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ROUND-OFF ERRORS IN CUTTING PLANE ALGORITHMS BASED ON THE REVISED SIMPLEX PROCEDURE

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16. ABSTRACT

This report statistically analyzes computational round-off errors associated with the cutting plane approach to solving linear integer programming problems. Cutting plane methods require that the inverse of a sequence of matrices be computed. The problem basically reduces to one of minimizing round-off errors in the sequence of inverses. Two procedures for minimizing this problem are presented, and their influence on error accumulation is statistically analyzed. One procedure employs a very small tolerance factor to round computed values to zero. The other procedure is a numerical analysis technique for "reinverting" or improving the approximate inverse of a matrix. The results indicate that round-off accumulation can be effectively minimized by employing a tolerance factor which reflects the number of significant digits carried for each calculation and by applying the reinversion procedure once to each computed inverse. If 18 significant digits plus an exponent are carried for each variable during computations, then a tolerance value of 0.1×10^{-12} is reasonable.

The prerequisite for reading this report is a working knowledge of the simplex method and the revised simplex algorithm in particular.

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ROUND-OFF ERRORS IN CUTTING PLANE ALGORITHMS BASED ON THE REVISED SIMPLEX PROCEDURE

INTRODUCTION

This report statistically analyzes computational round-off errors associated with the cutting plane approach to solving linear integer programming problems. Cutting methods typically employ the revised simplex procedure to augment fractional cuts (secondary constraints) during the search for the optimum integer solution. The revised simplex procedure requires that the inverse of the basic matrix be computed at each iteration. Since the inverse for each successive iteration is computed directly from the previous inverse, round-off errors tend to propagate. The accuracy of the computed inverse is very important because all the other information associated with an iteration is computed from the inverse and the original problem coefficients.

Two procedures for minimizing round-off error accumulation are presented and their influence is statistically analyzed. One procedure employs a very small tolerance factor to round computed values to zero. The justification for the procedure is that computed quantities having absolute values less than the tolerance are most likely accumulated errors. The other procedure is a numerical analysis technique for "reinverting" or improving the approximate inverse of a matrix. The value of the tolerance employed and the frequency at which the basic matrix is reinverted are both shown to have a statistically significant effect on round-off error. The results indicate that employing a tolerance factor which reflects the number of significant digits carried for each calculation and applying the reinversion procedure once to each computed inverse effectively minimizes round-off error accumulation. If 18 significant digits plus an exponent are carried for each computed quantity, then a tolerance factor of 0.1×10^{-12} is reasonable.

Readers of this report are assumed to have a working knowledge of the simplex method and the revised simplex algorithm.

LINEAR INTEGER PROGRAMMING AND CUTTING PLANE METHODS

Cutting methods seek to reshape the continuous solution space by consecutively imposing special constraints on it until the required optimum integer solution coincides with the optimum simplex solution. These methods capitalize on knowledge that the simplex solution to the continuous problem must occur at an extreme point. If the continuous solution is integer, it is the optimum solution. Otherwise, it is an infeasible point that can be eliminated. One or more secondary constraints can be computed from the current simplex tableau which, when augmented to the problem, "cut off" a portion of the feasible space which contains the current solution. A secondary constraint, referred to as a cut, does not violate any feasible integer points since it is nothing more than a necessary condition for integrality. The dual simplex method isolates another optimum extreme point as feasibility is recovered. Successive application of cuts produces a feasible space with an optimum extreme point satisfying the integrality conditions.

Figure 1 illustrates the application of cutting methods. Only two cuts were required in this example to permit the optimum simplex solution to coincide with the optimum integer solution.

Since most cuts contain fractional coefficients, the addition of each new cut adds to the round-off problem. Therefore, it is desirable to employ a simplex algorithm which has a mechanism for purging the tableau of round-off errors after several iterations have taken place. The revised simplex algorithm has such a mechanism. At any iteration of this algorithm, the entire tableau can be constructed from a knowledge of the inverse of the basic matrix and the original problem coefficients. This means that the tableau can be purged of round-off errors at any iteration by reinverting the basic matrix.

Figure 2 presents the primary steps required to use the revised simplex algorithm as an integral part of cutting algorithms. Each step is straightforward except for updating and inverting the basic matrix. The procedure for accomplishing this task depends on whether a redundant cut was dropped the previous iteration.

<u>Case 1</u>. No redundant cut was deleted the previous iteration. This means that the current problem contains $k \leqslant m + n$ cuts, where m and n are the original number of simplex constraints and variables respectively. Let

$$\alpha_1 \chi_1 + \alpha_2 \chi_2 + \ldots + \alpha_k \chi_k + S = \beta$$

be the computed cut in terms of the initial problem variables. The α_i are used to construct a row vector **C** to be augmented to the bottom of the current basic matrix $\mathbf{B}_c$ in forming the new basic matrix $\mathbf{B}_n$. C = $(c_1, c_2, \ldots, c_m)$, where

$$c_i = \begin{cases} \alpha_j \text{ if } j \leqslant k \text{ and } \chi_j \text{ is the ith basic variable} \\ 0 \text{ otherwise} \end{cases}$$

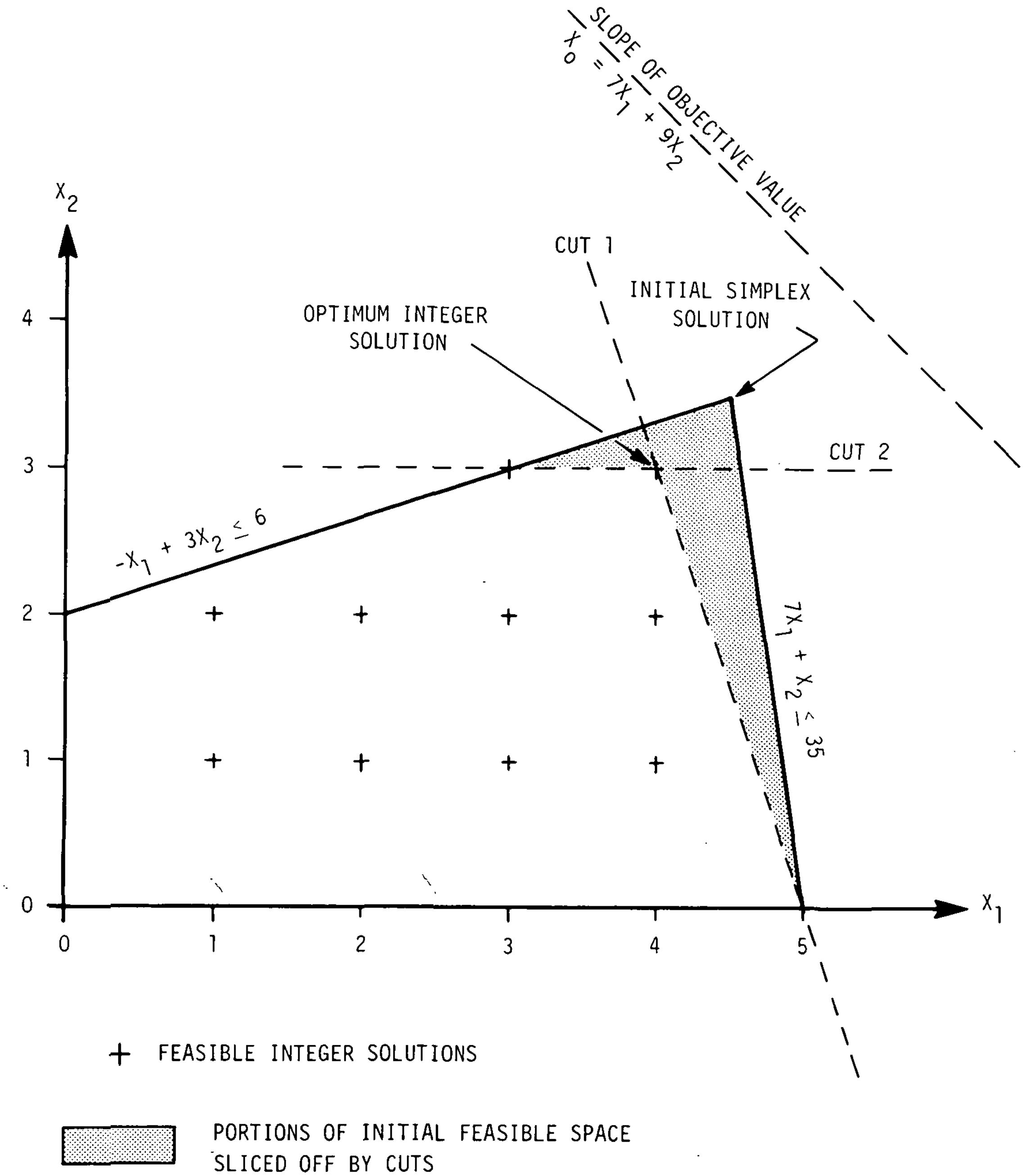


Figure 1. Illustration of cutting methods.

