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ALGEBRAICALLY SPECIAL SPACE–TIMES IN RELATIVITY, BLACK HOLES, AND PULSAR MODELS

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Abstract

The entire field of astronomy is now in very rapid flux, and at the center of interest are problems relating to the very dense, rotating, neutron stars now observed as pulsars, the hypothesized collapsed remains of stars known as black holes, and the not yet understood quasars. An understanding of these very exotic objects will undoubtedly require the use of rapidly developing observational tools such as infrared astronomy, ultraviolet astronomy, and x-ray astronomy, as well as the classical tools of optical and radio astronomy. On the theoretical side it will be necessary to understand the behavior of very intense gravitational fields, such as are expected to occur in these objects, and the behavior of matter and radiation in such fields.

We have used the so-called degenerate metric form, or Kerr-Schild metric form, to study several problems related to intense gravitational fields. These are as follows:

I. The fundamental problem of relativistic gravitational theory is to solve the Einstein equations to obtain metrics which generalize the Lorentz metric of special relativity,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{(special relativity)}$$

We have worked with "algebraically special" space-times in order to simplify the mathematical problem. The classification of space-times is therefore treated first, and a new relation between the Petrov and Debever-Penrose schemes is presented.

II. Our approach to the mathematical problem is to consider metrics of the specialized form

$$\eta_{\mu\nu} - 2m\, \ell_\mu \ell_\nu$$

where ℓ_μ is a null 4-vector, $\ell_\mu \ell^\mu = 0$, and m is an arbitrary parameter corresponding to the mass of the source of the field. We have studied time dependent metrics of this form and obtained very simplified equations and a number of solutions, including the Kerr metric which describes a spinning black hole. Although the algebraic problem is solved in nearly complete generality, several special new cases arise which appear to describe gravitational waves, and are being studied further.

III. The black hole solutions which arise in I. describe the exterior field of the final asymptotic state of a spinning star at the end of its evolution, if the mass exceeds a critical mass of about 2 solar masses. If the mass of a star is less than the critical mass, it may instead become a pulsar — a rapidly rotating neutron star of about the same density as an atomic nucleus, -10^{14} gm/cm^3. Such neutron stars are incredibly complicated in structure and extremely interesting, especially those with a mass near the critical mass. These stars have an exterior field that is given, to first order in the stellar rotation rate, by the solutions discussed in I. We have studied the interior field for an idealized model, which is tractable analytically yet possesses some of the most interesting properties of very complicated neutron star models. The model is composed of an incompressible fluid undergoing a small amount of differential rotation. It is attractive in its simplicity and has properties such as a constant spatial density which are very close to much more cumbersome models.

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"If only it were not so damnably difficult to find rigorous solutions"

A. Einstein - "The Born - Einstein Letters," p. 125;

Walker and Company 1971

1. Introduction.

In the decade following the publication of the field equations of general relativity[1], attention focussed on methods of solution that exploited one or other of two simplifying assumptions: symmetries in space-time, and linearization. This led to the Schwarzschild solutions[2], the Weyl solutions[3], the Einstein, De Sitter, Friedmann and other cosmological solutions and model universes[4], and to first-order corrections to Newtonian gravitational theory[5]. Much of the emphasis was on the mechanical implications of the theory, in the sense that generalizations were sought of Newtonian mechanics, including the conservation laws for momentum, angular momentum and energy.

In retrospect, an emphasis on analogies to classical mechanics looks to be too narrow. This point was emphasized in the late 1950's by Trautman, Pirani and others[6,7]. Instead of looking for a mechanical analogy, these authors argued that we should concentrate attention on analogies with other systems of field equations. In particular the electromagnetic field, which we know to be intimately related to the gravitational field[8], should be studied in a general Riemannian space-time.

This approach was introduced by Pirani, Trautman and Lichnerowicz[9], and developed further by Debever[10], Penrose[11], Robinson[12] and Bel[13], who were led to study the eigenbivectors of the Riemann tensor as the natural analogue of the eigenvectors of the energy-momentum tensor of the electromagnetic field. Using the Einstein field equations, the Riemann tensor was found to define four null vectors at a general point of space-time[10,11]. These four null vectors, termed the Debever-Penrose principal null vectors, can be related to a coordinate invariant algebraic classification of space-times developed by Petrov[14]. At any point of space-time, the space is defined to be <u>algebraically general</u> if the four Debever-Penrose null vectors are independent, and <u>algebraically special</u> if two or more of the Debever-Penrose null vectors coincide. We can regard the study of algebraically special spaces as being equivalent to a simplifying assumption in the search for solutions of the field equations, just as physical symmetries and linearization are also simplifying assumptions.

The first studies of algebraically special space-times emphasized applications to gravitational optics and to gravitational radiation theory[15,16,17]. However, the whole subject received a major new impetus in 1963, when Kerr[18] discovered a new solution of the field equations of great physical interest and importance. The Kerr solution, which appears to represent the exterior metric for the end-point of gravitational collapse of a massive rotating body (a "rotating black hole"[19]) has an algebraically special space-time structure and was found via an investigation of such space-times. Thus the latter were regarded with much greater interest and a search for other physically meaningful solutions of algebraically special type became logical. In the late 1960's, interest in the problem was again increased by the discovery of observational evidence that could be consistently interpreted as arising from rotating black holes[20,21]. The evidence still admits this interpretation, and additional supporting observations were made in 1972[22].

The main mathematical technique used so far to explore the structure of algebraically special space-times is differential geometry, couched in the tetrad formalism[23,24] or in the spinor formalism[25,26]. These have the advantages of generality and power, and the disadvantage that they are still outside the mathematical repertoire of many physicists and astronomers.

The plan in this work will be to study algebraically special space-times, using wherever possible and reasonable the most elementary and widely intelligible formulation. To accomplish this, we will find it necessary to compare and relate several alternative approaches, and in some cases to provide a completely new and elementary treatment. In Chapter 3 we discuss the classification of space-times, and in Chapters 4 and 5 we develop the relationship between two different methods of classification. After a short general discussion in Chapter 6, in Chapters 7 through 13 we present an elementary approach and solutions for a wide class of algebraically special space-times. In Chapter 14, we briefly discuss rotating solutions in the presence of matter.

Much of the material given here has been recently published or will soon be published in the Journal of Mathematical Physics, the Physical Review, and the Astrophysical Journal. Chapter 3, most of Chapter 4, and Chapter 6 are restatements of well-known material. The remaining work, except where references are quoted explicitly, are believed to represent new developments by the authors and co-authors[27,28,29,30,31].

2. Conventions and notation.

We work throughout in a space-time of signature (+,-,-,-), and with units such that $c = G = 1$.

In general, lower case Roman indices run from 1 to 3; upper case Roman indices run from 1 to 2 when associated with spinor quantities, and from 1 to 6 when associated with matrices; and Greek indices run from 0 to 3. Unless otherwise explicitly indicated, the summation convention applies to all indices.

We form the Ricci tensor by contraction on the first and third indices of the Riemann tensor, $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$. Covariant derivatives are denoted by a double line, $k_{\mu\|\nu}$, and ordinary derivatives by a single line, $k_{\mu|\nu}$. Square brackets denote antisymmetrization, $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$ and round brackets denote symmetrization, $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$. We associate a factor of $1/N!$ with symmetrized or antisymmetrized sums over N indices.

In general, the notation followed is that employed in Reference 8.

3. The Petrov Classification of Space-Times.

In 1954, Petrov published a method of classifying space-times in terms of algebraic properties at a single point[14]. This classification, which we will later show to be coordinate-independent, was made on a purely mathematical basis but was later found to be closely related to the physical problem of gravitational radiation[15,17].

The classification is based on the properties of the Riemann tensor in the vacuum case, and on the associated Weyl tensor in the non-vacuum case. For simplicity, we carry out the analysis for the case of a zero matter tensor, merely remarking that the non-vacuum case is handled in an exactly similar manner using the Weyl tensor in place of the Riemann tensor.

Consider the Riemann tensor $R_{\mu\nu\rho\lambda}$, which is antisymmetric in $\mu\nu$ and in $\rho\lambda$, and is symmetric in the pairs $\mu\nu$, $\rho\lambda$. We map the indices μ, ν, ρ, λ onto the indices $\bar{A}$, $\bar{B}$ by the homomorphism

$$\begin{bmatrix} \mu\nu & : & 23 & 31 & 12 & 10 & 20 & 30 \\ \bar{A} & : & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \tag{3.1}$$

Considered in terms of $\bar{A}$ and $\bar{B}$, $R_{\bar{A}\bar{B}}$ is a second order symmetric tensor with indices running from 1 to 6, and it can therefore be written as a 6x6 matrix of the form:

$$R_{\bar{A}\bar{B}} = \begin{bmatrix} M & N \\ N^T & Q \end{bmatrix} \tag{3.2}$$

where M and Q are real 3x3 symmetric matrices and N is a real 3x3 traceless matrix (we assume the use of real coordinates). The form of (3.2) follows directly from the symmetries of the Riemann tensor and $R_{\bar{A}\bar{B}}$ is isomorphic to the tensor $R_{\mu\nu\rho\lambda}$.

Raising the first index on $R_{\bar{A}\bar{B}}$ and employing a local Lorentz coordinate frame leads to the form:

$$R^{\bar{A}}{}_{\bar{B}} = \begin{bmatrix} M & N \\ -N^T & -Q \end{bmatrix} \qquad (3.3)$$

For a zero matter tensor, the field equations $R_{\mu\nu} = 0$ lead to the additional relations

$$\left.\begin{aligned} Q &= -M \\ N &= N^T \\ \mathrm{Tr}(M) &= 0 \end{aligned}\right\} \qquad (3.4)$$

In this case we therefore have the form

$$R^{\bar{A}}{}_{\bar{B}} = \begin{bmatrix} M & N \\ -N & M \end{bmatrix} \qquad (3.5)$$

where M and N are both symmetric real traceless 3x3 matrices. We now introduce the dual of the Riemann tensor, defined by[8]:

$$*R^{\mu\nu}{}_{\rho\lambda} = \tfrac{1}{2}\sqrt{-g}\, R^{\mu\nu}{}_{\gamma\delta}\, \varepsilon^{\gamma\delta\alpha\beta} g_{\alpha\rho} g_{\beta\lambda} \qquad (3.6)$$

where $\varepsilon^{\gamma\delta\alpha\beta}$ is the usual unit tensor density[8]. We observe that mapping μ,ν,ρ,λ onto $\bar{A}$, $\bar{B}$ as before and applying the field equations yields the form for the dual tensor

$$*R^{\bar{A}}{}_{\bar{B}} = \begin{bmatrix} N & -M \\ M & N \end{bmatrix} \qquad (3.7)$$

This result is most readily derived by element-by-element evaluation in the local Lorentz frame. We note here that the symmetries of the Riemann tensor lead to ${}^*R_{\mu\nu} = 0$, whereas the symmetries of ${}^*R_{\alpha\beta\gamma\delta}$ (which are the same as those of $R_{\alpha\beta\gamma\delta}$) follow from $R_{\mu\nu} = 0$.

Thus, now defining the self-dual tensor $R^{+\bar{A}}{}_{\bar{B}}$ below, we have:

$$R^{+\bar{A}}{}_{\bar{B}} = R^{\bar{A}}{}_{\bar{B}} + i\,{}^*R^{\bar{A}}{}_{\bar{B}} = \begin{bmatrix} M+iN & -i(M+iN) \\ i(M+iN) & M+iN \end{bmatrix} \tag{3,8}$$

This can be conveniently written in the direct product form:

$$R^{+\bar{A}}{}_{\bar{B}} = \begin{bmatrix} P & -iP \\ iP & P \end{bmatrix} = P \otimes J \tag{3.9}$$

where $P = M + iN$ (3.10)

and where $J = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$ (3.11)

P is a complex 3x3 traceless symmetric matrix and has 10 algebraically independent components.

The eigenvectors of $R^{+\bar{A}}{}_{\bar{B}}$ are the direct products of the eigenvectors of P and J, and the eigenvalues are the algebraic products of those of P and J. For J, we have eigenvalues 0 and 2 , and the corresponding eigenvectors (1, i) and (1, -i). Thus $R^{+\bar{A}}{}_{\bar{B}}$ has at least three zero eigenvalues. For the Petrov classification, it is necessary to consider only the eigenvalues and eigenvectors of $R^{+\bar{A}}{}_{\bar{B}}$ that are constructed using the non-zero eigenvalue of J. These eigenvalues of $R^{+\bar{A}}{}_{\bar{B}}$ may or may not be zero. Further, since the Petrov classification is in terms of invariant properties of the eigenvalues and eigenvectors of $R^{+\bar{A}}{}_{\bar{B}}$, the analysis applies in general even though the particular form given by (3.8) holds only in the special (Lorentz) frame.

The Petrov classification is performed according to the following simple set of statements about the eigenvalues and eigenvectors of the matrix P :-

(a) If P has three distinct eigenvectors and three distinct eigenvalues, then space-time is <u>Petrov Type I</u> (this clearly applies only to the particular <u>point</u> considered - the Petrov Type may vary from one part of space-time to another).

(b) If P has three distinct eigenvectors and two eigenvalues equal, then space-time is <u>Petrov Type I-D</u>.

(c) If P has just two distinct eigenvectors and two distinct eigenvalues, then space-time is <u>Petrov Type II</u>.

(d) If P has just two distinct eigenvectors and has two equal (and hence zero, since P is traceless) eigenvalues, then space-time is <u>Petrov Type II-N</u>.

(e) If P has only one eigenvector, then space-time is <u>Petrov Type III.</u>

These properties of P are most readily established by reducing P to Jordan canonical form[32], and this suggests that we should also state the Petrov classification in an equivalent way in terms of the elementary divisors of P, as follows:-

(a) If P has linear elementary divisors and three distinct eigenvalues then space-time is Petrov Type I.

(b) If P has linear elementary divisors and just two distinct eigenvalues then space-time is Petrov Type I-D.

(c) If P has just one linear elementary divisor and has two distinct eigenvalues then space-time is Petrov Type II.

(d) If P has just one elementary divisor and equal eigenvalues then space-time is Petrov Type II-N.

(e) If P has no linear elementary divisors then space-time is Petrov Type III.

Finally, the Petrov classification is sometimes stated in terms of the Segre characteristic of the matrix P, as follows[32]:-

Type:		Segre characteristic:
	I	(1,1,1)
	I-D	((1,1),1)
	II	(2,1)
	II-N	((2,1))
	III	(3)

As we mentioned earlier, the classification is independent of the coordinate system. For, consider the coordinate transformation: $x \longrightarrow \bar{x}$, so

$$\bar{R}^{\alpha\beta}{}_{\gamma\delta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\rho}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial \bar{x}^{\gamma}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\delta}} R^{\rho\lambda}{}_{\mu\nu} \tag{3.12}$$

With the index mappings $(\alpha, \beta) \longrightarrow A \ , \ (\gamma, \delta) \longrightarrow B$

$(\rho, \lambda) \longrightarrow L \ , \ (\mu, \nu) \longrightarrow M$

the coordinate transformation of (3.12) can be written as a matrix transformation of R :-

$$\bar{R}^{A}{}_{C} = T^{A}{}_{L} \bar{T}^{M}{}_{C} R^{L}{}_{M} \tag{3.13}$$

where $$T^{A}{}_{L} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\rho}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\lambda}} \quad \text{and} \quad \bar{T}^{n}{}_{C} = \frac{\partial x^{\mu}}{\partial \bar{x}^{\gamma}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\delta}} \tag{3.14}$$

We can write this as the matrix product

$$R = T R \bar{T} \tag{3.15}$$

But $$T \bar{T} = T^{A}{}_{L} \bar{T}^{L}{}_{C} = \delta^{\alpha}_{\gamma} \delta^{\beta}_{\delta} \tag{3.16}$$

Thus $\bar{T} = T^{-1}$ and so T is a similarity transformation on R. However, the Jordan canonical form, the eigenvalues and the number of linear elementary divisors are all unchanged under a similarity transformation, thus the Petrov classification of the space-time is invariant under coordinate transformations.

The 3x3 complex matrix P is the fundamental matrix of the Petrov classification. We show in the next chapter that P is related by a similarity transformation to another 3x3 complex matrix arising from another (spinor) approach, and thus either matrix can be used as the basis for the classification.

Although any coordinate independent representation of aspects of space-time is of interest, to this point the Petrov classification has appeared as a purely algebraic formalism, unrelated to the basic geometrical features of a space-time. In the next chapter, we introduce an alternative formulation in terms of spinors which leads to a direct and intimate connection with the space-time geometry.

4. The spinor approach and its relation to the Petrov classifications.

The Debever-Penrose principal null directions can also be used to classify space-times and in Chapter 5 we will do this explicitly. In this chapter, we wish to establish a spinor formalism and to relate certain fundamental spinor quantities to the Petrov classification matrix P. The Debever-Penrose principal null directions and their relationship to the Petrov classification can also be established by other methods(10,33), but the spinor approach appears to be the most revealing and the most elegant.

The notation we will use is that of References 11, 26 and 34, and we will also quote and use a number of theorems concerning spinors without offering proofs, which can all be found in the cited references.

Tensor and spinor quantities are transformed to each other via the general relationships:

$$\begin{aligned} T^{A\dot{B}}{}_{C\dot{D}} &= \sigma_{\lambda}{}^{A\dot{B}}\, T^{\lambda}{}_{\nu}\, \sigma^{\nu}{}_{C\dot{D}} \\ T^{\lambda}{}_{\nu} &= \sigma^{\lambda}{}_{A\dot{B}}\, T^{A\dot{B}}{}_{C\dot{D}}\, \sigma_{\nu}{}^{C\dot{D}} \end{aligned} \qquad (4.1)$$

where the quantities $\sigma_{\lambda}{}^{A\dot{B}}$ satisfy the equations:

$$\sigma_{\mu}{}^{A}{}_{\dot{C}}\, \sigma_{\nu}{}^{B\dot{C}} + \sigma_{\nu}{}^{A}{}_{\dot{C}}\, \sigma_{\lambda}{}^{B\dot{C}} = g_{\mu\nu}\, \varepsilon^{AB} \qquad (4.2)$$

The σ's relate spinor space to tensor space, and ε^{AB} is a skew-symmetric metric spinor for the two-dimensional complex spinor space, enabling spinor indices to be raised and lowered. Spinor indices take on the values 1 and 2. Equations (4.1) extend in the obvious way to any number of tensor indices. In a tangent flat space with signature (+,-,-,-), a suitable set of

σ's consists of multiples of the Pauli spin matrices, plus a multiple of the 2x2 identity matrix.

To any second rank tensor $F_{\mu\nu}$ there corresponds a 4-index spinor $F_{A\dot{B}C\dot{D}}$. If $F_{\mu\nu}$ is skew, $F_{A\dot{B}C\dot{D}}$ can be shown to have the unique spinor decomposition:

$$F_{A\dot{B}C\dot{D}} = \left(\phi_{AC}\, \varepsilon_{\dot{B}\dot{D}} + \varepsilon_{AC}\, \psi_{\dot{B}\dot{D}} \right) \qquad (4.3)$$

where ϕ_{AC} and $\psi_{\dot{B}\dot{D}}$ are symmetric in their spinor indices. Further, if $F_{\mu\nu}$ is real it may be shown that $\psi_{\dot{B}\dot{D}} = \overline{\phi}_{\dot{B}\dot{D}}$. (Note that (4.3) differs from the notation of Reference 11 by a factor of two).

