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MATHEMATICAL RESEARCH ON SPLINE FUNCTIONS

by

James M. Horner

Final Report

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INTRODUCTION

Spline functions are generally believed to have been introduced by I. J. Schoenberg in 1946. In the last decade, spline functions have attracted wide attention and the literature in this area has increased rapidly (approximately 350 known journal articles, books or dissertations in the period 1961-1970). Unfortunately, many of the recent developments are not in a form which is convenient and accessible for application - oriented users.

Spline functions, especially the cubic spline, have been a valuable addition to the fields of approximation theory and interpolation theory. Generally, they are much better than approximations which pass exactly through data points because they simultaneously approximate the function, its derivative and its integral.

Actually, spline functions have been known for more than two hundred years and were first introduced in attempts to mathematically model the elastic curve. Shoenberg did, however, introduce the name "spline functions" and was probably the first to systematically study the cubic spline and its generalizations and applications. There have been many generalizations of the cubic spline. But, for the most part, these generalizations had one common feature. They were generalizations of the cubic spline in terms of its properties, but not in terms of the original physical motivation.

One mathematical basis for the development of spline functions is the Euler-Lagrange differential equation which arises by applying the techniques of variational calculus to the problem of minimizing the integral

$$J = \int_a^b y''^2/(1+y'^2)^{5/2}\,dx$$

This integral represents the total strain energy in a relaxed, thin, elastic beam constrained to pass, without buckling, through a prescribed set of points. Because of the complexety and non-linear nature of the fourth order Euler-Lagrange differential equation which results from minimizing J, the exact problem has rarely been considered.

Generally, there are a variety of ways to approach the study of spline functions. One approach is to grossly estimate the integrand in J and exactly solve the resulting problem. If the integrand in J is approximated by y''^2, the resulting problem lends itself to exact solution - the familiar cubic spline. Another approach is to investigate various approximations to the integrand in J and attempt to solve the resulting problems. This is the approach taken by the author. The results are described in the following pages.

This report is a summary report on the progress of the author at the conclusion of approximately one half of the proposed project. Many questions remain unresolved. Most of the unresolved questions have received little or no attention - they were to have been considered in the latter stages of the project. The author does not feel that this report represents the report of a complete study of spline functions and has no intent of so implying.

MATHEMATICAL FOUNDATION

Given a set of points (x_n, y_n), $n = o, 1, 2, ---, N$, where the x_n, form a partition of the interval $[a,b]$ (i.e. $a = x_0 < x_1 < \ldots\ x_n = b$), we seek a family of methods (functions) for fitting a curve exactly through these points.

We desire further to have the functions from class $C^2[a,b]$, to have them satisfy a specified slope on curvature constrant at the end points, and to have them maintain the general global "shape" described by the specified points.

We will restrict our investigation to a family of methods (functions) which are, in a sense to be described, approximations to the elastic curve. Assuming a desirable "shape" to be that assumed by a thin beam (spline) constrained to pass through the specified points and meet the end conditions, the resulting function is that which minimizes, relative to all admissable functions, the strain energy integral

$$J = \int_a^b y''^2/\left(1+y'^2\right)^{5/2} dx$$

Our general method is to seek functions which minimize the integrals which result from replacing the exact intergrand in J by various approximations. The approximations which we will study are:

1. y''^2
2. $y''^2/(1+5y'^2/2)$
3. $y''^2(1-5y'^2/2)$
4. $y''^2/(1+5y'^2/4)^2$
5. $y''^2/(1-5y'^2/4)^2$
6. $y''^2/(1+y'^2)^{5/2}$ (the exact integrand)

For purposes of our study, we assume that y'^2 is very small when compared to unity and certainly that $y'^2 < 2/5$. The procedure will be to replace the intergrand in J by each approximation and apply methods from the calculus of variations to obtain a differential equation for the desired solution.

THE EULER-LAGRANGE EQUATIONS

It is well known that a necessary condition for a function y to extremize an integral of the form $I = \int_a^b \phi\ (x,y,y',y'')\,dx$ is that y be a solution of the Euler-Lagrange differential equation

$$\frac{d^2}{dx^2}\left(\frac{\partial \phi}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right) + \frac{\partial \phi}{\partial y} = 0$$

When viewed with the various integrands that we are studying, it would appear that little success would be expected. For, in each case, the resulting differential equation is fourth order and non-linear. For example, if $\phi = y''^2/(1+y'^2)^{5/2}$ the resulting differential equation is

$$2y^{IV}(1+y'^2)^2 - 5y''\left[(y'' + 4y'y''')(1+y'^2) - 7y'^2y''^2\right] = 0$$

There is little improvement in simplicity by taking ϕ to be one of the other approximating integrands (except $\phi = y''^2$, which is rather trivial).

Fortunately, in each of the cases we are considering, the approximating integrand does not explicitly contain terms involving y or x. This leads to a general method which permits two integrations of each of the fourth order equations.

Since, in each case, there are no terms involving y, the Euler-Lagrange equation can be written

$$0 = \frac{d^2}{dx^2}\left(\frac{\partial \phi}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right)$$

$$= \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{\partial \phi}{\partial y''}\right) - \frac{\partial \phi}{\partial y'}\right]$$

So, in any interval where y''' is continuous, there is some constant α so that

$$\frac{d}{dx}\left(\phi_{y''}\right) - \phi_{y'} = \alpha$$

where we have resorted to the usual subscript notation for partial derivatives.

Now,

$$\frac{d\phi}{dx} = \phi_y y' + \phi_{y'} y'' + \phi_{y''} y'''$$

But $\phi_y = o$ and $\phi_{y'} = \frac{d}{dx}\left(\phi_{y''}\right) - \alpha$. So,

$$\frac{d\phi}{dx} + \alpha y'' = y'' \frac{d}{dx}\left(\phi_{y''}\right) + \phi_{y''} y''' = \frac{d}{dx}\left(y'' \phi_{y''}\right)$$

So, for some constant β,

$$\phi + \alpha y' + \beta = y'' \phi_{y''}$$

So, given the integral $I = \int_a^b \phi(y', y'')\, dx$, where ϕ has continuous first partial derivatives with respect to y' and y'', the function y which extremizes I relative to all admissable functions (y''' continuous on $[a,b]$ except possibly at finitely many points $a = x_o < x_, < \ldots < x_N = b$ and $y \in C^2[a,b]$) must, in each interval $[x_{n-1}, x_n]$, $n = 1, 2, \ldots N$, satisfy the differential equation

$$y'' \phi_{y''} - \phi = \alpha_n y' + \beta_n$$

For the integrands which we are considering, this result can be simplified even further. In each of these cases, the integrand ϕ has the property that $y'' \phi_{y''} = 2\phi$ and the Euler-Lagrange equation becomes $\phi = \alpha_n y' + \beta_n$. Consequently, for the six integrands which we are considering, the respective differential equations are, in each interval $[x_{n-1}, x_n]$, $n = 1, 2, \ldots, N$:

1. $y''^2 = \alpha_n y' + \beta_n$

2. $y''^2 = (\alpha_n y' + \beta_n)(1+5y'^2/2)$

3. $y''^2 = (\alpha_n y' + \beta_n)(1-5y'^2/2)$

4. $y''^2 = (\alpha_n y' + \beta_n)(1+5y'^2/4)^2$

5. $y''^2 = (\alpha_n y' + \beta_n)/(1-5y'^2/4)^2$

6. $y''^2 = (\alpha_n y' + \beta_n)(1+y'^2)^{5/2}$

For mathematical interest we will add two additional equations:

2 (a) $y''^2 = \beta_n + \alpha_n y' + 5\beta_n y'^2/2$

and

4 (a) $y''^2 = \beta_n + \alpha_n y' + 5\beta_n y'^2/2 + 5\alpha_n y'^3/2 + 25\beta_n y'^4/16$

Note that the right hand members of 4, 4(a), 2, 2(a) and 1 are polynomials in y' such that each is obtained from the previous by eliminating the highest degree term. Similarly, the denominators of the right hand members of 5, 3 and 1 are polynominals in y' such that each is obtained from the previous by eliminating the highest degree term.

SOLUTIONS

Our attention is then focused to solving these eight non-linear, second order, differential equations. For convenience in notation during the discussion of solutions, we will assume that we are working in a specific interval $[x_{n-1}, x_n]$ and will delete the subscripts from the constants α_n and β_n.

The Equation $y''^2 = \alpha y' + \beta$

By differentiation of each side of this equation, we obtain, where $y'' \neq 0$, $2y''' = \alpha$. Or, $y^{(iv)} = 0$. Consequently, every solution is a third degree polynomial. Conversely, every third degree polynomial $y = ax^3 + bx^2 + cx + d$ satisfies an equation of this type with $\alpha = 12a$ and $\beta = 4b^2 - 12ac$.

This equation is then an alternate approach to the well known cubic spline. Results relative to the cubic spline are extensive and well known and will not be reproduced here.

