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RAPID ITERATIVE REANALYSIS FOR AUTOMATED DESIGN

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16. Abstract

A method for iterative reanalysis in automated structural design is presented for a finite-element analysis using the direct stiffness approach. A basic feature of the method is that the generalized stiffness and inertia matrices are expressed as functions of structural design parameters, and these generalized matrices are expanded in Taylor series about the initial design. Only the linear terms are retained in the expansions. The method is approximate because it uses static condensation, modal reduction, and the linear Taylor series expansions. The exact linear representation of the expansions of the generalized matrices is also described and a basis for the present method is established.

Results of applications of the present method to the recalculation of the natural frequencies of two simple platelike structural models are presented and compared with results obtained by using a commonly applied analysis procedure used as a reference. In general, the results are in good agreement. A comparison of the computer times required for the use of the present method and the reference method indicated that the present method required substantially less time for reanalysis. Although the results presented are for relatively small-order problems, the present method will become more efficient relative to the reference method as the problem size increases. An extension of the present method to static reanalysis is described, and a basis for unifying the static and dynamic reanalysis procedures is presented.

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RAPID ITERATIVE REANALYSIS FOR AUTOMATED DESIGN

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SUMMARY

A method for iterative reanalysis in automated structural design is presented for a finite-element analysis using the direct stiffness approach. A basic feature of the method is that the generalized stiffness and inertia matrices are expressed as functions of structural design parameters, and these generalized matrices are expanded in Taylor series about the initial design. Only the linear terms are retained in the expansions. The method is approximate because it uses static condensation, modal reduction, and the linear Taylor series expansions. The exact linear representation of the expansions of the generalized matrices is also described and a basis for the present method is established.

Results of applications of the present method to the recalculation of the natural frequencies of two simple platelike structural models are presented and compared with results obtained by using a commonly applied analysis procedure used as a reference. In general, the results are in good agreement. A comparison of the computer times required for the use of the present method and the reference method indicated that the present method required substantially less time for reanalysis. Although the results presented are for relatively small-order problems, the present method will become more efficient relative to the reference method as the problem size increases. An extension of the present method to static reanalysis is described, and a basis for unifying the static and dynamic reanalysis procedures is presented.

INTRODUCTION

Currently, there is considerable interest in automating as much of the structural design process as possible. A status report of the current situation in this area is presented in reference 1. The papers cited in this reference provide a good summary of the work to date in automated design. It is pointed out that the resizing methods for structures and the techniques for a systematic way of conducting analyses are two important aspects of any automated design system. The present study is concerned with systematic reanalysis procedures for automated design.

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The iterative nature of automated design requires that the individual analyses be repeated many times. Therefore, the economy, speed, and accuracy of the individual analyses are important factors in the success of an automated design process. A given computational technique may thus be well suited for a few analyses, such as in the detailed validation of a final design, but the technique may not be well suited for use in an automated design process.

Published work describing iterative reanalysis procedures is heavily oriented toward approximate static reanalysis, as evidenced by references 2 to 9. In contrast, there appears to be a scarcity of dynamic reanalysis procedures, which are obviously more complex and expensive than the static reanalysis procedures. An efficient approximate dynamic reanalysis is a valuable tool independently and in conjunction with a static reanalysis procedure.

The purpose of this paper is to present an approximate finite-element structural reanalysis procedure which is well suited for the iterative requirements of the automated design process. It is recognized that the initial formulation of the structural problem for iterative reanalysis should be made by taking into consideration the special nature of the iterative process, and that such a formulation can be different from the formulation for a single analysis. It is shown that the method can be applied to static and dynamic reanalysis, and a unified approach to static and dynamic reanalysis is presented but the development of the present method in the main body of the report emphasizes dynamic reanalysis. In particular, the method as developed herein provides a means of determining the natural frequencies and mode shapes of a structure as would normally be required during each cycle of an automated design process but without having to perform a complete structural reanalysis. The approximate nature of the method is examined in appendix A. In appendix B, an extension of the method to static reanalysis is presented, and a basis for unifying the static and dynamic reanalysis is described.

The method may be outlined as follows. The complete structural stiffness and inertia matrices are assembled by the direct stiffness method for the initial design. A set of stiffness design parameters and a set of inertia design parameters are defined so that the assembled stiffness matrix is a linear function of the stiffness design parameters and the assembled inertia matrix is a linear function of the inertia design parameters. The design parameters themselves can be nonlinear (and complicated) functions of the structural design variables. In the present study, the design variables are distinct from the design parameters and are restricted to structural element thicknesses, cross-sectional areas, elastic properties, and/or any combination of these. The complete structural matrices are then reduced by static condensation and an eigensolution obtained for the reduced system. A modal matrix is then constructed with the desired number of eigenvectors, and the generalized stiffness and inertia matrices for the initial design parameters are determined. The generalized matrices (for an arbitrary set of design variables) are expanded in two-term Taylor series expansions about the initial design by using the design parameters as the independent variables. The analytical expressions to evaluate the first derivatives appearing in the Taylor series expansions are derived. Once the Taylor series expansions are defined, the generalized stiffness and inertia matrices can be directly evaluated for any perturbations in the design variables from the initial design. An approximate reanalysis can then be performed by using the generalized matrices determined in this manner.

The method is illustrated for dynamic reanalysis and examples of application of this method to simple trapezoidal and rectangular plate models are presented and discussed. The results, in terms of natural frequencies, are compared with the results obtained by performing a complete structural analysis. A special-purpose digital computer program was written to obtain these results. This program is designated RITREAD (Rapid Iterative Reanalysis for Automated Design). Some of the features of this program are discussed in appendix C. The formulation of the method allows the derivatives of the generalized structural matrices with respect to design variables to be easily computed, as indicated in appendix D.

SYMBOLS

$\{D\}$ vector of structural design variables, $(NV \times 1)$

$\{F\}$ force vector

f_{ij} function of $\{D\}$ associated with $[K_{ij}]$

$[K]$ system stiffness matrix, $(n \times n)$

$[KG]$ generalized system stiffness matrix, $(NM \times NM)$

$[K_i]$ constant matrix of stiffness coefficients associated with ith stiffness design parameter, $(n \times n)$

$[\overline{K}_i]$ contribution of the ith element to the system stiffness matrix, $(n \times n)$

$[K_{ij}]$ constant matrix of stiffness coefficients corresponding to the jth subelement of ith finite element, $(n \times n)$

$[M]$ system inertia matrix, $(n \times n)$

$[MG]$ generalized system inertia matrix, (NM × NM)

$[M_i]$ constant matrix of inertia coefficients associated with ith inertia design parameter, (n × n)

NE total number of finite elements

NM number of natural vibration modes used in modal reduction

NP number of stiffness design parameters

$N\mathscr{P}$ number of inertia design parameters

NR number of degrees of freedom eliminated by static condensation

NV number of structural design variables

NZ number of degrees of freedom retained after static condensation

n total number of degrees of freedom, NZ + NR

n_i number of constant submatrices required to express $[\bar{K}_i]$ as a linear function of design parameters

P_{ij} stiffness design parameter associated with $[K_{ij}]$

$\{P\}$ vector of stiffness design parameters, (NP × 1)

$\{\mathscr{P}\}$ vector of inertia design parameters, ($N\mathscr{P}$ × 1)

$[Q]$ modal matrix, (NZ × NM)

$[Q_n]$ modal matrix, (n × NM)

$\{q_i\}$ eigenvector corresponding to ith natural mode of vibration, (NZ × 1)

$\{\bar{q}_i\}$ eigenvector corresponding to ith natural mode of vibration, (NM × 1)

$[S]$ final transformation matrix, (NR × NM)

$\mathscr{A}_m$ integer set associated with P_m, identifying the subelements for which $P_m = P_{ij}$, $j = 1,n_i$, $i = 1,NE$

$[T]$ transformation matrix relating the degrees of freedom eliminated to the degrees of freedom retained in static condensation, $(NR \times NZ)$

$\{U\}$ vector of structural displacements, $(n \times 1)$

$\{U_h\}$ vector of structural displacements corresponding to degrees of freedom retained in static condensation, $(NZ \times 1)$

$\{U_\alpha\}$ vector of structural displacements corresponding to degrees of freedom eliminated in static condensation, $(NR \times 1)$

$\{\Delta\}$ vector of static displacements, $(n \times 1)$

λ square of natural frequency, $(\mathrm{rad/sec})^2$

$\{\xi\}$ vector of generalized static displacements, $(NM \times 1)$

$\{\ \}$ column matrix

$[\ ]$ rectangular matrix

Subscript following a parenthesis denotes a partial derivative with respect to a design parameter, for example, $\left([K]\right)_i = \frac{\partial [K]}{\partial P_i}$, $\left([M]\right)_i = \frac{\partial [M]}{\partial \mathscr{P}_i}$. Double subscripted matrix represents a matrix partitioned or separated from a larger matrix. Superscript B denotes that the superscripted symbol corresponds to the initial design. Superscript T is used to denote a matrix transpose. All other symbols are locally defined.

A TYPICAL APPROACH FOR DYNAMIC ANALYSIS

A typical approach to dynamic analysis for determining the natural vibration characteristics of complex structures is described in this section and is used in the subsequent sections as a basis for the development of the reanalysis method.

In representing a complex structure by an analytical model, a large number of finite elements may be needed for a physically continuous structure with an infinite number of degrees of freedom to be adequately modeled with a finite number of degrees of freedom. The system stiffness matrix $[K]$ and the system inertia matrix $[M]$ can be determined for the complete structure by using the direct stiffness method of reference 10. If a large number of finite elements are used for the structural representation, $[K]$ and $[M]$ will be of a high order. The dynamic analysis is thus often hampered by difficulties in accurately and efficiently handling eigenvalue problems of large size. The usual practice is to reduce the size of the eigenvalue problem by static condensation in the manner suggested by Guyan (ref. 11) and Irons (ref. 12).