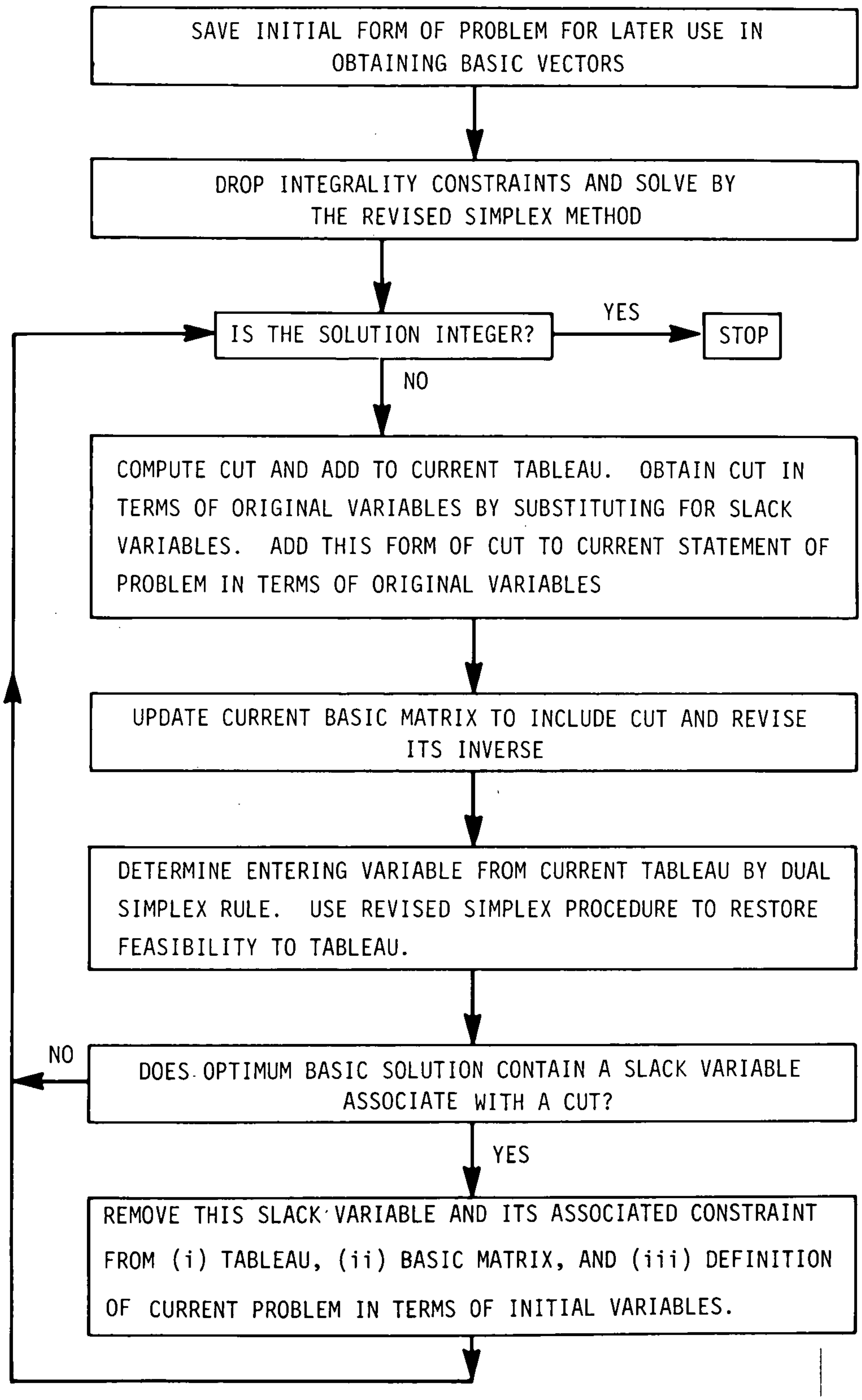


Figure 2. Cutting algorithms based on revised simplex.

Also, the column vector $(0, 0, \ldots, 0, 1)^T$ must be added to the right side of B_c in forming B_n. Then

$$B_n = \begin{bmatrix} B_c & 0 \\ & \\ C & 1 \end{bmatrix}$$

B_n^{-1} is obtained from B_c^{-1} by the partitioned matrix method [1].

$$B_n^{-1} = \begin{bmatrix} B_c^{-1} & 0 \\ & \\ -C\,B_c^{-1} & 1 \end{bmatrix}$$

Case 2. A redundant cut was deleted at the end of the last iteration. Deletion of a cut means that one row and one column of the current basic matrix B_c must be deleted. However, a new cut is being added to the problem which will add a row and a column to B_c. Rather than reduce the basic matrix by a row and a column and immediately add back a row and column, it is simpler to replace the old row and column with the new row and column. Note that the column being deleted and the column being added are identical. That is, this column corresponds to the slack variable being deleted and the slack variable being added. All the entries in the column are zero except for a one in the row being deleted/added. Thus the problem of finding the new inverse reduces to the following problem.

Given a $k \times k$ basic matrix A and its inverse A^{-1}, construct a new basic matrix B from A by replacing row r of A with a new row vector C. Compute the inverse of B from a knowledge of A^{-1} and the new row C. Here C is the row vector defined in Case 1.

B^{-1} can be identified by first computing the row vector

$$p = C A^{-1} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$$

Construct a new row vector ξ from p as follows:

$$\xi = (-\alpha_1/\alpha_r, -\alpha_2/\alpha_r, \ldots, 1/\alpha_r, \ldots, -\alpha_k/\alpha_r) \quad ,$$

where the ith element is $-\alpha_i/\alpha_r$ except for the rth element which is $1/\alpha_r$. Let **E** be a k X k identity matrix with its rth row replaced by $\boldsymbol{\xi}$. Then,

$$\mathbf{B}^{-1} = \mathbf{A}^{-1}\,\mathbf{E} \quad .$$

This procedure is only a slight variation of the revised method using the product form of the inverse. References 2 and 3 both provide a detailed discussion of the product form of the inverse.

ROUND-OFF ERRORS

Recall that at any iteration, the revised simplex procedure generates the tableau from the inverse of the basic matrix and the definition of the problem in terms of the original variables. This means that round-off can be minimized in the tableau by minimizing round-off accumulation in the inverse. An iterative procedure for improving the inverse of a matrix can be used for this purpose.

Let $\mathbf{A}$ be the current basic matrix and let $\mathbf{B}_i$ be a computed approximation to $\mathbf{A}^{-1}$. Then $\mathbf{B}_{i+1}$, the new approximation, is computed from $\mathbf{B}_i$ by the matrix equation

$$\mathbf{B}_{i+1} = \mathbf{B}_i (2\mathbf{I} - \mathbf{A}\,\mathbf{B}_i) \quad ,$$

where $\mathbf{I}$ is the identity matrix. The sequence $\mathbf{B}_i$, $\mathbf{B}_{i+1}$, $\mathbf{B}_{i+2}$, ... converges to $\mathbf{A}^{-1}$ quadratically if any norm (the Euclidean norm is usually used) of

$$\mathbf{R}_i = \mathbf{I} - \mathbf{A}\,\mathbf{B}_i$$

is less than one. A detailed presentation of this procedure is provided by Reference 4 and a discussion of its use with the revised simplex method is given by Reference 5.

Another technique which has proven to be computationally effective in reducing round-off accumulation is to round all computed values to zero that have an absolute value less than a given tolerance. That is, if α is a computed value and $|\alpha| < \xi$ where $\xi > 0$ is chosen to be very small, then set $\alpha = 0$. The value chosen for ξ should reflect the number of significant digits stored for the computational variables.

The computational success of these two techniques and the influence of the problem characteristics on round-off error have been statistically analyzed. The problem characteristics considered were number of variables, density of the constraint matrix, and the relative magnitude of the constraint coefficients. These three characteristics and the two round-off minimization techniques were considered as five factors in a nested-factorial experiment. The number of variables is considered to be nested within constraint matrix density which is considered to be nested within the relative magnitude of the constraint coefficients. A factorial experimental design is used since it provides an efficient means of obtaining information about the influence of each factor and about possible interactions among the factors.

The experiment contains 108 distinct combinations of the 5 factors. One iteration of the procedure for improving the inverse of the basic matrix was considered at three levels – after each simplex base change, after each fifth base change, and after each ninth base change. Three tolerance levels (0.1×10^{-6}, 0.1×10^{-12}, and 0.1×10^{-18}) were included for rounding computed values to zero. Ten variable, twenty variable, and thirty variable test problems were considered. All problems had 10 constraints of the type

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leqslant b_i \ , \quad i = 1, 2, \ldots, 10 \ .$$

All coefficients for the test problems were randomly chosen from uniform distributions. Table 1 provides the ranges for these distributions. The constraint matrices had either 75 or 40 percent densities. Also, the relative magnitude of the a_{ij} was either high (-2000 to 10 000) or low (-20 to 80). The 108 experimental conditions come from combining 3 factors having 3 levels and 2 factors having 2 levels.

Six observations were taken for each of the 108 experimental conditions. An observation was obtained from the application of 30 fractional Gomory cuts to a randomly generated test problem. Each test problem was screened to insure that 30 or more cuts would be required to solve it. All test problems were pure integer problems.

Since there exists no direct way of measuring accumulated round-off error, an indicator of this error is used. It is obtained as a byproduct of the computations for improving the inverse of the basic matrix. In this procedure

$$\mathbf{R}_i = \mathbf{I} - \mathbf{A}\,\mathbf{B}_i = \mathbf{A}[\mathbf{A}^{-1} - \mathbf{B}_i] = \mathbf{A}\,\mathbf{S}_i$$

represents the residual, while $\mathbf{S}_i$ represents the error matrix. Since $\mathbf{R}_i$ is available but $\mathbf{S}_i$ is not, the relationship of the norms

TABLE 1. COEFFICIENT RANGES FOR THE TEST PROBLEMS

Number of Variables	Density of Constraint Matrix	Range for the b_i
I. Problems with a_{ij} range of -20 to 80		
10	40%	30 to 110
10	75%	75 to 200
20	40%	80 to 220
20	75%	150 to 430
30	40%	100 to 330
30	75%	225 to 650
II. Problems with a_{ij} range of -2000 to 10 000		
10	40%	3 000 to 15 000
10	75%	4 000 to 25 000
20	40%	6 000 to 30 000
20	75%	15 000 to 55 000
30	40%	9 000 to 45 000
30	75%	22 500 to 80 000

$$\| R_i \| \leqslant \| A \| \; \| S_i \|$$

provides the only clue to the magnitude of the actual error. Note that $B_i = A^{-1}$ whenever R_i vanishes. In the absence of any better measure, the average value of the Euclidean norm $\|R_i\|$ obtained during the application of 30 fractional cuts to each test problem was selected as a quantitative indicator of the accumulative round-off error. All calculations were made in double precision by a UNIVAC 1108 computer which carries 18 digits plus an exponent for each double precision variable. As stated at the begining of this report the computer code used to generate the data is based on the revised simplex procedure.