The Equation $y''^2 = (\alpha y' + \beta)(1 + 5y'^2/2)$

For purposes of the general discussion, we assume that $\alpha \neq 0$ and change variables by letting

$$y = (2/5\alpha)(4\omega - 5\beta x/6)$$

The resulting equation is

$$\omega''^2 = 4(\omega' - e_2)(\omega' - e_1)(\omega' - e_3)$$

where

$$e_2 = -5\beta/12, \quad e_1 = 5\beta/24 + i(5\alpha^2/32)^{1/2}, \quad e_3 = \bar{e}_1$$

This equation is the well known equation of the Weierstrass elliptic function. So, we have

$$\omega' = \wp(x+\beta)$$

where the invariants of $\wp$ are

$$g_2 = -4(e_1e_2+e_1e_3+e_2e_3) = 12(5\beta/24)^2 - 4(5\alpha^2/32)$$

$$g_3 = 4e_1e_2e_3 = -8(5\beta/24)\left[(5\beta/24)^2+5\alpha^2/32\right]$$

and $\Delta = g_2^3 - 27g_3^2 < 0$

We now introduce the Weierstrass ξ - function ($\xi'(z) = -\wp(z)$), integrate the equation $\omega' = \wp(x+\beta)$, and change to our original variables. From this we obtain

$$y = A\,\xi\,(x+\beta;\, g_2,\, g_3) + Cx + D$$

where g_2 and g_3 are defined above, $A = -8/5\alpha$ and $C = -\beta/3\alpha$. Direct substitution verifies that this is indeed a solution to the original equation.

There is an interesting alternate approach to this particular equation in terms of Jacobi elliptic functions. Since $y'' = dy'/dx = y'\,dy'/dy$, we can write the equation in the equivallent integral form

$$x = s(2/5)^{1/2}\int\left[(\alpha u+\beta)(u^2+2/5)\right]^{-1/2}du$$

and

$$\alpha y + \beta x = s(2/5)^{1/2}\int\left[(\alpha u+\beta)/(u^2+2/5)\right]^{1/2}du$$

where $s = \pm 1 = \mathrm{sgn}(y'')$. By letting $u = \alpha v/|\alpha|$, we obtain

$$x = \lambda\int\left[(v-a)(v^2+b^2)\right]^{-1/2}dv$$

and

$$\alpha y + \beta x = |\alpha| \lambda \int \left[(v-a)/(v^2+b^2)\right]^{1/2} dv$$

where $\lambda = s(2/5)^{1/2} \alpha |\alpha|^{1/2}$, $a = -\beta/|\alpha|$ and $b = (2/5)^{1/2}$

Now, we let $A = (a^2+b^2)^{1/2}$, $k = \left[(A-a)/(2A)\right]^{1/2}$ and

$v = A\left[(1-cn(u)/(1+cn(u)\right] + a$, where cn(u) has parameter k. After some manipulation, we obtain

$$x = \lambda A^{-1/2} \int du$$

and

$$\alpha y + \beta x = \lambda A^{1/2} |\alpha| \int \left[(1-cn(u)/(1+cn(u)\right] du$$

Each of these integrals are known, and we have

$$x + \gamma^* = \lambda A^{-1/2} u$$

and

$$\alpha y + \beta x + \delta^* = |\alpha| \lambda A^{1/2} \left[u - 2E(u) + 2\,sn(u)\,dn(u)/(1+cn(u))\right]$$

Here, γ^* and δ^* are constants of integration and E(u) is the fundamental elliptic integral of the second kind. We now let $v = tu$ where $t = s\alpha/|\alpha| = \pm 1$. Then, since $t\,E(u) = E(v)$, $t\,sn(u) = sn(v)$, $dn(u) = dn(v)$, and $cn(u) = cn(v)$, we have

$$x + \gamma^* = \left[2/(5A|\alpha|)\right]^{1/2} v$$

and

$$\alpha y + \beta x + \delta^* = \left[2A|\alpha|/5\right]^{1/2} \left[v - 2E(v) + 2\,sn(v)dn(v)/(1+cn(v))\right]$$

With $c^{-1} = \left[2/(5A|\alpha|)\right]^{1/2}$ and $d^{-1} = (2A|\alpha|/5)^{1/2}$ (so $cd = (5/2)^{1/2}$)

we obtain

$$cx + \gamma = v$$

and

$$\alpha d y + (\beta d - c)x + \delta = -2\left[E(v) - sn(v)\,dn(v)/(1+cn(v))\right]$$

This provides a functional relation between x and y.

The Equation $y''^2 = \beta + \alpha y' + 5\beta y'^2/2$

First, we let $\lambda = (5\beta/2)^{1/2}$ (which may be complex) and change dependent variables by letting $y = ax + b\mu$, where $a = -\alpha/2\lambda^2$ and $\lambda b = (2/5 - a^2)^{1/2}$. This leads to the equation

$$\mu''^2 - \lambda^2 \mu'^2 = \lambda^4$$

whose general solution is $\mu' = \lambda \sinh \lambda(x+c)$. Consequently, $\mu = \cosh \lambda(x+c) + c^*$ and the desired solution is

$$y = ax + b\cosh\lambda(x+c) + d$$

In exactly the same manner, if we let $\lambda = (-5\beta/2)^{1/2}$, $a = \alpha/2\lambda^2$ and $\lambda b = (a^2 - 2/5)^{1/2}$ we obtain

$$y = ax + b\cos\lambda(x+c) + d$$

as the general solution. These are of course equivallent solutions in the context of complex constants.

The Equation $y''^2 = (\alpha y' + \beta)/(1 - 5y'^2/2)$

We assume that $\alpha\beta \neq 0$ and $y'^2 < 2/5$. Consequently, $\alpha y' + \beta > 0$. Now, we change variables by letting $y' = \alpha a^2 \omega'$, where $a = (5\alpha^2/2)^{-1/6}$, and obtain the equation

$$\omega''^2 = (\omega' - b)/(a^2 - \omega'^2)$$

where $b = -\beta/(\alpha^2 a^2)$. From the restrictions $y'^2 < 2/5$ and $\alpha y' + \beta > 0$, we obtain the corresponding restrictions $\omega' - b > 0$ and $a^2 - \omega'^2 > 0$, Also, $ab \neq 0$.

Then, there are three cases that must be considered. They are: (i) $b < -a < \omega' < a$, (ii) $-a < b < \omega' < a$, and (iii) $b = -a < \omega' < a$.

Case (i) $b < -a < \omega' < a$

In this case we assume that y'' (and hence ω'') has no zeroes in the interval being considered and let $s = \pm 1$ be a factor to indicate the sign of ω'' in the interval. The differential equation is then

$$s\,d\omega' = \left[(b-\omega')/(\omega'^2 - a^2)\right]^{1/2} dx$$

or,

$$s\omega' d\omega' = \left[(b-\omega')/(\omega'^2 - a^2)\right]^{1/2} d$$

So, we can write the two integrals as

$$x = s \int \left[(a-u)(u+a)/(u-b)\right]^{1/2} du$$

and

$$\omega = s \int \left[(a-u)(u+a)/(u-b\right]^{1/2} du .$$

For simplicity, we write these as

$$x = s \int \left[(a-u)(u+a)/(u-b)\right]^{1/2} du$$

and

$$\omega - bx = s \int \left[(a-u)(u+a)(u-b)\right]^{1/2} du$$

Now, we change variables of integration by employing Jacobi elliptic functions and letting

$$\mathrm{sn}(t) = \left[(a-u)/2a\right]^{1/2}$$

where the parameter k is given by $k = \left[2a/(a-b)\right]^{1/2}$.

We then obtain

$$x = -2sk(2a)^{3/2} \int \mathrm{sn}^2(t)\,\mathrm{cn}^2(t)\,dt$$

and

$$\omega - bx = -2\,sk^{-1}\,(2a)^{5/2}\int sn^2(t)\,dn^2(t)\,dt$$

Now, we simplify notation by taking $\lambda^{-1} = -2\,sk(2a)^{3/2}$ and $u^{-1} = -2\,sk^{-1}(2a)^{5/2}$.

Then, we have

$$\lambda x = \int sn^2(t)\,cn^2(t)\,dt$$

and

$$\mu\,(\omega - bx) = \int sn^2(t)\,cn^2(t)\,dn^2(t)\,dt$$

Since $dn^2(t) = 1 - k^2\,sn^2(t)$ and $cn^2(t) = 1 - sn^2(t)$, we obtain

$$\lambda x = A_2 - A_4$$

and

$$u(\omega - bx) = A_2 - (1+k^2)A_4 + k^2 A_6$$

where, again, $A_m = \int sn^m(t)\,dt$.

