurs directly in the two-body disturbing function. Next, the conditions which determine whether the procedure can be applied to the long period terms are enumerated. At this stage the Hamiltonian will have been reduced to the secular terms, terms which contain a multiple of the resonance angle, and other terms which cannot be eliminated by a perturbation expansion because their cosine arguments are very slow functions of time. Finally, an explicit calculation must be made to see whether the coefficient of the resonance term is much larger than the remaining terms. If the latter terms' effect on the partners of the resonance is comparatively small, then they can be ignored.

To reduce the procedure to its essentials: the disturbing function of the first body will be momentarily restricted to a single two-body potential of the form (from 2.2.12)

$$R(J,w;\ J',w';\ t) = \mu'\sum_i A_i(J,J')\cos\phi_i , \qquad (2.5.1)$$

and other effects such as those due to the presence of other satellites, other planets, the sun, and the primary planet's oblateness will be ignored. The subscript notation is here replaced by primed notation, where unprimed and primed variables refer to the first and second bodies respectively. (J,w) is shorthand for any

pair of conjugate action and angle variables of the first body. The above disturbing function can be formally separated into two parts:

$$R = R_s + R_c . \qquad (2.5.2)$$

R_s shall include the short periodic terms and R_c all the remaining terms. The Hamiltonian is therefore (2.4.1):

$$H + H_o(L) + R_s + R_c . \qquad (2.5.3)$$

Formally, R_s can be written as

$$R_s = \mu'\Sigma\ A_s(J,J')\cos\phi_s , \qquad (2.5.4a)$$

where

$$\phi_s = j\lambda + j'\lambda' + (\text{function of nodes and longitude of pericenter}). \qquad b)$$

The mean longitude λ is also equal to (2.1.6,2.2.3):

$$\lambda = nt + \epsilon . \qquad (2.5.5)$$

It follows that the mean longitude is an explicit function of its conjugate action variable L. We make this point now to avoid confusion later on.

The next step is to make a Hamilton-Jacobi transformation on the old Hamiltonian $H(J,w)$, with the demand that the new Hamiltonian generated, $H(\bar{J},\bar{w})$, does not contain the short period terms to first order. The transformation from the old set of action-angle variables can be accomplished with a generating function

$\bar{S}(J,\bar{w};\ J',w')$ defined by the relations:

$$\bar{J} = \frac{\partial\bar{S}}{\partial\bar{w}}\,(J,\bar{w};\ J',w'), \quad w = \frac{\partial\bar{S}}{\partial J}\,(J,\bar{w};\ J',w'), \tag{2.5.6}$$

which satisfy the Hamilton-Jacobi equation

$$\bar{H}(\bar{J},\bar{w};\ J',w') = H(J,w;\ J',w') + \frac{\partial\bar{S}}{\partial t}\,(J,\bar{w};\ J',w')\ . \tag{2.5.7}$$

A simultaneous transformation will be made on the action-angle variables of the second body and the corresponding Hamiltonian H', where R', H', $\bar{H}'$ and $\bar{S}'$ are defined by relations similar to (2.5.1-7). Although $\bar{H}(\bar{J},\bar{w};\ J'w')$ in the above equation is a function of the old variables belonging to the primed satellite, eventually the right hand side will be expanded in terms of the new variables of both satellites.

If the perturbations due to the short period terms in H are small, then the old Hamiltonian can be expanded in the difference between the old and new variables. A new generating function, S, can be defined which differs from $\bar{S}$ by the identity transformation

$$\bar{S} = J\bar{w} + S\ . \tag{2.5.8}$$

Then, by (2.5.4), S is directly related to the differences δJ, δw defined by

$$\delta J = J - \bar{J} = -\frac{\partial S}{\partial\bar{w}} \tag{2.5.9a}$$

$$\delta w = w - \bar{w} = +\frac{\partial S}{\partial J}\ . \qquad \text{b.}$$

Since $J,\bar{w}$ have no explicit time dependence, $\bar{S}$ can be replaced by S in the H-J equation. Again, the same transformation can be performed on S' with similar results.

Neither in H nor in H' does the time variable literally occur, yet both Hamiltonians are time dependent because of the appearance of the "external variables" belonging to their respective partners. Explicitly, we find that the equation of motion of H is:

$$\frac{dH}{dt}\,(J,w;\ J',w') = \frac{dJ}{dt}\frac{\partial H}{\partial J} + \frac{dw}{dt}\frac{\partial H}{\partial w} + \frac{dJ'}{dt}\frac{\partial H}{\partial J'} + \frac{dw'}{dt}\frac{\partial H}{\partial w'}$$

$$= \frac{dJ'}{dt}\ \frac{\partial H}{\partial J'} + \frac{dw'}{dt}\frac{\partial H}{\partial w'},$$

and since $\frac{dH}{dt}$ equals $\frac{\partial H}{\partial t}$, we have

$$\frac{\partial H}{\partial t} = \frac{dJ'}{dt}\frac{\partial H}{\partial J'} + \frac{dw'}{dt}\frac{\partial H}{\partial w'}\ . \tag{2.5.10}$$

The implication of this last equation (2.5.10) is that the partial time derivative occurring in each H-J equation acts both on the explicit time dependence which may occur in the respective generating functions and on the "external variables" which occur therein. Of course, S and S' can be chosen such that the time variable does not occur explicitly in either generating function. Therefore

$$\frac{\partial S}{\partial t}\,(J,\bar{w};\ J',w') = \frac{dJ'}{dt}\frac{\partial S}{\partial J'} + \frac{dw'}{dt}\frac{\partial S}{\partial w'}\ . \tag{2.5.11}$$

To help us determine the best choice for S, let's replace $\frac{dJ'}{dt}$ and $\frac{dw'}{dt}$ by their equivalents, using their respective equations of motion. We find:

$$\frac{\partial S}{\partial t} = -\frac{\partial H_o'}{\partial L'}\frac{\partial S}{\partial \lambda'} + \frac{\partial}{\partial t}(\delta S), \text{ where} \tag{2.5.12}$$

$$\frac{\partial}{\partial t}(\delta S) = \sum_{J',w'}\left(\frac{\partial(R_c' + R_s')}{\partial w'}\frac{\partial S}{\partial J'} - \frac{\partial(R_c' + R_s')}{\partial w'}\frac{\partial S}{\partial J'}\right)$$

We expect that the generating function S will be of $O(\mu)$ since the short period terms which will be eliminated by S are of $O(\mu)$. Therefore, the second sum of terms on the left hand side of (2.5.12) is of $O(\mu^2)$, while the function $-\frac{\partial H_o'}{\partial L'}\frac{\partial S}{\partial \lambda'}$ is of $O(\mu)$ and may contain short-period terms. Therefore we shall demand that this function not appear in $\bar{H}$. To see how this can be accomplished, expand $H_o + R_c$ in a power series of the differences δJ, δw, $\delta J'$ and $\delta w'$. In order to match the variable dependence of $S(J,\bar{w}; J',w')$ and $R(J,w; J',w')$, expand R_s in δw. The result of this expansion is:

$$H_o + R_c + R_s = H_o(\bar{L}) + \delta H_o + R_c(\bar{J},\bar{w}; \bar{J},\bar{w}') + \delta R_c + R_s (J,\bar{w}; J',w') + \delta R_s, \tag{2.5.13}$$

where δH_o, δR_c and δR_s are

$$\delta H_o \equiv H_o(L) - H_o(\bar{L}) = \overline{\frac{\partial H_o}{\partial L}}\left(-\frac{\partial S}{\partial \lambda}\right) + 1/2\,\overline{\frac{\partial^2 H_o}{\partial L^2}}(-\delta L)^2 + O(\mu^3); \tag{2.5.14a}$$

$$\delta R_c = R_c(J,w, J',w') - R_c(\bar{J},\bar{w}; \bar{J},\bar{w}) =$$

$$\delta R_c = \sum_{J,w;J,\bar{w}}\left(\frac{\partial R_c}{\partial J}\delta J + \frac{\partial R_c}{\partial w}\delta w + \frac{\partial R_c}{\partial J'}\delta J' + \frac{\partial R_c}{\partial w'}\delta w' + O(\mu^3)\right) \tag{b}$$

$$\delta R_s = R_s(J,w',J',') - R_s(J,\bar{w};J',w') = \sum_w\left(\frac{\partial R_s}{\partial w}\delta w + O(\mu^3)\right) \tag{c}$$

The term $\overline{\frac{\partial H_o}{\partial L}}$'L is similiar to that found in (2.5.12), and the factors $\left(-\overline{\frac{\partial H_o}{\partial L}}, -\frac{\partial H_o'}{\partial L'}\right)$ are equal to $\bar{n}$ and n', respectively (2.4.5). So far the equation for $\bar{H}$, in terms of the new variables, is:

$$\begin{aligned}\bar{H}(\bar{J},\bar{w}; \bar{J}'\bar{w}') = {} & H_o(\bar{L}) + \delta H + R_c(\bar{J}\bar{w};\bar{J}'\bar{w}') + \delta R_c + R_s(J,\bar{w}\, J',w') \\ & + \delta R_s + \bar{n}\frac{-\partial S}{\partial \lambda}(J,\bar{w};J',w') + n'\frac{\partial S}{\partial \lambda'}(J,\bar{w};J'w') \\ & + \frac{\partial}{\partial t}(\delta S).\end{aligned}$$

If the first order short period terms in $\bar{H}$ are to be eliminated to $O(\mu)$, the following equation must vanish at least to $O(\mu^2)$:

$$\bar{n}\frac{-\partial S}{\partial \lambda} + n'\frac{\partial S}{\partial \lambda'} + \mu'\sum_s A_s(J,J')\cos\phi_s(\bar{w},w) = O(\mu^2). \tag{2.5.15}$$

The angle $\phi_s(\bar{w},w')$ is a function of the new variables belonging to the unprimed partner and of the old variables belonging to the primed partner. Explicitly,

$$\phi_s(\bar{w},w) = j\bar{n} + j'n' + (\text{function of } \bar{\omega},\bar{\Omega};\omega',\Omega') \tag{2.5.16}$$

If S is chosen such that

$$S(J,\bar{w};J',w') = -\mu'\Sigma\frac{A_s(J,J')}{\nu_s(\bar{n},n')}\cos\phi_s(\bar{w},w') \tag{2.5.17}$$

where $\nu_s(\bar{w},w) = j\bar{n} + j'n'$,

then (2.5.15) vanishes identically.

There appears to be an inconsistency in that S depends on the old action variable J while, on the right hand side of (2.3.17), the coefficient of the cosine is explicitly dependent on $\bar{n}$ through $\nu(\bar{n},n')$. In reality, this only makes the relation between S and $\delta\lambda$ more complicated. Explicitly (2.5.9b):

$$\delta\lambda = \frac{\partial S}{\partial L} = -\mu'\Sigma\left(\frac{1}{\nu}\frac{\partial A_s}{\partial L} - \frac{A_s}{\nu^2}\frac{\partial \bar{L}}{\partial \bar{L}}\right)\cos\phi_s,$$

but by (2.5.9a)

$$\frac{\partial \bar{L}}{\partial L} = 1 + \frac{\partial}{\partial L}\frac{\partial S}{\partial \lambda} = 1 - \mu'\Sigma j\left(\frac{1}{\nu}\frac{\partial A_s}{\partial L}(J,J') - \frac{A}{\nu^2}\frac{\partial \bar{L}}{\partial L}\frac{\partial \nu}{\partial \bar{L}}\right)\sin\phi_s .$$

Solving for $\frac{\partial L}{\partial \bar{L}}$ we find

$$\frac{\partial \bar{L}}{\partial L} = \frac{1 - \mu'\sum_s \frac{1}{\nu}\frac{\partial A_s}{\partial L}\sin\phi_s}{1 - \mu\sum_s \frac{A}{\nu^2}\frac{\partial \nu}{\partial L}\sin\phi_s}$$

$$= 1 - \mu'\sum_s \frac{\partial}{\partial L}\left(\frac{A}{\nu}\right)\sin\phi_s + O(\mu^2).$$

We see that the difference between n and $\bar{n}$ in the generating function is a second order effect. In any case, the differences δJ, δw can be derived in an entirely consistent manner from the generating function given by (2.5.17). The only apparent drawback is that

$\delta\lambda$ may contain terms of $O(\mu^2)$ like those in R_c. But if we expanded the equations for $\{\delta J,\delta w\}$ in terms of the new variables, we would find, for example, that such products as $\delta J\frac{\delta^2 S}{\delta \Pi^2}$ in the expanded equation for δw would also lead to a second order contribution to the secular and long period terms contained in $(\tilde{H}_o + \tilde{R}_c)$.

The new Hamiltonian $\bar{H}$ is, to $O(\mu^3)$,

$$\bar{H} = \tilde{H}_o + \bar{R}_o + \delta^2 H_o + \delta R_c + \delta R_s + \delta\left(\frac{\partial S}{\partial t}\right), \qquad (2.5.18a)$$

where the second order remainders δR_c and δR_s are given by (2.5.14b,c), and $\delta^2 H_o$, $\delta(\frac{\partial S}{\partial t})$ are

$$\delta^2 H_o = 1/2\,\frac{\partial^2 H_o}{\partial \lambda^2}\left(\frac{\partial S}{\partial \lambda}\right)^2 + O(\mu^3), \qquad b)$$

$$\delta\left(\frac{\partial S}{\partial t}\right) = \sum_{J',w'}\left\{\frac{\partial(R_c' + R_s')}{\partial w'}\frac{\partial S}{\partial J'} - \frac{\partial(R_c' + R_s')}{\partial J'}\frac{\partial S}{\partial w'}\right. \qquad c)$$

Also, the unbarred variables are replaced by their barred counterparts in each of the remainders to obtain an expression for $\bar{H}$ accurate to $O(\mu^3)$.

Some elementary conclusions can be drawn about the qualitative nature of these remainders. Each of these terms is of $O(\mu'^2$ or $\mu'\mu)$. Obviously, the equivalent second order perturbations of the primed satellite due to the unprimed are also of $O(\mu^2$ or $\mu'\mu)$. If these contributions are to be negligible, then both μ and μ' must be relatively small compared to $\mu_o \equiv GM_o$. We shall find in section 2.7

that in the orbit-orbit problem, terms in R of $O(\mu^{3/2})$ will be neglected in order to reduce the Hamiltonian to one dimensional form. Thus, the second order coupling is a comparatively minor effect in this problem.

Next, the remainder $\delta^2 H_o$ involves products of short period terms in δL which can be re-expressed with a single cosine factor. Its argument will involve the sum and difference of the angles which occur in a given product. If the two angles are identical, the above will produce a second order contribution to the secular part of $\bar{H}$. If the sum or difference between two short period terms in a multiple of the resonance argument, then the above produces a second order contribution to the resonance term in $\bar{H}$. By a similar line of reasoning, δR_c contains only short period terms, since δw, δJ, etc., are all short periodic, while $\bar{R}_c$ contains only long period and resonance terms. On the other hand, the remainders δR_s and $\delta(\frac{\partial S}{\partial t})$ are much like $\delta^2 H_o$, although these remainders contain many more terms. Closer inspection of $\delta(\frac{\partial S}{\partial t})$ reveals that only the products

$$\frac{\partial R'_s}{\partial w'}\frac{\partial S}{\partial J'} \quad \text{and} \quad \frac{\partial R'_s}{\partial J'}\frac{\partial S}{\partial w'}$$

contain secular and long period terms. Therefore, the second order contribution to the secular and resonant terms in $\bar{H}$ are contained in

$$1/2\,\frac{\partial^2 H_o}{\partial\lambda^2}\left(\frac{\partial S}{\partial\lambda}\right)^2 + \sum_{\bar{w},\bar{J}',\bar{w}'}\left(\frac{\partial\bar{R}_s}{\partial w}\frac{\partial S}{\partial J} + \frac{\partial R'_s}{\partial w'}\frac{\partial S}{\partial J'} - \frac{\partial R'_s}{\partial J'}\frac{\partial S}{\partial w'}\right). \quad (2.5.19)$$

We can show that the contribution of the bracketed set of terms to the secular and resonant parts of $\bar{H}$ is equal to the following, to $O(\mu^{5/2})$:

$$\sum\left(\frac{\partial R'_s}{\partial w}\frac{\partial S}{\partial J} + \frac{\partial R'_s}{\partial w'}\frac{\partial S}{\partial J'} - \frac{\partial R'_s}{\partial J'}\frac{\partial S}{\partial w'}\right) =$$

$$\sum\left\{\frac{\partial R_s}{\partial w}\,\delta w + \frac{\partial R_s}{\partial J'}\,\delta J' + \frac{\partial R_s}{\partial w'}\,\delta w'\right\} + O(\mu^{5/2}). \quad (2.5.20)$$

The first set of terms on each side (i.e. $\frac{\partial R}{\partial w}\frac{\partial S}{\partial J}$ and $\frac{\partial R}{\partial w}\delta w$) agree since the only step has been to replace $\frac{\partial S}{\partial J}$ by its equivalent δw (2.5.9). The next two sets of terms on each side can be shown to agree to $O(\mu^{5/2})$ by first replacing R'_s by $(-\frac{\partial S'}{\partial t})$. Here it is understood that the partial with respect to t acts only on the λ and λ' dependences occurring in the cosine argument of each term. Taking the second set of terms on the left hand side, let's rewrite it as follows:

$$\frac{\partial R'}{\partial w'}\frac{\partial S}{\partial J'} = \frac{\partial^2 S'}{\partial w'\partial t}\frac{\partial S}{\partial J'} = +\frac{\partial S'}{\partial w'}\frac{\partial^2 S}{\partial J'\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial S'}{\partial w'}\frac{\partial S}{\partial J'}\right). \quad (2.5.21)$$

The product $(\frac{\partial S'}{\partial w'}\frac{\partial S}{\partial J'})$ can be expressed as a cosine series. Some of these terms will presumably contain cosine arguments identically equal to zero (i.e. secular) or equal to the resonance variable. The partial with respect to t of the secular part is identically zero. The effect of the partial on the resonance term is to multiply it by $(jn + j'n')$. Near resonance this frequence is of $O(\mu^{1/2})$, while the product of the disturbing functions is of $O(\mu^2)$.

Therefore the contribution of this term to the resonance is of $O(\mu^{5/2})$. The remaining step in demonstrating the equivalence is to replace $\frac{\partial S'}{\partial w'}$ by $(-\delta J)$, and $\frac{\partial^2 S}{\partial J \partial t}$ by $(-\frac{\partial F_s}{\partial J})$. At this point, we can see that the same operations can be applied to the third set of terms on each side in (2.5.20) to show that they also agree. This second formulation is used in section 4.2 to find the second order contribution caused by the indirect action of the sun.

There are a couple of drawbacks to the procedure outlined. First, if we wish to determine the orbital elements accurate to second order, then a second H-J transformation must be made on $\bar{H}$, transforming the variables from $\bar{J}$, $\bar{w}$, etc., to $\bar{\bar{J}}$, $\bar{\bar{w}}$ to eliminate the second order short period terms in $\bar{H}$. Second, the secular and resonant terms are not restricted to the Hamiltonian $\bar{H}$, but also occur in the differences δJ and δw. This means that in comparing the observed long period behavior of L with its theoretical motion calculated to second order, we must include the second order contribution in $\frac{\partial S}{\partial \lambda}$. Explicitly:

$$\left(\begin{matrix}\text{Long period}\\ \text{motion of } L\end{matrix}\right) = \bar{L} + \left(\begin{matrix}\text{long period}\\ \text{terms in:}\end{matrix} \sum_{J, J', w'} \frac{\partial^2 S}{\partial J \partial \lambda}\, \delta J\right.$$

$$\left. + \frac{\partial^2 S}{\partial J \partial \lambda}\, \delta J' + \frac{\partial^2 S}{\partial w' \partial \lambda}\, \delta w'\right)$$

where $\bar{L}$ is found from solving for the motion of $\vec{F}$. We should also point out the great power of the technique: first, the equations of motion of the transformed variables are still canonical; second,

differences δJ, δw, etc., are easily derived from the generating function; finally, the method tends to emphasize the point that the real problem we must solve is the long term behavior of $\bar{H}$.

Methods for the elimination of the long period terms for which $j = j'$ are less satisfactory. If the motion of the "slow" angle variables $\tilde{\omega}, \Omega$ are nearly linear except for a periodic perturbation due to the long periodic term in R_c, then this periodic motion of $\tilde{\omega}$ and Ω can be removed order by order using a modified version of the procedure already developed. The first step is to separate R_c into three parts. 1) secular; 2) long periodic; and 3) other terms, including the resonance term.

$$R_c = R_{sec} + R_{LP} + R_{res}. \qquad (2.5.22)$$

Taking our cue from (2.5.15), we can construct a generating function S_{LP} which will eliminate, to first order, these long period terms:

$$S_{LP}(J, \bar{w}; J', w') = -\mu' \sum_{LP} \frac{A_{LP}}{\nu} \sin\phi_{LP},$$

where

$$\nu(\bar{w}, w') = -k \frac{\partial \bar{R}_{sec}}{\partial \Gamma} - k' \frac{\partial R'_{sec}}{\partial \Gamma'} - i' \frac{\partial \bar{R}_{sec}}{\partial Z} - i' \frac{\partial R'_{sec}}{\partial Z'} . \qquad (2.5.23)$$

The old Hamiltonian is, as before, expanded in the perturbations, and second order coupling may occur which affects the resonance term. If all short periodic terms in H have been previously eliminated, and if R_{res} only contains resonant terms, then the coupling of the long period terms will produce either secular or other long period terms. Unless the resonance variable is also like the long period terms being eliminated ($j = j' = 0$), the second order coupling will only affect the secular part of the Hamiltonian.

It must be emphasized that if the above procedure is to work, then each ν must be 1) nonzero, and 2) large enough such that the perturbation expansion converges. Instead of μ, the effective expansion parameter for the long period terms is $\frac{n}{\nu}\mu$, and it is this parameter which must be small. The secular terms in the two-body interaction usually do not satisfy this criterion. For example, the motions of the planetary perihelion and nodes (which in some cases cannot be defined!) have periods 10^5 larger than the orbital period (Brouwer and Clemence, 1961b, p. 46). Still, the procedure can be applied to at least two of the satellite-satellite resonances of Saturn. The reason is that the combined perturbatins caused by the planet's oblateness, the sun, and the largest satellite Titan, lead to a relatively large motion in the pericenter and node of the inner satellits with periods 10^2 larger than their orbit periods (Jeffreys, 1953). It should be pointed out that the least satisfactory case is the Titan-Hyperion resonance. The prograde motion of the pericenter of Titan is only about 0.5°/year, while its orbital period is approximately 15 days. The pericenter motion of Hyperion is actually retrograde and caused entirely by the impressed resonance. In this case, we must appeal to the fact that the coefficient of the long

period term is $10^{-1} \Rightarrow 10^{-2}$ times smaller than the resonance term.

2.6 REDUCTION OF THE TIDE-FREE HAMILTONIAN TO A CONSTANT OF THE MOTION

We have reached the stage where each Hamiltonian that describes the motion of the appropriate partner of the resonance can be explicitly reduced to one degree of freedom. The two tide-free Hamiltonians which govern the motion of the first and second partners take the forms

$$H(J,w;J',w') = \sum_{\gamma=0}^{\infty} A_\gamma(J,J')\cos(\gamma\phi)$$

and

$$H'(J',w';J,w) = \sum_{\gamma=0}^{\infty} A'_\gamma(J',J)\cos(\gamma\phi),$$

where the angle ϕ is

$$\phi = j\lambda + j'\lambda' + k\varpi + k'\varpi' + l\Omega + l'\Omega'.$$

(Again, primed and unprimed variables refer to the first and second partners respectively.) Recall that the two-body interaction must satisfy the criteria that it can be expanded in powers of α or Laplace coefficients, and that the hypothetical resonance dominates the motion over a long time scale. This means that all short period terms ($j \neq 0$, $j' \neq 0$) and long period terms ($j = j' = 0$) are assumed negligible and that any terms in the disturbing function which may have very long periods must have relatively small coefficients compared to those for the variable ϕ. For example,

there may exist terms which nearly satisfy the commensurability relation (1.1) for a different set of integers $\{j,j'\}$ than occurs in ϕ. These very long period terms are neglected in our approximation. Implicit in the expansion criteria is the fact that the two orbits cannot intersect or make very close approaches. Then the magnitude of the perturbation acting on either satellite can be comparable to that of the primary.

We can see that the occurrence of the angle variables in each Hamiltonian is restricted to the combination of angles which comprises ϕ. The motion of the action variables is determined by the partial derivatives (acting on the appropriate Hamiltonian) with respect to their corresponding conjugate angle variables. Therefore their equations of motion must be proportional. Explicitly, the equations of motion of the first partner are

$$\frac{dL}{dt} = \frac{\partial H}{\partial \lambda} = -j\sum_{\gamma=0}^{\infty} \gamma A_\gamma \sin(\gamma\phi)$$

$$\frac{d\Gamma}{dt} = \frac{\partial H}{\partial \varpi} = -k\sum_{\gamma=0}^{\infty} \gamma A_\gamma \sin(\gamma\phi)$$

$$\frac{dZ}{dt} = \frac{\partial H}{\partial \Omega} = -l\sum_{\gamma=0}^{\infty} \gamma A_\gamma \sin(\gamma\phi)$$

Clearly, the variations of the action variables are proportional to each other. Therefore, a new variable x can be defined which simultaneously satisfies the relations

$$
\begin{aligned}
L &= jx + L_o \\
\Gamma &= kx + \Gamma_o \\
Z &= ix + Z_o .
\end{aligned}
\qquad (2.6.1)'
$$

Note that for a small eccentricity or inclination the fraction fluctuation of L(x), defined to be

$$\frac{\delta L(x)}{L_o} = \frac{(x_{max} - x_{min})}{L_o} = j\frac{\delta x}{L_o} , \qquad (2.6.2)$$

is of $O(e^2$ or $I^2)$ smaller than the fractional fluctuation in $\Gamma(x)$ or $Z(x)$ (2.4.4). The equation of motion of x is given by

$$\frac{dx}{dt} = \frac{\partial H(x,x',\phi)}{\partial \Phi} \qquad (2.6.3)$$

The three angle variables can be reduced to one also. Define y as

$$y = j\lambda + k\tilde{\omega} + i\Omega ;$$

The equation for $\frac{dy}{dt}$ is

$$\frac{dy}{dt} = -j\frac{\partial H}{\partial L} - k\frac{\partial H}{\partial \Gamma} - i\frac{\partial H}{\partial Z} = -\frac{\partial H}{\partial x} ,$$

and the Hamiltonian is reduced to one degree of freedom. The Hamiltonian of the other resonance partner can be reduced by a similar procedure. For our two-body resonance, the set of

Hamilton's equations reduces to

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial y}(x,y;x',y') & \frac{dx'}{dt} &= \frac{\partial H'}{\partial y'}(x',y';x,y) \\
\frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x,y;x',y') & \frac{dy'}{dt} &= -\frac{\partial H'}{\partial x'}(x',y';x,y),
\end{aligned}
\qquad (2.6.4)
$$

where $\phi = y + y'$.

Still, neither H nor H' is a constant of the motion. The reason is that the above Hamiltonians still contain an explicit time dependence because of the appearance of, say, the second partner's variables in the first partner's Hamiltonian. The next step is to establish some connection between the two sets of variables and find a new Hamiltonian which is a function of a single set of conjugate variables (x,ϕ). To do this, we must consider the form of the coefficients A_γ and A'_γ and determine how they are related.

The secular part in each Hamiltonian corresponds to the $\gamma = 0$ term and includes H_o. The coefficients A_o and A'_o can be formally separated into two parts

$$A_o(x,x') = s(x) + v(x,x')$$

$$A'_o(x',x) = s'(x') + v'(x',x).$$

The function s(x) contains H_o, the secular terms due to the oblateness of the primary, and the secular terms due to a host of other interactions, but not the secular part due to its resonance partner. The same holds for s'(x'). The mixed functions v(c,c') and (v'(x',x) are the secular parts derived from interaction with their respective partners, and are proportional to first order. This follows from the fact that the direct parts of the disturbing functions are proportional, and the indirect parts, to first order, have no secular term (2.1.8c). By inspection of (2.1.8c), we find that $v(x,x')$ is related to $v'(x',x)$ by

$$v'(x',x) = \frac{\mu}{\mu'} v(x,x'). \tag{2.6.5}$$

Comparing similar coefficients $A_\gamma(x,x')$ and $A'_\gamma(x',x)$, we find that they will also be proportional to first order, provided that the resonance variable ϕ is not contained in the indirect part of the disturbing function. From (2.2.10), the angles contained in the indirect part are restricted to the following set:

$$\begin{aligned}\varphi_{1,m,p,p',q,q'} = \{&(1 - 2p + q)\lambda - (1 - 2p' - q')\lambda' \\ &- q\tilde{\omega} - q'\tilde{\omega}' + (m - 1 + 2p)\Omega \\ &-(m - 1 + 2p')\Omega\},\end{aligned}$$

where the integers m,p,p',q,q' are restricted to the values: $m = \{-1,0,1\}$; $p = \{0,1\}$; $p' = \{0,1\}$; $-\infty \le q,q' \le +\infty$.

- The Enceladas-Dione commensurability,

$$\phi_{En-Di} = \lambda_{En} - 2\lambda_{Di} + \tilde{\omega}_{Di},$$

falls into the above class of resonance variables, dependent on the indirect part. This requires a slightly different approximation to reduce the Hamiltonian to one dimensional form. Except for cases such as this, however, the coefficients $A_\gamma(x,x')$ and $A'_{\gamma'}(x',x)$ are

proportional to first order. That is,

$$A'_{\gamma'}(x',x) = \frac{\mu}{\mu'} A_\gamma(x,x'). \tag{2.6.6}$$

Since the coefficients are proportional, the equations of motion for x and x' must also be proportional:

$$\frac{dx'}{dt} = \frac{\mu}{\mu'}\frac{dx}{dt}$$

In addition, the integration constant can be chosen such that x and x' are proportional. Explicitly,

$$x' = \frac{\mu}{\mu'} x. \tag{2.6.7}$$

To avoid confusion in taking partial derivatives, replace x' by $\frac{\mu}{\mu'}\bar{x}$ in both H and H'. Making use of the above relationships, we see that the two Hamiltonians take the forms

$$H(x,\bar{x}) = s(x) + v(x,\bar{x}) + \sum_{\gamma=1}^{\infty} A_\gamma(x,\bar{x})\cos(y + y')$$

$$H'(x,\bar{x}) = s'(\bar{x}) + \frac{\mu}{\mu'} v(x,\bar{x}) + \frac{\mu}{\mu'}\sum_{\gamma=1}^{\infty} A_\gamma(x,\bar{x})\cos(y + y').$$

The equation of motion for $\phi(= y + y')$ is

$$\frac{d\phi}{dt} = -\frac{\partial s(x)}{\partial x} - \frac{\mu'}{\mu}\frac{\partial s(\bar{x})}{\partial \bar{x}} - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \bar{x}}\right)v(x,\bar{x})$$

$$- \sum_{\gamma=1}^{\infty}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \bar{x}}\right)A(x,\bar{x})\cos\phi.$$

The sum of the partial derivatives with respect to x and $\bar{x}$ can be replaced by the partial with respect to x if the bar on "$\bar{x}$" is dropped. This means that $\frac{\partial}{\partial x}$ acts on both x and $\bar{x}$. The equations of motion for ϕ and x are now derivable from the Hamiltonian $\tilde{H}(x,\phi)$ given by

$$\tilde{H}(x,\phi) = \sum_{\gamma=0}^{\infty} \tilde{A}_\gamma(x)\cos(\gamma\phi), \qquad (2.6.8)$$

where $\tilde{A}_\gamma(x) = A_\gamma(x), \quad \gamma \neq 0,$

and $\tilde{A}_0(x) = s(x) + \frac{\mu'}{\mu}s'(x) + v(x); \quad \gamma = 0.$

$\tilde{H}(x,\phi)$ is a constant of the motion, which can be verified by taking the ordinary time derivative of $\tilde{H}(x,\phi)$ and relating that to a sum of partials. Explicitly,

$$\frac{d\tilde{H}}{dt}(x,\phi) = \frac{\partial\tilde{H}}{\partial t} + \frac{dx}{dt}\frac{\partial\tilde{H}}{\partial x} + \frac{d\phi}{dt}\frac{\partial\tilde{H}}{\partial\phi} = \frac{\partial\tilde{H}}{\partial t} = 0. \qquad (2.6.9)$$

A slightly different approximation shall be made to establish a proportionality relationship between $A_\gamma(x,x')$ and $A'_\gamma(x',x)$ for the En-Di case. From (2.2.10) and (2.2.11) we see that each part of each disturbing function contains the same leading factors, depending on the eccentricities and inclinations of the two bodies. We have already noted that for small eccentricities or inclinations the fractional fluctuation in L(x) is of $O(e^2$ or $I^2)$ smaller than in $\Gamma(x)$ or Z(x), respectively. This means that these same leading factors dominate the total fluctuation in $A_\gamma(x)$.

We shall explicitly demonstrate this assertion for the $\gamma = 1$

term of a simple e-type resonance (i.e. $\phi = j\lambda + j'\lambda + k\bar{\omega}$). From (2,2.12), $A_1(x)$ is:

$$A_1(x) = \frac{\mu'}{a_>(x)}C(x)e^{|k|}(x) \qquad (2.6.10)$$

The function C(x) is a polynomial series in α, I^2 and e^2, and terms which contain factors of e^2 or I^2 are at least of order e^2 or I^2 smaller than the leading terms which contain factors of α. After expanding C(x) and e(x) to first order in x, we have

$$C(x) = C(0) + (j\frac{\partial C}{\partial L} + j\frac{\mu}{\mu'}\frac{\partial C}{\partial L'} + k\frac{\partial C}{\partial \Gamma})x + O(x^2)$$

$$e^{k}(x) \doteq [(-kx - \Gamma_o)(\frac{2}{L_o})]^{\frac{|k|}{2}} \qquad = e(0)^{|k|} - k|k|e(0)^{|k|}\frac{x}{\Gamma_o}$$

$$(2.6.11)$$

The partials with respect to L, L' and Γ acting on each term in C reduce those terms which depend on L, L' or Γ by a factor of L, L' or Γ, respectively. Since the largest terms which depend on Γ multiply a polynomial in α of the same order as the leading term, we see that each of the partials of C are of $O(\frac{C(0)}{L_o})$. But in the expansion of $e(0)^{|k|}$, the coefficient of the linear term in x is of $O(\frac{e^k(0)}{\Gamma_o})$. Thus the variation of C(x) is of $O(e^2)$ smaller than the corresponding variation of $e^{|k|}(x)$, implying that this leading factor predominately determines the variation of $A_1(x)$ with x. It is also clear that the variation in x of the leading factors in $A_\gamma(x)$ determine the variation of $A_\gamma(x)$ with x to $O(e^2$ or $I^2)$ for the more general case.

The principal exceptions for the two body interaction are the synodic-type resonances, for which the expansion of the disturbing function in powers of α is invalid anyway. For a synodic resonance to occur, $\lambda \cong \lambda'$, or the satellites must be at approximately the same radius. Thus the whole concept of a resonance variable which can evolve slowly from rotation into libration is inapplicable to this case. However, synodic resonances involving three or more bodies can satisfy this evolutionary description.

The coefficients $A_\gamma(x,x')$ and $A'_\gamma(x',x)$ can be formally separated into their direct and indirect parts:

$$A_\gamma(x,x') = A_{\gamma D}(x,x') + A_{\gamma I}(x,x')$$
$$A'_\gamma(x',x) = A'_{\gamma D}(x',x) + A'_{\gamma I}(x',x).$$

Using (2.1.8c), we can establish the following relationships to $O(\frac{\delta x}{L_o})$:

$$A'_{\gamma D}(x',x) = \frac{\mu}{\mu'} A_{\gamma D}(x,x')$$
$$A'_{\gamma I}(x',x) = \frac{A'_{\gamma I}(0,0)}{A_{\gamma I}(0,0)} A_{\gamma I}(x,x')$$

Therefore, the relation between $A_\gamma(x,x')$ and $A'_\gamma(x',x)$ is

$$A'_\gamma(x',x) = K\frac{\mu}{\mu'} A_\gamma(x,x') \tag{2.6.12}$$

where $K = \frac{\mu'}{\mu}\frac{A'_{\gamma I}(0,0)}{A_{\gamma I}(0,0)}$

Unlike in the previous case, the proportionality constant differs for each value of γ. Since we have already presumed that e and I are small, the $\gamma = 2$ term in each Hamiltonian is of $O(e^{|k|}{}_e, {}^{|k'|}{}_I{}^{|i|}{}_I, {}^{|i'|})$ smaller than the $\gamma = 1$ term. If we restrict the Hamiltonians to the $\gamma = 1$ term, then the action variables x' and x are again proportional by the factor $K\frac{\mu}{\mu'}$. Still, this is not enough, because the mixed secular term $v(x,x')$ in H has a different proportionality constant than the $\gamma = 1$ term (cf. (2.6.5), (2.6.10)). If the system is to be described by a one dimensional Hamiltonian, then the mixed secular term must satisfy one of the following criteria: 1) The mixed term $v(x,x')$ is negligible compared to the unmixed secular terms and can be ignored. 2) The proportionality factor $K\frac{\mu}{\mu'}$ is either very large or very small, such that the effect of one of the satellites on the other is negligible. Of course, this implies that the fluctuations in x or x' are negligible (if $K\frac{\mu}{\mu'}$ is large or small, respectively) and the one dimensional form is immediately obtained. It should be pointed out that the Enceladas Dione case satisfies the first mentioned criterion. If this is the case, then this Hamiltonian is

$$\tilde{H}(x,\phi) = s(x) + \frac{\mu'}{K\mu} s'(x) + \tilde{A}_1(x)\cos\phi, \tag{2.6.13a}$$

and the equations of motion are

$$\frac{dx}{dt} = \frac{\partial\tilde{H}}{\partial\phi} = -A_1(x)\sin\phi, \tag{b}$$

We shall use the above Hamiltonian as an approximation to the motion of all two-body resonances, whether or not the indirect part contributes. This is not the end of approximations. Several more steps must be taken before the above Hamiltonian is reduced to the desired form.

2.7 FURTHER APPROXIMATIONS AND STABILITY ANALYSIS

The next step is to show that the secular term in (2.6.13) can be expanded in a rapidly decreasing polynomial series in x to $O(x^2)$, without increasing the order of approximation already established (roughly $\mu\frac{\delta x}{L_o}$). Again, the secular part of the disturbing function can be formally separated into two distinct pieces:

$$\text{(Secular part)} = \frac{\mu_o^2}{2L(x)^2} + \frac{\mu'}{K\mu}\frac{(\mu_o')^2}{2L'(x)^2} + s_p(x). \qquad (2.7.1)$$

The first two terms are the zero-order part of $\tilde{H}$. Recall that μ_o and μ_o' are shorthand for $\mu_o + \mu$ and $\mu_o + \mu'$, respectively. The parameter K equals one unless the indirect part contains a contribution to the resonance. In that case K is given by (2.6.10). The function $s_p(x)$ formally represents the sum of secular perturbations acting on both partners of the resonance. There is no problem expanding the first two terms, as long as the fractional fluctuation in L9x) and L'(x) is small. Whether $s_p(x)$ can be expanded is less certain. Recall that the secular part of the satellite-satellite disturbing function was a polynomial series in α, e^2, e'^2, I^2 and I'^2, or, equivalently, a polynomial series in L, Γ and Z (cf. 2.2.13). Part of the motion of ϕ is derived from the partial derivative with respect to x of $s_p(x)$, and the effect of this derivative is to reduce any given term in $s_p(x)$ by a factor of L, Γ or Z (cf. 2.5.11c). The reason that this is important is that if

$s_p(x)$ were to contain terms with factors of, say, ee', then those terms would provide relatively large contributions to the motion of $\tilde{\omega}$ or $\tilde{\omega}'$ if either e or e' were small, respectively. None of the possible gravitational perturbations have secular terms of the above type. A factor of ee' is always associated with a long period term whose cosine argument contains $(\tilde{\omega} - \tilde{\omega}')$. The conclusion we can draw is that $s_p(x)$ is not qualitatively different from the zero-order terms and can be neglected without seriously affecting the motion of either x or ϕ. The only exception is the case for angles ϕ which do not contain the mean longitude of either partner. For these angles, the zero-order part is independent of x (that is, $j = j' = 0$), and the contribution from $s_p(x)$ is uniquely important. Ignoring this exceptional case, we find the expansion of the secular part to be, to a good approximation,

$$\{\text{Secular Part}\} = A_0 + xA_{0x} + \frac{x^2}{2}A_{0xx}, \tag{2.7.2a}$$

where (cf. 2.1.7, 2.4.2)

$$A_0 = -\frac{\mu_0}{2a_0} - \frac{\mu'}{K\mu}\frac{\mu_0'}{2a_0'}, \tag{b}$$

$$A_{0x} = -jn_0 - j'n_0', \tag{c}$$

$$A_{0xx} = \frac{3j^2}{a_0^2} + \frac{\mu}{\mu'}\frac{3j'^2}{a_0'^2}. \tag{d}$$

Commensurability is associated with the vanishing of A_{0x}. The resonance angle ϕ can be constructed so that $\dot{\phi}$ and $(-A_{0x})$ are

positive as the system is pulled towards commensurability by the tidal accelerations.