If we apply (4.3) to the Riemann tensor $R_{\mu\nu\rho\lambda}$, which is skew in both pairs of indices $\mu\nu$ and $\rho\lambda$, the corresponding 8-index spinor may be shown to have the form:

$$R_{\mu\nu\rho\lambda} \longleftrightarrow R_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} = \left(\chi_{ABCD}\, \varepsilon_{\dot{E}\dot{F}}\, \varepsilon_{\dot{G}\dot{H}} + \varepsilon_{CD}\, \phi_{AB\dot{G}\dot{H}}\, \varepsilon_{\dot{E}\dot{F}} + \varepsilon_{AB}\, \overline{\phi}_{\dot{E}\dot{F}CD}\, \varepsilon_{\dot{G}\dot{H}} + \varepsilon_{AB}\, \varepsilon_{CD}\, \overline{\chi}_{\dot{E}\dot{F}\dot{G}\dot{H}} \right) \qquad (4.4)$$

The symmetry of $R_{\mu\nu\rho\lambda}$ in the pairs $\mu\nu$ and $\rho\lambda$ leads to the symmetries on χ :

$$\chi_{ABCD} = \chi_{BACD} = \chi_{ABDC} = \chi_{CDAB} \qquad (4.5)$$

In the vacuum case, applying the field equations leads to the additional conditions:

$$\phi_{AB\dot{C}\dot{D}} = 0 \quad ; \quad \chi_{ABCD} = \chi_{ADCB} \qquad (4.6)$$

Thus χ is <u>totally symmetric</u> in all its indices. This important property was apparently first remarked by Robinson[12]. However, it was Penrose who first exploited the full significance of this fact, in making it basic to the classification of space-times, in a manner that we will shortly display[11]. As Penrose remarks, it is curious and remarkable that the complete symmetry of χ follows only in a space with the signature $(+,-,-,-)$.

Using (4.6), equation (4.4) reduces to the form:

$$R_{\mu\nu\rho\lambda} \longleftrightarrow R_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} = \left(\chi_{ABCD}\,\varepsilon_{\dot{E}\dot{F}}\,\varepsilon_{\dot{G}\dot{H}} + \varepsilon_{AB}\,\varepsilon_{CD}\,\bar{\chi}_{\dot{E}\dot{F}\dot{G}\dot{H}}\right) \tag{4.7}$$

The dual tensor $^*R_{\mu\nu\rho\lambda}$ can similarly be shown to have the spinor equivalent form:

$$^*R_{\mu\nu\rho\lambda} \longleftrightarrow {}^*R_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} = -i\left(\chi_{ABCD}\,\varepsilon_{\dot{E}\dot{F}}\,\varepsilon_{\dot{G}\dot{H}} - \varepsilon_{AB}\,\varepsilon_{CD}\,\bar{\chi}_{\dot{E}\dot{F}\dot{G}\dot{H}}\right) \tag{4.8}$$

Thus the self-dual tensor $R^+_{\mu\nu\rho\lambda}$ has the spinor equivalent form:

$$R^+_{\mu\nu\rho\lambda} \longleftrightarrow R^+_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} = 2\chi_{ABCD}\,\varepsilon_{\dot{E}\dot{F}}\,\varepsilon_{\dot{G}\dot{H}} \tag{4.9}$$

Using (4.1) we then have explicitly:

$$R^+_{\mu\nu\rho\lambda} = \sigma_\mu{}^{A\dot{E}}\,\sigma_\nu{}^{B\dot{F}}\,\sigma_\rho{}^{C\dot{G}}\,\sigma_\lambda{}^{D\dot{H}}\left(2\chi_{ABCD}\,\varepsilon_{\dot{E}\dot{F}}\,\varepsilon_{\dot{G}\dot{H}}\right) \tag{4.10}$$

or, raising the μ and ν indices,

$$R^{+\mu\nu}{}_{\rho\lambda} = \sigma^{\mu A\dot{E}}\,\sigma^{\nu B\dot{F}}\,\sigma_\rho{}^{C\dot{G}}\,\sigma_\lambda{}^{D\dot{H}}\left(2\chi_{ABCD}\,\varepsilon_{\dot{E}\dot{F}}\,\varepsilon_{\dot{G}\dot{H}}\right) \tag{4.11}$$

Since $R^{+\mu\nu}{}_{\rho\lambda}$ is antisymmetric in $\mu\nu$ and in $\rho\lambda$ we write (4.11) in the explicitly antisymmetrized form:

$$R^{+\mu\nu}{}_{\rho\lambda} = \Sigma^{\mu\nu AB} \chi_{ABCD} \Sigma^{CD}_{\rho\lambda} \equiv \Sigma^{\mu\nu}_{AB} \chi^{AB}{}_{CD} \Sigma^{CD}_{\rho\lambda} \qquad (4.12)$$

where in (4.12) we define $\Sigma^{\mu\nu}_{AB}$ and $\Sigma^{CD}_{\rho\lambda}$ to be

$$\left.\begin{aligned} \Sigma^{\mu\nu}_{AB} &= 1/\sqrt{2}\left(\sigma^{\mu}{}_{A}{}^{\dot{E}}\sigma^{\nu}{}_{B}{}^{\dot{F}} - \sigma^{\nu}{}_{A}{}^{\dot{E}}\sigma^{\mu}{}_{B}{}^{\dot{F}}\right)\varepsilon_{\dot{E}\dot{F}} \\ \Sigma^{CD}_{\rho\lambda} &= 1/\sqrt{2}\left(\sigma_{\rho}{}^{C\dot{G}}\sigma_{\lambda}{}^{D\dot{H}} - \sigma_{\lambda}{}^{C\dot{G}}\sigma_{\rho}{}^{D\dot{H}}\right)\varepsilon_{\dot{G}\dot{H}} \end{aligned}\right\} \qquad (4.13)$$

Now we again introduce the tensor index mapping $\mu\nu \to \bar{A}$, $\rho\lambda \to \bar{B}$ as in equation (3.1). Bars distinguish matrix and spinor indices and we have from (4.12)

$$R^{+\bar{A}}{}_{\bar{B}} = \Sigma^{\bar{A}}{}_{AB}\, \chi^{AB}{}_{CD}\, \Sigma^{CD}{}_{\bar{B}} \qquad (4.14)$$

Now, from equation (3.8) we note that the upper left 3x3 submatrix of $R^{+\bar{A}}{}_{\bar{B}}$ is the matrix P. Thus if we restrict the range of the indices $\bar{A}$ and $\bar{B}$ to 1,2 and 3, equation (4.14) gives:

$$P^{\bar{A}}{}_{\bar{B}} = \Sigma^{\bar{A}}{}_{AB}\, \chi^{AB}{}_{CD}\, \Sigma^{CD}{}_{\bar{B}} \qquad (4.15)$$

where in (4.15) $\bar{A}$ and $\bar{B}$ take on the values 1,2 and 3, and A,B,C and D take on the values 1 and 2.

Equation (4.15) is an equation relating a 3x3 array, P, to a 4x4 array, χ . In general, the 4-index spinor will have four eigenspinors, whereas the matrix P can have at most three eigenvectors. Thus it would seem at first sight that χ is not directly relatable to the Petrov classifiaction given earlier and based on the properties of P. However, since χ is symmetric in all its spinor indices, any antisymmetric 2-index spinor, Z^{CD} , is an eigenspinor of χ , with zero eigenvalue, since

$$\chi_{ABCD} Z^{CD} = 0 \tag{4.16}$$

In addition, it can be shown that any antisymmetric 2-index spinor can be written as a scalar multiple of the skew metric spinor, ε^{AB} . Thus we need only consider the symmetric eigenspinors y^{AB} of χ , that are then at most three in number and satisfy:

$$\chi_{ABCD}\, y^{CD} = \lambda\, y^{AB} \tag{4.17}$$

Then since y and χ are both symmetric in their spinor indices, the 4x4 matrix system for the eigenspinors of χ can easily be written as an equivalent 3x3 system. For if we define the new variables Y_i and X_{ij} by the relations:

$$\left.\begin{aligned} & Y_1 = y_{11}\,, \quad Y_2 = \sqrt{2}\,y_{12} = \sqrt{2}\,y_{21}\,, \quad Y_3 = y_{22} \\ \text{and}\quad & X_{11} = \chi_{1111}\,, \quad X_{12} = X_{21} = \sqrt{2}\,\chi_{1112}\,, \quad X_{22} = 2\,\chi_{1122} \\ & X_{13} = X_{31} = \chi_{1122}\,, \quad X_{23} = X_{32} = \sqrt{2}\,\chi_{1222}\,, \quad X_{33} = \chi_{2222} \end{aligned}\right\} \tag{4.18}$$

then (4.17) takes the form

$$X_{ij}\, Y^j = \lambda\, Y_i \tag{4.19}$$

Thus the eigenvectors of the 3x3 matrix X are simply related to and in 1-1 correspondence with the symmetric eigenspinors of the 4-index spinor χ . Note that X, like P, is a symmetric matrix. Equations (4.17) and (4.19) may be related by the index mapping:

$$\begin{bmatrix} AB & : & 11 & 12,21 & 22 \\ i & : & 1 & 2 & 3 \end{bmatrix} \tag{4.20}$$

and we may then write (4.15) in the form:

$$P^{\bar{A}}{}_{\bar{B}} = \Sigma^{\bar{A}}{}_{i}\, X^{i}{}_{j}\, \Sigma^{j}{}_{\bar{B}} \tag{4.21}$$

where now all indices run from 1 to 3 and both P and X are traceless symmetric complex matrices. (Note: X is symmetric only if all indices are up or all are down).

We now show that equation (4.21) actually represents a similarity transformation between P and X. Thus either P or X may be used in performing the Petrov classification. Equally, because of the relation between X and χ, the Petrov classification can be done in terms of the symmetric eigenspinors and eigenvalues of χ.

The condition for (4.21) to be a similarity transformation is:

$$\Sigma^{\bar{A}}{}_{i}\, \Sigma^{i}{}_{\bar{B}} = \delta^{\bar{A}}{}_{\bar{B}} \tag{4.22}$$

This is most easily checked by again using the local Lorentz frame, in which the σ-matrices that enter the definition of Σ can be chosen as multiples of the Pauli spin matrices. Note that, since $\bar{A}$ and $\bar{B}$ range only over the values 1,2 and 3, the index 0 is not used in the set $\mu\nu\rho\lambda$. Thus we need only the forms of σ_1, σ_2 and σ_3, for which we will use the representations:

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.23}$$

Using the definitions (4.13) we have:

$$\Sigma^{\bar{A}}{}_{\dot{i}}\, \Sigma^{\dot{i}}{}_{\bar{B}} = \tfrac{1}{2}\left(\sigma^{\mu}{}_{A\dot{F}}\, \sigma^{\nu}{}_{B}{}^{\dot{F}} - \sigma^{\nu}{}_{A\dot{F}}\, \sigma^{\mu}{}_{B}{}^{\dot{F}}\right) \tag{4.24}$$

$$\times \left(\sigma_{\rho}{}^{A}{}_{\dot{H}}\, \sigma_{\lambda}{}^{B\dot{H}} - \sigma_{\lambda}{}^{A}{}_{\dot{H}}\, \sigma_{\rho}{}^{B\dot{H}}\right)$$

Consider one term of this product. Raising and lowering the spinor indices using the metric spinor allows us to write:

$$\sigma^{\mu}{}_{A\dot{F}}\, \sigma^{\nu}{}_{B}{}^{\dot{F}}\, \sigma_{\rho}{}^{A}{}_{\dot{H}}\, \sigma_{\lambda}{}^{B\dot{H}} = \sigma^{\mu P\dot{Q}}\, \varepsilon_{PA}\, \varepsilon_{\dot{Q}\dot{F}}\, \sigma^{\nu R\dot{F}}\, \varepsilon_{RB}\, \sigma_{\rho}{}^{A\dot{S}}\, \varepsilon_{\dot{S}\dot{H}}\, \sigma_{\lambda}{}^{B\dot{H}}$$

$$= \text{Trace}\left(\sigma_{\rho}\, \varepsilon\, \bar{\sigma}_{\lambda}\, \varepsilon\, \sigma^{\nu}\, \varepsilon\, \bar{\sigma}^{\mu}\, \varepsilon\right) \tag{4.25}$$

In (4.25) we have used the relation $\sigma^{\mu P\dot{Q}} = \overline{\sigma^{\mu \dot{Q}P}}$, and we note that since the trace is taken in spinor space, the tensor index positions can be either up or down. To evaluate the trace, we use the fact that with the representations of the σ - matrices used in (4.23), $\varepsilon\bar{\sigma}_{\lambda}\varepsilon = \sigma_{\lambda}$. Thus we have:

$$\text{Trace}\left(\sigma_{\rho}\, \varepsilon\, \bar{\sigma}_{\lambda}\, \varepsilon\, \sigma^{\nu}\, \varepsilon\, \bar{\sigma}^{\mu}\, \varepsilon\right) = \text{Trace}\left(\sigma_{\rho}\, \sigma_{\lambda}\, \sigma^{\nu}\, \sigma^{\mu}\right) \tag{4.26}$$

The Pauli spin matrices satisfy the well-known relation:[45]

$$\sigma_{\mu}\sigma_{\nu} = \tfrac{1}{2}\,\delta_{\mu\nu} + \tfrac{i}{\sqrt{2}}\,\varepsilon_{\mu\nu\lambda}\,\sigma_{\lambda} \tag{4.27}$$

Using (4.27), we find

$$\text{Trace}\left(\sigma_{\rho}\sigma_{\lambda}\sigma^{\nu}\sigma^{\mu}\right) = \tfrac{1}{2}\left(\delta_{\rho\lambda}\,\delta^{\mu\nu} - \delta_{\rho}{}^{\nu}\,\delta_{\lambda}{}^{\mu} + \delta_{\rho}{}^{\mu}\,\delta_{\lambda}{}^{\nu}\right) \tag{4.28}$$

Performing the same calculation for each term of (4.24), and combining the results, we find:

$$\sum^{\bar{A}}{}_{i} \sum^{i}{}_{\bar{B}} = \left(\delta_\lambda{}^\nu \delta_\rho{}^\mu - \delta_\lambda{}^\mu \delta_\rho{}^\nu \right) = \delta^{\bar{A}}{}_{\bar{B}} \qquad (4.29)$$

Note that since we require that $\delta^{\bar{A}}{}_{\bar{B}} = -1$ when $A = (\mu, \nu)$ and $B = (\nu, \mu)$, (4.29) is the appropriate form to use for $\delta^{\bar{A}}{}_{\bar{B}}$. Equation (4.29) confirms that the relation (4.15) is indeed a similarity transformation.

5. The eigenspinors of χ and the Debever-Penrose directions.

We have shown in the preceding two chapters that the Petrov classification can be described in terms of the properties of a complex traceless symmetric matrix P, and that this classification is invariant under a similarity transformation on P. Further, we have shown that P is related by a similarity transformation to the matrix X, and that the eigenvalues and eigenvectors of X are in 1-1 correspondence with the symmetric eigenspinors and associated eigenvalues of the wholly symmetric 4-index spinor χ . It now remains only to determine explicitly the symmetric eigenspinors and eigenvalues of χ and show how these relate to the Debever-Penrose directions, in order to complete the relationship of the Petrov method of classification to the geometric quantities of the Debever-Penrose principal null directions.

First, since χ_{ABCD} is symmetric in all its indices, there exists a unique (except for scale factors) decomposition of χ into a symmetrized product of 1-index spinors. This result is an immediate consequence of the fundamental theorem of algebra, which tells us that a general quartic form has a unique factorization into a product of linear factors(11). Thus we can write:

$$\chi_{ABCD} = k_{(A} m_B t_C s_{D)} \tag{5.1}$$

Each single index spinor such as k_A defines a vector in Riemannian space, from (4.1). This vector, say k^μ, is also readily shown to be null. Thus, the spinor χ will in general define a set of up to four such null vectors. These are the Debever-Penrose principal null vectors, and to relate them to the Petrov classification we need to obtain the symmetric eigenspinors of χ in terms of the single index spinors given in the decomposition of (5.1).

To find these eigenspinors, let us consider the product $\chi_{ABCD}\, k^{(C} m^{D)}$. Writing out in terms of symmetries over just two spinor indices, this gives:

$$\chi_{ABCD}\, k^{(C} m^{D)} = \Big[k_{(A} m_{B)}\, r_{(C} s_{D)} + k_{(A} r_{B)}\, m_{(C} s_{D)}$$

$$+ k_{(A} s_{B)}\, m_{(C} r_{D)} + r_{(A} m_{B)}\, k_{(C} s_{D)} + s_{(A} m_{B)}\, k_{(C} r_{D)}$$

$$+ r_{(A} s_{B)}\, k_{(C} m_{D)} \Big]\, k^{(C} m^{D)} \qquad (5.2)$$

Noting that $k_A s^A = k_1 s_2 - k_2 s_1 = -s_A k^A$ (5.3)

and similarly for the other spinor pairs, we will write $k_A s^A$ as $k_\wedge s$ and contract on C and D in (5.2) to give us:

$$\chi_{ABCD}\, k^{(C} m^{D)} = k_{(A} m_{B)} \left(r_\wedge k \; s_\wedge m + r_\wedge m \; s_\wedge k \right)$$

$$+ k_{(A} r_{B)}\, m_\wedge k \; s_\wedge m + k_{(A} s_{B)}\, m_\wedge k \; r_\wedge m$$

$$+ r_{(A} m_{B)}\, m_\wedge k \; k_\wedge s + s_{(A} m_{B)}\, m_\wedge k \; k_\wedge r$$

$$+ r_{(A} s_{B)}\, m_\wedge k \; k_\wedge m \qquad (5.4)$$

To simplify (5.4), we use the spinor identity:

$$\sum_P P\left(k_{(A} m_{B)}\, r_{[C} s_{D]} \right) = 0 \qquad (5.5)$$

This indicates that the sum over all cyclic permutations of the 1-index spinors k,m,r and s, symmetrized and anti-symmetrized as indicated, is identically zero. It is readily proved by symmetry arguments, but does not appear to be given in the standard reference works on spinor properties.