Now, from well known reduction formulas we know that

$$5k^2 A_6 = 4(1+k^2)\,A_4 - 3A_2 + sn^3(t)\,cn(t)\,dn(t)$$

Consequently

$$\lambda x = A_2 - A_4$$

and

$$5u(\omega - bx) = 2A_2 - (1+k^2)\,A_4 + sn^3(t)cn(t)dn(t)$$

Again, from known reduction formulas,

$$3k^2\,A_4 = 2(1+k^2)A_2 - A_0 + sn(t)\,cn(t)\,dn(t)$$

Then, we have

$$3\lambda k^2 x = A_0 - (1+2k^2) A_2 - \mathrm{sn}(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

and

$$15uk^2 (\omega - bx) = -(1+k^2) A_0 - 2(k^4 - k^2 + 1) A_2 - \mathrm{sn}(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t) \left[1 + k^2 - 3k^2 \mathrm{sn}^2(t) \right]$$

This result can be simplified further by writing it as

$$3\lambda k^2 x = A_0 - (1+2k^2) A_2 - \mathrm{sn}(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

and

$$15uk^2(\omega - bx) - 3\lambda k^2 (1+k^2) x = -2(1+k^2) A_0 - (k^2 - 1) A_2 + 3k^2 \mathrm{sn}^3(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

Now, $k^2 A_2 = A_0 - E(t) = t - E(t)$ where $E(t)$ is the fundamental elliptic integral of the second kind. Thus, we have

$$3\lambda k^4 x = (1+2k^2) E(t) - (1+k^2) t - k^2 \mathrm{sn}(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

and

$$-15uk^4 (\omega - bx) + 3\lambda k^4 (1+k^2) x = (1-k^2) E(t) - (2k^4 + 3k^2 - 1) t - 3k^4 \mathrm{sn}^3(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

Now, $\omega = (\alpha a^2)^{-1} y$, and constants of integration have not yet been added. So, we have the parametric solution

$$Ay + Bx + C = (1-k^2) E(t) - (2k^4 + 3k^2 - 1)t - 3k^4 \ \mathrm{sn}^3(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

and

$$Dx + F = (1+2k^2) E(t) - (1+k^2)t - k^2 \mathrm{sn}(t)\, \mathrm{cn}(t)\, \mathrm{dn}(t)$$

where α, β, C and F are integration constants, $A = -15\, uk^4 (\alpha a^2)^{-1}$, $B = 3\lambda k^4 (1+k^2) + 15\, ubk^4$, $D = 3\lambda k^4$, $k^2 = 2a/(a-b)$, $\lambda^{-1} = -2sk(2a)^{3/2}$, $u^{-1} = -2sk^{-1}(2a)^{5/2}$, $b = -\beta/(\alpha^2 a^2)$, $a = (5\alpha^2/2)^{-1/6}$, and $s = \pm 1$ (= sgn ω'').

Perhaps the more useful form of this solution is

$$\lambda x = \int sn^2(t)\, cn^2(t)\, dt$$

and

$$(\lambda + \mu b)\, x - \mu\omega = k^2 \int sn^4(t)\, cn^2(t)\, dt$$

Case (ii) $\quad -a < b < \omega' < a$

As in the previous case, we arrive at the integrals

$$x = s \int \left[(a-u)(u+a)/(u-b)\right]^{1/2} du$$

and

$$\omega - bx = s\int \left[(a-u)(u+a)(u-b)\right]^{1/2} du$$

Now, we change variables by taking

$$sn(t) = \left[(a-u)/(a-b)\right]^{1/2}$$

where the parameter k is given by $k = \left[(a-b)/2a\right]^{1/2}$. With this procedure we obtain

$$x = -2sk^2(2a)^{3/2} \int sn^2(t)\, dn^2(t)\, dt$$

and

$$\omega - bx = -2sk^4(2a)^{5/2} \int sn^2\, cn^2\, dn^2\, dt$$

or, with $\lambda^{-1} = -2sk^2(2a)^{3/2}$ and $\mu^{-1} = -2sk^4(2a)^{5/2} = 2ak^2\lambda^{-1}$,

$$\lambda x = \int sn^2(t)\, dn^2(t)\, dt$$

and

$$\mu(\omega - bx) = \int sn^2(t)\, dn^2(t)\, cn^2(t)\, dt$$

Since $dn^2(t) = 1-k^2\, sn^2(t)$ and $cn^2(t) = 1 - sn^2(t)$, we obtain

$$\lambda x = A_2 - k^2 A_4$$

and

$$u(\omega - bx) = A_2 - (1 + k^2)A_4 + k^2 A_6$$

where

$$A_m = \int sn^m(t)\,dt$$

Now, from well known reduction formulas, we know that

$$5k^2 A_6 = 4(1 + k^2) A_4 - 3A_2 + sn^3(t)\, cn(t)\, dn(t)$$

Consequently,

$$\lambda x = A_2 - k^2 A_4$$

and

$$5u(\omega - bx) = 2A_2 - (1 + k^2)A_4 + sn^3(t)\, cn(t)\, dn(t)$$

Again, from known reduction formulas,

$$3k^2 A_4 = 2(1 + k^2) A_2 - A_0 + sn(t)\, cn(t)\, dn(t)$$

Then, we have

$$3\lambda x = A_0 + (1 - 2k^2) A_2 - sn(t)\, cn(t)\, dn(t)$$

and

$$\begin{aligned} 15uk^2(\omega - bx) = (1+k^2)A_0 - 2(1 - k^2 + k^4) A_2 \\ - (1 + k^2)\, sn(t)\, cn(t)\, dn(t) \\ + 3k^2\, sn^3(t)\, cn(t)\, dn(t) \end{aligned}$$

This result can be simplified further by writing it as

$$3\lambda x = A_0 + (1 - 2k^2) A_2 - sn(t)\, cn(t)\, dn(t)$$

and

$$15uk^2(\omega - bx) - 3\lambda(1 + k^2)x = -3(1 - k^2) A_2 + 3k^2\, sn^3(t)\, cn(t)\, dn(t)$$

Now, $k^2 A_2 = A_0 - E(t)$ where $E(t)$ is the fundamental elliptic integral of the second kind. Thus, we have

$$3\lambda k^2 x = (1-k^2) A_0 - (1-2k^2) E(t) - k^2 \operatorname{sn}(t)\operatorname{cn}(t)\operatorname{dn}(t)$$

and

$$\lambda k^2 (1+k^2) x - 5u k^4 (\omega - bx) = (1-k^2) A_0 + (1-k^2) E(t) - k^4 \operatorname{sn}^3(t)\operatorname{cn}(t)\operatorname{dn}(t)$$

This result can be inproved further by subtracting the first equation from the second to obtain

$$3\lambda k^2 x = (1-k^2) A_0 - (1-2k^2) E(t) - k^2 \operatorname{sn}(t)\operatorname{cn}(t)\operatorname{dn}(t)$$

and

$$\lambda k^2 (k^2-2) x - 5u k^4 (\omega - bx) = (2-3k^2) E(t) + k^2 \operatorname{sn}(t)\operatorname{cn}(t)\operatorname{dn}^3(t)$$

Now

$Ao = t$, and we have not yet added constants of intergration. Also, $\omega = (\alpha a^2)^{-1} y$.
So, we have the parametric solution

$$Ay + Bx + C = (3k^2-2) E(t) - k \operatorname{sn}(t)\operatorname{cn}(t)\operatorname{dn}^3(t)$$

and

$$Dx + F = (2k^2-1) E(t) + (1-k^2) t - k^2 \operatorname{sn}(t)\operatorname{cn}(t)\operatorname{dn}(t)$$

where α, β, C and F are constants to be determined, $A = 5u k^4(\alpha a^2)^{-1}$

$B = -5u b k^4 - \lambda k^2 (k^2-2)$, $D = 3\lambda k^2$, $k^2 = 2a/(a-b)$, $\lambda^{-1} = -2sk^2(2a)^{3/2}$, $u = -2sk^4(2a)^{5/2}$,
$b = -\beta/(\alpha^2 a^2)$, $a = (5\alpha^{2}/2)^{-1/8}$ and $s = \pm 1$ ($=\operatorname{sgn}\omega''$).

Perhaps the more useful form of this solution is

$$\lambda x = A_2 - k^2 A_4 = \int \operatorname{sn}^2(t)\operatorname{dn}^2(t)$$

$$-u\omega + (ub+\lambda)x = A_4 - k^2 A_6 = \int \operatorname{sn}^4(t)\operatorname{dn}^2(t)\,dt$$

Case (iii) $\quad b = -a < \omega' < a$

In this case, the differential equation becomes $\omega''^2 (a-\omega') = 1$, which is satisfied whenever

$$\omega' = a - [\pm\, 3(x+\gamma)/2]^{2/3}$$

That is, whenever

$$(\alpha a^2)^{-1} y = a x + \delta - 2[\pm\, 3(k+\gamma)/2]^{5/3}/5$$

where α, γ and δ are constants of integration, $a = (5\alpha^2/2)^{-1/3}$ and $\beta = \alpha^2 a^3$.

Normally, these degenerate situations will not occur since there must be, in general, four undetermined constants in each solution interval. If the boundary conditions dictate the degenerate case, then the general case will in fact reduce to this case naturally.