The secular part can be expressed as a perfect square plus a constant

$$\{\text{Secular Part}\} = \tfrac{1}{2}A_{0xx}\left(x + \frac{A_{0x}}{A_{0xx}}\right)^2 + A_0 - \frac{A_{0x}^2}{2A_{0xx}}. \tag{2.7.3}$$

The Hamiltonian then equals

$$\bar{H}(x,\phi) = \tfrac{1}{2}A_{0xx}\left(x + \frac{A_{0x}}{A_{0xx}}\right)^2 + A_0 - \frac{A_{0x}^2}{2A_{0xx}} + A_1(x)\cos\phi.$$

Since the constant term $\left(A_0 - \frac{A_{0x}^2}{2A_{0xx}}\right)$ does not affect the equations of motion of x or ϕ, it can be absorbed into $\bar{H}$ without changing these equations. Even if this constant term were time dependent, it still would not affect these equations (although it would affect $\bar{H}$) and again this term could be eliminated by subtracting it from $\bar{H}$. The coefficient A_{0xx} can be effectively factored out by defining a new time variable $\bar{t}$ related to t by

$$\bar{t} = A_{0xx}t, \tag{2.7.4}$$

where $\bar{t}$ has the units time·length^{-2} (same as I^{-1}). The equations of motion then take the form

$$\frac{dx}{d\bar{t}} = \frac{dt}{d\bar{t}}\frac{\partial\bar{H}}{\partial\phi} = \frac{\partial(A_{0xx}^{-1}\bar{H})}{\partial\phi}, \tag{2.7.5}$$

$$\frac{d\phi}{d\bar{t}} = -\frac{dt}{d\bar{t}}\frac{\partial\bar{H}}{\partial x} = -\frac{\partial(A_{0xx}^{-1}\bar{H})}{\partial x}.$$

Finally, the equation of motion for $A_{oxx}^{-1}\tilde{H}$) takes the form

$$\frac{d}{d\bar{t}}(A_{oxx}^{-1}\tilde{H}) = \frac{dx}{d\bar{t}}\frac{\partial(A_{oxx}^{-1}\tilde{H})}{\partial x} + \frac{d\phi}{d\bar{t}}\frac{\partial(A_{oxx}^{-1}\tilde{H})}{\partial\phi} + \frac{\partial(A_{oxx}^{-1}\tilde{H})}{\partial\bar{t}}$$
$$= \frac{\partial(A_{oxx}^{-1}\tilde{H})}{\partial\bar{t}}. \quad (2.7.6)$$

Therefore, the equations of motion can be returned to canonical form by replacing $A_{oxx}^{-1}\tilde{H}$ by H_f. Even if A_{oxx} is time-dependent, a non-linear time $\bar{t}$ is defined,

$$\bar{t} = \int A_{oxx}(t)dt, \quad (2.7.7)$$

such that the equations of motion of x and ϕ are still canonical and derivable from the Hamiltonian, $A_{oxx}^{-1}\tilde{H}$. The form for the tide-free Hamiltonian is, then,

$$H_f(x,\phi) = 1/2(x + c)^2 + b(x)\cos\phi, \quad (2.7.8)$$

where the functions c and b(x) are given by

$$c = \frac{A_{co}}{A_{oxx}}, \quad b)$$

$$b(x) = \frac{A_1(x)}{A_{oxx}}. \quad c)$$

From the discussion in (1.1), we know that near resonance the fluctuation δx is roughly proportional to $|\beta|^{1/2}$ or to the square root of the small expansion parameter $\mu^{1/2}$. Neglecting terms of

$O(\delta x^3)$ in the secular part is equivalent to neglecting terms of $O(\mu\delta x)$, which is of the same order as the approximation imposed on the function b(x). Note that the above Hamiltonian is identical to that of pendulum plus torque except that the parameter c here is a constant and not a linear function of the time, and the coefficient b(x) depends on the action, not the time. The next major exercise is to include the effect of the tides, and make the parameter c time dependent. First, let's examine the above Hamiltonian for the existence and stability of libration centers.

The existence and location of libration centers can be determined either analytically or graphically from the phase-space curves generated by the tide-free Hamiltonian $H(x,\phi)$. A typical phase-space curve is shown in fig. 2.7.1. The asymmetry about the ϕ-axis is caused by the momentum dependence of the potential-like term in H_f. Also observe that in figure 2.7.1 there exist stable libration centers at ϕ equal to both even and odd multiples of π, and is the result of the momentum dependence of the potential term. By the way, this kind of behavior is not general and depends on the form of the momentum dependence and the value of the parameters of the system.

In the limit in which the momentum dependence vanishes, H_f is reduced to the Hamiltonian of a simple pendulum. In this case, the libration centers are at either even or odd multiples of π, depending on the sign of b.

The analytical definition of a libration center is as follows: Given there exists a point (x_p,ϕ_p) such that $(\dot{x}_p,\dot{\phi}_p)$ is zero for all

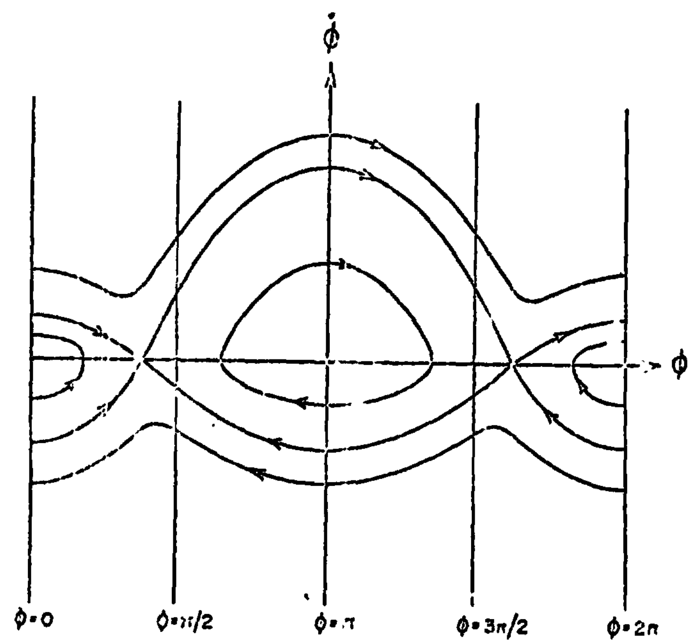


FIGURE 2.7.1

PHASE SPACE DIAGRAM

time, and if there exist closed curves about (x_p, ϕ_p) generated by $H(x_p + \delta x, \phi_p + \delta\phi)$ in some region surrounding the test point, then that point is a libration center. Inspection of (2.6.1, 2.6.11b,c) shows that x_p can be set equal to zero, and ϕ_p is either π or 2π. Next expand the Hamiltonian in a small region about the test point:

$$H(x,\phi) = H_r(x_0, \phi_0) + \delta x H_x + \delta\phi H_\phi + \tfrac{1}{2}\delta^2 x H_{xx} + \tfrac{1}{2}\delta^2\phi H_{\phi\phi} + \delta x\,\delta\phi H_{x\phi} + O(\delta^3).$$

This quadratic form defines a conic section (Oakley, 1949, p. 109), and an ellipse is defined by the condition $H_{x\phi}^2 - H_{xx}H_{\phi\phi} < 0$. For the π-libration center, the above condition reduces to

$$b(1 - b_{xx}) > 0. \qquad (2.7.9)$$

For the 2π-center the equivalent condition is

$$b(1 + b_{xx}) < 0. \qquad (2.7.10)$$

If $|b_{xx}|$ is less than one, then b must be positive for a π-libration center and negative for a 2π center. The sign of b determines the "normal" libration center, as it did for the simple pendulum examples discussed in (1 2). If $|b_{xx}|$ is greater than one, and has the opposite sign of b, then the system can librate in both its normal and inverted positions. But if $|b_{xx}|$ has the same sign as b, then niether libration center is stable. This unusual phenomenon is caused by the momentum dependence of the potential. We can determine which resonance variables can exhibit either two stable libration centers or perhaps none by explicitly expanding the coefficient, using (2.2.12). Recall that in e and I type resonances the function $b(x)$ has leading factors of $e^{|k|}$ and $I^{|l|}$, respectively. Comparing similar e and I type resonances for which $k = l$ (2.2.12, 2.6.1), we find that the Hamiltonians for each are formally the same. Therefore, $b(x)$ need only be expanded for a simple e-type resonance. The relative stability of the resonances can be determined as a function of k. The coefficients of the expansion are (2.2.12, 2.4.4, 2.6.1):

$$b(0) = \frac{\mu' C e_o^{|k|}}{a_> L_o} \qquad (2.7.11a)$$

$$b_x(0) = -\frac{\mu' C e_o^{(|k|-2)}}{2 a_> A_{oxx} L_o}\, |k|\, k \qquad b)$$

$$b_{xx}(0) = \frac{\mu' C e_o^{(|k|-4)}}{4 a_> A_{oxx} L_o^2}\, k^2 |k| (|k|-2). \qquad c)$$

Derivatives of $C(x)$ with respect to x have been neglected since their contribution to b_x and b_{xx} are of $O(e_o^2)$ smaller than the contribution from the expansion of $e(x)$. The factor $a_> A_{oxx} L_o^2$ is of $O(\mu_o)$, and therefore the coefficient b_{xx} is of $O(\frac{\mu'}{\mu_o} e_o^{(|k| - 4)})$. If the mean value of the eccentricity, e_o, is small enough, $|b_{xx}|$ can be greater than one, for values of $|k| < 4$. The coefficient b_{xx} vanishes for the $|k| = 2$ case, while b_{xx} has the same sign as b for $|k| > 2$. Only for the $|k| = 1$ case does b_{xx} have the opposite sign of b. The $|k| = 1$ case is therefore especially interesting, and the phenomena connected with two libration centers shall be thoroughly

investigated in (3.3).

At this point we make a brief digression to the subject of tides and their secular effect on the motions of satellites. A procedure is then outlined which introduces the tidal effect into the tide-free Hamiltonian just developed.

2.8 EFFECT OF TIDES

To understand the mechanics of tides, let's restrict ourselves to a specific example, the earth-moon system. Because the moon is a finite distance from the earth, there exists a gradient in the lunar force at the extended position of the earth. The earth is almost spherical, but not perfectly rigid or elastic. Thus the earth's shape is distorted into a football-like object which attempts to follow the moon's apparent motion. The maximum response to the distorting force lags behind the applied force, as with any oscillating system in which there is frictional loss and in which the forcing frequency is much less than the natural frequency. The magnitude of the energy dissipation may depend on the relative difference between the frequencies associated with the applied force and the natural frequencies of the affected body in a complicated way (Kaula, 1964). The quantitative effect of tidal friction on the earth's rotation is uncertain, but the qualitative effect is well understood (Munk and MacDonald, 1960, p. 136)

At present, the earth's rotation is much faster than the moon's orbital motion, and the "[illegible]" is carried ahead of the earth-moon axis. Since the gradient in the lunar gravitational force is symmetric along the earth-moon axis, it attempts to realign the distorted earth along that axis. The resulting torque despins the earth and accelerates the moon in its orbit. The earth also raises

a tide on the moon. But the moon's rotation is synchronous with its orbital mean motion, and the "radial" tide raised on the moon only weakly affects its orbit. This effect shall be discussed later.

The torque on the moon due to the tide raised on the earth not only affects the mean motion, but the eccentricity and inclination as well. If the lunar orbit is eccentric, then the torque on the moon (and earth) is stronger at perigee than at apogee, causing the orbit to become more eccentric as it expands. This positive change in the eccentricity depends on the earth's rotation being faster than the moon's orbital motion. The sign of this effect may be reversed once synchronous rotation with the moon's motion is achieved. The "averaged" torque exerted on the moon tends to be normal to the ecliptic plane and tends to increase the normal component to the ecliptic plane of the lunar orbital momentum, without changing the component in the plane. Therefore the lunar orbit is driven towards coincidence with the ecliptic plane, and the lunar orbital inclination is decreased. The results of a detailed calculation (Kaula, 1964) show that the fractional rates of change in a, e, and I are of the same magnitude. Explicitly,

$$\frac{1}{a}\frac{da}{dt} = \left(\frac{[illegible]}{19}\right)\frac{1}{e}\frac{de}{dt} = (-4)\,\frac{1}{I}\frac{dI}{dt}\,. \qquad (2.8.1)$$

The numerical coefficients in (2.8.1) were derived under the assumption that the individual phase lags associated with each frequency in the tidal disturbing potential have the same value.

From (2.4.2) and (2.8.1) the secular changes in the modified Delauney variables L, Γ, and Z are similarly proportional, obeying the relation (if e and I are small):

$$\frac{1}{a}\frac{da_T}{dt} = \frac{1}{2}\frac{1}{L}\frac{dL_T}{dt} \approx \left(\frac{1-e^2}{e^2}\right)\frac{1}{\Gamma}\frac{d\Gamma_T}{dt} \approx O(e^{-2})\left(\frac{1}{I^2}\right)\frac{dZ_T}{dt}. \qquad (2.8.2)$$

The variable Z is proportional to the component of orbital angular momentum in the ecliptic plane, and this component tends to be conserved. This is why $A^{-1}\frac{dZ_T}{dt}$ is comparatively small in (2.8.2). If the lunar orbital motion were faster than the earth's rotation, the "football" would be behind the earth-moon line, leading to a spin-up of the earth's rotation and reversing the secular effects on the lunar elements just discussed. In either case, the tendency is to synchronize the rotation of the earth with the lunar motion. The earth has not achieved that state, but obviously the moon has, as have many of the satellites of the other planets (Goldreich and Soter, 1966).

Once synchronous rotation is achieved, there is no longer an exchange of orbital and rotational momentum if, at the same time, the moon's orbit is circular. Any tidal deformation of the moon would remain stationary with respect to a fixed set of axes which co-rotated with it. Since there is no time-varying distortion with respect to a fixed mass element of the moon, there can be no friction which would result in a dissipation of energy.

But the moon is in an eccentric orbit. An observer on the earth sees a fixed point on the moon move toward and away from him.

This in-and-out motion results in a time-varying dissipative "radial" tide. Strictly speaking, this radial tide does not cause a torque along the direction of motion of the moon. Since there is no torque acting on the moon, orbital angular momentum is conserved while the orbital energy decreases. This energy is given by (2.5.1)

$$E_{orbit} = -\frac{\mu}{2a}.$$

Since E_{orbit} decreases, the semimajor axis must decrease. The relation between the tidal change in the lunar eccentricity and that in the semimajor axis can be derived from the constancy of the lunar orbital momentum $\mathcal{L}$:

$$\mathcal{L} = \sqrt{\mu a(1-e^2)}. \qquad (2.8.3)$$

Taking the time derivative of both sides we find:

$$\frac{1}{a}\frac{da_{Tr}}{dt} = e\,\frac{de_{Tr}}{dt}, \qquad (2.8.4)$$

where T_r refers to the radial tide component. The important point to recognize is that the induced change in e is opposite to that caused by the earth tide. $\frac{da_{Tr}}{dt}$ has the same functional form as the earth tide, except that it is multiplied by a factor of the square of the lunar eccentricity. All other things being equal, $\frac{da_{Tr}}{dt}$ is much smaller than $\frac{da_T}{dt}$. Kaula (1964) indicates that the secular change in e due to the radial tide is about two-thirds that caused by the earth tide, at the present time. For other satellite systems,

$\frac{de_{Tr}}{dt}$ appears to be larger than $\frac{de_T}{dt}$ (Goldreich, 1963). But the supporting arguments for either case are not absolutely convincing, especially since both depend on the relative dissipation in the primary and their satellites, of which little is known. Fortunately, the question of whether the total rate of change of the eccentricity is positive or negative is irrelevant for e-type resonances during transition and after capture into libration. The tidal torque acting on the mean motion induces a change in the eccentricity through the intervention of the resonance. For an e-type case, this resonance induced change in e is much larger than the tidally induced change. This assertion will be supported later. The basic fact is that the inelastic tidal response produces a secular change in the action variables, unlike the purely gravitational perturbation which tends to cause only periodic variations in the action elements. This point has not been proven, and the major theoretical support for this conclusion is "Poisson's theorem on the invariability of the semimajor axes" and the Laplace-Lagrange theory of secular perturbations (Hagihara, 1972, pp. 164-85).

We have not discussed the secular change in the angle variables because the tidal terms are not qualitatively different from those produced by conservative gravitational forces and are much smaller. By far the largest change occurs indirectly through the implied secular change in L,

$$L = L_o + \int \frac{dL}{dt}_T \, dt, \qquad (2.8.5)$$

which secularly changes H_o (cf. 2.4.1). This suggests that a method of introducing the tidal effect on the tide-free Hamiltonian would be to define new action variables which are constants in the absence of any other forces except the tides.

2.9 INTRODUCTION OF THE TIDES INTO THE TIDE-FREE HAMILTONIAN

In order to introduce tides into the Hamiltonian, let's return to the original Hamiltonian formulation before it was reduced to one degree of freedom (2.4.3). First, the secular tidal change in L, Γ and Z shall be formally added to the part due to conservative gravitational interactions already discussed. The result is

$$\frac{dL}{dt} = \frac{\partial H_f}{\partial \lambda} + \frac{dL_T}{dt} \qquad (2.9.1a)$$

$$\frac{d\Gamma}{dt} = \frac{\partial H_f}{\partial \beta} + \frac{d\Gamma_T}{dt} \qquad b)$$

$$\frac{dZ}{dt} = \frac{\partial H_f}{\partial \Omega} + \frac{dZ_T}{dt}, \qquad c)$$

where H_f is the tide-free Hamiltonian for the disturbed body (2.4.1), and $\frac{dL_T}{dt}$, $\frac{d\Gamma_T}{dt}$ and $\frac{dZ_T}{dt}$ are the rates of change of L, Γ and Z due to tides alone. The equations of motion for the angle variable are well approximated by the tide-free equations of motion, because the secular terms which arise from the inelastic tidal interaction are much smaller than the secular terms due to conservative gravitational interactions.

After adding the secular tidal change, the next step is to introduce a new set of action variables, $\bar{L}$, $\bar{\Gamma}$ and $\bar{Z}$, defined by the relations:

$$L = \bar{L} + \int \frac{dL_T}{dt}\, dt \qquad (2.9.2a)$$

$$\Gamma = \bar{\Gamma} + \int \frac{d\Gamma_T}{dt}\, dt \qquad b)$$

$$Z = \bar{Z} + \int \frac{dZ_T}{dt}\, dt. \qquad c)$$

The equations of motion of $\bar{L}$, $\bar{\Gamma}$ and $\bar{Z}$ are

$$\frac{d\bar{L}}{dt} = \frac{\partial H_f}{\partial \lambda}(J,w) \qquad (2.9.3a)$$

$$\frac{d\bar{\Gamma}}{dt} = \frac{\partial H_f}{\partial \beta}(J,w) \qquad b)$$

$$\frac{d\bar{Z}}{dt} = \frac{\partial H_f}{\partial \Omega}(J,w). \qquad c)$$

The elements $\bar{L}$, $\bar{\Gamma}$ and $\bar{Z}$ are constants in the absence of any gravitational perturbations other than the tides. Their explicit dependence on the tides has been eliminated, but the tide-free Hamiltonian depends on the unbarred action elements, not on the barred variables. We can substitute the barred variables for the unbarred variables in H_f, but the secular tidal motions ($\frac{dL_T}{dt}$, etc.) of the elements are also functions of the old elements, although they are approximately constant if the tidally induced changes in L, Γ and Z are relatively small. If this approximation is not possible, then the unbarred elements still can be successively eliminated in the Hamiltonian by a process of iteration.

There is another problem with the barred action variables in that the equations of motion of the angle variables depend upon the partial derivatives with respect to the corresponding unbarred action elements. We can replace the unbarred partial with the barred partial, with the restriction that the partial derivative does not operate on $\frac{dL_T}{dt}$, $\frac{d\Gamma_T}{dt}$ or $\frac{dZ_T}{dt}$ as it occurs in H_f. The equations of motion for each variable are now derivable from the same Hamiltonian. The procedure outlined in section 2.1-2.7 can be used to reduce the Hamiltonian to one degree of freedom. But this one-dimensional Hamiltonian is no longer a constant of the motion since an explicit time dependence is retained through the tidal interaction introduced into the Hamiltonian. We can see how this time dependence occurs by taking the time derivative of $H_f(J,w)$:

$$\frac{dH_f}{dt} = \sum_{J,w} \left(\frac{\partial J}{dt}\frac{\partial H_f}{\partial J} + \frac{d\phi}{dt}\frac{H}{\partial\phi}\right) + \frac{\partial H_f}{\partial t}.$$

Now replace J with $\bar{J}$ using (2.9.2), (2.9.3):

$$\frac{dH_f}{dt} = \sum_{\bar{J}} \frac{dJ_T}{dt}\frac{\partial H_f}{\partial \bar{J}} + \frac{\partial H_f}{\partial t}.$$

Since $\int \frac{dJ_T}{dt}dt$ always occurs in H_f in association with $\bar{J}$, we can absorb the first sum of terms directly into the partial derivative of H_f with respect to the time.

The secular part of the Hamiltonian has already been expanded in a power series in x. Therefore, a simpler procedure for introducing the tidal interaction into the tide-free Hamiltonian is to replace all the constants of integration (L_o, Γ_o, Z_o, etc.) with their tidal counterparts ($L_o + \int\frac{dL_T}{dt}dt$) etc., in the one-dimensional tide-free Hamiltonian (2.7.11a).

The next step is to determine just how these time-dependent

terms alter the Hamiltonian. The approximate tidal acceleration $\frac{dn_T}{dt}$ of the mean motion of a satellite, caused by the inelastic tide raised in the primary, is (Allan, 1969):

$$\frac{dn_T}{dt} = \frac{27}{4Q}n^2\left(\frac{\mu}{\mu_o}\right)\left(\frac{R}{a}\right)^5, \tag{2.9.4}$$

where Q is the dissipation function (MacDonald, 1964) and R is the radius of the primary. $\frac{dn_T}{dt}$ (hence $\frac{dL_T}{dt}$) is a rapidly decreasing function of the semimajor axis a. Since the fractional fluctuation in n near commensurability is small, the change in a and $\frac{dn_T}{dt}$ associated with transition will also be small. We can expand $\frac{dn_T}{dt}$ as a function of x to first order (2.1.7, 2.4.2, 2.6.1):

$$\frac{dn_T}{dt}(x) = \frac{dn_T}{dt}(0)\left(1 - 16j\frac{x}{L_o}\right). \tag{2.9.5}$$

Recall that the fractional fluctuation in x, $\frac{\delta x}{L_o}$, was of $O(\mu^{1/2})$. The change in $L(x)$ associated with transition into libration or into reverse rotation is also of $O(\mu^{1/2})$. Intuitively, we can see that if the tidal acceleration decreases appreciably during transition, then this effect by itself could lead to capture into the librational state. In order for the change in $\frac{dn_T}{dt}$ during transition to be of $O(10^{-1})$ of the constant term, μ must be of $O(10^{-4})$. The mass ratios of the more massive partner in each of the satellite-satellite resonances belonging to Saturn (♄) are (Jeffreys, 1953):

$$\frac{\mu_{Enceladus}}{\mu_{\odot}} = 1.27\times10^{-7}$$

$$\frac{\mu_{Tethys}}{\mu_{\odot}} = 1.14\times10^{-6} \qquad (2.9.6)$$

$$\frac{\mu_{Titan}}{\mu_{\odot}} = 2.41\times10^{-4}.$$

It appears that for the Titan-Hyperion case the change in $\frac{dn_T}{dt}$ may be an important factor during transition. It does not for the simple reason that the tidal acceleration of Hyperion is insignificant compared to that of Titan. The resonant flucturation in the semimajor axis of Titan (and $\frac{dn_T(x)}{dt_{Ti}}$) is governed by the mass ratio of Hyperion which is of $O(10^{-3})$ smaller than Titan's. For most cases, we shall find that $\frac{dn_T}{dt}$ can be well approximated by its mean value near the commensurability.

It is possible that the effective change in n during transition can be substantially increased if the ratio of the tidal motions of n and n' obeys approximately the same commensurability ratio found in the resonance variable ϕ. Rather surprisingly, this occurs in the Mimas-Tethys and Enceladas-Dione commensurabilities, increasing the effective change in the semimajor axis during transition by an order of magnitude (section 3.1). This will not substantially affect the process of transition from rotation into libration, but it can lead to an appreciable change in the long-term behavior of the resonance, especially in the amplitude of libration (Allan, 1969) as a function of the age of the resonance.

The effect of the time dependence of the "constants" is

greatest in the zero-order part of the Hamiltonian. Any time dependence occurring in the coefficient of the potential term is of $O(\mu)$ smaller. Let's return to the Hamiltonian $\bar{H}$ after the secular part had been expanded in a polynomial series in x. Replacing the constants of integration by their corresponding time-dependent terms, the coefficients of the expansion are (2.7.2):

$$A_0 = \frac{\mu_0^2}{2(L_0 + \int \frac{dL}{dt}\, dt)^2} + \frac{\mu'}{\mu}\frac{\mu'^2}{2(L_0' + \int \frac{dL'}{dt}\, dt)^2} \qquad (2.9.7a)$$

$$A_{0x} = -jn_0 - j'n_0' - j\int dt\frac{dn}{dt} - j'\int dt\frac{dn'}{dt} \qquad b)$$

$$A_{0xx} = \frac{-3j^2}{(L_0 + \int \frac{dL}{dt}\, dt)^2} + \frac{\mu}{\mu'}\frac{3j'^2}{(L_0' + \int \frac{dL'}{dt}\, dt)^2} \qquad c)$$

Earlier, these constants were manipulated to obtain a Hamiltonian which had the desired form. We shall now show that these operations are still legitimate.

If $\frac{dL_T}{dt}$, etc., are replaced by their mean values, then each of the above coefficients depends only on the time. Purely time-dependent terms can be added to the Hamiltonian without changing the equations of motion for either x or ϕ. Again, absorbing the term $(A_0 - A_{0x}^2 A_{0xx}^{-1})$ into the Hamiltonian is a legitimate operation. Introducing a new time variable $\tilde{t}$, (2.7.4), related to t by

$$\tilde{t} = \int A_{0xx}(t)\, dt, \qquad (2.9.8)$$

is quivalent to introducing a nonlinear time and is also a legitimate operation (see 2.7.6-7).

Near a commensurability, the constant term in A_{ox} is of $O(\mu^{1/2})$ which is abnormally small. Therefore the fractional change in A_{ox} over an equal time is of $O(\mu^{-1/2})$ larger than the change that occurs in A_o or A_{oxx}. Therefore, the coefficient $c(t)$, (2.7.8b), is well approximated by

$$c(t)\cdot A^2_{oxx}(0) \approx -A_{oxx}(jn_o + j'n'_o) - \left(j\frac{dn_T}{dt_o} + j'\frac{dn'_T}{dt_o}\right)\overline{(t - t_o)}. \tag{2.9.9}$$

Observe that the time derivative of $c(t)$ is proportional to the sum of the secular accelerations of the mean motions acting in the resonance variable. Explicitly, the time derivative is

$$A^2_{oxx}\frac{dc(t)}{dt} \approx -j\frac{dn_T}{dt_o} - j'\frac{dn'_T}{dt_o}. \tag{2.9.10}$$

The next question we shall resolve is the effect of the momentum dependence, contained in $\frac{dL_T}{dt}$, on these coefficients. The greatest dependence would seem to occur in the lowest order coefficient A_o. But this coefficient has no effect on the equations of motion for either x or ϕ since the partial derivative with respect to x cannot legitimately act on $\frac{dL_T}{dt}$, etc. Thus the function $(A_o - A^2_{ox}A^{-1}_{oxx})$ can still be abrobed in $\tilde{H}$ without changing the equations of motion. The coefficient in which the momentum dependence is largest is therefore A_{ox}. A_{ox} depends explicitly on the tidal torques as they occur inside terms such as $\int dt\frac{dn_T}{dt}(x,t)$. Expressing $\frac{dn_T}{dt}(x,t)$ as a function of $L(x,t)$ we find (2.9.4):

$$\frac{dn_T}{dt}(x,t) = \frac{-27}{4Q}\left(\frac{\mu}{\mu_o}\right)R^5\mu_o^9L^{-16}(x,t),$$

where

$$L(x,t) = jx + \int\frac{dL_T}{dt}\,dt + L_o$$

Instead of expanding $\frac{dn_T}{dt}$ in terms of x, we shall include the lowest order time dependence and expand in

$$\left(jx + \frac{dL_T}{dt_o}(t - t_o)\right).$$

To the first order terms in (2.9.9) we must add the term

$$-16j\frac{dn_{T_o}}{dt}\int dt\frac{\left(jx+\frac{dL_T}{dt_o}(t-t_o)\right)}{L_o} - 16j'\frac{dn'_{T_o}}{dt}\int dt\frac{\left(j'x + \frac{dL'_T}{dt_o}(t-t_o)\right)}{L'_o}. \tag{2.9.11}$$

The dependence on x in the coefficient A_{oxx} means that the operation of redefining the time (2.9.8) is no longer explicitly valid. In this case, the dependence on x will be ignored so that the simplest form for the Hamiltonian can be retained. Besides, it can be demonstrated that in those cases in which this dependence plays an important role in the evolutionary history of a resonance, the contribution from this term is insignificant.

The time dependence in the coefficient of the potential term plays a relatively minor role for e and I type resonances. This can be demonstrated by making a transformation of variables from

$\{x,\phi\}$ to another set $\{p,\Psi\}$, using the generating function $S(x,\Psi) = (x + c)\Psi$. These new variables are

$$p = \frac{\partial S}{\partial \Psi} = x + c; \quad \phi = \frac{\partial S}{\partial x} = \Psi, \qquad (2.9.12a)$$

and the new Hamiltonian $H(p,\phi,\bar{t})$ is

$$H(p,\phi,\bar{t}) = H(x,\phi,\bar{t}) + \frac{\partial S}{\partial \bar{t}} = \tfrac{1}{2}p^2 + b(p - c)\cos\phi + \frac{dc}{d\bar{t}}\phi . \qquad b)$$

Observe that the difference between $H(p,\phi,\bar{t})$ and $H(x,\phi,\bar{t})$ is equal to the potential term $\frac{dc}{d\bar{t}}\phi$. The leading variation of $b(p - c)$ with p is proportional to $\Gamma^{|k|/2}(p - c)$ for an e-type resonance. The time dependence of $(p - c)$ as it occurs in the element Γ is (2.4.3, 2.9.7b):

$$\Gamma(p - c) = k(p - c(t)) + \int \frac{d\Gamma_T}{dt}\, dt + \Gamma_o \qquad (2.9.13)$$

$$\doteq k(p - c(0)) + \Gamma_o + [kA_{oxx}(j\frac{dn_T}{dt_o} + j'\frac{dn'_T}{dt_o}) + \frac{d\Gamma_T}{dt}]t$$

The factor multiplying the time is proportional to

$$\tfrac{1}{2}e_o^2 \frac{1}{\Gamma_o}\frac{d\Gamma_T}{dt_o} + \frac{n_o}{L_o A_{oxx}}(j\frac{1}{n_o}\frac{dn_T}{dt_o} + j'\frac{n'_o}{n_o}\frac{1}{n'_o}\frac{dn'_T}{dt_o}) \qquad (2.9.14)$$

by (2.4.4, 2.9.9). By (2.8.1-2), the fractional rate of tidal change of each of the elements is of the same magnitude. The factor $\frac{n_o}{A_{oxx}}$

is of $O(n_o a_o^2)$, or equivalently of $O(L_o)$. Unless the secular tidal accelerations are commensurable, the first term in (2.9.14) is of $O(e^2)$ smaller than that associated with $a(t)$. The same result applies to the element Z. We have already shown that the x dependence in $b(x)$ can be approximated to $O(\mu^{1/2})$ by the leading factors of Γ and Z, and the tidal change in these elements has just been shown to be small compared with that related to $c(x,t)$. We shall now show that the above Hamiltonian equation agrees reasonably well with the second order equation of motion derived by Sinclair and also by Allan.

If the fractional fluctuation in Γ and Z are small, then the coefficient of the potential term $b(x)$ can be expanded to first order in x. The equations of motion for x and ϕ are

$$\frac{dx}{d\bar{t}} = -(b_o + xb_x)\sin\phi, \qquad (2.9.15a)$$

$$\frac{d\phi}{d\bar{t}} = -x - c - b_x\cos\phi. \qquad b)$$

Taking the time derivative of $\frac{d\phi}{d\bar{t}}$ and substituting for $\frac{dx}{d\bar{t}}$, we have:

$$\frac{d^2\phi}{d\bar{t}^2} - (b_o + xb_x)\sin\phi - b_x\sin\phi\frac{d\phi}{d\bar{t}} - \frac{dc}{d\bar{t}}(t,x) = 0. \qquad (2.9.16)$$

To $O(\mu^{1/2})$, $\frac{d\phi}{d\bar{t}}$ equals $(-x - c(t,x))$, and the two terms in (2.9.16) dependent on x and $\dot{\phi}$ tend to cancel. The function $\frac{dc}{d\bar{t}}$ includes the nearly constant torque term (2.9.10) plus a term which is suspiciously like a $\dot{\phi}$ term. If we eliminate x in favor of $\dot{\phi}$, change the time variable from $\bar{t}$ to t (2.9.8), and replace b_o by $A_{oxx}^{-1}A_1$, etc., the result is:

$$\frac{d^2\phi}{dt^2} + A_{oxx}(A_1 - c(t,x)A_{1x})\sin\phi + A_{oxx}^2\frac{d\sigma}{d\bar{t}}(t,0) - \rho A_{oxx}^{-1}\frac{d\phi}{dt} = 0 \qquad (2.9.17)$$

where the factor ρ in the dissipative term is defined by

$$\rho = 16j^2 L_o^{-1}\frac{dn_T}{dt} + \frac{K\mu}{\mu'}16j'^2 L_o'^{-1}\frac{dn_T'}{dt_o} < 0. \qquad (2.9.18)$$

If $\frac{dc(t)}{d\bar{t}}$ is small compared to b_o, then the time dependence of the torque will have little effect in the librational phase. In addition, the secular change in $c(t,x)$ is very nearly $c(t)$. The validity of the expansion is restricted to a time interval for which $c(t)b_x$ is small compared to b_o during transition. The first three terms (with $c(t,x)$ replaced by $c(t)$) constitute the equation Sinclair (1972) derived in his numerical calculation of transition probabilities for the Mimas-Tethys resonance. This restricted equation is identical to the second simple pendulum example treated in section (1.2). Allan neglected to consider the effect of the $c(t)$ term on the problem of capture into libration, although he did implicitly include it and the $\dot{\phi}$ term in his theory on the evolution of the Mi-Te system after transition.

2.10 SUMMARY

After a long sequence of approximations, we find that the satellite-satellite interaction can be approximated by the one-dimensional Hamiltonian

$$H(x,\phi,\bar{t}) = 1/2(x + c(c\ \bar{t}))^2 + b(x)\cos\phi. \qquad (2.10.1)$$

The equations of motion for x and ϕ are

$$\frac{dx}{d\bar{t}} = \frac{\partial H}{\partial x}, \qquad \frac{d\phi}{d\bar{t}} = -\frac{\partial H}{\partial \phi},$$

and it is understood that $\frac{\partial}{\partial x}$ does not act on $a(x,t)$. The explicit relations of each of the variables in H to the parameters defining each orbit follow. The angle variable is:

$$\phi = j\lambda + j'\lambda' + k\tilde{\omega} + k'\tilde{\omega}' + i\Omega + i'\Omega'. \qquad (2.10.2)$$

The most powerful restriction on the integers j, j', etc., derives from the fact that the interaction is independent of the coordinate system. Since each of the above angles is measured from a common reference, the sum of the integers must be zero. See (2.2.10) for further restrictions which apply in the two-body case.

Next, the momentum x is related to the elements e, I (e and I small) and a by (2.6.1) and (2.9.2):

$$L(x,t) = \sqrt{M_o a} = jx + L_o + \int\frac{dL}{dt}T\,dt,$$

$$\Gamma(x,t) \cong -\tfrac{1}{2}e^2 L_o \cong kx + \Gamma_o + \int\frac{d\Gamma}{dt}T\,dt, \qquad (2.10.3)$$

$$Z(x,t) \cong -\tfrac{1}{2}I^2 L_0 \cong ix + Z_0 + \int \frac{dZ_T}{dt}\,dt,$$

$$L'(x,t) = \sqrt{\mu' a'} = j\left(\frac{K\mu}{\mu'}\right)x + L_0 + \int \frac{dL'_T}{dt}\,dt, \text{ etc.}$$

Thus, the variable x is proportional to the fluctuation induced in the elements a, e, and I, etc., by the term in the disturbing function which contains the angle ϕ. Observe that the fractional fluctuation in L is of $O(e^2$ or $I^2)$ smaller than that of Γ or Z. This implies that the variation of $b(x)$ with x is principally determined by its leading factors of $\Gamma(x)$ and $Z(x)$ - if the inclination and eccentricity are small.

The function b(x) is related to the ϕ-dependent term in the disturbing function by (2.7.5c), i.e.:

$$b(x) = A_{oxx}^{-1} A_1(x), \tag{2.10.4}$$

where $\{A_1(x)\cos\phi\}$ is the corresponding term in R (2.2.10,11). A_{oxx} is the third coefficient in the Taylor expansion of the secular part of H and is approximately (2.7.2d),

$$A_{oxx} \cong \frac{3j^2}{a_o^2} + \frac{K\mu}{\mu'}\frac{3j'^2}{a'^2}. \tag{2.10.5}$$

The parameter K equals one if the indirect part of the disturbing function does not contribute to the resonance term. Otherwise it is given by (2.6.10).

The function c(x,t) is proportional to the acceleration of the motion of ϕ and is (2.9.6,11):

$$c(x,t) \doteq c(t) + \rho\int x d\bar{t} \tag{2.10.6}$$

where

$$c(t)A_{oxx}^2 \doteq -A_{oxx}(jn_o + j'n'_o) - \left(j\frac{dn_T}{dt_o} + j'\frac{dn'_T}{dt_o}\right)\overline{(t - t_o)},$$

$$\rho \doteq 16j^2 L_o^{-1}\frac{dn_{T}}{dt}a + \frac{K\mu}{\mu'}16j'^2 L_o'^{-1}\frac{dn'_{T}}{dt}a \quad .$$

The x-dependent term in (2.10.6) results from the fact that the tidal acceleration decreases as a function of the planet-satellite separation.