In particular, taking C = 1 and D = 2 in (5.5) we have:

$$\sum_P P\left(k_{(A} m_{B)} r_\wedge s \right) = 0 \tag{5.6}$$

To make use of this result, we regroup the terms of (5.4) to the form:

$$\begin{aligned}\chi_{ABCD}\, k^{(C} m^{D)} = {} & k_{(A} m_{B)} \left(r_\wedge k\, s_\wedge m + r_\wedge m\, s_\wedge k \right) \\ & + r_{(A} s_{B)}\, m_\wedge k\, k_\wedge m + m_\wedge k \left(k_{(A} r_{B)}\, s_\wedge m + s_{(A} m_{B)}\, k_\wedge r \right) \\ & + m_\wedge k \left(k_{(A} s_{B)}\, r_\wedge m + r_{(A} m_{B)}\, k_\wedge s \right)\end{aligned} \tag{5.7}$$

and now apply (5.6) with the symbols ordered as k,r,s,m to the third term of (5.7 and ordered as k,s,r,m to the fourth term. This gives:

$$\begin{aligned}\chi_{ABCD}\, k^{(C} m^{D)} = {} & k_{(A} m_{B)} \left(r_\wedge k\, s_\wedge m + r_\wedge m\, s_\wedge k \right) \\ & + 3\, r_{(A} s_{B)} \left(m_\wedge k\, k_\wedge m \right)\end{aligned} \tag{5.8}$$

Similarly, we find:

$$\begin{aligned}\chi_{ABCD}\, r^{(C} s^{D)} = {} & r_{(A} s_{B)} \left(r_\wedge k\, s_\wedge m + r_\wedge m\, s_\wedge k \right) \\ & + 3\, k_{(A} m_{B)} \left(s_\wedge r\, r_\wedge s \right)\end{aligned} \tag{5.9}$$

Taking linear combinations of (5.8) and (5.9) then at once gives us the eigenspinors of χ_{ABCD} :

$$y^{\pm}_{AB} = k_{(A} m_{B)}\, r_\wedge s \pm r_{(A} s_{B)}\, k_\wedge m \tag{5.10}$$

with associated eigenvalues

$$\begin{aligned}\lambda^{\pm} = {} & 2\left(r_\wedge k\, s_\wedge m + r_\wedge m\, s_\wedge k \right) \\ & \mp 6 \left(k_\wedge m\, r_\wedge s \right)\end{aligned} \tag{5.11}$$

Since we could just as well have started with $k^{(C}r^{D)}$ or $k^{(C}s^{D)}$ in (5.2), we have as the full set of eigenspinors:

$$y_1^{\pm} = k_{(A}m_{B)}\, r_\wedge s \pm r_{(A}s_{B)}\, k_\wedge m$$

$$y_2^{\pm} = k_{(A}r_{B)}\, m_\wedge s \pm m_{(A}s_{B)}\, k_\wedge r$$

$$y_3^{\pm} = k_{(A}s_{B)}\, r_\wedge m \pm r_{(A}m_{B)}\, k_\wedge s \qquad (5.12)$$

with their corresponding eigenvalues:

$$\lambda_1^{\mp} = 2\left(r_\wedge k\, s_\wedge m + r_\wedge m\, s_\wedge k\right) \mp 6\left(k_\wedge m\, r_\wedge s\right)$$

$$\lambda_2^{\mp} = 2\left(m_\wedge k\, s_\wedge r + m_\wedge r\, s_\wedge k\right) \mp 6\left(k_\wedge r\, m_\wedge s\right)$$

$$\lambda_3^{\mp} = 2\left(r_\wedge k\, m_\wedge s + r_\wedge s\, m_\wedge k\right) \mp 6\left(k_\wedge s\, r_\wedge m\right) \qquad (5.13)$$

The form of (5.12) suggests that there are six symmetric eigenspinors, but if we apply (5.6) we find that $y_1^+ = y_3^+$, $y_1^- = y_2^-$, and $y_2^+ = y_3^-$. Thus there are at most three independent symmetric eigenspinors, and we choose these to be $y_1 = y_1^+$, $y_2 = y_2^-$, and $y_3 = y_3^-$, with the corresponding eigenvalues λ_1^-, λ_2^+ and λ_3^+.

If now k,m,r and s are all distinct 1-index spinors, then in fact we have four distinct Debever-Penrose principal null directions, three independent symmetric eigenspinors of χ, and correspondingly three linear elementary divisors of P, so space-time is Petrov Type I. However, it is possible for k,m,r and s to coincide in various ways and we need to examine the behavior of the symmetric eigenspinors (5.12) and their eigenvalues (5.13) in such cases. The possible situations that can arise are the following:

a) k,m,r and s are all distinct. We have four independent Debever-Penrose directions; the symmetric eigenspinors of χ and their corresponding eigenvalues are all distinct; P has thus three linear elementary divisors and three distinct eigenvalues, and space-time is Petrov Type I.

b) If $k = r \neq m \neq s$, in this case, just two Debever-Penrose directions coincide. We have $k_\wedge r = 0$, so the symmetric eigenspinors are:

$$\begin{aligned} y_1 &= k_{(A} m_{B)}\, r_\wedge s + r_{(A} s_{B)}\, k_\wedge m \\ y_3 &= -y_2 = k_{(A} k_{B)}\, m_\wedge s \end{aligned} \tag{5.14}$$

with eigenvalues $\lambda_2 = \lambda_3 \neq \lambda_1$.

Thus we have two distinct eigenvalues and two symmetric eigenspinors, P has one linear elementary divisor and two distinct eigenvalues, and space-time is therefore Petrov Type II.

c) If $k = r = m \neq s$, in this case three Debever- Penrose directions coincide. We find that $y_1 = y_2 = y_3$, and $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus we have one symmetric eigenspinor, P has no linear elementary divisors, and space-time is Petrov Type III.

d) If $k = m \neq r = s$, then the Debever-Penrose directions coincide in pairs. It is necessary to be rather careful in applying (5.12) here, since the form of the eigenspinors becomes degenerate. It is then in fact easier to re-derive the eigenspinors directly from (5.8), and one finds the three symmetric eigenspinors:

$$y_1 = k_{(A} m_{B)} \qquad y_2 = k_{(A} k_{B)} \qquad y_3 = m_{(A} m_{B)} \tag{5.15}$$

with associated eigenvalues $\lambda_2 = \lambda_3 \neq \lambda_1$.
Thus χ has three distinct eigenspinors but only two distinct eigenvalues. P therefore has linear elementary divisors and two distinct eigenvalues, so space-time is <u>Petrov Type I-D.</u>

e) If k = m = r = s, all four Debever-Penrose directions coincide. In this case, we find the symmetric eigenspinors:

$$y_1 = k_{(A} k_{B)} \quad , \quad y_2 = k_{(A} j_{B)} \tag{5.16}$$

where j_B is any 1-index spinor independent of k_B. The eigenvalues in this case are zero, so P will have one linear elementary divisor and zero eigenvalues, which means that space-time is <u>Petrov Type II-N</u>.

For completeness, one final special case should be added. If χ is identically zero, then space is flat and we have no preferred directions. Anything is then an eigenspinor, and the classification ceases to have significance.

Although historically Petrov developed a formulation in which Type I-D is a special case of Type I, since both have linear elementary divisors, in terms of the Debever-Penrose directions Type I-D appears much more like a special case of Type II, in which a second pair of Debever-Penrose directions are allowed to move into coincidence.

This completes the general development in which we relate alternative ways of classifying space-times. For the remainder of this work, we will concentrate attention on algebraically special space-times, and utilize the structure that this implies to look for solutions of the field equations.

One final point should be made here. For ease of development, we have worked always with the vacuum case. When matter is present, the Weyl tensor is used in place of the Riemann tensor in the Petrov classification, and a corresponding gravitational spinor, ψ_{ABCD} , is used for the spinor formulation[11]. Then the analysis goes through in a way that exactly parallels the vacuum treatment. However, in terms of the utility of the results, the use of algebraic degeneracy has proved fruitful in looking for solutions mainly in the vacuum case. Although solutions of the Einstein-Maxwell equations have been found for algebraically special space-times[43,46], no solutions of great physical interest seem to have been discovered for such situations. We will discuss the non-vacuum case further in later chapters, and point out some of the added factors that complicate the solution of the field equations in such cases.

6. A summary discussion of degenerate space-times.

In any space-time which is not Petrov Type I, two or more Debever-Penrose principal null directions must coincide. Since this is a statement about the essential geometry of the space-time, it is natural to ask how this geometric statement relates to other geometric properties - for example, must an algebraically special space-time possess some physical symmetries (isometries)? Counterexamples show that physical isometries neither imply nor are implied by algebraic specialization of a space-time: algebraically special spaces exist that have no physical symmetries, and there are algebraically general spaces that possess one or more isometries[35].

The best insight into the meaning of algebraic specialization is gained by examining analogies with the electromagnetic field. However, the nonlinear nature of the gravitational field equations makes many of the methods often used in handling the Maxwell equations (such as Fourier decomposition) inappropriate[34]. To summarize a substantial body of work in a few sentences, it turns out that the most enlightening analogy is in terms of the eigenvectors (eigenbivectors, in the case of the Riemann tensor) of the field tensor. In the case of the Maxwell field, only two situations occur: the eigenvectors of the field tensor are independent, or they coincide. The former case occurs in any region with radiative sources present, the latter describes a pure radiation field asymptotically far from bounded sources[34,36,37].

The gravitational field tensor has a more elaborate possible structure for the eigenbivectors. Five cases can occur, depending how the four Debever-Penrose principal null vectors coincide. Thinking in terms of gravitational radiation from bounded sources, five different possible cases (which are related directly to their Petrov types) multiply different powers of a mean inverse distance from the bounded sources. This "peeling-off" property[34,38] is of interest to us here only because it indicates that gravitational radiation in regions containing sources cannot be Petrov Type II or Type III - i.e. space-time near radiating sources must be algebraically general.

This result suggests that solutions obtained for algebraically special space-times will either be non-radiating (like the Kerr and Schwarzschild solutions) or applicable only to pure gravitational radiation far from all sources. It is interesting to note that the Kerr and Schwarzschild solutions are of Type I-D, which does not occur in the "peeling-off" theorems, and seems to be associated with stationary solutions rather than with gravitational radiation. Very recently, other stationary asymptotically flat solutions that have non-zero angular momentum and are Petrov Type I have been developed, but their possible physical significance is not yet clear[39,40].

Despite the fact that algebraically special space-times cannot describe the most general physical situation, the problem is still sufficiently general that as yet no one has succeeded in constructing the most general form for the metric of an algebraically special space-time. It is quite easy to define a necessary and sufficient condition in terms of the Riemann tensor: the matrix P of Section 3, Equation 10 , must have at least two equal eigenvalues. Thus if the characteristic polynomial of the traceless matrix P is:

$$f(\lambda) = \lambda^3 - b\lambda + c \tag{6.1}$$

then space-time is algebraically special if and only if $f(\lambda) = 0$ has two equal roots, which requires that:

$$27\, c^2 = 4\, b^3 \tag{6.2}$$

This elementary and apparently attractive relation between b and c becomes an extremely complicated nonlinear partial differential equation in the metric tensor when we substitute the forms of b and c. As a result, the direct approach of (6.2) is quite useless. Instead, most efforts have been devoted to the analysis of particular classes of metrics that can be shown to correspond to <u>particular</u> algebraically special space-times, without seeking or claiming the most general possible form.

In the subsequent chapters, we develop a very elementary approach to just such a class of metrics, first studied using a tetrad formalism by Kerr and Schild[41].

7. A class of metrics with algebraically special space-times.

We now focus our attention on a particular class of space-times, with metrics of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} - 2m\, l_\mu l_\nu \tag{7.1}$$

where in (7.1) $\eta_{\mu\nu}$ is the Lorentz metric, m is any real arbitrary constant, and l_μ is a 4-vector that is null with respect to the Lorentz metric, so $l_\mu l_\nu \eta^{\mu\nu} = 0$.

It can be shown directly[41] that the metric (7.1) corresponds to an algebraically special space-time, but we prefer to establish this in the course of our general development.

A few comments are in order on the form of the metrics we are considering. First, if we make the choice

$$l_\mu = \left(1/r\right)^{1/2} \left(1,\ x/r,\ y/r,\ z/r\right) \tag{7.2}$$

where $r^2 = x^2+y^2+z^2$, then we obtain a line element that is the Eddinington form[42] of the Schwarzschild metric, obtained from the original form of the Schwarzschild metric by a change in the time coordinate. In this case, the arbitrary constant m appears as the mass of the body. Second, if we are interested in asymptotically flat space-times, l_μ must tend to zero at spatial infinity if x^μ are Cartesian coordinates for Minkowski space-time. Third, for small m and finite l_μ, (7.1) has the form of a perturbation on Minkowski space, thus we expect that the usual linearized theory will emerge for small m.

In the treatment given hereafter, we will treat the spatial coordinates in a symmetrical way. We will also make no assumption of stationary or axisymmetric solutions, nor do we introduce the assumption of algebraic degeneracy explicitly in our analysis. This differs from the treatment of Debney, Kerr and Schild[43], who in a tetrad formalism use the algebraic degeneracy from the outset, and from that of Misra[44], who assumes axial symmetry.

Using (7.1), a number of useful relations are readily established. We find

$$g^{\mu\nu} = \eta^{\mu\nu} + 2m\,\ell^\mu \ell^\nu \tag{7.3}$$

so that indices on ℓ_μ may be raised or lowered with either the full metric tensor or with the Lorentz metric tensor. Also, since ℓ_μ is null

$$\ell^\mu \ell_{\mu|\nu} = \ell_\mu \ell^\mu{}_{|\nu} = 0 \tag{7.4}$$

and from the product rule for covariant derivatives we also have

$$\ell^\mu \ell_{\mu||\nu} = \ell_\mu \ell^\mu{}_{||\nu} = 0 \tag{7.5}$$

The Christoffel symbols are easily calculated and we find

$$\left\{ {\alpha \atop \beta\mu} \right\} \ell^\mu = -m(\ell^\alpha \ell_\beta)_{|\mu}\, \ell^\mu \tag{7.6}$$

The field equations are greatly simplified by the choice of metric, since from (7.1) we find $\det(g) = -1$, and thus we have

$$\left\{ {\alpha \atop \beta\alpha} \right\} = \left(\log(\sqrt{-g})\right)_{|\beta} = 0 \tag{7.7}$$

Using (7.7), we find that the field equations reduce to only two terms

$$R_{\mu\nu} = -\left\{ {\alpha \atop \mu\nu} \right\}_{|\alpha} + \left\{ {\alpha \atop \beta\mu} \right\}\left\{ {\beta \atop \alpha\nu} \right\} = 0 \tag{7.8}$$

The metric tensor is linear in the arbitrary constant m, thus $R_{\mu\nu}$ is a fourth order polynomial in m. If we require that (7.1) shall lead to a solution of the free space field equations for any value of m, then each power of m in (7.8) must vanish separately, which gives rise to the four sets of equations:

order m:

$$\eta^{\alpha\rho}[\mu\nu,\rho]_{|\alpha} = 0 \tag{7.9}$$

order m^2 :

$$2m\left(\ell^{\alpha}\ell^{\rho}[\mu\nu,\rho]\right)_{|\alpha} - \eta^{\alpha\sigma}\eta^{\beta\lambda}[\beta\mu,\sigma][\alpha\nu,\lambda] = 0 \tag{7.10}$$

order m^3 :

$$\ell^{\beta}\ell^{\lambda}\eta^{\alpha\sigma}[\beta\mu,\sigma][\alpha\nu,\lambda] + \ell^{\alpha}\ell^{\lambda}\eta^{\beta\sigma}[\beta\mu,\lambda][\alpha\nu,\sigma] = 0 \tag{7.11}$$

order m^4 ;

$$\ell^{\alpha}\ell^{\sigma}\ell^{\beta}\ell^{\lambda}[\beta\mu,\sigma][\alpha\nu,\lambda] = 0 \tag{7.12}$$

Using (7.3) - (7.5), we can at once verify that the order m^4 equations are satisfied identically. The order m^3 equations, after expansion of the Christoffel symbols, can be written in the form

$$-m\ell_{\mu}\ell_{\nu}v^{\alpha}v_{\alpha} = 0 \tag{7.13}$$

where we have defined $v^{\alpha} \equiv \ell^{\beta}\ell^{\alpha}{}_{\|\beta}$. The latter is null, and it is also readily seen to be orthogonal to the null vector ℓ^{α} . This can happen only if v^{α} and ℓ^{α} are proportional to each other at every point, thus we can write

$$v^{\alpha} = \ell^{\mu}\ell^{\alpha}{}_{\|\mu} = -A\ell^{\alpha} \tag{7.14}$$

where A is a scalar field.

Note that (7.14) tells us that the vector field ℓ_μ is tangent to a family of geodesics[34], and ℓ_μ is parallel-propagated along the geodesics. If, as will happen later, we choose to perform a scalar change of variables $\ell_\mu = H k_\mu$ such that (7.14) becomes

$$k^\mu k^\alpha{}_{\|\mu} = k^\mu k^\alpha{}_{|\mu} = 0 \tag{7.15}$$

then we say that the vector field k_μ is <u>affinely parametrized</u>. There are many possible choices of the scalar H that will affinely parametrize a given family of geodesic tangents.

Now considering the order m equations, we define a new scalar L by:

$$L = -\ell^\alpha{}_{\|\alpha} = -\ell^\alpha{}_{|\alpha} \tag{7.16}$$

and then we can rewrite (7.9) in the convenient form:

$$\begin{aligned} -\Box^2(\ell_\mu \ell_\nu) &= -(\ell_\mu \ell_\nu)^{|\alpha}{}_{|\alpha} \\ &= \left[(L+A)\ell_\mu\right]_{|\nu} + \left[(L+A)\ell_\nu\right]_{|\mu} \end{aligned} \tag{7.17}$$

where the D'Alembertian operator is defined as $\Box^2 = \partial^2/\partial t^2 - \nabla^2$, so the upper index derivative in (7.17) is to be raised using $\eta^{\alpha\beta}$.

Defining $\quad G = (L + A) \quad$ (7.18)

we have as the final form for the order m equations:

$$-(\ell_\mu \ell_\nu)^{|\alpha}{}_{|\alpha} = 2\left(G\ell_{(\mu}\right)_{|\nu)} \tag{7.19}$$

These will be used extensively in subsequent chapters.

For the order m^2 equations, expanding the Christoffel symbols and using equation (7.14), equation (7.10) can be rewritten as

$$2(l^{\alpha}A)_{|\alpha} + l^{\alpha}{}_{|\beta}\, l^{\beta}{}_{|\alpha} - l^{\alpha}{}_{|\beta}\, l_{\alpha}{}^{|\beta} - A^2 = 0 \qquad (7.20)$$

Using the definitions of L and A, we readily derive

$$l^{\alpha}{}_{|\beta}\, l^{\beta}{}_{|\alpha} = \left[(L-A)\, l^{\alpha}\right]_{|\alpha} + L^2 \qquad (7.21)$$

and by using (7.19) to form $l^{\mu}(l_{\mu}l_{\nu})^{|\beta}{}_{|\beta}$ we also find

$$l^{\alpha}{}_{|\beta}\, l_{\alpha}{}^{|\beta} = (L^2 - A^2) + (G l^{\mu})_{|\mu} \qquad (7.22)$$

Substituting (7.21) and (7.22) in (7.20), we find that this gives an identity, thus the order m^2 equations contain no new information and are implied by the order m equations. This is expected on general grounds, since there are ten independent order m equations, and this should suffice to determine the metric completely. In the same way, we can also show that the relation (7.13) leading to the definition of v^{α} also follows from the order m equations, although the derivation from the order m^3 equations is shorter and simpler.