Tables 2 through 5 present the data. The sample medians and ranges are given by Tables 6 and 7. The sample means are not presented because many of the samples contained extreme values which caused the sample means to be poor indicators of central tendency. The large variations in the ranges clearly indicate that the variances of the different factorial combinations cannot be assumed to be equal. This precludes the use of the F-statistic in performing an analysis of variance on the data. However, Wilson [6] developed a Chi-square statistic which is distribution-free and does not require the equal variance assumption. This statistic can be used to test main factors and interaction effects in a factorial design.

Wilson's procedure was used to test the following hypotheses:

1. There is no difference in the round-off error accumulation in all 108 factorial combinations.

2. Round-off accumulation is the same for 10, 20, and 30 variable problems.

3. Round-off error is the same for all three frequencies at which the procedure for improving the inverse of the basic matrix was applied.

4. All three tolerance values used to round small computed values to zero have the same effect.

5. A 40 percent dense constraint matrix has the same effect as a 75 percent dense constraint matrix.

6. "High" relative magnitude constraint coefficients have the same effect as the "low" relative magnitude coefficients.

7. There is no significant interaction among the factors.

All the hypotheses were tested using an α (probability of rejecting a true hypothesis) of 0.05. Table 8 gives the Chi-square statistics associated with each hypothesis and the corresponding critical Chi-square value.

Having rejected all the null hypotheses, the following alternative hypotheses must be accepted.

TABLE 2. ACCUMULATIVE ROUND-OFF ERROR (LOW RELATIVE MAGNITUDE OF CONSTRAINT COEFFICIENTS 40 PERCENT DENSE CONSTRAINT MATRIX)

Basic Inverse Updated After								
First Base Change			Fifth Base Change			Ninth Base Change		
Number of Problem Variables								
10	20	30	10	20	30	10	20	30
Tolerance = 0.1×10^{-6}								
0.16×10^{-12}	0.38×10^{-12}	0.31×10^{-12}	0.35×10^{-13}	0.92×10^{-12}	0.14×10^{-11}	0.29×10^{-13}	0.37×10^{-11}	0.37×10^{-12}
0.40×10^{-13}	0.22×10^{-13}	0.10×10^{-7}	0.40×10^{-13}	0.88×10^{-13}	0.52×10^{-7}	0.14×10^{-12}	0.29×10^{-12}	0.51×10^{-13}
0.51×10^{-13}	0.55×10^{-10}	0.61×10^{-5}	0.14×10^{-13}	0.61×10^{-10}	0.50×10^{-10}	0.11×10^{-13}	0.14×10^{-3}	0.19×10^{-10}
0.74×10^{-13}	0.27×10^{-13}	0.15×10^{-7}	0.18×10^{-12}	0.59×10^{-12}	0.81×10^{-12}	0.14×10^{-11}	0.43×10^{-11}	0.13×10^{-11}
0.74×10^{-11}	0.42×10^{-9}	0.28×10^{-7}	0.40×10^{-11}	0.29×10^{-10}	0.85×10^{-11}	0.42×10^{-11}	0.11×10^{-7}	0.14×10^{-11}
0.26×10^{-13}	0.47×10^{-14}	0.42×10^{-9}	0.14×10^{-13}	0.26×10^{-13}	0.22×10^{-7}	0.86×10^{-13}	0.34×10^{-12}	0.24×10^{-10}
Tolerance = 0.1×10^{-12}								
0.16×10^{-12}	0.76×10^{-13}	0.39×10^{-12}	0.35×10^{-13}	0.47×10^{-12}	0.25×10^{-11}	0.46×10^{-13}	0.33×10^{-2}	0.31×10^{-12}
0.31×10^{-13}	0.22×10^{-13}	0.24×10^{-12}	0.15×10^{-10}	0.88×10^{-13}	0.16×10^{-10}	0.13×10^{-12}	0.32×10^{-3}	0.25×10^{-11}
0.11×10^{-12}	0.18×10^{-11}	0.14×10^{-12}	0.14×10^{-13}	0.23×10^{-9}	0.67×10^{-12}	0.11×10^{-13}	0.35×10^{-10}	0.71×10^{-1}
0.83×10^{-13}	0.56×10^{-12}	0.28×10^{-12}	0.50×10^{-11}	0.21×10^{-11}	0.26×10^{-11}	0.55×10^{-8}	0.63×10^{-12}	0.21×10^{-10}
0.17×10^{-12}	0.78×10^{-13}	0.44×10^{-12}	0.24×10^{-11}	0.13×10^{-12}	0.56×10^{-11}	0.19×10^{-2}	0.19×10^{-12}	0.30×10^{-12}
0.26×10^{-13}	0.30×10^{-13}	0.23×10^{-12}	0.14×10^{-13}	0.13×10^{-12}	0.10×10^{-12}	0.86×10^{-13}	0.11×10^{-9}	0.44×10^{-11}
Tolerance = 0.1×10^{-18}								
0.20×10^{-12}	0.85×10^{-14}	0.17×10^{-13}	0.70×10^{-1}	0.61×10^{-13}	0.13×10^{-13}	0.12×10^{-12}	0.40×10^{-9}	0.19×10^{-11}
0.16×10^{-13}	0.69×10^{-13}	0.14×10^{-13}	0.17×10^{-13}	0.82×10^{-13}	0.10×10^{-10}	0.29×10^{-12}	0.47×10^{-10}	0.45×10^{-1}
0.91×10^{-14}	0.54×10^{-14}	0.31×10^{-12}	0.13×10^{-13}	0.10	0.17×10^{-10}	0.49×10^{-13}	0.15×10^{-13}	0.35×10^{-11}
0.22×10^{-13}	0.84×10^{-8}	0.87×10^{-11}	0.34×10^{-11}	0.98×10^{-11}	0.76×10^{-12}	0.20×10^{-13}	0.16×10^{-12}	0.85×10^{-10}
0.10×10^{-12}	0.73×10^{-14}	0.38×10^{-13}	0.77×10^{-13}	0.11×10^{-12}	0.48×10^{-13}	0.18×10^{-12}	0.18	0.29×10^{-10}
0.12×10^{-12}	0.27×10^{-13}	0.54×10^{-13}	0.59×10^{-2}	0.40×10^{-12}	0.38×10^{-13}	0.10×10^{-12}	0.71×10^{-13}	0.29×10^{-15}

TABLE 3. ACCUMULATIVE ROUND-OFF ERROR (LOW RELATIVE MAGNITUDE OF CONSTRAINT COEFFICIENTS AND 75 PERCENT DENSE CONSTRAINT MATRIX)

Basic Inverse Updated After								
First Base Change			Fifth Base Change			Ninth Base Change		
Number of Problem Variables								
10	20	30	10	20	30	10	20	30
Tolerance = 0.1×10^{-6}								
0.22×10^{-13}	0.24×10^{-13}	0.22×10^{-13}	0.24×10^{-13}	0.38×10^{-12}	0.14×10^{-13}	0.17×10^{-9}	0.25×10^{-12}	0.30×10^{-12}
0.30×10^{-12}	0.89×10^{-13}	0.82×10^{-13}	0.16×10^{-12}	0.71×10^{-13}	0.86×10^{-13}	0.49×10^{-13}	0.14×10^{-11}	0.17×10^{-12}
0.33×10^{-8}	0.12×10^{-8}	0.63×10^{-12}	0.21×10^{-10}	0.54×10^{-7}	0.74×10^{-10}	0.17×10^{-9}	0.14×10^{-8}	0.35×10^{-10}
0.15×10^{-13}	0.43×10^{-11}	0.83×10^{-8}	0.16×10^{-12}	0.17×10^{-7}	0.57×10^{-10}	0.28×10^{-13}	0.94×10^{-12}	0.47×10^{-6}
0.15×10^{-14}	0.27×10^{-8}	0.11×10^{-12}	0.15×10^{-13}	0.18×10^{-8}	0.13×10^{-11}	0.11×10^{-11}	0.21×10^{-10}	0.19×10^{-11}
0.14×10^{-10}	0.10×10^{-12}	0.11×10^{-11}	0.31×10^{-11}	0.80×10^{-13}	0.82×10^{-11}	0.13×10^{-11}	0.57×10^{-12}	0.27×10^{-5}
Tolerance = 0.1×10^{-12}								
0.22×10^{-13}	0.24×10^{-12}	0.85×10^{-12}	0.70×10^{-13}	0.79×10^{-12}	0.14×10^{-12}	0.66×10^{-12}	0.22×10^{-11}	0.20×10^{-12}
0.48×10^{-12}	0.68×10^{-13}	0.83×10^{-13}	0.13×10^{-12}	0.89×10^{-13}	0.29×10^{-11}	0.12×10^{-12}	0.10×10^{-12}	0.21×10^{-8}
0.16×10^{-11}	0.31×10^{-11}	0.84×10^{-13}	0.44×10^{-12}	0.20×10^{-10}	0.13×10^{-10}	0.21×10^{-9}	0.14×10^{-10}	0.88×10^{-4}
0.15×10^{-13}	0.11×10^{-11}	0.26×10^{-10}	0.16×10^{-12}	0.10×10^{-9}	0.57×10^{-10}	0.28×10^{-13}	0.46×10^{-12}	0.12×10^{-8}
0.15×10^{-14}	0.40×10^{-12}	0.10×10^{-12}	0.15×10^{-13}	0.57×10^{-10}	0.12×10^{-12}	0.93×10^{-11}	0.22×10^{-11}	0.50×10^{-11}
0.14×10^{-10}	0.11×10^{-11}	0.25×10^{-12}	0.17×10^{-9}	0.12×10^{-10}	0.11×10^{-9}	0.16×10^{-9}	0.59×10^{-11}	0.40×10^{-10}
Tolerance = 0.1×10^{-18}								
0.15×10^{-14}	0.66×10^{-12}	0.22×10^{-13}	0.89×10^{-14}	0.22×10^{-12}	0.81×10^{-13}	0.12×10^{-12}	0.13×10^{-11}	0.11×10^{-13}
0.90×10^{-10}	0.64×10^{-12}	0.14×10^{-12}	0.22×10^{-11}	0.56×10^{-11}	0.26×10^{-10}	0.87×10^{-10}	0.61×10^{-13}	0.11×10^{-9}
0.12×10^{-13}	0.12×10^{-11}	0.15×10^{-13}	0.17×10^{-12}	0.56×10^{-11}	0.73×10^{-12}	0.89×10^{-13}	0.20×10^{-8}	0.96×10^{-13}
0.89×10^{-12}	0.16×10^{-13}	0.44×10^{-11}	0.26×10^{-13}	0.18×10^{-11}	0.73×10^{-12}	0.10×10^{-13}	0.25×10^{-12}	0.68×10^{-1}
0.26×10^{-10}	0.10×10^{-12}	0.65×10^{-14}	0.36×10^{-12}	0.42×10^{-12}	0.84×10^{-14}	0.20×10^{-11}	0.16×10^{-11}	0.78×10^{-11}
0.31×10^{-12}	0.13×10^{-8}	0.27×10^{-10}	0.17×10^{-13}	0.19×10^{-12}	0.64×10^{-12}	0.16×10^{-12}	0.33×10^{-12}	0.11×10^{-11}