Finally, $\bar{t}$ is related to ordinary time t by

$$\bar{t} = A_{oxx}t.$$

The most important restrictions and approximations imposed in deriving (2.10.1) are the following. 1) The disturbing function can be expanded in the ratio of the semimajor axes, and is valid in both the librational and rotational phases. Therefore, the above Hamiltonian (2.10.1) does not apply to a two-body resonance of the synodic type which is restricted to the 1:1 commensurability. 2) The fractional fluctuations in the semimajor axes (or in L and L') caused by the perturbations are small. This allows us to expand the secular part of H in a Taylor series in x. But there are no restrictions on the magnitude of the fraction fluctuations in either e or I. 3) There exist no terms in the disturbing function (such as the short period terms) which cannot be removed or ignored because of their high frequencies or comparatively small coefficients. 4) Another condition is necessary in some cases (2.9), namely, that the orbital inclinations and eccentricities be relatively small. The reason is that we impose the restriction that fractional fluctuation in $L(x)$ be small compared to [illegible] of $\Gamma(x)$ or $Z(x)$. One consequence is that the fluctuation in $A_1(x)$, [illegible] to $O(\frac{\delta x}{L_o})$, governed by its leading factors of i and e. This greatly simplifies the functional form of $b(x)$. The restriction to small inclinations and eccentricities can be lifted, but it means that the terms in $b(x)$ of $O(\frac{\delta x}{L_o})$ must be explicitly determined to find the coefficient b_x. This is a non-trivial exercise in the first-order expansion of $b(x)$.

It may be possible to improve the order of approximation to $O(\mu^2)$ if the indirect part of the disturbing function does not contain a ϕ-dependent term. Also, if the mass and/or angular momentum of one partner is very much larger than that of the second, then a better approximation may be obtained by neglecting the motion of the first partner due to the second. Unfortunately, the order of approximation may not be set by the expansion parameter, but by those very long period terms which had to be ignored because they could not be removed by any technique such as Brown's method.

Although we have used the specific example of the satellite-satellite gravitational interaction, the same methods can be employed on gravitational resonances involving more than two satellites. Of course, the many-body resonance must satisfy the same general restrictions outlined for the two-body case. In addition, gravitational resonances with the geopotential can be similarly reduced to the one dimensional Hamiltonian form if they

also satisfy the relevant criteria.

3.1 INTRODUCTION

Now that the necessary preparation has been completed, we are ready to turn to the problem of transition in orbit-orbit commensurabilities. The theoretical basis of our discussion of transition of pendulum-like systems subject to torques was laid earlier with a full investigation of two simple pendulum examples. Recall that the corresponding Hamiltonian of the first (I) and second (II) examples were (1.2.1, 1.2.2a):

$$\text{I:}\quad H(p,\theta,t) = \tfrac{1}{2}p^2 + b(t)\cos\theta,$$

$$\text{II:}\quad H(x,\phi,t) = 1/2(x + c(t))^2 + b(t)\cos\phi.$$

From a study of I, a picture of transition was developed which involved the motion of the turning points or roots of (R(p) = 0) obtained from the integral solution for x. Explicitly,

$$\int dp\frac{\mathrm{sign}(-b\sin\theta)}{\sqrt{R(p)}} = t-t_0; \quad R(p) = b^2(t)-(H-\tfrac{1}{2}p^2)^2.$$

The significant results were: 1) The roots were labeled in the rotation phase according to the value of ϕ at that root (either mod(π) or mod(2π)) and to the sign of $p(= -\dot{\phi})$. 2) The relative position of these roots in the complex plane qualitatively determined whether the system executed rotations or librations (fig. 2.1a-c). 3) Transition involved the motion of the interior roots towards the origin and then out along the imaginary axis.

From this picture, a "transition phase" (fig. 1.2.2) was defined in which our attention was focused on the motion of the interior roots. We found that their motion was well-defined except for a very small set of possible motions near the sticking motion.

The same methods were employed on II, with the following results: 1) We found that its transition phase was approximately the same as that defined for I to first order in the small parameters $b^{-1}\frac{dc}{dt}$ and $b^{-3/2}\frac{db}{dt}$. 2) A critical angle ϕ_{ic} was derived which defined the sticking motion in which the pendulum went over the top for the last time, reversed sign, and again slowly approached the top, tending to "stick" there. This particular motion separated transitions which led to libration of the pendulum from transitions which led to reverse rotation. 3) Also, a probability for capture was derived, given arbitrary kinetic energy of the pendulum as it went over the top for the last time.

The other major development involved the orbit-orbit interaction and how it related to these simple pendulum examples. A one-dimensional Hamiltonian was derived that is a valid approximation, to $O(\mu\frac{\delta x}{L_0})$, of the very long period motion, including the secular motion due to the tidal influence. This third Hamiltonian (III) has the form (see section 2.10 for definition of terms):

$$\text{III:}\quad H(x,\phi,\bar{t}) = 1/2(x + c(x,\bar{t}))^2 + b(x)\cos\phi, \tag{3.1.1a}$$

where it is understood that the partial derivative with respect to x does not operate on the x dependence in $c(x,t)$ (section 2.9). The equations of motion are:

$$\frac{dx}{d\tilde{t}} = \frac{\partial H}{\partial \phi} = -b(x)\sin\phi \ , \tag{3.1.1b}$$

$$\frac{d\phi}{d\tilde{t}} = -\frac{\partial H}{\partial x} = -x - c(x,\tilde{t}) - b_x(x)\cos\phi . \tag{c}$$

Moreover, the validity of III extends to both the librational and rotational phases of the resonance variable ϕ, so that it can be used to discuss transition. When restricted to small inclinations and eccentricities, the fractional fluctuation of the coefficient of the potential term$(\frac{\delta b}{b})$ is determined to $0(\frac{\delta x}{L_o})$ by its leading factors of e and I. In addition, the simple e and I type resonances for which k equals i had identical functional forms for b(x). This suggests that most of the physics of transition can be discovered from a thorough investigation of a few well-chosen examples.

The $|k| = 1,2$ cases of the simple e type (i.e. $\sigma = j\lambda + j'\lambda' + k\omega$) will be these examples. The reasons are: 1) the $|k| = 1$ case comprises most of the observed resonances; 2) the $|k| = 2$ case is equivalent to a first order Taylor expansion of b(x) and is therefore a practical generalization of all possible two-body resonances (except synodic) in the small fluctuation limit ($\delta x << L_o, \Gamma_o, Z_o$).

The principal topic of this chapter will be to understand the transition mechanism operating in the orbit-orbit interaction in terms of the picture developed in section 1.2. Although such

effects as the secular change in the roots as the system approaches the transition phase are relevant, we shall defer their explicit determination until section 3.4, along with other topics related to the secular change in the elements in each phase, including the amplitude of libration. The principal analytical tools will be 1) action integral and 2) averaged equations of motion for the turning points of x.

There are several steps we can perform preparatory to an explicit examination of transition, among these being the derivation of the equations of motion, and a specific labeling scheme to identify roots. To further simplify the problem, we shall adopt the following conventions, which in no way restrict the generality of the problem.

a) The "normal" libration center (obtained from $b(x) = b(0)$) is at mod(2π), or equivalently, $b < 0$. If, for a given resonance variable ϕ, the normal center is a mod(π), then a new resonance variable can be defined ($\Psi = \phi + \pi$), for which the normal center is at mod(2π).

b) The angle ϕ is constructed such that the tidal torque secularly decreases $\dot{\phi}$. This means that $\dot{\phi}$ is positive before the commensurability is established. It follows that, in the positive rotation phase far from transition, the function $c(x,\tilde{t})$ is less than zero and $\frac{dc}{dt}$ is greater than zero.

c) In the positive rotation phase far from transition, choose Γ_o such that the time average of x vanishes. Recall that the action $J_{pos.rot.}$ also vanishes given this particular choice. Another consequence is that $H_{pos.rot.} \cong 1/2c^2$.

From the integral solution for x, we find that the roots of the quartic polynomial R(x) bound the motion of x. This statement is true even if these roots are time (or momentum!) dependent. The roots are obtained from R(x) = 0, or

$$R(x) = b^2(x) - (H - 1/2(x + c)^2)^2 = 0 \qquad (3.1.2)$$

For the simple e-type resonance, the function b(x) has the form (2.7.8a, 2.4.4):

$$b(x) = \frac{\mu'}{a_>} A_{oxx}^{-1} Ce^{|k|}(x) \qquad (3.1.3)$$

$$e^{|k|}(x) = (2L_o^{-1}(-kx - \Gamma_o))^{|k|/2}$$

where

$$-\Gamma_o \cong 1/2e_o^2 L_o > 0. \qquad (3.1.3)$$

To simplify the algebraic manipulation, the variables x, c, H, b(x) and $\bar{t}$ will be rescaled such that the roots of (R(x) = 0) are dimensionless, while the equations of motion remain canonical. First, divide R(x) by Γ_o^2 and redefine x, c(x,t), p (which occurs in c(x,t)), and H by

$$\frac{x}{(-\Gamma_o)} \Rightarrow x; \quad \frac{c}{(-\Gamma_o)} \Rightarrow c; \quad \frac{p}{(-\Gamma_o)} \Rightarrow p; \quad \frac{H}{\Gamma_o^2} \Rightarrow H. \qquad (3.1.4)$$

The function $b(x)\Gamma_o^{-2}$ will be replaced by b(x) which in turn is to the following.

$$b(x)\Gamma_o^{-2} \Rightarrow b(x) = \beta(-kx + 1)^{|k|/2},$$

where

$$\beta = 4\frac{\mu'}{\mu_o} \frac{Ce_o^{(|k| - 4)}}{A_{oxx} a_> \;\; a_>} \; . \qquad (3.1.5)$$

Finally, the equations of motion are returned to canonical form by defining a new time $\bar{\bar{t}}$ related to the times $\bar{t}$ and t (2.9.8) by

$$\bar{\bar{t}} = (-\Gamma_o)\bar{t} = 1/2e_o^2 a^2 n A_{oxx} t. \qquad (3.1.6)$$

With this set of transformations, the equations of motion are.

$$\frac{dx}{d\bar{\bar{t}}} = +\frac{\partial H}{\partial \phi} = -\beta(-kx + 1)^{|k|/2}\sin\phi \qquad (3.1.7a)$$

$$\frac{d\phi}{d\bar{\bar{t}}} = -\frac{\partial H}{\partial x} = -x - c + \frac{|k|}{2}k\beta(-kx + 1)^{(|k|/2 - 1)}\cos\phi. \qquad b)$$

Hereafter, the double bar notation on the time (i.e. $\bar{\bar{t}}$) shall be dropped. Incidently, $c(x,\bar{\bar{t}})$ now takes the form

$$c(x,\bar{\bar{t}}) = c(0,\bar{\bar{t}}) + p\int x d\bar{\bar{t}} \qquad (3.1.8)$$

In addition, we see that e(x) is now related to x by the following:

$$\phi(x) = e_o(-kx + 1)^{1/2} \tag{3.1.9}$$

The four roots, obtained from $R(x) = C$, shall be labeled according to the following scheme.

a) First, if a given root x_r satisfies the equation

$$H = 1/2(x_r + c)^2 + b(x_r), \tag{2.1.10a}$$

then that root is a 2π-root ($x_{2\pi}$). Those roots which satisfy

$$H = 1/2(x_r + c)^2 - b(x_r) \tag{b}$$

shall be labeled π-roots (x_π).

b) The following rules shall be used to distinguish between π-roots:

1) If there exists only one π-root, the label is unique.

2) For the case where two π-roots are real, label them $x_{\pi+}$ and $x_{\pi-}$ with $x_{\pi+} > x_{\pi-}$. If these two roots are complex conjugate, again label them $x_{\pi+}$ and $x_{\pi-}$, with $\mathrm{Im}\ x_{\pi+} > \mathrm{Im}\ x_{\pi-}$.

3) There can exist three π-roots for the $|k| = 1$ case. Label them $x_{\pi-}$, $x_{\pi+}$, $x^*_{\pi+}$ if $x_{\pi-} < \mathrm{Re}\ x_{\pi+}$, or $x_{\pi+}$, $x_{\pi-}$, $x^*_{\pi-}$ if $x_{\pi+} > \mathrm{Re}\ x_{\pi-}$. For three real roots, label them $x_{\pi-}$, $x_{\pi+}$, $x^*_{\pi+}$ if $x_{\pi-} \le x_{\pi+} \le x^*_{\pi+}$.

Similar rules can be used to uniquely label the 2π-roots.

The next step is to derive the equations of motion of the roots.

A root x_r is an implicit function of the variables x and ϕ through the functions H and c. The equation of motion can be expressed as:

$$\frac{dx_r}{dt} = \frac{dc}{dt}\frac{\partial x_r}{\partial c} + \frac{dH}{dt}\frac{\partial x_r}{\partial H} \tag{3.1.11}$$

The partial derivatives of x with respect to c and H can be expressed as functions of the roots using (3.1.10):

$$\frac{\partial x_\pi}{\partial H} = -\frac{1}{\dot\phi(\pi)}, \qquad \frac{\partial x_\pi}{\partial c} = +\frac{x_\pi + c}{\dot\phi(\pi)},$$

also, the full time derivative of H equals its partial time derivative. Carrying out these steps, the equations of motion of the roots are:

$$\frac{dx_\pi}{dt} = \frac{dc(x,t)}{dt}\left(\frac{x_\pi - x}{\dot\phi(x_\pi)}\right), \tag{3.1.12a}$$

$$\frac{dx_{2\pi}}{dt} = \frac{dc(x,t)}{dt}\left(\frac{x_\pi - x}{\dot\phi(x_{2\pi})}\right), \text{ where} \tag{b}$$

$$\dot\phi(x_\pi) = -(x_\pi + c - b_x(x_\pi)), \tag{3.1.13a}$$

$$\dot\phi(x_{2\pi}) = -(x_{2\pi} + c + b_x(x_{2\pi})). \tag{b}$$

For the case where two π-roots are complex conjugate, their equations of motion can be separated into real and imaginary parts. The result is

$$x_\pi = \mathrm{Re}\, x_\pi + i\, \mathrm{Im}\, x_\pi, \tag{3.1.14a}$$

$$\frac{d\mathrm{Re}\, x_\pi}{dt} = \frac{dc(x,t)}{dt}\left[(x - \mathrm{Re}\, x_\pi)\frac{\partial}{\partial H}\mathrm{Re}\, x_\pi - \mathrm{Im}\, x_\pi \frac{\partial}{\partial H}\mathrm{Im}\, x_\pi\right], \qquad \text{b)}$$

$$\frac{d\mathrm{Im}\, x_\pi}{dt} = \frac{dc(x,t)}{dt}\left[(x - \mathrm{Re}\, x_\pi)\frac{\partial}{\partial H}\mathrm{Im}\, x_\pi - \mathrm{Im}\, x_\pi \frac{\partial}{\partial H}\mathrm{Re}\, x_\pi\right]. \qquad \text{c)}$$

The partial derivatives of $\mathrm{Im}\, x_\pi$ and $\mathrm{Re}\, x_\pi$ can be obtained from (3.1.9b) after separating these equations into their real and imaginary parts. This procedure works well for the $|k| = 2$ case, but not for the $|k| = 1$ case. A better method for this latter case is to separate the quartic polynomical $\{R(\mathrm{Re}\, x_\pi + i\, \mathrm{Im}\, x_\pi) = 0\}$ into its real and imaginary parts, and from these relationships determine the partial derivatives as functions of the real and imaginary parts of x_π.

Before proceeding, we shall consider the effect of approximating $c(x,t)$ by $c(0,t)$ during the transition phase. To estimate the effect of the action variable in $c(x,t)$ on the probability of capture into libration, let's approximate $b(x)$ by $b(0) \equiv \beta$, so that the pendulum motion is approximately that of II. The description of the transition phase will be qualitatively similar to that for I, to lowest order in the small parameter $\beta^{-1}\frac{dc}{dt}$ (fig. 1.2.2). The equation of motion of the imaginary part of x_π, during transition, is

$$1/2\, \frac{d}{dt}\mathrm{Im}x_\pi = -\frac{dc(x,t)}{dt}(x + c) \;. \tag{3.1.15}$$

Expressing $\frac{dc(x,t)}{dt}$ and $(x + c)$ in terms of $\dot\phi$, using (2.13.6), we find

$$1/2\, \frac{d\mathrm{Im}^2 x_\pi}{dt} = (\frac{dc_o}{dt} - \rho c)\dot\phi - \rho\dot\phi^2 \;. \tag{3.1.16}$$

At transition, the term $\rho c(t)$ is of $O(\frac{dc}{dt}\mu^{\frac{1}{2}})$ and will be neglected. The transition phase integral determines the critical angle ϕ_{ic} for which the sticking motion ($\dot\phi \to 0-$) occurs. The condition is that $\mathrm{Im}^2 x(f) = \mathrm{Im}^2 x(i) = 0$, or, after integrating (2.1.16),

$$0 = +\frac{dc}{dt}(\phi_f - \phi_{ic}) - \rho\int_{t_i}^{t_f}\dot\phi^2 dt \tag{3.1.17}$$

To lowest order, $\phi_f \cong \pi$ and $H \cong -\beta$ during transition. Therefore $\dot\phi$ is

$$\dot\phi^2 \simeq -2\beta(1 + \cos\phi),$$

and (3.1.17) reduces to

$$0 = \frac{dc}{dt}(\pi - \phi_{ic}) + 2\rho\beta\int_{t_i}^{t_f} dt(1 + \cos\phi). \tag{3.1.18}$$

Since ϕ_{ic} lies in the range $\pi \lesssim \phi_{ic} \lesssim 3\pi$, and $\beta < 0$, we can deduce that capture occurs only if $\rho < 0$. The above equation (3.1.18) is identical to the relation which defines the sticking motion for II, if the factor $(\frac{db}{dt})$ is replaced by $(2\rho\beta)$. Therefore the probability for capture can be immediately derived by replacing the same factor in (1.2.3a). The result is

$$P_c = \frac{2}{1 - \frac{\pi}{4}(|\beta|^{-1/2}\rho^{-1}\frac{dc}{dt}} \qquad (3.1.19)$$

It is understood that if $\epsilon > 0$, then $P_c = 0$, and if $P_c > 1$, capture occurs for all possible initial conditions, or has unit probability. The term

$$|\beta|^{-1/2}\rho^{-1}\frac{dc}{dt}$$

is, for the simple-e case (3.1.3, 2.10.6),

$$|\beta|^{-1/2}\rho^{-1}\frac{dc}{dt} = -\left|\frac{\mu'}{\mu_o}\frac{Ce_2|k|}{a_2 s_o \lambda_{oxx}}\right|^{-1/2} \frac{(j\frac{dn_T}{dt_o} + j'\frac{dn'_T}{dt_o})}{16(j^2\frac{dn_T}{dt_o} + j'^2 K\frac{\mu L_o}{\mu' L'_o}\frac{dn'_T}{dt_o})} \qquad (3.1.20)$$

The magnitude of the above term is of $O((\frac{\mu'_e|k|}{\mu_o})^{-1/2})$, which is, by assumption, very large. But this implies that P_c, due to the x dependence in $c(x,t)$ is very small.

Incidentally, if we choose $c(x,t)$ to equal

$$c(x,t) = \frac{dc}{dt}(t - t_o) + \rho\int(x + c(x,t))dt, \qquad (3.1.21)$$

then the second order equation for ϕ is:

$$\ddot{\phi} - b\sin\phi = \frac{dc_o}{dt} - \rho\dot{\phi}. \qquad (3.1.22)$$

The left hand side of this equation is formally identical to the

equation which describes the spin-orbit interaction, while the right hand side of this equation is one possible form for the momentum-dependent torque acting on the affected planet. Using this equation, Goldreich and Peale (1966) derived a probability for capture identical to (3.1.19).

3.2 TRANSITION THEORY FOR SIMPLE ECCENTRICITY-DEPENDENT RESONANCE: $(\phi = j\lambda + j'\lambda' + k\varpi)$ WHERE $|k| = 2$

Because of its mathematical simplicity, the $|k| = 2$ case will be examined first. The explicit solution of the roots as functions of c and H is particularly simple, since b(x) is linear in x. The explicit solutions for each root, along with their identification, are:

$$x_{\pi\pm} = -(c + k\beta) \pm\sqrt{(c + k\beta)^2 + 2(H + \beta - 1/2\, c^2)}$$

$$= -(c + k\beta) \pm\sqrt{A}; \qquad (3.2.1a)$$

$$x_{2\pi\pm} = -(c - k\beta) \pm\sqrt{(c - k\beta)^2 + 2(H - \beta - 1/2c^2)}$$

$$= -(c - k\beta) \pm\sqrt{A - 4\beta(kc + 1)}\ ; \qquad b)$$

while their equations of motion are:

$$\frac{dx_{\pi\pm}}{dt} = \frac{dc}{dt}\frac{(x - x_{\pi\pm})}{\mp\sqrt{A}}\ , \qquad (3.2.2a)$$

$$\frac{dx_{2\pi\pm}}{dt} = \frac{dc}{dt}\frac{(x - x_{2\pi\pm})}{\mp\sqrt{A - 4\beta(kc + 1)}}\ . \qquad b)$$

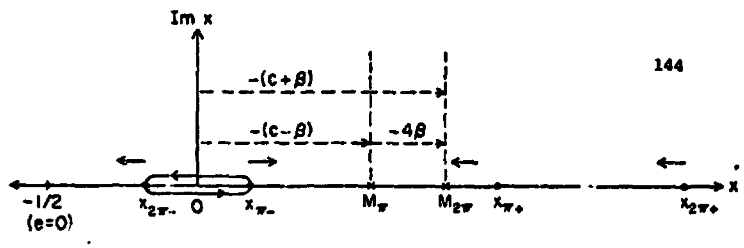
Since H is of $O(1/2c^2)$ in the positive rotation phase, far from transition, both the roots $x_{2\pi-}$ and $x_{\pi-}$ are of $O(\frac{\beta}{c})$ while $x_{2\pi+}$ and $x_{\pi+}$ are of $O(-2c)$. In the positive rotation phase, x is bounded

between the smaller pair of roots $x_{2\pi-}$ and $x_{\pi-}$. Inspection of the equation of motion for x reveals that the most negative value of x occurs at $\phi = \text{mod}(2\pi)$. Thus we can deduce $x_{2\pi-} < x_{\pi-}$ The relative location of $x_{\pi+}$, $x_{2\pi+}$ can be discovered by inspection of their difference:

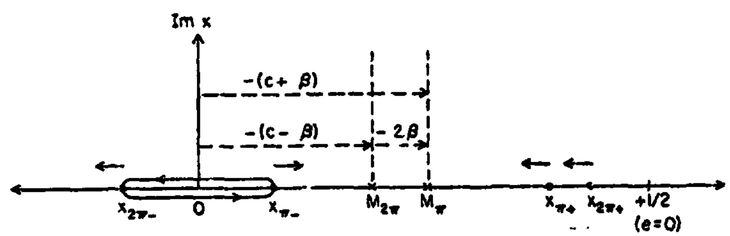
$$x_{2\pi+} - x_{\pi+} = 2k\beta + \sqrt{A - 2\beta(kc + 1)} - \sqrt{A}. \qquad (3.2.3)$$

for $k = -2$, $x_{2\pi+} > x_{\pi+}$ in the positive rotation phase. If $k = +2$, then $x_{\pi+} > x_{2\pi+}$ far from transition ($\langle\dot{\phi}\rangle >> |b_x|$). But near transition, these roots may interchange their relative positions. We should point out that in the positive rotation phase, the value of x for which b(x) vanishes (or $e = 0$) equals k^{-1} and must be less than $x_{2\pi-}$ when $k < 0$ and greater than $x_{\pi-}$ when $k > 0$ independent of the magnitude of k From an inspection of (3.1.9), we can deduce that for $|k| = 2$, a π and a 2π root are equal if and only if they equal k^{-1}. Conversely, if a π (or 2π) root equals k^{-1}, then there must exist at least one 2π (or π) root which equals k^{-1}. Therefore, we can conclude that 1) for $k = -2$, no root equals k^{-1} in either the positive rotation phase or the transition phase; 2) if $k = 2$, then the roots $x_{\pi+}$ and $x_{2\pi+}$ interchange where b(x) vanishes.

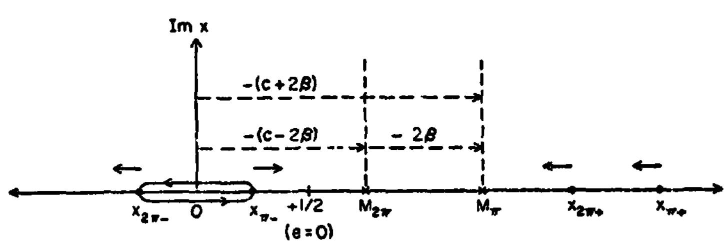
This information is sufficient to construct the possible rotation phase diagrams for each case (fig. 3.2.1a-c). Incidently, the principal qualitative difference in the positive rotation phase diagrams for this example and II is that in II (fig. 1.2.1a) the



a) Positive rotation phase for $k = -2$. M_π and $M_{2\pi}$ are the midpoints of the π and 2π pairs of roots, respectively.



b) Positive rotation phase for $k = +2$. Here $x_{2\pi+}$ and $x_{\pi+}$ are to the left of $x = +1/2$.



c) Positive rotation phase for $k = +2$. In this diagram the roots $x_{2\pi+}$ and $x_{\pi+}$ are to the right of $x = +1/2$, where $b(x)$ vanishes.

FIGURE 3.2.1

POSITIVE ROTATION PHASE DIAGRAMS FOR $|k| = 2$.

midpoints of the pair of π and 2π roots coincide, while in this example their respective midpoints, M_π and $M_{2\pi}$, are offset by an amount $(-2k\beta)$. There exists only one possible diagram for $k = 2$ (fig. 3.2.1a), while there are two possibilities for $k = +2$ (fig. 3.2.1b,c). Figures 3.2.1a,b are qualitatively similar to the diagram for the simple pendulum (fig. 1.2.1a). In addition, the transition phase for these diagrams will involve the π-roots which coincide, and then develop an imaginary component.

Figure 3.2.1c suggests that there exists a radically different form of transition directly into the libration phase, in which the roots $x_{\pi-}$ and $x_{2\pi+}$ exchange their relative positions on the real axis. The equations of motion of these roots (3.2.2) show that this automatic transition is an allowed motion of the roots. Whether it occurs depends explicitly on the value of the parameters c, H, and β at transition. Subtracting $x_{\pi-}$ and $x_{2\pi+}$ at coincidence we find:

$$x_{2\pi+} - x_{\pi-} = +2k\beta + \sqrt{A - 4\beta(kc + 1)} + \sqrt{A} = 0 . \qquad (3.2.4)$$

If $k = -2$ the above cannot vanish since β is negative. Transposing $(2k\beta)$ in the above equation and squaring, we find that H must satisfy the relation

$$H + 1/2\sqrt{A}\ \sqrt{A - 4\beta(kc + 1)} = 2\beta^2 \qquad (3.2.5)$$

at coincidence.

The condition that separates these two types of transition is for the two π-roots to coincide at the same moment that $x_{\pi-}$ and $x_{2\pi+}$ coincide. This requirement is satisfied if $A = 0$, $H = 2\beta^2$ and $c + k\beta = -1/2$. It turns out that this information is not sufficient to uniquely sepcify the parameters. Another relation can be obtained by evaluating the action in the positive rotation phase for the special case where $x_{\pi-} = x_{2\pi+} = x_{\pi+} = 1/2$ (see B. 19,20). The result is:

$$\beta = -1/8; \quad c = -1/4; \quad H = 1/32; \quad x_{2\pi-} = -1/2. \tag{3.2.6}$$

If $|\beta| \geq 1/8$ and $k = +2$, the system automatically makes a transition from the positive rotation phase into the libration phase. For $|\beta| < 1/8$, we expect that capture into libration depends on the initial conditions, much as it did for II. Transition of this type begins when the π-roots coincide and move into the complex plane.

The π-roots are complex if $A < 0$. From inspection of (3.2.1a), the real and imaginary components of $x_{\pi+}$ are easily identified and are

$$\mathrm{Re}\, x_{\pi\pm} = -(c + k\beta) \tag{3.2.7a}$$

$$\mathrm{Im}\, x_{\pi\pm} = \pm\sqrt{-A} \,. \tag{b}$$

Therefore, the equation of motion of the imaginary component is:

$$1/2 \frac{d\, \mathrm{Im}^2 x_\pi}{dt} = \frac{dc}{dt}(\mathrm{Re}\, x_\pi - x). \tag{3.2.8}$$

The next step is to integrate both sides between the time t_i that the

π-roots first coincide and the time t_f that $x = \mathrm{Re}\, x_\pi$. The result is

$$1/2(\mathrm{Im}^2 x_\pi(f) - \mathrm{Im}^2 x_\pi(i)) = \frac{dc}{dt}\int_{t_i}^{t_f} dt(\mathrm{Re}\, x_\pi - x), \quad \frac{dc}{dt} > 0 \tag{3.2.9}$$

By construction, $\mathrm{Im}^2 x_\pi(i) = 0$. The condition which separates transition into libration from transition into negative rotation is that $\mathrm{Im}^2 x_\pi(f)$ equals zero to first order in the small parameter $\beta^{-1}\frac{dc}{dt}$. Recall that this defines the sticking motion where $\dot\phi \Rightarrow 0-$. If $\mathrm{Im}^2 x_\pi(f) > 0$, then the imaginary components are nonzero as $x \Rightarrow x_{\pi+}$ and $\dot\phi$ reverses sign again. The conclusion is that the system has successfully entered the libration phase. But if the integral is negative, the implication is that the system has made the transition into the negative rotation phase.

The integration variable in (3.2.11) can be changed from t to x. Then the condition that the integral vanishes will determine the initial value of x, x_{ic}, which leads to the sticking motion $\dot\phi \Rightarrow 0-$. Unfortunately, it turns out that x_{ic} explicitly depends on the value of the roots at transition, which in turn depends, in a complicated way, on the parameter β. The dependence on β must be determined using the action J. Instead of pursuing this course, we shall determine an approximate condition, accurate in the small fluctuation limit $(\frac{\delta b}{3} \ll 1)$ and delay finding a more accurate relationship until the next section.

An instructive transformation is to observe that $x - \mathrm{Re}\, x_\pi$ is related to $\dot{\phi}$ by

$$(x - \mathrm{Re}\, x_\pi) = -\dot{\phi} + k\beta(1 + \cos\phi). \tag{3.2.10}$$

The transition criterion is then

$$0 = \frac{dc}{dt}[\phi_{ic} - \phi_f + k\beta\int_{t_{ic}}^{t_f} dt(1 + \cos\phi)], \tag{3.2.11}$$

To lowest order, $\phi(f) \cong \pi$ and ϕ_{ic} lies in the range $\pi \lesssim \phi_{ic} \lesssim 3\pi$. Thus for $k = -2$, the system completely evades capture into the libration phase. Observe that this relation is similar to (1.2.26) if the parameter $\frac{db}{dt}$ in (1.2.26) is replaced by $k\beta\frac{dc}{dt}$. If the fluctuation in b(x) is small, then the motion is nearly that of a simple pendulum. Given this approximation, the probability P_c for capture can be immediately obtained by replacing $\frac{db}{dt}$ by $k\beta\frac{dc}{dt}$ in (1.2.40). Furthermore, we can find a general result applicable in the small fluctuation limit if we replace $-k\beta$ by $\frac{db}{dx}$. The probability for capture for a system with an action-dependent potential is:

$$P_c = \frac{2}{1 + \pi/2\,(|\beta|^{-1/2} b_x)^{-1}}\,, \quad \frac{\delta b}{\beta} << 1. \tag{3.2.12}$$

The restrictions are that $P_c = 0$ if $b_x < 0$ and $P_c = 1$ if $|\beta|^{-1/2} b_x \geq 2/\pi$. For the e^2 type resonance, the condition that $P_c = 1$ (i.e. automatic transition) exactly corresponds to the requirement that $x_{\pi-} = x_{\pi+} = x_{2\pi+}$ at transition, or $\beta = -1/8$. The above formula predicts that $P_c = 1$ when $\beta = -\pi^2/16$. Therefore, the applicability of the above formula must be restricted to values of $|\beta|$ much smaller than 1/8 in order to keep δx and hence $\frac{\delta b}{\beta}$ small. For the $|k| = 2$ case, the "small fluctuation limit" is defined by the condition that $|\beta| << 1/8$. Conversely, the "large fluctuation limit" is defined by the condition that $|\beta| \gtrsim 1/8$. At the end of the following section a more accurate probability estimate is derived (see Fig. 3.3.9,10).

There is a more serious question concerning the validity of the Hamiltonian in the limit where b(x) vanishes. This problem is connected with the existence of terms proportional to e in the interaction which can cause very large fluctuations in $\bar{\omega}$ as η approaches zero, and is discussed more fully at the end of section 3.3. The indication from that analysis is that the automatic transition mechanism for the k = 2 case is seriously affected by such terms.

3.3 TRANSITION THEORY FOR SIMPLE ECCENTRICITY-DEPENDENT RESONANCE: $(\phi = j\lambda + j'\lambda' + k\tilde{\omega})$ WHERE $|k| = 1$

The case in which $b(x)$ is proportional to the first power of e is particularly interesting because of both its exotic behavior and the fact that the majority of the naturally occurring resonances are of this type. It includes the Enceladas-Dione $(\lambda_{En} - 2\lambda_{Di} + \tilde{\omega}_{Di})$ and the Titan-Hyperion $(-4\lambda_{Hy} + 3\lambda_{Ti} + \tilde{\omega}_{Hy})$ examples. The bag of tricks used on this case shall be more qualitative than the rather straightforward method applied on the simple pendulum and $|k| = 2$ cases. The principal difficulty is that although the quartic equation $(R(c) = 0)$ can be solved for its roots, the solutions are too complicated to derive from them the position of the roots in the complex plane as functions of the parameters c, β and H (see App. C). Fortunately, there are other ways to answer the important questions. Our past experience with the k = 2 case should suggest that this example should behave similarly in the small fluctuation limit. That is, capture into libration may occur, depending on the initial conditions, if k is positive, while it is evaded if k is negative. The first really interesting question is, how does this system behave in the large fluctuation limit? From analogous behavior for k = 2, we expect to find that for k = +1 the system automatically enters the libration phase if the parameter β is large enough. A second question of concern is, what condition

separates the two types of behavior? For $|k| = 2$, the condition was that three roots equal 1/2 (3.2.6). The final question is, how does the system enter the inverted libration phase - if it can? As with previous examples, the starting point of this discussion shall be an investigation of the motion of the system in the positive rotation phase.

Far from transition, the motion of the real variable x in the complex plane must be bounded on the left by a 2π-root $(x_{2\pi-})$ and on the right by a π-root $(x_{\pi-})$, while the magnitude of these roots is of $O(\frac{\beta}{c})$. The relative magnitude and position of the remaining pair of roots can be discovered from an inspection of the equation $(R(x_r) = 0)$. For $|k| = 1$, we find

$$R(x_r) = [H - 1/2(x_r + c)^2]^2 - \beta^2(-kx_r + 1) = 0. \qquad (3.3.1)$$

Since H is of $O(1/2\ c^2)$, the first term in (3.3.1) is small only if x_r is approximately equal to either 0 or -2c. The first case corresponds to the relatively small bounding roots, $x_{2\pi-}$ and $x_{\pi-}$. If c is small compared to 1, then the pair of roots of O(-2c) is also real, since the second term in (3.3.1) is then negative. But if (-2kc) > 1, then both terms in (3.3.1) are positive. The conclusion must be that if (-2kc) >> 1, then the large roots are complex. Thus, in the positive rotation phase, where -c >> 1, the large roots are real if k = -1 and complex if k = +1, while in the negative rotation phase, the converse is true.

If the roots are real, then their labeling in the positive rotation phase must be similar to that of the simple pendulum. That is, the large roots are $x_{\pi+}$ and $x_{2\pi+}$, and the two π-roots are interior to the 2π-roots, the same as in II. This description always applies to the $k = -1$ case and would appear to lead to uninteresting behavior during transition in that it always enters the negative rotation phase. Although this is true, as shall presently be demonstrated, this peculiar pendulum can, for a time, librate in an inverted fashion.

For $k = +1$, these large roots are complex, if $-c >> 1$. They can be identified by solving for the roots in the positive rotation phase for typical values of c, H and β, and then substituting them into (3.1.9), which defined the 2π and π-roots, respectively. The result of this exercise is that these complex roots are both π-roots and, in the positive rotation phase, can be labeled $x_{\pi+}$ and $x^*_{\pi+}$. Another piece of information that could be useful later would be to have some idea of the magnitude of the imaginary parts of these "exterior" π-roots. We expect the imaginary part to be small compared to the real part (this shall be rigorously demonstrated later). If this is so, then we can substitute $x_\pi = \mathrm{Re}\, x_\pi + i\,\mathrm{Im}\, x_\pi$ directly in the relation defining the π-roots (3.1.9), expanding the factor $(-kx_\pi + 1)^{1/2}$ and separating out the lowest-order imaginary component. We find $\mathrm{Im}\, x_\pi$ is, approximately,

$$2c\, \mathrm{Im}\, x_\pi \approx 1/2\, \beta\, |\mathrm{Re}\, x_\pi|^{-3/2}, \qquad (3.3.2)$$

which is small if $|\mathrm{Re}\, x_\pi| >> (\frac{\beta}{c})^{2/3}$.

For the case where all roots are real prior to entering the transition phase, we expect that transition is initiated by the coincidence of the interior π-roots. If the roots $x_{\pi+}$ and $x^*_{\pi+}$ remain complex through transition, then something analogous to the automatic transition from rotation into libration found for the $k = +2$ case must occur. Recall that this phenomenon happened when in the large fluctuation limit both $c(t)$ and β were relatively large parameters.

To discover the equivalent phenomenon for the $|k| = 1$ case, we shall assume transition occurs when $|c| >> 1$. The maximum fluctuation of x must be of $O(1)$, since $b(x) \propto (-kx + 1)^{1/2} \geq 0$. This implies that the Hamiltonian can be approximated by neglecting the x^2 term for values of $|c| >> 1$, since the real part of the complex roots is of $O(-2c)$. Physically, this means that the variation in the mean longitude of either resonance partner is small compared to the varition in the pericenter of the unprimed partner. The result is

$$H - {}^{1}/2c^2 \approx h = xc + \beta(-kx + 1)^{1/2}\cos\phi. \qquad (3.3.3)$$

If one of the turning points in the motion of x occurs at $x = k$, then h equals kc. This condition defines the value of h for which b(x) periodically vanishes whenever x equals k. The values of h for which the system either librates or rotates can be determined by substituting σck for h in (3.3.3) and solving for $\cos\phi$. The result is

$$\cos\phi = \frac{kc(-kx + \sigma)}{\beta(-kx + 1)^{1/2}} \quad (3.3.4)$$

where σ is an arbitrary dimensionless parameter. The physical solution must be such that $kx < 1$ for all allowed values of x. If the param er σ is < 1, then it appears that the right hand side of (3.3.3) can t on both positive and negative values, with the implication that the system has either rotations or librations of amplitude > 90°. (Later we shall discover that only the former is allowed.) But i. σ > 1, then the left hand side cannot change sign as x varies. Si ce $\frac{c}{\beta} > 0$, the turning points in the motion of x must occur at $\cos\phi = k$. The implication is that this system librat about the 2π-center for k = +1 and about the π-center for k = -1 with amplitude ≤ 90°.

To better understand the behavior of this unusual pendulum, let's reconsider the integral solution of x:

$$\int\frac{dx}{\sqrt{R(x)}}\,\mathrm{sign}(-\beta\sin\phi) = t - t_o. \quad (3.3.5a)$$

The function R(x) now equals

$$R(x) = \beta^2(-kx + 1) - (h - cx)^2, \quad b)$$

and is a quadratic polynomial in ~ instead of a quartic polynomial. That is, there are only two bounding roots, x_+ and x_-, given by

$$x_\pm = \frac{1}{2c^2}[(2ch - \kappa\beta^2) \pm|\beta|\sqrt{\beta^2 + 4c(c - kh)}]. \quad (3.3.6)$$

In the positive rotation phase, the roots x_+, x_- can be equivalently labeled $x_{\pi-}$ and $x_{2\pi-}$, respectively. Since the labeling is unique until h = ka, we must conclude that the system must rotate up to the critical value of h = ka.