We now work exclusively on the order m equations. From (7.19)

$$-(l_{\mu}l_{\mu})^{|\alpha}{}_{|\alpha} = 2(G l_{\mu})_{|\mu} \qquad \text{(no sum on } \mu\text{)} \qquad (7.23a)$$

$$-(l_{\nu}l_{\nu})^{|\alpha}{}_{|\alpha} = 2(G l_{\nu})_{|\nu} \qquad \text{(no sum on } \nu\text{)} \qquad (7.23b)$$

using (7.23), we can now eliminate the second order terms from (7.19), which leads to the six independent first order equations:

$$2(l_{\mu|\alpha}l_\nu{}^{|\alpha})l_\mu l_\nu - l_{\mu|\alpha}l_\mu{}^{|\alpha}l_\nu^2 - l_{\nu|\alpha}l_\nu{}^{|\alpha}l_\mu^2$$

$$= -G\left[2l_{(\mu|\nu)}l_\mu l_\nu - l_{\mu|\mu}l_\nu^2 - l_{\nu|\nu}l_\mu^2\right] \tag{7.24}$$

(no sum on μ and ν)

These first order equations, together with (7.23a), are completely equivalent to (7.19). If we now take

$$l_\mu = \mathrm{H}\,k_\mu \tag{7.25}$$

where H is any scalar field, we find the form:

$$2(k_{\mu|\alpha}k_\nu{}^{|\alpha})k_\mu k_\nu - k_{\mu|\alpha}k_\mu{}^{|\alpha}k_\nu^2 - k_{\nu|\alpha}k_\nu{}^{|\alpha}k_\mu^2$$

$$= -\frac{G}{H}\left[2k_{(\mu|\nu)}k_\mu k_\nu - k_{\mu|\mu}k_\nu^2 - k_{\nu|\nu}k_\mu^2\right] \tag{7.26}$$

(no sum on μ and ν)

This has exactly the same form as (7.24), but k_μ has replaced l_μ, and G/H has replaced G. We also readily confirm that k_μ is null and that (7.4) and (7.5) are satisfied with k_μ replacing l_μ. If now in (7.26) we make the particular choice $\mathrm{H} = l_0$, so that $k_0 = 1$, then we have:

$$2k_{i|\alpha}k_j{}^{|\alpha}k_i k_j - k_{i|\alpha}k_i{}^{|\alpha}k_j^2 - k_{j|\alpha}k_j{}^{|\alpha}k_i^2$$

$$= -\frac{G}{l_0}\left[2k_{(i|j)}k_i k_j - k_{i|i}k_j^2 - k_{j|j}k_i^2\right] \tag{7.27a}$$

(no sum on i and j)

$$k_{j|\alpha}k_j{}^{|\alpha} = \frac{G}{l_0}\left[k_{j|0}k_j - k_{j|j}\right] \tag{7.27b}$$

(no sum on j)

Roman indices in (7.27) and subsequently run from 1 to 3.

Using (7.27b) to simplify (7.27a), we find a form completely equivalent to equations (7.27):

$$k_{i|\alpha} k_j{}^{|\alpha} = -\frac{G}{\ell_0}\left[k_{(i|j)} - k_{(i} k_{j)||0} \right] \tag{7.28}$$

Finally, using (7.25) with H $= \ell_0$ in (7.14), we have the relations:

$$\ell^\mu \ell_{0||\mu} = -A \ell_0 \tag{7.29a}$$

$$\ell^\mu (\ell_0 k^\alpha)_{||\mu} = -A \ell_0 k^\alpha \tag{7.29b}$$

Eliminating A between (7.29a) and (7.29b) at once gives:

$$k^\mu k^\alpha{}_{||\mu} = k^\mu k^\alpha{}_{|\mu} = 0 \tag{7.30}$$

Thus k_μ is an affinely parametrized null vector. We note that the choice H $= \ell_s$, where ℓ_s is any component of ℓ_μ, also leads to an affine parametrization. However, the choice of the time component is the most natural, since it permits a subsequent analysis that is symmetric in its treatment of the spatial coordinates.

8. The reduction of the first order field equations.

Equations (7.23a) and (7.28) comprise four second order and six first order differential equations, which together are equivalent to the entire set (7.19) of second order field equations. To solve the field equations, we first address the reduction of (7.28) to a simpler form. Noting that we can lower indices on both ℓ^μ and k^μ using $\eta_{\mu\nu}$, we can write $k^j = -k_j$. We further define $p = G/2\ell_0$, so that equations (7.28) become:-

$$k_{i|0}\, k_{j|0} - k_{i|\ell}\, k_{j|\ell} = -p\left[k_{i|j} + k_{j|i} - (k_i k_j)_{|\ell}\, k_\ell \right] \tag{8.1}$$

where we are now summing over all repeated indices.

Noting that (7.30) can be written as:

$$k_{i|0} = k_{i|\ell}\, k_\ell \tag{8.2}$$

we write (8.1) as:

$$(\delta_{\ell r} - k_\ell k_r)\, k_{i|\ell}\, k_{j|r} = -p\left[(k_i k_j)_{|\ell}\, k_\ell - (k_{i|j} + k_{j|i}) \right] \tag{8.3}$$

If we now define a 3x3 matrix M and a 3-vector $\vec{k}$ by:

$$M_{ij} = k_{i|j} \tag{8.4a}$$

$$(\vec{k})_i = k_i \tag{8.4b}$$

then (8.3) can be written as the matrix equation:

$$\frac{1}{p}\left[M(I-\vec{k}\vec{k}^T)M^T\right] =$$

$$M + M^T - (M\vec{k})\vec{k}^T - \vec{k}(M\vec{k})^T \tag{8.5}$$

We now perform a rotation of the coordinate axes, so that $\vec{k}$ goes to the form $\vec{k}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

In these coordinates, using the relation

$$M'^T\vec{k}' = M^T\vec{k} = k_i l_{ij} k_i = 0 \tag{8.6}$$

we find that M' must have the form:

$$M'^T = \left(\begin{array}{c|cc} 0 & x & y \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{m'}} \\ 0 & & \end{array}\right) \tag{8.7}$$

where m' is a 2x2 submatrix. Hence:

$$M'\vec{k}' = \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \tag{8.8}$$

Using (8.7) and (8.8) to form $M'M'^T$, $(M' + M'^T)$ and $M'\vec{k}'\ (M'\vec{k}')^T$, and substituting in (8.5), we find that x and y cancel from the equation, and we are left with the remarkably simple result:

$$m'm'^T = p\left(m' + m'^T\right) \tag{8.9}$$

Equation (8.9) implies that m' can be written in terms of a real 2x2 unitary matrix U, as:

$$U = I - m'/p \tag{8.10}$$

For our main development, we will assume that U is proper, with positive determinant, and so can be written in terms of a single real parameter θ in the form

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{8.11}$$

The cases of improper U, vanishing p and vanishing θ are discussed in Chapter 13.

Using (8.11), M' can then be written as:

$$M' = p \begin{pmatrix} 0 & 0 & 0 \\ u & 1-\cos\theta & -\sin\theta \\ v & \sin\theta & 1-\cos\theta \end{pmatrix} \tag{8.12}$$

where we define pu = x and pv = y.

To get back to the original coordinate system, we must form

$$M = R^T M' R \tag{8.13}$$

where R is a real 3x3 orthogonal matrix. Using the orthonormality of the rows and columns of R, and noting in particular that

$$R_{1i} = \varepsilon_{ijk} R_{2j} R_{3k} \tag{8.14}$$

we find the form for M:

$$M_{ij} = p(1-\cos\theta)(\delta_{ij} - R_{1i}R_{1j}) + p\sin\theta\,\varepsilon_{ijl}R_{1l} + p(uR_{2i} + vR_{3i})R_{1j} \tag{8.15}$$

Now defining

$$T_i = p\left(uR_{2i} + vR_{3i}\right) \tag{8.16}$$

we note that since R_{2i} and R_{3i} are rows of R and therefore orthogonal to R_{1i}, T_i is orthogonal to R_{1i}. Further, using $M_{ij}k_i = 0$, we also have

$$T_i k_i = 0 \tag{8.17}$$

so $\vec{T}$ is orthogonal to $\vec{k}$.

Using (8.17) in (8.15) and multiplying by $k_i k_j$, we find that $k_i R_{1i} = 1$. Since both $\vec{k}$ and $\vec{R}_1$ are unit vectors, we must have $\vec{k} = \pm \vec{R}_1$, and we can choose the plus sign since the overall sign of R is arbitrary. Then using (8.4a) we can write (8.15) as:

$$k_{i|j} = \alpha\left(\delta_{ij} - k_i k_j\right) + \beta\, \varepsilon_{ije} k_e + T_i k_j \tag{8.18}$$

where in (8.18) we have defined α and β by:

$$\alpha = p\left(1 - \cos\theta\right) \tag{8.19a}$$

$$\beta = p \sin\theta \tag{8.19b}$$

Multiplying by k_j and using (8.2) we have at once

$$T_i = k_{i|0} \tag{8.20}$$

so (8.18) can be written as

$$k_{i|j} = \alpha\left(\delta_{ij} - k_i k_j\right) + \beta\, \varepsilon_{ije} k_e + k_{i|0} k_j \tag{8.21}$$

or, using (8.2), in the alternative form:

$$k_{i|\ell}(\delta_{\ell j} - k_\ell k_j) = \alpha(\delta_{ij} - k_i k_j) + \beta\, \varepsilon_{ij\ell}\, k_\ell \qquad (8.22)$$

The forms (8.21) and (8.22) will be used extensively. They replace the six nonlinear first order equations of (7.28) by an explicit expression for the derivatives $k_{i|j}$. Before studying these equations further, which is our task in the next chapter, we first note an important feature of (8.22). The matrix ($\delta_{\ell j} - k_\ell k_j$) is singular, thus although (8.22) is a general set of relations between the $k_{i|j}$ and the vector k_i, we cannot use (8.22) to solve for $k_{i|j}$ in terms of k_i. The only exception to this statement occurs if $k_{i|j}\, k_j = 0$, when (8.22) becomes soluble for $k_{i|j}$. However, from (8.2) we see that this occurs only when $k_{i|0} = 0$, which is the stationary case.

For the general non-stationary case, the additional information needed to solve (8.22) for $k_{i|j}$ will come from the equations (7.23a) which we have not yet considered.

9. Properties of the first order equations and the optical scalars.

The equations (8.21) lead to a remarkable number of interesting relations involving α and β , but we will confine our attention to the main results needed for subsequent analysis.

Using $k_i \, k_{i|0} = 0$, (8.21) at once gives

$$k_{i|i} = 2\alpha \tag{9.1}$$

Also we find, directly from (8.21), the following:-

$$\varepsilon_{ijl} k_{i|j} k_l = 2\beta \tag{9.2}$$

$$k_{i|j} k_{i|j} - k_{i|0} k_{i|0} = 2(\alpha^2+\beta^2) \tag{9.3}$$

$$k_{i|j} k_{j|i} = 2(\alpha^2-\beta^2) \tag{9.4}$$

$$\varepsilon_{ijl} k_{i|j} = 2\beta k_l + \varepsilon_{ijl} k_{i|0} k_j \tag{9.5}$$

In the study of geometrical optics in a gravitational field, the quantities known as the optical scalars play a fundamental role[15,17,34]. Two of these scalars, the expansion θ and the twist ω , are defined in terms of an affinely parametrized ray vector field k^μ by[34]:-

$$\theta = \tfrac{1}{2} k^\mu{}_{||\mu} \tag{9.6}$$

$$\omega = \tfrac{1}{2}\left[(k_{\mu||\nu} - k_{\nu||\mu}) k^{\mu||\nu}\right]^{1/2} \tag{9.7}$$

Using (9.1), we readily show that:

$$\tfrac{1}{2} k^\mu{}_{||\mu} = \tfrac{1}{2} k^\mu{}_{|\mu} = \alpha \tag{9.8}$$

and also that:

$$\tfrac{1}{2}\left(k_{\mu\|\nu} - k_{\nu\|\mu}\right) k^{\mu\|\nu} = \tfrac{1}{2}\left(k_{\mu|\nu} - k_{\nu|\mu}\right) k^{\mu|\nu}$$

$$= -\tfrac{1}{2}\, k_{i|0} k_{i|0} + \tfrac{1}{2}\left(k_{i|j} - k_{j|i}\right) k_{i|j} \qquad (9.9)$$

Now using (9.3) and (9.4), we have:

$$\tfrac{1}{2}\left(k_{\mu\|\nu} - k_{\nu\|\mu}\right) k^{\mu\|\nu} = 2\beta^2 \qquad (9.10)$$

Comparison of (9.6) and (9.8), and of (9.7) and (9.9) indicates that except for a choice of sign α and β are exactly the optical scalars θ and ω . It is interesting to see that the optical scalars enter the solution in a natural and fundamental way, without being introduced at the outset.

The third optical scalar is the shear, σ . For metrics of the form (7.1), the Goldberg-Sachs theorem tells us at once that the shear must be zero, if the space-time represented by (7.1) is algebraically special. Conversely, a direct calculation of σ for the metric of the form (7.1) yields σ = 0, which confirms directly that these metrics correspond to algebraically special space-times.

(The shear is defined in terms of an affinely parametrized ray congruence k^μ to be:[34]

$$|\sigma| = \left(k_{(\mu\|\nu)} k^{\mu\|\nu} - 2\theta^2\right)^{1/2} / \sqrt{2} \qquad (9.11)$$).

Using (8.2), (9.1) and (9.4) to calculate $\alpha_{|0}$ leads to an important relation for α :-

$$\alpha_{|\nu} k^\nu = \left(\alpha^2 - \beta^2\right) \qquad (9.12)$$

Similarly, using (8.2) and (9.2) and calculating $\beta_{|0}$ and $\beta_{|p} k_p$ leads to an analogous relation for β :-

$$\beta_{|\nu} k^{\nu} = 2 \alpha \beta \tag{9.13}$$

Introducing the variable

$$\gamma = \alpha + i \beta \tag{9.14}$$

(9.12) and (9.13) combine to the single equation:

$$\gamma_{|\nu} k^{\nu} = \gamma^2 \tag{9.15}$$

The complex variable γ plays a fundamental role in the following discussion, and (9.15) will be used frequently.

One of major objectives in the subsequent development will be to derive differential relations involving k_i and γ . However, there is an alternative method of seeking a solution, which directly explores the integrability conditions of (8.21). We will not discuss this method in detail, since it forces us to abandon a symmetric treatment for the spatial coordinate variables, but in Appendix 1 we give a brief discussion of the approach for the stationary case when $k_{i|0} = 0$.

To develop differential relations, the 3-gradients of α and β also prove useful. To obtain, these, we use (8.21) to form $k_{i|j|j}$, and we also form $\varepsilon_{lps} (\varepsilon_{ijp} k_{i|j})_{|s}$ using (9.5), to give us a second equation involving $k_{i|j|j}$. Eliminating $k_{i|j|j}$ between these equations, using (9.1) and applying (8.21) to evaluate $k_{i|0}\, k_{j|i}$ we find the form:

$$\alpha_{l\ell} = \alpha_{l0} k_\ell + (\beta^2 - \alpha^2) k_\ell + 2\beta\, \varepsilon_{\ell i p} k_p k_{i|0} + \varepsilon_{\ell p j} \beta_{lj} k_p \tag{9.16}$$

Forming $\varepsilon_{t\ell n} k_n \alpha_{l\ell}$ from (9.16) and using (9.15) then leads to:

$$\beta_{l\ell} = \beta_{l0} k_\ell - 2\alpha\beta k_\ell + 2\beta k_{\ell|0} - \varepsilon_{\ell p j} \alpha_{lj} k_p \tag{9.17}$$

Equations (9.16) and (9.17) cannot at once be combined to a simple equation involving only γ , and we will later show that we must introduce a slightly different γ to achieve such an equation for the gradient. For the moment, we simply note that it is only the terms of (9.16) and (9.17) involving $k_{i|0}$ that fail to combine to a simple form in γ , and thus an analysis of the stationary case should be possible without redefinition of this complex scalar. This is precisely the path that we followed in treating the stationary case in Reference 27.

10. Reduction of the second order field equations.

We now turn our attention to the manipulation of the second order field equations (7.23a). We look to these equations to provide the information we need to resolve the indeterminacy of (8.21) and (8.22), thus we expect that some relation between the time and space derivatives of k_i should be provided from the second order equations. Using $l_\mu = l_0 k_\mu$, equations (7.23a) become:

$$-(l_0^2)_{|\alpha}{}^{|\alpha} = 2(Gl_0)_{|0} \tag{10.1a}$$

$$-(l_0^2 k_i^2)_{|\alpha}{}^{|\alpha} = 2(Gl_0 k_i)_{|i} \tag{10.1b}$$

(no sum on i)

If we add the three equations of (10.1b) and subtract (10.1a) we have after using (9.1):

$$(Gl_0)_{|\mu} k^\mu = (Gl_0) 2\alpha \tag{10.2}$$

Consistent with our comment following equation (8.11), we may now assume that $\alpha \neq 0$. Comparing (10.2) and (9.13), it is natural to take

$$Gl_0 = D\beta \tag{10.3}$$

which at once leads to the relation:

$$D_{|\mu} k^\mu = \left(\frac{Gl_0}{\beta}\right)_{|\mu} k^\mu = 0 \tag{10.4}$$

If we now take (7.27b) and sum over j, we have:

$$k_{i|0} k_{i|0} - k_{i|j} k_{i|j} = -\frac{2G\alpha}{l_0} \tag{10.5}$$

Using (9.3) now at once gives

$$G = \frac{\ell_0}{\alpha}\left(\alpha^2+\beta^2\right) \tag{10.6}$$

Now using (10.4) and (10.6) together with (9.12) and (9.13) gives:

$$\left(\ell_0^2/\alpha\right)_{|\mu} k^\mu = 0 \tag{10.7}$$

This suggests that we should write, defining W :

$$W\alpha = \ell_0^2 \tag{10.8}$$

and then

$$W_{|\mu} k^\mu = 0 \tag{10.9}$$

where W is a real scalar variable.