The Equation $y''^2 = (\alpha y' + \beta)(1 + 5y'^2/4)^2$

This equation is one of the more interesting of the equations we are studying. The general approach will be to obtain parametric representations for x and y and then eliminate the parameter to determine the functional relationship between x and y.

We assume that $\alpha \neq o$ and note that $\alpha y' + \beta > o$ with equality only at points where $y'' = o$. First of all, we change variables by the substitution $\omega = \alpha y + \beta x$. This leads to the equation

$$\omega''^2 = \lambda^2 \omega' (\omega'^2 - 2\beta\omega' + k^4)^2$$

where, for simplicity, we have let $\lambda = 5/4\alpha$ and $k = (\beta^2 + 4\alpha^2/5)^{1/4}$. We note that the quadratic equation $t^2 - 2\beta + k^4 = 0$ has no real roots. Consequently, $\omega'^2 - 2\beta\omega' + k^4$ has constant sign throughout the interval of consideration. We make the further assumption that ω'' (and hence y'') has no zeroes in the interval of consideration.

$$s\lambda \; d\omega'/dx = (\omega')^{1/2}(\omega'^2 - 2\beta\omega' + k^4)$$

where $s = \pm 1$. Now, we may also write (since $\omega'' = d\omega'/dx = \omega' d\omega'/d\omega$)

$$s\lambda\omega' \; d\omega'/d\omega = (\omega')^{1/2} \; (\omega'^2 - 2\beta\omega' + k^4)$$

From these two forms, we conclude that

$$x = \lambda s \int \frac{du}{\sqrt{u}\,(u^2 - 2\beta u + k^4)}$$

and

$$\omega = \lambda s \int \frac{\sqrt{u}\; du}{(u^2 - 2\beta u + k^4)}$$

where the parameter u is actually ω'. Now, we change variables of integration by letting $v = (u)^{1/2}$ and obtain

$$x = 2\lambda s \int (v^4 - 2\beta v^2 + k^4)^{-1} \, dv$$

and

$$\omega = 2\lambda s \int v^2 (v^4 - 2\beta v^2 + k^4)^{-1} dv$$

In this representation, the parameter v is actually $(\omega')^{1/2} = (\alpha y' + \beta)^{1/2}$. Fortunately, each of these integrals can be integrated in closed form in terms of well known functions. After considerable effort, we arrive at the result that

$$x + c' = \frac{\lambda s}{2k^3}\left[\frac{1}{2\cos(\theta/2)}\left(\frac{v^2 + 2kv\cos(\theta/2) + k^2}{v^2 - 2kv\cos(\theta/2) + k^2}\right) + \frac{1}{\sin(\theta/2)}\tan^{-1}\left(\frac{2kv\sin(\theta/2)}{k^2 - v^2}\right)\right]$$

and

$$\omega + d' = \frac{\lambda s}{2k}\left[\frac{-1}{2\cos(\theta/2)}\left(\frac{v^2 + 2kv\cos(\theta/2) + k^2}{v^2 - 2kv\cos(\theta/2) + k^2}\right) + \frac{1}{\sin(\theta/2)}\tan^{-1}\left(\frac{2kv\sin(\theta/2)}{k^2 - v^2}\right)\right]$$

where $\cos\theta = \beta/k^2$, $0 < \theta < \pi$, $v = (\omega')^{1/2}$, and c' and d' are constants of integration. Consequent, we obtain

$$\omega + d' + k^2(x + c') = \left(\lambda s/k\sin(\theta/2)\right)\tan^{-1}\left(2kv\sin(\theta/2)/(k^2 - v^2)\right)$$

and

$$\omega + d' - k^2(x + c') = \left(\lambda s/2k\cos(\theta/2)\right)\left[\frac{v^2 - 2kv\cos(\theta/2) + k^2}{v^2 + 2kv\cos(\theta/2) + k^2}\right]$$

Now, for breuity, we write

$$F(\omega, x) = \exp\left[f(\omega, x)\right]$$

and

$$G(\omega, x) = \tan\left[g(\omega, x)\right]$$

where

$$f(\omega, x) = \left[2k\cos(\theta/2)\right]\left[\omega + d' - k^2(x + c')\right]/\lambda s$$

and

$$g(\omega, x) = \left[k\sin(\theta/2)\right]\left[\omega + d' + k^2(x + c')\right]/\lambda s$$

Then, we have

$$v^2 - 2kv\cos(\theta/2) + k^2 = (v^2 + 2kv\cos(\theta/2) + k^2)\ F(\omega, x)$$

and

$$2kv\sin(\theta/2) = (k^2 - v^2)\ G(\omega, x)$$

By simplifying each, we obtain

$$v^2 + 2k\cos(\theta/2)\coth(f)v + k^2 = 0 \quad (*)$$

and

$$v^2 + 2k\sin(\theta/2)\cot(g)v - k^2 = 0 \quad (**)$$

If we add (*) and (**), we obtain

$$v^2 + vk\ [\cos(\theta/2)\coth(f) + \sin(\theta/2)\cot(g)] = 0$$

so (since $v = o$ is not a solution)

$$v = -k\ [\cos(\theta/2)\coth(f) + \sin(\theta/2)\cot(g)]$$

If we subtract (*) from (**) we obtain

$$v = -k/[\cos(\theta/2)\coth(f) - \sin(\theta/2)\cot(g)]$$

Since $v/k > o$ and $\cos(\theta/2) > o$, we conclude that $\coth(f) < o$ (and hence $f < o$). Equating the latter two expressions, we eliminate v and see that

$$[\cos(\theta/2)\coth(f)]^2 - [\sin(\theta/2)\cot(g)]^2 = 1$$

Consequently (since $\coth(f) < o$ and $\cos(\theta/2) > o$) a necessary condition is

$$-\cos(\theta/2)\coth(f) = [1 + \sin(\theta/2)\cot(g^2]^{1/2}$$

where

$$f = 2k\cos(\theta/2)\ [\alpha y + (\beta - k^2)x + c]\ /\lambda s,$$

$$g = k\sin(\theta/2)\ [\alpha y + (\beta + k^2)x + d]\ /\lambda s,$$

$$\sin(\theta/2) + [(k^2 - \beta)/2k^2]^{1/2},$$

$$\cos(\theta/2) = [(k^2 + \beta)/2k^2]^{1/2},$$

$$k = (\beta^2 + 4\alpha^2/5)^{1/4},$$

$\lambda = 4\alpha/5$, and $s = \operatorname{sgn}(y'') = \pm 1$

The constants α, β, c, and d are arbitrary constants to be determined by boundry conditions.

We note that the above result gives a necessary condition. For sufficiency, assume that x and y are related as above. Define a function $\mu(x,y)$ by

$$\mu(x,y) = \sinh^{-1}\left[\sin(\theta/2)/\tan g\right]$$

Then, $\sinh\mu = \sin(\theta/2)/\tan g$ and $\cosh\mu = -\cos(\theta/2)/\tanh f$.
Now, we let $v = k e^{-u} > o$ and determine that

$$\tan g = 2kv\sin(\theta/2)/(k^2 - v^2)$$

and

$$\tanh f = -2kv\cos(\theta/2)/(k^2 + v^2)$$

Now, we solve for f and g and differentiate with respect to v. The result is

$$\alpha\dot{y} + (\beta + k^2)\dot{x} = 2\lambda s(v^2 + k^2)/(v^4 - 2k^2\cos\theta\, v^2 + k^4)$$

and

$$\alpha\dot{y} + (\beta - k^2)\dot{x} = 2\lambda s(v^2 - k^2)/(v^4 - 2k^2\cos\theta\, v^2 + k^4)$$

Then,

$$\alpha\dot{y} + \beta\dot{x} = 2\lambda s v^2/(v^4 - 2\beta v^2 + k^4)$$

and

$$\dot{x} = 2\lambda s/(v^4 - 2\beta v^2 + k^4)$$

Then, we have

$$\alpha y' + \beta = v^2$$

and

$$y'' = (dy'/dv)/\dot{x} = v(v^4 - 2\beta v^2 + k^4)/\alpha\lambda s$$

So,

$$(\alpha y' + \beta)^{1/2}(1 + 5y'^2/4) = 5v(v^4 - 2\beta v^2 + k^4)/4\alpha^2$$

Since $\alpha\lambda = 4\alpha^2/5$,

$$sy'' = 5v(v^4 - 2\beta v^2 + k^4)/4\alpha^2$$

Consequently,

$$y'' = s(\alpha y' + \beta)^{1/2}(1 + 5y'^2/4)$$

The Equation $y''^2 = \beta + \alpha y' + 5\beta y'^2/2 + 5\alpha y'^3/2 + 25\beta y'^4/16$

As in other cases, we assume $\alpha, \beta \neq 0$ and let $s = \mathrm{sgn}(y'') = \pm 1$. Since $y'' = dy'/dx = y' dy'/dy$, we obtain the integrals

$$x = s\int (a_0 u^4 + 4a_1 u^3 + 6a_2 u^2 + 4a_3 u + a_4)^{-1/2} du$$

and

$$y = s\int u(a_0 u^4 + 4a_1 u^3 + 6a_2 u^2 + 4a_3 u + a_4)^{-1/2} du$$

where $a_0 = 25\beta/16$, $a_1 = 5\alpha/8$, $a_2 = 5\beta/12$, $a_3 = \alpha/4$ and $a_4 = \beta$.