TABLE 4. ACCUMULATIVE ROUND-OFF ERROR (HIGH RELATIVE MAGNITUDE OF CONSTRAINT COEFFICIENTS AND 40 PERCENT DENSE CONSTRAINT MATRIX)

Basic Inverse Updated After								
First Base Change			Fifth Base Change			Ninth Base Change		
Number of Problem Variables								
10	20	30	10	20	30	10	20	30
Tolerance = 0.1×10^{-6}								
0.10×10^{-8}	0.22×10^{-1}	0.24×10^{-11}	0.20×10^{-11}	0.70×10^{-2}	0.12×10^{-10}	0.76×10^{-12}	0.29×10^{-4}	0.25×10^{-9}
0.35×10^{-8}	0.84×10^{-2}	0.21×10^{-1}	0.12×10^{-8}	0.78×10^{-1}	0.99×10^{-14}	0.61×10^{-8}	0.13×10^{-14}	0.16×10^{-2}
0.13×10^{-11}	0.10×10^{-2}	0.60×10^{-4}	0.18×10^{-11}	0.17×10^{-3}	0.32×10^{-1}	0.58×10^{-11}	0.46×10^{-3}	0.11×10^{-3}
0.61×10^{-3}	0.43×10^{-2}	0.65×10^{-5}	0.50×10^{-4}	0.25×10^{-1}	0.33×10^{-9}	0.34×10^{-1}	0.11	0.26×10^{-10}
0.19×10^{-12}	0.57×10^{-4}	0.29×10^{-11}	0.39×10^{-11}	0.13×10^{-3}	0.15×10^{-10}	0.70×10^{-11}	0.30×10^{-3}	0.11×10^{-11}
0.39×10^{-5}	0.35×10^{-2}	0.57×10^{-4}	0.15×10^{-9}	0.14×10^{-1}	0.25×10^{-3}	0.96×10^{-10}	0.10	0.99×10^{-10}
Tolerance = 0.1×10^{-12}								
0.30×10^{-10}	0.15×10^{-10}	0.28×10^{-11}	0.35×10^{-10}	0.52×10^{-11}	0.27×10^{-8}	0.28×10^{-10}	0.12×10^{-10}	0.48×10^{-10}
0.22×10^{-8}	0.23×10^{-11}	0.22×10^{-10}	0.45×10^{-6}	0.92×10^{-3}	0.12×10^{-7}	0.22×10^{-7}	0.11×10^{-10}	0.88×10^{-11}
0.13×10^{-11}	0.18×10^{-10}	0.76×10^{-11}	0.21×10^{-11}	0.11×10^{-9}	0.15×10^{-9}	0.58×10^{-11}	0.53×10^{-7}	0.35×10^{-10}
0.63×10^{-12}	0.58×10^{-12}	0.23×10^{-10}	0.17×10^{-11}	0.33×10^{-11}	0.32×10^{-5}	0.94×10^{-11}	0.42×10^{-11}	0.45×10^{-10}
0.24×10^{-12}	0.30×10^{-10}	0.29×10^{-11}	0.24×10^{-11}	0.41×10^{-10}	0.15×10^{-10}	0.45×10^{-11}	0.23×10^{-11}	0.56×10^{-11}
0.82×10^{-11}	0.98×10^{-11}	0.58×10^{-11}	0.58×10^{-10}	0.26×10^{-4}	0.13×10^{-9}	0.91×10^{-11}	0.29×10^{-6}	0.35×10^{-9}
Tolerance = 0.1×10^{-18}								
0.64×10^{-12}	0.22×10^{-11}	0.35×10^{-2}	0.44×10^{-11}	0.14×10^{-11}	0.56×10^{-11}	0.66×10^{-12}	0.24×10^{-8}	0.11×10^{-10}
0.39×10^{-11}	0.73×10^{-12}	0.68×10^{-10}	0.11×10^{-11}	0.10×10^{-10}	0.23×10^{-10}	0.14×10^{-11}	0.22×10^{-10}	0.56×10^{-9}
0.70×10^{-12}	0.17×10^{-11}	0.49×10^{-10}	0.22×10^{-9}	0.11×10^{-10}	0.85×10^{-11}	0.33×10^{-11}	0.33×10^{-11}	0.37×10^{-10}
0.78×10^{-12}	0.29×10^{-11}	0.28×10^{-11}	0.75×10^{-9}	0.40×10^{-12}	0.13×10^{-10}	0.99×10^{-11}	0.16×10^{-10}	0.11×10^{-11}
0.65×10^{-12}	0.22×10^{-11}	0.29×10^{-11}	0.54×10^{-11}	0.55×10^{-10}	0.15×10^{-11}	0.30×10^{-12}	0.13×10^{-9}	0.51×10^{-12}
0.25×10^{-9}	0.16×10^{-9}	0.39×10^{-11}	0.16×10^{-11}	0.19×10^{-4}	0.44×10^{-11}	0.66×10^{-11}	0.16×10^{-10}	0.30×10^{-9}

TABLE 5. ACCUMULATIVE ROUND-OFF ERROR (HIGH RELATIVE MAGNITUDE OF CONSTRAINT COEFFICIENTS AND 75 PERCENT DENSE CONSTRAINT MATRIX)

Basic Inverse Updated After								
First Base Change			Fifth Base Change			Ninth Base Change		
Number of Problem Variables								
10	20	30	10	20	30	10	20	30
Tolerance = 0.1×10^{-6}								
0.67×10^{-11}	0.25×10^{-10}	0.96×10^{-11}	0.53×10^{-10}	0.18×10^{-9}	0.10×10^{-9}	0.13×10^{-10}	0.38×10^{-5}	0.18×10^{-9}
0.42×10^{-2}	0.15×10^{-10}	0.55×10^{-5}	0.27×10^{-3}	0.61×10^{-10}	0.38×10^{-8}	0.37×10^{-8}	0.17×10^{-7}	0.14×10^{-9}
0.32×10^{-9}	0.54×10^{-9}	0.20×10^{-4}	0.67×10^{-11}	0.16×10^{-9}	0.67×10^{-4}	0.41×10^{-10}	0.37×10^{-8}	0.69×10^{-7}
0.32×10^{-11}	0.67×10^{-2}	0.56×10^{-6}	0.15×10^{-10}	0.78×10^{-11}	0.11×10^{-10}	0.32×10^{-10}	0.28×10^{-9}	0.36×10^{-8}
0.17×10^{-10}	0.46×10^{-11}	0.17×10^{-4}	0.12×10^{-10}	0.99×10^{-10}	0.99×10^{-10}	0.21×10^{-9}	0.75×10^{-9}	0.60×10^{-9}
0.17×10^{-11}	0.70×10^{-5}	0.14×10^{-2}	0.62×10^{-11}	0.86×10^{-2}	0.13×10^{-1}	0.25×10^{-10}	0.49×10^{-10}	0.48×10^{-11}
Tolerance = 0.1×10^{-12}								
0.66×10^{-11}	0.16×10^{-11}	0.68×10^{-11}	0.20×10^{-9}	0.23×10^{-10}	0.18×10^{-3}	0.29×10^{-9}	0.59×10^{-9}	0.22×10^{-7}
0.36×10^{-11}	0.92×10^{-11}	0.37×10^{-9}	0.15×10^{-10}	0.32×10^{-10}	0.20×10^{-7}	0.32×10^{-9}	0.14×10^{-10}	0.89×10^{-7}
0.36×10^{-11}	0.47×10^{-12}	0.12×10^{-10}	0.11×10^{-10}	0.11×10^{-9}	0.15×10^{-8}	0.13×10^{-10}	0.31×10^{-6}	0.38×10^{-2}
0.32×10^{-11}	0.27×10^{-9}	0.15×10^{-11}	0.48×10^{-11}	0.12×10^{-10}	0.27×10^{-11}	0.34×10^{-10}	0.66×10^{-9}	0.12×10^{-10}
0.16×10^{-10}	0.43×10^{-11}	0.23×10^{-9}	0.24×10^{-10}	0.42×10^{-9}	0.11×10^{-10}	0.76×10^{-9}	0.22×10^{-9}	0.93×10^{-9}
0.17×10^{-11}	0.17×10^{-10}	0.17×10^{-11}	0.62×10^{-11}	0.98×10^{-9}	0.28×10^{-11}	0.28×10^{-10}	0.49×10^{-10}	0.15×10^{-10}
Tolerance = 0.1×10^{-18}								
0.15×10^{-10}	0.38×10^{-11}	0.28×10^{-11}	0.51×10^{-2}	0.37×10^{-11}	0.26×10^{-10}	0.71×10^{-10}	0.40×10^{-11}	0.23×10^{-11}
0.18×10^{-9}	0.23×10^{-10}	0.64×10^{-10}	0.54×10^{-11}	0.37×10^{-6}	0.23×10^{-9}	0.41×10^{-11}	0.35×10^{-10}	0.16×10^{-9}
0.58×10^{-11}	0.23×10^{-9}	0.10×10^{-10}	0.18×10^{-11}	0.49×10^{-10}	0.15×10^{-10}	0.10×10^{-9}	0.10×10^{-9}	0.15×10^{-9}
0.47×10^{-12}	0.16×10^{-10}	0.74×10^{-11}	0.57×10^{-12}	0.30×10^{-10}	0.60×10^{-11}	0.11×10^{-10}	0.31×10^{-11}	0.16×10^{-11}
0.69×10^{-12}	0.40×10^{-12}	0.51×10^{-10}	0.15×10^{-9}	0.32×10^{-11}	0.98×10^{-11}	0.96×10^{-11}	0.21×10^{-10}	0.97×10^{-10}
0.63×10^{-12}	0.37×10^{-11}	0.41×10^{-10}	0.31×10^{-12}	0.38×10^{-10}	0.25×10^{-10}	0.28×10^{-11}	0.85×10^{-9}	0.21×10^{-9}