Making use of (10.6) and (10.8), the field equations (7.19) take on the form:

$$\begin{aligned}\left(W\alpha k_\mu k_\nu\right)_{|\rho}{}^{|\rho} &= \left[W\left(\alpha^2+\beta^2\right)k_\mu\right]_{|\nu} \\ &\quad + \left[W\left(\alpha^2+\beta^2\right)k_\nu\right]_{|\mu}\end{aligned} \tag{10.10}$$

Taking $\mu =$ 0 then leads to the four equations

$$-\left(W\alpha\right)_{|\rho}{}^{|\rho} = 2\left[W\left(\alpha^2+\beta^2\right)\right]_{|0} \tag{10.11a}$$

$$-\left(W\alpha k_i\right)_{|\rho}{}^{|\rho} = \left[W\left(\alpha^2+\beta^2\right)\right]_{|i} + \left[W\left(\alpha^2+\beta^2\right)k_i\right]_{|0} \tag{10.11b}$$

Using (7.28), it is then straightforward to show that (10.1) and (10.11) can each be obtained from the other and we will therefore work with (10.11) since they have the advantage of being linear in k_i . Subtracting (10.11a) times k_i from (10.11b) then gives us the equations:

$$W\alpha k_{i|\rho}{}^{|\rho} + 2W_{|\rho} k_i{}^{|\rho}\alpha + 2W\alpha_{|\rho} k_i{}^{|\rho}$$
$$= \left[W(\alpha^2+\beta^2)\right]_{|0} k_i - \left[W(\alpha^2+\beta^2)\right]_{|i}$$
$$- W(\alpha^2+\beta^2) k_{i|0} \tag{10.12}$$

We now evaluate each term of the left hand side. Using (8.21), (9.12), (9.5) and (9.17) we find:

$$\alpha_{|\rho} k_i{}^{|\rho} = -\tfrac{1}{2}(\alpha^2+\beta^2)_{|i} + \tfrac{1}{2} k_i(\alpha^2+\beta^2)_{|0}$$
$$+ k_{i|0}(\alpha^2+\beta^2) - k_i\alpha(\alpha^2+\beta^2) \tag{10.13}$$

Similarly, using (10.9) leads at once to:

$$W_{|\rho} k_i{}^{|\rho} = -\alpha W_{|i} + \alpha k_i W_{|0} - \beta \varepsilon_{ijl} W_{|j} k_l \tag{10.14}$$

and finally by differentiating (8.21), and then using (9.12), (9.5) and (9.16), we find:

$$k_{i|\rho}{}^{|\rho} = 2(\alpha^2+\beta^2) k_i \tag{10.15}$$

Using (10.13), (10.14) and (10.15) in (10.12) at once gives:

$$(\beta^2-\alpha^2)(W_{|0} k_i - W_{|i}) + 2\alpha\beta\varepsilon_{ijl} W_{|j} k_l \tag{10.16}$$
$$= 3W(\alpha^2+\beta^2) k_{i|0}$$

or, putting $W = P^{-3}$,

$$(\beta^2 - \alpha^2)(P_{|0} k_i - P_{|i}) + 2\alpha\beta\, \varepsilon_{ijl} P_{|j} k_l$$

$$= -P(\alpha^2 + \beta^2) k_{i|0} \qquad (10.17)$$

These are three first order linear equations in P. We will use solutions of these equations to provide the additional information needed in (8.22) to determine $k_{i|j}$ in terms of k_i .

In particular, we note that if P is a function of k_μ alone, and has no explicit dependence on the coordinates, then (10.9) is automatically satisfied since

$$W_{|\mu} k^\mu = \frac{dW}{dP} \frac{\partial P}{\partial k_i} \left(k_{i|\mu} k^\mu \right) = 0 \qquad (10.18)$$

With the assumption that $P = P(k_\mu)$, we can simplify (10.17) using (8.21) to the form:

$$\frac{\partial P}{\partial k_r} \left[k_{i|r} - k_{i|0} k_r \right] = -P k_{i|0} \qquad (10.19)$$

Consider now the possible forms for P. It is to be a real scalar function, depending only on k_μ . We cannot construct a scalar by contracting k_μ on itself, since k_μ is null, and if we contract k_μ with any other 4-vector, the latter must be independent of position. This strongly suggests that P must be a function in which k_μ is contracted with a set of constant 4-vectors, thus:

$$P = P\left(a^{(1)\mu} k_\mu ,\ a^{(2)\mu} k_\mu , \ldots\ldots \right) \qquad (10.20)$$

where $a^{(n)\mu}$ is a vector with constant components. Substitution of any such form as (10.20) into (10.19) will lead to a relation between $k_{i|r}$ and $k_{i|0}$. We now seek the simplest such relation, by supposing that P can be written as a Laurent expansion in terms of a single scalar $a^{\mu}k_{\mu}$, thus:

$$P = \sum_{n=-\infty}^{\infty} (a^{\mu}k_{\mu})^{n} \tag{10.21}$$

Each term in the expansion, used in (10.19), leads to the form:

$$-n\left(a^{r}k_{i|r} - a^{r}k_{r}k_{i|0}\right) = \left(a^{0} + k_{r}a^{r}\right)k_{i|0} \tag{10.22}$$

Thus any value of n gives a relation between $k_{i|r}$ and $k_{i|0}$, but we see from (10.22) that if and only if n = +1, we obtain a relation that does not involve k_r itself. Taking n = +1 gives the very simple result:

$$-a^{r}k_{i|r} = a^{0}k_{i|0} \tag{10.23}$$

or

$$a^{\rho}k_{i|\rho} = 0 \tag{10.24}$$

We then have for P and W :

$$P = a^{\mu}k_{\mu} \tag{10.25}$$

$$W = \left(a^{\mu}k_{\mu}\right)^{-3} \tag{10.26}$$

Since P is real, it follows that a^{μ} also has real components.

We will shortly return to consider (8.21) for the particular form of P given by (10.26), but before doing so we make one important observation. In order for a space to admit a Killing vector a^{μ} with <u>constant</u> components, we must have :-

(10.27)

For the metric we are using, from (10.8) we see that (10.27) will be satisfied if:

$$(W\alpha)_{|\rho}\, a^{\rho} = 0 \tag{10.28a}$$

$$(W\alpha k_i)_{|\rho}\, a^{\rho} = 0 \tag{10.28b}$$

Using (9.1), (9.2) and (10.23) we readily derive the following:

$$W_{|\rho}\, a^{\rho} = 0 \tag{10.29a}$$

$$\gamma_{|\rho}\, a^{\rho} = 0 \tag{10.29b}$$

Thus (10.28a) is at once seen to be satisfied, and using this and (10.23) in (10.28b) we see that (10.28b) is satisfied.

This implies that a^{μ} is a Killing vector of both $g_{\mu\nu}$ and of the background metric $\eta_{\mu\nu}$, a fact that we will use later to simplify the final equations for γ .

11. Defining differential relations for the function γ.

We are now in a position to establish the basic differential equations that the complex scalar γ must satisfy. Using (10.23) in (8.21) gives us:

$$a^{0} k_{i|j} + k_{i|t}\, a^{t} k_{j} = a^{0}\left[\alpha\left(\delta_{ij} - k_i k_j\right) + \beta\, \varepsilon_{ijl} k_l\right] \tag{11.1}$$

Assuming for the moment that $a^{0} \neq 0$, an assumption that we will examine later, equations (11.1) become:

$$k_{i|t}\left(a^{0}\delta^{t}{}_{j} + a^{t} k_{j}\right) = a^{0}\alpha\left(\delta_{ij} - k_i k_j\right) + a^{0}\beta\, \varepsilon_{ijl} k_l \tag{11.2}$$

The solution of (11.2) for $k_{i|r}$ is simple, since the inverse of the matrix $(a^{0}\delta^{t}{}_{j} + a^{t}k_{j})$ is just $(\delta^{t}{}_{j} - a^{t}k_{j}/P)/a^{0}$, readily verified by direct multiplication.

Thus (11.2) is equivalent to:

$$k_{i|j} = \left[\alpha\left(\delta_{it} - k_i k_t\right) + \beta\, \varepsilon_{itl} k_l\right] \times \left[\delta^{t}{}_{j} - a^{t} k_{j}/P\right] \tag{11.3}$$

As we remarked following (9.17), the scalars α and β have the disadvantage that their gradients do not conveniently combine to a form in the complex scalar γ. We therefore find it convenient to introduce at this point new scalars $\bar{\alpha}$ and $\bar{\beta}$, which we will show have suitable gradients that combine to a form in $\bar{\gamma} = \bar{\alpha} + i\bar{\beta}$. We define the new variables by:

$$\begin{aligned} \alpha &= P\bar{\alpha} \\ \beta &= P\bar{\beta} \end{aligned} \tag{11.4}$$

We will also define a lower index form of a^μ by the relation

$$\tilde{a}_\mu = a^\nu \eta_{\mu\nu} \tag{11.5}$$

We should note here that indices on a^μ cannot be raised and lowered in general using $\eta_{\mu\nu}$. However, we will later find that all the results we derive have exactly the form they would have were we to use $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$ to raise and lower the indices on a^μ . As a practical point, <u>all</u> raising and lowering operations with the metric of the form (7.1) seem to be accomplished just as well with $\eta_{\mu\nu}$ as with $g_{\mu\nu}$, which again emphasizes the very special form of this metric.

Using (11.3), (11.4) and (11.5) then gives us:

$$\begin{aligned} k_{i|j} = {} & P(\bar{\alpha}\delta_{ij} + \bar{\beta}\varepsilon_{ijl}k_l) \\ & + \bar{\alpha}(\tilde{a}_i - k_i a^0)k_j + \bar{\beta}\varepsilon_{irl}\tilde{a}_r k_l k_j \end{aligned} \tag{11.6}$$

In reaching equations (11.6), we assumed that $a^0 \neq 0$. However, if $a^0 = 0$, direct calculation from (10.23) and (11.2) gives us exactly the same form as (11.6), as is shown in Appendix 2. Thus we can proceed to use (11.6) in all cases.

Equation (11.6) serves as a replacement for equation (8.21), and will allow us to obtain governing equations for $\bar{\gamma}$. We will make heavy use of it in what follows.

Using (9.1), we now consider:

$$2(P\bar{\alpha})_{|\ell} = k_{i|i|\ell} \tag{11.7}$$

After a rather lengthy calculation, given in Appendix 3, we derive the result:

$$\begin{aligned} \vec{\nabla}\bar{\alpha} = & \left(\bar{\alpha}_{|0} + a^{0}(\bar{\beta}^{2}-\bar{\alpha}^{2})\right)\vec{k} + (\bar{\alpha}^{2}-\bar{\beta}^{2})\vec{a} \\ & + 2\bar{\alpha}\bar{\beta}\,(\vec{a}\times\vec{k}) + (\vec{k}\times\vec{\nabla}\bar{\beta}) \end{aligned} \tag{11.8}$$

where we have defined $\vec{a} = \tilde{a}_i$.

Forming $\vec{\nabla}\bar{\alpha} \times \vec{k}$, we at once find:

$$\begin{aligned} \vec{\nabla}\bar{\beta} = & (\bar{\beta}_{|0} - a^{0}.\,2\bar{\alpha}\bar{\beta})\vec{k} + 2\bar{\alpha}\bar{\beta}\,\vec{a} \\ & -(\bar{\alpha}^{2}-\bar{\beta}^{2})(\vec{a}\times\vec{k}) + \vec{\nabla}\bar{\alpha}\times\vec{k} \end{aligned} \tag{11.9}$$

Using (9.12) and (9.13), the last two equations can be written:

$$\begin{aligned} \vec{\nabla}\bar{\gamma} = & (\bar{\gamma}_{|0} - a^{0}\bar{\gamma}^{2})\vec{k} + \bar{\gamma}^{2}\vec{a} \\ & + i\,(\vec{\nabla}\bar{\gamma}\times\vec{k}) - i\,\bar{\gamma}^{2}(\vec{a}\times\vec{k}) \end{aligned} \tag{11.10}$$

The above very useful result is the anticipated replacement for (9.16) and (9.17), and takes on a somewhat simpler form in terms of the new variable

$$\bar{\omega} = 1/\bar{\gamma} \tag{11.11}$$

thus:

$$\begin{aligned} \vec{\nabla}\bar{\omega} &= \left(\bar{\omega}_{|0} + a^{0}\right)\vec{k} - \vec{a} \\ &\quad + i\left(\vec{\nabla}\bar{\omega} \times \vec{k}\right) + i\left(\vec{a} \times \vec{k}\right) \end{aligned} \tag{11.12}$$

Now using the relations

$$\vec{\nabla}\bar{\omega} \cdot \vec{a} = \bar{\omega}_{|0}\, a^{0} \tag{11.13}$$

$$\vec{\nabla}\bar{\omega} \cdot \vec{k} = \bar{\omega}_{|0} + P \tag{11.14}$$

which are derived easily from (9.15) and (10.29), we find:

$$(\Box\bar{\omega})^{2} \equiv \bar{\omega}_{|\nu}\,\bar{\omega}^{|\nu} = -a^{\mu}a^{\nu}\eta_{\mu\nu} \tag{11.15}$$

The second defining relation satisfied by $\bar{\gamma}$ and $\bar{\omega}$ is found by taking the divergence of (11.10). Again, after a rather lengthy calculation given in Appendix 4 we find the simple result

$$\Box^{2}\bar{\gamma} \equiv \bar{\gamma}_{|\nu}{}^{|\nu} = 0 \tag{11.16}$$

or in terms of $\bar{\omega}$

$$\bar{\omega}\,\Box^{2}\bar{\omega} = -2\left(a^{\mu}a^{\nu}\eta_{\mu\nu}\right) \tag{11.17}$$

Equations (11.15) and (11.17) are the two fundamental equations satisfied by our complex scalar generating function. Before considering the equations further, we will confirm that $\bar{\omega}$ and a^μ in fact determine the metric completely. From the form of the metric, and from equations (10.8) and (10.26), it is clear that a^μ and k^μ determine the metric completely, thus it is sufficient to show that $\bar{\omega}$ and a^μ completely determine k^μ.

Writing (11.11) in the form:

$$(\nabla\bar{\omega} + \vec{a}) = (\bar{\omega}_{,0} + a^0)\vec{k} + i(\nabla\bar{\omega} + \vec{a}) \times \vec{k} \tag{11.18}$$

and now writing

$$\vec{Q} = \nabla\bar{\omega} + \vec{a} \tag{11.19a}$$

$$S = \bar{\omega}_{,0} + a^0 \tag{11.19b}$$

we readily verify that

$$\vec{Q} \times \vec{Q}^* = -i(S^*\vec{Q} + S\vec{Q}^*) + B\vec{k} \tag{11.20}$$

where B is a function of $\vec{k}$ and $\bar{\omega}$. To find B, we use

$$0 = (\vec{Q} + \vec{Q}^*).(\vec{Q} \times \vec{Q}^*) =$$
$$-i(S^*\vec{Q} + S\vec{Q}^*).(\vec{Q} + \vec{Q}^*) + B(\vec{Q} + \vec{Q}^*).\vec{k} \tag{11.21}$$

From (11.18) we find that

$$(\vec{Q} + \vec{Q}^*).\vec{k} = S + S^* \tag{11.22}$$

thus we have at once that:

$$B = i\left[(S^*\vec{Q}.\vec{Q} + S\vec{Q}^*.\vec{Q}^*) + \vec{Q}.\vec{Q}^*(S+S^*)\right] \div (S+S^*) \tag{11.23}$$

and thus finally

$$\vec{k} = \left[\vec{Q} \times \vec{Q}^* + i\left(S\vec{Q}^* + S^*\vec{Q}\right)\right] / B \qquad (11.24)$$

Since B, S and Q depend only on $\bar{\omega}$ and a^μ, we have confirmed that (11.24) determines k^μ uniquely in terms of $\bar{\omega}$ and a^μ.

12. Use of the Killing vectors and discussion of solutions.

The equations for the complex generating function $\bar{\gamma}$ can be simplified by use of the Killing vector a^μ. There are only three cases to consider:

a) If a^μ is timelike, then a Lorentz transformation must exist that takes it to the form (1,0,0,0). In this case, equations (10.26), (11.15) and (11.17) reduce to the forms:

$$W = 1 \tag{12.1}$$

$$(\vec{\nabla}\bar{\omega})^2 = 1 \tag{12.2}$$

$$\bar{\omega}\,\nabla^2\bar{\omega} = 2 \tag{12.3}$$

Since there is now no dependence on the time, we would expect to obtain this case directly from (8.21) with $k_{i|0}$ set equal to zero. As we remarked earlier, in this case the first order equations are then explicitly soluble without using the second order field equations of (7.23a). This approach has been carried through in detail[27] and does in fact yield the exact equations (12.2) and (12.3). Note that, since $P = 1$, the redefinition of the variable γ in this case is the identity transformation, and hence $\bar{\gamma}$ is the same as γ.

The most important known solutions of the stationary vacuum case are now obtained very easily. First, let us note that the general solution of (12.2) can be written using the theory of envelopes[47] in the form:

$$\bar{\omega} = A_i x_i + b(A_i) \tag{12.4}$$

where we define:

$$A_i = \left(x_i + \partial b/\partial A_i\right) . N \qquad (12.5)$$

where $b(A_i)$ is any function of the A_i , and where $A_i . A_i = 1$.

The simplest possible choice we can make for $b(A_i)$ is to set it equal to zero. Then we have

$$A_i = x_i / r \qquad (12.6)$$

where $r^2 = x_i . x_i$. This gives us $\bar{\omega} = r$, so $\alpha = r^{-1}$ and thus

$$\ell_0^{\,2} = W\alpha = r^{-1} \qquad (12.7)$$

We now calculate k_i immediately from (11.18), using $a^0 = 1$. (Normally we would use (11.23) and (11.24). However, when $\bar{\omega}$ is real, we can see at once from (11.18) that we must have

$$\vec{\nabla}\bar{\omega} + \vec{a} = \left(\bar{\omega}_{|0} + a^0\right)\vec{k} \qquad (12.8)$$

and thus in this case

$$\vec{k} = \vec{\nabla}\bar{\omega} = \vec{r}/r \qquad (12.9)\ )$$

From (12.7) and (12.9) we then have

$$\ell_\mu = \left(1/r\right)^{1/2}\left(1\ ,\ x/r\ ,\ y/r\ ,\ z/r\right) \qquad (12.10)$$

and comparing this with (7.2) we see at once that we have produced the Schwarzschild solution with the Eddington form for the metric tensor.

By direct substitution, we also readily verify that the choice we have made for $\bar{\omega}$ in this case also satisfies the second equation (12.3).

A second simple choice for $b(A_i)$ is to make it depend on just one component of A, thus:

$$b(A_i) = K A_3 \tag{12.11}$$

where K is any constant. Using (12.5) then gives us

$$\begin{aligned} A_1 &= x/N , \qquad A_2 = y/N \\ A_3 &= (z+K)/N \\ N &= \left(x^2 + y^2 + (z+K)^2\right)^{1/2} \end{aligned} \tag{12.12}$$

Thus using (12.4) we find

$$\omega = \bar{\omega} = \left(x^2 + y^2 + (z+K)^2\right)^{1/2} \tag{12.13}$$

and again we readily verify that this choice satisfies (12.3) also.