The system displacement vector $\{U\}$ and the system stiffness matrix $[K]$ are partitioned as

$$\underset{(n \times 1)}{\{U\}} = \left\{\begin{array}{c} \{U_h\} \\ \hdashline \{U_\alpha\} \end{array}\right\} \tag{1}$$

and

$$\underset{(n \times n)}{[K]} = \left[\begin{array}{c:c} [K_{hh}] & [K_{h\alpha}] \\ \hdashline [K_{\alpha h}] & [K_{\alpha\alpha}] \end{array}\right] \tag{2}$$

The physical coordinates to be retained are denoted by the subscript h and those to be eliminated are denoted by the subscript α. One may, for example, either intuitively decide to eliminate some degrees of freedom or use the method of reference 13 to determine more rationally the degrees of freedom to be eliminated. However, the method of selecting the degrees of freedom to be eliminated is an important practical consideration but will not be discussed herein. By the methods of references 11 and 12 a transformation matrix $[T]$ relates $\{U_\alpha\}$ to $\{U_h\}$ so that

$$\underset{(NR \times 1)}{\{U_\alpha\}} = [T]\{U_h\} \tag{3}$$

where

$$\underset{(NR \times NZ)}{[T]} = -[K_{\alpha\alpha}]^{-1}[K_{\alpha h}] \tag{4}$$

The reduced stiffness matrix $[K_h]$ is defined by

$$\underset{(NZ \times NZ)}{\left[K_h\right]} = \left[K_{hh}\right] + \left[K_{h\alpha}\right]\left[T\right] \tag{5}$$

The reduced inertia matrix $\left[M_h\right]$ can be determined as in references 11 and 12, or the same expression can be obtained from the equivalence of virtual work as in reference 14 (pp. 291-292), and is given by

$$\underset{(NZ \times NZ)}{\left[M_h\right]} = \left[M_{hh}\right] + \left[M_{h\alpha}\right]\left[T\right] + \left[\left[M_{h\alpha}\right]\left[T\right]\right]^T + \left[T\right]^T\left[M_{\alpha\alpha}\right]\left[T\right] \tag{6}$$

where $\left[M_{hh}\right]$, etc., are the inertia matrix partitions defined by

$$\left[M\right] = \left[\begin{array}{c:c} \left[M_{hh}\right] & \left[M_{h\alpha}\right] \\ \hdashline \left[M_{\alpha h}\right] & \left[M_{\alpha\alpha}\right] \end{array}\right]$$

The reduced structural matrices are used to solve the eigenvalue problem

$$\left[\left[K_h\right] - \lambda_i\left[M_h\right]\right]\left\{q_i\right\} = \left\{0\right\} \qquad (i = 1,NZ) \tag{7}$$

where $\left\{q_i\right\}$ is the eigenvector defining normal-mode displacements for the NZ degree of freedom system and $\left\{\begin{matrix}\left\{q_i\right\} \\ \left[T\right]\left\{q_i\right\}\end{matrix}\right\}$ defines the normal-mode displacements for the unreduced system with n degrees of freedom.

The solution of equation (7) completes the process of obtaining the natural vibration frequencies and mode shapes of primary interest. A substantial saving is usually realized by reducing the order of the eigenvalue problem from n to NZ as described. The order of the reduced structural matrices NZ may still, however, be unwieldy if either flutter or dynamic response analyses are to be performed. In order to further reduce the order of the stiffness and inertia matrices in such instances, a transformation to a smaller number of modal coordinates is effected by means of the familiar normal mode approach. A modal matrix $\left[Q\right]$ is constructed from the first NM eigenvectors corresponding to the lowest NM frequencies of vibrations obtained from the solutions of equation (7). The generalized structural matrices are then determined from the following equations:

$$\underset{(NM \times NM)}{\left[KG\right]} = \left[Q\right]^T\left[K_h\right]\left[Q\right]$$
$$= \left[Q\right]^T\left[K_{hh}\right]\left[Q\right] + \left[Q\right]^T\left[K_{h\alpha}\right]\left[S\right] \tag{8a}$$

and

$$\underset{(NM \times NM)}{\left[MG\right]} = \left[Q\right]^T\left[M_h\right]\left[Q\right]$$
$$= \left[Q\right]^T\left[M_{hh}\right]\left[Q\right] + \left[Q\right]^T\left[M_{h\alpha}\right]\left[S\right] + \left[\left[Q\right]^T\left[M_{h\alpha}\right]\left[S\right]\right]^T + \left[S\right]^T\left[M_{\alpha\alpha}\right]\left[S\right] \tag{8b}$$

where $\left[S\right] = \left[T\right]\left[Q\right]$. The generalized matrices given by equations (8a) and (8b) are diagonal matrices and are usually of an order much smaller than the system matrices, that is, $NM << n$. The eigenvalue problem using the generalized matrices is of order NM, and is expressed by

$$\left[\left[KG\right] - \lambda_i\left[MG\right]\right]\left\{\bar{q}_i\right\} = \left\{0\right\} \qquad (i = 1,NM) \tag{9a}$$

where

$$\left\{q_i\right\} = \left[Q\right]\left\{\bar{q}_i\right\} \tag{9b}$$

In the method for dynamic reanalysis developed in the following section, the order of the eigenvalue problem to be solved, if required, is NM.

DEVELOPMENT OF THE PRESENT METHOD

The typical dynamic analysis procedure described in the previous section is not well suited for efficient reanalysis because (1) evaluation or reassembly of $\left[K\right]$ and $\left[M\right]$ is required for each reanalysis, (2) the transformation matrix $\left[T\right]$ has to be recomputed, and $\left[K_h\right]$ and $\left[M_h\right]$ have to be determined for each reanalysis, and (3) reduction of $\left[K_h\right]$ and $\left[M_h\right]$ to $\left[KG\right]$ and $\left[MG\right]$ is required during each reanalysis. The modal matrix $\left[Q\right]$ determined for the initial analysis could be used as an assumed-mode matrix during reanalysis and the order of the eigenvalue problem required to be solved for natural-vibration analysis reduced from NZ (as in eq. (7)) to NM (as in eq. (9a)). If the natural frequencies and mode shapes of the structure are not required, and only $\left[KG\right]$ and $\left[MG\right]$ are required, the eigenvalue problem of equation (9a) need not be solved during reanalysis. Even then the complete reanalysis procedure as described can be very expensive.

The method presented herein can economically determine the generalized stiffness and inertia matrices during reanalysis following their generation during the initial analysis. The present method involves some modifications in the formulation of system stiffness and inertia matrices during the initial analysis. The complete system stiffness and inertia matrices are expressed during the initial analysis as linear functions of the stiffness and inertia design parameters, respectively. The generalized matrices and their derivatives with respect to the design parameters are then determined. The generalized stiffness and inertia matrices are expressed by linear Taylor series expansions in the design parameters about the initial design. For subsequent analyses, the generalized matrices are determined from the Taylor series expansions. This procedure is developed subsequently.

Formulation of System Structural Matrices

In the direct stiffness method of finite-element analysis, the system stiffness matrix for the complete structure can be generated by properly assembling all the elemental stiffness matrices. This assembly is indicated by the relation

$$\underset{(n \times n)}{\left[K\right]} = \sum_{i=1}^{NE} \left[\overline{K}_i\right] \tag{10}$$

When only a single analysis is required, there is no further need for $\left[\overline{K}_i\right]$ matrices after they have been added to the system stiffness matrix $[K]$, and these elemental matrices are not normally saved. When some of the design variables are changed in an automated design process, a reanalysis is normally performed. As a consequence of the change in design variables, many of the elemental stiffness matrices will have to be recomputed before a reanalysis. Considerable computational efficiency can be obtained if the recomputation of the affected elemental matrices can be avoided. This can be done if the elemental stiffness matrices are expressed in the following form:

$$\underset{(n \times n)}{\left[\overline{K}_i\right]} = \sum_{j=1}^{n_i} f_{ij}\left(\{D\}\right)\left[K_{ij}\right]$$

$$= \sum_{j=1}^{n_i} P_{ij}\left[K_{ij}\right] \tag{11}$$

where $P_{ij} = f_{ij}(\{D\})$ is defined as a stiffness design parameter and is a uniquely defined function of the design variables $\{D\}$; $[K_{ij}]$ where $j = 1, n_i$ are constant submatrices corresponding to the ith element, and n_i is the number of constant submatrices required for the ith element to represent $[\overline{K}_i]$ as a linear function of the design parameters. Substituting equation (11) into equation (10) yields

$$\underset{(n \times n)}{[K]} = \sum_{i=1}^{NE} \sum_{j=1}^{n_i} P_{ij}[K_{ij}] \tag{12}$$

The system stiffness matrix as given by equation (12) is a linear function of the stiffness design parameters P_{ij}.[2]

The stiffness design parameters are not required to be linear functions of the design variables $\{D\}$. The concept of the stiffness design parameters and design variables is illustrated in the following example for a beam element which includes the effect of shear deformation (ref. 15). The stiffness matrix for this element is

$$[\overline{K}_i] = \left[\begin{array}{c|c|c|c} R & -\frac{l}{2}R & -R & \frac{l}{2}R \\ \hline & \frac{l^2}{4}R + \frac{EI_y}{l} & \frac{l}{2}R & \frac{l^2}{4}R - \frac{EI_y}{l} \\ \hline & & R & \frac{l}{2}R \\ \hline & \text{Symmetric} & & \frac{l^2}{4}R + \frac{EI_y}{l} \end{array}\right]$$

where

l — length of the beam element

EI_y — flexural stiffness of beam element

---

[2] The form of the stiffness matrix given in equation (12) is similar to that published by Kavlie and Powell (ref. 4), but was independently developed by the author.

$$R = \left(\frac{l}{k_z AG} + \frac{l^3}{12EI_y}\right)^{-1}$$

A	cross-sectional area of beam element
G	shear modulus
k_z	constant defining fraction of cross-sectional area of beam effective in shear

The element matrix may be expressed in the form of equation (11) by writing

$$\left[\overline{K}_i\right] = \sum_{j=1}^{2} P_{ij}\left[K_{ij}\right]$$

where

$$P_{i1} = EI_y$$

$$P_{i2} = R = \left(\frac{l}{k_z AG} + \frac{l^3}{12EI_y}\right)^{-1}$$

$$\left[K_{i1}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & \frac{1}{l} & 0 & -\frac{1}{l} \\ & & 0 & 0 \\ & \text{Symmetric} & & \frac{1}{l} \end{bmatrix}$$

$$
\left[K_{i2}\right] = \left[\begin{array}{c|c|c|c}
1 & -\frac{l}{2} & -1 & \frac{l}{2} \\
\hline
 & \frac{l^2}{4} & \frac{l}{2} & \frac{l^2}{4} \\
\cline{2-4}
 & & 1 & \frac{l}{2} \\
\cline{3-4}
 & & & \frac{l^2}{4}
\end{array}\right]
$$

The design variables for this beam element could be E, G, I_y, A, b, and h or any combination of these variables, and P_{i1} and P_{i2} are the design parameters which may be nonlinear functions of the design variables.

Although a large number of finite elements may be required for adequate representation of complex structures, some of the design parameters P_{ij} may be identical to each other. It would be advantageous to distinguish the number of unique design parameters from the number of elements as suggested in reference 16. Let $\mathscr{A}_m$, where $m = 1,NP$, be a set associated with a stiffness design parameter P_m such that all the submatrices for which $P_m = P_{ij}$ $\left(\text{where } i = 1,NE,\ j = 1,n_i\right)$ are uniquely identified, and there is at least one $P_{ij} = P_m$. In other words, the various constant submatrices associated with any P_m (where $m = 1,NP$) can be summed into a single matrix $\left[K_m\right]$ and the system stiffness matrix expressed in the form

$$
\underset{(n \times n)}{\left[K\right]} = \sum_{m=1}^{NP} P_m\left[K_m\right] \tag{13}
$$

where

$$
\underset{(n \times n)}{\left[K_m\right]} = \sum_{i \epsilon \mathscr{A}_m} \sum_{j \epsilon \mathscr{A}_m} \left[K_{ij}\right]
$$

and $\sum\limits_{i \in \mathscr{A}_m} \sum\limits_{j \in \mathscr{A}_m}$ indicates that the summation is performed only for those values of i and j which are identified by the set $\mathscr{A}_m$.

The system inertia matrix can also be expressed as

$$\underset{(n \times n)}{[M]} = \sum_{i=1}^{N\mathscr{P}} \mathscr{P}_i [M_i] \tag{14}$$

where $\mathscr{P}_i$, $i = 1,N\mathscr{P}$, are inertia design parameters, and $[M_i]$, $i = 1,N\mathscr{P}$, are constructed in the same manner as $[K_i]$, $i = 1,NP$. The design parameters for the stiffness and inertia matrices are defined independently of each other and, in general, will not be the same.