The parameter u is, in fact, y'. Now, we assume that $y'' = 0$ when $y' = b_o$ (i.e. b_o is a root of the quritic); and let $u = b_o + \omega^{-1}$ in the integrals. After considerable algebraic manipulation, we obtain

$$x = -s \int (4A_1 \omega^3 + 6A_2 \omega^2 + 4A_3 \omega + A_4)^{-1/2} du$$

$$y - b_o x = -s \int \omega^{-1}(4A_1 \omega^3 + 6A_2 \omega + A_4)^{-1/2} du$$

where $y''^2 = f(y')$ and

$$A_1 = a_o b^3 + 3a_1 b_o^2 + 3a_2 b_o + a_3 = f'(b_o)/4$$

$$A_2 = a_o b_o^2 + 2a_1 b_o + a_2 = f''(b_o)/12$$

$$A_3 = a_o b_o + a_1 = f'''(b_o)/24$$

$$A_4 = a_o = f^{IV}(b_o)/24$$

Now, we again change variables in the integral by taking $\omega = A_1^{-1}(t - A_2/2)$. The result is

$$x = -s \int (4t^3 - g_2 t - g_3)^{-1/2} dt$$

and

$$(y - b_o x)/A_1 = -s \int (t - A_2/2)^{-1} (4t^3 - g_2 t - g_3)^{-1/2} dt$$

where

$$g_2 = 3A_2^2 - 4A_1 A_3 = a_o a_4 - 4a_1 a_3 + 3a_2^2$$

and

$$g_3 = 2A_1 A_2 A_3 - A_2^3 - A_1^2 A_4 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4$$

Note that g_2 and g_3 are, in fact, the fundamental invariants of the original quartic $f(y')$. In terms of α and β,

and

$$g_2 = (25\beta^2/12) - (5\alpha^2/8)$$

$$g_3 = (5\beta/6)^3 - (5\beta/6)(5\alpha^2/8)(11/16)$$

Once more we change variables by introducing the Weierstrass Pe function $\wp(z; g_2, g_3) = \wp(z)$ and letting $t = \wp(z)$. This leads to the integrals

$$x = s\int dz$$

and

$$(y-b_0 x)/A_1 = s\int(\wp(z) - A_2/2)^{-1} dz$$

The second integral is the Weierstrass elliptic integral of the second kind. With b chosen so that $A_2 = 2\wp(b)$, we have

$$x + \gamma = sz$$

and

$$\delta + (y-b_0 x)/A_1 = [\wp'(b)]^{-1} \left\{ \ln\left[\sigma(z-b)/\sigma(z+b)\right] + 2z\,\zeta(b) \right\}$$

Here σ and ζ are, respectively, the sigma and zeta functions of Weierstrass. Because the sigma function is an odd function (i.e. $\sigma(-u) = -\sigma(u)$) we obtain, for either value of s,

$$\delta + (y - b_o x)/A_1 = [\wp'(b)]^{-1} \left\{ \ell n \left[\sigma(x+\gamma-b)/\sigma(x+\gamma+b) \right] + 2\zeta(b)(x+\gamma) \right\}$$

and y has been expressed as a function of x and four constants (α, β, γ, and δ) of integration. The sigma and zeta functions are formed using the invariants g_2 and g_3. The appearance of the numbers b and b_o in the solution will present great difficulty in fitting boundary conditions as will the dependence of g_2 and g_3 on the boundary conditions.

For sufficiency, we note that

$$(y' - b_o) = [A_1/\wp'(b)] \left\{ [\sigma'(x+\gamma-b)/\sigma(x+\gamma-b)] - [\sigma'(x+\gamma+b)/\sigma(x+\gamma+b)] + 2\sigma'(b)/\sigma(b) \right\}$$

$$= A_1/[\wp(x+\gamma) - \wp(b)] = A_1/[\wp(x+\gamma) - A_2/2]$$

So,

$$y'' = -A_1 \wp'(x+\gamma)/[\wp(x+\gamma) - A_2/2]^2$$

Consequently, we have

$$y''^2 = (y' - b_o)^4 \, [\wp'(x+\gamma)]^2/A_1^2$$

But,

$$[\wp'(x+\gamma)]^2 = 4\wp^3(x+\gamma) - g_2 \wp(x+\gamma) - g_3$$

So,

$$y''^2 = A_1^{-2}(y' - b_o)^4 \left[4\wp^3(x+\gamma) - g_2\wp(x+\gamma) - g_3\right]$$

However,

$$\wp(x+\gamma) = (A_2/2) + A_1/(y'-b_o)$$

and we substitute to obtain

$$y''^2 = A_1^{-2}(y'-b_o)^4\Big\{ 4\left[A_1/(y'-b_o)\right]^3 + 6A_2\left[A_1/(y'-b_o)\right]^2$$

$$+ (3A_2^2 - g_2)\left[A_1/(y'-b_o)\right] + 4(A_2/2)^3 - (g_2 A_2/2) - g_3 \Big\}$$

Now, $3A_2^2 - g_2 = 4A_1A_3$ and $4(A_2/2)^3 - (g_2A_2/2) - g_3 = A_1^2 A_4$.

So, we have

$$y''^2 = A_4(y'-b_o)^4 + 4A_3(y'-b_o)^3 + 6A_2(y'-b_o)^2 + 4A_1(y'-b_o)$$

Expansion and simplification gives the original differential equation (since $f(b_o) = 0$).

The Equation $y''^2 = (\alpha y' + \beta)/(1 - 5y'^2/4)^2$

This equation lends itself well to a parametric solution. However, no method has been found to eliminate the parameter and finda functional relation between y and x.

As in previous cases, we assume $\alpha\beta \neq 0$ and that y'' has no zeroes in the interval of consideration.

Then,

$$y'' = s(\alpha y' + \beta)^{1/2}/(1 - 5y'^2/4)$$

where $s = \pm 1 = \mathrm{sgn}(y'')$

We now change variables by letting $\omega = \alpha y + \beta x$. Then,

$$\omega'' = 4\alpha^3 s(\omega')^{1/2}/(4\alpha^2 - 5\beta^2 + 10\beta\omega' - 5\omega'^2)$$

Then, we can write

$$x = -(4\alpha^3 s)^{-1}\int\left[5\mu^2 + 10\beta\mu - (4\alpha^2 - 5\beta^2)\right]\ \mu^{-1/2}du$$

and, since $d\omega'/dx = (d\omega'/d\omega)\omega'$,

$$\omega = -(4\alpha^3 s)^{-1}\int\left[5\mu^2 - 10\beta\mu - (4\alpha^2 - 5\beta^2)\right]\ \mu^{1/2}du$$

where μ is actually ω'

Now, we change variables in the integrals by taking $t = \mu^{1/2}$, and integrate, to obtain the parametric solution.

$$x + \gamma = -(2\alpha^3 s)^{-1}\ \left[t^5 - 10\beta t^3/3 - (4\alpha^2)t\right]$$

and

$$\omega + \alpha = -(2\alpha^3 s)^{-1}\ \left[5t^7/7 - 2\beta t^5 - (4\alpha^2 - 5\beta^2)t^3/3\right]$$

where the parameter t is actually $(\alpha y' + \beta)^{1/2}$

We note that, regardless of any knowledge of the significance of the parameter t, $t^2\dot{x} = \dot{\omega}$. From this relationship we can easily verify that these parametric solutions are indeed sufficient to satisfy the original differential equation (assuming that $\dot{t}$ is positive). The significance of this parameter will be of considerable interest when we consider the boundary conditions and the problems of joining solutions over adjacent intervals.