If K is a real quantity, then the solution corresponds to the Schwarzschild case, in which we have merely displaced the origin of coordinates along the z-axis. However, if K is imaginary, then use of (11.23) and (11.24) gives us a new result. Writing $K = ia$, we find:

$$\left.\begin{aligned} k_1 &= (\rho x + a y)/(a^2 + \rho^2) \\ k_2 &= (\rho y - a x)/(a^2 + \rho^2) \\ k_3 &= z/\rho \end{aligned}\right\} \tag{12.14}$$

where ρ is the real part of ω and is a solution of the equation[18]:

$$\rho^4 - \rho^2(r^2 - a^2) - a^2 z^2 = 0 \tag{12.15}$$

Similarly, we find

$$\ell_o^2 = \rho^3 / (\rho^4 + a^2 z^2) \tag{12.16}$$

and using (12.14) and (12.16) in (7.1), we find the form for the line element to be :

$$\begin{aligned} ds^2 = \; & dt^2 - dx^2 - dy^2 - dz^2 - \frac{2m\rho^3}{\rho^4 + a^2 z^2} \Big[dt \\ & + (x\,dx + y\,dy)\rho / (a^2 + \rho^2) + z\,dz/\rho \\ & + (y\,dx - x\,dy) a / (a^2 + \rho^2) \Big]^2 \end{aligned} \tag{12.17}$$

This is exactly the form for the line element that was given in Kerr's original paper[18]. A great deal of work has subsequently been done on both the geometry and the implied physics of this solution, and it would be inappropriate to reproduce much of that here. We merely remark that the form (12.17) is not the most revealing form for the line element, and for future reference we will quote an alternate form derived by Boyer and Lindquist[48], in which the axial symmetry of the solution is quite explicit:

$$\begin{aligned} ds^2 = \; & dt^2 - (r^2 + a^2)\sin^2\theta \, d\phi^2 - \frac{2mr(dt + a\sin^2\theta \, d\phi)^2}{(r^2 + a^2\cos^2\theta)} \\ & - (r^2 + a^2\cos^2\theta)\left(d\theta^2 + dr^2/(r^2 - 2mr + a^2)\right) \end{aligned} \tag{12.18}$$

This form is obtained from the Kerr form by transformations of the time and the azimuthal angle variables(48,49,50) and we will find it very useful when we consider interior solutions in Chapter 14.

As is readily seen from (12.5), most choices that can be made for $b(A_i)$ lead to nonlinear equations from which A_i must be determined, and to date no other solutions of physical interest have been found for the case with timelike Killing vector.

b) If a^μ is <u>spacelike</u>, then a Lorentz transformation exists that takes it to the form (0,0,0,1). In this case, equations (10.26), (11.15) and (11.17) reduce to the forms:

$$W = 1/k_3^3 \tag{12.19}$$

$$\left(\frac{\partial\bar{\omega}}{\partial t}\right)^2 - \left(\frac{\partial\bar{\omega}}{\partial x}\right)^2 - \left(\frac{\partial\bar{\omega}}{\partial y}\right)^2 = 1 \tag{12.20}$$

$$\bar{\omega}\left[\frac{\partial^2\bar{\omega}}{\partial t^2} - \frac{\partial^2\bar{\omega}}{\partial x^2} - \frac{\partial^2\bar{\omega}}{\partial y^2}\right] = 2 \tag{12.21}$$

This case has no dependence on z and thus represents a 2-dimensional problem spatially. Solutions of this form are not candidates for the representation of gravitating bodies, though they are appropriate to certain cases of gravitational radiation, and cylindrical radiation has been extensively studied[51].

c) If a^μ is <u>lightlike</u>, then a Lorentz transformation exists that takes a^μ to the form (1,0,0,1). We then have, from (10.26), (11.15) and (11.17), the equations:

$$W = 1/(1+k_3)^3 \tag{12.22}$$

$$\left(\frac{\partial\bar{\omega}}{\partial x}\right)^2 + \left(\frac{\partial\bar{\omega}}{\partial y}\right)^2 = 0 \tag{12.23}$$

$$\frac{\partial^2\bar{\omega}}{\partial x^2} + \frac{\partial^2\bar{\omega}}{\partial y^2} = 0 \tag{12.24}$$

where
$$\bar{\omega} = \bar{\omega}(x, y, t-z) \tag{12.25}$$

From (12.23) and (12.24), $\bar{\omega}$ must be a function of (x + iy) or (x - iy). Using this in (11.23) and (11.24) permits an explicit calculation of k_i, and leads to:

$$k_1 = k_2 = 0 ; \qquad k_3 = -1 \tag{12.26}$$

However, this means that W as defined by (12.22) becomes indeterminate. Thus we are led to conclude that there are no solutions of this form with a lightlike Killing vector.

This is a surprising result, since previous papers by Kerr and Schild[41] and by Debney, Kerr and Schild[43] give classes of solutions for metrics of the form (7.1), and these solutions include cases with lightlike Killing vectors. At first sight, the results given in these references are quite unrelated to the governing equations given here. However, one can show (see Appendix 5) that the basic generating function F_Y used by Debney, Kerr and Schild in fact satisfies equations (11.15) and (11.17), although their Killing vector is written in a rather different form. To within a constant multiplying factor, one can then exactly identify the variable $\bar{\omega}$ with F_Y, and thus the choice of the form (10.26) for W leads to all the vacuum solutions derived in References 41 and 43.

A closer inspection of the solutions with lightlike Killing vector set forth in these references actually reveals that no non-trivial solutions exist when $\alpha \neq 0$. A recent paper by Debney[52] establishes the general result that <u>any</u> spacetime with a lightlike Killing vector must have $\alpha = 0$, thus there are, as we found, no solutions of (12.22)-(12.25).

However, the formulation given here leads to a Killing vector only for the particular choice of W given by (10.26). It is not clear that there are no other possible choices that lead to vacuum solutions with a metric of the form (7.1) and non-zero complex expansion. For example, consider the form:

$$W = W(x^\mu k_\mu) \tag{12.27}$$

This depends explicitly on the coordinates rather than on k_μ alone, but it is readily shown to satisfy (10.9) and (10.16). It leads to equations similar to (11.1), but with the 4-vector x^μ replacing a^μ, thus:

$$\begin{aligned} &x^0 k_{i|j} + k_{i|r} x^r k_j = \\ &\quad x^0 \left[\alpha (\delta_{ij} - k_i k_j) + \beta \varepsilon_{ijl} k_l \right] \end{aligned} \tag{12.28}$$

These equations require further study.

13. The improper case and the case of vanishing expansion.

We now return to consider the cases where we have improper U, vanishing p or vanishing θ , cases which we explicitly excluded in our development following equation (8.11). The cases of zero complex expansion have been considered from the point of view of gravitational radiation theory in a paper by Kundt[53].

Case I - $\theta = 0$.

Referring to (8.12), we see that M' will have the form:

$$M' = p\begin{pmatrix} 0 & 0 & 0 \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix} \tag{13.1}$$

Rotating back to the original coordinate system, exactly as in Chapter 8, and noting the first row of the matrix R must be the vector $\vec{k}$, since R transforms $\vec{k}$ to the form (1,0,0) , we have at once the result analogous to (8.21) :

$$k_{i|j} = k_{i|0}\, k_j \tag{13.2}$$

Since $\alpha = \beta = 0$, we then have at once that

$$k_{i|i} = 0 \tag{13.3}$$

Further, since the twist is zero, we know that the ray congruence k_μ must be hypersurface orthogonal, and thus that we can write k_μ in the form:

$$k_\mu = A\,\phi_{|\mu} \tag{13.4}$$

where ϕ is a scalar field and λ is a scalar field (see Reference 34, pages 334 et seq.).

Using (13.3) and their definitions (7.14) and (7.16), we readily find:

$$L \quad = \quad A \quad = \quad -\ell_{0|\nu}\, k^{\nu} \tag{13.5}$$

and thus, from (7.18), if we define $S = G\ell_0$, we have:

$$S \quad = \quad -\ell_0^2{}_{|\nu}\, k^{\nu} \tag{13.6}$$

By the use of (13.2), and changing the order of derivative operators, we find the important result:

$$k_{i|\alpha}{}^{|\alpha} \quad = \quad 0 \tag{13.7}$$

so that k_i satisfies the wave equation in the background space.

Further, from (10.2), which is directly applicable, we have:

$$S_{|\mu}\, k^{\mu} \;=\; 0 \tag{13.8}$$

and the second order field equations for $\nu = 0$ can therefore be written:

$$-\left(\ell_0^2\right)_{|\alpha}{}^{|\alpha} \quad = \quad 2S_{|0} \tag{13.9}$$

$$-\left(\ell_0^2 k_i\right)_{|\alpha}{}^{|\alpha} \;=\; \left(S k_i\right)_{|0} + S_{|i} \tag{13.10}$$

Expanding the left hand side of (13.10), and subtracting k_i times (13.9) from it, we find with the use of (13.7) the important result:

$$S k_{i|0} \quad = \quad S_{|i} \;-\; S_{|0}\, k_i \tag{13.11}$$

which we can write $\quad (S k_i)_{|0} \;=\; S_{|i} \qquad (13.12)$

Thus, forming $S_{|i|i}$ and using (13.3), we have:

$$S_{|i|i} = S_{|0|0} \tag{13.13}$$

so that S also satisfies the wave equation in the background space.

Now, since $k_0 = 1$, from (13.4) we must have:

$$A \phi_{|0} = 1 \tag{13.14}$$

and so

$$k_i = \phi_{|i} / \phi_{|0} \tag{13.15}$$

Since k is a unit vector, (13.15) at once gives:

$$\phi_{|\alpha} \phi^{|\alpha} = 0 \tag{13.16}$$

Using (13.15) in (13.12) gives us

$$\left(S \phi_{|i} / \phi_{|0} \right)_{|0} = S_{|i} \tag{13.17}$$

and we will therefore satisfy (13.17) at once if we choose:

$$S = \phi_{|0} \tag{13.18}$$

Finally, to determine ℓ_0, we note that (13.9) now takes the form

$$-\left(\ell_0^2\right)_{|\alpha}{}^{|\alpha} = 2 \phi_{|0|0} \tag{13.19}$$

The solutions of this case are therefore characterized by:

a) the scalar function $\phi_{|0}$ satisfies the wave equation (13.13) and ϕ the eikonal-type equation (13.16)

b) ℓ_0^2 satisfies the wave equation (13.19) with a source function $\phi_{|0|0}$

c) since ϕ determines $\vec{k}$ by (13.15) and ℓ_0^2 by (13.19), the whole metric is defined by the scalar function ϕ

d) since the solutions are hypersurface orthogonal, solutions with rotation, such as the Kerr solution, cannot occur. However, solutions with wave properties are not excluded and are in fact strongly suggested by the form of the governing equations. As one would expect in view of the nonlinear nature of the field equations, solutions have a nonlinear constraint given by (13.16).

<u>Case II</u> - $p = 0$.

This case is similar to Case I so far as the treatment of the first order field equations is concerned, and equations (13.2)-(13.7) still hold, except that we now have the added constraint:

$$S = L = A = 0 \tag{13.20}$$

As before, (13.4) leads to (13.15), and (13.16) again follows from it. However, since we now have (13.20) satisfied, we have the constraint

$$\ell_0^2{}_{|\nu} k^\nu = 0 \tag{13.21}$$

and (13.19) reduces to the homogeneous form:

$$\left(\ell_0^2\right)_{|\alpha}{}^{|\alpha} = 0 \tag{13.22}$$

Thus the relevant features of the governing equations in this case may be summarized as:

a) the scalar function ϕ satisfies the eikonal equation (13.15)

b) the vector k_i satisfies the free space wave equation in the background space

c) the scalar ℓ_0^2 satisfies the free space wave equation in the background space

d) the vector k_i is defined by (13.15)

e) the vector k_i and the scalar ϕ are constrined by the relation (13.21).

<u>Case III</u> - U is improper.

We now explore the case where U is chosen to have negative determinant, which means that (8.11) can be written in the form:

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \tag{13.23}$$

Thus, using (8.10), M' can be written as:

$$M' = p \begin{pmatrix} 0 & 0 & 0 \\ u & 1-\cos\theta & \sin\theta \\ v & \sin\theta & 1+\cos\theta \end{pmatrix} \tag{13.24}$$

We find it convenient to split M' into two parts, a symmetric part from the lower 2x2 matrix, and a part depending only on u and v, thus:

$$M' = p \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\cos\theta & \sin\theta \\ 0 & \sin\theta & 1+\cos\theta \end{pmatrix} + p \begin{pmatrix} 0 & 0 & 0 \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix} \tag{13.25}$$

In rotating back to the original coordinate system, as in (8.13), we note that the second term of (13.25) is exactly as in the case of proper U, and thus we use that analysis to show that we must have the second term as $k_{i/0}\, k_j$. For the first term of (13.25), we note that it is symmetric, and thus after rotation remains symmetric. Further, it is of rank 1 and thus remains of rank 1 after rotation. Finally, the trace of the matrix is not changed by the rotation, so the matrix after rotation has trace 2p. Noting that there exists a unique dyad decomposition of any symmetric matrix into the form:

$$M_{ij} = \sum_{n} \lambda_n e_i^{(n)} e_j^{(n)} \tag{13.26}$$

where λ_n are the eigenvalues of M and $e_i^{(n)}$ are the eigenvectors, we have in this case only one non-zero eigenvalue and thus can write the matrix M in the original coordinate system in the form:

$$M_{ij} = 2p\, e_i e_j + k_{i|0} k_j \tag{13.27}$$

Use of the form (13.27) now enables us to describe the most important features of the solutions of the improper case without fully solving the field equations. For, consider the twist β as defined by (9.2). From (13.27), we readily calculate that the twist is zero, and thus all solutions of the improper case admit a hypersurface orthogonal ray congruence. As noted in treating the case where $\theta = 0$, this means that we can write k_i in the form (13.15) where ϕ is a scalar field. Solutions of this type, with non-zero expansion but zero twist, have been studied by Robinson and Trautman[54].

In the particular case where the solutions are independent of the time, (13.27) leads at once to:

$$k_{i|j} - k_{j|i} = 0 \tag{13.28}$$

and thus k_i is the gradient of a scalar field, ψ.

From (13.27) we have

$$k_{i|i} = 2p \qquad (13.29)$$

and then from (7.14) and (7.16) we have

$$A = \ell_{0|i} k_i \qquad (13.30)$$

$$L = \ell_{0|i} k_i + 2p\ell_0 \qquad (13.31)$$

But since $(L + A) = 2p\ell_0$, from (13.30) and (13.31) we have

$$L = 2p\ell_0 \qquad (13.32)$$

$$A = 0 \qquad (13.33)$$

$$\ell_{0|i} k_i = 0 \qquad (13.34)$$

so that k_i is perpendicular to the gradient of ℓ_0.

From (13.27), we note the eigenvalue relation:

$$k_{i|j} e_j = 2p\, e_i \qquad (13.35)$$

Now consider the second order field equations (13.9) and (13.10). In the stationary case these become:

$$(\ell_0^2)_{|j|j} = 0 \qquad (13.36)$$

$$(\ell_0^2 k_i)_{|j|j} = (2p\ell_0^2)_{|i} \qquad (13.37)$$

Expanding the left hand side of (13.37) and using (13.36), we have

$$2\ell^2_{0|j} k_{i|j} + \ell_0^2 k_{i|j|j} = k_{j|j|i}\ell_0^2 + 2p\,\ell^2_{0|i} \qquad (13.38)$$

However, using the fact that k_i is the gradient of ψ, we note that $k_{i|j|j} = k_{j|j|i}$, so that (13.38) reduces to:

$$k_{i|j}\, \ell_{o|j} = p\,\ell_{o|i} \tag{13.39}$$

Comparing (13.35) and (13.39), we see that we must have

$$e_i = \ell_{o|i} \tag{13.40}$$

since M has only one non-zero eigenvalue in this case. However, this demands that either $p = 0$, or that $\ell_{o|i} = 0$.

If $p = 0$, (13.27) requires that $\vec{k}$ be a constant vector, and thus ψ is a two-dimensional harmonic function, constant in the direction of $\vec{k}$.

If $\ell_{o|i} = 0$, then ℓ_o is constant. Absorbing it into the constant term m of the metric (7.1), we see that the metric in this case must be of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} - 2m\, \psi_{|\mu}\, \psi_{|\nu} \tag{13.41}$$

where, since k_i is a unit vector, we must have ψ constrained by the eikonal equation:

$$\psi_{|i}\, \psi_{|i} = 1 \tag{13.42}$$

We note that every case analyzed in this chapter has zero twist, thus only cases with proper U and non-zero complex expansion can lead to solutions having the rotational properties of the Kerr solutions.

14. Rotating solutions in the non-vacuum case.

All the analysis and discussion to this point has dealt with the case where we are in vacuum and the governing field equations have zero matter tensor. However, it is natural to seek solutions that can apply within extended bodies of matter, analogous to the interior Schwarzschild solution in an incompressible spherical body[2]. The difficulties of solution then become formidable, mainly because much of the elegant structure of the field equations disappears when matter is present. Several approaches have been adopted in the search for "interior Kerr" solutions, that bear the same relation to the known exterior Kerr solution as does the interior Schwarzschild to the exterior Schwarzschild case. We will briefly discuss the method used in such searches, and then concentrate our attention on slow rotation, where as we shall see analytic solutions are attainable. In practice, "slow rotation" means having surface velocities small compared with the speed of light and this does not constitute a serious practical restriction for known stellar models.

To generate interior Kerr solutions, Jackson[55,56] employs a method first given by Newman and Janis as a trick for generating the exterior Kerr solution from the exterior Schwarzschild solution[57]. The essence of the method is the replacement of certain real coordinates by complex coordinates in a somewhat ad hoc manner. As Newman and Janis point out, it is not clear that such a procedure will even result in a solution of the field equations. Jackson uses a very similar method to produce solutions which are "complexified" forms of the interior Schwarzschild and the Newman, Unti and Tamburino solutions[58]. It is verified that the procedure leads to solutions that have several of the expected properties of interior Kerr solutions, but unfortunately the results must be regarded as non-physical, since the resulting matter tensor is not positive definite, with respect to timelike 4-vectors. Thus there must exist observers for whom the energy density appears negative, a most unsatisfactory situation. We have been unable to modify the prescription suggested by Newman and Janis and employed by Jackson so as to define a solution with the energy always positive, thus this line of attack does not appear to us to hold much promise for the generation of new interior solutions.

Instead, we now limit our attention to cases where the rotation rate is small. As Kerr suggested, and Boyer and Price[59] subsequently verified, the product ma in the Kerr solution corresponds to the angular momentum of a rotating body. Thus, let us consider the Kerr solution in the limit of small a, where we can neglect all powers of a higher than the first. Expanding the Boyer and Lindquist form of (12.18) in this way, we derive:

$$ds^2 = dt^2(1 - 2m/r) - dr^2(1 - 2m/r)^{-1} - r^2(d\theta^2 + r^2\sin^2\theta\, d\phi^2) - \frac{4ma}{r}\sin^2\theta\, d\phi\, dt \tag{14.1}$$

This has a very suggestive form. It is simply the usual Schwarzschild exterior solution, with one additional cross-term added to it. We thus are led to investigate a metric consisting of an <u>interior</u> Schwarzschild form, modified by a cross-term of a form corresponding to that of (14.1). Using the notation of Reference 8, Chapter 9, we write such a metric as:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2\Omega\, r^2\sin^2\theta\, d\phi\, dt \tag{14.2}$$

where we assume that λ, ν and Ω are functions of r alone. The choice of this form, and the reasons lying behind it, deserve some discussion. First, we expect that to first order in Ω, spherically symmetric configurations will remain spherical, since we certainly do not expect that the direction of rotation, involving the sign of Ω, should be relevant to the shape of the body. Second, we know that for the spherically symmetric case, the most general time independent metric can be written in the form of (14.2), without the term in Ω.[8] The functions λ and ν are functions of r alone.

Thus the metric chosen for the investigation of interior solutions has the main properties:

1) When $\Omega = 0$, the form is that of an interior Schwarzschild metric. Thus we can regard the added term of (14.2) as a perturbation to such a metric.