Equations (13) and (14) express the system stiffness and inertia matrices as linear functions of two different sets of design parameters $\{P\}$ and $\{\mathscr{P}\}$, respectively. The recalculation of $[K]$ and $[M]$ for any set of design parameters is facilitated if the constant matrices $[K_i]$ and $[M_i]$ are saved during initial analysis for subsequent use.

In addition to equations (13) and (14), the expressions for the partial derivatives of $[K]$ and $[M]$ will be subsequently used to develop the present method. These expressions are given by

$$\left([K]\right)_j = \frac{\partial [K]}{\partial P_j} = [K_j] \tag{15}$$

$$\left([M]\right)_j = \frac{\partial [M]}{\partial \mathscr{P}_j} = [M_j] \tag{16}$$

Equations (13) to (16) will be subsequently used in the derivation of the expressions for the generalized stiffness and inertia matrices, and their derivatives with respect to the design parameters.

Determination of Generalized Stiffness and Inertia Matrices During Reanalysis

The system stiffness and inertia matrices can be expressed as functions of the stiffness and inertia design parameters (eqs. (13) and (14)), respectively, and the corresponding generalized matrices (eqs. (8a) and (8b)) can be determined by the method described earlier. The generalized matrices can be expressed in terms of the design parameters, and equations (8a) and (8b) are rewritten as

$$\underset{(NM \times NM)}{\left[KG\right]} = \sum_{i=1}^{NP} P_i \left[\left[Q\right]^T \left[K_{hh,i}\right] \left[Q\right] + \left[Q\right]^T \left[K_{h\alpha,i}\right] \left[S\right] \right] \tag{17}$$

and

$$\underset{(NM \times NM)}{\left[MG\right]} = \sum_{i=1}^{N\mathscr{P}} \mathscr{P}_i \left[\left[Q\right]^T \left[M_{hh,i}\right] \left[Q\right] + \left[Q\right]^T \left[M_{h\alpha,i}\right] \left[S\right] \right.$$
$$\left. + \left[\left[Q\right]^T \left[M_{h\alpha,i}\right] \left[S\right] \right]^T + \left[S\right]^T \left[M_{\alpha\alpha,i}\right] \left[S\right] \right] \tag{18}$$

where $\left[K_{hh,i}\right]$, $\left[M_{hh,i}\right]$, etc., are defined by

$$\underset{(n \times n)}{\left[K_i\right]} = \left[\begin{array}{c|c} \left[K_{hh,i}\right] & \left[K_{h\alpha,i}\right] \\ \hline \left[K_{\alpha h,i}\right] & \left[K_{\alpha\alpha,i}\right] \end{array} \right]$$

and

$$\underset{(n \times n)}{\left[M_i\right]} = \left[\begin{array}{c|c} \left[M_{hh,i}\right] & \left[M_{h\alpha,i}\right] \\ \hline \left[M_{\alpha h,i}\right] & \left[M_{\alpha\alpha,i}\right] \end{array} \right]$$

The generalized matrices are now expanded in Taylor series about the initial design, and the series expansions are given by

$$\underset{(NM \times NM)}{\left[KG\right]} = \left[KG^B\right] + \sum_{j=1}^{NP} \left(\left[KG\right]\right)_j \left(P_j - P_j^B\right) + \ldots \tag{19}$$

and

$$\underset{(NM \times NM)}{[MG]} = [MG^B] + \sum_{j=1}^{N\mathscr{P}} ([MG])_j (\mathscr{P}_j - \mathscr{P}_j^B) + \ldots \tag{20}$$

where the superscript B refers to the initial design.

The following assumptions are now made:

(1) The generalized matrices are linear in the design parameters. This linearity allows neglecting the second and higher order terms in the Taylor series expansions. Consequently the generalized derivatives $([KG])_j$, $j = 1,NP$, and $([MG])_j$, $j = 1,N\mathscr{P}$, are constants.

(2) The modal transformation matrix $[Q]$ is a constant matrix. Therefore during the reanalysis, the modal reduction (that is, from NZ to NM coordinates) is essentially the same as in the assumed-mode method except that in the present method the assumed modes are the normal modes of vibration (obtained from eq. (7)) of the initial structure. It is known that by itself the assumed-mode method during reanalysis would give good approximations for the lower modes of the vibration (ref. 17).

(3) The transformation matrix $[T]$ is a constant for the purpose of calculating $([MG])_i$, $i = 1,N\mathscr{P}$. This assumption simplifies the expression for $([MG])_i$ and will be further discussed in appendix A.

In order to derive an expression for $([KG])_j$, equation (17) is differentiated with respect to the design parameter P_j. The resulting equation is

$$\underset{(NM \times NM)}{([KG])_j} = [Q]^T [K_{hh,j}][Q] + [Q]^T [K_{h\alpha,j}][S] + [Q]^T [K_{h\alpha}]([S])_j \tag{21}$$

where $[Q]$ is a constant matrix. Since $\underset{(NR \times NM)}{[S]} = [T][Q]$,

$$([S])_j = ([T])_j [Q] \tag{22}$$

From equation (4),

$$\underset{(NR \times NZ)}{[T]} = -[K_{\alpha\alpha}]^{-1}[K_{\alpha h}]$$

Therefore

$$\left([T]\right)_{,j} = -\left([K_{\alpha\alpha}]^{-1}\right)_{,j}[K_{\alpha h}] - [K_{\alpha\alpha}]^{-1}\left([K_{\alpha h}]\right)_{,j}$$

or,

$$\left([T]\right)_{,j} = [K_{\alpha\alpha}]^{-1}\left[\left([K_{\alpha\alpha}]\right)_{,j}[K_{\alpha\alpha}]^{-1}[K_{\alpha h}] - \left([K_{\alpha h}]\right)_{,j}\right]$$

$$= -[K_{\alpha\alpha}]^{-1}\left[[K_{\alpha\alpha,j}][T] + [K_{\alpha h,j}]\right] \tag{23}$$

where $[K_{\alpha\alpha,j}] = \left([K_{\alpha\alpha}]\right)_{,j}$, etc., since $[K]$ is a linear function of the design parameters. Substituting equation (23) into equation (22) yields

$$\left([S]\right)_{,j} = -[K_{\alpha\alpha}]^{-1}\left[[K_{\alpha\alpha,j}][S] + [K_{\alpha h,j}][Q]\right] \tag{24}$$

Then

$$[Q]^T[K_{h\alpha}]\left([S]\right)_{,j} = [S]^T[K_{\alpha\alpha,j}][S] + [S]^T[K_{\alpha h,j}][Q]$$

Substituting this expression into equation (21) results in

$$\underset{(NM \times NM)}{\left([KG]\right)_{,j}} = [Q]^T[K_{hh,j}][Q] + [Q]^T[K_{h\alpha,j}][S] + \left[[Q]^T[K_{h\alpha,j}][S]\right]^T + [S]^T[K_{\alpha\alpha,j}][S] \tag{25}$$

Equation (25) gives the expression for evaluating the partial derivatives of the generalized stiffness matrix. Since the matrix $\left([KG]\right)_{,j}$ calculated from this expression is symmetric, the generalized stiffness matrix calculated from equation (19) will also be symmetric.

An expression for the partial derivatives of the inertia matrix can be similarly derived by differentiating equation (9) and is given by

$$
\underset{(NM \times NM)}{\left(\left[MG\right]\right)_{,j}} = \left[Q\right]^{T}\left[M_{hh,j}\right]\left[Q\right] + \left[Q\right]^{T}\left[M_{h\alpha,j}\right]\left[S\right] + \left[\left[Q\right]^{T}\left[M_{h\alpha,j}\right]\left[S\right]\right]^{T} + \left[S\right]^{T}\left[M_{\alpha\alpha,j}\right]\left[S\right] + \left[Q\right]^{T}\left[M_{h\alpha}\right]\left(\left[S\right]\right)_{,j} + \left[\left[Q\right]^{T}\left[M_{h\alpha}\right]\left(\left[S\right]\right)_{,j}\right] + \left[S\right]^{T}\left[M_{\alpha\alpha}\right]\left(\left[S\right]\right)_{,j} + \left[\left[S\right]^{T}\left[M_{\alpha\alpha}\right]\left(\left[S\right]\right)_{,j}\right]^{T} \tag{26}
$$

where $\left(\left[S\right]\right)_{,j}$ is given by equation (24). If $\left[T\right]$ is assumed to be a constant,[3] $\left(\left[MG\right]\right)_{,j}$ is simplified to

$$
\left(\left[MG\right]\right)_{,j} = \left[Q\right]^{T}\left[M_{hh,j}\right]\left[Q\right] + \left[Q\right]^{T}\left[M_{h\alpha,j}\right]\left[S\right] + \left[\left[Q\right]^{T}\left[M_{h\alpha,j}\right]\left[S\right]\right]^{T} + \left[S\right]^{T}\left[M_{\alpha\alpha,j}\right]\left[S\right] \tag{27}
$$

Further discussion of the justification for assuming $\left[T\right]$ a constant matrix is presented in appendix A.

Equations (19) and (20), in conjunction with equations (25) and (27), provide the basis for an efficient procedure for calculating $\left[KG\right]$ and $\left[MG\right]$ during an iterative reanalysis. The generalized matrices corresponding to the initial design and their derivatives with respect to the design parameters can be determined and saved during the initial design cycle. For any perturbation from the initial design, the new generalized matrices can be easily determined.

APPLICATION OF THE PRESENT METHOD

The results of application and the efficiency of the present method are discussed in this section. During the initial analysis cycle, the generalized structural matrices and their partial derivatives with respect to the design parameters, are calculated by using equations (8a), (8b), (25), and (27). In evaluating the partial derivatives of the generalized inertia matrix by equation (27), the transformation matrix $\left[T\right]$ is assumed to be a constant matrix. The generalized matrices have been derived by using static condensa-

---

[3] Equation (26) can be simplified if $\left[T\right]$ is assumed to be a constant. It is noted that the last four terms in equation (26) contain the product $\left[M_{h\alpha}\right]\left[K_{\alpha\alpha}\right]^{-1}$ or $\left[M_{\alpha\alpha}\right]\left[K_{\alpha\alpha}\right]^{-1}$, and intuitively their sum would be (numerically) substantially smaller than the sum of the first four terms. This condition led to the assumption of constant $\left[T\right]$ for the purpose of evaluating $\left(\left[MG\right]\right)_{,j}$.

tion, modal reduction, and the assumption that they can be expressed by linear Taylor series expansions. The generalized matrices during reanalysis are, therefore, approximate, and the eigensolutions obtained with these matrices are also approximate. In order to evaluate the results from the new method, the natural frequencies determined from the approximate generalized matrices are compared with the natural frequencies determined from a "reference method."

In the reference method, the system structural matrices are formulated as in equations (13) and (14) and are assembled during each reanalysis from the $\left[K_i\right]$ and $\left[M_i\right]$ matrices corresponding to the various design parameters. The typical procedure described earlier is then employed to determine the eigenproblem expressed by equation (7). The natural frequencies obtained from the solution of equation (7) are then taken as a basis for the comparison of the natural frequencies obtained from the present method. It is noted that the reference method is also approximate but it is widely used; therefore it provides a rational basis for the comparison of the new method. The computer central processing unit (CPU) times required for reanalysis by the present and reference methods are compared to determine the efficiency of the present method.