The Equation $y''^2 = (\alpha y' + \beta)(1 + y'^2)^{5/2}$

In this case we assume that, in the interval under consideration, $\alpha\beta \neq 0$. We effect a rotation of coordinates by making the change of variables

$$\mu = (\alpha x - \beta y)/(\alpha^2 + \beta^2)^{1/2}$$

$$v = (\beta x + \alpha y)/(\alpha^2 + \beta^2)^{1/2}$$

Then, $\dot{\mu} = d\mu/dv = (\alpha - \beta y')/(\alpha y' + \beta)$. Interestingly, this gives $y' = (\alpha - \beta\dot{\mu})/(\alpha\dot{\mu} + \beta)$. So,

$$y''^2 = \ddot{\mu}^2 (\alpha^2 + \beta^2)^3/(\alpha\dot{\mu} + \beta)^6,$$

$$\alpha y' + \beta = (\alpha^2 + \beta^2)/(\alpha\dot{\mu} + \beta) \qquad > 0,$$

and

$$1 + y'^2 = (\alpha^2 + \beta^2)(1 + \dot{\mu}^2)/(\alpha\dot{\mu} + \beta)^2$$

Hence, the differential equation becomes

$$\ddot{\mu}^2 = (\alpha^2 + \beta^2)^{1/2}(1 + \dot{\mu}^2)^{5/2}$$

Now, we let

$$\mu = 2\omega/(\alpha^2 + \beta^2)^{1/4}$$

$$v = 2T/(\alpha^2 + \beta^2)^{1/4}$$

Then, $\dot{\omega} = d\omega/dT = d\mu/dv$ and $\ddot{\mu} = d^2\mu/dv^2 = (d^2\omega/dT^2)(\alpha^2 + \beta^2)^{1/4}/2$ and we have the differential equation

$$\ddot{\omega}^2 = 4(1 + \dot{\omega}^2)^{5/2}.$$

By direct substitution, we discover that the equation is satisfied if

$$\dot{\omega}^2 = (\omega + \gamma)^{-4} - 1$$

where γ is a constant of integration.

Now, we again change variables by introducing the Jacobi elliptic functions and letting

$$(\omega + \gamma)^2 = cn^2(s,k); \quad k = 2^{-1/2}$$

We then obtain the differential equation

$$(dT/ds)^2 = k^2 cn^4(s,k)$$

So,

$$T = \pm k \int cn^2(s,k)\,ds = \pm k \int (2dn^2(s,k) - 1)\,ds$$

Hence, $\quad T + \gamma = \pm k(2E(s,k) - s)$

where E (s,k) is the fundamental elliptic integral of the second kind.

By writing $a = \alpha/2(\alpha^2+\beta^2)^{1/4}$ and $b = \beta/2(\alpha^2+\beta^2)^{1/4}$ we can write this solution in the form

$$(bx + ay + \gamma)^2 = k^2(2E(s) - s)^2$$

where $E(s) = \int_0^s dn^2(t,k)\,dt$, $cn^2(s,k) = (ax - by + \alpha)^2$ and $k = 2^{-1/2}$

To determine sufficiency, we note that $d(2E(s) - s)/ds = cn^2(s,k)$ so that

$$(b + ay')(bx + ay + s) = k^2(2E(s) - s)\,cn^2(s,k)\,(ds/dx)$$

But, $-cn(s,k)\,sn(s,k)\,dn(s,k)\,(ds/dx) = (ax - by + \gamma)(a - by')$.

So,

$$(b+ay')^2 = k^2 cn^4(s,x)(ax-by+\gamma)^2(a-by')^2/cn^2(s,k)\,sn^2(s,k)\,dn^2(s,k)$$

$$= k^2 cn^4(s,x)(a-by')^2/(1-cn^2(s,k))(1-k^2 sn^2(s,k)$$

$$= cn^4(s,x)(a-by')^2/(1-cn^4(s,k))$$

Or,

$$(a-by')^2/(b+ay')^2 = L^{-4}-1$$

where $L = ax-by+\gamma$. From this equation, we obtain algebraically

$$(a^2+b^2)(1+y'^2) = L^{-4}(b+ay')^2$$

and by differentiation

$$(a^2+b^2)y'' = 2L^{-5}(b+ay')^3$$

Consequently,

$$y''^2(1+y'^2)^{-5/2} = 4(a^2+b^2)^{1/2}(ay'+b) = \alpha y'+\beta$$

Several remarkable results can be observed by rewriting the above solution as:

$$ax-by+\gamma = f(t) = t\,cn(t,k)$$
$$bx+ay+\gamma = kg(t) = k\int_0^t cn^2(u,k)\,du$$

where, $g = f^2(t)$, $k = 2^{-1/2}$, $a = \alpha/2(\alpha^2+\beta^2)^{1/4}$ and $b = \beta/2(\alpha^2+\beta^2)^{1/4}$.

We again check to verify solution of the differential equation by observing that:

$$a\dot{x} - b\dot{y} = \dot{f}$$
$$b\dot{x} + a\dot{y} = kf^2$$
$$\dot{f}^2 = k^2(1 - f^4)$$
$$\ddot{f} + f^3 = 0$$
$$\dot{x} = (a\dot{f} + bkf^2)/(a^2 + b^2)$$
$$\dot{y} = (-b\dot{f} + akf^2)/(a^2 + b^2)$$
$$\alpha y' + \beta = kf^2(\alpha a + \beta b)/(bkf^2 + a\dot{f})$$
$$bkf^2 + a\dot{f} > 0$$
$$y'' = kf(a^2 + b^2)^2(bkf^2 + a\dot{f})^{-3}$$
$$\operatorname{sgn}(f) = \operatorname{sgn}(y'')$$
$$1 + y'^2 = k^2(a^2 + b^2)(bkf^2 + a\dot{f})^{-2}$$

We note especially that the curvature κ is given by $\kappa = y''/(1 + y'^2)^{3/2} = 2(a^2+b^2)^{1/2} f = 2(a^2+b^2)^{1/2}(ax - by + \gamma)$. Of more interest, and of considerable importance, is the significance of the parameter t. If s is arc length, then $\dot{s}^2 = \dot{x}^2 + \dot{y}^2 = (\dot{f}^2 + k^2 f^4)/(a^2 + b^2) = k^2/(a^2 + b^2)$. That is, the parameter t is, in each interval, a constant multiple of arc length. We have then derived a parametric representation of the true elastic curve with arc length as the parameter.

MINIMIZATION OF INTEGRALS OF POWERS OF CURVATURE

The solution process for the previous equation, which arises from minimizing $\int k^2 ds$ (where k is curvature as a function of s), suggests a method of approaching a much more general problem. Though not directly a part of the present study, this generalization will be presented.

Suppose that we wish to find a function $y \in C^2[a,b]$ which extremize the integral $\int k^n ds$ where k is curvature, n is a positive integer and $n \neq 1$. From our earlier work, we see that the Euler-Lagrange equation is

$$(y'')^n = (n-1)^{-1} (\alpha y' + \beta)(1+y'^2)^{(3n-1)/2}$$

Now, we change variables by making the substitution

$$a\mu = \alpha x - \beta y$$

$$a v = \beta x + \alpha y$$

$$a = (n-1)^{1/n} (\alpha^2 + \beta^2)^{(n-1)/2n}$$

With $\dot{u} = du/dv$, we obtain

$$(\ddot{u})^n = (-1)^n (1+\dot{u}^2)^{(3n-1)/2}$$

By direct substitution, we find that this equation is satisfied whenever

$$\dot{u}^2 = \left[(n-1)\mu/n + \gamma^*\right]^{2n/(1-n)} - 1$$

Now, we put $k = 2^{-1/2}$ and

$$\left[(n-1)u/n + \gamma^*\right]^2 = \left[cn^4(t,k)\right]^{(n-1)/n}$$

Then, we obtain

$$(dv/dt)^2 = 4k^2 \left[cn^2(t,k)\right]^{(3n-2)/n}$$

For sufficiency, we write

$$\alpha x - \beta y + \gamma = An/(n-1)\left[f(t)\right]^{(n-1)/n}$$

and

$$\beta x + \alpha y + \delta = 2k B \int_0^t \left[f(t)\right]^{(3n-2)/2n} dt$$

where $f(t) = cn^2(t,k)$, $k = 2^{-1/2}$, $A^n = (n-1)(\alpha^2 + \beta^2)^{(n-1)/2}$ and $B = \pm A$

We observe that

$$\alpha \dot{x} - \beta \dot{y} = A \dot{f} f^{-1/n}$$

$$\beta \dot{x} + \alpha \dot{y} = 2kBf^{(3n-2)/2n}$$

$$\dot{f}^2 = 4k^2 f(1-f^2)$$

$$\ddot{f} = 2k^2 (1-3f^2)$$

$$(\alpha^2 + \beta^2) f^{1/n} \dot{x} = \alpha A \dot{f} + 2k\beta B f^{3/2}$$

$$(\alpha^2 + \beta^2) f^{1/n} \dot{y} = -\beta A \dot{f} + 2k\beta B f^{3/2}$$

$$\alpha y' + \beta = 2kB(\alpha^2 + \beta^2) f^{3/2} / (\alpha A \dot{f} + 2k\beta B f^{3/2})$$

$$1 + y'^2 = 4(\alpha^2 + \beta^2) B^2 k^2 f / (\alpha A f + 2k\beta B f^{3/2})^2$$

$$y'' = (2k)^2 AB (\alpha^2 + \beta^2)^2 f^{(3n+2)/2n} / (\alpha A \dot{f} + 2k\beta B f^{3/2})^3$$

Direct substitution shows that indeed, $(n-1)(y'')^n = (\alpha y' + \beta)(1 + y'^2)^{(3n-1)/2}$.