The [illegible]tegral solution for x, assuming that the parameter c is consta[illegible]n particularly simple in that the integral is propo[illegible]e arcsine of a linear function of x. The solution fo[illegible]ion of t is:

$$\ldots(x_+ - x_-)\sin|c|(t - t_o) - \frac{k\beta^2}{2c^2} + \frac{h}{c}\ . \quad (3.3.7)$$

Observe hat the frequency of the motion equals |c| and becomes small as $|c| \to 0$. But transition is already presumed to occur when $|c| \gg 1$. Therefore, the secular behavior of the system as a unction of c can be obtained from t ction integral J, since J is adiabatically conserved when the oscillation frequency is large compared to the changes in the system's parameters.

In the positive rotation phase, the appropriate integral is:

$$J_{pos.rot.} = \int_0^{2\pi} x\,d\phi. \quad (3.3.8)$$

The first step in evaluating this integral is to express x as a function of ϕ using (3.3.3):

$$x = \frac{1}{2c^2}[2ch - \kappa\beta^2\cos^2\phi \pm|\beta\cos\phi|\sqrt{\beta^2\cos^2\phi + 4c(c - kh)}]. \quad (3.3.9)$$

Since the turning points in the motion of x occur at $|\cos\phi| = 1$, the two solutions ... implicit in (3.3.9) must be matched at either ... or when the argument of the radical in (3.3.9) vanishes. The first choice applies if the pendulum rotates and the second if it librates. *In the positive rotation phase, the matching of the two solutions* leads to the following result:

$$x_{pos.rot.} = \frac{1}{2c^2}\left\{2ch - k\beta^2\cos^2\phi + \beta\cos\phi\sqrt{\beta^2\cos^2\phi + 4c(c - kh)}\right\}. \tag{3.3.10}$$

In the libration phase, the argument of the radical vanishes where $\dot\phi$ reverses sign. Also, $x_{lib.}$ is a minimum at $\phi = 2\pi$, $\dot\phi > 0$ for the $k = 1$ case, while it is a maximum at $\phi = \pi$, $\dot\phi > 0$ for $k = -1$. This implies that the contribution from the second term in (3.3.9) must be multiplied by $\mathrm{sign}(-k\dot\phi)$. Explicitly:

$$x_{lib.} = \frac{1}{2c^2}\left\{2ch - k\beta^2\cos^2\phi + \mathrm{sign}(-k\dot\phi)\left|\beta\cos\phi\right|\sqrt{\beta^2\cos^2\phi + 4\beta(\beta - kh)}\right\}. \tag{3.3.11}$$

Observe that the first term in (3.3.10) for $x_{pos.rot.}$ is an even function while the second is an odd function in ϕ over the interval $(0, 2\pi)$. This means that the integral of the second term over the range $(0 \leq \phi \leq 2\pi)$ must vanish. Thus, $J_{pos.rot.}$ is given by

$$J_{pos.rot.} = \frac{\pi}{2c^2}(4ch - k\beta^2). \tag{3.3.12}$$

Since $J_{pos.rot.}$ vanishes, the average behavior of h as a function of c in the positive rotation phase is given by

$$h_{pos.rot.} = \frac{k\beta^2}{4c}. \tag{3.3.13}$$

At transition, $h = kc$, implying that $c = 1/2\beta$.

The next step is to evaluate J in the librational phase:

$$J_{lib.} = \oint x_{lib.}\,d\phi = \int_{\phi_{min}}^{\phi_{max}} x_{lib.}(\dot\phi > 0)\,d\phi + \int_{\phi_{max}}^{\phi_{min}} x_{lib.}(\dot\phi < 0)\,d\phi; \tag{3.3.14}$$

$(\phi_{max}\phi_{min})$ are the angles at which $\dot\phi$ vanishes. Clearly, the only nonzero contribution to $J_{lib.}$ involves the sign $(\dot\phi)$ dependent part of $x_{lib.}$. The result of integrating (3.3.14) is

$$J_{lib.} = \frac{k\pi c^{-2}}{2}(\beta^2 + 4c(c - kh)). \tag{3.3.15}$$

Evaluating $J_{lib.}$ at $h = kc$ and $c = 1/2\beta$, we find that $J_{lib.} = 2\pi k$ and that h is given by

$$h_{lib.} = \frac{k\beta^2}{4c}. \tag{3.3.16}$$

This expression is identical to the secular dependence of h found in the rotation phase. This behavior supports our suspicion that the distinction between rotation and libration is weaker for this example than it is for an ordinary pendulum.

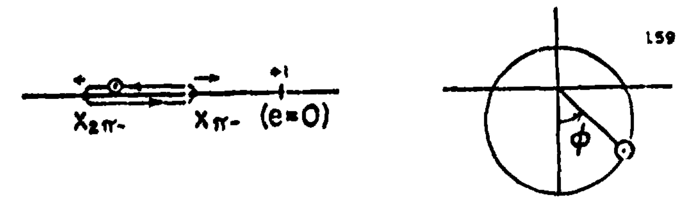
Another piece of useful information is the variation of the amplitude of libration ϕ_m as a function of the parameter c. The extrema in the motion of ϕ ($\phi_m = \phi_{max} = -\phi_{min}$) are obtained from the condition that the arguments of the radical in (3.3.9) vanish. This condition reduces to the following relation for ϕ_m:

$$\sin\phi_m = \frac{2c}{\beta}, \tag{3.3.17}$$

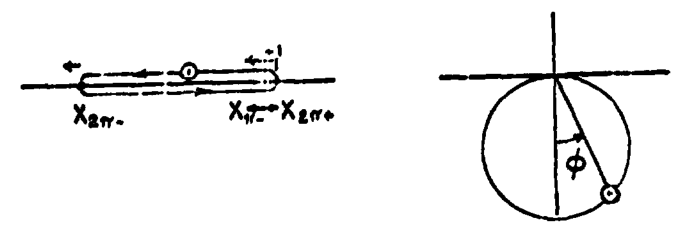
or $\sin\phi_m$ decreases linearly with c(t).

The following description of transition can be deduced from these facts. For k = +1, the root $x_{\pi-}$ approaches − 1 as c approaches the value, 1/2β. This means that the function b(x) which is proportional to $(-x + 1)^{1/2}$ is smaller in magnitude at ϕ = mod(π) than at ϕ = mod(2π). At transition, when c = 1/2β, the function b(x) vanishes at ϕ = mod(2π), while for $|c| < 1/2|\beta|$, the pendulum librates with maximum amplitude ϕ_m = 90° about the 2π center. Figure 3.3.1 shows this sequence of events for the k = +1 case, while figure 3.3.2 shows the equivalent sequence for the k = −1 case, which for $|c| < \frac{|\beta|}{2}$ librates about the π center.

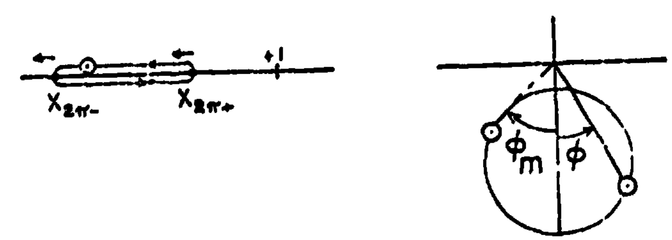
By the way, it is no accident in figures 3.3.1,2 that the path traced out by the pendulum in each phase is a circle of constant radius. Greenberg (1972b) observed this phenomenon in his analysis of a similar problem. Furthermore, he found the angular velocity, as measured from the center of that circle, is constant and equal to $|c|$. To prove these assertions let's consider figure 3.3.3 which is a diagram of the path traced out by the pendulum in the positive rotation phase. The center of the figure is at D where d = C(t) is given by



a) Positive rotation phase. Here $|c| > 2|\beta|$ and (−c) is a monotonically decreasing function of time.



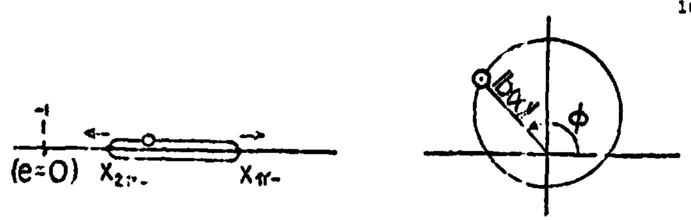
b) Here the root $x_{\pi-}$ equals +1, while the function $|b(x)|$, vanishes at c = mod(π). In addition, the root $x_{\pi-}$ changes from a π to a 2π root.



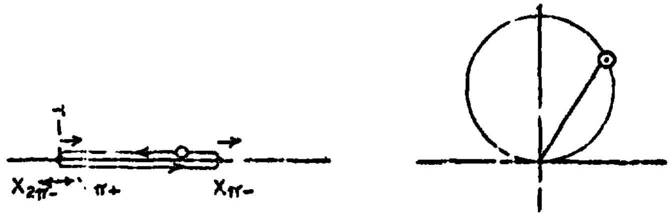
c) Libration phase; the pendulum librates about ϕ = mod(2π) with amplitude $\phi_m \leq 90°$.

FIGURE 3.3.1

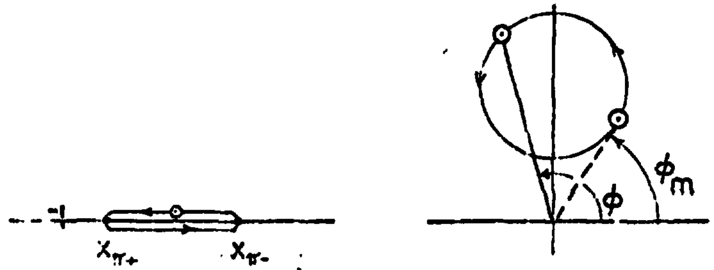
TRANSITION IN LIMIT $|\beta| \gg 1$ FOR k = +1.



a) Positive rotation phase.



b) The root $x_{\pi-}$ equals -1 and changes its label from a 2π to a π root



c) The pendulum librates about $\phi = \pi$ with amplitude $\phi_m \leq 90°$.

FIGURE 3.3.2

TRANSITION IN LIMIT $|\beta| >> 1$ FOR $k = -1$.

$$d = \frac{-\beta}{2}((-kx_+ + 1)^{1/2} - (-kx_- + 1)^{1/2}). \qquad (3.3.18)$$

Thus, D is the midpoint of the maximum and minimum values of $b(x)\cos\phi$. Substituting h as given by (3.3.12) in the equations defining x_+ and x_-, we find

$$x_\pm = \frac{k}{4c^2}(-\beta^2 \pm [\text{illegible}]\beta c), \quad (c, \beta < 0), \qquad (3.3.19)$$

so that d is given by

$$d = \frac{k\beta^2}{2c} \quad . \qquad (3.3.20)$$

The distance r from the center of the figure to the point P is related to $b(x)$, ϕ and Ψ, the angle measured from the center O to the point P, by the law of cosines:

$$r^2 = d^2 + \beta^2(-kx + 1) + 2d\beta(-kx + 1)^{1/2}\cos\phi. \qquad (3.3.21)$$

Eliminating the $\cos\phi$ dependence using (3.3.2), we find that the explicit x dependence occurring in each term cancels such that $r^2 = \beta^2$ = constant. That is, the figure is a circle of constant radius. It is no problem to show that this result also holds in the libration phase. This means that the absolute variation in $b(x)$ remains constant although its mean value grows as $|c(t)|$ decreases. Inspection of the figures depicting the motion of the pendulum through transition reveals that the relative damping of the fluctuation of $b(x)$ is proportional to $c(t)$, vanishing as $c(t) \to 0-$.

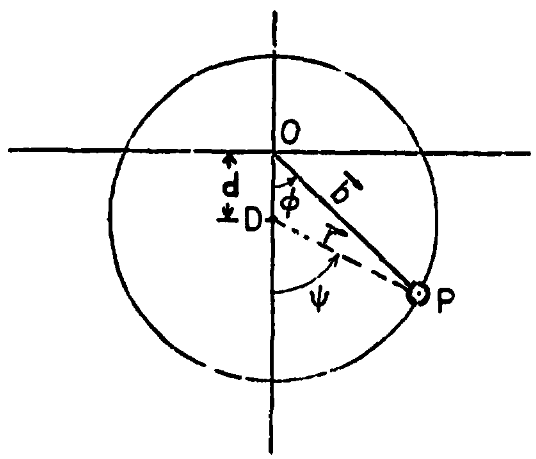


FIGURE 3.3.3
Path traced out by pendulum governed by (3.3.3) in positive rotation phase for k = +1 case. The parameter d is the midpoint between the minimum and maximum values of b(x). Ψ is the angle made by vector $\vec{r}$ with respect to the origin at D while ϕ is the angle made by vector $\vec{b}(x)$ with respect to the origin at O.

The fact that Ψ is a linear function of the time follows directly from the law of sines. Explicitly:

$$\frac{\sin\Psi}{\beta(-kx+1)} = \frac{-\sin\phi}{r} . \tag{3.3.22}$$

Using the equation of motion for x (3.1.1b), and (3.3.6), we find

$$|\beta|\sin\Psi = \frac{dx}{dt} = \beta\cos|c|(t - t_o), \tag{3.3.23a}$$

or

$$\Psi = |c|(t - t_o) + \mathrm{mod}(\pi). \qquad \text{b)}$$

Eventually $|c(t)|$ decreases enough so that the Hamiltonian h(x,ϕ,t) (3.3.3) is no longer a valid approximation. We expect that the approximation breaks down when the x^2 term in H is of the same magnitude as the cx term, or until $x \sim |c|$. From (3.3.19), the mean value of x is

$$\langle x\rangle = 1/2(x_+ + x_-) = \frac{k\beta}{4c^2} . \tag{3.3.24}$$

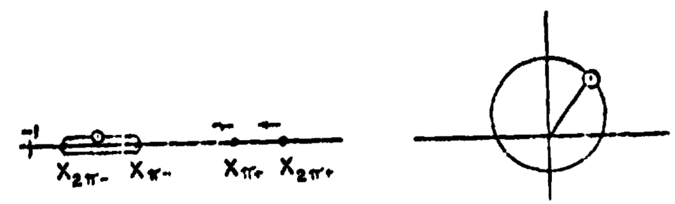
Therefore, the minimum value of $|c(t)|$ for which the Hamiltonian h is a useful approximation of the motion is

$$\frac{c}{\beta} \sim |\beta|^{-1/3} . \tag{3.3.25}$$

This means that $|\beta|$ must be a very large number if there is to be significant damping of the amplitude of libration via this mechanism.

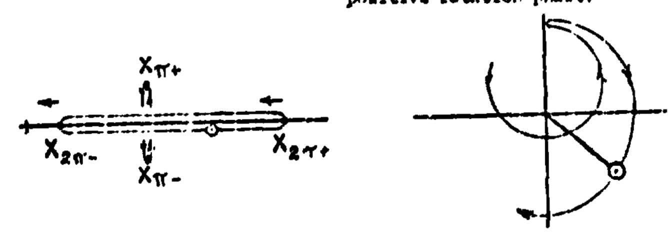
So far we have a qualitative picture of how the pendulum behaves in both the small and large fluctuation limits for the $|x| = 1$ case. The next stage of the development will be to determine the critical values of the parameters which separate the two types of behavior and to develop a more accurate probability argument. First we shall discuss the complete qualitative behavior for the $k = -1$ case, since it can be easily deducud from the previous arguments.

Far from transition, the relative position of the four roots will be as shown in figure 3.3.4a. As the system evolves, two possibilities for the transition phase exist. If the left bounding root $x_{2\pi-}$ does not reach the value -1 prior to the coincidence of the roots, then transition will involve the temporary motion of the π-roots off the real axis. As with the $k = +2$ case, the variable x will move past the point where $x = \mathrm{Re}\, x_\pi$. Thereafter the components $\pm \mathrm{Im}\, x_\pi$ will move back toward the real axis, reaching it before x returns to the value $\mathrm{Re}\, x_\pi$. This means that the system has entered the negative rotation phase without any possibility of librating (see fig. 3.34b,c). The other possibility is that the root $x_{2\pi-}$ does reach the value -1 prior to the coincidence of the interior π-roots. The pendulum then begins librating about the inverted position. Eventually the π-roots will coincide and become complex. The motion of $\dot\phi$ will vanish and reverse sign somewhere near the top π-position (fig. 3.3.5a,b). Since x is increasing as



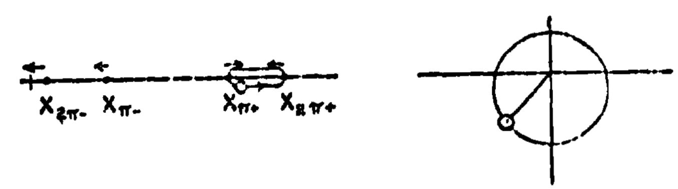
a) Diagram of positive rotation phase in complex x-plane

Diagram of path traced out in polar coordinates $(|b(x)|, \phi)$ in positive rotation phase.



b) Diagram of the motion of x just after π-roots coincide.

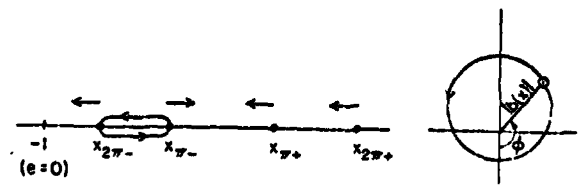
Motion during period when $\dot\phi$ reverses sign.



c) Negative rotation phase.

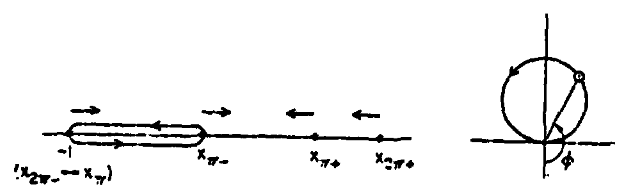
FIGURE 3.3.4

TRANSITION DIRECTLY INTO NEGATIVE ROTATION PHASE FOR $k = -1$ AND $|\beta| < 13.89$.

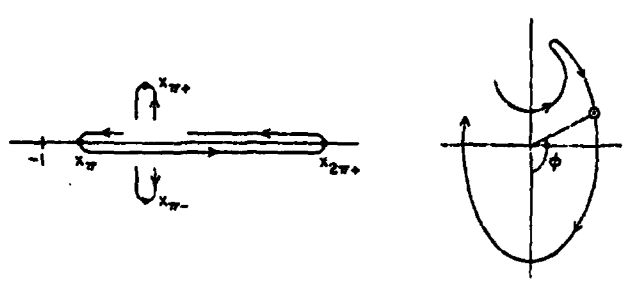


a) Positive rotation phase in complex plane.

Path of motion in polar coordinates $(|b(x)|,\phi)$ in the positive rotations.



b) Temporary transition into inverted libration phase.



c) Transition into negative rotation.

FIGURE 3.3.5

T[illegible]RY[illegible]ITION IN[illegible]V[illegible]

it moves past the position where it equals Re x_{π}, the coefficient of the pendulum potential $b(x)$ which is proportional to $(x + 1)^{1/2}$ also increases, after $\dot{\phi}$ reverses sign. After x reaches the root $x_{2\pi+}$, the variable x then moves back toward the position Re x_{π}. Meanwhile the imaginary component of x_{π} has vanished and again the system enters the negative rotation phase.

The special situation which separates these two types of behavior is the following: Let the root $x_{2\pi-}$ equal -1 at the same instant that the two interior π-roots coincide. From the action integral (App. B.23) the values of the parameters for this special case are:

$$\beta = -13.89; \quad [illegible] = -6.670; \quad H = 217.3;$$
$$x_{2\pi-} = -1; \quad x_{\pi} = +4.780; \quad x_{2\pi+} = 18.12. \tag{3.3.26}$$

If $|\beta| < 13.89$, the pendulum behaves normally during transition. For $|\beta| > 13.89$, the system temporarily enters the inverted libration phase.

Now that the $k = -1$ case has been disposed of, we shall concentrate on the more subtle behavior of the $k = +1$ case during transition. With just one more piece of information, we can outline the complete qualitative behavior of the pendulum during transition for the $k = +1$ case. We assert (and will prove later) that the function $\dot{\phi}(\pi)$ (3.1.8a) vanishes when the exterior π-roots, $x_{\pi+}$ and $x^*_{\pi+}$, reach the real axis and coincide. Recall that $\dot{\phi}(\pi)$ occurred in

[illegible] ...plicitly for [illegible] π-roots we have

$$\frac{dx_\pi}{dt} = \frac{dc}{dt}\,\frac{(x_\pi - x)}{\dot{\phi}(\pi)}\,, \tag{3.3.27a}$$

$$\dot{\phi}(\pi) = -x_\pi - c - \frac{k}{2}\beta(-kx_\pi + 1)^{-1/2}. \qquad \text{b)}$$

From the above equation, the only allowed motion after coincidence is for the two roots $x_{\pi+}$ and $x^*_{\pi+}$ to separate. One root $(x_{\pi+})$ moves towards the left bounding root $x_{\pi-}$ while the other $(x^*_{\pi+})$ moves towards the $x = +1$ position on the real axis. From our earlier discussion we expect that when the root $x^*_{\pi+} = +1$, it changes from a π to a 2π root $(x_{2\pi+})$ and thereafter decreases. We also know that when the interior π-roots coincide, $\dot{\phi}(x_\pi)$ again vanishes. This means that this function must be double valued in x_π. From the equation $(\dot{\phi}(\pi) = 0)$, the following relation can be derived:

$$\frac{k}{2}\beta = -(-kx_{\pi+} + 1)^{1/2}(x_{\pi+} + c). \tag{3.3.28}$$

Figure 3.3.6 is a graph of $|\beta|$ versus $x_{\pi+}$ for the special value of $c = -1/2$ and $k = +1$. This special value results from evaluating the action integral at the instant $x_{\pi+}$ and $x^*_{\pi+}$ coincide (B.23). In addition, the position of the left bounding root $x_{\pi-}$ is graphed as a function of β, using (C.2a) and (B.16a). Observe that the position of $x_{\pi-}$ is to the left of the second zero of $\dot{\phi}(\pi)$. This is to be expected since the motion of $x_{\pi-}$ is toward the position where $\dot{\phi}(x_\pi,c)$ vanishes. (Note that this will occur for a more positive value of c than $c = -1/2$). According to figure 3.3.6, there exists

a maximum allowed value of $|\beta|$ for which the equation (3.3.28) is satisfied. This critical value for β and x_π can be determined by maximizing (3.3.26) with respect to x_π. The results are

$$\beta_{c1} = \frac{-2}{3\sqrt{6}} = -0.2722; \quad x_{\pi c1} = \frac{5}{6} = 0.8333. \tag{3.3.29a}$$

Incidently, the remaining values of the parameters for this special case are (B.21):

$$H = \frac{1}{6}; \quad x_{2\pi-} = -1/2\ . \qquad \text{b)}$$

For $|\beta| < |\beta_{c1}|$, the value of x_π where the exterior π-roots reach the real axis is in the range $\frac{5}{6} \le x_\pi \le 1$, while the second zero in $\dot{\phi}(\pi)$ for $c = -1/2$ lies in the range $\{1/2 \le x_\pi \le \frac{5}{6}\}$. Clearly, at the critical value of $\beta = \beta_{c1}$, the three π-roots coincide. For values of $|\beta| > |\beta_{c1}|$ the implication is that the exterior π-roots never do reach the real axis during subsequent evolution of the system. Therefore transition is qualitatively similar to the earlier discussed approximation. The situation is more complex for $|\beta| < |\beta_{c1}|$. As the system evolves, the roots $x_{\pi-}$ and $x_{\pi+}$ move towards coincidence. If these roots coincide before $x^*_{\pi+} = +1$, then the system can either temporarily evolve into an inverted libration phase or remain in the positive rotation phase as shown in fig. 3.3.7-8. Which occurs depends on the motion of the imaginary components of the interior π roots during the transition phase. If the imaginary components first move off the real axis and then return while x is to the right of Re x_π, then the motion of x is eventually trapped between $x_{\pi+}$

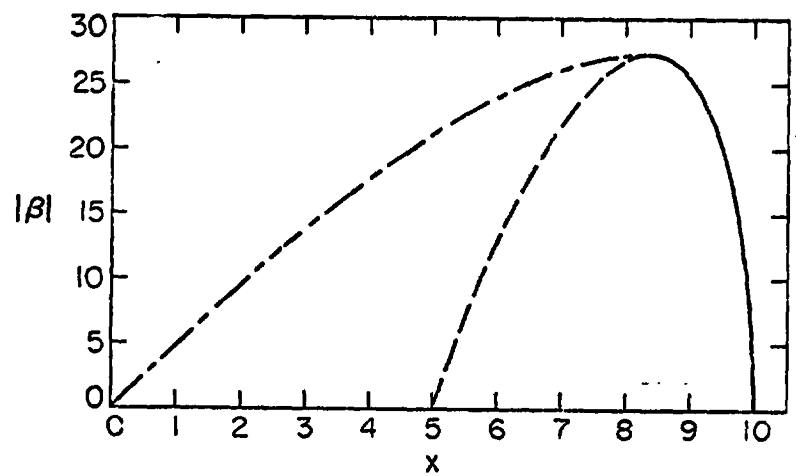
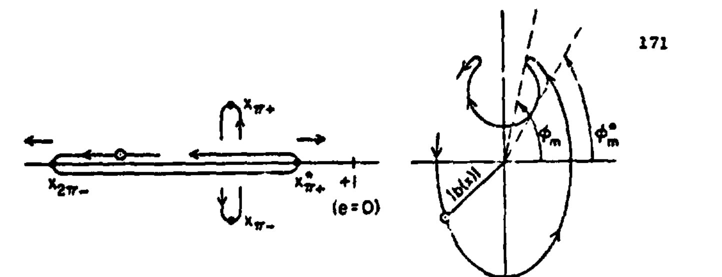


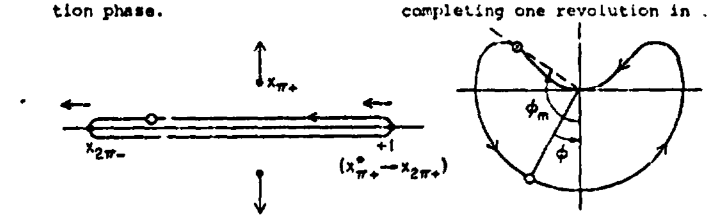
FIGURE 3.3.6

Graph of $|\beta|$ versus $x_{\pi+}$ for the special value of $c = -1/2$ and $k = +1$, obtained from the $\{\dot{\phi}(\pi) = 0\}$ equation. From the solid line (——), one finds the value of the roots $x_{\pi+}$ and $x^*_{\pi+}$ at coincidence as a function of $|\beta|$. The dash line (---) gives the location of the second zero of the $\{\dot{\phi}(\pi) = 0\}$ equation at $c(t) = -1/2$, while the dash-dot line (-·-) represents the location of the $x_{\pi-}$ root at $c(t) = -1/2$. During subsequent evolution, the dash and dash-dot curves move toward coincidence.



a) Transition is initiated by coincidence of interior π-roots and continues as they move off real axis. The motion of x is first towards $x_{2\pi-}$, next towards Re $x_{\pi\pm}$, near where $\dot{\phi}$ vanishes (near top π position) then back through Re $x_{\pi\pm}$ where $\dot{\phi}$ again reverses sign. This motion involves transition from positive rotation phase to yet another "temporary" positive rotation phase.

The pendulum moves toward the top π position where $\dot{\phi}$ vanishes at ϕ_m and reverses direction. The angle then increases to the maximum ϕ^*_m where $\dot{\phi}$ again vanishes and reverses direction and then moves toward the bottom π position. After passing through the π position, ϕ decreases to the value $(-\phi^*_m)$ where it again reverses direction. Finally it moves toward the π position and $\dot{\phi}$ vanishes for the fourth time, completing one revolution in ..



b) Second transition, in which the system automatically evolves into the libration phase as the $x_{\pi\pm}$ root equals +1, then reverses direction, becoming a 2π-root.

Here, the pendulum moves into libration phase as b(x) tends to vanish. The maximum amplitude of libration is in the range $90° \leq \phi_m \leq 180°$.

FIGURE 3.3.7

TRANSITION INTO LIBRATION PHASE FOR β

IN THE RANGE $(-0.2104 \geq \beta \geq -0.2722)$ AND $k = +1$.

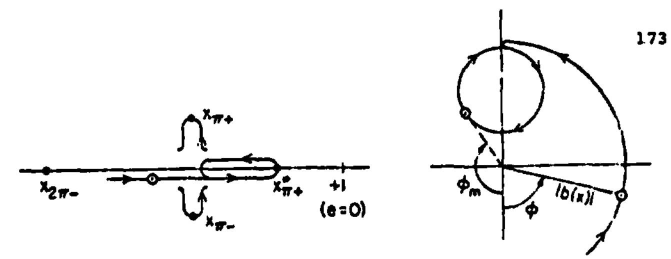
and $x^*_{\pi+}$. Therefore, the pendulum has made the transition from positive rotation into inverted libration. But if these components never return to the real axis, the motion of x is trapped between $x_{2\pi-}$ and $x^*_{\pi+}$. Although the curve traced out by the pendulum is more complex, and $\dot{\phi}$ *reverses sign four times during one rotation*, it still executes positive rotations. In either case, subsequent evolution of the pendulum tends to increase $x^*_{\pi+}$ (and decrease $\beta(-x^*_{\pi+} + 1)^{1/2} \Rightarrow 0$) until it reaches the value 1. If the system has made the transition into the inverted libration phase, then as $|c|$ decreases the pendulum eventually enters the negative rotation phase (see fig. 3.3.8b). But if the system has remained in the positive rotation phase, it will automatically enter the libration phase with amplitude of libration $\leq 180°$.

Whether or not the pendulum executes these exotic motions during transition depends, intuitively, on the parameter β. If its magnitude is too small, then $x_{\pi+}$ equals +1, then decreases and changes labels prior to the *coincidence* of the interior π-roots. The condition which separates the ordinary transition from the two-stage type just discussed is for $x^*_{\pi+} = 1$ when $x_{\pi-}$ and $x_{\pi+}$ coincide. From the action integral (B.23), we find that the values of the parameters are:

$$\beta_{c2} = -0.2104; \quad c = -0.4705;$$

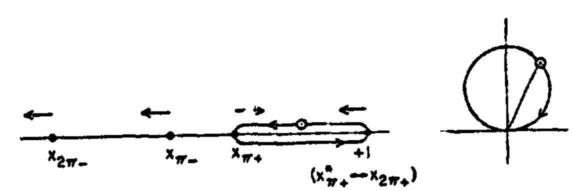
$$x_{\pi\pm} = +0.6463; \quad x_{2\pi-} = -0.4108; \tag{3.3.30}$$

$$H = 0.1407 .$$



a) Transition in which interior π-roots first move off, then return to real axis, trapping x between $x_{\pi\pm}$ and $x^*_{\pi+}$. This is a state of inverted libration.

The pendulum moves towards top π-position ($x_{\pi+}$) where $\dot{\phi}$ reverses sign, next moving through the bottom π-position ($x^*_{\pi+}$) and again towards and through the top π-position. The amplitude of libration ϕ^* about the π-center is $\leq 90°$.



b) Here $x^*_{\pi+} = +1$ and becomes a 2π-root. At this point the system automatically enters the negative rotation phase.

The libration amplitude increases to 90° and thereafter executes negative rotations.

FIGURE 3.3.8

TRANSITION INTO NEGATIVE ROTATION PHASE THROUGH A "TEMPORARY" INVERTED LIBRATION PHASE FOR β IN RANGE $(-0.2104 \leq |\beta| \leq -0.2722)$ AND $k = +1$.

In summary, we have found that there are three distinct modes of transition depending on the magnitude of the parameter β. If $|\beta| < |\beta_{c2}|$, then transition is qualitatively similar to II in that it involves the coincidence of interior π-roots. For β in the range $\{|\beta_{c2}| \le |\beta| \le |\beta_{c1}|\}$, transition is a two-staged mechanism; the first involves the coincidence of the interior π-roots, after which the system evolves either into the inverted libration phase or a complex positive rotation phase. In the second stage the pendulum evolves directly into either the negative rotation phase or the libration phase, respectively. During this stage $b(x)$ tends to vanish (i.e. $e(x) \Rightarrow 0$). The final mode involves direct transition from positive rotation into libration if $|\beta| \ge |\beta_{c1}|$. Now that the qualitative behavior for the $k = +1$ case has been thoroughly discussed, the next step is to back up some of our assertions and to develop a more accurate probability estimate than that developed in section 3.2.

First let's obtain a more explicit form for the equations of motion of the real and imaginary parts of the x_π-roots (3.1.9). The partials of the real and imaginary parts with respect to H can be determined as explicit functions of the parameters c, β, H and the π-root parts by separating the equation $\{R(\mathrm{Re}\, p_\pi + i\mathrm{Im}\, p_\pi) = 0\}$ into its real and imaginary components. After some simple manipulations of these two components, the following relations are obtained:

$$\mathrm{Im}\, p_\pi = \mathrm{Im}(x_\pi + c) = \mathrm{Im}\, x_\pi = \pm\sqrt{-2H + \frac{k\beta^2}{\mathrm{Re}\, p_\pi} + \mathrm{Re}^2 p_\pi}\ ; \qquad (3.3.31a)$$

$$\frac{+k^2\beta^4}{\mathrm{Re}^2 p_\pi} - 4\mathrm{Re}^4 p_\pi{}^4 + 8H\,\mathrm{Re}^2 p_\pi - 4\beta^2(kc + 1) = 0. \qquad b)$$

The second relation (3.3.31b) can be used to determine the partial with respect to H of the real part of x_π. The result is:

$$\frac{\partial x_\pi}{\partial H} = \frac{1}{\mathrm{Re}\, p_\pi}\,((1 - \eta)^2 + \varepsilon^2)^{-1}\ , \qquad (3.3.32a)$$

where η and ε are

$$\eta = \frac{k\beta^2}{2\mathrm{Re}^3 p_\pi}\ , \quad \varepsilon = \frac{\mathrm{Im}\, x_\pi}{\mathrm{Re}\, p_\pi}\ . \qquad b)$$

The first relation can be used to determine the partial of the imaginary part in terms of the partial of the real part of x_π. We find

$$\frac{\partial \mathrm{Im}\, x_\pi}{\partial H} = \frac{\eta(1 - \eta) - \varepsilon^2}{\mathrm{Im}\, x_\pi ((1 - \eta)^2 + \varepsilon^2)}\ .$$

Therefore the equations of motion for the complex parts of x_π take the form:

$$1/2\frac{d}{dt}\mathrm{Im}^2 x_\pi = \frac{dc}{dt}((1 - \eta)^2 + \varepsilon^2)^{-1}\{(\eta(1 - \eta) - \varepsilon^2)(x - \mathrm{Re}\, x_\pi) - \varepsilon^2\,\mathrm{Re}\, p_\pi\} \qquad (3.3.33a)$$

$$\frac{d}{dt}\mathrm{Re}\, x_\pi = \frac{dc}{dt}((1 - \eta)^2 + \varepsilon^2)^{-1}\{\mathrm{Re}^{-1} p_\pi(x - \mathrm{Re}\, x_\pi) - \eta(1 - \eta) + \varepsilon^2\}. \qquad b)$$

We should point out that the above equations are valid only if $Im^2 x_\pi > 0$. Otherwise (3.3.27) governs the motion of the real roots.

Incidently, the condition that Im x_π vanishes implies that $\dot{\phi}(\pi)$ also vanishes. This can be simply demonstrated by substituting $\dot{H}$ evaluated at $x = x_\pi$ into (3.3.31a) and observing that the resulting expression is consistent with $\dot{\phi}(\pi) = 0$ (3.3.27b).

It appears as the system evolves towards transition (for k = +1), that the imaginary part of the exterior π-roots first gradually increases (since they are $\sim 1/2\beta\,|\mathrm{Re}\ x_{\pi+}|^{-1/2}$ for $(-c) \gg 1$), reach a maximum and then decrease towards zero. For $\varepsilon^2 \gg \eta$ and $\eta < 1$, the motion of Im x_π is definitely towards zero. The value of the parameter η when the $x_{\pi+}$ roots coincide, after using (3.3.28) to eliminate the β dependence, is

$$\eta = \frac{2(-x_{\pi+} + 1)}{(+x_{\pi+} - 1/2)} . \qquad (3.3.34)$$

Since $x_{\pi+}$ lies in the range $(\frac{5}{6} \le x_{\pi+} \le 1)$, η is bounded between 1 and zero, having its largest value when the three π-roots coincide. One can also show that the parameter η is greater than or equal to 1 when the interior π-roots coincide. This is to be expected, since otherwise the imaginary part of $x_{\pi+}$ could not move off the real axis while the variable x is to the left of Re $x_{\pi+}$ (see 3.3.30a). During the normal transition phase (i.e. coincidence of interior π-roots!) the parameter ε is of $O(\beta^{-1}\frac{dc}{dt})$ and is small compared to η. Therefore the equation governing the motion of Im $x_{\pi+}$ during transition can be

approximated by

$$1/2\,\frac{d}{dt}\mathrm{Im}^2 x_\pi \doteq \frac{dc}{dt}\left(\frac{\eta}{\eta - 1}\right)(\mathrm{Re}\ x_\pi - x), \qquad (3.3.35)$$

which is similar to the equation of motion for the k = 2 case. The parameter η is a slowly varying quantity. Therefore the factor $(\frac{\eta}{\eta - 1})$ is also slowly varying unless $(\eta - 1)$ very [illegible]ly vanishes. Therefore, the transition integral can be defined as before, and evaluated to lowest order by replacing the slowly varying parameters η and Re x_π by their values at coincidence. After integrating the above equation we have

$$1/2[\mathrm{Im}^2 x_\pi(f) - 1/2\,\mathrm{Im}^2 x_\pi(i)] \doteq \frac{dc}{dt}\left(\frac{\eta}{\eta - 1}\right)\int_{t_i}^{t_f}(\mathrm{Re}\ x_\pi - x)\,dt. \qquad (3.3.36)$$

By construction, Im $x_\pi(i) = 0$. This time, let's change the integration variable from t to x, keeping in mind that t is a monotonically increasing function. The right hand side of (3.3.36) then becomes

$$\frac{dc}{dt}\left(\frac{\eta}{\eta - 1}\right)\oint_{x_i}^{x_f}\frac{|dx|(\mathrm{Re}\ x_\pi - x)}{\sqrt{R(x)}} . \qquad (3.3.37)$$

The above is understood to be the path integral defined by the transition phase diagram (fig. 1.2.2). At coincidence, R(x) is

$$R(x) = 1/4(x - x_{2\pi-})(x_{\pi\pm} - x)^2(x_4 - x). \qquad (3.3.38)$$

(Here the root x_4 is either $x^*_{\pi+}$ or $x_{2\pi+}$, depending on whether $|\beta|$ is greater or less than $|\beta_{c2}|$). Therefore, the above integral reduces to the following:

$$2\frac{dc}{dt}\left(\frac{n}{n-1}\right)\oint_{x_i}^{x_f} \frac{|dx|\,\mathrm{sign}(\mathrm{Re}\,x_\pi - x)}{\sqrt{(x - x_{2\pi-})(x_4 - x)}} \quad . \tag{3.3.39}$$

From the transition phase diagram (fig. 1.2.2), we find that $x_f = \mathrm{Re}\,x_{\pi\pm}$ and x_i lies between $x_{2\pi-}$ and $\mathrm{Re}\,x_\pi$. Although this integral can be explicitly found in terms of arcsine functions, this step will be deferred until later.