2) The added term chosen to represent rotation has the same form as the term which represents the exterior Kerr solution to first order in a, added to the exterior Schwarzschild metric.

3) The functions occurring in the metric, which will be determined in terms of the matter distribution, are functions of r alone, consistent with the assumption that the matter distribution remains spherical for a treatment linear in Ω.

Since the functions λ and ν depend only on r, we see that they will be determined exactly as in the usual spherically symmetric Schwarzschild case, and cannot depend on Ω, but only on the matter distribution that we assume. We then expect to have an equation that Ω must satisfy, in order to be consistent with the matter distribution and with the functions

λ and ν .

Suppose now that the matter is assumed to be a perfect fluid, so that the energy-momentum tensor has the form:(8)

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p\left(u_\alpha u_\beta - g_{\alpha\beta}\right) \tag{14.3}$$

where u_α is the 4-velocity, with index lowered using the metric of (14.2). Then the 4-velocity u^α is given by:

$$u^\alpha = (g_{00})^{-1/2}\left(1, 0, 0, \omega\right) \tag{14.4}$$

where we define $\omega = d\phi/dt$, and assume that this is small, and comparable in size with Ω . The field equations that we require for this case are then (again see Reference 8, Chapter 9):

$$R_{\alpha\beta} = -8\pi\left(T_{\alpha\beta} - g_{\alpha\beta}\, T^\mu{}_\mu/2\right) \tag{14.5}$$

The evaluation of the Riemann tensor in contracted form from (14.2) is a laborious but straightforward procedure, made somewhat easier by the fact that we can neglect quadratic or higher terms in Ω . We find that the diagonal terms of the contracted Riemann tensor are exactly those of the usual interior Schwarzschild solution, so that we can at once assume and use several of the standard formulae that apply to that problem. In particular, we have from the diagonal terms of the field equations, the following:

$$8\pi\rho = e^{-\lambda}\left(1/r^2 - \lambda'/r\right) - 1/r^2 \tag{14.6}$$

$$8\pi p = 1/r^2 - e^{-\lambda}\left(1/r^2 + \frac{\nu'}{r}\right) \tag{14.7}$$

$$8\pi p = e^{-\lambda}\Big[1/4\,\nu'\lambda' - 1/4\,\nu'^2 - 1/2\,\nu'' - 1/2\,(\nu' - \lambda')/r\Big] \tag{14.8}$$

The off-diagonal terms of the field equations (14.3) are identically satisfied to first order in Ω, except for the term R_{03}, which serves, with the assistance of (14.6) - (14.8), to provide a single scalar differential equation for Ω in terms of the pressure and density. Since all the variables are functions of r alone, this serves to confirm that the conjecture made earlier, that (14.2) represents a suitable form for an interior metric to first order in the rotation rate. The governing differential equation for Ω is:

$$e^{-\lambda}\left[-r^2/_2\,\Omega'' - 2r\Omega' - \Omega + \Omega r\lambda' + \frac{(\lambda'+\nu')}{4}\left(r^2\Omega' - 2r\Omega\right)\right]$$

$$+\ \Omega = 8\pi r^2\left[\tfrac{1}{2}\left(\rho + 3p\right)\Omega - \left(\rho + p\right)\omega\right] \qquad (14.9)$$

If we now define m(r) to be the mass of the body enclosed within the radius r from its center, then (14.9) can be written in a form that depends only on the pressure p, density ρ, and mass m, the variables λ and ν having been eliminated using the relations (14.6) - (14.8), thus:

$$\left(1 - \frac{2m(r)}{r}\right)\left(\frac{\Omega''}{2} + \frac{2\Omega'}{r}\right) - 2\pi\left(\rho + p\right)r\Omega'$$

$$= 8\pi\left(\Omega - \omega\right)\left(\rho + p\right) \qquad (14.10)$$

The procedure is then in general the following: from the assumed equation of state for the fluid, the equations of Tolman, Oppenheimer and Volkoff[60] or some other equivalent set of relativistic equations are used to solve for the configuration of the body, and hence the mass distribution m(r). Next, the equation (14.10) is solved for the variable Ω, for an assumed matter rotation ω. Finally, the boundary conditions are applied at the edge of the rotating body, to couple to an exterior solution and hence to conditions at spatial infinity. It is important to remember that the calculation of the mass distribution from the equation of state is done assuming spherical symmetry, so that a great deal of previous analysis of such problems is available, and provides inputs to the problem with rotation. When ω = constant, the entire problem can be solved in closed form for the case where the equation of state is that of any perfect gas[29,31]. For the remainder of this section, we will discuss a different case, which corresponds to the best-known form of the interior Schwarzschild solution where the fluid is assumed to be <u>incompressible</u>. Several other cases, corresponding to matter shells, have been studied by Cohen and Brill[61,62] who also point out that the function Ω is a measure of the "dragging" of the metric, i.e., a measure of the rotation of the inertial frame induced by the rotation of the body.

For the incompressible case, we have the relations[8]:

$$1 - 2m/r = \mathcal{R}^2 \tag{14.11}$$

$$p + \rho = 2\rho\,\mathcal{R}_0 / (3\mathcal{R}_0 - \mathcal{R}) \tag{14.12}$$

where we define

$$\mathcal{R}^2 = \left(1 - r^2/\hat{R}^2\right) \quad ; \quad \hat{R} = 3/8\pi\rho \tag{14.13}$$

$$\mathcal{R}_0 = \mathcal{R}(r_0) \tag{14.14}$$

Also, solving (14.10) for ω in terms of Ω, we have:

$$\omega(r) = \tfrac{1}{4} r\Omega' + \Omega - \frac{1}{8\pi}\,\frac{\left(1 - \frac{2m}{r}\right)\left(\frac{\Omega''}{2} + \frac{2\Omega'}{r}\right)}{(p+\rho)} \tag{14.15}$$

which by virtue of (14.11) and (14.12) can be written:

$$\omega(r) = \tfrac{1}{4} r\Omega' + \Omega - \tfrac{1}{8\pi} \frac{r^2(3R_0 - R)}{2\rho R_0}\left(\frac{\Omega''}{2} + 2\frac{\Omega'}{r}\right) \tag{14.16}$$

At the boundary of the body, where $r = r_0$, the metric must join in a continuous way to the exterior Kerr metric, expanded to first order in a, thus we must have from the use of (14.1) and (14.2) the relation:

$$\Omega = -2ma/r^3 \tag{14.17}$$

We define[59] the total angular momentum of the body by:

$$J = -ma = \Omega(r_0)\, r_0^3/2 \tag{14.18}$$

Finally, let us consider the boundary conditions that must be applied at $r = r_0$. In order that the rotation Ω'' should remain finite at the boundary, we require that Ω and Ω' should be continuous there. We also require, for solutions of physical interest, that ω and Ω should be finite at $r = 0$, and that $\omega(r)$ should be monotonic in the range $(0, r_0)$. (We do not want a star to exhibit a different sense of rotation at different depths within it).

These boundary conditions, taken in conjunction with (14.16), place some restrictions of the form of Ω. Thus, if Ω = constant, we see at once from the requirement that Ω' be continuous at $r = r_0$ that J and hence Ω must be zero everywhere. Also, if Ω contains a linear term in r, we find from (14.16) that ω becomes infinite at the origin. Finally, if we suppose that Ω can be expanded as a power series in r, and that there is no quadratic term, use of (14.16) tells us that we must have

$\omega(0)=\Omega(0)$, which is the case of perfect dragging, in which the inertial frame at the origin is rotating at the same speed as the star center. This provides an unreasonable limitation on the solutions, thus we will require that Ω should contain a term quadratic in r.

For practical application of this method, we would like ω to be not only monotonic in $(0,r_o)$, but slowly varying also. Thus, for a given stellar density and radius, we wish to choose a functional form for Ω , thus:

$$\Omega = c_o + \sum_{n=2}^{\infty} c_n r^n \tag{14.19}$$

in which the constants c_n are chosen so as to give an acceptable slowly varying monotonic function $\omega(r)$.

To illustrate this let us choose a fourth order function, which we write for convenience as:

$$\Omega(r) = \Omega(o)\left[1 - b\left(\frac{r}{r_o}\right)^2 - b\tau\left(\frac{r}{r_o}\right)^4\right] \tag{14.20}$$

If we match Ω and Ω' at the stellar boundary we obtain

$$b = \frac{3}{5+7\tau},\ J = \Omega(o)r_o^{\,3}\left(\frac{1+2\tau}{5+7\tau}\right) \tag{14.21}$$

that is, the two conditions determine b and J in terms of the arbitrary parameter τ; $\Omega(o)$ plays the role of a scaling factor. The expression for $\omega(r)$ becomes

$$\begin{aligned}\omega/\Omega(o) = 1 &- \frac{9}{2(5+7\tau)}\left(\frac{r}{r_o}\right)^2 - \frac{6\tau}{5+7\tau}\left(\frac{r}{r_o}\right)^4 \\ &+ \frac{\left(1-\frac{2m}{r}\right)}{\left(1+\frac{p}{e}\right)}\frac{r_o}{2m_T}\left[\frac{5}{5+7\tau} + \frac{14\tau}{5+7\tau}\left(\frac{r}{r_o}\right)^2\right]\end{aligned} \tag{14.22}$$

The ratio $r_o/2m_T$ is a convenient parameter describing the density of the star; for a typical neutron star it is about 5. In figure 14.1 we have plotted $\omega(r)/\Omega(o)$ for the cases $\tau = 0$ (no 4th order term) and $\tau = .1$. For $\tau = 0, \omega$ decreases by about 20% between $r = 0$ and $r = r_o$, while for $\tau = .1$ the decrease is only about 4%, with $\omega(r) \simeq$ const. $= 4.7\ \Omega(o)$.

The power series has been investigated also for $\ell = 0,2,4,$ and 6, and as expected the variation in ω is extremely small.[63]

Appendix 1. An alternative approach to the solution of the first order field equations.

In Chapter 9, we mentioned an alternative method of seeking solutions of the first order field equations. We develop here such a method, and confine ourselves to the stationary case, where as we remarked in Chapter 9 the first order field equations serve to determine solutions completely, and where the analysis that follows is particularly simple and elegant.

From (8.21), in the stationary case where $k_{i|0} = 0$, we have

$$k_{i|j} = \alpha(\delta_{ij} - k_i k_j) + \beta\,\varepsilon_{ijl} k_l \tag{A1.1}$$

Using this form, and calculating the quantity $\varepsilon_{ipl} k_{l|j} k_p$, we find:

$$\varepsilon_{ipl} k_{l|j} k_p = -\alpha\,\varepsilon_{ijl} k_l + \beta(\delta_{ij} - k_i k_j) \tag{A1.2}$$

This can be written concisely in terms of the complex scalar γ, thus:

$$(\delta_{il} - i\,\varepsilon_{ilp} k_p) k_{l|j} = \gamma(\delta_{ij} - k_i k_j - i\,\varepsilon_{ijl} k_l) \tag{A1.3}$$

Multiplying by dx_j, we then have:

$$(\delta_{il} - i\,\varepsilon_{ilp} k_p)\, dk_l = \gamma(\delta_{ij} - k_i k_j - i\,\varepsilon_{ijl} k_l)\, dx_j \tag{A1.4}$$

Now let us define the function F_i by:

$$F_i = \phi_i + (\delta_{ij} - k_i k_j - i\,\varepsilon_{ijl} k_l)\, x_j \tag{A1.5}$$

where ϕ_i is a function of k_i alone. Further, let us use the equations

$$F_i = 0 \tag{A1.6}$$

to determine k_i as a function of the coordinates x_i.

If we require $F_i = 0$, then $dF_i = 0$, and thus

$$\frac{\partial F_i}{\partial k_j} dk_j + (\delta_{ij} - k_i k_j - i\varepsilon_{ijl} k_l) dx_j = 0 \tag{A1.7}$$

Multiplying by γ (which we assume to be non-zero, consistent with the treatment of Chapter 8), we then have:

$$\gamma \frac{\partial F_i}{\partial k_j} dk_j + \gamma(\delta_{ij} - k_i k_j - i\varepsilon_{ijl} k_l) dx_j = 0 \tag{A1.8}$$

and so from (A1.4) we must have:

$$\left[\gamma \frac{\partial F_i}{\partial k_j} + (\delta_{ij} - i\varepsilon_{ijp} k_p)\right] dk_j = 0 \tag{A1.9}$$

In order that these differential relations in the dk_j should be satisfied, we must have that:

$$\det\left(\gamma \frac{\partial F_i}{\partial k_j} + (\delta_{ij} - i\varepsilon_{ijp} k_p)\right) = 0 \tag{A1.10}$$

which thus provides us with a relation from which we can determine γ. However, noting that we must also have

$$k_j \, dk_j = 0 \tag{A1.11}$$

since $\vec{k}$ is a unit vector, we see that we cannot specify three functions ϕ_i. Rather, we can choose two functions, say ϕ_1 and ϕ_2, and then the equations that we must solve for γ become:

$$\begin{vmatrix} \gamma \dfrac{\partial F_1}{\partial k_j} + \delta_{1j} - i\,\varepsilon_{1jp} k_p \\ \gamma \dfrac{\partial F_2}{\partial k_j} + \delta_{2j} - i\,\varepsilon_{2jp} k_p \\ k_j \end{vmatrix} = 0 \qquad \text{(A1.12)}$$

Then $\vec{k}$ determined from (A1.6) and γ (and thus ℓ_0) determined from (A1.12) completely specify the metric, as can be seen from (12.1) - (12.3). If we choose complicated forms for ϕ_i in (A1.5), the solution for $\vec{k}$ will be very cumbersome. However, for simple ϕ_i this method provides a simple alternative to the methods developed in the main body of the discussion. For example, let us take the simplest case, $\phi_i = 0$. Then from (A1.5) and (A1.6) we have at once that

$$k_i = x_i / r \qquad \text{(A1.13)}$$

Also noting that we then have:

$$\frac{\partial F_i}{\partial k_j} = -\left(k_i x_j + (k_p x_p)\delta_{ij} + i\,\varepsilon_{ipj} x_p \right) \qquad \text{(A1.14)}$$

we find for the determinantal condition:

$$\begin{vmatrix} 1 - \gamma(k_1 x_1 + \vec{k}.\vec{x}) & -(k_1 x_2 - i x_3)\gamma - i k_3 & -(k_1 x_3 + i x_2)\gamma + i k_2 \\ -(k_1 x_2 + i x_3)\gamma + i k_3 & 1 - \gamma(k_2 x_2 + \vec{k}.\vec{x}) & -(k_2 x_3 - i x_1)\gamma - i k_1 \\ k_1 & k_2 & k_3 \end{vmatrix} = 0 \qquad \text{(A1.15)}$$

Using (A1.13), we see at once that both the real and imaginary parts are satisfied if we have $\gamma = 1/r$, since in this case all imaginary terms cancel and the determinant of the remaining real coefficients is zero.

Comparing the results

$$k_i = x_i/_r \quad ; \quad \ell_0^2 = R_e(\gamma) = 1/_r \tag{A1.16}$$

with Chapter 12, we see that we have obtained the Schwarzschild solution.

Appendix 2. A direct calculation for the case where the Killing vector has a vanishing time component.

We remarked in Chapter 11 that an important relation, given by equation (11.6), was valid even when $a^0 = 0$, although the result was derived explicitly assuming that $a^0 \neq 0$. We demonstrate this assertion here.

When $a^0 = 0$, from (10.23) we have

$$a^t k_{i|t} = 0 \tag{A2.1}$$

Also, from (8.22) we have

$$(\delta_{tj} - k_t k_j) k_{i|t} = \alpha(\delta_{ij} - k_i k_j) + \beta \varepsilon_{ijl} k_l \tag{A2.2}$$

We will now use the information provided by (A2.1) to calculate $k_{i|j}$ from (A2.2). The direct approach would be to use (A2.1) and two of the three equations of (A2.2), to give us three linear equations to be solved for $k_{i|j}$. This becomes very messy and has to be performed for each value of i as a separate calculation. However, a much more elegant method is available, as follows:

Since (A2.1) holds, we also have:

$$a^t a^j k_{i|t} = 0 \tag{A2.3}$$

Adding this to (A2.2) gives us:

$$\begin{aligned}(\delta_{tj} - k_t k_j + a^t a^j) k_{i|t} \\ = \alpha(\delta_{ij} - k_i k_j) + \beta \varepsilon_{ijl} k_l\end{aligned} \tag{A2.4}$$

Now, as is readily verified by multiplication, the inverse of the matrix $(\delta_{rj} - k_r k_j + a^r a^j)$ is

$$\left(\delta_{jp} + (1 + a^n a^n) k_j k_p / P^2 - (a^j k_p + a^p k_j)/P\right) \qquad \text{(A2.5)}$$

where we define $P = a^i k_i$.

Multiplying (A2.4) by the inverse (A2.5), we have:

$$\begin{aligned} k_{i|p} \quad = \quad & \alpha\left(\delta_{ip} - a^i k_p / P\right) \\ & + \beta\left(\varepsilon_{ip\ell} k_\ell - \varepsilon_{ij\ell}\, a^j k_\ell k_p / P\right) \end{aligned} \qquad \text{(A2.6)}$$

Now noting that $\tilde{a}_i = -a^i$, from (11.5), and that P here is the same as that of (10.25) when $a^0 = 0$, we see that (A2.6) is exactly (11.6) with $a^0 = 0$.

Appendix 3. Derivation of the gradient of the complex scalar $\overline{\gamma}$.