Alternately, we can write

$$\alpha x - \beta y + \gamma = [An/(n-1)] \; [f(t)]^{(n-1)/2n}$$

and

$$\beta x + \alpha y + \delta = A \int_0^t [f(u)]^{(n-1)/2n} \, du$$

where $f(t) = \cos^2 t$ and $A = (n-1)(\alpha^2 + \beta^2)^{(n-1)/2}$

We observe that

$$\alpha \dot{x} - \beta \dot{y} = 1/2 \, A f^{-(n+1)/2n} \dot{f}$$

$$\beta \dot{x} + \alpha \dot{y} = A f^{(n-1)/2n}$$

$$\dot f^2 = 4f(1-f)$$

$$\ddot f = 2(1-2f)$$

$$f^{(n+1)/2n}(\alpha^2+\beta^2)\dot x = (1/2)\alpha A\dot f + \beta A f$$

$$f^{(n+1)/2n}(\alpha^2+\beta^2)\dot y = -(1/2)\beta A\dot f + \alpha A f$$

$$\alpha y' + \beta = (\alpha^2+\beta^2) f/((1/2)\alpha\dot f + \beta f)$$

$$1 + y'^2 = (\alpha^2+\beta^2) f/((1/2)\alpha\dot f + \beta f)^2$$

$$A y'' = (\alpha^2+\beta^2)^2 f^{(3n+1)/2n}/((1/2)\alpha\dot f + \beta f)^3$$

Direct substitution shows that indeed

$$(y'')^n = (n-1)^{-1}(\alpha y' + \beta)(1+y'^2)^{(3n-1)/2}$$

For the special case $n = 2$, the parameter t is related to curvature k by the relation $k^2 = \cos t$. Arc length s is related to the parameter t by the relation $\dot s^2 = (\cos t)^{-1}$, and strain energy $E(t)$ at any point is related to curvature by the relation $\dot E = K$.

Each of the previous cases is a special case of a more general approach. In general, we can let

$$\alpha x - \beta y + \gamma = [An/(n-1)]F(t)$$

and

$$\beta x + \alpha y + \delta = [An/(n-1)]G(t)$$

where $F(t)$ is arbitrary, $\dot G^2 = \dot F^2 F^{2n/(n-1)}/(1-F^{2n/(n-1)})$ and $A^n = (-1)^n(n-1)(\alpha^2+\beta^2)^{(n-1)/2}$

We observe then that

$$\alpha\dot{x} - \beta\dot{y} = [An/(n-1)]\,\dot{F}$$

$$\beta\dot{x} + \alpha\dot{y} = [An/(n-1)]\,\dot{G}$$

$$(\alpha^2 + \beta^2)\dot{x} = [An/(n-1)]\,(\alpha\dot{F} + \beta\dot{G})$$

$$(\alpha^2 + \beta^2)\dot{y} = [An/(n-1)]\,(-\beta\dot{F} + \alpha\dot{G})$$

$$\alpha y' + \beta = (\alpha^2 + \beta^2)\,\dot{G}/(\alpha\dot{F} + \beta\dot{G})$$

$$\dot{G}\ddot{F} - \dot{F}\ddot{G} = [n/(n-1)]\dot{G}^3 F^{(3n-1)/(1-n)}$$

$$1 + y'^2 = (\alpha^2 + \beta^2)(\dot{F}^2 + \dot{G}^2)/(\alpha\dot{F} + \beta\dot{G})^2$$

$$y'' = -(\alpha^2 + \beta^2)^2\,\dot{G}^3 F^{(3n-1)/(1-n)}/A(\alpha\dot{F} + \beta\dot{G})^3$$

$$\dot{F}^2 + \dot{G}^2 = \dot{G}^2 F^{2n/(1-n)}$$

and direct substitution verifies that $(y'')^n = (n-1)^{-1}(\alpha y' + \beta)(1 + y'^2)^{(3n-1)/2}$

This permits an infinite variety of parametric representations. We need only choose F arbitrarily and let G be given by

$$G = \pm \int \frac{F^{n(n-1)}\,dF}{(1-F^{2n/(n-1)})^{1/2}}$$

BOUNDARY CONDITIONS

The problem of matching the solutions of the foregoing differential equations was to have been the major emphasis of the second phase of this project. Consequently, as of the writing of this summary report, very little attention has been devoted to the boundary condition problem.

Basically, the problem is as follows: in each of the intervals (x_{n-1}, x_n) $n = 1, 2, ---, N$ we have a solution of the form $f(x, y, \alpha_n, \beta_n, \gamma_h, \delta_n) = 0$. Consequently, we have 4N constants of integration to be determined. With these constants, we have the conditions that the solutions pass through the specified points (2N conditions), that y' be continuous at each joint (N-1 conditions) and that y" be continuous at each joint (N-1 conditions). In addition, we impose additional condition at each end (at x_o and at x_N). Consequently, we have 4N specified conditions to determine the 4N constants of integration. While, in theory, this is sufficient, the practical situation is that this is a formidable problem indeed. For example, in several of the situations the solutions involve elliptic functions whose invariants are dependent upon the constants of integration. In these cases, functional values cannot be determined except as transcendental functions of the constants of integration. So, in general, the problem reduces to one of solving 4N non-linear equations in 4N unknowns. Detailed study of individual cases might well result in effecient individual procedures for attacking this problem. But, as of this report, no such detailed study has been attempted.

It is observed that, in each case, the differential equation provides an additional route for investigation of half the constants of integration (the α_n and β_n). In this process, however, one must temporarily introduce additional unknown quantities (the unknown values of the slope and the second derivative at each joint).

The following very general approach may indicate a possible process for determining the constants of integration. Of the eight differential equations we have considered, the six which result from approximate integrands are of the form

$$y'' = (\alpha_n y' + \beta_n)\, f(y') \qquad x \in (x_{n-1}, x_n) \quad , \quad n = 1, 2, ---, N$$

If we assume that y" has no zeroes, except possibly at the joints, and introduce the quantities

$$m_n = y'(x_n) \qquad n = 0,\ 1,\ 2,\ ---,\ N$$

and

$$M_n = y''(x_n) \qquad n = 0,\ 1,\ 2,\ ---,\ N$$

we obtain the "parametric solutions" (for $n = 1, 2, ---, N$)

$$x - x_{n-1} = s_n \int_{m_{n-1}}^{y'} \Big[(\alpha_n u + \beta_n) f(u) \Big]^{-1/2} du$$

and

$$y - y_{n-1} = s_n \int_{m_{n-1}}^{y'} u\left[(\alpha_n u + \beta_n)f(u)\right]^{-1/2} du$$

where sn = ± 1 is the sign of y" in (x_{n-1}, x_n). For the determination of the constants we have

$$x_n - x_{n-1} = s_n \int_{m_{n-1}}^{m_n} \left[(\alpha_n u + \beta_n)f(u)\right]^{-1/2} du \qquad n = 1,2, --, N$$

$$y_n - y_{n-1} = s_n \int_{m_{n-1}}^{m_n} u\left[(\alpha_n u + \beta_n)f(u)\right]^{-1/2} du \qquad n = 1,2, --, N$$

$$M_{n-1}^2 = (\alpha_n M_n + \beta_n)f(M_{n-1}) \qquad n = 1,2, --, N$$

$$M_n^2 = (\alpha_n M_n + \beta_n)f(M_n) \qquad n = 1,2, --, N$$

and two end conditions. This, in theory, would permit one to determine the 4N+2 unknown quantities if the values of s_n were supplied in advance. Some obvious simplifications are possible but it is not presently possible to state whether or not this general process would be either effecient or fruitful. We can only conclude that the boundary condition problem must be given considerable attention before it is possible to make a valid conclusion as to the utility of our solutions for applications oriented users.

APPENDIX A: SPECIAL CASES

In our discussion of the eight differential equations given on page 5, we have generally excluded the cases where $\alpha \cdot \beta = 0$. The purpose for this exclusion was that, generally, four arbitrary constants were required in each interval in order to satisfy the imposed boundary conditions. In the case that a particular constant is zero by viture of boundary conditions, then the solution will appropriately reduce to fit the boundary conditions. In many physical applications, continuity of y" at the joints is essential (e.g. in automotive or ship design problems, the mathematical curves are reproduced by mechanical means. The milling (or drafting) machine will not operate in such a way as to allow the second derivative (and hence the currvature) to vary in a discontinuous manner. Such variance would require machinery to achieve instantaneous acceleration.) However, in many curve fitting applications, continuity of second derivatives may not be an essential feature. This fact has led us to briefly examine the results of setting $\alpha = 0$ in the differential equations in question.

The Special Cases $\alpha = 0$

The cases which arise by putting $\alpha = 0$ in the differential equations have a special interest since they can also be interpreted as the results of minimization of the strain energy integral when continuity of y'' is ignored. We saw earlier that the Euler-Lagrange equations for extremizing the integral $\int_a^b \varphi(y',y'')dx$, when $y'' \varphi_{y''} = 2\varphi$, was $\varphi = \alpha y' + \beta$. If we put $y' = u$ in the integral, we seek to extremize $\int_a^b \varphi(u,u')dx$. The Euler-Lagrange equation is then $\varphi = u' \varphi_{u'} - \beta$. But $u' \varphi_{u'} = 2\varphi$. So, we have $\varphi = \beta$ as the differential equation. This is the same result that is obtained by putting $\alpha = 0$ in original case.