Results of Application

The present dynamic reanalysis method is applied to simple trapezoidal and rectangular plates with various boundary conditions. The structure is represented in all cases as an assemblage of triangular-plate bending elements (ref. 14, pp. 111-115) whereas inertia properties are described by a consistent mass representation. Each element has a total of nine degrees of freedom – three normal displacement degrees of freedom and six rotational degrees of freedom. The rotational degrees of freedom were chosen to be eliminated from the assembled system structural matrices by static condensation.

For the first set of example problems, the trapezoidal plate structure shown in figure 1 is represented by 35 grid points defining 48 finite elements. The free structure has a total of 105 degrees of freedom. Thickness of each element is taken to be a design variable. There are thus 48 design variables and the same number of design parameters. All percent changes in design variables are with reference to their initial values. The various cases studied are described below, and the generalized matrices are 10×10 in each case unless otherwise mentioned.

Case 1.- The grid points 1 to 5 are rigidly fixed and represent a cantilevered boundary condition at $X = 0$. During the first reanalysis, all the design variables were uniformly increased by 300 percent, and during the second reanalysis they were increased by 1500 percent from their respective initial values. The results are given in table I. The frequencies from the present method are seen to be identical to the frequencies

INITIAL PLATE THICKNESSES		
Elements	Thickness	
	cm	in
1,5,9,13,17,21, 25,29,33,37,41,45	0.47625	0.1875
4,8,12,16,20,24, 28,32,36,40,44,48	0.47625	0.1875
All remaining elements	0.9525	0.375

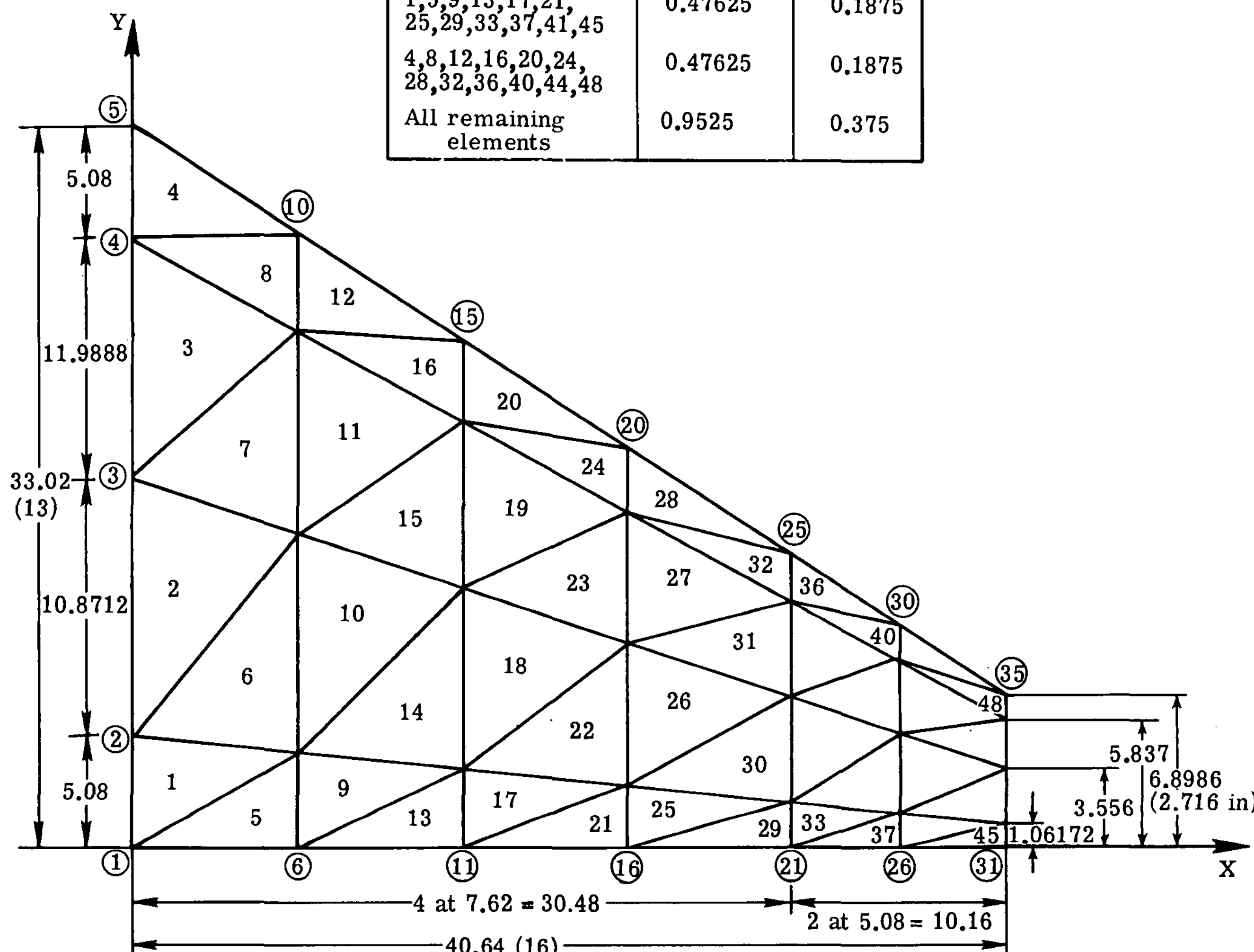

Figure 1.- Swept, tapered plate, finite-element idealization. Circled numbers are grid points; plain numbers are elements. All dimensions are in centimeters (inches).

TABLE I.- COMPARISON OF NATURAL FREQUENCIES FOR THE CANTILEVERED SWEPT, TAPERED PLATE (CASE 1)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	55.713	222.85	222.85	0.0	891.40	891.40	0.0
2	276.62	1106.5	1106.5		4 426.0	4 426.0	
3	364.77	1459.1	1459.1		5 836.2	5 836.2	
4	720.95	2883.8	2883.8		11 535	11 535	
5	909.96	3639.8	3639.8		14 559	14 559	
6	1390.9	5563.5	5563.5		22 254	22 254	
7	1470.6	5882.4	5882.4		23 530	23 530	
8	1689.0	6755.8	6755.8		27 023	27 023	
9	2245.7	8982.6	8982.6		35 930	35 930	
10	2459.5	9837.8	9837.8		39 351	39 351	

*All design variables increased uniformly by 300 percent of their respective initial values for the first reanalysis; all design variables increased uniformly by 1500 percent of their respective initial values for second reanalysis.

from the reference method; thus, it is demonstrated that the present method does indeed give exact natural frequencies for uniform changes in the design parameters as shown in appendix A.

Case 2.- The boundary conditions for this case are the same as those for case 1. The thicknesses for the elements numbered 2, 3, and 7 were fixed at their initial values, and were not changed during reanalysis. The rest of the design variables were increased by 25 percent during the first reanalysis, and by 56.25 percent during the second reanalysis. The results are presented in table II. The maximum error for the first reanalysis is about 1.78 percent in the ninth mode, and for the second reanalysis it is about 5.2 percent in the tenth mode. Thus, the results obtained from the present method can be considered to be good even when a few design variables remain unchanged while most of the design variables are changed uniformly.

Case 3.- The boundary conditions are the same as in the previous two cases. During the first reanalysis, the thickness of element number 23 was set to zero. In the second reanalysis, the thicknesses for two elements, element numbers 18 and 23, were set to zero. From the results in table III, the maximum error occurs in the second mode; for the first reanalysis it is 1.905 percent and for the second reanalysis it is 5.749 percent. When three elements were removed, the present method gave an error of about 13 percent in the second mode and about 14 percent in the tenth mode; these results are not tabulated. The results show that the present method gives reasonably good results even for drastic changes in the structure. This type of application could possibly be used to study the effect of structural cutouts.

Case 4.- The boundary conditions are again the same as in the previous three cases. A constant concentrated mass of 0.45359 kilogram (1 lbm weight) is placed at the grid point number 35. The design variables were uniformly increased by 25 percent and 56.25 percent of their respective initial values. The results are presented in table IV. The frequencies calculated from the present method are almost identical to those computed from the reference method. Since the concentrated mass remains constant during reanalysis, the inertia matrix is effectively experiencing a nonuniform change in the inertia design parameters. But the uniform change in the stiffness design parameters causes the transformation matrix $[T]$ to remain a constant, and therefore the frequencies from the present and reference methods can be expected to be nearly identical.

Case 5.- The swept, tapered plate structure is restrained only at grid point number 3; a 451.939 newton-meter/radian (4000 lb-in./radian) spring restrains rotation about the X-axis, and the normal displacement in the Z-direction and the rotation about the Y-axis are specified as zero. This configuration simulates the arrangement of an all-movable control surface. The results presented in table V indicate that the frequencies from the present method are almost the same as those from the reference method.

TABLE II.- COMPARISON OF NATURAL FREQUENCIES FOR THE CANTILEVERED SWEPT, TAPERED PLATE (CASE 2)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	55.713	61.785	62.076	0.471	68.982	69.980	1.447
2	276.62	322.70	322.70	.000	378.07	381.39	.878
3	364.77	439.22	440.05	.291	533.22	536.92	.694
4	720.95	863.67	866.06	.277	1032.3	1052.7	1.976
5	909.96	1103.6	1105.9	.208	1343.3	1356.2	.960
6	1390.9	1654.9	1663.6	.526	1962.5	2018.7	2.864
7	1470.6	1734.3	1742.8	.490	2034.1	2112.9	3.874
8	1689.0	2084.8	2092.4	.365	2574.5	2604.8	1.177
9	2245.7	2658.6	2705.9	1.779	3162.9	3324.2	5.100
10	2459.5	2919.1	2968.7	1.699	3473.3	3654.4	5.214

*Design variables for element numbers 2, 3, and 7 fixed at their respective initial values; all others increased uniformly by 25 percent of their respective initial values for first reanalysis and 56.25 percent of their respective initial values for second reanalysis.

TABLE III.- COMPARISON OF NATURAL FREQUENCIES FOR THE CANTILEVERED SWEPT, TAPERED PLATE (CASE 3)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	55.713	55.301	55.584	0.512	52.281	54.466	4.179
2	276.62	269.81	274.95	1.905	257.44	272.24	5.749
3	364.77	359.85	361.66	.503	345.60	351.04	1.574
4	720.95	716.38	717.59	.169	688.44	708.51	2.843
5	909.96	881.85	889.70	.890	859.77	875.66	1.848
6	1390.9	1384.9	1406.4	1.553	1342.1	1372.5	2.265
7	1470.6	1457.7	1459.2	.103	1417.1	1454.2	2.618
8	1689.0	1668.7	1669.0	.018	1668.7	1684.0	.917
9	2245.0	2221.0	2223.1	.095	2148.9	2200.8	2.378
10	2459.5	2432.4	2453.6	.872	2484.3	2615.6	5.285

*For the first reanalysis, thickness of element number 23 set to zero and all other design variables were retained equal to their respective initial values. For the second reanalysis, thicknesses of element numbers 18 and 23 set to zero and all other design variables were retained equal to their respective initial values.