TABLE 6. SAMPLE MEDIANS

Low Relative Magnitude of Constraint Coefficients								
40 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
Basic Inverse Updated After								
First Base Change			Fifth Base Change			Ninth Base Change		
Number of Problem Variables								
10	20	30	10	20	30	10	20	30
0.63×10^{-13}	0.20×10^{-12}	0.13×10^{-7}	0.37×10^{-13}	0.76×10^{-12}	0.29×10^{-10}	0.11×10^{-12}	0.40×10^{-11}	0.14×10^{-11}
Tolerance = 0.1×10^{-12}								
0.95×10^{-13}	0.77×10^{-13}	0.26×10^{-12}	0.12×10^{-11}	0.30×10^{-12}	0.26×10^{-11}	0.11×10^{-12}	0.75×10^{-10}	0.35×10^{-11}
Tolerance = 0.1×10^{-18}								
0.61×10^{-13}	0.18×10^{-13}	0.46×10^{-13}	0.17×10^{-11}	0.26×10^{-12}	0.40×10^{-12}	0.11×10^{-12}	0.24×10^{-10}	0.16×10^{-10}
75 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
0.16×10^{-12}	0.22×10^{-11}	0.37×10^{-12}	0.16×10^{-12}	0.89×10^{-9}	0.48×10^{-11}	0.12×10^{-11}	0.12×10^{-11}	0.18×10^{-10}
Tolerance = 0.1×10^{-12}								
0.25×10^{-12}	0.75×10^{-12}	0.17×10^{-12}	0.14×10^{-12}	0.16×10^{-10}	0.78×10^{-11}	0.50×10^{-11}	0.22×10^{-11}	0.61×10^{-9}
Tolerance = 0.1×10^{-18}								
0.60×10^{-12}	0.65×10^{-12}	0.81×10^{-13}	0.98×10^{-13}	0.11×10^{-11}	0.68×10^{-12}	0.14×10^{-12}	0.80×10^{-12}	0.45×10^{-11}
High Relative Magnitude of Constraint Coefficients								
40 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
0.22×10^{-8}	0.39×10^{-2}	0.32×10^{-4}	0.76×10^{-10}	0.11×10^{-1}	0.17×10^{-9}	0.52×10^{-10}	0.38×10^{-3}	0.17×10^{-9}
Tolerance = 0.1×10^{-12}								
0.47×10^{-11}	0.12×10^{-10}	0.67×10^{-11}	0.19×10^{-10}	0.73×10^{-10}	0.14×10^{-8}	0.92×10^{-11}	0.11×10^{-10}	0.40×10^{-10}
Tolerance = 0.1×10^{-18}								
0.74×10^{-12}	0.22×10^{-11}	0.26×10^{-10}	0.49×10^{-11}	0.10×10^{-10}	0.71×10^{-11}	0.23×10^{-11}	0.19×10^{-10}	0.24×10^{-10}
75 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
0.12×10^{-10}	0.28×10^{-9}	0.11×10^{-4}	0.13×10^{-10}	0.13×10^{-9}	0.20×10^{-8}	0.36×10^{-10}	0.22×10^{-8}	0.39×10^{-9}
Tolerance = 0.1×10^{-12}								
0.36×10^{-11}	0.67×10^{-11}	0.92×10^{-11}	0.13×10^{-10}	0.69×10^{-10}	0.74×10^{-9}	0.16×10^{-9}	0.41×10^{-9}	0.11×10^{-7}
Tolerance = 0.1×10^{-18}								
0.32×10^{-11}	0.10×10^{-10}	0.25×10^{-10}	0.36×10^{-11}	0.34×10^{-10}	0.20×10^{-10}	0.10×10^{-10}	0.28×10^{-10}	0.13×10^{-9}

TABLE 7. SAMPLE RANGES

Low Relative Magnitude of Constraint Coefficients								
40 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
Basic Inverse Updated After								
First Base Change			Fifth Base Change			Ninth Base Change		
Number of Problem Variables								
10	20	30	10	20	30	10	20	30
0.74×10^{-11}	0.42×10^{-9}	0.61×10^{-5}	0.40×10^{-11}	0.61×10^{-10}	0.52×10^{-7}	0.42×10^{-11}	0.14×10^{-3}	0.23×10^{-10}
Tolerance = 0.1×10^{-12}								
0.15×10^{-12}	0.18×10^{-11}	0.30×10^{-12}	0.15×10^{-10}	0.23×10^{-9}	0.16×10^{-10}	0.19×10^{-2}	0.33×10^{-2}	0.71×10^{-1}
Tolerance = 0.1×10^{-18}								
0.19×10^{-12}	0.84×10^{-8}	0.87×10^{-11}	0.70×10^{-1}	0.10	0.17×10^{-10}	0.27×10^{-12}	0.18	0.45×10^{-1}
75 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
0.33×10^{-8}	0.27×10^{-8}	0.83×10^{-8}	0.21×10^{-10}	0.54×10^{-7}	0.74×10^{-10}	0.17×10^{-9}	0.14×10^{-8}	0.27×10^{-5}
Tolerance = 0.1×10^{-12}								
0.14×10^{-10}	0.31×10^{-11}	0.26×10^{-10}	0.17×10^{-9}	0.10×10^{-9}	0.11×10^{-9}	0.21×10^{-9}	0.14×10^{-10}	0.88×10^{-4}
Tolerance = 0.1×10^{-18}								
0.90×10^{-10}	0.13×10^{-8}	0.27×10^{-10}	0.22×10^{-11}	0.55×10^{-11}	0.26×10^{-10}	0.87×10^{-10}	0.20×10^{-8}	0.68×10^{-1}
High Relative Magnitude of Constraint Coefficients								
40 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
0.61×10^{-3}	0.22×10^{-1}	0.21×10^{-1}	0.50×10^{-4}	0.78×10^{-1}	0.32×10^{-1}	0.34×10^{-1}	0.11	0.16×10^{-2}
Tolerance = 0.1×10^{-12}								
0.22×10^{-8}	0.29×10^{-10}	0.20×10^{-10}	0.45×10^{-6}	0.92×10^{-3}	0.32×10^{-5}	0.22×10^{-7}	0.29×10^{-6}	0.34×10^{-9}
Tolerance = 0.1×10^{-18}								
0.25×10^{-9}	0.16×10^{-9}	0.35×10^{-2}	0.75×10^{-9}	0.19×10^{-4}	0.21×10^{-10}	0.96×10^{-11}	0.24×10^{-8}	0.56×10^{-9}
75 Percent Dense Constraint Matrix Tolerance = 0.1×10^{-6}								
0.42×10^{-2}	0.67×10^{-2}	0.14×10^{-2}	0.27×10^{-3}	0.86×10^{-2}	0.13×10^{-1}	0.37×10^{-8}	0.38×10^{-5}	0.69×10^{-7}
Tolerance = 0.1×10^{-12}								
0.14×10^{-10}	0.26×10^{-9}	0.37×10^{-9}	0.20×10^{-9}	0.97×10^{-9}	0.18×10^{-3}	0.75×10^{-9}	0.31×10^{-6}	0.38×10^{-2}
Tolerance = 0.1×10^{-18}								
0.18×10^{-9}	0.23×10^{-9}	0.62×10^{-10}	0.51×10^{-2}	0.37×10^{-6}	0.22×10^{-9}	0.99×10^{-10}	0.85×10^{-9}	0.21×10^{-9}

TABLE 8. RESULTS OF FACTORIAL EXPERIMENT

Hypothesis Number	Degrees of Freedom	Computed χ^2	Critical χ^2	Decision
1	107	249.33	133.26	Reject
2	2	15.82	5.99	Reject
3	2	15.26	5.99	Reject
4	2	19.70	5.99	Reject
5	1	4.17	3.84	Reject
6	1	124.47	3.84	Reject
7	8	69.91	15.51	Reject