Recall that if the above integral is positive, the system has made the transition into the libration phase. But if it is negative the converse is true. The value of x_i for which the above integral vanishes (x_{ic}) separates these two events. Recall that the probability measure was defined in terms of the value of $\dot{\phi}^2$ as the system went over the top for the last time, and was directly related to the function $\mathrm{Im}^2 x_\pi$ evaluated between fixed limits (1.2.36-38). Specifically,

$$P_c = -\frac{\mathrm{Im}^2 x_\pi \Big|_{\phi=\pi}^{\phi=3\pi} - \mathrm{Im}^2 x_\pi \Big|_{\phi_{ic}}^{\phi=3\pi}}{\quad x_\pi \Big|_{\phi=\pi}^{\phi=3\pi}} \quad . \tag{3.3.40}$$

Note that of the terms is evaluated in that part of the transition phase where $\dot{\phi} \gtrless 0$. The correspondin units in the variable x are x_π and x_{ic}. Again the appro path integral in (3.3.40) over x is understood. The relation which defined x_{ic} can be

to eliminate the specific dependence in P_c on ϕ_{ic}. From (3.3.36) we find:

$$\mathrm{Im}^2 x_\pi(f) = \mathrm{Im}^2 x_\pi(\dot{\phi} \geq 0)\Big|_{\phi_{ic}}^{\phi=3\pi} + \mathrm{Im}^2 x_\pi(\dot{\phi} \leq 0)\Big|_{\phi=3\pi}^{\phi=\pi} = 0. \tag{3.3.41}$$

Therefore the probability P_c takes the form

$$P_c = \frac{\mathrm{Im}^2 x_\pi(\dot{\phi} \geq 0)\Big|_{\pi}^{3\pi} + \mathrm{Im}\ x_\pi(\dot{\phi} \leq 0)\Big|_{3\pi}^{\pi}}{\mathrm{Im}^2 x_\pi(\dot{\phi} \geq 0)\Big|_{\pi}^{3\pi}} . \tag{3.3.42}$$

(3.3.39) can be used to evaluate each of the terms in (3.3.42) by substituting the appropriate limits and the value of $\dot{\phi}$ in the former equation. We find

$$\mathrm{Im}^2 x_\pi(\dot{\phi} \geq 0)\Big|_{\pi}^{3\pi} = \frac{\pi}{2} + \arcsin\delta; \tag{3.3.43a}$$

$$\mathrm{Im}^2 x_\pi(\dot{\phi} \leq 0)\Big|_{3\pi}^{\pi} = \frac{\pi}{2} - \arcsin\delta; \tag{b}$$

where

$$\delta = \frac{2x_\pi - x_{2\pi-} - x_{2\pi+}}{x_{2\pi+} - x_{2\pi-}} \tag{c}$$

Therefore P_c is given by

$$P_c = \frac{2}{1 + \frac{\pi}{2}(\arcsin\delta)^{-1}} . \tag{3.3.44}$$

This formula is valid only for the $k = 1,2$ cases with δ in the range $(0 \le \delta \le 1)$. P_c vanishes if $k = -1,-2$.

If the 2π-roots were symmetrically placed about the coinciding π-roots, then β and the arcsin of β would vanish, implying that P_c would also vanish. But this case corresponds to $b(x)$ = constant for which capture into libration never occurs. The position of the roots at transition is implicitly a function of the parameter β. Therefore, the interesting relationship is how P_c depends on β. Again, the action integral can be used to uniquely determine the parameters of the system as a function of β (B.16-18). Figure 3.3.9 is a plot of P_c versus (β/β_{c1}) for both the $k = 1$ and $k = 2$ cases. Observe that the small fluctuation limit for P_c where it is proportional to $|\beta|^{1/2}$ (3.2.12) is only valid for $P_c \lesssim 0.1$ for the $k = 2$ case, and $P_c \lesssim 0.5$ for the $k = 1$ case. Another interesting fact directly derivable from (B.18) is that $P_c = 1/2$ when $\beta = \beta_{c2}$ and decreases to zero as β approaches β_{c1}. Observe that the probability that the pendulum may temporarily enter the inverted libration phase ($k = +1$) is reasonably large. A more revealing graph (fig. 3.3.10) is a plot of P_c versus the mean eccentricity e_o far from transition or, equivalently, $(\frac{e_o}{e_{c1}})$. From (3.1.3), this ratio is

$$\frac{e_o}{e_{c1}} = \left(\frac{\beta_{c1}}{\beta}\right)^{\frac{1}{(4 - |k|)}} . \qquad (3.3.45)$$

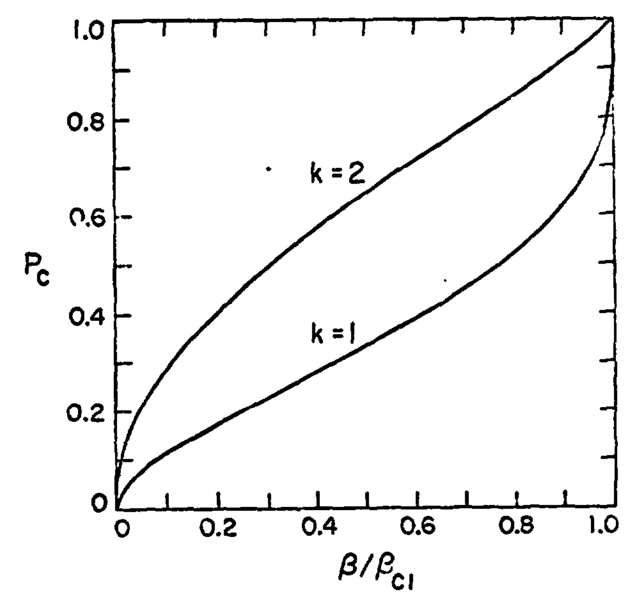


FIGURE 3.3.9

Graph of probability for capture into libration. P_c versus β/β_{c1} for the $k = +1, +2$ cases.

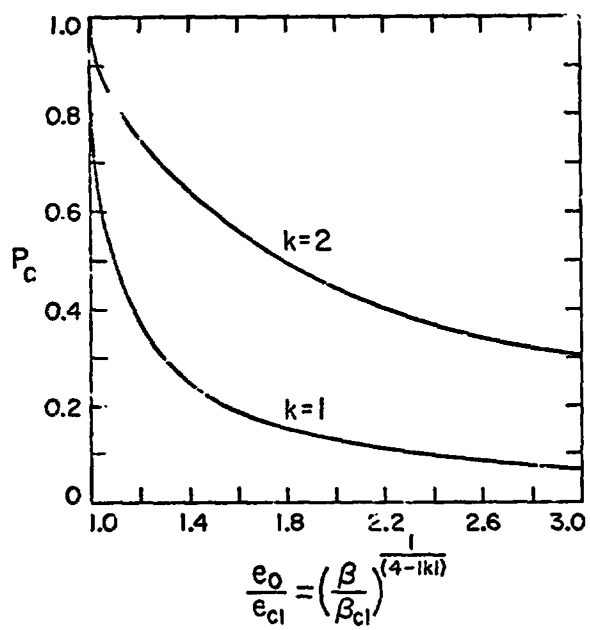


FIGURE 3.3.10

GRAPH OF P_c VERSUS e_o/e_{c1} FOR k = +1, + 2 CASES.

Note how dramatically P_c decreases as a function of e_o for the k = +1 case. P_c is less than 0.1 for e_o no more than $3e_{c1}$. Furthermore, the two-stage transition occurs only if e_o lies in the narrow range

$$e_{c1} \le e_o \le 1.09e_{c1}. \tag{3.3.46}$$

Before turning to questions concerning the secular behavior of the system before and after transition, there is a serious problem concerning the applicability of the theory of transition developed for the e-type resonances, which must be examined. Recall that in developing the one-dimensional Hamiltonian which describes the resonance interaction, the second order effect of non-resonant and non-secular terms on the interaction was of $O(\mu')$ smaller than the first order terms. These included an infinity of terms for which the coefficients of their respective cosine arguments were proportional to the first power of the eccentricity e. If e were very small, then such terms would tend to produce very large fluctuations in the motion of the perihelion, $\tilde{\omega}$. But, as we have seen, a major part of the motion of the resonance variable ϕ is due to the motion of $\tilde{\omega}$ if e_o is very small. In fact, e must very nearly vanish if transition involves the automatic entry into libration phase. The question is, could these other terms proportional to e effectively inhibit this automatic transition mechanism? To get a qualitative idea of their effect, let's add the following term to the Hamiltonian equation:

$$\lambda\beta(-kx + 1)^{1/2}\cos(\omega t)\cos\phi. \qquad (3.3.47)$$

Here ω is constant and is assumed large compared to the mean value of $\dot{\phi}$, and λ is a dimensionless parameter of $O(1)$. The Hamiltonian is now given by:

$$H(x,\phi,t) = 1/2(x + a(t))^2 + b(-kx + 1)^{1/2}[(-kx + 1)^{\frac{|k| - 1}{2}} +$$

$$\lambda\cos(\omega t)]\cos\phi.$$

We see immediately that the effect of this extra term on the motion of the roots is to add a high frequency oscillation. For $|k| = 1$ case, this rapid oscillation would not radically change the average position even for x very near the value k^{-1}. The indication is that the coupling of the short period terms to the resonance term does not radically effect the $|k| = 1$ case. But for the $|k| = 2$ case, this extra term clearly dominates the motion of ϕ for x very near k^{-1}, leading us to believe that the automatic transition mechanism described in section 3.2 for this case is seriously inhibited by similar terms occurring in the expansion of the disturbing function.

A more rigorous discussion of this point is beyond the scope of this thesis. We should also mention that (3.3.47) does not faithfully represent the complete effect of a high frequency term occurring in the disturbing function. In addition, the following term

$$\lambda'\beta(-kx + 1)^{1/2}\sin(\omega t)\cos\phi; \quad \lambda' \neq \lambda$$

also occurs in the expansion of the disturbing function, associated with the frequencies $(\dot{\phi} \pm \omega)$.

3.4 SECULAR BEHAVIOR OF THE AMPLITUDE OF LIBRATION

The most interesting behavior of any pendulum-like system governed by a time dependent Hamiltonian is the damping of the libration amplitude ϕ_m. The damping in the satellite-satellite resonances to be discussed in the next chapter is presently beyond the limits of measurement. This is also true of the mechanism which is the root cause of the supposed damping, the tidal torque. As we have already seen, there appears to be a direct link between the rate of change in the libration amplitude with a given change in $c(t,x)$ and the value of the parameter β at transition. The most interesting case is the $k = +1$, e-type resonance, and most of our effort shall be concentrated on fully understanding this example.

Incidently, two of the three resonances among Saturn's satellites (discussed in 4.1) are e-type with $k = +1$. But the Mimas-Tethys resonance is a mixed I type (i.e. $b(x)$ is proportional to $I_{Mi}(x)I_{Te}(x)$). It happens that the variation of $I_{Te}(x)$ with x is of $O(\frac{\mu_{Mi}}{\mu_{Te}} \sim \frac{1}{17})$ smaller than the variation of $I_{Mi}(x)$. Therefore, the behavior of this particular example closely mimics that of an e-type. Thus, we are fully justified in restricting this discussion to this one case. Furthermore, we shall neglect the x dependent term associated with the asymmetry of the applied tidal torque and assume here that $c(t,x) \doteq c(t)$. In the course of this investigation, we shall indicate how it affects the damping of ϕ_m.

There are two limits in which information concerning the adiabatic damping of ϕ_m is readily obtained, and they depend on whether the magnitude of β is >> or << than 1. The former case is oddly enough, the simplest, and many of the important results have already been obtained.

For the case where $|\beta| >> 1$, transition is governed by the approximate Hamiltonian $h(x,\phi,t)$ (3.3.3). Recall that "transition" involves the secular evolution of the system from positive rotation directly into libration ($\phi_{max} = 90°$) without the possibility of the system entering the negative rotation phase. This unusual mechanism is distinguished by two facts. Unlike the simple pendulum the instantaneous frequency associated with the pendulum motion remains finite and varies smoothly during transition. In addition, the eccentricity (and the function $b(x)$) tend to vanish during transition. This even occurs when the parameter $c(t) = 1/2\beta$. The libration amplitude thereafter decreases rapidly as $c(t) \to 0-$. Explicitly (3.3.17)

$$\sin\phi_m = \frac{2c(t)}{\beta} \tag{3.4.1}$$

Conversely, the mean value of the action rapidly blows up as $c(t) \to 0-$. From (3.3.7,13) we find

$$\langle x \rangle = \frac{\beta^2}{4c^2} \tag{3.4.2}$$

Incidently, the above relation is valid in both the positive rotation and libration phases, indicating that $\langle x \rangle$ tends to vanish as $c \to -\infty$. Of course, this is the expected behavior.

The eccentricity is related to the action variable by the following relation (3.1.3):

$$\frac{e}{e_o} = (-x + 1)^{1/2} \qquad (3.4.3)$$

where e_o is the mean eccentricity in the positive rotation phase, far from transition. In the libration phase, the average eccentricity ($\equiv 1/2(e_{max} + e_{min})$) is inversely proportional to $c(t)$, the exact relation being (3.3.18):

$$\frac{e_{ave}}{e_o} = \frac{\beta}{2c(t)} \qquad (3.4.4)$$

On the other hand, the fluctuation $\delta e \equiv (e_{max} - e_{min})$ remains constant in the libration phase, and equals $2e_o$. By the way, the behavior of e_{ave} and δe is reversed in the positive rotation phase. The average eccentricity is then equal to e_o, while δe is given by

$$\delta e_{pos.rot.} = \frac{\beta}{c(t)} e_o \qquad (3.4.5)$$

This behavior directly follows from (3.3.18-19), and a careful inspection of the appropriate diagram describing transition (fig. 3.3.1).

The validity of the above results extend into the libration phase until

$$\frac{c(t)}{\beta} \sim |\beta|^{-1/3} \qquad (3.4.6)$$

The parameter β is a function of the mean orbital elements of the resonance partners evaluated in the positive rotation phase far from transition (3.1.5). But the orbital elements available to us are those that the system possesses at the present time in its evolution. What we would like to know is how the above limitation on this (3.4.6) approximation's validity translates in terms of the parameter β, evaluated with the presently observed mean orbital elements.

We have already seen that in an e-type resonance, any change in the eccentricity caused by the applied torque is of $O(e^{-2})$ larger than similar changes in the semimajor axis of either partner. Therefore, the important question is how does the parameter $\underline{\beta}$ scale as a function of $\langle e(t) \rangle$. Recall that in deriving the dimensionless form for the Hamiltonian (sec. 3.3.4-6) which described the motion for an e-type resonance, H was divided by the factor

$$(-\Gamma_o)^2 \doteq (1/2e_o^2 L_o)^2 \qquad (3.4.7)$$

Since the coefficient of the pendulum term is proportional to $e^{|k|}$, the parameter β tends to scale like $e^{|k|-4}$. For the $k = +1$ case, this implies that the relation between β, evaluated in the positive rotation phase (β_o), and the same parameter evaluated with the present mean value of $e(\beta_{now})$ is:

$$\beta_{now} = \beta_o\left(\frac{e_{now}}{e_o}\right)^3 \qquad (3.4.8)$$

Thus, by (3.4.4,6,8), $h(x,\phi,t)$ is a valid approximation for

$$|\beta_{now}| \approx O(1). \qquad (3.4.9)$$

This is significant since for the Enceladus-Dione resonance, $|\beta_{now}| \sim 10$, which means that the complete tidal evolution of this example can be determined from the simplified Hamiltonian. Furthermore, it appears that its presently small amplitude of libration of $O(1°)$ can only be explained as the result of tidal damping via the mechanism just outlined.

The last question connected with this topic is: How does the dissipative term in $\frac{dc}{dt}(x,t)$ affect the damping in the limit $|\beta| >> 1$? In this case it happens that the contribution of this term is easily obtained by taking the time average of the equation of motion for h over one libration period. The equation of motion for h is:

$$\frac{dh}{dt} = \frac{dc}{dt}(t,x)x \qquad (3.4.10)$$

The tidal torque is (2.10.6):

$$\frac{dc}{dt}(c,t) = \frac{dc}{dt}(0,t) + \rho x = \frac{dc(0,x)}{dt}(1 + \lambda\langle x\rangle + \lambda(x - \langle x\rangle)),$$

$$\lambda = \frac{\rho}{\frac{dc}{dt}(0,t)} < 0.$$

We have added an subtracted a term proportional to $\langle x\rangle$ in $\frac{dc}{dt}(x,t)$ since we want to emphasize that it is the fluctuation in x and not its actual value which contributes to the damping of ϕ_m. The parameter λ is a dimensionless negative-definite constant of $O(e_o^2)$, which means it is small compared to one. Therefore the term $(1 + \lambda\langle x\rangle)$ is equal to one to $O(e_o^2)$.

Taking the time average of (3.4.10), we find

$$\left\langle\frac{dh}{dt}\right\rangle \doteq \frac{dc}{dt}(\langle x\rangle + \lambda(\langle x^2\rangle - \langle x\rangle^2)). \qquad (3.4.12)$$

The time averages of x and x^2 are obtained from (3.3.7). The results are

$$\langle x\rangle = -\frac{kb^2}{4a^2}, \qquad (3.4.13a)$$

$$\langle x^2\rangle = + 1/2\,\frac{b^2}{a^2} + \frac{b^4}{(2a)^4} \qquad b)$$

Thus, the average equation of motion of h(t) after changing the independent variable from t to the dimensionless parameter c(t) is:

$$\frac{dh}{dc} = -(1 - 2\lambda k)\frac{kb^2}{4a^2}$$

Integrating the above equation and choosing the integration constant so that $h = kc$ when $c = 1/2\beta$, we find

$$h(c) = (1 - 2\lambda k)\frac{k\beta^2}{4c} + \lambda k\beta \qquad (3.4.15)$$

Going back to the relation which defines $\sin\phi_m$ in terms of h (3.3.9.17), we find that

$$\sin\phi_m \doteq \sqrt{\frac{4c^2}{\beta^2} + \lambda k(2 - \frac{4c}{\beta})} \quad . \qquad (3.4.16)$$

As long as λ is small, the effect of the x-dependent in the tidal torque is minimal.

At the opposite extreme, the magnitude of β_{now} for the Mimas-Tethys case is $\sim 10^{-4}$. This fact, coupled with its presently large libration amplitude (97°) indicates that β_o was also small at transition. For this case, a different approximation can be invoked to determine its secular behavior which depends on the fact that the fluctuation in x is always small, compared to one, if β_o is small. This follows from the observation that the maximum fluctuation in x is of $O(|\beta|^{1/2})$ in the small fluctuation limit. The next step is to consider the time average of ϕ in the libration phase:

$$\langle\dot\phi\rangle = -\langle x\rangle - \langle c\rangle - \langle b_x(x)\cos\phi\rangle \qquad (3.4.17)$$

This time average identically vanishes in the absence of a tidal torque $\frac{dc}{dt}$, and must approximately vanish in the adiabatic sense if the changes induced in $b_x(x)$ are slow compared with the libration frequency. Therefore to $O(|\beta|^{1/2})$ we find,

$$\langle x\rangle \doteq c(t) \qquad (3.4.18)$$

Since the fractional fluctuation in h(x) is small, x can be replaced by its mean value $\langle x\rangle$ in the function b(x). The resulting Hamiltonian is then identical to example II of simple pendulum plus constant applied torque discussed in section 1.2. Therefore, in the libration phase, $H(x,\phi,t)$ for the k = +1 e-type resonance is:

$$H(x,\phi,t) \doteq 1/2(x + c(t))^2 + \beta(c(t) + 1)^{1/2}\cos\phi \ ,$$

while the equation of motion of H is:

$$\frac{dH}{dt} = \frac{dc}{dt}(x + c(t)) + 1/2\beta\frac{dc}{dt}(c(t) + 1)^{-1/2}\cos\phi. \qquad (3.4.19)$$

Observe that the dependent variable can be changed from t to c(t).

We can either take the time average of the above equation or use the action integral to determine the secular behavior of H with c(t). The action integral represents the simplest approach for this example. If the x-dependent term in c(x,t) is included, then it appears that the averaged equation of motion must be used and that numerical integration is required to find H as a function of c(x,t) (Allan, 1969).

The evaluation of the action integral in the libration phase is a straightforward exercise and, rather than repeat it here, we shall observe that it agrees with the result of Best (1968):

$$J_{lib.} = 4\beta(c + 1)^{1/2}[E(K) - (1 - K^2)K(K)] \qquad (3.4.20)$$

where the parameter K is given by

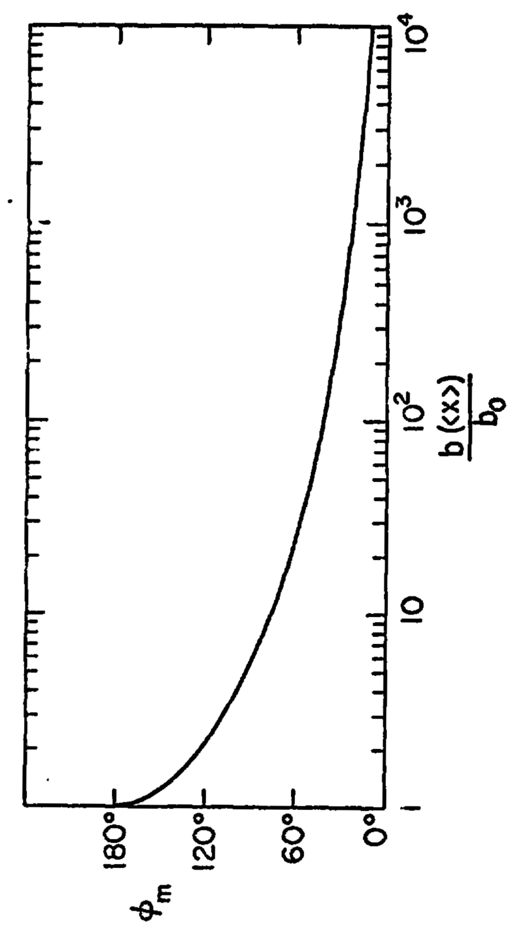


FIGURE 3.4.1
ADIABATIC DAMPING OF ϕ_m IN THE LIMIT $|\beta| << 1.$

$$K = -\frac{H - \beta(c+1)^{1/2}}{2\beta(c+1)^{1/2}} < 1$$

K(K) and E(K) are the complete elliptic integrals of the first and second types, and are defined as follows:

$$K(K) = \int_0^{\frac{\pi}{2}} (1 - K^2 \sin^2\theta)^{-\frac{1}{2}} d\theta \qquad (3.4.21a)$$

$$E(K) = \int_0^{\frac{\pi}{2}} (1 - K^2 \sin^2\theta)^{+\frac{1}{2}} d\theta \qquad b)$$

The most interesting properties of the above functions are the following (Byrd and Friedman, 1971):

$$K(0) = E(0) = \frac{\pi}{2}; \qquad (3.4.22a)$$

$$E(1) = 1; \qquad b)$$

$$\lim K \to 1 \qquad K(K) = \ln\left(\frac{4}{1-K^2}\right) \qquad c)$$

The amplitude of libration ϕ_m, as a function of c(t), is obtained by evaluating H where $\dot{\phi}$ vanishes. We find that ϕ_m can be directly related to the parameter K:

(3.4.23)

Using (3.4.20.23), ϕ_m is graphed in Fig. 3.4.1 as a function of

$$\frac{b(\langle x\rangle)}{b(0)} \approx (c(t) + 1)^{1/2}.$$

In the limit of small librations the simple pendulum can be approximated by a harmonic oscillator. The adiabatic constant for a harmonic oscillator is proportional to $\phi_m^2|b(\langle x\rangle)|^{1/2}$, or

$$\phi_m \propto (c(t) + 1)^{-1/8}, \quad \phi_m << 1 \qquad (3.4.24)$$

This means that adiabatic damping in the small libration limit (when $|\beta| << 1$) is an extremely slow process compared to the adiabatic damping found for the approximation obtained when $|\beta| >> 1$. For the Mimas-Tethys case we find that ϕ_m equals 130° when the function b(x) was approximately five times larger than at present. This implies that the dimensionless parameter β_o was approximately 5^3 times larger than β_{now}. Including the momentum dependence in the tidal torque increases the rate of damping as a function of the inclination of Mimas such that the function b(x) was about three times smaller than at present (Allan, 1969). In any case, the tidal evolution of Mimas-Tethys can be adequately determined using this approximtion since the implied magnitude of β_o at transition is of $O(10^{-2})$.

Again, we should observe that this contribution to the damping from the dissipative term is magnified in the Mimas-Tethys resonance by the unusual situation that the ratio of the tidal

torques of Mimas and Tethys approximately obey the same commensurability relation as the resonance variable. The effect of this is to greatly magnify the ratio of the x-dependent term and the effective torque, $\frac{dc}{dt}$, acting on the resonance variable.

If β_o happens to be of $O(1)$, then no approximation appears to be available. It happens that for the Titan-Hyperion resonance, which we shall examine, has $\beta_{now} = 0.058$ and $\phi_m(\text{now}) = 36°$. To treat this case we shall use the action integral to determine the secular behavior of the system as a function of c(t). The action integral in the libration phase takes the following form.

$$J_{lib.} = \oint x d\phi = -k^{-1}\int(-kx + 1)d\phi + k^{-1}\oint d\phi \qquad (3.4.25)$$

We know that $(-kx + 1)$ is positive definite since $e(x) \propto (-kx + 1)^{1/2}$ and $e(x)$ is real. For a libration, the initial and final values of ϕ are identical. Therefore the term $\oint d\phi$ vanishes. A more explicit form for the above integral is:

$$J_{lib.} = -k\int_{\phi_{min};\,\dot\phi > 0}^{\phi_{max}} (-kx + 1)d\phi + k\int_{\phi_{min};\,\dot\phi < 0}^{\phi_{max}} (-kx + 1)d\phi \qquad (3.4.26)$$

Since the Hamiltonian is symmetric about $\phi = n\pi$ (n being an integer), the minimum and maximum amplitude are equal and opposite (i.e. $\phi_m = \phi_{max} = -\phi_{min}$). Corresponding to ϕ equal to ϕ_m, the action variable x equals x_m where x_m is defined by the condition

$\dot{\phi}_m = 0$ or

$$\dot{\phi}_m = -x_m - a - \frac{k|k|}{2}\beta(-kx + 1)^{-k/2}\cos\phi \tag{3.4.27}$$

If the libration center is at $\phi = \mathrm{mod}(2\pi)$, then x_m lies between the two roots $x_{2\pi-}(\dot{\phi} > 0)$ and $x_{2\pi+}(\dot{\phi} < 0)$. We can change the integration variable f· m ϕ to x in (3.4.26). The result is

$$J_{lib.} = -2k\int_{x_{2\pi-}}^{x_m} (-kx + 1)\frac{d\phi}{|dx|}\,dx + 2k\int_{x_m}^{x_{2\pi+}} (-kx + 1)\frac{d\phi}{|dx|}dx \tag{3.4.28}$$

The function $\frac{d\phi}{|dx|}$ can be obtained from $\frac{d\phi}{dt}/\frac{|dx|}{dt}$ and expressed as a function of just x, H, β and c (see B.6).

The roots $x_{2\pi-}$ and $x_{2\pi+}$ can be expressed as functions of H, β and c. (App. C). The limit x_m can be expressed in terms of c and H by using the Hamiltonian evaluated at x_m, ϕ_m,

$$H = 1/2(x_m + c) + \beta(-kx_m + 1)^{k/2}\cos\phi_m \tag{3.4.29}$$

to eliminate the $\cos\phi_m$ dependence occurring in 3.4.27. From these two equations, we find that $p_m(\equiv x_m + c)$ satisfies the following equation:

$$k(1 - \frac{|k|}{4})p_m^2 - (kc + 1)p_m + \frac{k|k|}{2}H = 0 \tag{3.4.30}$$

The solution for the function $(-kx_m + 1)$ for the $|k| = 1$ case is:

$$(-kx_m + 1) = 1/3(kc + 1) \pm\sqrt{\frac{4}{9}(kc + 1)^2 - (\frac{2}{3})H} \tag{3.4.31}$$

In fig. 3.3.7, there are two distinct angles, ϕ_m and ϕ_m^*, for which $\dot{\phi}$ vanishes. Observe that as the system evolves towards transition into the libration phase, the angle ϕ_m^* increases to 90°. Thereafter ϕ_m^* no longer corresponds to a value for which $\dot{\phi}$ vanishes. On the other hand, ϕ_m does correspond to a real libration amplitude in the libration phase. The important question is: Which solution (±) corresponds to this "normal" solution? *The right hand side of* (3.4.31) is positive definite and the above equation must be valid for all possible values of β. In the limit $|\beta| >> 1$, the function $c(t)$, at transition, equals $1/2\beta$ which is presumed to be a large negative definite number. Therefore, for the $k = +1$ case, we should choose the + sign in (3.4.31). In the limit $|\beta| << 1$, $x_m \approx -c(t)$ in the libration phase. Again the + solution of (3.4.31) agrees with the expected behavior. The "normal" solutions for $(-x_m + 1)$ and $\cos\phi_m$ are:

$$(-x_m + 1) = 1/3(c + 1) + \sqrt{4/9(c + 1)^2 - (2/3)H}; \tag{3.4.32a}$$

$$b\cos\phi_m = 2p_m(-x_m + 1)^{+1/2} \tag{b)}$$

The action integral can be solved in terms of standard elliptic integrals, but its form is exceedingly complex. A simpler procedure is to numerically evaluate the integral for a given value of c and a test value of H. Then H can be varied until the numerically calculated value of $J_{lib.}$ agrees with its appropriate initial value. $J_{lib.}$ is easily calculated at transition and is found to be (B.30,31);

$$J_{lib.} = \begin{cases} 4\pi c \ ; & |\beta| < |\beta_{c1}| \\ -2\pi k \ ; & |\beta| > |\beta_{c1}| \end{cases}$$

Figure 3.4.2 is a plot of the initial value $c(t_i)$ versus the magnitude of β. The break in the curve occurs at $\beta_{c1} \cong -0.2722$. It results from that fact that transition for $|\beta| < |\beta_{c1}|$ involves the coincidence of two π-roots while for $|\beta| > |\beta_{c1}|$, it involves the vanishing of b(x). Figure 3.4.3 is a graph of $\sin\phi_m$ versus the parameter $(1 - 2c(t)/\beta)$ for several values of the parameter $|\beta| > |\beta_{c1}|$. This graph clearly supports the analytic approximation developed in the limit $|\beta| >> 1$. Note that the initial slope of the curves approaches the straight line generated by ploting $2c(t)/\beta$ versus $(1 - 2c(t)/\beta)$. Furthermore, the value of c(t) for which the slope begins to flatten out is approximately equal to $|\beta|^{1/3}$. Figure 3.4.4 is a graph of ϕ_m versus $(c(t) + 1)^{1/2}$ for values of $|\beta| \leq 2$. The important observation is that the curve rapidly approaches the limiting curve generated by the approximation obtained when $|\beta| << 1$.

In the course of this development, various kinds of stability have been mentioned or implied. In studying the neighborhood of stationary solutions of the Hamiltonian (Section 2.7), we inferred

that these solutions may or may not be stable against small perturbations. If dynamically stable, these stationary solutions were designated as libration centers. The system of simple pendulum plus constant applied torque suggested that librations were possible only if the magnitude of the applied tidal torque is smaller than maximum value of the pendulum torque or is "tidally stable."

Finally, in investigating example III in which the coefficient b(x) of the pendulum term was momentum dependent, we found that if the system made a permanent transition into libration, the magnitude of b(x) had to tend to increase thereafter. Since it is the slow change in the parameter c(t) which indirectly causes the magnitude of b(x) to adiabatically increase or decrease, we shall use the terms "adiabatically stable" and "adiabatically unstable" in alluding to this kind of behavior in the next chapter.

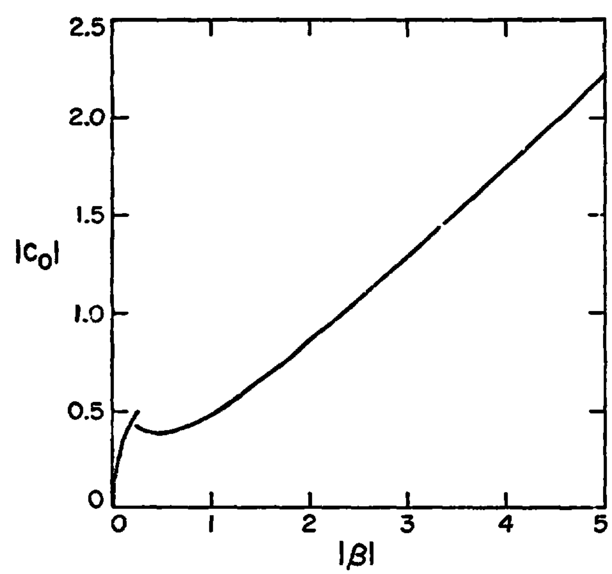


FIGURE 3.4.2

Graph of the initial value of c at transition, c_o, versus the magnitude of β for $|\beta| \leq 2$.

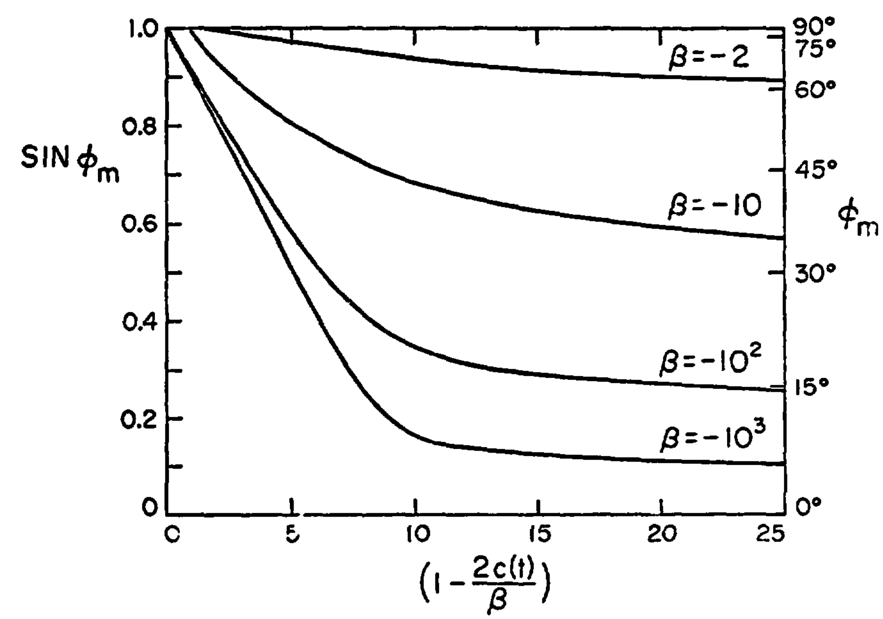


FIGURE 3.4.3

Plot of the sine of the amplitude of libration versus the parameter $(1 - \frac{2c(t)}{\beta})$ for several values of β above the critical value of β_{c1}.

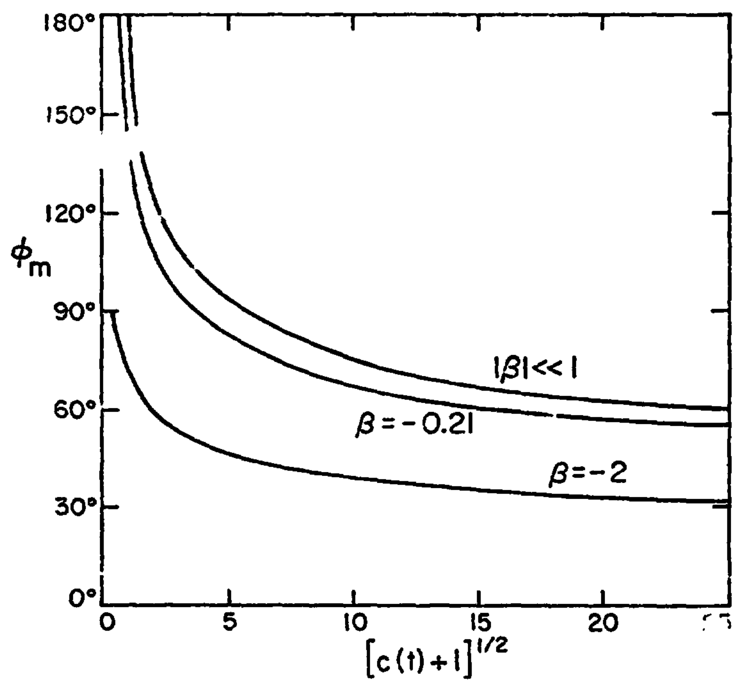


FIGURE 3.4.4

Plot of ϕ_m versus the parameter $(c(t) + 1)^{1/2}$ for values of $|\beta| \geq 2$.

4.1 THE SATELLITE-SATELLITE RESONANCES OF SATURN

There exist three examples of two-body resonance interactions in the ten-satellite system of Saturn. Observations of Mimas ($\frac{a_{Mi}}{R_\theta}$ = 3.11) and Tethys (4.94) reveal that the ratios of their mean motions is 2:1 and that conjunction of the two satellites tends to librate about the midpoint of their nodes with amplitude 46.5°. An investigation of their mutual gravitational interaction (Tisserand, 1896) shows that this phenomenon can be explained as a gravitational resonance in which the angle $\phi \equiv 4\lambda_{Mi} - 2\lambda_{Te} + \Omega_{Mi} + \Omega_{Te}$ librates about zero with maximum amplitude 97° and with a period of 70.78 years. The coefficient of the term in the expansion of the disturbing function specifically responsible for the observed behavior is proportional to the product of the inclinations: $I_{Mi}I_{Te}$. Thus the resonance can be classed as a mixed [illegible] type (2.3).

The conjunctions of Enceladus (3.99) and Dione (6.33) are observed to librate about the pericenter of the inner satellite with period of approximately twelve years ($\phi \equiv \lambda_{En} - 2\lambda_D + \omega_{En}$). The amplitude of libration is very small, quoted values ranging from 20' (Goldreich, 1965) to 1.5° (Sinclair, 1972). Ob[illegible]tions of the resonant perturbations in the mean longitudes of t[illegible]ir are even smaller; 1.4' in Enceladus and 0.9' in Dione (Brouwe[illegible]nd Clemence, 1961, p. 133). This means that the librat[illegible] the resonance variable is principally governed by the [illegible] of the pericenter of Enceladus.

The pair of satel' ;es farthest removed from Saturn is also involved in an e-type resonance. In this case, conjunction of Titan (20.48) and Hyperion (24.83) librates about the apocenter of the outer satellite with amplitude of 36° and period equal to $\sim$ 2 years. The commensurability ratio is 3:4 and the resonance variable is: $(\phi = 3\lambda_{Ti} - 4\lambda_{Hy} + \tilde{\omega}_{Hy})$.

One novel aspect of this resonance phenomenon is that it tends to keep the satellites as far apart as possible at conjunction. This behavior is seen as enhancing stability among the participating satellites. We should also mention that the masses of the satellites can be determined from a knowledge of the period of libration and the ratio of the libration amplitude of the mean longitude of each satellite (Jefferies, 1953).

This is the set of information available concerning these satellites. Any speculation concerning the existence of appreciable tidally-induced torques acting on any of Saturn's satellites has not as yet been supported with visual evidence of either a secular change of orbital periods or, in the case of resonances, a displacement of the center of libration away from mod(π) or mod(2π). Goldreich suggested that tidal torques are acting on the sa[illegible] and offered two arguments to support his thesis: 1) The exi[illegible] so many resonances cannot be a chance affair and must be due to some mechanism. He suggested that significant tidal evolution of the inner satellites of Saturn and Jupiter must have taken place over the

age of the solar system ($\sim 4 \times 10^9$ years). A lower bound on the dissipation function Q^{-1} was calculated by integrating (2.9.4) and demanding that the closest satellite (Mimas) was at the planet's surface 4×10^9 years ago. The tidal dissipation function Q^{-1} is defined by the relation

$$Q^{-1} = \frac{1}{2\pi E_o} \oint \left(\frac{dE}{dt}\right) dt,$$

where E_o is the maximum energy stored in the tidal distortion a[illegible] the integral is over one complete cycle.

2) A theoretical calculation of the dissipation function was attempted for Jupiter and Saturn and was found to be in rough agreement with its lower bound (Goldreich and Soter, 1966). Q^{-1} for Saturn is estimated to be $\sim 1.5' \times 10^5$. Accepting this estimate of Q and assuming that it applies to all the other satellites, we find, for example, that Titan, which is the most massive of Saturn's satellites, has increased the radius of its orbit by only 1/4% over the age of the solar system!