TABLE IV.- COMPARISON OF NATURAL FREQUENCIES FOR THE CANTILEVERED SWEPT, TAPERED PLATE WITH CONCENTRATED MASS (CASE 4)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	44.813	58.130	58.130	0.000	74.997	74.997	0.000
2	240.17	306.38	306.38		390.26	390.26	
3	339.78	427.17	427.17		537.23	537.23	
4	675.88	853.37	853.38	.001	1076.8	1076.8	
5	790.46	996.43	996.43	.000	1257.9	1258.0	.010
6	1312.1	1655.1	1655.1		2088.9	2089.1	.010
7	1401.9	1753.9	1753.9		2195.2	2195.2	.000
8	1532.5	1919.9	1919.9		2406.7	2406.8	.005
9	2054.2	2579.1	2579.1		3240.7	3241.3	.020
10	2418.4	3029.4	3029.6	.006	3796.4	3797.7	.030

*All design variables increased by 25 percent of their respective initial values for the first reanalysis; all design variables increased by 56.25 percent of their respective initial values for the second reanalysis.

TABLE V.- COMPARISON OF NATURAL FREQUENCIES FOR THE SPRING-MOUNTED, SWEPT, TAPERED PLATE (CASE 5)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	16.094	14.658	14.689	0.212	13.233	13.360	0.960
2	74.788	92.133	92.143	.011	114.31	114.34	.026
3	251.14	313.93	313.93	.000	392.41	392.41	.000
4	430.36	537.49	537.53	.007	671.58	671.64	.009
5	532.73	665.81	665.81	.000	832.19	832.21	.002
6	730.35	912.92	912.92	.000	1141.1	1141.1	.000
7	989.87	1237.1	1237.2	.008	1546.2	1546.3	.007
8	1279.5	1599.3	1599.4	.006	1999.1	1999.2	.005
9	1375.4	1719.1	1719.1	.000	2148.8	2148.8	.000
10	1670.3	2087.5	2087.7	.010	2609.2	2609.5	.012

*All design variables increased by 25 percent of their respective initial values for the first reanalysis; all design variables increased by 56.25 percent of their respective initial values for the second reanalysis.

In this example, even though the design parameters are uniformly changed, the transformation matrix [T] does not remain a constant because the spring stiffness remains constant during the reanalysis. The results of this example show that the present method can be expected to provide good results for the spring-support type of boundary conditions.

In a variation of case 5, a concentrated mass of 0.45359 kilogram (1 lbm weight) was placed at the grid point number 35. The percent errors in frequencies by the present method were almost the same as those for the example without the concentrated mass. To study the effect of the number of modes, the example cases 4 and 5 were rerun by using the first five modes of natural vibration for the generalized matrices instead of the first ten modes of natural vibration. The results showed that there was no appreciable increase in the percent errors (five modes compared with ten modes). For the example without the concentrated mass, the maximum errors for 56.25 percent uniform change in the design variables was 1.096 when using five modes compared with 0.960 when using ten modes.

In the second series of examples, a rectangular-plate structure idealized into 48 triangular finite elements was considered. This structure is illustrated in figure 2. Two types of boundary conditions were considered and these are described by the following two cases. In all the examples discussed below, the generalized matrices were calculated from the first seven modes of natural vibration.

Case 6.- The rectangular plate is rigidly supported along its four edges. The thicknesses were uniformly increased for all the elements by 25 percent and then by 56.25 percent of their respective initial values. The natural frequencies calculated from the present method were identical to those calculated from the reference method, and therefore the results for this method are not presented. In a subsequent study, the thicknesses for elements 17 to 32 were increased by 25 percent and 56.25 percent from their respective initial values whereas the remainder of the element thicknesses were unchanged from their respective initial values. The results for this example are presented in table VI. The maximum error in frequency is 2.265 percent in the third mode for the 25 percent increase, and about 10 percent in the seventh mode for the 56.25 percent increase. The results are considered to be acceptable for reanalysis during automated design.

Case 7.- The rectangular plate of figure 2 is simply supported along its edges. For uniform increase in thickness for all the elements, the natural frequencies calculated from the present method were identical to those from the reference method. The results for a case when the thicknesses for element numbers 17 to 32 are increased, are given in table VII. The percent errors for this case are somewhat smaller than those for the plate with the fixed edges.

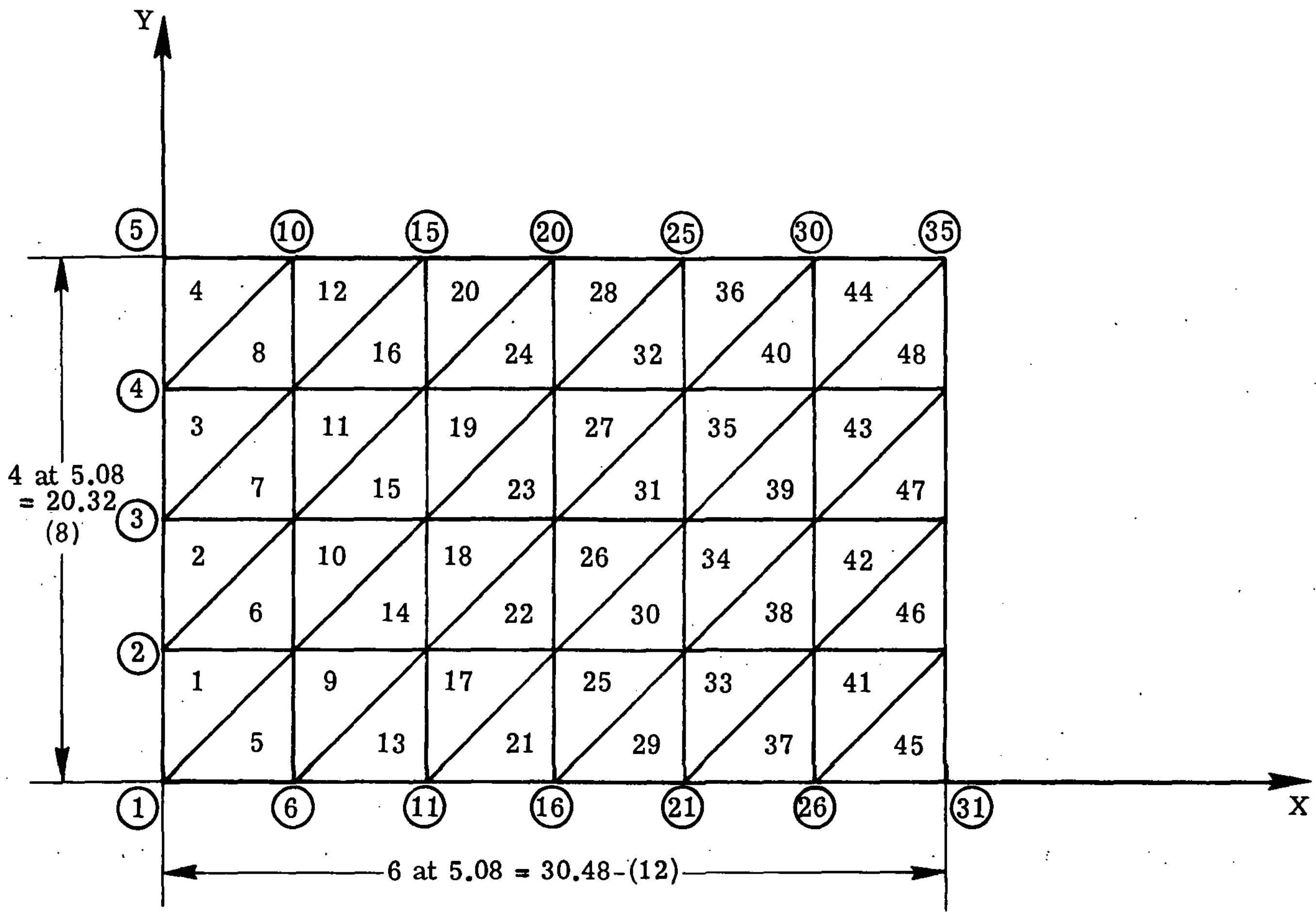

Figure 2.- Rectangular-plate, finite-element idealization. Initial thickness for all elements, 0.254 cm (0.1 in.). Circled numbers are grid points; plain numbers are elements. All dimensions are in centimeters (inches).

TABLE VI.- COMPARISON OF NATURAL FREQUENCIES FOR THE RECTANGULAR PLATE WITH FIXED EDGES (CASE 6)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	385.22	434.20	435.13	0.214	503.20	507.40	0.835
2	579.48	631.22	637.30	.963	699.95	714.18	2.033
3	938.29	1020.0	1043.1	2.265	1123.2	1219.4	8.565
4	1076.4	1231.0	1232.2	.098	1416.7	1432.8	1.136
5	1218.4	1338.1	1363.8	1.921	1486.7	1612.8	8.482
6	1441.1	1555.3	1554.3	-.064	1659.9	1733.3	4.422
7	1544.5	1662.4	1693.9	1.895	1797.2	1976.9	9.999

*Design variables for element numbers 17 to 32 increased by 25 percent of their respective initial values for the first reanalysis; design variables for element numbers 17 to 32 increased by 56.25 percent of their respective initial values for the second reanalysis.

TABLE VII.- COMPARISON OF NATURAL FREQUENCIES FOR THE RECTANGULAR PLATE WITH SIMPLY SUPPORTED EDGES (CASE 7)

Mode	Initial analysis	First reanalysis*			Second reanalysis*		
	Frequency, Hz	Frequency (reference method), f_1, Hz	Frequency (present method), f_2, Hz	Percent error, $\left(\frac{f_2}{f_1} - 1\right) \times 100$	Frequency (reference method), f_3, Hz	Frequency (present method), f_4, Hz	Percent error, $\left(\frac{f_4}{f_3} - 1\right) \times 100$
1	187.51	203.29	203.89	0.295	226.55	228.79	0.989
2	359.52	391.38	394.1	.695	433.77	439.20	1.252
3	603.96	681.69	682.37	.100	788.83	793.41	.581
4	663.52	721.7	733.13	1.584	794.62	853.17	7.368
5	748.79	818.62	831.70	1.598	909.77	971.77	6.815
6	1017.5	1100.9	1122.8	1.990	1194.1	1301.3	8.978
7	1135.7	1218.9	1232.5	1.116	1288.9	1404.4	8.961

*Design variables for element numbers 17 to 32 increased by 25 percent of their respective initial values for the first reanalysis; design variables for element numbers 17 to 32 increased by 56.25 percent of their respective initial values for the second reanalysis.

TABLE VIII.- COMPUTER TIMES FOR THE PRESENT AND REFERENCE METHODS

[N = Total number of analyses (including initial analysis)]

Total number of degrees of freedom	Number of degrees of freedom retained after condensation	Number of generalized coordinates	Number of design parameters	CDC 6600 CPU time, sec				Present* method time as percent of reference method for reanalysis
				Reference method		Present method		
				Eigensolution only	Total	Derivative calculation*	Analysis	
90	30	10	48	7.728	13.044	11.356	0.444	$3.4 + \frac{187.1}{N}$
103	34	10	48	10.592	17.290	12.488	.470	$2.7 + \frac{172.2}{N}$
103	34	5	48	10.572	18.292	3.294	.178	$1.0 + \frac{117.91}{N}$
45	15	7	48	1.038	1.990	3.400	.258	$13.0 + \frac{270.9}{N}$
85	15	7	48	.992	4.846	5.138	.258	$5.3 + \frac{206.0}{N}$

*Derivative calculations performed once only during the initial analysis.