1. All five factors have a significant effect on round-off error accumulation.

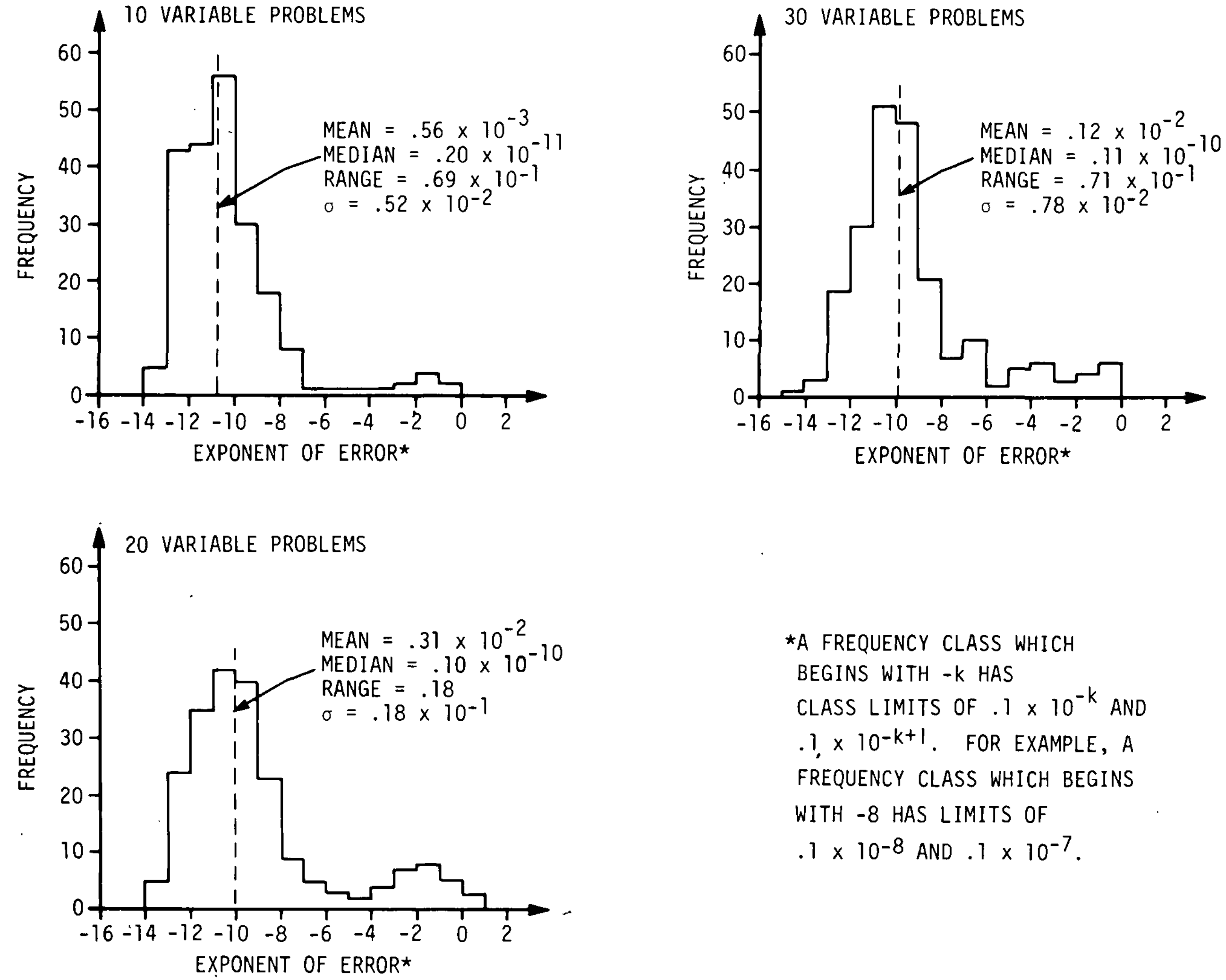
2. There exists a significative interaction among the factors.

The remainder of this report is devoted to an analysis of the individual factor and interaction effects. The results of this analysis should indicate which combination of the two round-off error minimization techniques produces the best results. The individual factor effects will be discussed first.

The 648 observations of round-off error, on which the above analysis of variance was based, can be divided in different ways and used to develop histograms for the different levels of each factor. The observations were first divided into three groups according to the number of problem variables. Figure 3 gives the histograms developed from these groups. Note that in all three histograms the median provided a much better estimate of central tendency than did the mean. This is a common characteristic of data collected in this experiment.

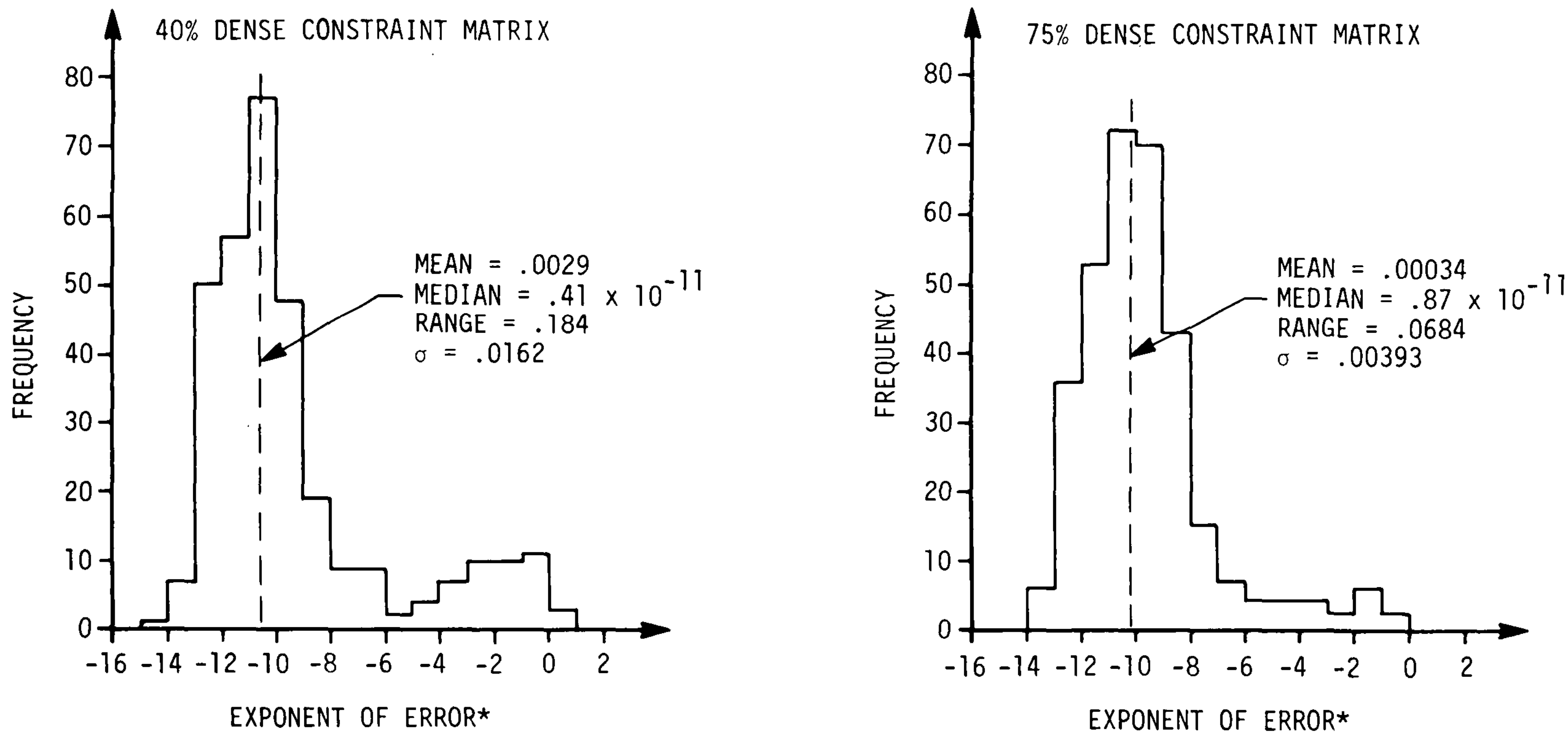
One would expect round-off error to increase with the number of problem variables. Figure 3 indicates a significant increase in both central tendency and variation between 10 variable and 20 variable problems. There appears to be no noticable change between 20 and 30 variable problems. These results can be partially explained by noting that round-off accumulation is more directly related to the size of the basic matrix than to the number of variables. This is true because the entire simplex tableau is generated from the original data and the inverse of the basic matrix. Since all the test problems had 10 original constraints, then the average size of the basic matrix for the 20 and 30 variable problems was not significantly different.

Next, the error observations were divided according to density of the constraint matrix and histograms were constructed. Figure 4 presents the results. Larger errors appear to occur more frequently in the 40 percent density problems. Also, the mean and standard deviation of the histogram for the 40 percent density problems are larger than the mean and standard deviation of the histogram for the 75 percent density problems.



*A FREQUENCY CLASS WHICH BEGINS WITH -k HAS CLASS LIMITS OF .1 x 10^{-k} AND .1 x 10^{-k+1}. FOR EXAMPLE, A FREQUENCY CLASS WHICH BEGINS WITH -8 HAS LIMITS OF .1 x 10^{-8} AND .1 x 10^{-7}.

Figure 3. Histograms of round-off error associated with number of problem variables.



*A FREQUENCY CLASS WHICH BEGINS WITH -k HAS CLASS LIMITS OF .1 x 10^{-k} AND .1 x 10^{-k+1}. FOR EXAMPLE, A CLASS WHICH BEGINS WITH -8 HAS CLASS LIMITS OF .1 x 10^{-8} AND .1 x 10^{-7}.