We should mention that using this value of Q and (2.9.4), the tidal deceleration of the mean motion of Mimas is 1.4×10^{-22} sec^2, or equivalently, 0.04° century^{-2}. The magnitude of the resonance torque can be estimated from the period of libration (i.e., 70.78 yrs. for Mi-Te). We find that the ratio of the tidal torque acting on Mimas to the parameter β is $\sim 10^{-5}$. Because the torques acting on each body tend to cancel in the resonance variable, the ratio of

$b(x)$ to the sum of tidal torques in $\dot{\phi}$ (2.9.10) is an order of magnitude larger. Since the torques acting on the other satellite resonances are weaker, the ratio is even larger for these systems. Therefore, the hypothetical tidal evolution of these satellites should be well described by the theory developed in chapters one and three. Recall that its quantitative accuracy is set by the parameter $\beta^{-1}\frac{dc}{dt}$.

As mentioned earlier the effect of dissipative tides raised by a given satellite on its primary is to cause a torque parallel to its angular velocity. This is true if the spin of the planet and the orbital motion of its satellite are in the same direction and the planet's rotation period is shorter than the satellite's orbital period. This torque tends to increase the size of the orbit and decrease its period. One expects that in a many-satellite system, after a time, some pairs will approach a commensurability and then evolve through a succession of related resonances. Table 4.1.1 is a list of the strongest resonances associated with a 2:1 commensurability. (Unprimed variables refer to the inner satellite, primed variables to the outer body.)

They have been ordered in the same sequence in which the pair would encounter them under the following assumptions: 1) The tidal acceleration of the inner satellite is at least twice that of the outer one. That is, the tidal torque acting on the inner satellite determines the sign of the torque acting on the resonance variable. Therefore, in the absence of a resonance $\dot{\phi}$ tends to decrease

2) The motion of the perihelion $\tilde{\omega}$ is prograde while that of the node Ω is retrograde. This kind of behavior is caused by the secular terms in the disturbing function. 3) The motions of $\tilde{\omega}'$ and Ω' of the outer satellite are smaller than the corresponding motions of $\tilde{\omega}$ and Ω.

We should observe that all the resonances listed in Table 4.1.1 are "adiabatically stable" (i.e., the tidal torque tends to secularly increase the coefficient $b(x)$ in the libration phase and decrease the libration amplitude). This pathological result is related to the two body resonance. We have not rejected any tidally unstable resonance variables outright. The strongest tidally unstable resonance must have at least three leading factors of e and I. For example, the angle $\phi = \lambda - 2\lambda - \tilde{\omega} + 2\tilde{\omega}'$ has a leading factor of ee'^2. It is adiabatically unstable if, in the libration phase the tidally induced decrease in e is greater than the corresponding increase in e'^2. In the three body interaction, both kinds of resonance variables appear with comparable coefficients $b(x)$ (see Table 4.2.4 for specific examples).

Table 4.1.2 lists the pertinent data now available on the masses and orbital elements of these satellites. This information shall be called upon during the course of this discussion and is collected here for convenience.

Table 4.1.3 lists the observed angular frequencies of these satellites, the libration period of the resonance and the observed periods associated with the secular motion of $\tilde{\omega}$ and Ω. The

Leading Dependence of b(x)	Resonance Angle
I'^2	$2\lambda-4\lambda'+2\Omega'$
II'	$2\lambda-4\lambda'+\Omega+\Omega'$
I^2	$2\lambda-4\lambda'+2\Omega$
e'	$\lambda-2\lambda'+\varpi'$
e	$\lambda-2\lambda'+\varpi$
e'^2	$2\lambda-4\lambda'+2\varpi'$
e^2	$2\lambda-4\lambda'+2\varpi$

TABLE 4.1.1

Table of the major resonance frequencies associated with a 2:1 commensurability of the inner (unprimed) satellite with the outer (primed) one. The frequencies are ordered in the same sequence in which this pair of satellites encounters them.

observed secular motion of ϖ and Ω includes both the contribution from the secular and resonance terms of the disturbing function. Explicitly

$$\left(\frac{d\varpi}{dt}\right)_{ob} = \left(\frac{d\varpi}{dt}\right)_{sec} + \left(\frac{d\varpi}{dt}\right)_{res} . \qquad (4.1.1)$$

The separate contributions from the secular and resonant parts of the motion of the appropriate pericenter or node have been calculated by Jefferies (1953). In some cases data is missing either because it is inappropriate or because the corresponding eccentricities and inclinations are very small and variable and the corresponding observed average motion of the pericenter or node has not been determined.

Except for Hyperion, the sign of the motion of perihelion and node indicate that contribution from the secular term in the disturbing function is much larger than that of the resonance term.

These frequencies appear to be well spaced for both the Mimas-Tethys and Enceladus-Dione examples so that the evolutionary picture involving a single resonance variable tentatively applies. In the case of Hyperion, the motion of the perihelion is retrograde and large compared to the expected prograde motion due to the secular terms of the disturbing function. Compare $\frac{d\varpi_T}{dt} \simeq +0.5°/yr$, with $\frac{d\varpi}{dt}_{Hy} \sim -19°/yr$. Both should be approximately the same if the principal contribution to the motion of ϖ came from the secular terms. The present ordering of the resonance frequencies for the Titan

Satellites	Semi-major axis (10^3km)	Period (days)	Eccentricity	Inclination to Saturn's Equator	Mass ($\frac{m}{M_♆}$)
Mimas	186	0.942422	0.0201	1°517	$6.7\text{X}10^{-8}$
Enceladus	238	1.370218	0.00445	1:4	$1.27\text{X}10^{-7}$
Tethys	295	1.887802	0.0	1°093	$1.14\text{X}10^{-6}$
Dione	377	2.734915	0.0022	1:4	$1.8\text{X}10^{-6}$
Titan	1222	15.945452	0.0290	0°3	$2.4\text{X}10^{-4}$
Hyperion	1481	21.27666	0.104	0°5	$2\text{X}10^{-7}$

TABLE 4.1.2

Data on six satellites of Saturn involved in resonances. Most of the information comes from Allen's Astrophysical Quantities (1963), except for the masses (Jefferies, 1953) and the inclinations of Enceladus and Dione (Sinclair, 1972).

Satellite	Mean Motion (°/day)	Libration Frequency °/yr	$(\frac{d\varpi}{dt})_{sec}$	$(\frac{d\varpi}{dt})_{res}$	$\frac{d\varpi}{dt}$ °/yr	$(\frac{d\Omega}{dt})_{sec}$	$(\frac{d\Omega}{dt})_{res}$	$\frac{d\Omega}{dt}$ °/yr
Mimas	381.994	5.09		---	365.60			-365.23
Tethys	190.697			---				-72.2
Enceladus	262.732	32.4	123.4	-29.3	153.8		---	
Dione	131.535			---	30.7		---	
Titan	22.577	205	0.50	---	0.50	-0.49	---	-0.49
Hyperion	16.919		3.20	-22.14	-18.53	---	---	

TABLE 4.1.3

Table including the relevant frequencies of each of Saturn's satellites involved in a two body resonance. Data not included is either inappropriate or unavailable. The primary source for the above values is Jefferies (1953).

Hyperion example for the 4:3 commensurability indicates that the first resonance variable encountered is the presently observed example. This is due entirely to the large negative (retrograde) motion of $\tilde{\omega}_{Hy}$ resulting from the resonance term. Before we proceed we should ask if this same situation prevailed at transition for the other two examples.

In the libration phase, an adiabatically stable resonance tends to cause a retrograde motion of either perihelion or node, depending on whether the angle variable is an e or I type, respectively. This can be seen by inspection of the appropriate equation of motion for $\tilde{\omega}$ or Ω. In the case of an e-type $(\frac{d\tilde{\omega}}{dt})_{res}$ is: (2.4.3; 3.1.7b):

$$(\frac{d\tilde{\omega}}{dt})_{res} = \frac{k|k|}{2}\,\beta(-kx+1)\frac{(|k|-4)}{2}\,\cos\phi\,, \qquad (4.1.2)$$

where the parameter β for an e-type resonance variable is given by

$$\beta = \frac{4\mu'}{\mu_o}\frac{Ce^{(|k|-4)}}{A_{oxx}a_>a_o} \quad .$$

If ϕ librates about mod(2π), then β is negative. For an adiabatically stable resonance, k is positive, hence $(\frac{d\tilde{\omega}}{dt})_{res}$ is negative. The angular frequency associated with this impressed motion, in dimensionless time unit $\bar{t}(\equiv A_{oxx}x_o t)$, is of $O(b)$. In terms of real time t, this angular frequency is approximately given by:

$$(\frac{d\tilde{\omega}}{dt})_{res} \approx n_o\,\frac{\mu'}{\mu_o}(\frac{a_o}{a_>})\,Ce_o^{(|k|-2)} \quad . \qquad (4.1.3)$$

We see explicitly that the impressed motion of $\tilde{\omega}$ tends to blow up as $e_o \to 0$ only for the $|k| = 1$ case. If e_o is very small at transition, then the motion of $\tilde{\omega}_{res}$ may be both large and retrograde such that the first resonance variable encountered is the k = +1 e-type.

The mixed-I type can also lead to a large retrograde motion of $(\Omega + \Omega')$ if one but not the other inclination is very small. Even so, we expect that the impressed motion of the nodes is O(I) smaller than the similar motion of the perihelion of the lighter satellite. This means that the e-type (k = +1) resonance variable can still be the first encountered unless I_o is very much smaller than e_o.

From Table 4.1.2 we can see that the motions of $\tilde{\omega}$ and Ω, in which the secular term predominates, tend to be equal and opposite, and that the retrograde motion of the node of the inner satellite is greater than that of the outer one. (Specifically, inspect the Mimas-Tethys case to assure yourself that the above statement is true.) If this is the case, we should determine the critical value of e_o such that the impressed retrograde motion of $\tilde{\omega}$ (or for that matter, $\tilde{\omega}'$) is greater than twice the retrograde motion of the node Ω of the inner satellite.

The present value of the eccentricity for Mimas is exceptionally large compared to that for other inner satellites (Compare $e_{Mi} = 0.020$ with the next largest eccentricity found among the inner satellites: $e_{En} = .00445$). Presumably the reason lies in the relative importance of the tide raised by Mimas with the radial tide raised on Mimas by Saturn and their opposite effects on the eccentricity (see 2.8 for a discussion). Another possibility is that e_{Mi} was driven to a large value through a previously established e-type resonance with, say, Enceladus which since has been disrupted. In either case, it means that we should look at the e-type resonance involving the pericenter to Tethys to determine the maximum value of $e_{o_{Te}}$ such that the variable $\{\phi = \lambda_{Mi} - 2\lambda_{Te} - \tilde{\omega}_{Te}\}$ is encountered first. Using (4.1.3) and setting $C = -1$ (which is approximately correct), we find that the impressed angular frequency $(\dot{\tilde{\omega}}_{Te}) \approx (2\dot{\Omega}_{Mi})_{sec} \doteq -730°/yr$ is

$$e_{o_{Te}} \sim \frac{m_{Mi}}{M_{\theta}} \frac{n_{Te}}{\left|\frac{2d\Omega_{Mi}}{dt}\right|} \sim 10^{-5} . \tag{4.1.4}$$

This value is extremely small and shall receive more comment later.

For the Enceladus-Dione example the largest possible retrograde motion results from the resonance with the perihelion $\tilde{\omega}$ of the inner satellite. Making the same approximations, but this time setting $\left(\frac{d\tilde{\omega}_{En}}{dt}\right)_{res}$ equal to $(-2 \times 152.57°)$, we find that if

$$e_{o_{En}} \sim \frac{m_{Di}}{M_{\theta}} \frac{n_{En}}{\left(\frac{d\tilde{\omega}}{dt}\right)_{res}} \sim 10^{-3} , \tag{4.1.5}$$

then the first resonance encountered was the one in which the pair of

satellites is presently captured. We shall discover later that a reasonable estimate for $e_{o_{En}}$ at transition is $\sim 10^{-4}$.

We now see that the earlier description of tidal evolution through an ordered sequence of well-spaced resonance variables only applies if the eccentricities of either partner of the resonance are not too small. Otherwise, it may happen that a variable such as $(\lambda - 2\lambda' + \tilde{\omega})$ is encountered first. Whether or not capture occurs shall be our next topic. But before we proceed, we should observe that the one dimensional model so carefully constructed may fail under certain circumstances.

Consider the implication that the relative order of the resonances may be interchanged depending on the parameters e_o, I_o, etc. It may happen that two resonance frequencies may nearly overlap. It is no longer necessarily true that the resonance system can be described by a one dimensional Hamiltonian. Normally one expects that an e-type resonance (i.e. $b(x) = e$) is stronger than, say, a mixed I type (i.e. $b(x) = II'$). Then, hopefully, a reasonable approximation would be to ignore the mixed I type. But if e_o is very small, then the dominant resonance may be the mixed I type.

Another possibility is that e_o is so small that the corresponding resonance variable is well-spaced from any I type resonance <u>at transition</u>. If this value is less than the critical value, then the system will automatically enter the libration phase with an initial amplitude of 90°. Since the resonance is tidally stable, the mean value of the eccentricity will thereafter increase. As the mean value of e increases, the magnitude of the impressed retrograde motion of the perihelion must decrease. Eventually it will overlap with an I-type resonance. It may happen that this I type disrupts the established resonance, depending on their relative strengths. Exactly what may transpire in either case would require a rigorous examination of a two-resonance variable system subject to a constant applied torque.

Does either case have relevance to the previously discussed examples? In the Mimas-Tethys case, compare the value of $e_{o_{Te}}$ for which these two variables overlap (4.1.4) and the present value of $I_{Mi} I_{Te}$. We find that the latter is of O(10) larger. For this example, it appears that if the two resonances overlap, the mixed I type still predominantly determines the fluctuations in the mean longitudes, λ and λ'. On the other hand, if we make a similar comparison of the variables $(\lambda_{En} - 2\lambda_{Di} + \tilde{\omega})$ and $(2\lambda_{En} - 4\lambda_{Di} + \Omega_{Di} + \Omega_{En})$, we find that the coefficient of the e-type is O(10[illegible]) greater than the mixed I type.

It seems unlikely that $e_{o_{Te}}$ is or was ever as small as 10^{-5}. If this be true, then Mimas-Tethys evolved through the sequence found in Table 4.1.1. The first resonance encountered was the I^2 resonance. Presumably because of unfavorable initial conditions at transition, the system evolved past this resonance and later approached the II' resonance. The corresponding resonance term in the disturbing function R has the form:

$$(R)_{res.} = \frac{\mu_{Te}}{a_{Te}} I_{Mi}(x) I_{Te}(x) C \cos\phi, \tag{4.1.6}$$

where the dependence of the inclinations on the action variable x is (2.4.4; 2.6.1):

$$\frac{I_{Mi}(x)}{I_{o_{Mi}}} = (-x + 1)^{1/2},$$

$$\frac{I_{Te}(x)}{I_{o_{Te}}} = (\frac{m_{Mi}}{m_{Te}}) x + 1)^{1/2} \tag{4.1.7}$$

The mass ratio $\frac{m_{Mi}}{m_{Te}} \sim \frac{1}{17}$. Thus the variation in $I_{Te}(x)$ with x is $(\frac{1}{17})$ times smaller than a corresponding change in I_{Mi}. This also applies to any secular change in the inclinations after libration is established. The factor C can be expressed in terms of Laplace coefficients (2.2.15) and numerically evaluated (Tisserand, vol. 14, p. 100, 1896):

$$C = -\tfrac{1}{4}\alpha b_{3/2}^{(2)}(\alpha) = -0.4086. \tag{4.1.8}$$

The probability of capture is determined by the dimensionless parameter β which occurs as a factor in the pendulum-like term of the Hamiltonian (3.1.5). The appropriate parameter in this case is (obtained by dividing by $Z = 1/2\, I_{o_{Mi}}^2 L_{o_{Mi}}$):

$$b = 4\frac{m_{Te}}{M_\theta}\frac{CI_{o_{Te}}I_{o_{Mi}}^{-3}}{A_{oxx}a_{Te}a_{Mi}}\,; \quad A_{oxx} = \frac{12}{a_{Mi}^2} + \frac{m_{Mi}}{m_{Te}}\frac{48}{a_{Te}^2}\,. \tag{4.1.9}$$

The mass ratio occurring in A_{oxx} is $\sim \frac{1}{17}$ while $\frac{a_{Mi}}{a_{Te}} = (\frac{n_{Mi}}{n_{Te}})^{-2/3} = 2^{-2/3}$. Evaluating β with the present values of the orbital parameters, we find

$$\beta_{now} = -1.02 \times 10^{-4}. \tag{4.1.10}$$

Compare this with the critical value $\beta_{c1} \cong -0.27$ for which the system automatically enters the libration phase. Incidently, the inclination of Mimas would have had to be 31° for this to occur. The present libration amplitude (97°) and the small value of β_{now} indicate that β, evaluated at transition, will also be small. Therefore the approximation developed in section 3.4 in the limit $|\beta_o| << 1$ can be applied to this resonance.

Neglecting the effect of the x-dependent term in the tidal torque (2.10.6), we find that transition occurred when $b(x)_{now} \cong 0.2\beta_o$, or when the mean inclination of Mimas was one-fifth its present value. Allan (1969) included the effects of the x-dependent term and determined the value of the semimajor axis and the inclination for each satellite when $\phi_m = 180°$ numerically. His results are:

$$
\begin{aligned}
a_{o_{Mi}} &= 0.9922\, a_{Mi}\,(\text{now}); \\
I_{o_{Mi}} &= 0.277\, I_{Mi}(\text{now}) = 0.°415'6; \\
a_{o_{Te}} &= 0.9922 \quad a_{Te} = 0.9922\, a_{Te}(\text{now}) \\
I_{o_{Te}} &= 0.945\, I_{Te}(\text{now}) = 1°046.
\end{aligned} \tag{4.1.11}
$$

Calculating β using the above values, we find

$$\beta_o = -4.4 \times 10^{-3}. \tag{4.1.12}$$

Since the value of the parameter β_o is quite small compared to one, (3.2.12) can be used to approximate the probability P_c for capture into libration.

$$P_c \cong \frac{4}{\pi}|\beta|^{-1/2}b_x(o) \cong \frac{2}{\pi}(1 + \frac{1}{17})|\beta|^{1/2} = 4.4\%. \tag{4.1.13}$$

Note that $b(x)$ is proportional to $(-x + 1)^{1/2}(\frac{1}{17}x + 1)^{1/2}$ and that the mass ratio of the inner to the outer satellite is $\sim \frac{1}{17}$. Thus $b_x(0)$ equals $1/2(1 + \frac{1}{17})|\beta|$ where β equals β_o. Sinclair found through numerical calculation that $P_c \sim 4\%$, which is in agreement with the above analytic result. The probability for the first resonance encounter can be estimated by comparing the appropriate function $b(x)$ for an I^2 resonance with that for an II' type. For the k = +2 case b(x) is:

$$b(x) = \beta(-2x + 1) = 4\frac{M_{Te}}{M_{\oplus}} \frac{C(I^2) I_{o_{Mi}}^{-2}}{A_{oxx} a_{Te} a_{Mi}} (-2x + 1). \qquad (4.1.14)$$

For the I^2 resonance, $b_x(0)$ equals $2|\beta|$. The coefficient $C(I^2)$ can be determined by comparing the lowest order contributions in the disturbing function to the I^2 and II' resonance variable. We find that $C(I^2) = -1/2C(II')$. (Note: to obtain this result set $\ell_{min} = 4$, $m = 4$, $p_1 = 1$ and $p_2 = 0$ for the I^2 term. For the II' term set $m = 3$.) By the way, since $C(I^2)$ is positive, the I^2 resonance variable librates about the mod(π) position. Finally, the relation between $P_c(I^2)$ and $P_c(II')$ can be found by comparing the equations (4.1.9, 4.1.13, and 4.1.14). Specifically:

$$P_c(I^2) \doteq \frac{4}{\sqrt{2}}\left(\frac{I_{o_{Mi}}}{I_{o_{Te}}}\right)^{1/2} P_c(II') \simeq 7.9\%. \qquad (4.1.15)$$

Sinclair's estimate for this case is $P_c(I^2) \sim 7\%$.

Incidently, Allan found that transition occurred $\sim 2.2 \times 10^8$ years ago using (2.9.4) and Goldreich's estimate of the dissipation function $(Q_{\oplus}^{-1} = 1.5 \times 10^{-5})$. This resonance appears to have been established well within the age of the solar system ($\sim 4 \times 10^9$ years).

The next case we shall discuss is the resonance involving Enceladus and Dione. The relevant part of the disturbing function is:

$$(R)_{res} = \frac{\mu_{Di}}{a_{Di}} Ce_{En} \cos\phi. \qquad (4.1.16)$$

From Jefferies (1953), the factor C approximately equals -0.753 while the coefficient C' belonging to R' equals $\sim -11?$. The relevant parameter β for this type is:

$$\beta = 4\frac{\mu_{Di}}{\mu_{\oplus}} \frac{Ce_{o_{En}}^{-3}}{A_{oxx} a_{Di} a_{En}}, \quad A_{oxx} = \frac{3}{a^2_{En}} + \frac{KM_{En}}{M_{Di}} \frac{12}{a^2_{Di}}. \qquad (4.1.17)$$

The parameter K is obtained from 2.6.10 and equals 1.59. The mass ratio of the satellites is $\frac{m_{En}}{m_{Di}} \doteq \frac{1}{14}$. Calculating β_{now}, we find

$$\beta_{now} = -10. \qquad (4.1.18)$$

Since the critical value $\beta_{cl} \doteq -0.27$, and $|\beta_o|$ must have been larger than $|\beta_{now}|$, we can conclude that the system automatically entered the libration phase with maximum amplitude of 90°. Since $|\beta_{now}| >> 1$, the relevant approximation of the Hamiltonian applies (3.3.3). Using Sinclair's quoted value for the libration amplitude ($\phi_m \simeq 1°5'$), the change in e since transition is obtained from (3.4.1). We find

$$e_o = e_{now} \sin\phi_m = \frac{1}{42} e_{now} \doteq 1.1 \times 10^{-4}. \qquad (4.1.19)$$

The relation between the initial and the present values of the parameter $c(t)$ occurring in the Hamiltonian equation (3.4.4) is given by

$$c_{now} = \frac{1}{42} c_o . \tag{4.1.20}$$

The change in $c(t)$ is directly related to the change in the commensurability relation by (2.9.9). Explicitly:

$$c(t)A_{oxx}(-\Gamma_o) \cong -(n_{o_{En}} - 2n_{o_{Di}}) - T(\frac{dn_T}{dt_{En}} - 2\frac{dn_T}{dt_{Di}}); \tag{4.1.21}$$

T is the time since transition. Of course $c(T) = c_{now}$.

The change in the commensurability relation can be expressed in terms of the present average value for e_{En} ($e_{ave.} = 1/2(e_{max} + e_{min})$) using (3.4.8, 4.1.20). Namely,

$$a(T)A_{oxx}x_o = n_{En}\frac{\mu_{Di}}{\mu_{\oplus}}(\frac{a_{En}}{a_{Di}})Ce_{En}^{-1}(now), \tag{4.1.22}$$

where we have assumed that the mean motions (and semimajor axes) have changed little since transition. Observe that the right hand side tends to vanish as e_{En} increases. The implication is that the approximate commensurability tends toward an exact one as the system evolves. Goldreich was apparently the first to make this observation (1965).

The next step is to determine how much the mean motion and the semimajor axis of Enceladus have changed since transition. Assuming the same dissipation function for both satellites we find

$$(\frac{dn_T}{dt_{En}} - 2\frac{dn_T}{dt_{Di}}) \cong 0.28 \frac{dn_T}{dt_{En}} , \tag{4.1.23}$$

Since the resonance only weakly affects the motion of the mean longitudes in the limit $|\beta| >> 1$, we find, for example, that the change in the mean motion of Enceladus since transition, Δn_{En}, is the following.

$$\Delta n_{En} \cong \frac{dn_T}{dt_{En}} T = -3.6\Delta c(T)A_{oxx}(-\Gamma_o) \cong -0.032\, n_{En}(T) \tag{4.1.24}$$

where $\Delta c(T)$ equals $c(T) - c_o = 41\, c(T)$. This corresponds to a change in the semimajor axis Δa_{En} of Enceladus given by (using the approximate relation: $\frac{\Delta a}{a} \cong -\frac{2}{3}\frac{\Delta n}{n}$):

$$\Delta a_{En} = 0.021\, a_{En}(T). \tag{4.1.25}$$

Using Goldreich's estimate for Q we find that the transition into 90° libration occurs when

$$T = 1.4 \times 10^9 \text{ years.} \tag{4.1.26}$$

Again, this appears to be an estimate within the age of the solar system.

Titan and Hyperion are also presently engaged in an e-type resonance. The relevant part of the disturbing function is:

$$(R)_{res} = \frac{\mu_{Ti}}{a_{Hy}} Ce_{Hy} \cos\phi. \tag{4.1.27}$$

From Tisserand (vol. 4, p. 107, 1896), we find

$C = +3.26.$

As C is positive, this resonance variable librates about the mod(π) instead of the mod(2π) position. The mass of Titan is $O(10^3)$ greater than that of Hyperion which implies that 1) the tidal evolution of the resonance is caused almost entirely by the tidal torque acting on Titan and 2) the effect of the resonance on the orbit of Titan is almost nil.

We mentioned in section 2.9 that the x-dependent term associated with the variation in $\frac{dn_T(x)}{dt_{Ti}}$ with x is small although a naive estimate of its effect seemed unusually large. The reason the estimate was wrong is that we assumed that the fluctuations in the tidal torques were of $O(\frac{\delta x}{L_o})$ and ignored the fact that one tidally driven partner might be much more massive than the other. The fluctuations in $\frac{dn_T(x)}{dt_{Ti}}$ are of $O(\frac{x'}{L_o})$ where $x' = \frac{m_{Hy}x}{m_{Ti}}$. Since the mass ratio $\frac{m_{Hy}}{m_{Ti}}$ is of $O(10^{-3})$, the contribution from this term to the tidal evolution of this resonance can be safely ignored.

The parameter β_{now} for this case is

$$\beta_{now} = 4\frac{m_{Ti}}{M_\theta}\frac{e_{Hy}^{-3}(now)}{A_{oxx}a^2_{Hy}}\,; \quad A_{oxx} = \frac{48}{a^2_{Hy}} + \frac{m_{Hy}}{m_{Ti}}\,\frac{27}{a^2_{Ti}}\ .$$

Using $\frac{m_{Ti}}{M_\theta} = 2.4 \times 10^{-4}$ and $e_{Hy}(now) = 0.104$, we find

$$\beta_{now} = -0.0575. \tag{4.1.28}$$

Incidently, the value of e_{Hy} corresponding to $\beta_{cl} = -27$ is:

$$e_{cl_{Hy}} = 0.057. \tag{4.1.29}$$

The above value for β_{now} is certainly close enough to the critical value for automatic transition that we should expect that it did *occur*. *This mode of transition can be inferred from the following* argument.

Assuming for the moment that the limit $|\beta| >> 1$ can be applied to determine the mean value of the eccentricity at transition, we obtain the result

$$e_{o_{Hy}} = \sin(\phi_m = 36.2°)e_{Hy}(now) = 0.061 \tag{4.1.30}$$

$$= e_{Hy}(now)\ \sin(\phi_m = 36°).$$

This limit represents the fastest possible damping of ϕ_m for the smallest possible change in the eccentricity of Hyperion. The actual change in e_{Hy} since transition must be greater than the above value. Thus this value (4.1.30) represents an absolute maximum for the value of $e_{o_{Hy}}$ at transition. After comparing $e_{cl_{Hy}}$ with (4.1.30), we can infer that capture into 90° libration occurred. Unfortunately, we also see that neither approximation to the Hamiltonian (i.e. $|\beta| >> 1$ and $|\beta| << 1$) can be rigorously applied to determine the evolution of the system. But we can attempt to match the solutions found *for the two limits to obtain* an estimate on the evolution of the system since transition. The most reasonable value of β to match the two solutions is β_{c1}. That is, we shall demand that the system evolve backward in time according to the solution obtained for $|\beta| \ll 1$ until $b(x) = \beta_{c1}$. Before that time have this resonance evolve according to the solution found in the limit $|\beta| \gg 1$. A more rigorous derivation using the action integral is given, following this argument.

The present values of β and ϕ_m are 0.0575 and 36°, respectively. Going back in time, we find that when $\bar{b}(x) = \beta_{c1}$ at time $\bar{T}$, $\bar{b}(x)$ has decreased by a factor of 0.21. This means that, since $b(x)$ is proportional to $e_{Hy}(x)$, the value of $\bar{e}_{Hy}$ at time $\bar{T}$ is

$$\bar{e}_{Hy} = 0.211 e_{Hy}(\text{now}) = 0.022. \tag{4.1.31}$$

From figure 3.4.1, the libration amplitude at time $\bar{T}$ is

$$\bar{\phi}_m = 54°. \tag{4.1.32}$$

The relation between $\bar{e}_{Hy}$, $e_{Hy}(\text{now})$ and the parameter $a(T)$ is

$$\frac{e_{Hy}(\text{now})}{\bar{e}_{Hy}} = (a(T) + 1)^{1/2} \approx (a(T))^{1/2}, \tag{4.1.33}$$

where we have chosen $a(\bar{T}) \to 0$ at time $\bar{T}$. Since $e_{Hy}(\text{now})$ is much larger than $\bar{e}_{Hy}$, $(a(T) + 1) \approx a(T)$.

The mass of Hyperion is much smaller than that of Titan, implying that the resonance has not affected the tidal evolution of Titan. Therefore, the change in the mean motion of Titan since

time $\bar{T}$ is

$$\Delta n_{Ti} = \frac{dn_T}{dt_{Ti}}(T - \bar{T}). \tag{4.1.34}$$

Additionally, the change in the commensurability relation is

$$\Delta c(T)\, A_{oxx}\Gamma_{o_{Hy}} = 3\frac{dn_T}{dt_{Ti}}(T - \bar{T}), \tag{4.1.35}$$

where $A_{oxx}\Gamma_{o_{Hy}} = -24 e^2_{o_{Hy}} n_{Hy} = -18 e^2_o n_T$;

The fractional change in n_{Ti} can be related to the present value of e_{Hy} using (4.1.33,34,35):

$$\Delta n_{Ti} = -6\Delta c(T) e^2_{oHy}\, n_{Ti} \doteq -0.065\, n_{Ti}(\text{now}). \tag{4.1.36}$$

The change in the semimajor axis of Titan is:

$$\Delta a_{Ti} \doteq 0.043\, a_{Ti}(\text{now}). \tag{4.1.37}$$

Finally, the age $(T - T^*)$ corresponding to $\bar{b}(x) = \beta_{c1}$ is

$$(T - \bar{T}) = 7 \times 10^{10} \text{ years}. \tag{4.1.38}$$

We see that this event occurs before the formation of the solar system. Pressing backward with the second approximation (i.e., $|\beta| \gg 1$) we find from (3.4.1,4) that the mean value of the eccentricity at transition is

$$e_{o_{Hy}} = \sin 54° \cdot e^*_{Hy} \doteq 0.018. \tag{4.1.39}$$

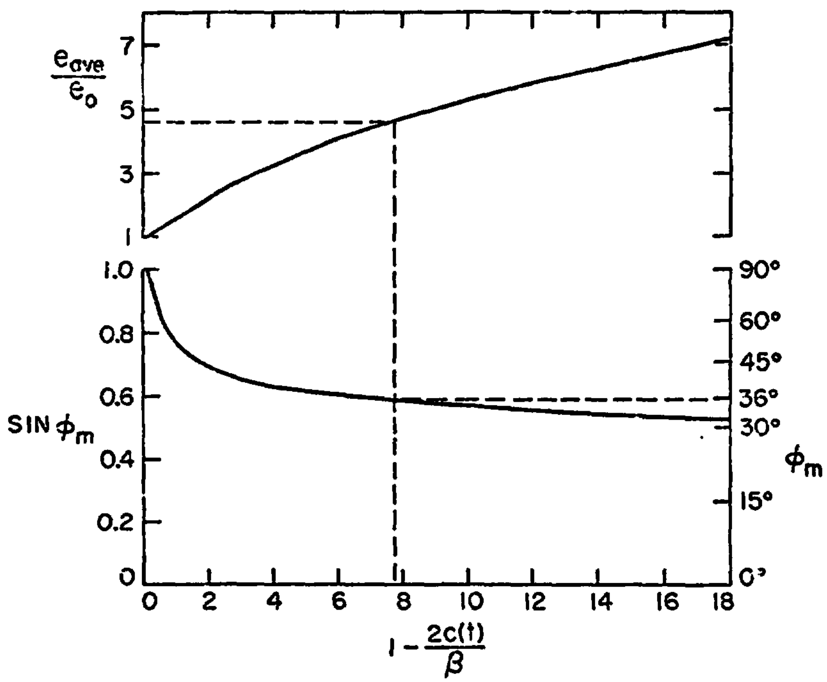


FIGURE 4.1.1

Plot of e_{ave} and $\sin\phi_m$ versus $(\frac{1 - 2c(t)}{\beta})$ for the Titan-Hyperion resonance. The β dashed line indicates the present values for these parameters.

Rather surprisingly, the above value for $e_{o_{Hy}}$ agrees reasonably well with Greenberg's (1972) estimate of 0.015. The change in Titan's semimajor axis up to the time $\bar{T}$, a·' the value for $\bar{T}$, can be determined with arguments similar to those applied in the Enceladus-Dione case. The result is:

$$\Delta a_{Ti} \doteq 0.0125 a_{Ti}; \quad \bar{T} \doteq 2 \times 10^{10} \text{ years.}$$

Thus the age of the resonance and the total change in Titan's semimajor axis are:

$$\Delta a_{Ti} \doteq 0.07 a_{Ti}(\text{now}); \quad T \doteq 9 \times 10^{10} \text{ years.} \qquad (4.1.40)$$

Greenberg estimated that transition occurred 4×10^{10} years ago.

A more accurate estimate of both the age and the change in the orbital parameters can be obtained via the action integral. To make use of the action integral, we must first reduce its dependence to a single unknown parameter. If we evaluate $\dot{\phi}$, H and e at ϕ equal to ϕ_m = 36° where $\dot{\phi}$ vanishes at time T (corresponding to the present), we find

$$\dot{\phi}\big|_{\phi=\phi_m} = 0 = -x_m - c(T) + 1/2\beta(-x_m + 1)^{1/2} \cos\phi_m , \qquad (4.1.41)$$

$$H\big|_{\phi=\phi_m} = 1/2(x_m + c(T))^2 + \beta(-x_m + 1)^{1/2} \cos\phi_m ,$$

$$\left(\frac{e_m}{e_o}\right)_{Hy} = (-x_m + 1)^{1/2}$$

The explicit dependence on x_m as it occurs in H can be eliminated in favor of e_m. The result is

$$H = \frac{1}{8}\left(\frac{\quad}{e_o}\right)^{-2} \beta^2 \cos^2\phi_m + \frac{e_m}{e_o}\beta\cos\phi_m . \qquad (4.1.42)$$

Both $\beta(\equiv\beta_o)$ and e_o are unknown quantities. They are related through the relation

$$\beta_{now} = \left(\frac{e_o}{e_m}\right)^3 \beta_o . \qquad (4.1.43)$$

Thus (4.1 42) can be written as

$$H = \frac{1}{8}\left(\frac{e_m}{e_o}\right)^4 \beta_{now}^2 \cos^2\phi_m + \left(\frac{e_m}{e_o}\right)^4 \beta_{now} \cos\phi_m , \qquad (4.1.44)$$

or H is proportional to $\left(\frac{m}{e_o}\right)^4$. If $|\beta_o| > 0.27$, then $J_{lib.}$ equals (-2π). Therefore, the equation we wish to minimize as a function of e_o is:

$$J_{lib.} + 2\pi = 0 = \oint x d\phi + 2\pi . \qquad (4.1.45)$$

The action integral $J_{lib.}$ can be numerically calculated as a function of e_o and (4.1.45) minimized. The resulting values for the parameters are:

$$e_{o_{Hy}} = 0.0224, \quad \beta_o = -5.85 ,$$

$$c_o = -2.62 , \quad c(T) = 20.1 .$$

The change in the parameter c(t) since transition is related to the change in n_{Ti} by

$$\Delta n_{Ti} = \frac{1}{3}(c(T) - c_o)A_{oxx}\Gamma_o$$

$$= n_{Hy} - n_{Ti} .$$

The corresponding age T is 6×10^{10} years which is in reasonable agreement with our earlier heuristic calculation. The evolution of the parameters as a function of c(t) can be found using the same procedure as outlined in (3.4). Figure 4.1.1 is a plot of the average eccentricity ($\equiv 1/2(e_{max} - e_{min})$) and the amplitude of libration as a function of the parameter $(1 - 2c(t)/\beta)$.

The reasonableness of these calculations is naturally conditioned by the initial assumptions. Clearly the most crucial of these is the dependence of the tides on a constant Q which is the same for all satellites. If Mimas could have risen from the seas of Saturn, Enceladus and Titan would have been nearly motionless spectators to the event. Only a 6% and a 1/4% change can occur, respectively, in the orbits of Enceladus and Titan. Thus "significant" tidal evolution is limited to the closest satellites.

Perhaps one way out is to say that Q has an amplitude dependence (let's guess that it's proportional to the height of the tides raised on Saturn's surface). Because of its greater mass, we might expect that the Q for Titan is significantly larger than it is for other satellites. The amplitude of the tide is roughly proportional to the gradient of the force at Saturn's surface. Explicitly:

$$\text{Tide Height} \propto \mu \left(\frac{R}{a}\right)^3 . \tag{4.1.42}$$

Comparing the tide height raised by Titan and by the next strongest case, Tethys, we find

$$\frac{\text{Tide Height by Titan}}{\text{Tide Height by Tethys}} \sim 10 \tag{4.1.43}$$

from which we might conclude that it is at least possible that Q_{Titan} is significantly larger and that the age estimate for this resonance is within reasonable bounds. But if Titan's tidal torque is greater, then why not Tethys'? The tide height of Tethys is about four times greater than that for Mimas. The problem is that the two tidal torques nearly cancel in the commensurability relation

$$\frac{dn_T}{dt_{Mi}} \quad \gtrsim \frac{dn_T}{dt_{Te}} \approx \frac{3}{20} \frac{dn_T}{dt_{Mi}} , \tag{4.1.44}$$

with Mimas just barely winning the battle. If $\frac{dn_T}{dt_{Te}}$ were just a few percent larger, the resonance variable becomes tidally unstable!

We should mention that the probability for capture is increased slightly ($\lesssim$ 1%) by the x dependent term in $\frac{dc}{dt}(x,t)$. But it also happens that the ages and the capture probabilities are significantly affected by the assumption that both tidal torques

obey (2.9.4), independent of the value of Q.

In the Mimas-Tethys case the values of the orbital parameters was significantly affected by the fact that the x dependent term is an important factor during evolution. The relative damping due to this term is large only because the two torques tend to cancel in the commensurability relation (41.142). If $\left|\frac{dn_T}{dt_{Mi}}\right| >> \left|\frac{dn_T}{dt_{..}}\right|$, then the x-dependent term is much less of a factor. The evolution of the resonance is then accurately described by the solution obtained for the limit $|\rho| << 1$. Of course this means that the initial inclinations were less than predicted by Allan. Going back and repeating the calculation we would find that the probability for capture for the II' resonance is increased to 7-8%, while that for the I^2 resonance is increased to about 10%. Contrary to expectations, the a[illegible] sonance is dec eased significantly by about one fifth. [illegible] tuation holds in the Enceladus-Dione case if

$$\left|\frac{dn_T}{dt_{En}}\right| >> \left|\frac{dn_T}{dt_{Di}}\right| .$$

We've already noted that the dissipative term plays a minor role during capture and evolution in the limit $|\beta| >> 1$. But the cancellation of the tidal torques does affect the commensurability relation in the original problem. However, if $\frac{dn_T}{dt_{En}} >> \frac{dn_T}{dt_{Di}}$, the age of the resonance is decreased by a factor of 0.28. Thus, the age of the Enceladus-Dione resonance may be as small as

4×10^9 years.