Comparison of Computer Times for Reanalysis

The computer cost saved by using the present method for iterative reanalysis is a complicated function of the number of (a) degrees of freedom n, (b) design parameters NP and $N\mathscr{P}$, (c) degrees of freedom eliminated NR, (d) generalized coordinates NM, and (e) total number of reanalysis cycles. Table VIII presents a comparison of the computer CPU times required with the present and reference methods. The computer time listed for the eigensolution in the reference method is for complete solution of equation (7), and all the NZ eigenvalues and eigenvectors were calculated even though only the first NM eigenvectors were needed. The computer subroutine used for the eigensolution is based on Jacobi's method and finds the solution for all NZ modes. However, for the cases in table VIII, the reference method CPU times are not heavily penalized since most of the methods that would be used for partial modal solution would still require determination of the symmetric matrix $\left[\left[M_h\right]^{-1/2}\right]^T \left[K_h\right]\left[M_h\right]^{-1/2}$ or $\left[\left[K_h\right]^{-1/2}\right]^T \left[M_h\right]\left[K_h\right]^{-1/2}$. Thus, the CPU times in table VIII can be used for the comparison and show the efficiency of the present method. Each "analysis" (table VIII) for the present method involved evaluating the generalized matrices for the new set of design variables and solving the eigenvalue problem of order NM. It is seen that the present method yields substantial time savings and hence cost savings when a large number of analyses are required. For example, from the first entry in table VIII, the present method would require about 3.6 percent CPU time for the reference method if 100 reanalyses are required, and about 22 percent CPU time if 10 analyses are required. This statement assumes that generalized derivatives are calculated only once. If the generalized derivatives are required to be recomputed, the additional time required for each recomputation is of the order of the time required for one analysis by the reference method.

A comparison of the number of multiplications required in the present method to the number required in the reference method indicates that as the problem size and/or the number of reanalysis cycles increase, the computer cost savings achieved by using the present method relative to the reference method increase.

CONCLUDING REMARKS

A method for iterative reanalysis in automated structural design has been presented for a finite-element analysis using the direct stiffness approach. A basic feature of the method is that the generalized stiffness and inertia matrices are expressed as functions of structural design parameters, and these generalized matrices are expanded in Taylor series about the initial design. Only the linear terms are considered in the expansions.

The method is approximate because it uses static condensation, modal reduction, and the linear Taylor series expansions. The exact linear representation of the expansions of the generalized matrices is also described, and a basis for the present method is established. The formulation of the present method provides a simple means of evaluating the derivatives of the structural matrices with respect to structural design variables.

The method has been illustrated for dynamic reanalysis by applications to simple trapezoidal and rectangular plate models. The results in terms of natural frequencies were compared with the results obtained from a complete commonly applied structural analysis. Several support boundary conditions were considered, and the results obtained from the present method were generally in good agreement with the results from the reference method. A comparison of the computer times required for the use of the present method and the reference method indicated that the present method required substantially less time for reanalysis. The present method will become more efficient relative to the reference method as the problem size increases.

The application of the present method to static reanalysis has been described and the basic equations have been derived. A procedure for combining the static and dynamic reanalysis into a unified approach is offered, but this procedure has not been demonstrated by actual application.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., July 9, 1973.

APPENDIX A

THE APPROXIMATE AND EXACT LINEAR REPRESENTATION OF GENERALIZED MATRICES

It was assumed that the generalized stiffness and inertia matrices are linear functions of their respective design parameters, and that the transformation matrix $[T]$ is a constant for the purpose of deriving expressions for the derivative of the generalized inertia matrix with respect to an inertia design parameter. It is shown in this appendix that $[T]$ remains a constant if all the stiffness design parameters are uniformly changed, that is, the design parameters are changed by a constant percentage from their respective initial values. The exact linear representation of the generalized matrices is then discussed, and it is shown that the assumption of a constant $[T]$ in deriving the expression for the generalized inertia matrix derivatives is justified.

The transformation matrix $[T]$ for the initial analysis is given by equation (3) and is

$$
\begin{aligned}
[T] &= -\left[K_{\alpha\alpha}\right]^{-1}\left[K_{\alpha h}\right] \\
&= -\left[P_i^B\left[\overline{K}_{\alpha\alpha}\right]\right]^{-1}\left[P_i^B\left[\overline{K}_{\alpha h}\right]\right] \\
&= -\left[\overline{K}_{\alpha\alpha}\right]^{-1}\left[\overline{K}_{\alpha h}\right]
\end{aligned}
\tag{A1}
$$

where an arbitrary initial stiffness design parameter P_i^B is factored out from $\left[K_{\alpha\alpha}\right]$ and $\left[K_{\alpha h}\right]$, and each element of $\left[\overline{K}_{\alpha\alpha}\right]$ $\left(\text{or } \left[\overline{K}_{\alpha h}\right]\right)$ is obtained by dividing the corresponding element of $\left[K_{\alpha\alpha}\right]$ $\left(\text{or } \left[K_{\alpha h}\right]\right)$ by P_i^B. Since $\left[K_{\alpha\alpha}\right]$ and $\left[K_{\alpha h}\right]$ are linear in $\{P\}$, each element of $\left[\overline{K}_{\alpha\alpha}\right]$ and $\left[\overline{K}_{\alpha h}\right]$ is either a constant multiplied by the ratio of a design parameter to P_i^B or a summation of several such terms. After a uniform percent change in all the stiffness design parameters from their initial values, the ratio of any design parameter to the ith design parameter will be the same as the corresponding ratio at the initial design, and $[T]$ will remain unchanged.[4] In this case, the expression for $\left(\left[MG\right]\right)_j$ in equation (27) is exact. In addition, $[KG]$ and $[MG]$ are linear in design parameters, and can be exactly represented by the constant and the linear term of the Taylor series expansion.

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[4] This relationship was pointed out to the author by William C. Walton, Jr., and Jerrold M. Housner of NASA Langley Research Center.

For a nonuniform change in the stiffness design parameters, the transformation matrix $[T]$ does not remain a constant, and the generalized matrices obtained from the two-term Taylor series expansions are not exact. An exact linear representation of the generalized matrices can be obtained if the static condensation is not performed, and the eigenvalue problem associated with the complete system matrices $[K]$ and $[M]$ is solved to determine the modal transformation matrix $[Q_n]$. Then

$$\underset{(NM \times NM)}{[KG]} = \underset{(NM \times n)}{[Q_n]^T} \; \underset{(n \times n)}{[K]} \; \underset{(n \times NM)}{[Q_n]} \tag{A2}$$

and

$$\underset{(NM \times NM)}{[MG]} \quad \underset{(NM \times n)}{[Q_n]^T} \; \underset{(n \times n)}{[M]} \; \underset{(n \times NM)}{[Q_n]} \tag{A3}$$

The generalized matrices thus obtained are linear with respect to the design parameters, and the Taylor series expansions obtained by using only the first two terms, are exact. The exact series representation is, however, achieved at the cost of solving a considerably larger eigenproblem, specifically one of order n. As an alternative, one may perform the static condensation and solve equation (7) of the order NZ to form the $(NZ \times NM)$ modal matrix $[Q]$. Since the transformation matrix $[T]$ has already been determined, the modal matrix for the unreduced system $[Q_n]$ can be calculated. The matrix $[Q_n]$ determined by this method would be, strictly speaking, an assumed-mode matrix approximating the exact modal matrix. In any case, $[Q_n]$ can be used to determine the generalized matrices which are linear in the design parameters. The generalized matrices can be expressed as

$$[KG] = \left[\begin{array}{c:c} [Q]^T & [S]^T \end{array}\right] \left[\begin{array}{c:c} [K_{hh}] & [K_{h\alpha}] \\ \hdashline [K_{\alpha h}] & [K_{\alpha\alpha}] \end{array}\right] \left[\begin{array}{c} [Q] \\ \hdashline [S] \end{array}\right] \tag{A4}$$

and

$$\left[MG\right] = \left[\begin{array}{c:c} \left[Q\right]^T & \left[S\right]^T \end{array}\right] \begin{bmatrix} \left[M_{hh}\right] & \left[M_{h\alpha}\right] \\ \left[M_{\alpha h}\right] & \left[M_{\alpha\alpha}\right] \end{bmatrix} \begin{bmatrix} \left[Q\right] \\ \left[S\right] \end{bmatrix} \tag{A5}$$

where $\left[S\right] = \left[T\right]\left[Q\right]$ as before, and

$$\left[Q_n\right] = \begin{bmatrix} \left[Q\right] \\ \left[S\right] \end{bmatrix}$$

Equations (A4) and (A5) are identical to equations (8a) $\left(\text{for } \left[KG\right]\right)$ and equation (8b) $\left(\text{for } \left[MG\right]\right)$, respectively. Differentiating equations (A4) and (A5) with respect to the design parameters P_j and $\mathcal{P}_j$, respectively, and assuming $\left[Q_n\right]$ to be a constant matrix gives expressions for $\left(\left[KG\right]\right)_j$ and $\left(\left[MG\right]\right)_j$. These expressions are identical to equations (25) and (27). It should be recalled that in deriving equation (27) for $\left(\left[MG\right]\right)_j$, $\left[T\right]$ was assumed to be a constant. Therefore, the assumption of a constant $\left[T\right]$ for the generalized inertia matrix derivatives is equivalent to obtaining an assumed-mode matrix $\left[Q_n\right]$ from the normal-mode matrix $\left[Q\right]$ for the initial design.

APPENDIX B

STATIC REANALYSIS

The application of the present method to static reanalysis is described in this appendix. The system of equations required to be solved for the structural displacements is

$$[K]\{\Delta\} = \{F\} \tag{B1}$$

where $\{\Delta\}$ is the displacement vector, and $\{F\}$ is the applied-force vector. The stiffness matrix $[K]$ is formulated as a linear function of the stiffness design parameters P_i, $i = 1,NP$, as in the dynamic reanalysis. Equation (B1) can be written in partitioned form as

$$\left[\begin{array}{c|c} [K_{11}] & [K_{12}] \\ \hline [K_{21}] & [K_{22}] \end{array}\right] \left\{\begin{array}{c} \{\Delta_1\} \\ \hline \{\Delta_2\} \end{array}\right\} = \left\{\begin{array}{c} \{F_1\} \\ \hline \{F_2\} \end{array}\right\} \tag{B2}$$

Let $\{\Delta_1\}$ be the set of displacements which are to be retained, and $\{\Delta_2\}$ be the set of displacements which are to be eliminated. For example, $\{\Delta_1\}$ could be the vector of normal displacements for a wing structure whereas $\{\Delta_2\}$ could be the vector of rotational displacements. In general, the force vector $\{F_2\}$ corresponding to $\{\Delta_2\}$, will not be identically zero.