Figure 4. Histograms of round-off error associated with density of the problem constraint matrix.

The two histograms associated with relative magnitude of the constraint coefficients are presented by Figure 5. They clearly indicate that round-off error increases with the relative magnitude of the constraint coefficients.

Figure 6 presents the histograms associated with the procedure for improving the inverse of the basic matrix. Both central tendency and variation decrease as the procedure is applied more frequently. However, the range of these data was chosen to narrow to show the importance of the procedure. The experience of this writer indicates that round-off error accumulates very fast when the procedure is not applied at all.

Finally, the observations were divided according to the tolerance factor applied. The histograms are presented by Figure 7. A tolerance value of 0.1×10^{-6} appears to be too large. It apparently causes valid data to be lost, thereby adding to the round-off problem. However, a review of Tables 2 through 5 reveals that it is an acceptable tolerance for problems which have low relative magnitude constraint coefficients. That is, only one of the 108 error observations for problems with low relative magnitude constraint coefficients was greater than 0.7×10^{-5}.

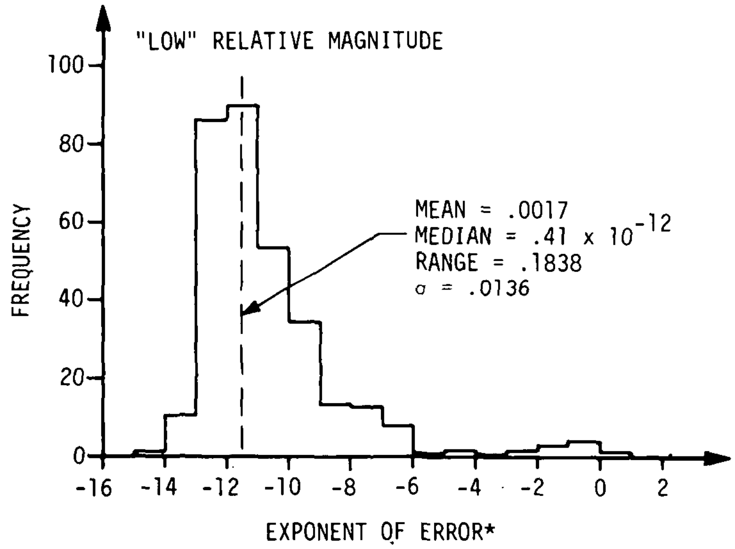
A tolerance of 0.1×10^{-18} creates a problem that is not revealed by Figure 7. It allows very small round-off errors to remain in the tableau. When one of these small values is selected as the pivot element, the problem completely degenerates. This phenomenon occurred in over half of the problems that had high relative magnitude constraint coefficients and 40 percent dense constraint matrices. A total of 73 random problems with high relative magnitude constraint coefficients and 40 percent dense constraint matrices had to be generated to obtain 18 test problems that would not totally degenerate before 30 cuts were applied when a tolerance of 0.1×10^{-18} was used. The phenomenon occurred very infrequently in the other test problems when a tolerance of 0.1×10^{-18} was applied.

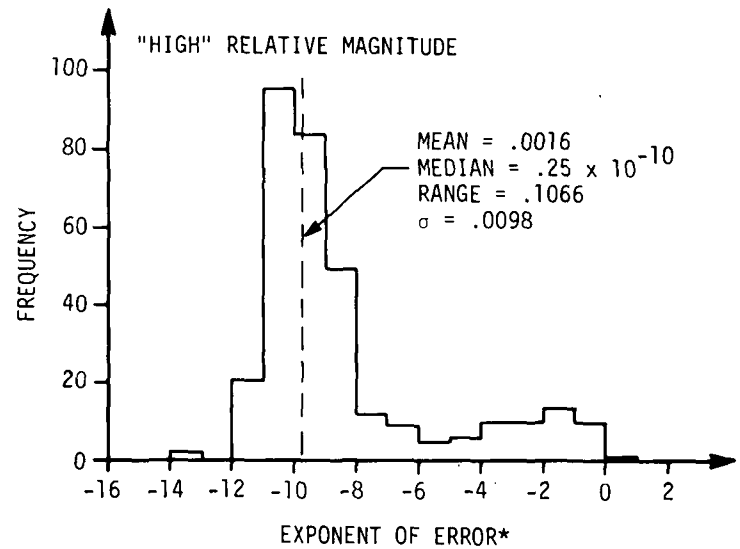
Figure 7 indicates that 0.1×10^{-12} is an acceptable tolerance value. It is small enough to prevent valid data from being lost, yet it is large enough to prevent the phenomenon discussed above.

The discussion will now turn to an analysis of the interactions among the factors. Wilson's Chi-square statistic can also be used to test first order interactions. Since the number of problem variables and the density of the constraint matrix were considered to be nested factors, only the interaction among relative magnitude of constraint coefficients, tolerance value, and the procedure for improving the inverse of the basic matrix will be analyzed. Table 9 gives the Chi-square statistics assuming an α of 0.05. Each null hypothesis assumes that no interaction exists.

Having accepted the alternative hypothesis that all three interactions are significant, a detailed inspection of each interaction is in order. An interaction can best be inspected by dividing the data into subsamples according to factorial combinations of the levels of the factors involved. The interaction effect can then be analyzed by comparing the statistical characteristics of the subsamples.

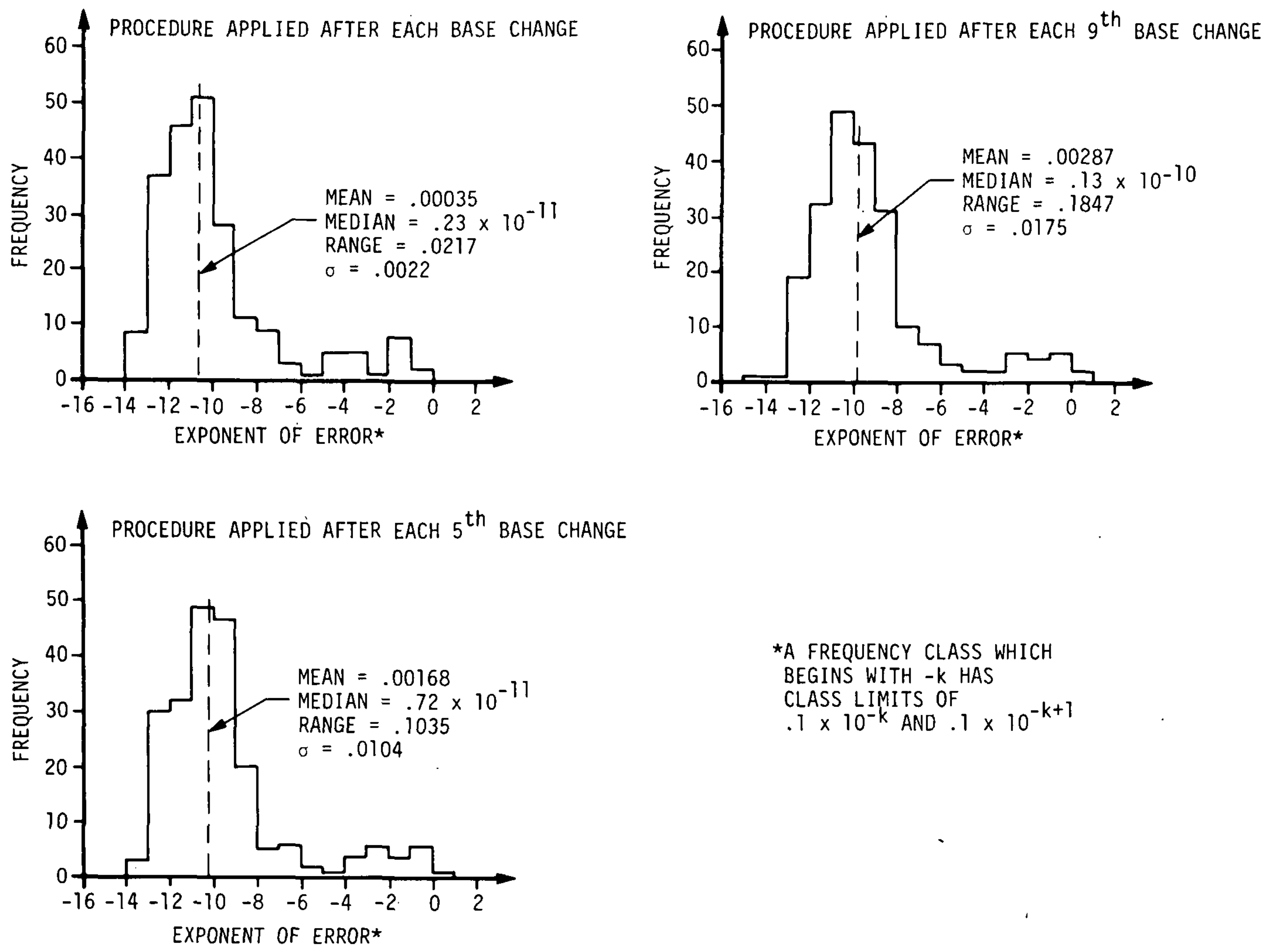
Table 10 presents the medians and ranges of the subsamples for the interaction of tolerance value and relative magnitude of the constraint coefficients. These data indicate that a tolerance value of 0.1×10^{-12} is more desirable than the other two values





*A FREQUENCY CLASS WHICH BEGINS WITH -k HAS CLASS LIMITS OF $.1 \times 10^{-k}$ AND $.1 \times 10^{-k+1}$. FOR EXAMPLE, A CLASS WHICH BEGINS WITH -12 HAS CLASS LIMITS OF $.1 \times 10^{-12}$ AND $.1 \times 10^{-11}$.