In Chapter 11 we wrote the formula for the gradient of $\overline{\alpha}$ without providing any derivation. This is given here. First, from (9.1) we have:

$$2\alpha = 2(P\overline{\alpha}) = k_{i|i} \qquad \text{(A3.1)}$$

In taking the gradient of this, we will use the fact that $\vec{a}$ and $\vec{k}$ are independent 3-vectors, and write the general form:

$$\vec{\nabla}\overline{\alpha} = A\vec{k} + B\vec{a} + C\,\vec{a}\times\vec{k} \qquad \text{(A3.2)}$$

Multiplying by k_i, and using (9.12), which from (10.9) we note can be written as:

$$\overline{\alpha}_i k_i = \overline{\alpha}_{|0} - P(\overline{\alpha}^2 - \overline{\beta}^2) \qquad \text{(A3.3)}$$

we see that A can easily be found if we know B and C. Thus, in writing all expressions for the gradient of $\overline{\alpha}$, we will drop all terms in k_j, and recover these later by use of (A3.2). Introducing the notation "$\doteq$" to mean "equal to modulo k_j", we then have using (11.6):

$$\begin{aligned} 2\alpha_{|j} &= k_{i|i|j} = (k_{i|j})_{|i} \\ &\doteq \alpha_{|j} + P\overline{\beta}_{|i}\,\varepsilon_{ijl}k_l - a_r k_{r|i}\overline{\beta}\,\varepsilon_{ijl}k_l \\ &\quad + P\overline{\beta}\,\varepsilon_{ijl}k_{l|i} + \overline{\beta}\,\varepsilon_{irl}a_r k_l k_{j|i} \end{aligned} \qquad \text{(A3.4)}$$

and thus:

$$\begin{aligned} P\overline{\alpha}_{|j} \doteq\; & a_r k_{r|j}\overline{\alpha} + P\overline{\beta}_{|i}\,\varepsilon_{ijl}k_l + P\overline{\beta}\,\varepsilon_{ijl}k_{l|i} \\ & - a_r k_{r|i}\overline{\beta}\,\varepsilon_{ijl}k_l + \overline{\beta}\,\varepsilon_{irl}a_r k_l k_{j|i} \end{aligned} \qquad \text{(A3.5)}$$

We now use (11.6) again to evaluate the terms involving $k_{r|j}$, $k_{l|i}$, $k_{r|i}$, and $k_{j|i}$, in each case dropping terms in k_j. This gives us:

$$a_r k_{r|j} \bar{\alpha} \doteq P\bar{\alpha}^2 a_j + P\bar{\alpha}\bar{\beta}\,\varepsilon_{rjp}\, a_r k_p \qquad (A3.6)$$

$$P\bar{\beta}\,\varepsilon_{ijl}\, k_{l|i} \doteq P\bar{\alpha}\bar{\beta}\,\varepsilon_{ijl}\, a_l k_i - P\bar{\beta}^2 a_j \qquad (A3.7)$$

$$-\bar{\beta} a_r\,\varepsilon_{ijl}\, k_l k_{r|i} \doteq -P\bar{\alpha}\bar{\beta}\, a_r \varepsilon_{rjl} k_l + P\bar{\beta}^2 a_j \qquad (A3.8)$$

$$\bar{\beta}\,\varepsilon_{irl}\, a_r k_l k_{j|i} \doteq P\bar{\alpha}\bar{\beta}\,\varepsilon_{jrl}\, a_r k_l - P\bar{\beta}^2 a_j \qquad (A3.9)$$

Using the above forms in (A3.5) then leads to:

$$\bar{\alpha}_{|j} \doteq a_j(\bar{\alpha}^2 - \bar{\beta}^2) + \bar{\beta}_{|i}\,\varepsilon_{ijl}\, k_l + 2\bar{\alpha}\bar{\beta}\,\varepsilon_{jrl}\, a_r k_l \qquad (A3.10)$$

Now multiplying by k_j and using (A3.3) gives us the coefficient A, as:

$$A = \bar{\alpha}_{|0} - (\bar{\alpha}^2 - \bar{\beta}^2) \qquad (A3.11)$$

Thus finally from (A3.2), (A3.10) and (A3.11) we have:

$$\begin{aligned} \bar{\alpha}_{|j} &= (\bar{\alpha}_{|0} - (\bar{\alpha}^2 - \bar{\beta}^2))k_j + a_j(\bar{\alpha}^2 - \bar{\beta}^2) \\ &\quad + \beta_{|i}\,\varepsilon_{ijl}\, k_l + 2\bar{\alpha}\bar{\beta}\,\varepsilon_{jrl}\, a_r k_l \end{aligned} \qquad (A3.12)$$

This can be written in the vector form:

$$\begin{aligned} \vec{\nabla}\bar{\alpha} &= (\bar{\alpha}_{|0} - (\bar{\alpha}^2 - \bar{\beta}^2))\vec{k} + (\bar{\alpha}^2 - \bar{\beta}^2)\vec{a} \\ &\quad + 2\bar{\alpha}\bar{\beta}(\vec{a} \times \vec{k}) + (\vec{k} \times \vec{\nabla}\bar{\beta}) \end{aligned} \qquad (A3.13)$$

which is exactly equation (11.8).

Appendix 4. Demonstration that the complex scalar $\bar{\gamma}$ satisfies the wave equation.

In Chapter 11 we stated without proof that the complex function $\bar{\gamma}$ satisfies the wave equation. To establish this result, we first write (11.10) in component form:

$$\bar{\gamma}_{|i} = \bar{\gamma}^2(\tilde{a}_i - k_i a^0 + i\varepsilon_{ij\ell} k_j \tilde{a}_\ell) + \bar{\gamma}_{|0} k_i + i\varepsilon_{ij\ell}\bar{\gamma}_{|j} k_\ell \tag{A4.1}$$

Differentiating this, we have:

$$\begin{aligned}\bar{\gamma}_{|i|i} = {} & 2\bar{\gamma}\bar{\gamma}_{|i}(\tilde{a}_i - k_i a^0 + i\varepsilon_{ij\ell} k_j \tilde{a}_\ell) \\ & + \bar{\gamma}^2(-k_{i|i} a^0 + i\varepsilon_{ij\ell} k_{j|i}\tilde{a}_\ell) \\ & + \bar{\gamma}_{|0|i} k_i + \bar{\gamma}_{|0} k_{i|i} + i\varepsilon_{ij\ell}\bar{\gamma}_{|j} k_{\ell|i}\end{aligned} \tag{A4.2}$$

To simplify this expression, we consider the reduction of each term. First, putting $\gamma = P\bar{\gamma}$ in (9.15) gives us:

$$\bar{\gamma}_{|\nu} k^\nu = P\bar{\gamma}^2 \tag{A4.3}$$

which we write:

$$\bar{\gamma}_{|i} k_i = \bar{\gamma}_{|0} - P\bar{\gamma}^2 \tag{A4.4}$$

Also, using (11.15) and putting $\bar{\gamma} = 1/\bar{\omega}$, we find:

$$\bar{\gamma}_{|i}\bar{\gamma}_{|i} = \bar{\gamma}_{|0}^2 + (a^{0\,2} - \tilde{a}_i\tilde{a}_i)\bar{\gamma}^4 \tag{A4.5}$$

Using (A4.1) to substitute for one of the $\bar{\gamma}_{|i}$ in the left hand side of (A4.5), and applying (A4.4), we find:

$$2\bar{\gamma}_{|i}\bar{\gamma}\left(\tilde{a}_i - k_i a^0 + i\varepsilon_{ije}k_j\tilde{a}_e\right)$$
$$= 2\left(a^{0^2} - \tilde{a}_i\tilde{a}_i\right)\bar{\gamma}^3 + 2P\bar{\gamma}\bar{\gamma}_{|0} \qquad \text{(A4.6)}$$

Next, using (A4.1) and differentiating to form $\bar{\gamma}_{|i|0}$, we find:

$$\bar{\gamma}_{|i|0}k_i = 2\bar{\gamma}\bar{\gamma}_{|0}\left(\tilde{a}_i k_i - a^0\right) + \bar{\gamma}_{|0|0}$$
$$+ i\bar{\gamma}^2\varepsilon_{ije}k_i k_{j|0}\tilde{a}_e - i\varepsilon_{ije}k_i k_{j|0}\bar{\gamma}_{|e} \qquad \text{(A4.7)}$$

Forming $\varepsilon_{ije}k_{j|i}$ using (8.21) gives:

$$\varepsilon_{ije}k_{j|0}k_i = \varepsilon_{ije}k_{j|i} + 2P\bar{\beta}k_e \qquad \text{(A4.8)}$$

and using this in (A4.7) then leads to:

$$\bar{\gamma}_{|i|0}k_i = -2P\bar{\gamma}\bar{\gamma}_{|0} + \bar{\gamma}_{|0|0}$$
$$+ i\bar{\gamma}^2\varepsilon_{ije}k_{j|i}\tilde{a}_e + 2P\bar{\beta}i\bar{\gamma}^2\left(a^0 - P\right)$$
$$- i\varepsilon_{ije}k_{j|i}\bar{\gamma}_{|e} - i2P\bar{\beta}\left(\bar{\gamma}_{|e}k_e\right) \qquad \text{(A4.9)}$$

We can also use (11.6) to form $\varepsilon_{ije}k_{j|i}$, and this leads to:

$$\bar{\gamma}^2\tilde{a}_e\varepsilon_{ije}k_{j|i} = -2P\bar{\beta}\bar{\gamma}^2\left(a^0 - P\right)$$
$$+ \bar{\beta}\bar{\gamma}^2\tilde{a}_i\tilde{a}_i - \bar{\beta}\bar{\gamma}^2\left(a^0 - P\right)^2 \qquad \text{(A4.10)}$$

Finally, using (11.6) again to form $\varepsilon_{ije} k_{j|i}$, multiplying by $\gamma_{|e}$, and using (A4.4) to reduce the result gives us:

$$\varepsilon_{ije} k_{j|i} \gamma_{|e} = -P\bar{\beta}\gamma_{|0} + P^2\bar{\beta}\bar{\gamma}^2 + a^0 P\bar{\beta}\bar{\gamma}^2 + \bar{\alpha}\varepsilon_{ije}\bar{\gamma}_{|e} k_i \tilde{a}_j \tag{A4.11}$$

We now substitute the forms (A4.6), (A4.9), (A4.10) and (A4.11) into (A4.2), which gives the result:

$$\begin{aligned}\gamma_{|i|i} = {} & \gamma_{|0|0} + 2i\varepsilon_{ije} k_{j|i}(\bar{\gamma}^2\tilde{a}_e - \bar{\gamma}_{|e}) \\ & + 2(a^{02} - \tilde{a}_i\tilde{a}_i)\bar{\gamma}^3 - 2P\bar{\alpha}\bar{\gamma}^2 a^0 \\ & + 2Pi\bar{\beta}a^0\bar{\gamma}^2 - 2iP\bar{\beta}\bar{\gamma}_{|0} + 2P\bar{\alpha}\bar{\gamma}_{|0}\end{aligned} \tag{A4.12}$$

To complete the reduction, we observe that (10.29) leads to:

$$a^0\bar{\gamma}_{|0} = \tilde{a}_i\bar{\gamma}_{|i} \tag{A4.13}$$

and forming $\tilde{a}_i\bar{\gamma}_{|i}$ with the aid of (A4.1) then gives:

$$\begin{aligned}a^0\bar{\gamma}_{|0} = {} & \bar{\gamma}^2(\tilde{a}_i\tilde{a}_i + a^0(a^0 - P)) + \bar{\gamma}_{|0}(a^0 - P) \\ & + i\varepsilon_{ije}\bar{\gamma}_{|j} k_e \tilde{a}_i\end{aligned} \tag{A4.14}$$

Thus (4.11) can be written as:

$$\begin{aligned}\varepsilon_{ije} k_{j|i}\gamma_{|e} = {} & -P\bar{\beta}\bar{\gamma}_{|0} + P^2\bar{\beta}\bar{\gamma}^2 + P\bar{\beta}\bar{\gamma}^2 a^0 \\ & + i\bar{\alpha}\bar{\gamma}^2(\tilde{a}_i\tilde{a}_i - a^0(a^0 - P)) - i\bar{\alpha}P\bar{\gamma}_{|0}\end{aligned} \tag{A4.15}$$

which means in turn that we can write the difference of equations (A4.10) and (A4.15) as:

$$\varepsilon_{ije}\, k_{j|i}\left(\bar{\gamma}^2 \tilde{a}_\ell - \bar{\gamma}_{|\ell}\right) = \bar{\beta}\bar{\gamma}^2\left(\tilde{a}_i \tilde{a}_i - a^{\circ 2}\right)$$
$$+ \; P\bar{\beta}\left(\bar{\gamma}_{|o} - \bar{\gamma}^2 a^\circ\right) - i\bar{\alpha}\bar{\gamma}^2\left(\tilde{a}_i \tilde{a}_i - a^{\circ 2} + P a^\circ\right)$$
$$+ \; i\bar{\alpha} P \bar{\gamma}_{|o} \qquad \text{(A4.16)}$$

Now substituting this result in (A4.12), we have the final result, after combining terms:

$$\bar{\gamma}_{|i|i} = \bar{\gamma}_{|o|o} \quad ; \quad \bar{\gamma}_{|\alpha}{}^{|\alpha} = 0 \qquad \text{(A4.17)}$$

Thus $\bar{\gamma}$ satisfies the wave equation.

Appendix 5. The relation of the present treatment to alternative formulations.

Debney, Kerr and Schild[43] have also considered metrics of the type (7.1) using a tetrad approach. Their results can be summarized as follows:

The line element of the metric is written as:

$$ds^2 = 2d\zeta d\bar{\zeta} + 2du\,dv + P^{-2}\,2\,\mathrm{Re}\left(1/F_Y\right) \times \left[du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv\right]^2 \tag{A5.1}$$

where the variables u,v, ζ and $\bar{\zeta}$ are defined by:

$$\begin{aligned} \sqrt{2}\,\zeta &= x + iy & \sqrt{2}\,u &= z + t \\ \sqrt{2}\,\bar{\zeta} &= x - iy & \sqrt{2}\,v &= z - t \end{aligned} \tag{A5.2}$$

The function F is defined by:

$$\begin{aligned} F = \phi &+ (qY + c)(\zeta - Yv) \\ &- (pY + \bar{q})(u + Y\bar{\zeta}) \end{aligned} \tag{A5.3}$$

where Y is determined by solving the equation F = 0. ϕ is any analytic function of Y, and P is defined by:

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c \tag{A5.4}$$

Noting that the first two terms of (A5.1) are merely a flat space metric, comparison with the form (7.1), and use of $\ell_\mu = \ell_0 k_\mu$ shows that for consistency of the two approaches we must require:

$$F_Y = \bar{\omega} \tag{A5.5}$$

We confirm this conjecture, by establishing that F_Y satisfies the two equations (11.15) and (11.17).

First, since Y is a function of the coordinates, we note that we must have

$$0 = dF = F_Y\, dY + (qY+c)\,d\zeta - Y(pY+\bar{q})\,d\bar{\zeta} - (pY+\bar{q})\,du - Y(qY+c)\,dv \tag{A5.6}$$

which we write as

$$F_Y\, dY = -\vec{DF}\cdot d\vec{q} \tag{A5.7}$$

where we define

$$\vec{q} = (\zeta, \bar{\zeta}, u, v)$$
$$\vec{DF} = (qY+c, -Y(pY+\bar{q}), -(pY+q), -Y(qY+c)) \tag{A5.8}$$

Directly from (A5.8) we have, where D denotes differentiation at constant Y, the relation:

$$\frac{DF}{D\zeta}\frac{DF}{D\bar{\zeta}} + \frac{DF}{Du}\frac{DF}{Dv} = 0 \tag{A5.9}$$

and thus using (A5.7)

$$\frac{\partial F_Y}{\partial \zeta}\frac{\partial F_Y}{\partial \bar{\zeta}} + \frac{\partial F_Y}{\partial u}\frac{\partial F_Y}{\partial v} = (q\bar{q} - pc) \tag{A5.10}$$

Finally, changing from the variables $\zeta, \bar{\zeta}, u, v$ to t,x,y,z, we find the result:

$$(\Box F_Y)^2 = 2(q\bar{q} - pc) \tag{A5.11}$$

where $(q\bar{q} - pc)$ is a real constant, since p and c are real.

In an exactly similar way, we can calculate the second derivative of F_Y, using the variables $\zeta, \bar{\zeta}, u, v$.

We find:

$$F_Y\left[\frac{\partial^2 F_Y}{\partial\zeta\partial\bar{\zeta}}+\frac{\partial^2 F_Y}{\partial u\partial v}\right]=2(q\bar{q}-pc) \tag{A5.12}$$

which in terms of the variables t,x,y,z gives:

$$F_Y\,\Box^2 F_Y = 4(q\bar{q}-pc) \tag{A5.13}$$

Thus we have confirmed that F_Y satisfies the same governing equations as $\bar{\omega}$, with the constants arising from the Killing vector written in rather different form. This completes the demonstration that we can take $\bar{\omega} = F_Y$, relating the two approaches to solving the field equations.

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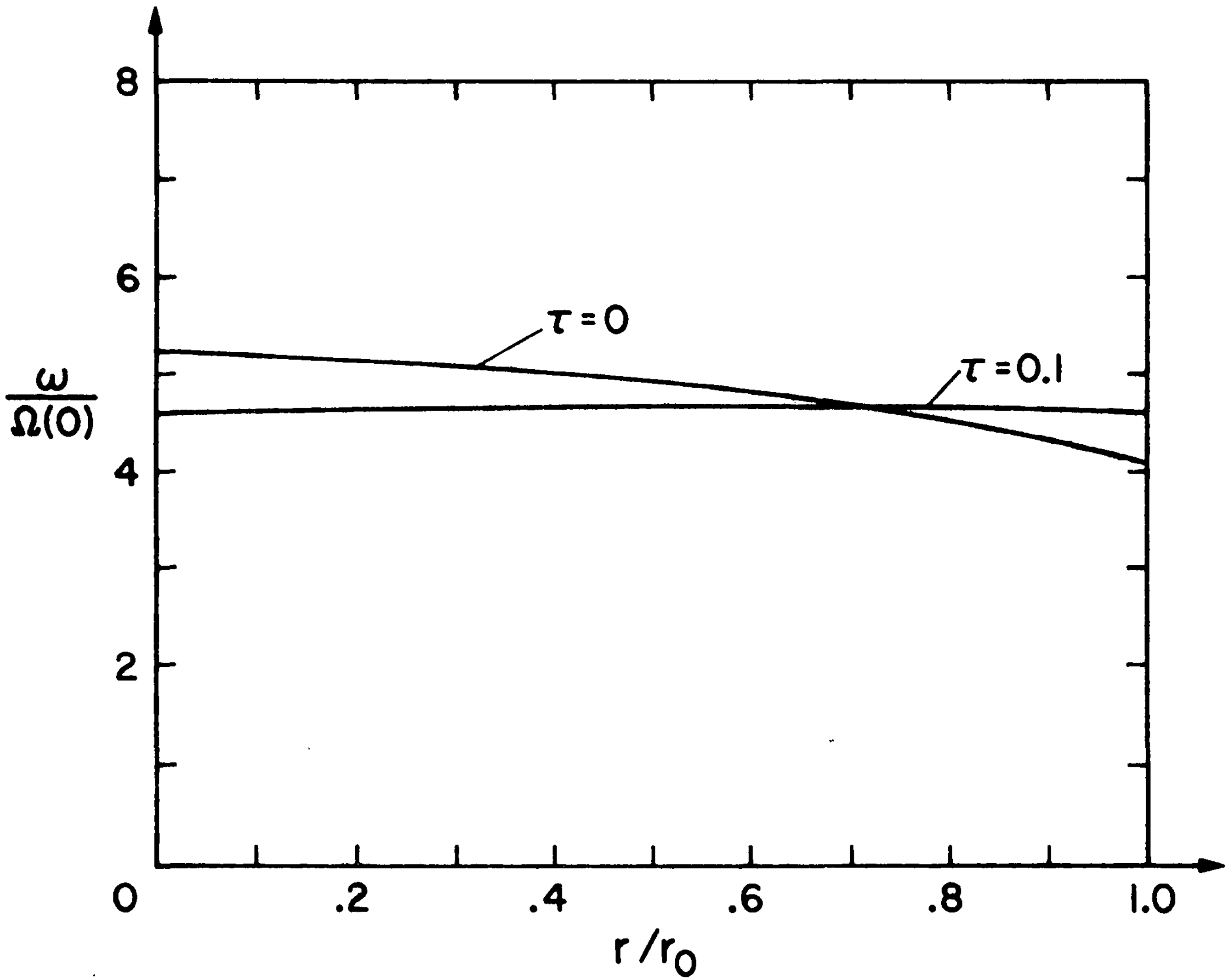

Fig. 1. Rotation rate as a function of radius.