From equation (B2), $\{\Delta_2\}$ can be expressed in terms of $\{\Delta_1\}$ and $\{F_2\}$ as follows:

$$\{\Delta_2\} = [T_S]\{\Delta_1\} + [K_{22}]^{-1}\{F_2\} \tag{B3}$$

where

$$[T_S] = -[K_{22}]^{-1}[K_{21}] \tag{B4}$$

Equation (B1) can then be written as

$$[\hat{K}]\{\Delta_1\} = \{\hat{F}\} \tag{B5}$$

where

$$\left[\hat{K}\right] = \left[K_{11}\right] + \left[K_{12}\right]\left[T_S\right] \tag{B6}$$

and

$$\left\{\hat{F}\right\} = \left\{F_1\right\} + \left[T_S\right]^T \left\{F_2\right\} \tag{B7}$$

The order of equation (B5) is less than the order of original displacement equation (B1). For the purpose of reanalysis, $\left[\hat{K}\right]$ and $\left\{\hat{F}\right\}$ can be expanded in Taylor series about the initial design. If only the first-order terms are retained, that is, if $\left[\hat{K}\right]$ and $\left\{\hat{F}\right\}$ are assumed to be linear in P_i, then

$$\left[\hat{K}\right] = \left[\hat{K}^B\right] + \sum_{i=1}^{NP} \left(\left[\hat{K}\right]\right)_i \left(P_i - P_i^B\right) \tag{B8}$$

and

$$\left\{\hat{F}\right\} = \left\{\hat{F}^B\right\} + \sum_{i=1}^{NP} \left(\left\{\hat{F}\right\}\right)_i \left(P_i - P_i^B\right) \tag{B9}$$

where the superscript B denotes the initial design point, P_i are the stiffness design parameters, NP is the number of stiffness design parameters, and a subscript following a parenthesis denotes a partial derivative with respect to the design parameter corresponding to the subscript. The partial derivatives are given by the following expressions:

$$\left(\left[\hat{K}\right]\right)_i = \left(\left[K_{11}\right]\right)_i + \left(\left[K_{12}\right]\right)_i \left[T_S\right] + \left[K_{12}\right]\left(\left[T_S\right]\right)_i \tag{B10}$$

$$\left(\left\{\hat{F}\right\}\right)_i = \left(\left[T_S\right]^T\right)_i \left\{F_2\right\} \tag{B11}$$

and

$$\begin{aligned}\left(\left[T_S\right]\right)_i &= -\left[K_{22}\right]^{-1}\left(\left[K_{22}\right]\right)_i\left[T_S\right] - \left[K_{22}\right]^{-1}\left(\left[K_{21}\right]\right)_i \\ &= -\left[K_{22}\right]^{-1}\left[\left(\left[K_{22}\right]\right)_i\left[T_S\right] + \left(\left[K_{21}\right]\right)_i\right]\end{aligned} \tag{B12}$$

During reanalysis, $\left[\hat{K}\right]$ and $\left\{\hat{F}\right\}$ can be evaluated from equations (B8) and (B9). As a consequence of assuming $\left[\hat{K}\right]$ and $\left\{\hat{F}\right\}$ to be linear, the partial derivatives $\left(\left[\hat{K}\right]\right)_i$ and $\left(\left\{\hat{F}\right\}\right)_i$, i = 1,NP, are constants and need to be calculated only at the initial design. The displacements $\left\{\Delta_1\right\}$ can be solved from equation (B5), and $\left\{\Delta_2\right\}$ can be determined from equation (B3).

In order to obtain $\left\{\Delta_2\right\}$ from equation (B3) during reanalysis, matrices $\left[T_S\right]$ and $\left[K_{22}\right]^{-1}$ are required. Several alternatives are described to obtain these matrices during reanalysis. By virtue of equation (B11) and the fact that the $\left\{\hat{F}\right\}$ are taken to be constants, $\left[T_S\right]$ is linear with respect to $\left\{P\right\}$, and is given by the following expression:

$$\left[T_S\right] = \left[T_S{}^B\right] + \sum_{i=1}^{NP} \left(\left[T_S\right]\right)_i \left(P_i - P_i{}^B\right) \tag{B13}$$

Note that $\left[K_{22}\right]^{-1}$ is not linear in $\left\{P\right\}$, but may be approximated by assuming it to be linear with respect to $1/P_i$, i = 1,NP, in a manner similar to reference 7. Thus

$$\begin{aligned}\left[K_{22}\right]^{-1} &= \left[K_{22}{}^B\right]^{-1} + \sum_{i=1}^{NP} \frac{\partial\left[K_{22}\right]^{-1}}{\partial\left(\frac{1}{P_i}\right)}\left(\frac{1}{P_i} - \frac{1}{P_i{}^B}\right) \\ &= \left[K_{22}{}^B\right]^{-1} + \sum_{i=1}^{NP} \left[\left[K_{22}{}^B\right]^{-1}\left(\left[K_{22}\right]\right)_i\left[K_{22}{}^B\right]^{-1}\right]\frac{P_i{}^B\left(P_i{}^B - P_i\right)}{P_i}\end{aligned} \tag{B14}$$

The matrix triple product $\left[K_{22}{}^B\right]^{-1}\left(\left[K_{22}\right]\right)_i\left[K_{22}{}^B\right]^{-1}$ is a constant matrix and has to be evaluated only during the initial analysis. It is apparent from equation (B3) that if $\left\{F_2\right\}$ is identically zero, $\left[K_{22}\right]^{-1}$ will not be required. If $\left\{F_2\right\}$ is small in magnitude relative to $\left\{F_1\right\}$, then $\left[K_{22}\right]^{-1}$ could be assumed to be constant for the purpose of determining $\left\{\Delta_2\right\}$. Alternatively, it may be argued that assuming $\left[T_S\right]$ to be linear in the stiffness design parameters implies that $\left[K_{22}\right]^{-1}$ is a constant. This condition is evident from equation (B4) and the fact that $\left[K_{21}\right]$ is formulated as a linear function of the stiffness design parameters. Therefore, assuming $\left[K_{22}\right]^{-1}$ to be a constant may be satisfactory in practice.

The procedure outlined above – static condensation followed by Taylor series expansion – is probably more efficient than reassembling a new $\left[K\right]$ matrix for each reanalysis and solving equation (B1) for $\{\Delta\}$. The procedure is, however, not as efficient as the method for dynamic reanalysis presented in the main body of this report because the static reanalysis does not incorporate modal reduction. In order to achieve the same degree of efficiency for the static reanalysis as for the dynamic reanalysis, it is proposed that the static reanalysis be performed by using an approach parallel to that of the dynamic reanalysis, in which case the static and dynamic reanalyses can be integrated into one unified approach. A procedure to accomplish this integration is described.

The degrees of freedom to be eliminated by static condensation in the static reanalysis are selected to be the same as in the dynamic reanalysis. The transformation matrix $[T_S]$ will now be identical to the transformation matrix $[T]$ for the dynamic reanalysis, and $\{\Delta_1\}$ and $\{\Delta_2\}$ will be identical to $\{U_h\}$ and $\{U_\alpha\}$, respectively. If the modal transformation matrix $[Q]$ determined for the dynamic reanalysis is used to transform the physical displacements $\{U_h\}$ to a set of generalized displacements $\{\xi\}$, then the generalized matrix $[KG]$ is the same for the static and dynamic reanalyses. The generalized displacements $\{\xi\}$ can be obtained from

$$[KG]\{\xi\} = \{F_g\} \tag{B15}$$

where

$$\{F_g\} = [Q]^T\{F_h\} + [S]^T\{F_\alpha\} \tag{B16}$$

and

$$[S] = [T][Q]$$

The vector $\{F_g\}$ is assumed to be expressed by a two-term Taylor series expansion

$$\{F_g\} = \left\{F_g{}^B\right\} + \sum_{i=1}^{NP} \left(\{F_g\}\right)_i \left(P_i - P_i{}^B\right) \tag{B17}$$

where

$$\left(\{F_g\}\right)_i = \left([S]^T\right)_i \{F_\alpha\}$$

and $\left([S]\right)_i$ is given by equation (24). If $\{F_\alpha\}$ is identically zero, then $\{F_g\}$ is a constant.

Equation (B15) can be solved to obtain $\{\xi\}$, and $\{U_h\}$ and $\{U_\alpha\}$ can be determined by back substitution; thus

$$\{U_h\} = [Q]\{\xi\} \tag{B18}$$

and

$$\{U_\alpha\} = [S]\{\xi\} + [K_{\alpha\alpha}]^{-1}\{F_\alpha\} \tag{B19}$$

Then $[S]$ is determined during reanalysis from

$$[S] = [S^B] + \sum_{i=1}^{NP} \left([S]\right)_i \left(P_i - P_i^B\right) \tag{B20}$$

where $\left([S]\right)_i$, given by equation (24), is determined only during the initial analysis. $[K_{\alpha\alpha}]^{-1}$ can be determined in the same manner as $[K_{22}]^{-1}$ in equation (B14), and is given by

$$[K_{\alpha\alpha}]^{-1} = [K_{\alpha\alpha}^B]^{-1} + \sum_{i=1}^{NP} \left[[K_{\alpha\alpha}^B]^{-1} \left([K_{\alpha\alpha}]\right)_i [K_{\alpha\alpha}^B]^{-1} \right] \frac{P_i^B\left(P_i^B - P_i\right)}{P_i} \tag{B21}$$

The procedure described for the approximate static reanalysis has not been programed and evaluated. However, the rationale for the static case is similar to that for the dynamic reanalysis which is shown to be economical and does give fairly accurate results.

APPENDIX C

COMPUTER PROGRAM RITREAD

The special purpose computer program RITREAD (Rapid Iterative Reanalysis for Automated Design), which was developed for solving the example problems presented in the main text, is described in this appendix. The program was written in Fortran IV for use on the Control Data 6000 series digital computers at the Langley Research Center. A simplified flow chart and a brief discussion of some of the salient features of RITREAD are included in this appendix. The program listing and the usage procedure are not described.

RITREAD has only a triangular plate bending element and the corresponding consistent mass representation, and each element has nine degrees of freedom associated with it. The finite-element idealizations for the plate configurations of the type shown in figures 1 and 2, can be automatically generated by RITREAD from a simple input.

RITREAD has been written for a maximum of 35 grid points and 48 elements. Each element is assigned one stiffness design parameter and one inertia design parameter, and each element thickness is taken to be a design variable. Therefore, the number of design variables, the number of stiffness design parameters, and the number of inertia design parameters are identical to the number of elements. The stiffness design parameter for each element is the element thickness cubed, and the inertia design parameter is equal to the element thickness. Each grid point has three degrees of freedom – one normal displacement and two rotations. The rotational degrees of freedom are eliminated by Guyan reduction, and the reduced stiffness and inertia matrices are used to solve for the normal vibration modes. A maximum of ten modes can be used to construct the modal matrix, and the maximum size of the generalized matrices is thus 10 by 10.

Figure 3 shows a simplified flow diagram of RITREAD. The program is arranged in multiple overlays with each overlay performing a logically discernible task. Overlay (0,0) monitors the execution sequence and transfers the required data to different overlays. The large amount of data required for element matrices, and the generalized matrices are written on the two disk files to save the in-core memory required, and to have the capability to eliminate recomputation of the element matrices, the base generalized matrices, and the generalized derivatives during a future run for the same structure.