Figure 5. Histograms of round-off error associated with *relative magnitude of the constraint coefficients.*



*A FREQUENCY CLASS WHICH BEGINS WITH -k HAS CLASS LIMITS OF $.1 \times 10^{-k}$ AND $.1 \times 10^{-k+1}$

Figure 6. Histograms of round-off error associated with the procedure for improving the inverse of basic matrix.

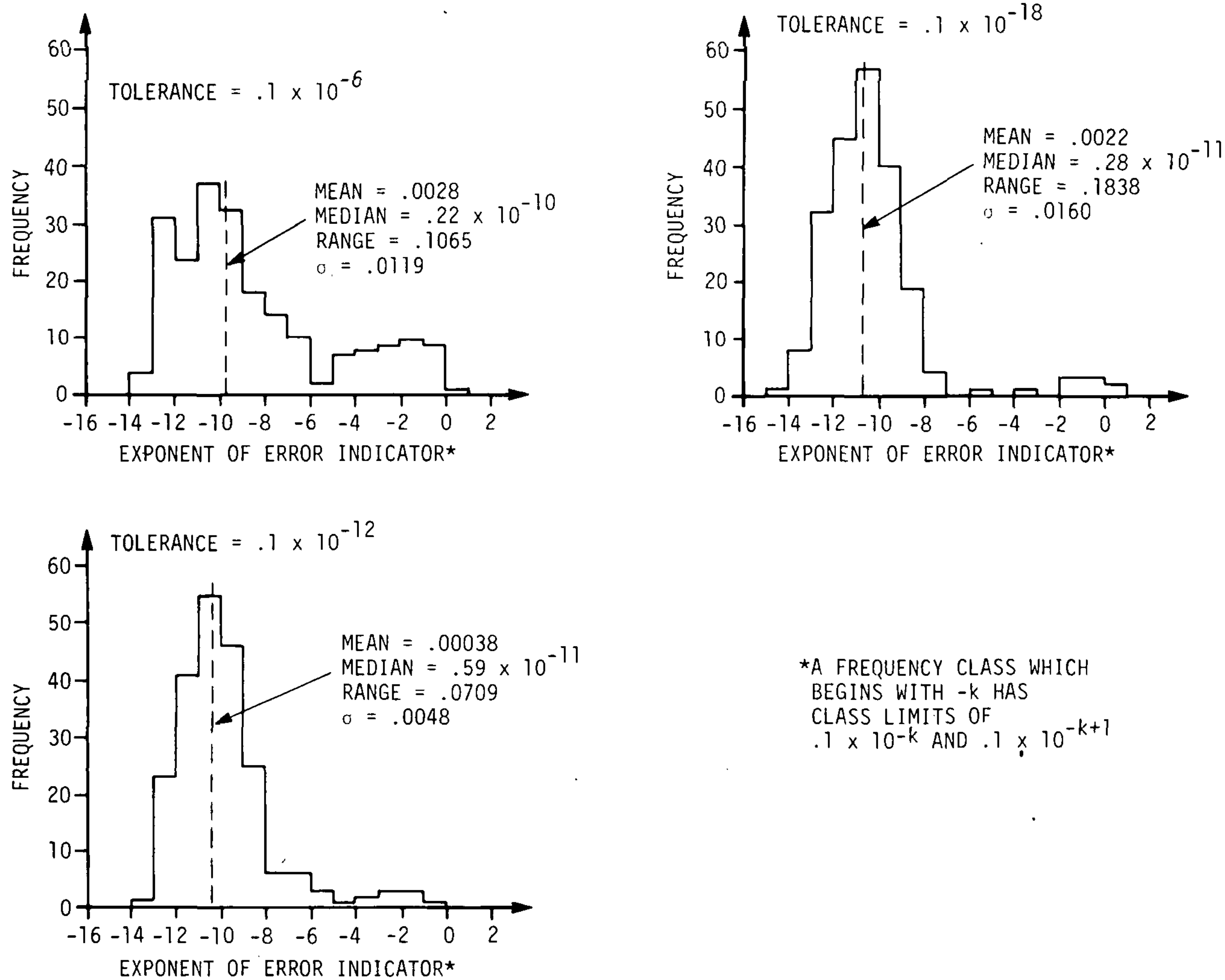


Figure 7. Histograms of round-off error associated with the tolerance value used to round small computed values to zero.

TABLE 9. CHI-SQUARE VALUES FOR INTERACTIONS AMONG THE FACTORS

Interaction Hypothesis[a]	Degrees of Freedom	Computed χ^2	Critical χ^2	Decision
R X T	5	148.07	11.07	Reject
R X P	5	140.07	11.07	Reject
T X P	8	44.67	15.51	Reject

a. R means relative magnitude of constraint coefficients; T means tolerance value; and P means procedure for improving the inverse of the basic matrix.

TABLE 10. ROUND-OFF ERRORS ASSOCIATED WITH DIFFERENT COMBINATIONS OF TOLERANCE VALUES AND CONSTRAINT COEFFICIENT RELATIVE MAGNITUDES

Relative Magnitude of Constraint Coefficients	Tolerance Values		
	1.0×10^{-6}	0.1×10^{-12}	0.1×10^{-18}
Low	0.93×10^{-12} [a] (0.14×10^{-3})	0.47×10^{-12} (0.0709)	0.20×10^{-12} (0.1838)
High	0.57×10^{-9} (0.1066)	0.17×10^{-10} (0.0038)	0.97×10^{-11} (0.0051)

a. Median (range) from sample of 108 observations of round-off error.

because it has the smallest range for high relative magnitude coefficients and the second smallest range for low relative magnitude coefficients. Also, its median is acceptable in both cases. The ranges for the other two tolerances are not consistent between the two coefficient cases. As noted previously, 0.1×10^{-6} is an acceptable tolerance to apply to problems that have low relative magnitude constraint coefficients.

The interaction between the frequency at which the basic matrix "reinversion" procedure is applied and the relative magnitude of the constraint coefficients is indicated by the data in Table 11. These data clearly indicate that decreasing the frequency at which the reinversion procedure is applied increases both the median and range of the round-off error.

Table 12 shows the interaction between tolerance value and the frequency at which the basic matrix reinversion procedure is applied. Once again, a tolerance of 0.1×10^{-12} is shown to be superior to the other two values. Also, applying the reinversion procedure after each base change holds the range of the round-off error well within acceptable bounds. However, each application of this procedure increases the amount of computer time required to solve a problem. Some compromise between

TABLE 11. ROUND OFF ERRORS ASSOCIATED WITH DIFFERENT COMBINATIONS OF CONSTRAINT COEFFICIENT RELATIVE MAGNITUDES AND FREQUENCIES FOR APPLYING THE PROCEDURE WHICH IMPROVES THE COMPUTED INVERSE OF THE BASIC MATRIX

Relative Magnitude of Constraint Coefficients	Reinversion Procedure Applied After		
	First Base Change	Fifth Base Change	Ninth Base Change
Low	0.15×10^{-12} [a] (0.61×10^{-5})	0.65×10^{-12} (0.1035)	0.13×10^{-11} (0.1838)
High	0.99×10^{-11} (0.0217)	0.31×10^{-10} (0.0779)	0.47×10^{-10} (0.1066)

a. Median (range) from sample of 108 observations of round-off error.

TABLE 12. ROUND-OFF ERRORS ASSOCIATED WITH DIFFERENT COMBINATIONS OF TOLERANCE VALUES AND FREQUENCIES FOR APPLYING THE PROCEDURE WHICH IMPROVES THE COMPUTED INVERSE OF THE BASIC MATRIX

Tolerance Values	Reinversion Procedure Applied After		
	First Base Change	Fifth Base Change	Ninth Base Change
0.1×10^{-6}	0.21×10^{-10} [a] (0.0217)	0.15×10^{-10} (0.0779)	0.26×10^{-10} (0.1066)
0.1×10^{-12}	0.14×10^{-11} (0.22×10^{-8})	0.12×10^{-10} (0.92×10^{-3})	0.18×10^{-10} (0.0709)
0.1×10^{-18}	0.84×10^{-12} (0.0035)	0.36×10^{-11} (0.1035)	0.37×10^{-11} (0.1838)

a. Median (range) from sample of 72 observations of round-off error.

computer time and allowable range for the round-off error may be desirable. Table 12 indicates that the reinversion procedure should be applied at least after each fifth base change, because the round-off error in the table is only the amount which accumulated during the application of 30 cuts to each test problem. Several hundred cuts may be required to solve difficult problems if they can be solved by cutting methods at all.

SUMMARY

The first part of this report shows how the revised simplex procedure can be used in cutting algorithms. The second part defines and statistically analyzes two procedures for minimizing round-off error accumulation in a cutting plane code. The following five factors are shown to have a statistically significant effect on round-off error:

1. The number of problem variables.

2. The density (percent of non-zero elements) of the problem constraint matrix.

3. Relative magnitude of the coefficients in the problem constraint matrix.

4. The tolerance value used to round small computed values to zero.

5. The frequency at which a matrix procedure is applied to remove round-off from the computed inverse of the basic matrix.

Round-off error accumulation was shown to increase with (1) an increase in the number of problem variables, (2) a decrease in the density of the constraint matrix, (3) an increase in the relative magnitude of the constraint coefficients, (4) a tolerance value of 0.1×10^{-5} or greater, and (5) a decrease in the frequency at which the basic matrix reinversion procedure is applied. Round-off error can be minimized by applying the matrix procedure to improve the computed inverse after each base change and selecting an appropriate tolerance which is related to the number of significant digits carried for each variable. When all calculations are done in double precision (16 to 18 digits plus an exponent) a tolerance value of 0.1×10^{-12} is an acceptable value. The selection of a significantly larger tolerance tends to add to the round-off problem. Selection of a significantly smaller tolerance reduces the effectiveness of the tolerance by allowing division by very small insignificant values.

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