Although we've generated many numbers concerning the ages and evolution of the various resonances discussed, we find that good numbers are hard to find. This must be accepted as an excerise which demonstrates that Goldreich's hypothesis is probably correct but that it generates more puzzles than it solves.

4.2 THE LUNAR-PLANETARY RESONANCE HYPOTHESIS

The moon, like the satellites of Saturn and Jupiter, is spiraling away from the earth due to a tidally-induced torque. If a simple tidal model is invoked, and if the present value of the tidal acceleration is used to determine a constant dissipation function or Q-number, then several investigations have shown that the moon was within the Roche Limit less than two billion years ago (see Goldreich, 1966). As the earth-moon system appears to be much older, something else must be invoked to resolve the time-scale paradox. The most plausible solution is that complex factors influence the tidal torque and that, contrary to expectations, the energy dissipation factor may have been considerably less in the past. Pannella, MacClintock and Thompson (1968) examined the tidally induced periodicities in the daily growth structures of various types of shells of widely different ages. The implication they drew (fig. 4.1.1) is that the tidal torque was both variable and also probably less in the past, beyond approximately 70 million years ago.

R. G. Hipkin made the novel suggestion that perhaps the time-scale paradox could be resolved if the moon were trapped for an appreciable time in the past in a resonance with Venus. Goldreich had already shown that partners of a resonance, subject to tidal torques, tended to maintain their near commensurabilities through

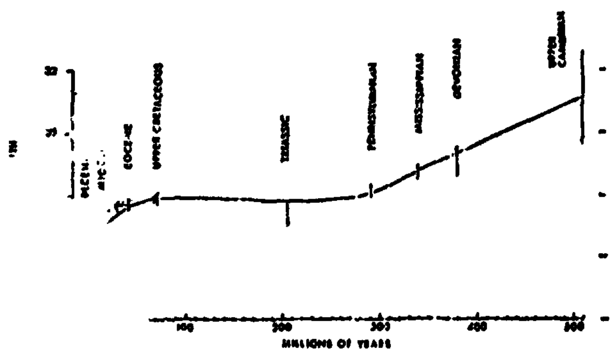


FIGURE 4.2.1

Plot of the number of days per synodic lunar month versus estimated age of each sample. This graph is taken from an article by Pannella, MacClintock and Thompson (Science, Vol. 162, pages 792-96, 1968). Hazel and Waller determined that some of the age estimates were in error, and this was acknowledged by the above authors (Science, Vol. 164, page 201, 1969). The dashed line is the result of these corrections.

a transfer of angular momentum from one partner to another. Therefore, a hypothetical lunar-Venusian resonance could transfer angular momentum from the lunar orbit into the much larger orbit of Venus. Hipkin reasoned that such a process would negligibly affect the Venusian orbit, and would effectively trap the lunar orbit at a fixed radius over the lifetime of the resonance. He is presently working on this problem, basically using the same approach as outlined here, except that his solution is apparently much more precise. It should be mentioned that in his solution (and here also) the planetary orbits are assumed to be both circular and coplanar since otherwise the expansion and evaluation of the relevant coefficients become inordinately difficult. But in calculating the effect of second and higher order coupling, Hipkin does take into account the coupling of terms in the expansion proportional to the eccentricity of one of the resonance partners before setting the planetary eccentricities to zero (Peale, private communication). We shall find later that the approximation of circular planetary orbits affects the maximum value of the resonant torque. On the other hand, the magnitude of the tidal torque in the past is not well established either. Therefore, the first step should be to calculate the relevant gravitational torque with approximations which simplify the calculation as much as possible, and compare the magnitude of the resonant torque with the present value for the tidal torque. If the tidal torque is much greater than a given resonant torque, then there is no need to further refine the calculation. The implication is that the tidal evolution of the lunar orbit could not be arrested by the given resonant interaction.

Venus is chosen as the most likely partner in any resonance because it induces large purturbations in the mean motion of the moon due to its relative nearness to the earth. Although Jupiter is much more massive than Venus, a counterbalancing factor of $\left(\frac{a_<}{a_>}\right)^c$ enters in the development of the disturbing function, where the integer c is the ratio of the synodic mean motion of the moon to the synodic mean motion of the disturbing planet. The ratio c is ~ 14 for Jupiter and ~ 20 for Venus. A trivial calculation shows that $\frac{m_{♃}}{m_{\odot}}(.2)^{14} \ll \frac{m_{♀}}{m_{\odot}}(.72)^{20}$. A plausible partner, not considered by Hipkin, is Mercury, which has the smallest ratio c of any planet (~ 4). Hipkins' original argument applied only to *resonances* of the synodic type, but as we have seen, the simple e-type should also be considered, because of capture considerations if for no other reason.

The method followed in determining the first and second order contributions of the disturbing functions acting on the moon is the procedure outlined in sections 2.2 and 2.5. Naturally, the first step is the expansion of the relevant disturbing functions. They are:

Disturbing Function Acting on the Moon by a Planet.

$$R(p \to ☽) = \mu_p\left(\frac{1}{\Delta} - \frac{\vec{r}_☽ \cdot \vec{D}}{D^3}\right); \quad \Delta = \left|\vec{r}_☽ + \vec{D}\right|; \quad \vec{D} = \vec{r}_☽ - \vec{r}_p. \qquad (4.2.1a)$$

Disturbing Function Acting on the Earth by a Planet.

$$R(p \to \oplus) = \mu_p\left(\frac{1}{D} - \frac{\vec{r}_☽ \cdot \vec{r}_p}{r_p^3}\right). \qquad b)$$

Disturbing Function Acting on the Moon by the Sun.

$$R(\odot \to ☽) = \mu_{\odot}\left(\frac{1}{\Delta_1} - \frac{\vec{r}_☽ \cdot \vec{r}_{\odot}}{r_{\odot}^3}\right); \quad \Delta = \left|\vec{r}_☽ + \vec{r}_{\odot}\right|. \qquad c)$$

Figure 4.2.2 shows all the relevant radius vectors and angles. To simplify the expansion, we shall make the approximations that all orbits are coplanar and, except for the lunar orbit, are circular. In addition, the motion of the earth about the barycenter shall be neglected in each of the disturbing function. This motion can be included in the expansion by expanding the relevant vectors about the barycenter of the earth-moon system (Plummer, 1960), but the expected error introduced by neglecting it is of $O\left(\frac{m_☽}{m_{\oplus}} \approx \frac{1}{81}\right)$ and is small compared to other approximations to be invoked later. Finally, the interaction of the moon on the planetary partner will be ignored because of the planet's relatively larger mass and angular momentum. Since the interaction between the planet and moon will tend to conserve angular momentum, any change in the lunar angular momentum produced by the planet will be balanced by an equal and opposite change in the planet's momentum by the moon. Therefore the ratio of the fractional change is:

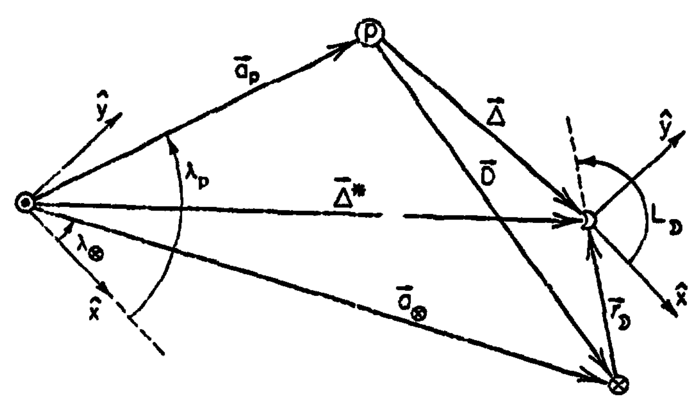


FIGURE 4.2.2

Vector diagram of the planet, earth and lunar positions with respect to the sun.

$$\frac{\Delta L_p}{L_{D}}(p\to D) : \frac{\Delta L_p}{L_p}(D\to p) = -\frac{L_p}{L_{D}} \sim -\frac{m_p a_p^2 n_p}{m_{D} a_{D}^2 n_{D}}.$$

The above ratio indicates that the perturbation of the planet's orbital elements by the moon is quite negligible. Therefore, we only need determine the first and second order contributions of the previously mentioned disturbing functions. Starting with the lunar-solar disturbing function, $R(\odot\to D)$, let's first expand the direct part Δ^{-1} (2.3.1).

$$\Delta^{-1} = \sum_{\ell=0}^{\infty}{}' \left(\frac{r_{D}}{a_\oplus}\right)^{\ell} P_\ell(\hat{r}_{D}\cdot\hat{r}_\oplus).$$

The indirect part of $R(\odot\to D)$ will exactly cancel the $\ell = 1$ term in Δ^{-1}, while the $\ell = 0$ term does not contain any of the lunar elements and can be dropped. The result is (2.2.1)

$$R(\odot\to D) = \mu_\odot \sum_{\ell=2}^{\infty} \frac{(-)^{\ell}}{a_\oplus}\left(\frac{r_{D}}{a_\oplus}\right)^{\ell} P_\ell(\hat{r}_{D}\cdot\hat{r}_\oplus).$$

The factor of $(-)^{\ell}$ is introduced because Δ is the <u>sum</u> rather than the difference of $\vec{r}_{D}$ and $\vec{r}_\oplus$. The Legendre function can first be expanded in terms of the spherical harmonic functions, and the spherical harmonics in terms of the inclination function $F_{\ell,m,p}(I)$ (2.2.5). The result of this pair of expansions, under the assumption of co-planar orbits, is:

$$R(\odot\to D) = \mu_\odot \sum_{\ell=2}^{\infty}{}' \sum_{m=-\ell}^{+\ell} (-)^{\ell} \frac{(\ell-m)!}{(\ell+m)!}\, \frac{1}{a_\oplus}\left(\frac{a_{D}}{a_\oplus}\right)^{\ell} \left|F_{\ell,m,\frac{\ell-m}{2}}(0)\right|^2 \left(\frac{r_{D}}{a_{D}}\right)^{\ell} e^{im(L_{D}-\lambda_\oplus)},$$

$$L_{D} = f_{D} + \omega_{D}.$$

The final expansion is of the function $(\frac{r_D}{a_D})^l e^{imf}$ in terms of the mean anomaly M (2.2.8). The complete expansion of $R(\odot \to D)$ is:

$$R(\odot \to D) = \mu_\odot \sum_{l,m} (-)^l \frac{(l-m)!}{(l+m)!} \frac{1}{a_\odot} \left(\frac{a_D}{a_\odot}\right)^l \left| F_{l,m,\frac{l-m}{2}}(0) \right|^2 \sum_{q=-\infty}^{+\infty} X^q_{l,m}(e_D) \tag{4.2.2}$$

$$\cdot e^{im(\lambda_D - \lambda_\odot) + iqM_D}.$$

The expansion of $R(p \to \odot)$ is even simpler, except that D^{-1} shall be expanded in terms of Laplace coefficients (2.2.15):

$$R(p \to \odot) = \mu_p \sum_{j=-\infty}^{+\infty} \left\{ \frac{1}{a_>} b^j_{\frac{1}{2}}(\alpha) - \frac{\delta_{j,1}}{a_p} \left(\frac{a_\odot}{a_p}\right) \right\} \frac{e^{i(j(\lambda_p - \lambda_\odot))}}{2}, \tag{4.2.3}$$

where

$$\alpha = \frac{a_<}{a_>}.$$

Note that all terms in the above expansion, except j = 0 are of short period.

The expansion of $R(p \to D)$ involves the same procedure, with one extra step. The result of the sequence of expansions just outlined is:

$$R(p \to D) = \mu_p \sum_{l=2}^{\infty} \left(\frac{-}{D}\right)^l \left(\frac{r_D}{D}\right)^l P_l(\hat{r}_D \cdot \hat{D})$$

$$= \mu_p \sum_{l,m} \frac{(l-m)!}{(l+m)!} \frac{1}{D} \left(\frac{r_D}{D}\right)^l \left| F_{l,m,\frac{l-m}{2}}(0) \right|^2 e^{i(m(L_D - \phi_D))}$$

$$= \mu_p \sum_{l,m,q} \frac{(l-m)!}{(l+m)!} \left| F_{l,m,\frac{l-m}{2}}(0) \right|^2 X^q_{l,m}(e_D) \frac{1}{D} \left(\frac{a_D}{D}\right)^l e^{i(m(\lambda_D - \phi_D) + qM_D)}.$$

The angle ϕ_D is just the angle made by the vector D with respect to the reference frame vector in the orbital plane, $\hat{x}$ (fig. 4.2.2). This angle is related to λ_p and $\lambda_\odot$ through the following relations:

$$\cos\phi_D = \hat{D} \cdot \hat{x} = a_\odot D^{-1} (\hat{a}_\odot - \frac{a_p}{a_\odot} \hat{a}_p) \cdot \hat{x}$$

$$= a_\odot D^{-1} (\cos\lambda_\odot - \frac{a_p}{a_\odot} \cos\lambda_p),$$

$$\sin\phi_D = \hat{D} \cdot \hat{y} = a_\odot D^{-1} (\sin\lambda_\odot - \frac{a_p}{a_\odot} \sin\lambda_p),$$

from which can be derived, after a binomial expansion:

$$e^{-im\phi_D} = (a_\odot D^{-1})^{|m|} \sum_{s=0}^{m} (-)^s \left(\frac{a_p}{a_\odot}\right)^s \binom{m}{s} e^{-is\lambda_p} e^{-i(m-s)\lambda_\odot}.$$

The final form of $R(p \to D)$, after expanding $D^{-(l+|m|+1)}$, is

$$R(p \to D) = \frac{\mu_p}{a} \sum_{\substack{m=-l \\ j=0 \\ j,q=-\infty}}^{+\infty,\,+l} (-)^l \frac{(l-m)!}{(l+m)!} \left| F_{l,m,\frac{l-m}{2}}(0) \right|^2 X^q_{l,m}(e_D) \tag{4.2.4a}$$

$$\cdot \left(\frac{a_D}{a_>}\right)^l \left(\frac{a_<}{a_>}\right)^{|m|} b^j_{\frac{l+|m|+1}{2}}(\alpha) \sum_{s=0}^{m} (-)^s \left(\frac{a_p}{a_\odot}\right)^s \binom{m}{s} \frac{1}{2} e^{i\psi_{m,q,s,j}},$$

where

$$\psi_{m,q,s,j} = m(\lambda_D - \lambda_\odot) + q(\lambda_D - \tilde{\omega}_D) + (j-s)(\lambda_p - \lambda_\odot). \tag{4.2.4b}$$

From the symmetry properties established in section ? ? the above will collapse to a cosine series. The exponential can be replaced by

$$\lambda_{e}^{i\phi_{m,q,s,j}} = K_{m,q,j-s} \cos\phi_{m,q,s,j};$$

$$K_{m,q,j-s} = \begin{cases} H(m), & \text{if } m \neq 0, \\ H(q), & \text{if } m = 0,\ q \neq 0, \\ H(j-s), & \text{if } m = q = 0,\ j-s \neq 0, \\ 2, & \text{if } m = q = j-s = 0; \end{cases} \tag{4.2.5}$$

$$H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0 . \end{cases}$$

A given angle in the series has fixed m,q and (j - s). Those terms which have the same ϕ are made more obvious if a new summation variable, p, is introduced,

$$p = j - s,$$

and the variable s is eliminated. Since the resonance partners of greatest interest are inferior to the earth, choose $a_{\flat} = a_{\oplus}$. The result of these operations is:

$$R(p \to \flat) = \frac{\mu_p}{a_\oplus} \sum_{\substack{m=0\\ \ell=0\\ p=-\infty}}^{+\infty} (-)^{\ell} K_{m,q,p} \frac{(\ell-m)!}{(\ell+m)!} \left|F_{\ell,m,\frac{\ell-m}{2}}(0)\right|^2 X^q_{\ell m}(e)$$

$$\cdot \left(\frac{a_\flat}{a_\oplus}\right)^{\ell} \sum_{j=p}^{p+m} (-)^{(p-j)} \alpha^{p-j} \binom{|m|}{p-j} b^{j}_{\frac{\ell+m+1}{2}}(\alpha) \cos\phi_{m,q,p}. \tag{4.2.6}$$

The ratio $\left(\frac{a_\flat}{a_\oplus}\right) \cdot \frac{1}{400}$, implies that the $\ell = 2$ term should give the dominant contribution. The inclination function $F_{\ell,m,\ell-m}(0)$ (2.2.5)

is

$$F_{\ell,m,\frac{\ell-m}{2}}(0) = \frac{(\ell+m)!}{2^{\ell}\left(\frac{\ell-m}{2}\right)!\left(\frac{\ell+m}{2}\right)!} \cos\left(\pi \frac{(\ell-m)}{2}\right), \tag{4.2.7}$$

which vanishes for (ℓ - m) odd. Given $\ell = 2$, $|m|$ is restricted to the values (0,2). For a synodic type resonance (q = 0), the $\ell = 2$ term in the above expansion corresponds to twice the commensurability ratio, or a <u>double angle</u> resonance. The disturbing function contains a factor of $\left(\frac{a_p}{a_\oplus}\right)^p$ for a given $\phi_{m,q,p}$ and would be raised to the eighth(p-8) and fortieth(p-40) powers for Mercury and Venus, respectively. The lowest order contribution to the single angle resonance is contained in the $\ell = 3$ term of the expansion. A rough comparison of the coefficients of the single and double angle terms is given by

$$\frac{(\text{double angle})}{(\text{single angle})} \sim \left(\frac{a_\oplus}{a_\flat}\right)\left(\frac{a_p}{a_\oplus}\right)^{c}.$$

Calculation of the above ratio indicates that it is within an order of magnitude of unity for both planets. Therefore, both terms shall be explicitly calculated for the synodic case. For the strongest possible e-type resonance, q = 1. As |m| = 0,2 for the $\ell = 2$ term, the lowest order contribution for the e-type corresponds to the single angle case.

The lowest order contribution to a given angle can be written down directly, using the explicit, lowest order expansion of the eccentricity function $X^{q}_{\ell,m}(e)$ (Table 4.2.1) and the equation $F_{\ell,m,\frac{\ell-m}{2}}(0)$ found from (2.2.7b). The results of these operations for the following four cases are:

Double Angle Synodic Type, $\phi_{2,0,p}$:

(4.2.8a)

$$(R(p\to\text{☽}))_{2,0,p} \cong \frac{3}{8}\frac{\mu_p}{a_\odot}\left(\frac{a_\text{☽}}{a_\odot}\right)^2\left\{b^{p}_{5/2}(\alpha)-2\alpha b^{p+1}_{5/2}(\alpha)+\alpha^2 b^{p+2}_{5/2}(\alpha)\right\}\cos\phi_{2,0,p};$$

Single Angle Synodic Type, $\phi_{1,0,p}$: b)

$$(R(p\to\text{☽}))_{1,0,p} \cong \frac{-3}{16}\frac{\mu_p}{a_\odot}\left(\frac{a_\text{☽}}{a_\odot}\right)^3\left\{b^{p}_{5/2}(\alpha)-\alpha b^{p+1}_{5/2}(\alpha)\right\}\cos\phi_{1,0,p};$$

e-type, $\phi_{2,-1,p}$: c)

$$(R(p\to\text{☽}))_{2,-1,p} \cong \frac{-9}{8}\frac{\mu_p}{a_\odot}\, e_\text{☽}\left(\frac{a_\text{☽}}{a_\odot}\right)^2\left\{b^{p}_{5/2}(\alpha)-2\alpha b^{p+1}_{5/2}(\alpha)+\alpha^2 b^{p+2}_{5/2}(\alpha)\right\}$$

$$\cdot\cos\phi_{2,-1,p};$$

e-type, $\phi_{0,1,p}$: d)

$$(R(p\to\text{☽}))_{0,1,p} \cong -\frac{1}{4}\frac{\mu_p}{a_\odot}\, e_\text{☽}\left(\frac{a_\text{☽}}{a_\odot}\right)^2 b^{p}_{3/2}(\alpha)\cos\phi_{0,1,p}.$$

The Laplace coefficients can be evaluated using the equivalent polynomial expansion (2.2.15). Before calculating the above, let's determine the important second order contributions.

The indirect influence of the sun on a given resonance cannot be neglected because of its relatively great mass. Unfortunately, the effect of the sun on the lunar orbit is so large that an ordinary perturbation expansion of the disturbing function in powers of the

ℓ	m	q	$X^{q}_{\ell,m}(e)$
2	0	1	$-e$
2	2	-1	$-\frac{1}{2}e$
2	2	0	1

Definitions

$$F_{\ell,m,\frac{\ell-m}{2}}(I) = i^{(\ell-m)}\frac{(\ell-m)!}{2^{\ell}\left(\frac{\ell-m}{2}\right)!\left(\frac{\ell+m}{2}\right)!}\sum_k(-)^k\binom{2\ell-p}{k}\binom{2p}{\ell-m-k}\gamma^{2\ell-\nu}\sigma^{\nu},$$

where

$$\gamma=\cos\frac{I}{2},\qquad \sigma=\sin\frac{I}{2},\qquad \nu=m-\ell+2p+2k,$$

and k is summed over all nonnegative factorials.

$$F_{\ell,m,\frac{\ell-m}{2}}(0) = \cos\left(\frac{\pi(\ell-m)}{2}\right)\frac{(\ell+m)!}{2^{\ell}\left(\frac{\ell-m}{2}\right)!\left(\frac{\ell+m}{2}\right)!}.$$

$$a_>^{s}D^{-s} = \sum_{q=-\infty}^{+\infty} b^{j}_{s/2}(\alpha)\,\frac{e^{i(j(\lambda_p-\lambda_\oplus))}}{2},\qquad \alpha=\frac{a_<}{a_>}.$$

TABLE 4.2.1

Table of relevant Hansen's coefficients, $X^{q}_{\ell,m}$. Also included are the explicit definitions of the inclination function $F_{\ell,m,p}(I)$.

mass, or equivalently, in powers of $\frac{n_\odot}{n_{☽}}$, converges slowly (Brown and Shook, 1964, Lunar Theory). Since our object is only to determine the plausibility of Hipkin's hypothesis, a calculation of the lowest order coupling will be sufficient. We found in section 2.5 that the second order contribution to a given term in $\hat{R}_c$ (2.6.2) occurred in δR_s, the coupling of short periodic terms in R_s, and δH_o, the coupling of short periodic terms in the unperturbed Hamiltonian H_o. The function δR_s, in this case, is composed of $\delta R_s(p \to ☽)$ and $\delta R_s(\odot \to ☽)$. The perturbation in δH_o only involves the perturbation $\delta L_{☽}$, which here is composed of $\delta L_{☽}(\odot \to ☽)$ and $\delta L_{☽}(p \to ☽)$. Formally, each of these terms is (2.5.14):

$$\delta H_o = \frac{1}{2}\frac{\partial^2 H_o}{\partial L^2}\left(\delta^2 L(p \to ☽) + 2\delta L(p \to ☽)\delta L(\odot \to ☽) + \delta^2 L(\odot \to ☽)\right). \tag{4.2.9}$$

The only significant second order contribution to the lunar-planetary resonance is in the cross term, $\delta L(p \to ☽)\delta L(\odot \to ☽)$. The significant second order contributions from coupling of the short period terms are the following:

$$\delta R_{s'}(p \to ☽) = \sum_{w_{☽}}\left\{\frac{\partial R_{s'}(p \to ☽)}{\partial w_{☽}}\delta w_{☽}(\odot \to ☽) + O(\mu_p \mu_\odot)\right\}; \tag{4.2.10a}$$

$$\delta R_{s''}(\odot \to ☽) = \sum_{w_{☽}, w_\odot, J_\odot}\left\{\frac{\partial R_{s''}(\odot \to ☽)}{\partial w_{☽}}\delta w_{☽}(p, ☽) + \frac{\partial R_{s''}}{\partial w_\odot}\delta J_\odot(p, \odot)\right\}. \tag{b}$$

Here (w, J) represents the complete set of conjugate action and angle variables. The functions $\delta L_{☽}$, $\delta w_{☽}$, etc., are the short periodic first order perturbations of the canonical variables and are obtained from the appropriate generating function through the relations (2.5.9):

$$\delta w = \frac{\partial S}{\partial J}(J, \bar{w}); \quad \delta J = -\frac{\partial S}{\partial \bar{w}}(J, \bar{w}). \tag{4.2.11}$$

The generating function $S(J, \bar{w})$ is related to the appropriate short periodic part (R_s) of each individual disturbing function (2.5.2) acting on the moon. If each R_s is formally expressed as a transcendental series in ϕ (2.5.4a),

$$R_s = \mu \sum_s A_s e^{i\phi_s}, \text{ etc.},$$

then the generating functions are (2.5.17):

$$S(p \to ☽) = -\mu_p \sum_{s'} \frac{A_{s'}(p \to ☽)}{i\nu_{s'}(p \to ☽)} \cdot e^{i\phi_{s'}(p \to ☽)}, \quad \nu_s = \frac{\partial \phi_s}{\partial t}; \tag{4.2.12a}$$

$$S(\odot \to ☽) = -\mu_\odot \sum_{s''} \frac{A_{s''}(\odot \to ☽)}{i\nu_{s''}} \cdot e^{i\phi_{s''}}; \tag{b}$$

$$S(p \to \odot) = -\mu_p \sum_{s'''} \frac{A_{s'''}}{i\nu_{s'''}} \cdot e^{i\phi_{s'''}}. \tag{c}$$

This allows us to write down δH_o and δR_s in terms of the above formal expansions for S and R_s, and determine which contributions are significant:

$$\delta R_s(p\to\mathbb{D}) = \mu_{\mathbb{D}}\mu_{\odot} \sum_{\substack{s',s''\\ w_{\mathbb{D}}}} -\left(\frac{\partial\phi_{s'}}{\partial w_{\mathbb{D}}}\right) A_{s'} \frac{\partial}{\partial J_{\mathbb{D}}}\left(\frac{A_{s''}}{\nu_{s''}}\right) e^{i(\phi_{s'}+\phi_{s''})}; \quad (4.2.13a)$$

$$\delta R_s(\odot\to\mathbb{D}) = \mu_{\mathbb{D}}\mu_{\odot} \sum_{\substack{s',s'',s'''\\ w_{\mathbb{D}}, w_{\oplus}, J_{\oplus}}} \left\{-\left(\frac{\partial}{\partial w_{\mathbb{D}}}\phi_{s''}\right) A_{s'} \frac{\partial}{\partial J_{\mathbb{D}}}\left(\frac{A_{s''}}{\nu_{s''}}\right) e^{i(\phi_{s'}+\phi_{s''})}\right\} \quad b)$$

$$+\left\{-\frac{\partial\phi_{s''}}{\partial w_{\oplus}} A_{s''} \frac{\partial}{\partial J_{\oplus}}\left(\frac{A_{s'''}}{\nu'''}\right) + \frac{1}{\nu_{s''}}\frac{\partial\phi_{s'''}}{\partial w_{\oplus}} A_{s'''} \frac{\partial A_{s''}}{\partial J_{\oplus}}\right\} e^{i(\phi_{s''}+\phi_{s'''})};$$

$$\delta H_o \cong \frac{3\mu_{\mathbb{D}}\mu_{\odot}}{2a_{\mathbb{D}}^2} \sum_{s',s''} \left\{\frac{\partial\phi_{s'}(p\to\mathbb{D})}{\partial\lambda_{\mathbb{D}}}\frac{\partial\phi_{s''}(\odot\to\mathbb{D})}{\partial\lambda_{\mathbb{D}}} \frac{A_{s'}}{i\cdot\nu_{s'}}\frac{A_{s''}}{i\cdot\nu_{s''}} e^{i(\phi_{s'}+\phi_{s''})}\right\}$$

where c)

$$\frac{\partial^2 H_o}{\partial L^2} = 3a_{\mathbb{D}}^{-2}.$$

Recall that ν is a function of L (cf. 2.2.5). Therefore the partial derivative with respect to L explicitly acts on the frequency ν in the above equations.

There is the obvious requirement that each of the above angles $\phi_{s'}$, $\phi_{s''}$, or $\phi_{s'''}$ has nonzero frequency, which is derived from the short periodic nature of $\bar{R}_s$. Because of the assumption of coplanar, circular (except $\mathbb{D}$) orbits, the only action variables which enter in the above are $L_{\mathbb{D}}$, $\Gamma_{\mathbb{D}}$, $L_{\oplus}$ and L_p. In the case of synodic resonance, the set can be restricted still further with the assumption that the lunar orbit is circular. The explicit angles which occur in the above are (4.2.2,3)

$$\phi_{s'}+\phi_{s''} = (m'+m'')(\lambda_{\mathbb{D}}-\lambda_{\oplus}) + (q'+q'')M_{\mathbb{D}} + (j'-s')(\lambda_p-\lambda_{\oplus});$$

$$\phi_{s''}+\phi_{s'''} = m''(\lambda_{\mathbb{D}}-\lambda_{\oplus}) + q''M_{\mathbb{D}} + j'''(\lambda_p-\lambda_{\oplus}). \quad (4.2.14a)$$

Given an angle $\phi_{m,q,p}$, the second order coupling corresponding to $\phi_{s''} + \phi_{s'''}$ will be restricted to a single term. That is, the indices m", q", and J'" are fixed. The second order coupling corresponding to $\phi_{s'} + \phi_{s''}$ will involve several terms, which can be restricted to the terms of lowest order by minimizing the number of factors of $\left(\frac{a_>}{a_{\oplus}}\right)$ and $e_{\mathbb{D}}$ which occur in the expansion, consistent with the given angle. The ratio of the relative magnitude of second order coupling of perturbations of the lunar orbit with that of the earth's orbit in δR_s is roughly

$$\frac{(\text{second order}:\mathbb{D})}{(\text{second order}:\oplus)} \sim \frac{L_{\oplus}}{L_{\mathbb{D}}}\left(\frac{a_{\mathbb{D}}}{a_{\oplus}}\right)^2 = \frac{n_{\oplus}}{n_{\mathbb{D}}} \sim \frac{1}{13}, \quad (4.2.15)$$

Assuming that the sum of Laplace coefficients is approximately the same. The same ratio tends to hold for contributions from δH_o. Given that the important contributions involve frequencies of $O(n_{\mathbb{D}})$, the approximate ratio is

$$\frac{(\delta H)}{(\text{second order}:\oplus)} \sim \frac{L_{\oplus}}{a_{\mathbb{D}}^2 n_{\mathbb{D}}}\left(\frac{a_{\mathbb{D}}^2}{a_{\oplus}}\right)^2 \approx \frac{n_{\oplus}}{n_{\mathbb{D}}}. \quad (4.2.16)$$

Thus it appears that the perturbations in the earth's position give the principal second order contribution to a given angle in $\bar{R}_c(p\to\mathbb{D})$, which suggests that the second order contributions be restricted to just these terms. Yet another reason is that only the coupling due to the first order terms of the solar-lunar disturbing function have been considered. The second order term of the solar-lunar disturbing function coupled with the first order perturbations of the earth will produce a contribution of $0((\frac{n_\oplus}{n_\leftmoon})^2)$. But this is the same order as those terms involving first order coupling of the lunar coordinates. If these approximations are accepted, then the principal contribution to a given angle $\phi_{m,q,p}$ is

$$(\delta R_s)_{m,q,p} \cong \mu_p \mu_\oplus \Big\{ -\frac{\partial \phi_{m,q}}{\partial \lambda_\oplus}(\oplus \to \leftmoon) A_{m,q}(\oplus \to \leftmoon) \cdot \frac{\partial}{\partial L_\oplus}\Big(\frac{A_p(p,\oplus)}{\nu_p(p,\oplus)}\Big) + \frac{\partial A_{m,q}}{\partial L_\oplus} \frac{\frac{\partial \phi_p}{\partial \lambda_\oplus}}{\nu_p} A_p(p,\oplus)\Big\} \tfrac{1}{2} e^{i\phi_{m,q,p}} \qquad (4.2.17a)$$

Substituting the explicit expressions for $A_{m,q}$, $\phi_{m,q}$, A_p, ϕ_p, we get

$$(\delta R_s)_{m,q,p} \cong \mu_p \mu_\oplus (-)^{\ell} \frac{(\ell-m)!}{(\ell+m)!} |F_{\ell,m,\frac{\ell-m}{2}}(0)|^2 \chi^q_{\ell,m}(e_\leftmoon) a^\ell_\leftmoon \frac{m}{p} \cdot a_\oplus^{-(\ell+1)} \frac{\partial}{\partial L_\oplus}\Big(\frac{a_\oplus^{-1} b^p_{\frac{1}{2}}(\alpha)}{n_p - n_\oplus}\Big) - (n_p - n_\oplus)^{-1} a_\oplus^{-1} b^p_{\frac{1}{2}}(\alpha) \cdot \frac{\partial}{\partial L_\oplus} a_\oplus^{-(\ell+1)} \tfrac{1}{2} e^{i\phi_{m,q,p}} \qquad \text{b)}$$

The partial with respect to $L_\oplus$ can be replaced by any of the following:

$$\frac{\partial}{\partial L_\oplus} = \frac{2}{n_\oplus a_\oplus}\frac{\partial}{\partial a} = -\frac{3}{a_\oplus^2}\frac{\partial}{\partial n_\oplus} = -\frac{2\alpha}{n_\oplus a_\oplus}\frac{\partial}{\partial \alpha} \qquad (4.2.18)$$

where $\alpha = \frac{a_p}{a_\oplus}$, $L_\oplus = n_\oplus a_\oplus^2$.

Next, the exponential $\{1/2e^{i\phi_{m,q,p}}\}$ can be replaced by $\cos\phi_{m,q,p}$, and finally, the explicit forms for $F_{\ell,m,\frac{\ell-m}{2}}(0)$ and $\chi^q_{\ell,m}(e\ \ )$ can be substituted into the above. Using the relation for $\mu_\oplus$,

$$\mu_\oplus \equiv GM_\oplus = n_\oplus^2 a_\oplus^3, \qquad (4.2.19)$$

and the approximation

$$p(n_p - n_\oplus) = -(m+q)n_\leftmoon + mn_\oplus \cong -(m+q)n_\leftmoon, \qquad (4.2.20)$$

the contributions from δR_s for each of the angles already considered are:

Double Angle, $\phi_{2,0,p}$:

$$(\delta R_s)_{2,0,p} \cong \frac{3}{8}\frac{\mu_p}{a_\oplus}\Big(\frac{a_\leftmoon}{a_\oplus}\Big)^2 \frac{n_\oplus}{n_\leftmoon}\Big\{\Big(\frac{3n_\oplus}{(n_p - n_\oplus)}\Big) + 2 - 3p + 2\alpha\frac{\partial}{\partial \alpha}\Big) b^p_{\frac{1}{2}}(\alpha)\Big\} \cos\phi_{2,0,p}; \qquad (4.2.21a)$$

Single Angle, $\phi_{1,0,p}$:

$$(\delta R_s)_{1,0,p} \cong \frac{-3}{16}\frac{\mu_p}{a_\oplus}\Big(\frac{a_\leftmoon}{a_\oplus}\Big)^2 \frac{n_\oplus}{n_\leftmoon}\Big\{\Big(\frac{3n_\oplus}{(n_p - n_\oplus)}\Big) + 2 - 8p + 2\alpha\frac{\partial}{\partial \alpha}\Big) b^p_{\frac{1}{2}}(\alpha)\Big\} \cos\phi_{1,0,p}; \qquad \text{b)}$$

e-type, $\phi_{2,-1,p}$:

$$(\delta R_s)_{2,-1,p} \cong \frac{-9}{16} c_\flat \frac{\mu_p}{a_\oplus} \left(\frac{a_\flat}{a_\oplus}\right)^2 \frac{n_\oplus}{n_\flat} \left\{ \left(\frac{6n_\oplus}{(n_p - n_\oplus)} + 4 - 6p + 4\alpha \frac{\partial}{\partial \alpha} \right) b_{\frac{1}{2}}^{p}(\alpha) \right\} \cos\phi_{2,-1,p}; \qquad (4.2.21c)$$

e-type, $\phi_{0,1,p}$:

$$(\delta R_s)_{0,1,p} = -\frac{1}{4} \frac{\mu_p}{\varepsilon_\oplus} c_\flat \left(\frac{a_\flat}{a_\oplus}\right)^2 \frac{n_\oplus}{n_\flat} \left\{ -6p\, b_{\frac{1}{2}}^{p}(\alpha) \right\} \cos\phi_{0,1,p}. \qquad (4.2.21d)$$

The integer p is negative, implying that each of the bracketed terms, which are sums of Laplace coefficients, is positive. Comparing the above results with the direct part determined earlier, we see that both contributions have the same sign for each angle.

The next step is to compare the lunar tidal torque with the maximum torque due to resonance interaction with the given planet. If the torque due to the action of the planet is conspicuously larger than the tidal torque, the tentative conclusion is that the resonance is "tidally stable." For this comparison, the second order equation of motion for the angle variable ϕ (2.9.17) is most useful. This equation is approximately

$$\frac{d^2\phi}{dt^2} + A_{oxx} A_1 \sin\phi + A_{oxx}^2 \frac{dc}{dt} \cong 0. \qquad (4.2.22)$$

The units of the coefficient $A_{oxx}A_1$ are time^{-2}, and the scale of this coefficient is set by the factor

$$\frac{m_p}{M_\oplus} n_\oplus^2 .$$

The factor A_{oxx} is the second order coefficient in the expansion of the zero order Hamiltonian (2.7.2d). Since the resonance has little effect on the planets involved, A_{oxx} is approximately given by:

$$A_{oxx} \cong \frac{3(m+q)^2}{a_\flat^2}. \qquad (4.2.23)$$

The tidal torque term $(A_{oxx}^2 \frac{dc}{dt})$ acting on ϕ is (2.9.10,17):

$$A_{oxx}^2 \frac{da}{dt} \cong -(m+q)\frac{dn_{T\flat}}{dt} > 0. \qquad (4.2.24)$$

The oldest determination of $\frac{dn_{T\flat}}{dt}$ is that of Spencer Jones.

$$\left(\frac{dn_{T\flat}}{dt}\right)_{S.J.} = -21'' \text{ century}^{-2} = 1.1 \times 10^{-23} \text{ rad/sec}^2 \qquad (4.2.25)$$

Recent investigations by Newton (1969), van Flandern (1970), Morrison (1971) and Oesterwinter and Cohen (1972) indicate that about twice Spencer Jones value is a more reasonable estimate.

The function $A_1\cos\phi$ is a term in the disturbing function related to the resonance, and is the sum of the contributions from the direct and indirect parts. Explicitly,

$$A_1 = A_{1I} + A_{1D} = (R_I + R_{II})_{\phi = 2\pi}. \qquad (4.2.26)$$

Another interesting question is how long ago the resonances were established. A rough estimate of the time, $T_{m,q,p}$ that $\dot{\phi}_{m,q,p}$ approached zero and then reversed sign is given by (Allan, 1969).

$$T_{m,q,p} = \frac{-3}{13} n_o(☽) \frac{dn_T(☽)}{dt_0}^{-1} [1-(\frac{a(☽)_{mqp}}{a_o(☽)})^{13/2}] \cdot \tag{4.2.27}$$

$T_{m,qp}$ is calculated using Spencer Jones value for $\frac{dn_T(☽)}{dt_0}$ (4.2.25) since it appears that the more recent determination applies only to the present value and rot what it may have been in the distant past.

Table 2 contains the relevant parameters needed to calculate $A_{oxx}A_1$. Table 3 has the numerical values of the relevant sums of Laplace coefficients. Table 4 has the numerical evaluations of A_{1D} and A_{1I}, the estimated age of the particular resonance, and an evaluation of their tidal stability.