In overlay (1,0), each unrestrained degree of freedom is assigned an identifying number determined from the finite-element definitions which are input into the program or automatically generated within the program from a simple input. The constant matrices of stiffness and inertia coefficients for each element are determined. Each constant matrix is $n \times n$, but has only 81 nonzero elements since the plate element has nine degrees

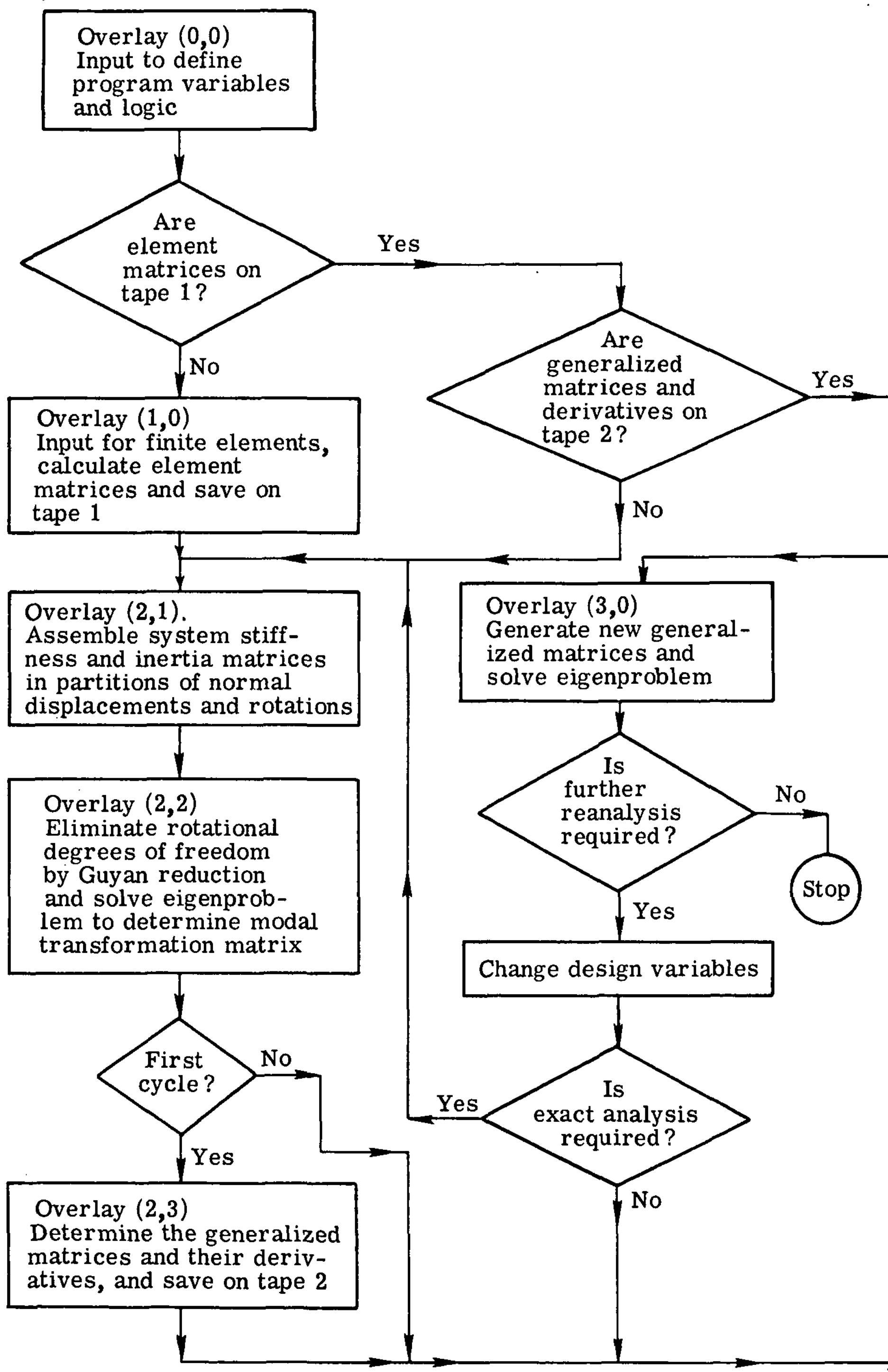

Figure 3.- Simplified flow diagram of program RITREAD.

of freedom. These matrix coefficients are written on a disk file as two matrices of size 9×9 along with a 9×1 vector $\{LG\}$ identifying the element degrees of freedom with the system degrees of freedom. For the restrained element displacements, the corresponding element in $\{LG\}$ is zero. This procedure minimizes the disk space required, and tape 1 can be copied on an external tape and saved for future use, if desired.

In overlay (2,1), the matrices of the stiffness and inertia coefficients and the associated vector $\{LG\}$ are read (one element per reading operation) from tape 1. Each coefficient matrix is multiplied by the appropriate initial design parameter, and the element stiffness and inertia matrices are thus determined for each element. The contributions from all elements are added to assemble directly the matrix partitions $[K_{hh}]$, $[K_{h\alpha}]$, $[K_{\alpha\alpha}]$, $[M_{hh}]$, $[M_{h\alpha}]$, and $[M_{\alpha\alpha}]$. These are passed by means of a labeled common block to overlay (2,2).

The transformation matrix $[T]$ is determined in overlay (2,2) by solving the matrix equation $[K_{\alpha\alpha}][T] = -[K_{h\alpha}]^T$ by use of Cholesky's method. The reduced matrices $[K_h]$ and $[M_h]$ are then determined by the expressions in equations (5) and (6). The modal matrix $[Q]$ is constructed from the solution of the eigenproblem of equation (7). The eigenproblem is solved by Jacobi's method, and all the NZ eigenvalues and eigenvectors are determined. The matrices $[T]$ and $[Q]$ are passed to overlay (2,3) by means of a labeled common block.

The generalized matrices $[KG^B]$ and $[MG^B]$ corresponding to the initial design, and the derivatives of the generalized matrices with respect to the design parameters are calculated in overlay (2,3). The matrix $[S]$ is first determined from the matrix product $[T][Q]$. The determination of $[KG^B]$ and $[MG^B]$, and the derivative calculations are performed in the same pass through the program. The value of $[KG^B]$ is obtained by multiplying the first two terms in the expression for $\left([KG]\right)_j$ in equation (25), by the corresponding stiffness design parameter, and summing the resulting matrix over all the NP design parameters. Similarly, $[MG^B]$ is obtained by simply evaluating $\sum_{i=1}^{N\mathscr{P}} \left([MG]\right)_i \mathscr{P}_i$, where $\left([MG]\right)_i$ is given by equation (27).

In determining the derivatives of the generalized matrices by equations (25) and (27), the triple matrix products of the form $[Q]^T[K_{hh,j}][Q]$, $[Q]^T[K_{h\alpha,j}][S]$, etc., are required. An optimum procedure for obtaining these products is desirable since the number of such triple products is $4(NP + N\mathscr{P})$. The general procedure used in RITREAD is illustrated for determining $[Q]^T[K_{hh,i}][Q]$. The 9×9 element stiffness matrix and the 9×1 vector $\{LG\}$ for the ith element are read from tape 1. The element stiffness matrix is rearranged into matrix partitions $[KZ]$, $[KZR]$, and $[KR]$ where $[KZ]$ is the 3×3 partition for the normal displacements (retained displacements), $[KR]$ is the 6×6 partition for the rotational displacements (eliminated displacements), and $[KZR]$ is the 3×6 partition of the normal displacements – rotations coupling. The vector $\{LG\}$ is also rearranged so that the first three terms identify the normal displacement degrees of freedom corresponding to $[KZ]$, and the last six terms identify the rotational degrees of freedom corresponding to $[KR]$. Any zeros in $\{LG\}$ indicate that during input the corresponding displacements associated with the particular grid points have been specified to be zero. The number of nonzeros for the normal displacements IZ and for the rotational displacements IR in $\{LG\}$ are counted. If $IZ < 3$, the subsequent elements to any zero element in the first three elements of $\{LG\}$ are moved up and the zero elements eliminated. If $IR < 6$, the last six elements of $\{LG\}$ are also similarly rearranged. The rows and columns in the three partitioned matrices, corresponding to the zeros, if any, in $\{LG\}$ are eliminated, and the subsequent rows and columns are moved to their proper position to correspond to the newly arranged $\{LG\}$. The matrix triple products can now be efficiently performed as illustrated by the following expression for A(I,J), where $[A] = [Q]^T[K_{hh,i}][Q]$:

$$A(I,J) = \sum_{l=1}^{IZ} Q\big(LG(l),I\big) \sum_{m=1}^{IZ} KZ(l,m)Q\big(LG(m),J\big)$$

Therefore, determination of $[Q]^T[K_{hh,i}][Q]$ requires $(NM^2)(IZ^2)$ multiplications instead of the $(NM^2)(NZ^2)$ multiplications required if the advantage of the zero elements is not taken. The other triple products are also carried out in the manner shown for the product $[Q]^T[K_{hh,i}][Q]$.

The matrices $[KG^B]$, $[MG^B]$, the design variables $\{D\}$, and the generalized stiffness and inertia derivatives are written on tape 2 in overlay (2,3). The generalized matrices are calculated for any set of design variables $\{D\}$ in overlay (3,0). The required matrices are read from tape 2, and the generalized stiffness and inertia matrices are determined from the expressions for the Taylor series expansion. Once $[KG]$ and $[MG]$ are determined, an eigenvalue problem of the order NM is solved for NM eigenvalues and eigenvectors. Tape 2 can be copied on an external tape if the same problem with different design variables is required to be solved at a future date.

This brief program description outlines the computational procedure used to implement the iterative reanalysis method presented in this paper for solution of the sample problems. Although this particular program is limited in size and is restricted to only one type of finite element, experience with its use has indicated that the basic program organization and solution flow is good and should form a basis for the development of a more comprehensive program.

APPENDIX D

DERIVATIVES OF STRUCTURAL MATRICES WITH RESPECT TO DESIGN VARIABLES

In automated structural design procedures based on mathematical programing, the derivatives of the structural matrices with respect to the design variables are often required. It is shown in this appendix that these derivatives can be obtained in a simple manner when the structural formulation is based on the method presented in this report.

The system stiffness and inertia matrices are expressed in equations (13) and (14) as linear functions of the design parameter. The design parameters themselves can be nonlinear functions of the structural design variables. Therefore the derivatives of the system matrices with respect to a design variable are

$$\frac{\partial[K]}{\partial D_j} = \sum_{i=1}^{NP} [K_i] \frac{\partial P_i}{\partial D_j} \qquad (j = 1,NV) \qquad \text{(D1)}$$

and

$$\frac{\partial[M]}{\partial D_j} = \sum_{i=1}^{N\mathscr{P}} [M_i] \frac{\partial \mathscr{P}_i}{\partial D_j} \qquad (j = 1,NV) \qquad \text{(D2)}$$

The derivatives of the generalized structural matrices with respect to a design variable are obtained by noting that the derivatives of the generalized matrices with respect to design parameters are constant. Thus,

$$\frac{\partial[KG]}{\partial D_j} = \sum_{i=1}^{NP} \left([KG]\right)_i \frac{\partial P_i}{\partial D_j} \qquad (j = 1,NV) \qquad \text{(D3)}$$

and

$$\frac{\partial[MG]}{\partial D_j} = \sum_{i=1}^{N\mathscr{P}} \left([MG]\right)_i \frac{\partial \mathscr{P}_i}{\partial D_j} \qquad (j = 1,NV) \qquad \text{(D4)}$$

The design parameters are functions of the design variables, and these functional definitions are known. Therefore, $\partial P_i/\partial D_j$, $\partial \mathscr{P}_i/\partial D_j$, etc., are easily determined. The higher derivatives can also be computed on the same basis.

The equations (C1) to (C4) are further simplified if the design variable D_j, $j = 1,NV$, is associated with only one or a few of the design parameters $\{P\}$ and $\{\mathscr{P}\}$. In such a case, the summations do not have to be performed over all the design parameters.

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