In this case, the equations from the six integrands given on page 3 become, respectively,

1. $y''^2 = \beta_n$
2. $y''^2 = \beta_n (1 + 5y'^2/2)$
3. $y''^2 = \beta_n/(1 - 5y'^2/2)$
4. $y''^2 = \beta_n (1 + 5y'^2/4)^2$
5. $y''^2 = \beta_n/(1 - 5y'^2/4)^2$
6. $y''^2 = \beta_n (1 + y'^2)^{5/2}$

we note that, in each case, $\beta_n > 0$. For simplicity of notation, we delete the subscript n and change the notation of the arbitrary constant by, respectively, taking $\beta_n = \beta^2$, $\beta_n = 2\beta^2/5$, $\beta_n = 5\beta^2/2$, $\beta_n = (4\beta/5)^2$, $\beta_n = (5\beta/4)^2$, and $\beta_n = \beta^2$. This gives the six equations

1. $y''^2 = \beta^2$

2. $y''^2 = \beta^2 (a^2 + y'^2)$; $a^2 = 2/5$

3. $y''^2 = \beta^2/(a^2 - y'^2)$; $a^2 = 2/5$

4. $y''^2 = \beta^2(a^2 + y'^2)^2$; $a^2 = 4/5$

5. $y''^2 = \beta^2/(a^2 - y'^2)^2$; $a^2 = 4/5$

6. $y''^2 = \beta^2(1 + y'^2)^{5/2}$

As in previous studies, we assume that y'' has no zeroes in the interval in question. Consequently, we may assume that β and u'' have the same sign and $\beta \neq 0$.

<u>The equation $y''^2 = \beta^2$</u>

With our assumptions, this gives $y'' = \beta$ and y is a second degree polynomial in x. Note that there are three arbitrary constants to be determined

<u>The equation $y''^2 = \beta^2(a^2 + y'^2)$</u>

In this case we have $y'' = \beta(a^2 + y'^2)^{1/2}$ whose solution is $\beta y + \delta = a \cosh(\beta x + \gamma)$ (where $a > 0$).

<u>The equation $y''^2 = \beta^2/(a^2 - y'^2)$</u>

In this case we have $y''(a^2 - y'^2)^{1/2} = \beta$. Now, since $y'' = dy'/dx$ and $y'' = y'dy'/dy$, we have

$$\beta x + \delta = \int (a^2 - u^2)^{1/2} du$$

and

$$\beta y + \gamma = \int (a^2 - u^2)^{1/2} u du$$

where δ and γ are constants of integration and the parameter u is, in fact, y'. Now, in each integral we change variables by letting $u = a \sin\theta$ (where $-\pi/2 < \theta < \pi/2$). We obtain then

$$\beta x + \delta = a^2 \int \cos^2 \theta \, d\theta$$

and

$$\beta y + \gamma = a^3 \int \cos^2 \theta \sin\theta \, d\theta$$

We can readily integrate these equations to obtain

$$\beta x + \delta = (a^2/2)(\theta + \sin\theta \cos\theta)$$

and

$$\beta x + \gamma = -(a^3/3) \cos^3 \theta$$

It is easily verified that this parametric solution does indeed satisfy the differential equation. It is also possible to eliminate the parameter to obtain a functional relation between y and x. Since $a > 0$ and $\cos\theta > 0$, we must have $(\beta y + \gamma) < 0$. Then

$$a \cos\theta = -\left[3(\beta y + \gamma)\right]^{1/3},$$

$$\theta = s \cos^{-1} \left[(-3/a^3)(\beta y + \gamma)\right]^{1/3},$$

and

$$a \sin\theta = s \left\{ a^2 - \left[3(\beta y + \gamma)\right]^{2/3} \right\}^{1/2}$$

where $s = \pm 1 = \mathrm{sgn}(y')$. Consequently,

$$2s(\beta x + \delta) = a^2 \cos^{-1} \left[(-3/a^3)(\beta y + \gamma)\right]^{1/3} - \left[3(\beta y + \gamma)\right]^{1/3} \left\{a^2 - \left[3(\beta y + \gamma)\right]^{2/3}\right\}^{1/2}$$

The Equation $y''^2 = \beta^2 (a^2 + y'^2)^2$

In this case, we have $y'' = \beta (a^2 + y'^2)$. Again, since $y'' = dy'/dx$ and $y'' = y' \, dy'/dx$, we have

$$\beta x + \delta = \int (a^2 + u^2)^{-1} du = (1/a) \tan^{-1} (u/a)$$

and

$$\beta y + \gamma^* = \int (a^2 + u^2)^{-1} \, u du = (1/2) \ln (a^2 + u^2)$$

where δ and γ^* are constants of integration and the parameter u is, in fact, y'. We note that these equations can be written

$$u/a = \tan \left[a(\beta x + \delta)\right]$$

and

$$(1/2) \ln (1 + u^2/a^2) = \beta x + \gamma$$

Thus,

$$\beta y + \gamma = (1/2) \ln \left[\sec^2 a(\beta x + \delta)\right]$$

or,

$$\beta y + \gamma = - \ln \cos a(\beta x + \delta)$$

The Equation $y''^2 = \beta^2/(a^2 - y'^2)^2$

In this case, we have $y''(a^2 - y'^2) = \beta$. Again, since $y'' = dy'/dx$ and $y'' = y' \, dy'/dx$, we have

$$\beta x + \delta = \int (a^2 - u^2) du = a^2 u - u^3/3$$

and

$$\beta y + \gamma = \quad (a^2 - u^2) \quad du = (a^2 u^2/2) - u^4/4$$

where δ and γ are constants of integration and the parameter u is, in fact, y'.

Now, from the second equation, we find that $u^2 = a^2 \pm \left[a^4 - 4(\beta y + \gamma)\right]^{1/2}$.

However, $u^2 - a^2 < 0$ so that $u^2 = a^2 - \left[a^4 - 4(\beta y + \gamma)\right]^{1/2}$. Now, we find by combining the original two equations that $u(\beta x + \gamma)/2 - (\beta y + \gamma) = u^4/12$. But,

$u^4 = 2a^4 - 4(\beta y + \gamma) - 2a^2 \left[a^4 - 4(\beta y + \gamma)\right]^{1/2}$. So, we have

$$u^2 = a^2 - \left[a^4 - 4(\beta y + \gamma)\right]^{1/2}$$

and

$$3u\,(\beta x + \delta) = a^4 + 4(\beta y + \gamma) - a^2\left[a^4 - 4(\beta y + \gamma)\right]^{1/2}$$

If the second equation is squared, and then u^2 is replaced by the first expression, we obtain a functional relation between x and y.

<u>The Equation $y''^2 = \beta^2(1 + y'^2)^{5/2}$</u>

In this case we have $y'' = \beta(1 + y'^2)^{5/4}$. By inspection, we see that equation is satisfied when ever

$$y'^2 = \left[(-\beta y/2) + \gamma\right]^{-4} - 1$$

We may assume, with no loss of generality that $0 < (-\beta y/2) + \gamma \leq 1$. Now, we put $(-\beta y/2) + \gamma = \mathrm{cn}\,(s,k)$ where $k = 2^{-1/2}$. With this substitution we obtain

$$(\beta\, dx/ds)^2 = 2\, \mathrm{cn}^4(s)$$

so that

$$A x + \delta = + 2^{1/2} \int cn^{2}(s)\, ds$$

So,

$$A x + \delta = + 2^{3/2} \left[E(s) - s/2\right]$$

where E(s) is the fundamental elliptic integral of the second kind.

There does not appear to be a method of eliminating the parameter s so as to achieve a functional relation between x and y.

BOUNDARY CONDITIONS

In the preceeding special cases, the boundary condition problem is quite different from that of the general case. In these cases, there are three constants per interval. Generally, two of the constants (say δ and γ) can be eliminated by requiring the solution to fit through the prescribed joints. This leaves, for N + 1 intervals, N + 1 β's to be determined. The requirement of continuous first derivatives at the joints will provide N conditions on these N + 1 constants and there remains only one additional condition that can be imposed. In this case then, we cannot provide the usual two end conditions. Since the approximated integral can be integrated and expressed in terms of the β's one possible approach is to take a fixed value for the (approximated) total strain energy as the final condition. This might be of particular interest in computer graphics where one could generate, and compare, an infinite family of curves by varying this one variable.

As mentioned earlier, there is every indication that the boundary condition problem will be a formidable problem. As of the writing of this report, very little attention has been devoted to this aspect of the general problem.

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* This extensive bibliography lists more than 350 items related to spline functions and names several other sources of rather elaborate bibliographies (including our references 2, 8 and 15). No attempt is made here to duplicate these bibliographical efforts.