The resonance angle $\phi_{m,q,p}$ has been constructed so that it secularly decreases in the absence of the resonance. Therefore the analysis of capture and the stability developed in chapter three (3) directly applies. At the end of section 3.4, we mention that there are various types of stability, the most important here being the "tidal" and "adiabatic" stability. Recall that adiabatic stability is governed by whether the magnitude of A_1 is increased or decreased by the long term tidal interaction such that the amplitude of libration decreases with time. For e-type resonances, the leading factor is $e(x)$, where

$$\frac{e}{e_o} = (\alpha x+1)^{\lambda}, \quad q=-k. \tag{4.2.28}$$

Planet or Satellite.	Semimajor Axis (A.U.)	Mean motion n $10^{-7}sec^{-1}$	Mass Ratio $M_p/M_\odot (10^{-6})$	Eccen-tricity	Inclination to Ecliptic
Mercury ☿	0.3871	8.396	0.165	0.206	$7^o0'$
Venus ♀	0.7233	3.230	2.44	0.0068	$3^o23'$
Earth ⊕	1.000	1.974	3.04	0.0168	-----
Moon ☽	$2.57\ 10^{-3}$	28.37	0.0376	0.0549	$5^o.8'$

$\frac{d}{dt}\tilde{\omega}_☽ = .00845 n_☽ = .113 n_\oplus$ (prograde).

$\frac{d}{dt}\Omega_☽ = -.00401 n_☽ = -0.0536 n_\oplus$ (retrograde).

TABLE 4.2.2

ORBITAL ELEMENTS FOR MOON, EARTH, VENUS AND MERCURY.

$\theta_{m,q,p}$	MERCURY coefficient	numerical value	VENUS coefficient	numerical value
2,0,p	$b_{5/2}^{8}-2\alpha b_{5/2}^{7}+\alpha^2 b_{5/2}^{6}$	0.290 -03	$b_{5/2}^{40}-2\alpha b_{5/2}^{39}+\alpha^2 b_{5/2}^{6}$	0.259 -05
	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{8}$	0.175 -02	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{40}$	0.2484-04
	$(\frac{3n_{☽}}{n_{☿}-n_{☉}}+2+24+2\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{8}$	0.977 -02	$(\frac{3n_{☽}}{n_{♀}-n_{☉}}+2+120+2\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{40}$	0.‥27 -03
1,0,p	$b_{5/2}^{4}-\alpha b_{5/2}^{3}$	0.172 00	$b_{5/2}^{20}-\alpha b_{5/2}^{19}$	0.107 00
	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{4}$	0.549 -01	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{20}$	0.116 -01
	$(\frac{3n_{☽}}{n_{☿}-n_{☉}}+2+32+2\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{4}$	0.572 00	$(\frac{3n_{☽}}{n_{♀}-n_{☉}}+2+160+2\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{20}$	0.116 00
1,0,p	$b_{5/2}^{5}-\alpha b_{5/2}^{4}$	0.738 00	$b_{5/2}^{21}-\alpha b_{5/2}^{20}$	0.788 -01
	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{5}$	0.24 -01	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{21}$	0.859 -02
	$(\frac{3n_{☽}}{n_{☿}-n_{☉}}+2+40+2\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{5}$	0.246 00	$(\frac{3n_{☽}}{n_{♀}-n_{☉}}+2+168+2\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{21}$	0.854 -01
2,-1,p	$b_{5/2}^{4}-2\alpha b_{5/2}^{3}+\alpha^2 b_{5/2}^{2}$	0.176 -01	$b_{5/2}^{19}-2\alpha b_{5/2}^{18}+\alpha^2 b_{5/2}^{17}$	0.323 -02
	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{4}$	0.549 -01	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{19}$	0.157 -01
	$(\frac{6n_{☽}}{n_{☿}-n_{☉}}+4+24+4\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{4}$	0.615 00	$(\frac{6n_{☽}}{n_{♀}-n_{☉}}+4+114+4\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{19}$	0.162 00
2,-1,p	$b_{5/2}^{5}-2\alpha b_{5/2}^{4}+\alpha^2 b_{5/2}^{3}$	0.620 -02	$b_{5/2}^{20}-2\alpha b_{5/2}^{19}+\alpha^2 b_{5/2}^{18}$	0.231 -02
	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{5}$	0.238 -01	$\alpha\frac{\partial}{\partial\alpha}b_{1/2}^{20}$	0.116 -01
	$(\frac{6n_{☽}}{n_{☿}-n_{☉}}+4+30+4\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{5}$	0.261 00	$(\frac{6n_{☽}}{n_{♀}-n_{☉}}+4+120+4\alpha\frac{\partial}{\partial\alpha})b_{1/2}^{20}$	0.120 00
0,1,p	$b_{1/2}^{4}$	0.145 00	$b_{1/2}^{20}$	0.4983-01
	$+24b_{1/2}^{4}$	0.318 00	$+120b_{1/2}^{20}$	0.662 -01
0,1,p	$b_{1/2}^{5}$	0.6133-01	$b_{1/2}^{21}$	0.3681-01
	$+30b_{1/2}^{5}$	0.1384 00	$+126b_{1/2}^{21}$	0.492 -01

TABLE 4.2.3

Numerical values of sums of Laplace coefficients occurring in the expansion.

Disturbing Planet	Resonance Angle	$\frac{1}{m+q}b_D(x_0)$ (10^{-23} sec^{-2})	$\frac{1}{m+q}b_I(x_0)$ (10^{-23} sec^{-2})	Total $\frac{1}{m+q}b(x_0)$ (10^{-23} sec^{-2})	$T_{m,q,p}$ 10^6 yrs B.C.	$\frac{n_{m,q,p}}{n_{☉}}$	$\frac{a_{m,q,p}}{R_0}$ (radius of 0)	Adiabatic/Tidal Stability
☿ MERCURY	$2(\lambda_{☽}-\lambda_{☉})-8(\lambda_{☿}-\lambda_{☉})$	0.427	1.056	+ 1.48	131	13.61	59.56	Yes/Yes
	$(\lambda_{☽}-\lambda_{☉})-4(\lambda_{☿}-\lambda_{☉})$	-0.164	-0.0399	-0.204	131	13.61	59.56	Yes/No
	$2(\lambda_{☽}-\lambda_{☉})-(\lambda_{☽}-\tilde{\omega}_{☽})-4(\lambda_{☿}-\lambda_{☉})$	-1.07	-2.57	-3.64	524	14.50	57.10	Yes/?
	$2(\lambda_{☽}-\lambda_{☉})-(\lambda_{☽}-\tilde{\omega}_{☽})-5(\lambda_{☿}-\lambda_{☉})$	-0.376	-0.897	-1.27	1,240	17.65	50.88	Yes/No
	$(\lambda_{☽}-\tilde{\omega}_{☽})-4(\lambda_{☿}-\lambda_{☉})$	-3.91	-0.674	-4.58	427 AD	12.72	62.31	No/Yes
	$(\lambda_{☽}-\tilde{\omega}_{☽})-5(\lambda_{☿}-\lambda_{☉})$	-1.65	-0.235	1.89	868	15.87	53.76	No/No
♀ VENUS	$2(\lambda_{☽}-\lambda_{☉})-40(\lambda_{♀}-\lambda_{☉})$	0.056	0.204	+0.260	78.0	13.51	59.85	Yes/No
	$(\lambda_{☽}-\lambda_{☉})-20(\lambda_{♀}-\lambda_{☉})$	-1.500	-0.120	-1.62	78.0	13.57	59.85	Yes/?
	$(\lambda_{☽}-\lambda_{☉})-21(\lambda_{♀}-\lambda_{☉})$	-1.10	-0.084	-1.18	381	14.14	58.06	Yes/?
	$2(\lambda_{☽}-\lambda_{☉})-(\lambda_{☽}-\tilde{\omega}_{☽})-19(\lambda_{♀}-\lambda_{☉})$	-2.30	-10.6	-14.5	212	13.77	59.10	Yes/Yes
	$2(\lambda_{☽}-\lambda_{☉})-(\lambda_{☽}-\tilde{\omega}_{☽})-20(\lambda_{♀}-\lambda_{☉})$	-2.07	-7.48	-9.55	487	14.40	57.36	Yes/Yes
	$(\lambda_{☽}-\tilde{\omega}_{☽})-20(\lambda_{♀}-\lambda_{☉})$	-19.8	-2.09	-21.9	503 AD	12.62	62.63	No/Yes
	$(\lambda_{☽}-\tilde{\omega}_{☽})-21(\lambda_{♀}-\lambda_{☉})$	-14.7	-1.48	-15.2	704 AD	13.25	60.63	No/Yes
	$(\lambda_{☽}-\tilde{\omega}_{☽})-22(\lambda_{♀}-\lambda_{☉})$	-10.9	-1.05	-12.0	260	13.87	58.81	No/Yes

TABLE 4.2.4

Numerical values of the maximum torque $(m+q)^{-1}A_{0xx}A_1$ on the moon due to the given resonance. These values are to be compared with the present estimates of $\frac{dn(☽)}{dt}T$ which range from $1 \to 2 \times 10^{-23}$ rad/sec^2, the higher value being the more recent determination.

For small librations $\langle x(t) \rangle \approx \frac{dc}{dt}(t - t_o)$. Since $\frac{dc}{dt}$ is positive (4.2.24), e increases or decreases, depending on whether q is negative or positive, respectively. Thus $\phi_{2,-1,p}$ is adiabatically stable while $\phi_{0,1,p}$ is not. The "tidal stability" of a given resonance is determined by whether the tidal torque is greater (unstable) or less than (stable) the maximum amplitude of the resonance torque.

We should observe that the adiabatic stability of the synodic resonances is not governed as much by whether b(x) increases or decreases, since this is a relatively small effect, but by the asymmetry of the tidal torque. This asymmetry arises from the fact that the torque is a rapidly decreasing function of the planet-satellite distance. To lowest order, this asymmetry will add a $\dot{\phi}$ term to the right hand side of the second order equation of motion for ϕ (2.9.17). This term is

$$-\rho A_{oxx} \frac{d\phi}{dt}, \quad \rho A_{oxx}^2 \sim \frac{16 j^2}{n_{☽} a_{☽}^2} \frac{dn_{T☽}}{dt}. \tag{4.2.29}$$

As the coefficient ρ is negative, this asymmetry implies a dissipative mechanism which tends to damp out oscillations. The results in Table 4.2.4 indicate marginal tidal stability for some of the resonances. This qualitatively agrees with Hipkin's results. The strongest <u>adiabatically stable</u> resonance involves Venus and the resonance variable $\phi_{2,-1,19}$. Mercury also has surprisingly strong resonances.

Unfortunately, there are some serious flaws in either the approximations or the supposed effect of a given type resonance which drastically change the results so far obtained. Also, the question of the likelihood of capture into libration needs to be answered. But first, let's examine the approximations more closely.

We could discuss the effect of second and higher order terms which have been neglected. Although these terms may be sizable, it it unlikely that they will critically change the order of magnitude of the coefficient A_1. It appears that the grossest approximation is the circularity of planetary orbit although on the surface it seems to be fairly reasonable, at least for Venus and the earth. After all, the present eccentricity of the earth is ~ 0.017 while that of Venus is ~ 0.007 which are both quite small. (Mercury's eccentricity is ~.2). But in the past (Brower and Clemence, 1961b), these eccentricities have varied considerably due to the long period perturbations of one planet on another (Table 4.2.5). The moon itself is indirectly affected by such perturbations, especially by the long period fluctuation of the earth's eccentricity. These fluctuations have periods ranging from 50,000 years to approximately two million years. The libration periods of all the planetary-lunar resonances are all roughly given by

$$T_{period} \sim \frac{2\pi}{\sqrt{A_1 A_{oxx}}} \sim 2\pi 10^{+12} \text{sec.} \sim 60,000 \text{ yrs.} \tag{4.2.30}$$

So the important question is, how does the fact that the planetary orbits are eccentric <u>and</u> have long period fluctuations affect the coefficient A_1?

There are two separate effects. The first is due to the short period averaging, in which the eccentricity is treated as a "constant". To estimate how it changes A_1, a rough estimate of the Hansen coefficient relevant to either planet is required. The definition of $X^q_{l,m}(e)$ is (2.2.8):

$$(\frac{r}{a})^s \cos(tf) = \sum X^q_{s,t}(e)\cos(q+t)M. \qquad (4.2.31)$$

Since none of the resonance variables contains the perihelion of either planetary resonance partner, the relevant coefficient is $X^o_{t,f}(e)$. To evaluate $X^o_{s,t}(e)$, relations which connect r and f with M are needed. The radius r is related to a and f by

$$\frac{r}{a} = \frac{1}{1-e\cos f}, \qquad (4.2.32a)$$

while f is related to M by the equation of center (Smart, 1953), which is, to lowest order,

$$f \cong M+2e\sin M. \qquad b)$$

Therefore the function cos(tf) is given by

$$\cos(tf) = \cos(tM+2et\sin M) \qquad (4.2.33)$$

Expanding the sine and cosine of 2etsinM, using Bessel function expansions, we find (Dwight, p. 198, 1969),

$$\cos(2et\sin M) \cong J_o(2te)+2J_2(2te)\cos(2M), \qquad (4.2.34a)$$

$$\sin(2et\sin M) \cong 2J_1(2te)\sin M. \qquad b)$$

Next, expand $(\frac{r}{a})^s$ to second order in e. Also use the fact that $s \gg 1$ to simplify the coefficients. The result is

$$(\frac{r}{a})^s = 1+(\frac{se}{2})^2-se\cos f+(\frac{se}{2})^2\cos(f) \quad . \qquad (4.2.35a)$$

Next express the above in terms of M, accurate to second order in e:

$$(\frac{r}{a})^s = 1+(\frac{se}{2})^2-se(\cos(tM)-2J_1\sin(tM)\sin M \quad . \qquad b)$$

Multiplying the factors together and keeping only lowest order secular terms, we have

$$X^o_{s,t}(e) \cong J_o(2te)+(\frac{se}{2})^2 \cong 1-(te)^2+\tfrac{1}{4}(se)^2. \qquad (4.2.36)$$

In the lu[illegible]netary disturbing function, the minimum exponent of the ra[illegible] a([illegible]) is approximately $|p|$, where $|p| \gg 1$. Therefore set t [illegible] p in the above equation for $X^o_{s,t}(e)$. Thus, to lowest order, we can deduce that taking the eccentricities into account will multiply A_1 by the following factor;

$$X^o_{p,p}(e_e)X^o_{-p,p}(e_p) \sim (1-\tfrac{3}{4}(pe_e)^2)(1-\tfrac{3}{4}(pe_p)^2). \qquad (4.2.37)$$

The data in Table 5 is taken from an article by Brouwer and van Woerkom (1950), and gives the maximum variation of the eccentricity

Planet	Eccentricity		Period of ϖ (Millennia)	sin i		Period of Ω (Millennia)
	Max.	Min.		Max.	Min.	
Mercury	0.241	0.109	220	0.170	0.079	[illegible]
Venus...	.074	....	....	.049	....	....
Earth.. ...	.067	....	....	.051	....	....
Mars	.141	.004	72	.102	....	....
Jupiter. ...	.063	.027	300	.005	.004	30
Saturn	.056	.012	47	.018	.014	20
Uranus ...	.067	....	....	.019	.016	430
Neptune..	0.013	0.005	2000	0.011	0.010	1000

TABLE 4.2.5

The secular elements of the planets as determined by Brouwer and Van Woerkom (1950) using the Laplace-Lagrange approximation.

due to the interplanetary perturbations just discussed (also see Brouwer and Clemence, 1961b) using the Laplace-Laguange approximation. It appears that the fluctuations in b(x) for Venus must be of at least an order of magnitude and may actually reverse the sign of b(x). Mercury, on the other hand, fluctuates only by a factor of two or three for the single angle synodic resonance and the e-type resonance.

This implies that the potential associated with the zero-eccentricity approximation is now split among several "side-band" frequencies which differ by the motions of the perihelions of the earth and the disturbing planet. The motions of the inner planetary perihelions are of order 5×10^3 sec. of arc per century, or

$$\frac{d\tilde{\omega}}{dt}\,{}_{\text{planets}}^{\text{inner}} \sim 5 \times 10^3{}''/\text{century} \sim 10^{-11}\,\text{rad/sec.} \qquad (4.2.38)$$

Compare this with the maximum value of $\dot{\phi}$ that the resonance can withstand without being disrupted:

$$\dot{\phi}_{max} \sim A_{oxx}A_1 \sim 3 \times 10^{-12}. \qquad (4.2.39)$$

Obviously the side band freqiencies differ from the libration frequency by some small multiple and should exert strong accelerations on the resonance variable. It is possible that the moon could be trapped not at the single frequency, but among the several side band frequencies, but such a motion, ultimately, appears to be highly unstable to tidal disruption. The important point is that an eccentric planetary orbit effectively destroys the phase relationship between the planet and moon at conjunction because of the high commensurability ratio. This is especially true for Venus since $c \sim 20$.

There is yet another long period effect associated with the varying planetary eccentricity. A varying eccentricity implies a time varying motion of the planetary perihelion or a long period "secular" acceleration. This acceleration is of O(sec. of arc/century^2), and because the associated periods are up to several multiples of the libration period, they would appear over a few oscillations as a secular acceleration in the equations governing the side band frequencies. A more significant effect arises from the fact that these long period variations (primarily that of the earth) induce a corresponding effect on the lunar eccentricity and perihelion which would tend to disrupt the eccentricity-dependent resonances.

The next problem is to estimate the probability P_c that any one of these resonance variables made the transition from positive rotation into libration. Admittedly, the theory developed in chapter three applied to the case where the tidal torque is small compared to the maximum pendulum torque. Nor did we not consider how long periodic fluctuations in the coefficient $b(x)$ of the resonance term affected capture. Therefore, any numbers calculated should be indicative of the general magnitude of P_c. For an e-type resonance ($|k| = 1$, P_c is approximately given by (3.2.12)

$$P_c = \frac{2}{\pi}|\beta|^{1/2}, \qquad (4.2.40)$$

if $|\beta| \ll 1$. The parameter β (3.1.5) can be related to earlier defined functions if we identify $\mu' a_{☽}^{-1} C e^{|k|}$ as A_1 and μ_o as $n_{☽}^2 a_{☽}^3$. Explicitly,

$$\beta = \frac{4(A_{oxx} A_1)}{9 n_{☽}^2 e_{☽}^4}, \qquad (4.2.41)$$

where we have multiplied by (A_{oxx}/A_{oxx}) and have used $A_{oxx} \approx -\frac{3}{a^2}$. The strongest possible <u>adiabatically</u> <u>stable</u> resonance belongs to Venus. From Table 5 we find

$$\beta = \frac{4(A_{oxx} A_1)}{9 n_{☽}^2 e_{☽}^4}. \qquad (4.2.42)$$

Calculating β, with present values for the lunar orbital element, we find that

$$\beta_{now} \simeq -10^{-6}, \quad P_c \sim 10^{-3}. \qquad (4.2.43)$$

For an e-type resonance where $b(x)$ is proportional to e, β tends to scale like e^{-3}. Therefore, if the lunar eccentricity were ten times smaller at transition than it is at present, the probability for capture would be increased to $\sim 3\%$.

In the case of the synodic resonance, capture appears to be mainly due to the ϕ-dependent term associated with the tidal torque. An estimate for P_c can be obtained from (3.1.14,15):

$$P_c = \frac{32(m+q)}{\pi}\left|\frac{A_{max} A_L}{9(m-q)^4 a_{D}^2}\right|^{\frac{1}{2}}. \qquad (4.2.44)$$

For a double angle synodic resonance, $(m + q) = 2$, while for a single angle resonance $(m + q) = 1$. The strongest possible synodic resonance involves Venus with

$$[\Phi_{1,0,2} = \lambda_{D} - \lambda_{\oplus} - 2(\lambda_{♀} - \lambda_{\oplus})].$$

The probability for capture is $\sim 0.5 \times 10^{-5}$, and unlike the previous case, there appears to be no way to substantially increase it. The conclusion to be drawn is that capture is a highly unlikely event if only tidal forces are involved.

The long term variations in the pendulum torque associated with a given resonance may cause temporary capture. Since the variation in the coefficient $b(x)$ is a significant fraction of its mean value, we should expect that the probability for temporary capture is much larger than the numbers calculated for permanent capture. But the maximum time that the libration can last is of $O(10^5$ yrs), the time scale being set by the periods associated with the so-called secular variations of the planets. There is one final argument to be leveled against our earlier results. It is that only the synodic type resonances tend to "lock" the moon at a fixed radius without appreciably affecting the other orbital parameters. Recall that the stable e-type resonances increase e if there is a tidal torque. The

mean motion, and therefore the semimajor axis, is fixed if the moon is trapped in one of these resonances, but the mean radius actually tends to increase. The short period average of r to lowest order in e is

$$\frac{r}{a} \sim 1 + \tfrac{1}{2}e^2. \qquad (4.2.45)$$

The present value of e_{D} puts an upper bound on the lifetime of an e-type resonance. This fraction is qualitatively small compared to the change in a_{D} due to the tidal acceleration over the lifetime of the earth-moon system. Thus, only the synodic type resonances effectively trap the moon at a fixed radius. But the synodic resonances are, at best, marginally stable to tidal disruption, are subject to long period disruptive torques due to the "secular" interplanetary perturbations and the associated "secular" change in the lunar mean motion, and have a very low probability of capture. We are forced to conclude that it is unlikely that the moon was ever trapped in an orbit-orbit resonance with a planet.

There is one last question to ask. Could the orbital elements have been significantly changed either by passage through the resonance or by temporary capture? The change in the mean value of the momentum variable due to passage through resonance is approximately given by (1.2.47):

$$\Delta x \sim \frac{8}{\pi}|\beta|^{1/2}.$$

We've already seen that β is a very small parameter for each of the lunar resonances. For an e-type resonance, Δx is roughly proportional to the fractional change in the eccentricity. The above effect cannot be very large unless e_p were very small before transition.

The phenomenon of temporary capture for an e-type resonance could lead to a much larger change in the eccentricity, although still only a small fraction of its present value. This can be demonstrated by comparing the estimated lifetime of the resonance ($\sim 10^5$ yrs.) with the time necessary to have tidally changed the fractional radius of the moon by an amount equal to $1/2e_p^2$ (4.2.45). This time is approximately

$$\frac{\frac{1}{2}e_p^2 R}{\left|\left(\frac{\dot{R}}{R}\right)_T\right|} \sim [illegible]\times 10^7 \text{ years.}$$

At most, temporary capture may have changed the lunar eccentricity by as much as 1%.

We should note that the inclination-type resonances have not been mentioned since their strengths are an order of magnitude smaller than that of the e-types. This follows from the observation that all I or mixed-I type resonance variables have a coefficient $b(x)$ proportional to I^2 or II'.

4.3 ON THE THREE-SATELLITE RESONANCE OF JUPITER

The theory developed in the first three chapters has met with varying success when applied to specific examples. The principal problem is that whereas the Hamiltonian invoked is one-dimensional, the real world is not. There always seemed to be a complication: it was side-band frequencies in the case of the lunar and planetary resonance. In the example of the two-body resonance, it was the possibility that sometime in their evolution two resonance variables associated with the same commensurability might overlap and destroy the simple one-dimensional description of the phenomena. Whether such complex phenomena will yield their secrets so readily as the Hamiltonian we derived with the analytical tools developed to describe transition is best left to future investigation.

There is a more specific example of a satellite resonance in which the necessity of a multi-resonance-variable theory to explain both capture and its present libration amplitude seems unavoidable. This is the three body Laplace relation satisfied by the three inner Galilean satellites of Jupiter. ($\lambda_{JI} - 3\lambda_{JII} + 2\lambda_{JIII}$) (Hagihara, 1972). The following observations about this relation are especially interesting. 1) No libration amplitude has been observed, 2) the satellites also nearly satisfy a 2:1 commensurability between the inner pair (JI and JII) and the outer pair (JII and JIII); 3) at the present time, a tidal torque which acts principally on the inner satellites drives them toward this commensurability; and 4) the observed eccentricities are small and variable and the masses are within an order of magnitude of one another.

In the lunar theory we found that synodic frequencies could be found in the expansion of the disturbing function. It happened because the principal disturbing function $R(p + 1)$ contained explicitly the orbital elements of the earth as well as of the perturbing planet and the moon. This occurs only because there exist two distinct "primary" bodies in the problem: the sun for the planets and the earth for the moon. In Jupiter's three-satellite resonance there exists one common primary. Any inter-satellite gravitational interaction contains the orbital elements of just two of them. Since the supposed resonance involves explicitly the mean longitudes of three bodies, this angle must occur as a second order term in a perturbation expansion. This implies that perturbations in the mean longitude or semimajor axis of any one of these satellites is of $O((\mu^2)^{1/2} = \mu)$. A typical mass ratio is 10^{-4}. We found that in the case of lunar synodic resonances, both capture and damping were extremely slow, being governed by the magnitude of the fluctuations in the mean motion. So what is going on in the Jupiter satellite case? Did we miss something in the development which might radically affect the results?

Recall that in the generating function from which the short periodic perturbations were obtained, there occurred in the denominator of each term a factor nearly equal to the frequency of

the corresponding cosine argument (2.5.23). It so happens that Souillart (see Hagihara, 1972) has developed an analytic theory along the same lines as outlined in chapter three, and he finds that the dominant contributions involve the coupling of e-type angle variables which nearly satisfy the near commensurabilities among the inner and outer pairs of satellites. We should also point out that the synodic frequency occurs as the difference of two e-type frequencies. Specifically, they are

$$\phi_a = \lambda_I - 2\lambda_{II} + \tilde{\omega}_{II};$$

$$\phi_b = \lambda_{II} - 2\lambda_{III} + \tilde{\omega}_{II};$$

$$\phi_a - \phi_b = \lambda_I + 3\lambda_{II} - 2\lambda_{II}.$$

The frequency occurring in the second order coupling of these dominant terms is

$$\nu = n_I - 2n_{II} - \left(\frac{d\tilde{\omega}_{II}}{dt}\right)_{sec}.$$

We should note that the true frequency associated with the related e-type angle variable differs from the above by an amount $(\frac{d\tilde{\omega}}{dt})(e\cos\phi)$ and is a natural result of the perturbation expansion. The fluctuations which may occur in ν due to the resonance appear to be restricted to the mean motions. Another possibility is that if the $e\cos\phi$ term did contribute a large retrograde motion to $\tilde{\omega}$, its absence in ν may indicate a breakdown in the perturbation method.

The differences of the mean motions in the term $(n_I - 2n_{II})$ are:

$n_I = 203^{\circ}.489/\text{day};$

$n_{II} = 101^{\circ}.374/\text{day};$

$n_I - 2n_{II} = 0^{\circ}.740/\text{day} = 270^{\circ}.1/\text{year}.$

One of the possible implications of previous remarks is that the interaction of one or more e-type variables with the synodic variable was crucial in both capture and subsequent damping of the resonance. If these variables were significant, then the impressed retrograde motion of, say, $\frac{d\tilde{\omega}_I}{dt}$ must have been very large in the past (greater than $2 \times 270^{\circ}.1$/year). From (4.1.3) we find that if e_I were as large as 10^{-2}, the impressed retrograde motion of $\tilde{\omega}$ is still of O(degrees/day). My guess is that the critical evolution which explains the present small librations appears to be tied to an interaction involving synodic and e-type angle variables. How those variables came to overlap can be explained by either of the following two scenarios:

1) Transition into the three-body synodic resonance occurred first. The system then evolved through the e-type resonance. These variables' complex interaction led to a rapid damping of the amplitude of libration of the synodic variable.
2) First, a pair of satellites established an e-type resonance. Subsequently the system evolved toward the synodic commensurability. Somehow the original e-type resonance was disrupted while allowing capture into the synodic to occur.

Because of capture considerations the second seems the most plausible. Whether the above remarks have any relevance must await a more rigorous examination of this type of problem.

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APPENDIX A

THE CANONICAL EQUATIONS OF MOTION BY THE JACOBEAN METHOD*

A transformation from a set of coordinate variables $\{q_i, p_i\}$ to a new set $\{\alpha_i, \beta_i\}$ can be accomplished by a generating function $S(q_i, \alpha_i, t)$ defined by the relation

$$a)\ p_i = \frac{\partial S}{\partial q_i}\ ; \qquad b)\ \beta_i = \frac{\partial S}{\partial \alpha_i} \tag{A.1}$$

where S is a solution of the H-J equation

$$H'(\alpha_i, \beta_i) = H(p_i, q_i) + \frac{\partial S}{\partial t}\ . \tag{A.2}$$

Since the disturbing function R is considered small compared to H_o, and can be expanded in terms of α_i, β_i, and t, set $H' = R$. The equations of motion of the new variables become

$$\frac{d\beta_i}{dt} = -\frac{\partial R}{\partial \alpha_i}\ ; \qquad \frac{d\alpha_i}{dt} = \frac{\partial R}{\partial \beta_i}\ . \tag{A.3}$$

In the limit $R = 0$, $\{\alpha_i, \beta_i\}$ are constants. Therefore set $R = 0$ to determine $\{\alpha_i, \beta_i\}$. In spherical coordinates the two-body Hamiltonian H_o is

$$H_o = \frac{1}{2}\left(p_r^2 + \left(\frac{p_\phi}{r\sin\theta}\right)^2 + \left(\frac{p_\theta}{r}\right)^2\right) - \frac{\mu}{r} = -\frac{\mu}{2a}\ . \tag{A.4}$$

H_o is the total energy of the two-body system and can be chosen to

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*The material presented here is drawn from an exposition by E. W. Brown in PLANETARY THEORY, Ch. 4.

be one of the constants of motion. Also, the time dependent part of $S(\alpha_i, q_i, t)$ can be chosen such that H' is identically zero. If we write

$$S(\alpha_i, q_i, t) = \alpha_1 t + S^*(\alpha_i, q_i, t),$$

then α_1 is equal to H_o, or

$$\alpha_1 = -\frac{\mu}{2a},$$

if we demand that the new generating function $s^*(\alpha_i, q_i, t)$ is independent of t.

The next step is to substitute p_i for $\frac{\partial S}{\partial q_i}$ in (A.2). The result is

$$\frac{1}{2}\left(\left(\frac{\partial S^*}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S^*}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\right)^2\left(\frac{\partial S^*}{\partial \phi}\right)^2\right) - \frac{\mu}{r} - \alpha_1 = 0 \qquad \text{(A.7)}$$

The only form of $S^*(\alpha_i, q_i)$ for which (A.7) is separable is

$$S^* = S_2(\phi) + S_3(\theta) + S_4(r). \qquad \text{(A.8)}$$

Since ϕ does not explicitly appear in (A.7) set S_2 equal to

$$S_2 = \alpha_3 \phi. \qquad \text{(A.9)}$$

Separate the remaining terms in (A.7) into those which depend on θ and those which depend on r, and equate each to a constant α_2^2. The solution for S is

$$S = -\alpha_1 t + \alpha_3 \phi + \int_{r_o}^{r}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}} dr + \int_{\theta_o}^{\theta}\left(\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\theta}\right)^{\frac{1}{2}} d\theta \qquad \text{(A.10)}$$

The constants r_o, θ_o are at our disposal. Choose the values

$$r_o = r_{o\,@\,pericenter} = a(1 - e); \quad \theta_o = 0 \qquad \text{(A.11)}$$

β_i are defined by relation (A.1b):

$$\text{a)}\quad \beta_1 = \frac{\partial S}{\partial \alpha_1} = -t + \int_{r_1}^{r} \frac{dr}{\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}}} \qquad \text{(A.12)}$$

$$\text{b)}\quad \beta_2 = \frac{\partial S}{\partial \alpha_2} = -\int_{r_1}^{r} \frac{\alpha_2\, dr}{r^2\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}}} + \int_0^{\theta} \frac{\alpha_2\, d\theta}{\left(\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\theta}\right)^{\frac{1}{2}}}$$

$$\text{c)}\quad \beta_3 = \frac{\partial S}{\partial \alpha_3} = \phi - \int_0^{\theta} \frac{\alpha_3\, d\theta}{\cos^2\theta\left(\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\theta}\right)^{\frac{1}{2}}}$$

Beginning with (A.12a), we know that the integral must be equal to $\beta_1 + t$ and that β_1 is a constant. This suggests that the integration variable be changed from r to M since M is a linear function of t. Using the relations

$$r = a(1 - e\cos E)$$

and

$$\frac{dr}{dM} = \frac{ea^2}{r}\sin E,$$

the integral becomes

$$\int_{r_o}^{r} \frac{dr}{\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}}} = \int_0^{M} \frac{ea^2 \sin E\, dM}{\left((\mu a(1-e^2) - \alpha_2^2) + \mu e^2 a \sin^2 E\right)^{\frac{1}{2}}} \qquad \text{(A.13)}$$

Since the integral must be proportional to M, α_2 is forced to take the value

$$\alpha_2 = \sqrt{\mu a(1-e^2)} \tag{A.14}$$

and

$$\beta_1 = -t + \frac{M}{n} = t_0 = \frac{\varepsilon - \tilde{\omega}}{n} \tag{A.15}$$

Proceeding on to the equation defining β_2: the integral

$$\int_{r_1}^{r} \frac{\alpha_2\, dr}{r^2\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}}}$$

is a linear function if transformed to the variable f. Making use of the previous transformation and

$$\frac{dM}{df} = \frac{r^2}{a^2\sqrt{1-e^2}} ,$$

the above reduces to $\int_0^f df = f$. In the second integral, if we make the variable transformation

$$\sin u = \sqrt{\frac{\alpha_2^2}{\alpha_3^2 - \alpha_2^2}}\ \sin\theta$$

the integral reduces to $\int_0^u du = u$. The functions u and α_3 can be discovered by inspection of the spherical triangle (Fig. A.1) from which can be derived the relation

$$\sin\theta = \sin i \sin(f+\omega),$$
$$\alpha_3 = \alpha_2 \cos i , \tag{A.16}$$

Thus

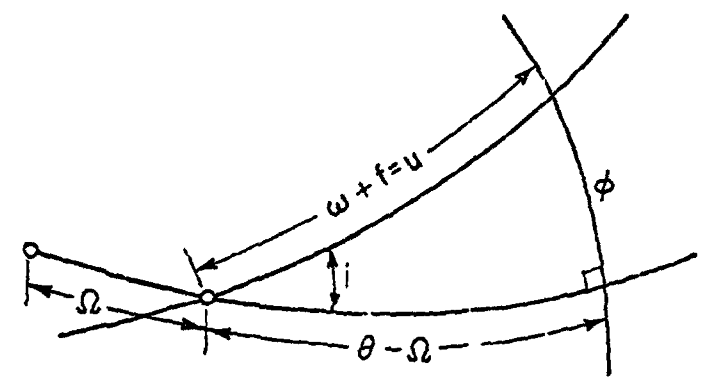

FIGURE A.1 SPHERICAL TRIANGLE

$$\beta_2 = \omega = \tilde{\omega} - \Omega \tag{A.17}$$

The integral in the β_3 equation can be reduced by a similar transformation to

$$\frac{\alpha_3}{\alpha_2}\int_0^u \frac{du}{\left(1 - \frac{\sin^2 u}{\sin^2 i}\right)} = \tan^{-1}(\cos i \tan u)$$

But this integral equals $\phi - \beta_3$ by (A.12c). Therefore,

$$\tan(\phi - \beta_3) = \cos i \tan u.$$

Again we appeal to the relations derivable from Fig. A.1; we find

$$\beta_3 = \Omega . \tag{A.18}$$

Collecting the results, we have

$$\begin{aligned} \alpha_1 &= -\frac{\mu}{2a} & \beta_1 &= \frac{(\epsilon - \tilde{\omega})}{n} \\ \alpha_2 &= \sqrt{\mu a(1 - e^2)} & \beta_2 &= \tilde{\omega} - \Omega \\ \alpha_3 &= \cos I \sqrt{\mu a(1 - e^2)} & \beta_3 &= \Omega \end{aligned} \tag{A.19}$$

The equation of motion for the set of orbital elements $\{a, e, I, \epsilon, \tilde{\omega}, \Omega\}$ can be derived from the above and (A.3) as a purely algebraic exercise.

The above set is not the most useful for our purposes. Make another H - J transformation on $\{\alpha_i, \beta_i\}$ to a new set $\{J_i, \omega_i\}$ and demand that the new Hamiltonian satisfy

$$H'' = H' + \frac{\partial S^*}{\partial t} = R + \frac{\mu}{2a} = R - \alpha_1 \tag{A.20}$$

The transformation is defined by the relations

$$\begin{aligned} &a)\quad \beta_i = \frac{\partial S^*}{\partial \alpha_i}(\alpha_i, \omega_i) \\ &b)\quad J_i = \frac{\partial S^*}{\partial \omega_i} . \end{aligned} \tag{A.21}$$

Again, let R = 0 to determine the new set. Since H' = 0 if R = 0, we can demand

$$S^* = -\alpha_1 t + S^{**}(\alpha_i, \omega_i) \tag{A.22}$$

where the new generating function $S^{**}(\alpha_i, \omega_i)$ is explicitly independent of the time. We have some freedom in choosing $S^{**}(\alpha_i, \omega_i)$, and shall demand that it be equal to the identity transformation in the old variables $\{\alpha_i, \beta_i\}$

$$S^{**}(\alpha_i, \omega_i) = \frac{\alpha_1}{n}(\epsilon - \tilde{\omega}) + \alpha_2(\tilde{\omega} - \Omega) + \alpha_3 \Omega .$$

Now we can write down the original $S^*(\alpha_i, \beta_i)$ and use (A.15) to eliminate t. The result is

$$S^* = \frac{\alpha_1}{n} M + \alpha_2(\tilde{\omega} - \Omega) + \alpha_3 \Omega . \tag{A.23}$$

There is obviously quite a bit of freedom in choosing the new variables. One choice for ω_i is

$$\begin{aligned} \omega_1 &= M \\ \omega_2 &= \tilde{\omega} - \Omega \\ \omega_3 &= \Omega \end{aligned} \tag{A.24a}$$

For which the conjugate action variables are

$$
\begin{aligned}
J_1 &= -\frac{\alpha_1}{n} = \sqrt{\mu a} \\
J_2 &= \alpha_2 = \sqrt{\mu a(1 - e^2)} \\
J_3 &= \alpha_3 = \cos I \sqrt{\mu a(1 - e^2)}
\end{aligned}
\qquad \text{(A.24b)}
$$

This set constitutes the well-known Delauney system of elements. Several similar sets of conjugate variables can be obtained by rearrangement of the angle variables in (A.23). A modified set used in this paper is

$$
\begin{aligned}
L &= \sqrt{\mu a} &&; \ \lambda = M + \varpi \\
\Gamma &= (\sqrt{1 - e^2} - 1)\sqrt{\mu a} &&; \ \varpi \\
Z &= (1 - \cos I)\sqrt{\mu a(1 - e^2)} &&; \ \Omega .
\end{aligned}
\qquad \text{(A.25)}
$$

APPENDIX B

EVALUATION OF THE ACTION INTEGRAL

The fact that the action J is an adiabatic constant as long as the instantaneous frequency is fast compared to the slow changes induced in a Hamiltonian system, provides a means of obtaining the secular motions of H and of the roots as a function of c. We are especially interested in evaluating $J_{pos.rot.}$ at transition. Transition in the small fluctuation limit involves the coincidence of the two interior π-roots, while in the large fluctuation limit the condition is that $b(x_\pi)$ vanishes. Each of these will be calculated separately.

In the positive rotation phase, the action is chosen to vanish. The explicit integrand we evaluate is

$$J_{pos.rot.} = 0 = \int_0^{2\pi} x\,d\phi = -2k^{-1}\int_0^{\pi}(-kx+1)\,d\phi + 2\pi k^{-1}. \qquad \text{(B.1)}$$

The integrand $(-kx + 1)$ is positive definite for all physical values of x, implying that the associated integral is positive definite. Changing the integration variable from x to ϕ, and integrating over the range $x_{2\pi-} < x \le x_{\pi-}$ where $x_{2\pi-}$ and $x_{\pi-}$ are the left and right bounding roots respectively, we obtain

$$\int_0^{\pi}(-kx+1)\,d\phi = \int_{x_{\pi-}}^{x_{2\pi-}}(-kx+1)\frac{d\phi}{|dx|}dx. \qquad \text{(B.2)}$$

where $\frac{d\phi}{|dx|}$ is obtained from $\left(\frac{d\phi}{dt}/\left|\frac{dx}{dt}\right|\right)$ and is: