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A REVIEW OF DEVELOPMENTS IN
THE THEORY OF ELASTO-PLASTIC FLOW

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16. Abstract

The theory of elasto-plastic flow is developed in a fairly general manner so that it may accommodate features such as work-hardening, anisotropy, plastic compressibility, non-continuous loading including local or global unloading, and others. A complete theory is given in quasi-linear form; as a result, many useful attributes are accessible. Several integral theorems may be written, finite deformations may be incorporated, and efficient methods for solving problems may be developed; these and other aspects are described in some detail. The theory is reduced to special forms for 2-space, and extensive experience in solving such problems is cited.

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FOREWORD

This report summarizes research in the theory of elasto-plastic

flow performed at Carnegie-Mellon University for Langley Research

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April 1973. Notes for this report are maintained in file SM-73-8;

it is one of several issued in conjunction with this effort, the

others including references 4-6, 10, 13, 15-19, 21, and 22 following

the text. In addition, several articles intended for journal

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The author is most appreciative of the valuable technical

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TABLE OF CONTENTS

INTRODUCTORY REMARKS

In a variety of situations of engineering interest, materials behave in a nonlinear manner, that is to say, the response to excitation deviates from a simple linear relationship. In years past, it was sufficient to ignore this behavior or, in some instances, to approximate it in what is now viewed as a crude manner. More recently, however, there has been impetus to look closely at nonlinearities. The motivation derives from several sources. On the one hand, design requirements have grown more stringent as the result of the need for improved performance of materials, the use of more expensive materials, and the desire for better prediction of response to high excitation. On the other hand, our capabilities have grown not only with the overall advances of technology but also and perhaps more importantly, with the availability of large high-speed computers. These tools have provided researchers and engineers both with the capacity to solve problems that once were posed largely for intellectual interest. Computers have also served to reduce the disparity of language between researcher and practicing engineer; the one can translate complex mathematical results into computer programs, and the other can use the code to solve problems directly.

One of the kinds of nonlinearity that is of extensive interest occurs in metals and their alloys where excitation is sufficient to induce plastic flow. Such behavior is expected in structures designed

for high strength and is sometimes observed in structures otherwise intended to be quite stiff. Plastic flow also occurs in regions of high stress concentrations such as holes, cutouts, and notches, sometimes quite unintentionally. The issue in such cases becomes that of determining the integrity of the structure after plastic flow has occurred. In addition, there is frequent interest in the mechanics of the flow process itself partly to determine whether the process can either be avoided or be turned to advantage.

The theory or, more accurately, theories of plasticity are hardly new to students and practioners of mechanics. The literature dates from the last century, and elements of plasticity are to be found in engineering curricula, both undergraduate and graduate, throughout the country. Since 1950, the publication date of Hill's book [1]*, there has been a resurgence of interest and effort in plasticity. Yet, in certain respects, such attention has been accompanied by inherent limitations which may be of consequence in a variety of applications. The use, for example, of 'perfect' plasticity fails to model many materials of technological importance; rigid plasticity is inappropriate to structural situations in which elastic and plastic strains are both of interest; and slipline techniques tell little of the *process* that precedes and engenders flow.

Of more recent origin is what we term here the theory of elasto-plastic flow. Developed by the author, his colleagues and students

*Numbers in brackets denote entries in the list of references following the text.

over the past few years, elasto-plasticity results from a special formulation of information already in hand. This formulation is believed to serve two purposes. First, it provides means for solving a broad variety of problems involving plastic flow, a few of which are indicated in later sections of this report. Second, it suggests an approach for formulating theories of other non-linear phenomena. While we do not discuss the latter feature in this report, we fully expect it to offer a significant potential to other researchers and engineers.

This report is divided into several sections which give general and special forms of the constitutive relations, together with other relations which provide the full theory; reductions to two spatial dimensions; solutions to elementary problems; numerical methods for solving more complex problems; and a review of some solutions that have been obtained.

CONSTITUTIVE RELATIONS IN GENERAL

As suggested above, there are several theories of plasticity, distinguished by a variety of relationships between strain or other kinematic measures, and stress. Accompanying this range, there are many approaches available for deriving or developing these relationships, and the interested reader may care to peruse some of the texts now available to see these approaches. Since our discussion is not as heuristic as that found in a text, we proceed on a more axiomatic basis, relying on just two primary assumptions.

In addition, we omit dynamic effects so that the flow process to be described is taken to be slow, in the sense that no wave phenomena occur. For the most part we deal with infinitesimal strains, although attention is given below to finite strain effects. While body forces will appear in the formulation of the next section, they are regarded largely in a formalistic sense and are excluded in specific problems.

The first primary assumption is that there is a loading function f which depends upon the stress field* σ_{ij}, the plastic strains $\varepsilon_{ij}^{(p)}$, and the plastic strain energy density $W^{(p)}$. At each point of the body, if $f < 0$, no flow occurs; where $f = 0$, flow may occur and we write this

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*Standard indicial notation and its associated conventions are used throughout, except where noted. The range of roman indices is 1,2,3 and δ_{ij} is tne Kronecker delta.

condition on the loading function as

$$f(\sigma_{ij}, \varepsilon_{ij}^{(p)}, W^{(p)}) < 0 \tag{1}$$

This statement may be visualized somewhat more easily if we write
the particular form

$$f(\sigma_{ij}, \varepsilon_{ij}^{(p)}, W^{(p)}) = \phi(\sigma_{ij}) - \psi(\varepsilon_{ij}^{(p)}, W^{(p)}) \leq 0 \tag{2}$$

which is used throughout most of our development. In (2) $\phi(\sigma_{ij})$ is
a function which may be regarded as having two roles. In the hyperspace*
whose axes are the stresses σ_{ij}, ϕ is a measure of the vector from the
origin to the point describing the stress field at a point in the body.
If this measure is less than the scalar ψ, no flow ensues. Alterna-
tively, the relation

$$\phi(\sigma_{ij}) - \psi = 0 \tag{3}$$

is a surface in stress space; if the stress vector is such that its tip
lies on the surface, flow may occur depending upon how the stress vector
changes as a result of additional excitation. Note that flow will
alter ψ so that the stress vector will never pierce the surface.

The second primary assumption is that hypothesized by Drucker [2].
Without reviewing the arguments in full detail, we note that the
hypothesis concerns a prestressed body subject to a small increment in
load. This self-equilibrated loading is applied and then removed.

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*Alternatively, one may think in terms of a 3-space whose axes are the
principal stresses σ_1, σ_2, σ_3. While the alternate is more easily
visualized, the form in (2) is more useful in formulation of the
constitutive equations.

The resultant stress increment $\delta\sigma_{ij}$ engenders an increment in plastic strains $\delta\epsilon_{ij}^{(p)}$ and, largely on thermodynamic grounds, Drucker requires that

$$\delta\sigma_{ij}\delta\epsilon_{ij}^{(p)} \geq 0 \tag{4}$$

That the quantity on the left of (4) is nonnegative requires that no work may be extracted from the process. We may also interpret (4) to imply that the motion in the process corresponds to the loading, and vice versa.

The utility of (4) lies in the conclusions that may be drawn from it. The first is that the surface described in (3) is convex, and the second that the direction of $\delta\epsilon_{ij}^{(p)}$ is normal to that surface. Hence we have

$$\delta\epsilon_{ij}^{(p)} = \lambda\partial f/\partial\sigma_{ij} \tag{5a}$$

or

$$\delta\epsilon_{ij}^{(p)} = \lambda\partial\phi/\partial\sigma_{ij} \tag{5b}$$

depending on whether (1) or (2) is the appropriate form of the loading function. The third conclusion is that the increments of stress and plastic strain are proportional to one another and, as that result is central to our formulation, we consider it in further detail below. The arguments necessary to develop convexity and normality may be found in Drucker's paper [2] or Fung's book [3], chapter 6.

Consider a point in a body whose loading history is such that (3) applies. Let an increment in excitation be imposed such that flow occurs; it follows that ϕ and ψ will change in concert:

$$\delta\phi - \delta\psi = 0$$

or, to first order terms,

$$(\partial\phi/\partial\sigma_{ij})\delta\sigma_{ij} - [\partial\psi/\partial\epsilon_{ij}^{(p)} + (\partial\psi/\partial W^{(p)})\sigma_{ij}]\delta\epsilon_{ij}^{(p)} = 0 \qquad (6)$$

where we have used the differential definition of plastic strain energy density in the increment:

$$\delta W^{(p)} = \sigma_{ij}\delta\epsilon_{ij}^{(p)}$$

Substituting (5b) into (6) and solving for λ gives*

$$\lambda = \frac{(\partial\phi/\partial\sigma_{ij})\delta\sigma_{ij}}{[\partial\psi/\partial\epsilon_{mn}^{(p)} + (\partial\psi/\partial W^{(p)})\sigma_{mn}](\partial\phi/\partial\sigma_{mn})} \qquad (7)$$

so that there results

$$\delta\epsilon_{ij}^{(p)} = \frac{(\partial\phi/\partial\sigma_{ij})(\partial\phi/\delta\sigma_{k\ell})\delta\sigma_{k\ell}}{[\partial\psi/\partial\epsilon_{mn}^{(p)} + (\partial\psi/\partial W^{(p)})\sigma_{mn}](\partial\phi/\partial\sigma_{mn})} \qquad (8a)$$

and the proportionality between plastic strain and stress increments is clear. Had we used (5a), we would find

$$\delta\epsilon_{ij}^{(p)} = \frac{(\partial f/\partial\sigma_{ij})(\partial f/\partial\sigma_{k\ell})\delta\sigma_{k\ell}}{[\partial f/\partial\epsilon_{mn}^{(p)} + (\partial f/\partial W^{(p)})\sigma_{mn}](\partial f/\partial\sigma_{mn})} \qquad (8b)$$

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*An obvious substitution of dummy indices is required in this step.

Either of (8a,b) is often termed the flow rule for incremental plasticity.

It is useful next to introduce an intrepretation of the function ϕ. As we have described it, it is suggestive of an equivalent stress, and we now specify that

$$\phi \equiv \tau_{eq}$$

and it should be noted that eq are *not* indices. There is then a corresponding equivalent plastic strain increment defined by

$$\tau_{eq} \delta \varepsilon_{eq}^{(p)} = \sigma_{ij} \delta \varepsilon_{ij}^{(p)} = \delta W^{(p)} \tag{9}$$

and we infer a functional relationship between the two equivalent quantities. In particular, we have at any value of τ_{eq} such that flow is in progress

$$\delta \tau_{eq} / \delta \varepsilon_{eq}^{(p)} = 2\mu_{eq}^{(p)} \tag{10}$$

It may be observed that, once suitable experimentation is performed to find the form of the function $\phi(\sigma_{ij})$, the (scalar) equivalent stress and plastic strain relationship may be identified, and that the modulus of that relationship may be determined. If we combine (8a), (9), and (10) we find that

$$\delta \varepsilon_{ij}^{(p)} = \frac{\phi (\partial \phi / \partial \sigma_{ij}) (\partial \phi / \partial \sigma_{k\ell})}{2\mu_{eq}^{(p)} (\partial \phi / \partial \sigma_{mn}) \sigma_{mn}} \delta \sigma_{k\ell} \tag{11}$$

We have thus a form of the flow rule that depends on ϕ and the more physical $\mu_{eq}^{(p)}$, rather than ψ. It does not appear possible to reduce (8b) to an analogous form since f is not dependent only on σ_{ij}.

The total strain increment comprises both elastic and plastic portions, and we have so far dealt with the latter. The former follows Hooke's law written in incremental form, viz:

$$2\mu\delta\varepsilon_{ij}^{(p)} = \delta\sigma_{ij} - [\nu/(1+\nu)]\delta\sigma_{kk}\delta_{ij} \tag{12}$$

where μ is the elastic shear modulus and ν is Poisson's ratio. Combining (11) and (12) we have the total flow rule

$$2\mu\delta\varepsilon_{ij} = \delta\sigma_{ij} - [\nu/(1+\nu)]\delta\sigma_{kk}\delta_{ij} + \Gamma(\partial\phi/\partial\sigma_{ij})(\partial\phi/\partial\sigma_{k\ell})\delta\sigma_{k\ell} \tag{13}$$

where we have set

$$\Gamma = [\mu/\mu_{eq}^{(p)}] \phi/[(\partial\phi/\partial\sigma_{mn}) \sigma_{mn}]$$

as a matter of convenience. The inverse to (13) is

$$\delta\sigma_{ij}/2\mu = \delta\varepsilon_{ij} + [\nu/(1-2\nu)]\delta\varepsilon_{kk}\delta_{ij} \tag{14}$$

$$- \frac{\{\partial\phi/\partial\sigma_{ij} + [\nu/(1-2\nu)]\partial\phi/\partial\sigma_{rs}\delta_{rs}\delta_{ij}\}\{\partial\phi/\partial\sigma_{k\ell} + [\nu/(1-2\nu)]\partial\phi/\partial\sigma_{pq}\delta_{pq}\delta_{k\ell}\}\delta\varepsilon_{k\ell}}{(\partial\phi/\partial\sigma_{mn})\{\partial\phi/\partial\sigma_{mn} + [\nu/(1-2\nu)]\partial\phi/\partial\sigma_{tu}\delta_{tu}\delta_{mn}\} + 1/\Gamma}$$

Terms of the form $(\partial\phi/\partial\sigma_{rs})\delta_{rs}$ are to be observed in (14). The degree to which such terms do not vanish is a measure of compressibility of the

plastic strains. That is, in (11), we have the plastic dilatation which so far is not presumed to vanish. If it is required, however, that plastic strains are incompressible, (14) reduces to

$$\delta\sigma_{ij}/2\mu = \delta\varepsilon_{ij} + [\nu/(1-2\nu)]\delta\varepsilon_{kk}\delta_{ij} - \frac{(\partial\phi/\partial\sigma_{ij})(\partial\phi/\partial\sigma_{k\ell})\delta\sigma_{k\ell}}{(\partial\phi/\partial\sigma_{mn})(\partial\phi/\partial\sigma_{mn}) + 1/\Gamma}$$

a somewhat more compact form and one which corresponds to the usual assumption (in metals) of incompressibility of plastic strains.

While (8b) does not reduce to a form analogous to (11), it may be inverted to a form similar to (14); the result is

$$\delta\sigma_{ij}/2\mu = \delta\varepsilon_{ij} + [\nu/1-2\nu)]\delta\varepsilon_{kk}\delta_{ij}$$

$$- \frac{\{(\partial f/\partial\sigma_{ij}) + [\nu/(1-2\nu)]\partial f/\partial\sigma_{rs}\delta_{rs}\delta_{ij}\}\{\partial f/\partial\sigma_{k\ell} + [\nu/(1-2\nu)]\partial f/\partial\sigma_{pq}\delta_{pq}\delta_{k\ell}\}\delta\varepsilon_{k\ell}}{(\partial f/\partial\sigma_{mn})\{\partial f/\partial\sigma_{mn} + [\nu/(1-2\nu)]\partial f/\partial\sigma_{tu}\delta_{tu}\delta_{mn}\} + 1/\Gamma^*}$$

where

$$\Gamma^* = -2\mu/\{[\partial f/\partial\varepsilon_{mn}^{(p)} + (\partial f/\partial W^{(p)})\sigma_{mn}](\partial f/\partial\sigma_{mn})\}$$

Note that utility of the last expression awaits a definite form for f, although one may use this result in a formal sense in the development below.

In each of these cases, we find relationships that show the proportionality between the increments in stress and strain. Symbolically

we may therefore write

$$2\mu\delta\epsilon_{ij} = C_{ijk\ell}\delta\sigma_{k\ell} \tag{15}$$

and

$$\delta\sigma_{ij}/2\mu = E_{ijk\ell}\delta\epsilon_{k\ell} \tag{16}$$

where $C_{ijk\ell}$ and $E_{ijk\ell}$ are a generalized compliance and modulus whose specific forms depend upon the loading function selected or otherwise taken to model the material. To the extent that other models may be put into this same form, we may proceed to develop the complete set of equations for elasto-plastic flow, as shown in the next section.

GOVERNING EQUATIONS

The derivation of a flow equation of the form of (15) and the existence of its inverse (16) permits development of the full theory to proceed. We take as a starting point the fact that the constitutive equations relate increments of stress and strain in an explicit manner. It follows that the remaining elements of the theory should deal with incremental quantities directly; to do otherwise leads perforce to nonlinearities in the theory and therefore difficulty in solving problems.

Experience has shown that the simplest formulation will be in terms of displacement increments rather than potential functions (or their increments), such as any of several types of stress functions. Accordingly, we write the strain-displacement equations in the form

$$\delta\varepsilon_{ij} = (\delta u_{i,j} + \delta u_{j,i}) \ / \ 2 \qquad (17)$$

where δu_i are the displacement increments. These quantities will become the dependent variables of prime interest. Note that (17) is tensorially correct when the differentiation is interpreted as covariant inasmuch as the development here presumes small deformations (or displacement gradients). This feature characterizes the remaining development except where noted in the following sections.

The equilibrium equations are also written in incremental form. Formally, we have

$$\delta\sigma_{ij,j} + \delta X_i = 0$$

where δX_i are increments in the body force. For most problems of interest, however, we drop reference to body force and write simply

$$\delta\sigma_{ij,j} = 0 \tag{18}$$

as the equations of equilibrium.

Note that both (17) and (18) are in effect differential equations in both space and some time-like parameter that reflects the sequential nature of the loading and the process of elasto-plastic flow. It follows that (17) and (18) must be accompanied by initial conditions. One might specify, for example, the position of the body at an initial 'time', along with its then-existing state of stress and strain. While in many problems, these field quantities will be null, it is clear that problems involving, say, a residual stress state may come under consideration; the mathematical need for initial conditions provides the proper means for inserting such physical information into the problem.

We now assemble the foregoing. (17) is inserted into (16) and, owing to the symmetry of the generalized modulus, we have

$$\delta\sigma_{ij}/2\mu = E_{ijk\ell}\delta u_{k,\ell} \tag{19}$$

Equilibrium is enforced to obtain

$$(E_{ijk\ell}\delta u_{k,\ell})_{,j} = 0 \tag{20}$$

which is an extended or elasto-plastic form of Navier's equations. It is this (set of three) equation(s) which governs the flow of an elasto-plastic medium.

Boundary Conditions: It should be reasonably clear that (20) is an analogue to the equations of classical elasticity; the differences are that the modulus is variable rather than constant, and that the variables are the increments of displacement rather than their instantaneous values. We may use this analogue to write boundary conditions. At the boundary of the domain in which (20) is to be integrated, one specifies either increments in displacement δu_i or increments in traction δt_i, the latter being given by an adaptation of Cauchy's equations

$$\delta t_i = \delta\sigma_{ij} n_j = 2\mu E_{ijk\ell} \delta u_{k,\ell} n_j \tag{21}$$

having employed (19). In (21) n_j is the unit outward normal to the boundary. It is further possible to specify a mixed boundary-value problem in exactly the same manner as is done in elasticity.

Integral Theorems: Because the theory is formulated in a manner to identify incremental quantities as dependent variables, it is straightforward to devise integral theorems that may be of use in solving problems. Some of these have been described elsewhere, and we repeat them for completeness, In order to avoid complexity of notation, we replace the incremental notation by that usually associated with rates; thus

$$\delta u_i \rightarrow \dot{u}_i$$
$$\delta\varepsilon_{ij} \rightarrow \dot{\varepsilon}_{ij}$$
$$\delta\sigma_{ij} \rightarrow \dot{\sigma}_{ij}$$

so that the variables of interest are instantaneous rates of change (with respect to the inferred time-like parameter noted above) rather than increments.

We have first an analogue to the theorem of minimum potential energy; consider the functional

$$\Pi = \mu \int_V E_{ijk\ell}\dot{\epsilon}_{ij}\dot{\epsilon}_{k\ell}dV - \int_V \dot{X}_i\dot{u}_i dV - \int_{S_\sigma} \dot{t}_i\dot{u}_i dS$$

in which V is the interior of the domain of interest, S is its surface or boundary, and S_σ is that portion on which the traction rates t_i are specified. It may be shown [4] that, if all displacement rates that satisfy stated boundary conditions on S_u, where $S = S_\sigma + S_u$, those which satisfy the equations of equilibrium in rate form are distinguished by a stationary value of Π; moreover, that value of Π is a minimum. The procedure by which this theorem is established is given in [4] and by the pattern set in [3], chapter 10. Basically, one sets the first variation of Π to zero, i.e.,

$$\delta\Pi = 2\mu \int_V E_{ijk\ell}\dot{\epsilon}_{ij}\delta\dot{\epsilon}_{k\ell}dV - \int_V \dot{X}_i\delta\dot{u}_i dV - \int_{S_\sigma} \dot{t}_i\delta\dot{u}_i dS$$

$$= 0$$

and integrates terms in the first integral by parts, using (17). The results are (20), modified to include body forces, and (21), in rate form.

We consider next the functional

$$\Pi^* = \frac{1}{4\mu} \int_V C_{ijk\ell}\dot{\sigma}_{ij}\dot{\sigma}_{k\ell}dV - \int_V \dot{u}_i\dot{X}_idV - \int_{S_u} \dot{u}_i\dot{t}_idS$$

Using the same type of procedure, it may be shown that, of all stress rates that satisfy the equations of equilibrium rate and boundary conditions on S_σ, those which lead to a compatible strain rate field are distinguished by a stationary value of Π^*; moreover, that value of Π^* is a minimum. This theorem is less useful than the one above, since its proof and use require introduction of stress (rate) functions which we view, as noted above, as awkward.

Third, we write the functional

$$P = \int_V [\mu E_{ijk\ell}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{k\ell} - \dot{X}_i\dot{u}_i]dV - \int_V \dot{\sigma}_{ij}[\dot{\varepsilon}_{ij}-(\dot{u}_{i,j}+\dot{u}_{j,i})/2]dV$$

$$- \int_{S_\sigma} \dot{t}_i\dot{u}_idS - \int_{S_u} n_j\dot{\sigma}_{ij}(\dot{u}_i-\dot{u}_i^*)dS$$

in which $\dot{u}_i^*$ are the displacement rates specified on S_u. We compute the first variation of P, allowing $\dot{u}_i$, $\dot{\varepsilon}_{ij}$, and $\dot{\sigma}_{ij}$ to vary in V, $\dot{u}_i$ to vary on S_σ, and $\dot{\sigma}_{ij}$ to vary on S_u. Note that $\dot{X}_i$, $\dot{t}_i$, and $\dot{u}_i^*$ are prescribed. Computing δP, integrating by parts those terms involving

$\delta \dot{u}_{i,j}$, and setting the result to zero, we find

$$\dot{\sigma}_{ij} = 2\mu E_{ijk\ell}\dot{\varepsilon}_{k\ell} \qquad \text{in } V$$

$$\dot{\sigma}_{ij,j} + \dot{X}_i = 0 \qquad \text{in } V$$

$$\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2 \qquad \text{in } V$$

$$\dot{t}_i = \dot{\sigma}_{ij}n_j \qquad \text{on } S_\sigma$$

$$\dot{u}_i = \dot{u}_i^* \qquad \text{on } S_u$$

Thus this analogue to Reissner's theorem gives a recapitulation of the various equations derived above. A complementary form of the theorem derives from the functional

$$P^* = -\int_V [\frac{1}{4\mu} C_{ijk\ell}\dot{\sigma}_{ij}\dot{\sigma}_{k\ell} + \dot{X}\dot{u}_i]dV + \int_V [\dot{\sigma}_{ij}(\dot{u}_{i,j} + \dot{u}_{j,i})/2]dV$$

$$- \int_{S_\sigma} \dot{u}_i \dot{t}_i dS - \int_{S_u} (\dot{u}_i - \dot{u}_i^*)\dot{\sigma}_{ij}n_j dS$$

but is of limited utility as noted above for Π^*.

Finally, we note the existence of a form of Betti's reciprocal work theorem. It has been shown [5] that

$$\int_{V-V'} \sigma'_{ij}\dot{\varepsilon}_{ij}^{(p)}dV = \int_{V-V'} \varepsilon'_{ij}\dot{\sigma}_{ij}dV$$

The primed stress and strain fields are those associated with an elastic field; in particular we consider Kelvin's problem of a point

load in an infinite elastic body. V' is a ball surrounding the
load point, deleted from the integrals owing to the singular nature
of the local stress and strain fields. Note that rates or equivalently,
increments, form the basis of reciprocity. Due to the inferred
ellipticity of the governing equations [5,6], the $\dot{u}_i$ are continuous
and should possess continuous second derivatives. Thus the divergence
theorem holds and the reciprocal theorem may be written as

$$\int_{S+S'} t_i' \ddot{u}_i \, dS = \int_{S+S'} u_i' \dot{t}_i \, dS + \int_{V-V'} \sigma_{ij}' \dot{\varepsilon}_{ij}^{(p)} \, dV$$

where S' is the bounding surface of V'. Taking the limit as $V' \to 0$,
we have

$$\dot{u}_i + \int_S T_{ij} \dot{u}_j \, dS = \int_S U_{ij} \dot{t}_j \, dS + \int_V \Sigma_{ijk} \dot{\varepsilon}_{jk}^{(p)} \, dV$$

as an extended form of Somigliana's identity. In this relationship,
note that the plastic strain rate is proportional to the stress rate
and therefore the total strain rate, through either of (8), and (10).
Thus this integral statement may be written to relate surface tractions
and displacements may be eliminated [5]. The kernals T_{ij}, U_{ij}, and
Σ_{ijk} are given in [5] and need not be repeated here.

Undoubtedly, other relationships analogous to those familiar in
elasticity may be developed for elasto-plastic flow. The reason for

this, and the central feature of the present theory, is the attention given to increments (or equivalently, rates) of field quantities as the dependent variables of interest when a problem is to be solved. What this type of formulation does is to shift the nonlinearities of the physical process into a slightly subordinate position. Mathematically, *the governing equations are not nonlinear*. Instead, they are quasilinear; the nonlinear aspects of the process show up as coefficients in the governing equations and, as described below, the solution of problems is greatly facilitated. Moreover, the equations themselves are accessible to study; we have, for example, looked in varying degrees of detail at the character of the equations [5,6] and found them to be elliptic so long as there is some work-hardening. This nature of the equations has enormous implications for the means to be used in solving problems. We know, for example, that so long as there is some work-hardening, no slip lines will occur. Thus many of the solution techniques developed for rigid and perfectly plastic materials are demonstrably inappropriate. In their place, however, one is free to choose methods appropriate to elastic problems; the chief numerical methods thus available are finite elements, finite differences, and boundary integral equations. Each of these offers certain advantages and disadvantages to the analyst, depending upon the particular problem at hand, and he has access to the extensive experience relating to each as he proceeds to attack his problem. We discuss some aspects of solution strategy in a subsequent section of this report.

SPECIAL CONSTITUTIVE RELATIONS

In the penultimate section, we discussed constitutive relations in general, arriving at relations of the form

$$2\mu\delta\varepsilon_{ij} = C_{ijk\ell}\delta\sigma_{k\ell} \tag{15}$$

and the inverse

$$\delta\sigma_{ij}/2\mu = E_{ijk\ell}\delta\varepsilon_{k\ell} \tag{16}$$

It is important that relations of this type be used so that the remaining development of the governing equations may be followed. Within these limits, however, a considerable variety of specific constitutive relations may be employed and, in this section, we indicate a few that may be useful in analysis. Fung [3], Chapter 6, lists many such possibilities by indicating the appropriate form of the loading function, and his bibliography cites much of the literature in this area. Both isotropic and anisotropic forms are given, the latter deriving from Koiter's work [7], and various examples are cited that provide for kinematic and isotropic hardening, and for a Bauschinger effect. Indeed, Fung provides an informative overview of this aspect of plasticity without becoming involved in the details of individual relations or solution methods.

It is useful here, however, to examine such details for a few situations. In so doing, we are altogether mindful of the dual needs for computational convenience and physical sensibility. Thus, for

example, we do not discuss loading functions that reflect a
Bauschinger effect; the experimentation required to infer the form
of such a function alone is prohibitive in the absence of compelling
reasons for such work. Still, the procedures outlined are easily
adapted to such representations when warranted.

Anisotropic Relations: Following Koiter's generalization [7],
we consider not one but a set of η load functions

$$f_\alpha(\sigma_{ij},\varepsilon_{ij}^{(p)},W^{(p)}) \leq 0 \qquad \alpha = 1,2,\ldots,\eta \qquad (22)$$

The condition $f = 0$ describes a segmented surface in stress space
comprising η mutually exclusive segments. These pieces of course
are contiguous and, since the stress vector corresponding to any
point in the physical domain must be single-valued, yield proceeds by
activating only one load function at any instant of time. Thus, if

$$f_\beta = 0$$

then

$$f_\alpha < 0 \qquad \alpha = 1,2,\ldots, \beta-1, \beta+1,\ldots,\eta$$

Of course, different f_α may be operative at different points of the
physical domain at a given time, or at the same point but at different
times. If for convenience we let

$$f_\alpha(\sigma_{ij},\varepsilon_{ij}^{(p)},W^{(p)}) = \phi_\alpha(\sigma_{ij})-\psi_\alpha(\varepsilon_{ij}^{(p)},W^{(p)}) \qquad (23)$$

it is a direct calculation to arrive at the flow rule

$$\delta\varepsilon_{ij}^{(p)} = \sum_{\alpha=1}^{\eta} \frac{\phi_\alpha(\partial\phi_\alpha/\partial\sigma_{ij})(\partial\phi_\alpha/\partial\sigma_{k\ell})}{2\mu_\alpha^{(p)}\ (\partial\phi_\alpha/\partial\sigma_{mn})\sigma_{mn}}\ \delta\sigma_{k\ell} \tag{24}$$

as an analogue to (11). In (24) $\mu_\alpha^{(p)}$ is a set of plastic moduli analogous to $\mu_{eq}^{(p)}$ in (11). Note that since flow is proceeding on only the βth surface, the summation in (24) is not a collection of processes that occur simultaneously, but of those only *one* of which may occur at any time and position in the physical domain; (24), however, represents the totality of processes that may go forward at all times and points.

The simplification (23) is non-essential and an analogue to (8b) may be written. It would involve replacement of f by f_α and a summation, as in (24), and need not be written here explicitly. To either such a result or (24) may be added the necessarily linear relations between the elastic strain increments and the stress increments; the result assumes the form (15) which is necessary for the remaining development of the theory. It should be noted that the elastic relations may be fully anisotropic with no conceptual difficulty. There is certainly the potential tedium of inverting such equations to arrive at the analogue to (16), but, to the extent that we envisage extensive use of the computer in solving problems, no practical impediment to an anisotropic theory of elasto-plastic flow is apparent.

Isotropy and Incompressibility: Up to this point, we have allowed the function $\phi(\sigma_{ij})$ (or the functions $\phi_\alpha(\sigma_{ij})$ to be arbitrary. Experientially, however, it is clear that specific functions are very attractive from the standpoints of computational convenience and physical sensibility. In particular, we may now require flow to be both isotropic and incompressible by restricting ϕ to depend only on the stress deviator

$$S_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\,\delta_{ij}$$

and, more particularly, its invariants

$$J_2 = S_{ij}S_{ji}/2$$

$$J_3 = S_{ij}S_{jk}S_{ki}/3$$

so that, for example

$$\phi = \phi(J_2, J_3) \tag{25}$$

This form is appropriate to the observations made by Osgood [8] and discussed by Drucker [9] wherein

$$\phi = \{J_2^3 - (9/4)J_3^2\}^{1/6} \tag{25a}$$

is suggested as an equivalent stress. The resulting equations pertinent to (25) were given in some detail in [6] and need not be repeated here. It is a straightforward matter to use the specific form (25a) in these formulae, althouth we have not pursued such an effort.

Alternatively, one may revert to the simpler and more popular form

associated with the name of von Mises. In this case

$$\phi = \sqrt{(2J_2/3)} \tag{26}$$

and the equivalent stress τ_{eq} is indistinguishable from the octahedral

shear stress τ_o. Using an octahedral plastic strain increment in

place of the equivalent quantity defined in (9), we find that if

$$\delta\tau_o/\delta\epsilon_o^{(p)} = 2\mu_o^{(p)}$$

then

$$\mu_o^{(p)} \equiv 3\mu_{eq}^{(p)}$$

It is then easy to show [5] that the flow rule is

$$2\mu\delta\epsilon_{ij} = \delta\sigma_{ij} - [\nu/(1+\nu)]\delta\sigma_{kk}\delta_{ij} + (\mu/\mu_o^{(p)})S_{ij}S_{k\ell}\delta\sigma_{k\ell}/3\tau_o^2$$

and its inverse is

$$\delta\sigma_{ij}/2\mu = \delta\epsilon_{ij} + [\nu/(1-2\nu)]\delta\epsilon_{kk}\delta_{ij} - S_{ij}S_{k\ell}\delta\epsilon_{k\ell}/[3\tau_o^2(1+\mu_o^{(p)}/\mu)]$$

The extended form of Navier's equations is, in the absence of body forces,

$$[1/(1-2\nu)]\delta u_{j,ji} + \delta u_{i,jj} - 2\{S_{ij}S_{k\ell}\delta u_{k,\ell}/[3\tau_o^2(1+\mu_o^{(p)}/\mu)]\}_{,j} = 0 \tag{27}$$

and it is again useful to note the quasilinear form of this differnetial

equation: The dependent variable δu_i appears linearly in (27), while

the nonlinearities of the problem are confined to the coefficients of

the last term. The operational result of this important feature is

discussed below as well as in separate publications dealing with special aspects [5,10]. Finally, we note that, as $\mu_o^{(p)}/\mu \to 0$ (perfect plasticity), the flow rate in the form shown loses meaning; however, both the inverse and the differential equation (27) go smoothly to the limit and exist and are tractable.

FINITE DEFORMATIONS

Up to this point we have developed the theory of elasto-plastic flow wholly within the context of infinitesimal strains, that is, for ε_{ij} sufficiently small for (17) to apply. As Rice [11] and Hill [12] have noted, however, there may be instances in which the magnitude of the deformations and the implied assumption $\sigma_{ij}/\mu_{eq}^{(p)} << 1$ are such that this framework is inappropriate. For these and other reasons, the theory was extended to the case of finite deformations. The development is presented in considerable detail elsewhere [13], and we give below just its main points.*

The total deformation is described in a fixed reference frame by the mapping

$$x^i = x^i(X^J, t) \quad , \quad |x^i,_J| \neq 0 \tag{28}$$

In (28), x^i are spatial coordinates of material particles comprising domain B at time $t \to t_o$ which were located in B_o at t_o. We differentiate (28) with respect to time to obtain a velocity field $v^i = v^i(x^j, t)$ and compute the velocity gradient

$$v_{i;j} = d_{ij} + \omega_{ij}$$

---

*For this purpose, we use in this section general tensor notation. covariant (contravariant) character is denoted by subscript (super-script) indices; a comma (semicolon) indicates partial (covariant) differentiation; repeated indices in subscript-superscript pairs implies summation over the range 1,2,3; x^i and X^I are coordinates in a single, fixed, orthogonal curvilinear system y^i; g_{ij} is the metric tensor of y^i; and δ^i_j is the Kronecker delta.

where d_{ij} is the (symmetric) deformation rate tensor and ω_{ij} is the (skew-symmetric) spin tensor. These tensors are of particular significance in that they both provide a full description of motion, and are linearly dependent upon v^i.

We next invoke the assumptions of the existence of a load function and Drucker's hypothesis. In his original work [13], Osias used a loading function of the form of (25) — in the context of (3) — but it seems clear that far more general forms are altogether acceptable. The statement of Drucker's hypothesis is written as*

$$\hat{\sigma}^{ij} d_{ij}^{(p)} \geq 0 \qquad (29)$$

where $\hat{\sigma}^{ij}$ is the Jaumann stress rate given by

$$\hat{\sigma}^{ij} = \dot{\sigma}^{ij} + \sigma^i_{k}\omega^{kj} - \sigma_k^{j}\omega^{ik}$$

$$\dot{\sigma}^{ij} = \partial\sigma^{ij}/\partial t + (\partial\sigma^{ij}/\partial x^k)v^k$$

and $d_{ij}^{(p)}$ is the plastic component of the deformation rate. It may be shown that, independent of any rotation, (29) requires that work done by the external agency in producing $d_{ij}^{(p)}$ is nonnegative, consistent with the sense of the original hypothesis. The Jaumann rate is used in (29) to provide an objective measure of the change in stress viewed from a frame rotating with the material, as follows from the work of Prager [14].

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*Note that (e), (p), and eq are not indices.

With these bases, Osias derives a flow rule of the form (15);
the details, however, are such that explicit notation is in order.
The flow rule is

$$2\mu d_{ij} = B_{ijk\ell} \hat{\sigma}^{k\ell} \tag{30}$$

$$B_{ijk\ell} = (g_{ik}g_{j\ell} + g_{i\ell}g_{jk})/2 - [\nu/(1+\nu)]g_{ij}g_{k\ell}$$

$$+ \gamma(\partial\phi/\partial\sigma^{ij})(\partial\phi/\partial\sigma^{k\ell})$$

$$\gamma = (\mu/\mu_{eq}^{(p)})\{\phi/[(\partial\phi/\partial\sigma^{ij})\sigma^{ij}]\}$$

and the inverse is

$$\hat{\sigma}^{ij}/2\mu = D^{ijk\ell}d_{k\ell} \tag{31}$$

$$D^{ijk\ell} = (g^{ik}g^{j\ell} + g^{i\ell}g^{jk})/2 + [\nu/(1-2\nu)]g^{ij}g^{k\ell}$$

$$- \frac{(\partial\phi/\partial\sigma^{ab})(\partial\phi/\partial\sigma^{cd})g^{ai}g^{bj}g^{ik}g^{d\ell}}{1/\gamma + (\partial\phi/\partial\sigma^{mn})(\partial\phi/\partial\sigma^{rs})g^{mr}g^{ns}}$$

and g^{ij} is the associated metric. Note that the elastic deformation
rates and the Jaumann stress rate are presumed to be related in precisely
the same fashion as are infinitesimal strains and stresses — or their
increments — in classical isotropic elasticity. To that extent,
(30) and (31) are tailored to materials for which a shear modulus and
and a Poisson's ratio may be defined in the elastic range. It should

also be noted that (30) and (31) do not involve strains per se; such quantities are computed *a posteriori* in analysis.

In requiring equilibrium, Osias adopts the rate or increment viewpoint which has proven so useful. He observes that the total force acting on a body must vanish, i.e.,

$$\int_S t^i dS = 0$$

where t^i are surface tractions, and requires that the time rate of change of this integral also vanishes. Hence

$$\int_S (\dot{t}^i + \sigma^{ij} v^k_{;k} n_j) dS = 0 \tag{32}$$

so that the traction rate is

$$\dot{t}^i = (\dot{\sigma}^{ij} - \sigma^{ki} v^j_{;k}) n_j \tag{33}$$

and the equations of equilibrium are

$$\dot{\sigma}^{ij}_{;j} - \sigma^{ki}_{;j} v^j_{;k} = 0 \tag{34}$$

with no body forces acting, of course. In the interior of B, (34) apply. On the boundary S, however, there is some choice. One may specify velocities in the usual way; one may specify traction rates (33) when, for example, a pressurization is known; or one may specify a force in terms of an integral as on the left of (32) when such a

quantity is known over some portion of the boundary S.

Combining (34) and (31), along with the definition of deformation rates in terms of velocity gradients gives the final equations for the latter as

$$[D^{ijk\ell} v_{j;\ell}]_{;i} + g^{i\ell} \sigma^{kj} v_{i;j\ell}$$

$$+(1/2)[\delta_n^i(\delta_m^\ell g^{jk} - \delta_m^j g^{\ell k}) - \delta_m^k \delta_n^j g^{i\ell} - \delta_n^k \delta_m^\ell g^{ij}][\sigma^{mn} v_{j;\ell}]_{;i} = 0 \qquad (35)$$

Inspection of (31) and (35) shows the latter to be quasilinear in the velocity v_i so that, aside from increased complexity, (35) is operationally of the same form as (20) or (27).

As a result we have a form of Navier's equations further extended to the case of elasto-plastic flow of bodies permitting finite deformations, together with boundary conditions appropriate to their integration. The various comments in preceding sections of this report concerning the utility of this sort of formulation pertain equally here. While we have not examined such matters as integral theorems, we have used a planar counterpart of (35) to solve problems, and these results are described fully elsewhere [13].

REDUCTIONS TO 2-SPACE

While derivation of the theory is easily prosecuted using indicial notation, it is useful to reduce the final relations to a form akin to that employed when actually solving problems. Accordingly, we devote this section to the governing equations for a series of situations in 2-space, that is, in more familiar extended representations. We perform this reduction only for Mises or J_2 theory where ϕ is given explicitly by (26). Forms appropriate to (25) are given in [6].

Torsion: There are two cases of torsion, one for prismatic bars or bars of constant cross-sectional shape, and the other for axisymmetric rods whose shape is otherwise variable. The first may be described in rectangular cartesian coordinates (x,y,z), and the dependent variables are the corresponding displacement increments $(\delta u, \delta v, \delta w)$. It may be shown that, if the z-axis lies along the center of rotation,

$$\delta u = -yz\delta\vartheta$$

$$\delta v = xz\delta\vartheta$$

$$\delta w = \delta w(x,y)$$

where $\delta\vartheta$ is an increment in the twist per unit bar length. It follows that the stresses are given by

$$\sigma_x = \sigma_y = \sigma_z = \sigma_{xy} = 0$$

$$\sigma_{xz} = \tau_x(x,y)$$

$$\sigma_{yz} = \tau_y(x,y)$$

and that equilibrium requires

$$\partial \delta \tau_x / \partial x + \partial \delta \tau_y / \partial y = 0$$

We define

$$\beta = \mu / \mu_o^{(p)}$$

and observe that

$$J_2 = \tau_x^2 + \tau_y^2$$

to write the flow rule

$$\mu \delta w,_x = y \mu \delta \vartheta + (1 + \beta \tau_x^2 / J_2) \delta \tau_x + (\beta \tau_x \tau_y / J_2) \sigma \tau_y$$

$$\mu \delta w,_y = -x \mu \delta \vartheta + (\beta \tau_y \tau_x / J_2) \delta \tau_x + (1 + \beta \tau_y^2 / J_2) \tau_y$$

and its inverse

$$[(1+\beta)/\mu] \delta \tau_x = [(1+\beta \tau_y^2 / J_2)(\delta w,_x - y \delta \vartheta) - (\beta \tau_x \tau_y / J_2)(\delta w,_y + x \delta \vartheta)]$$

$$\tag{36}$$

$$[(1+\beta)/\mu] \delta \tau_y = [-(\beta \tau_y \tau_x / J_2)(\delta w,_x - y \delta \vartheta) + (1 + \beta \tau_x^2 / J_2)(\delta w,_y + x \delta \vartheta)]$$

If we apply equilibrium to the inverse of the flow rule, we obtain

$$(1+\beta \tau_y^2 / J_2) \delta w,_{xx} - 2(\beta \tau_x \tau_y / J_2) \delta w,_{xy} + (1 + \beta \tau_x^2 / J_2) \delta w,_{yy}$$

$$\tag{37}$$

$$+ \alpha_1 \delta w,_x + \alpha_2 \delta w,_y = (y \alpha_1 - x \alpha_2) \delta \vartheta$$

where

$$\alpha_1 = (1+\beta)\{ [(1+\beta \tau_y^2 / J_2)/(1+\beta)],_x - [(\beta \tau_x \tau_y / J_2)/(1+\beta)],_y \}$$

$$\alpha_2 = (1+\beta)\{-[(\beta \tau_y \tau_x / J_2)/(1+\beta)],_x + [(1+\beta \tau_x^2 / J_2)/(1+\beta)],_y \}$$

Boundary conditions for (37) are

$$\delta\tau_x \cos\psi + \delta\tau_y \sin\psi = 0$$

where ψ is the angle between the unit outward normal and the x-axis,

and the stress increments relate to δw and its derivatives through

(36). Note that, where no flow occurs, $\beta = 0$ and (37) reduces to the

familiar

$$\nabla^2 \delta w = 0$$

The alternative formulation derives from elimination of δw from

the flow rule, having introduced an extended Prandtl stress function as

$$\tau_x = \partial\zeta/\partial y \quad , \quad \tau_y = -\partial\zeta/\partial x$$

with $\zeta = \zeta(x,y)$. The differential equation is

$$(1+\beta\zeta_{,x}^2/J_2)\delta\zeta_{,xx} + 2(\beta\zeta_{,x}\zeta_{,y}/J_2)\delta\zeta_{,xy} + (1+\beta\zeta_{,y}^2/J_2)\delta\zeta_{,yy}$$

$$+ \alpha_3\delta\zeta_{,x} + \alpha_4\delta\zeta_{,y} = -2\mu\delta\vartheta \tag{38}$$

where

$$J_2 = |\nabla\zeta|^2 = \zeta_{,x}^2 + \zeta_{,y}^2$$

$$\alpha_3 = (1+\beta\zeta_{,x}^2/J_2)_{,x} + (\beta\zeta_{,x}\zeta_{,y}/J_2)_{,y}$$

$$\alpha_4 = (\beta\zeta_{,y}\zeta_{,x}/J_2)_{,x} + (1+\beta\zeta_{,y}^2/J_2)_{,y}$$

and $\delta\zeta = 0$ on the boundary of the domain so long as it singly connected.

This last restriction renders (38) less useful than (37), for multiply

connected domains are excluded from study. Still, the quasilinear form

of (38) is evident, as is true for (37), and solution procedures may be developed to exploit this feature.

Torsion of axisymmetric bodies is described in a similar fashion. In a system of cylindrical coordinates, the only active displacement is in the circumferential direction and we denote it by $v(r,z)$. The only active stresses are

$$\sigma_{r\theta} = \tau_r \quad , \quad \sigma_{\theta z} = \tau_z$$

and

$$J_2 = \tau_r^2 + \tau_z^2$$

Using the same definition of β, the flow rule becomes

$$\mu\delta,_z = (1+\beta\tau_z^2/J_2)\delta\tau_z + (\beta\tau_z\tau_r/J_2)\delta\tau_r$$
$$\mu(\delta v,_r - \delta v/r) = (\beta\tau_r\tau_z/J_2)\delta\tau_z + (1+\beta\tau_r^2/J_2)\delta\tau_r$$

and its inverse is

$$[(1+\beta)/\mu]\delta\tau_z = (1+\beta\tau_r^2/J_2)\delta v,_z - (\beta\tau_z\tau_r/J_2)(\delta v,_r - \delta v/r)$$

$$[(1+\beta)/\mu]\delta\tau_r \quad -(\beta\tau_r\tau_z/J_2)\delta v,_z - (1+\beta\tau_z^2/J_2)(\delta v,_r - \delta v/r)$$

(39)

If we apply equilibrium

$$\partial\delta\tau_r/\partial r + 2\delta\tau_r/r + \partial\delta\tau_z/\partial z = 0$$

we have

$$(1+\beta\tau_z^2/J_2)\delta v,_{rr} - 2(\beta\tau_r\tau_z/J_2)\delta v,_{rz} + (1+\beta\tau_r^2/J_2)\delta v,_{zz}$$

(40)

$$+ \, \alpha_1'(\delta v,_r - \delta v/r) + \alpha_2'\delta v,_z = 0$$

where

$$\alpha_1' = (1+\beta)\{ [(1+\beta\tau_z^2/J_2)/(1+\beta)]_{,r} - [(\beta\tau_z\tau_r/J_2)/(1+\beta)]_{,z}\}+(1+\beta\tau_z^2/J_2)/r$$

$$\alpha_2' = (1+\beta)\{-[(\beta\tau_r\tau_z/J_2)/(1+\beta)]_{,r} + [(1+\beta\tau_r^2/J_2)/(1+\beta)]_{,z}\}-(\beta\tau_r\tau_z/J_2)/r$$

Boundary conditions for (40) reduce to

$$\delta\tau_r\cos\psi + \delta\tau_z\sin\psi = 0$$

where ψ is the angle between the unit outward normal and the r-axis

(i.e., the plane z=0), and the stress increments relate to δv and its

derivatives through (39). Note that, for purely elastic behavior, (40)

reduces to

$$\delta v_{,rr}+(1/r)\delta v_{,r} - (1/r^2)\delta v + \delta v_{,zz} = 0$$

which of course is the standard elastic result.*

Alternatively we may introduce the stress function $\zeta(r,z)$ through

$$r^2\tau_r = -\zeta_{,z} \qquad , \qquad r^2\tau_z = \zeta_{,r}$$

and eliminate reference to δv in the flow rule. The result is easily

written terms of the ratios

$$\zeta_r = \zeta_{,r}/|\nabla\zeta| \qquad , \qquad \zeta_z = \zeta_{,z}/|\nabla\zeta|$$

with $\qquad |\nabla\zeta| = \sqrt{(\zeta_{,r}^2 + \zeta_{,z}^2)}$

---

*A slightly altered form of both this relation and (40) is obtained if
the dependent variable δv is replaced by an angle of twist $\delta\vartheta(r,z) = \delta v(r,z)/r$.

as

$$(1+\beta\zeta_r^2)[\delta\zeta,_{rr}-(3/r)\delta\zeta,_r] + 2\zeta_r\zeta_z\delta\zeta,_{rz}+(1+\beta\zeta_z^2)\delta\zeta,_{zz}$$

$$+ \alpha_3'\delta\zeta,_r + \alpha_4'\delta\zeta,_z = 0 \tag{41}$$

with

$$\alpha_3' = (1+\beta\zeta_r^2),_r + (\beta\zeta_r\zeta_z),_z$$

$$\alpha_4' = (\beta\zeta_z\zeta_r),_r + (1+\beta\zeta_z^2),_z - (3/r)\beta\zeta_z\zeta_r$$

The boundary condition on (41) is of course constancy of $\delta\zeta$, as in elasticity. In both (40) and (41) the quasilinear nature of the differential equation is evident as noted for other cases in this section.

Axisymmetric Extension: Still operating in cylindrical coordinates (r,θ,z), the non-zero displacements are observed to be

$$u = u(r,z)$$

$$w = w(r,z)$$

and the strains are

$$\varepsilon_r = \partial u/\partial r$$

$$\varepsilon_\theta = u/r$$

$$\varepsilon_z = \partial w/\partial z$$

$$\gamma_{rz} = \partial u/\partial z + \partial w/\partial r$$

to which there are corresponding stresses σ_r, σ_θ, σ_z, and τ_{rz}. The flow rule is conveniently written in terms of normalized deviators

$$s_r = (2\sigma_r - \sigma_\theta - \sigma_z)/(3\sqrt{3}\tau_o)$$

$$s_\theta = (2\sigma_\theta - \sigma_z - \sigma_r)/(3\sqrt{3}\tau_o)$$

$$s_z = (2\sigma_z - \sigma_r - \sigma_\theta)/(3\sqrt{3}\tau_o)$$

$$s_{rz} = \tau_{rz}/(\sqrt{3}\tau_o)$$

where

$$\tau_o^2 = (2/9)(\sigma_r^2 + \sigma_\theta^2 + \sigma_z^2 - \sigma_\theta\sigma_z - \sigma_z\sigma_r - \sigma_r\sigma_\theta + 3\tau_{rz}^2)$$

We then have

$$2\mu\delta\epsilon_r = [1/(1+\nu) + \beta s_r^2]\delta\sigma_r + [-\nu/(1+\nu) + \beta s_r s_\theta]\delta\sigma_\theta + [-\nu/(1+\nu) + \beta s_r s_z]\delta\sigma_z$$
$$+ 2s_r s_{rz}\delta\tau_{rz}$$

$$2\mu\delta\epsilon_\theta = [-\nu/(1+\nu) + \beta s_\theta s_r]\delta\sigma_r + [1/(1+\nu) + \beta s_\theta^2]\delta\sigma_\theta + [-\nu/(1+\nu) + \beta s_\theta s_z]\delta\sigma_z$$
$$+ 2s_\theta s_{rz}\delta\tau_{rz}$$

$$2\mu\delta\epsilon_z = [-\nu/(1+\nu) + \beta s_z s_r]\delta\sigma_r + [-\nu/(1+\nu) + \beta s_z s_\theta]\delta\sigma_\theta + [1/(1+\nu) + \beta s_z^2]\delta\sigma_z$$
$$+ 2s_z s_{rz}\delta\tau_{rz}$$

$$\mu\delta\gamma_{rz} = \beta s_{rz} s_r \delta\sigma_r + \beta s_{rz} s_\theta \delta\sigma_\theta + \beta s_{rz} s_z \delta\sigma_z + (1 + \beta s_{rz}^2)\delta\tau_{rz}$$

as the flow rule. The inverse is written using Young's modulus E
and the Lamé constant λ, where

$$\lambda = 2\nu\mu/(1-2\nu) \qquad , \qquad E = 2(1+\nu)\mu$$

and we have

$$\delta\sigma_r/E = [(\lambda+2\mu)/E - s_r^2/(1+1/\beta)]\delta\epsilon_r + [\lambda/E - s_r s_\theta/(1+1/\beta)]\delta\epsilon_\theta$$

$$+ [\lambda/E - s_r s_z/(1+1/\beta)]\delta\epsilon_z - [s_r s_{rz}/(1+1/\beta)]\delta\gamma_{rz}$$

$$\delta\sigma_\theta/E = [\lambda/E - s_\theta s_r/(1+1/\beta)]\delta\epsilon_r + [(\lambda+2\mu)/E - s_\theta^2/(1+1/\beta)]\delta\epsilon_\theta$$

$$+ [\lambda/E - s_\theta s_z/(1+1/\beta)]\delta\epsilon_z - [s_\theta s_{rz}/(1+1/\beta)]\delta\gamma_{rz} \qquad (42)$$

$$\delta\sigma_z/E = [\lambda/E - s_z s_r/(1+1/\beta)]\delta\epsilon_r + [\lambda/E - s_z s_\theta/(1+1/\beta)]\delta\epsilon_\theta$$

$$+ [(\lambda+2\mu)/E - s_z^2/(1+1/\beta)]\delta\epsilon_z - [s_z s_{rz}/(1+1/\beta)]\delta\gamma_{rz}$$

$$\delta\tau_{rz}/E = -[s_{rz} s_r/(1+1/\beta)]\delta\epsilon_r - [s_{rz} s_\theta/(1+1/\beta)]\delta\epsilon_\theta$$

$$- [s_{rz} s_z/(1+1/\beta)]\delta\epsilon_z + [\mu/E - s_{rz}^2/(1+1/\beta)]\delta\gamma_{r\theta} .$$

Navier's equations are obtained by combining (42) with the equations of equilibrium, viz.

$$\partial\delta\sigma_r/\partial r + (\delta\sigma_r - \delta\sigma_\theta)/r + \partial\delta\tau_{rz}/\partial z = 0$$

$$\partial\delta\tau_{rz}/\partial r + \delta\tau_{rz}/r + \partial\delta\sigma_z/\partial z = 0$$

having expressed the strain increments in terms of displacement increments in the usual manner. Boundary conditions are given in terms of δu, δw, or tractions which are linear combinations of (42). The result is more involved than (40) or (41), say, but the operational nature is very much the same.

Planar Behavior: It is useful to consider two types of problem under this heading, plane strain, and combined plane stress and (Kirchhoff) bending. The first may be established by formally requiring

$$\varepsilon_z = \partial w/\partial z = 0$$

$$\gamma_{yz} = \partial v/\partial z + \partial w/\partial y = 0$$

$$\gamma_{xz} = \partial u/\partial z + \partial w/\partial x = 0$$

where the domain of interest lies in the x-y plane. We then operate on (27) and its associated relations to develop the requisite equations. From (17) we have

$$\delta\varepsilon_x = \partial\delta u/\partial x$$

$$\delta\varepsilon_y = \partial\delta v/\partial y$$

$$\delta\gamma_{xy} = \partial\delta u/\partial y + \partial\delta v/\partial x$$

and the flow rule is again written in terms of normalized deviators:

$$s_x = (2\sigma_x - \sigma_y - \sigma_z)/(3\sqrt{3}\tau_o)$$

$$s_y = (2\sigma_y - \sigma_z - \sigma_x)/(3\sqrt{3}\tau_o)$$

$$s_z = (2\sigma_z - \sigma_x - \sigma_y)/(3\sqrt{3}\tau_o)$$

$$s_{xy} = \tau_{xy}/(\sqrt{3}\tau_o)$$

with

$$\tau_o^2 = (2/9)(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_y\sigma_z - \sigma_z\sigma_x - \sigma_x\sigma_y + 3\tau_{xy}^2)$$

We note that the transverse stress σ_z accumulates from observing the constraint

$$0 = (1+\beta s_z^2)\delta\sigma_z + (-\nu + \beta s_z s_x)\delta\sigma_x$$

$$+ (-\nu + \beta s_z s_y)\delta\sigma_y + (2\beta s_z s_{xy})\delta\tau_{xy}$$

which, together with $\delta\tau_{xz} = \delta\tau_{yz} = 0$, insures the plane strain condition.
The flow rule then becomes

$$2\mu(1+\beta s_z^2)\delta\epsilon_x = [1-\nu+\beta(s_x^2+2\nu s_x s_z+s_z^2)]\delta\sigma_x$$

$$+[-\nu+\beta(s_x s_y-2\nu s_z^2)]\delta\sigma_y$$

$$+2[\beta(s_x+\nu s_z)s_{xy}]\delta\tau_{xy}$$

$$2\mu(1+\beta s_z^2)\delta\epsilon_y = [-\nu+\beta(s_y s_x-2\nu s_z^2)]\delta\sigma_x$$

$$+[1-\nu+\beta(s_y^2+2\nu s_y s_z+s_z^2)]\delta\sigma_y$$

$$+2[\beta(s_y+\nu s_z)s_{xy}]\delta\tau_{xy}$$

$$\mu(1+\beta s_z^2)\delta\gamma_{xy} = [\beta s_{xy}(s_x+\nu s_z)]\delta\sigma_x$$

$$+[\beta s_{xy}(s_y+\nu s_z)]\delta\sigma_y$$

$$+\{1+2\beta[s_{xy}^2+(1+\nu)s_z^2]\}\delta\tau_{xy}$$

with the inverse

$$\delta\sigma_x/E = [(\lambda+2\mu)/E-s_x^2/(1+1/\beta)]\delta\epsilon_x+[\lambda/E-s_x s_y/(1+1/\beta)]\delta\epsilon_y$$

$$-[s_x s_{xy}/(1+1/\beta)]\delta\gamma_{xy}$$

$$\delta\sigma_y/E = [\lambda/E-s_y s_x/(1+1/\beta)]\delta\epsilon_x+[(\lambda+2\mu)/E-s_y^2/(1+1/\beta)]\delta\epsilon_y$$

$$-[s_y s_{xy}/(1+\lambda/\beta)]\delta\gamma_{xy}$$

$$\delta\tau_{xy}/E =-[s_{xy}s_x/(1+1/\beta)]\delta\epsilon_x-[s_{xy}s_y/(1+1/\beta)]\delta\epsilon_y$$

$$+[\mu/E-s_{xy}^2/(1+1/\beta)]\delta\tau_{xy}$$

which are then combined with the equilibrium equations

$$\partial\delta\sigma_x/\partial x + \partial\delta\tau_{xy}/\partial y = 0$$

$$\partial\delta\tau_{xy}/\partial x + \partial\delta\sigma_y/\partial y = 0$$

to produce the final result.

The combined bending-stretching equations are derived by using the analogue to to the theorem of minimum potential energy, outlined above. We consider the displacements and their increments to be functions of the coordinates (x,y) only, but write the strain increments as

$$\delta\varepsilon_x = \partial\delta u/\partial x - z\partial^2\delta w/\partial x^2$$

$$\delta\varepsilon_y = \partial\delta v/\partial y - z\partial^2\delta w/\partial y^2 \qquad (43)$$

$$\delta\gamma_{xy} = \partial\delta u/\partial y + \partial\delta v/\partial x - 2z\partial^2 w/\partial x\partial y$$

and insert these into the functional Π under the condition that $\tau_{xz} = \tau_{yz} = \sigma_z = 0$. Jones [4] has worked this out in full detail, so we shall not repeat his results here. It may be noted that he recovers a coupled theory, and this coupling occurs both in the field equations and in the boundary conditions. The former are in terms of the three displacement increments ($\delta u, \delta v, \delta w$) and are of course quasilinear. The latter are in terms either of these three quantitites or resultant increments of force (for in-plane loading) and of moment and equivalent shear (for out-of-plane loading).

He also demonstrates appropriate 'corner conditions', in terms of twisting moment and corner angle. Through solutions to problems, he shows that, although the strains remain linearly distributed through the plate's thickness per (43), the stresses become nonlinear as yield progresses. He also makes clear the manner in which bending and stretching couple in the presence of plastic flow, although the two events are distinct for purely elastic behavior. We do not reproduce his findings here both because they are accessible elsewhere and because their form is evident from the foregoing developments.

- 43 -

SOME ELEMENTARY SOLUTIONS

It may be useful, in order to see more clearly the workings of the theory, to consider some elementary examples. We therefore look in this section at torsion of an axisymmetric rod of constant radius R, and at bending and extension of a rectangular bar. For the first case, the circumferential displacement increment is simply

$$\delta v = rz\delta\vartheta$$

where, as in the last section, ϑ is the twist per unit rod length. The flow rule is simply

$$\mu r\delta\vartheta = (1+\beta)\delta\tau_z$$

having used the notation of the last section. Since the only active stress is τ_z and

$$\beta = \mu/\mu_o^{(p)}$$

where $\mu_o = \mu_o^{(p)}(\tau_o)$ and $\tau_o = \sqrt{(2/3)}\tau_z$, the flow rate may be easily integrated. The result is

$$\mu r\vartheta = \tau_z + 2\mu\varepsilon_o^{(p)}[\sqrt{(2/3)}\tau_z] \qquad (44)$$

and $\varepsilon_o^{(p)}$ is a given function of τ_o or, as implied, $\sqrt{(2/3)}\tau_z$. Next, we observe that the twisting moment is simply

$$M_t = 2\pi \int_o^R r^2\tau_z dr$$

so that moment and twist may be related. Let us suppose, for example, that

$$\varepsilon_o^{(p)} = \begin{cases} 0 & \tau_o \leq \tau_\ell \\ \varepsilon^*(\tau_o/\tau_\ell - 1)^{1/n} & \tau_o > \tau_\ell \end{cases}$$

where ε^*, τ_ℓ, and $n \leq 1$ are constants. Then if

$$\tau_p = (2/3)\tau_\ell$$

we have

$$\mu\vartheta r = \begin{cases} \tau_z & 0 \leq r \leq \tau_p/\mu\vartheta \\ \tau_z + 2\mu\varepsilon^*(\tau_z/\tau_p - 1)^{1/n} & \tau_p/\mu \leq r \leq R \end{cases} \tag{45}$$

Thus the elastic relation is recovered for $R\vartheta \leq \tau_p/\mu$ and

$$M_t = \mu J \vartheta , \qquad J = \pi R^4/2$$

but as yield progresses, (45) obtains. In that event (45) must be evaluated for τ_z as a function of r to find M_t. As an illustration, let $n = 1$; we have then

$$M_t = \mu J \frac{\vartheta + 8\varepsilon^*/3R}{1 + 2\mu\varepsilon^*/\tau_p}$$

which gives a moment-twist relationship that is asymptotic to the implied bilinear 'curve'. The case of perfect plasticity may be represented by $\varepsilon^* \to \infty$, in which case

$$M_{t-lim} = (2\pi/3)\tau_p R^3$$

is a limiting moment that may be applied to the rod.

For rectangular beams, Jones [4] derives the governing equations with the aid of the analogue to the theorem of minimum potential energy. The beam occupies the region $0 < x < L$, $-y_o/2 < y < y_o/2$, $-z_o/2 < z < z_o/2$; the end at $x = 0$ is cantilevered; the end at $x = L$ is subject to increments in an axial extension force δF, a shear force δQ, and a bending moment δM; the length of the beam is loaded transversely by δq. Since the flow rule is

$$\delta\sigma = (1/E + 1/3\mu_o^{(p)})\delta\sigma$$

where subscripts on $\delta\varepsilon$ and $\delta\sigma$ are suppressed, and the inverse (including transverse straining) is

$$\delta\sigma = E[1 - 1/(1 + 3\mu_o^{(p)}/E)]\delta\varepsilon$$

he takes the strain-displacement equations in the form

$$\delta\varepsilon = d\delta u/dx - y d^2\delta v/dx^2$$

and computes the functional Π. Operating to find a minimum value, he obtains the equations

$$d(C_1 d\delta u/dx)/dx - d(C_2 d^2\delta v/dx^2)/dx = 0$$

$$d^2(C_3 d^2\delta v/dx^2)/dx^2 - d^2(C_2 d\delta u/dx)dx^2 = \delta q$$

(46)

where

$$C_1 = z_o \int_{-y_o/2}^{y_o/2} E' \, dy$$

$$C_2 = z_o \int_{-y_o/2}^{y_o/2} y E' \, dy$$

$$C_3 = z_o \int_{-y_o/2}^{y_o/2} y^2 E' \, dy$$

and $E' = E$ where no yield occurs, otherwise

$$E' = E[1 - 1/(1 + 3\mu_o^{(p)}/E)]$$

The boundary conditions (at $x = L$) are

$$C_1 d\delta u/dx \qquad - \quad C_2 d^2 \delta v/dx^2 \quad = \quad \delta F \quad \text{or} \quad \delta u \qquad \text{specified}$$

$$C_3 d^2 \delta v/dx^2 \quad - \quad C_2 d\delta u/dx \qquad = \quad \delta M \quad \text{or} \quad d\delta v/dx \qquad \text{specified} \qquad (47)$$

$$d(C_3 d^2 \delta v/dx^2)/dx - d(C_2 d\delta u/dx) dx = -\delta Q \quad \text{or} \quad \delta v \qquad \text{specified}$$

It is seen that C_2 couples bending and stretching in both the differential equations and the boundary conditions. Furthermore, all of the coefficients (C_1, C_2, C_3) are problem dependent so that solution to a given problem may not procede in a simple manner.

If the bar is loaded in simple tension, then it is easy to show that the original stress-strain relationship is recovered, albeit in

uniaxial quantitites. On the other hand, if the bar is put into pure bending, matters are more complex in that there is an elastic core sandwiched between two elasto-plastic outer layers. The strain distribution remains linear through the beam's thickness, i.e., with respect to y, but the bending stress distribution becomes nonlinear as a reflection of the stress-strain curve.

For the particular case of pure bending, C_1 is of no interest; C_2 is null owing to the symmetry in E'; C_3 has different values in the core and outer layers, the latter depending on the shape of the curve; the increment in curvature is given by

$$\delta\kappa = d^2\delta v/dx^2$$

and the (incremental) moment-curvature relation is

$$C_3\delta\kappa = \delta M$$

which satisfies both (46) and (47). So long as σ is everywhere less than the yield stress Y, we have

$$(Ey_o^3/z_o/12)\delta\kappa = \delta M$$

and the deflection increment is

$$\delta v = 6[\delta M/(Ey_o^3 z_o)]x^2$$

which are the familiar elastic results, and they obtain for

$$M \leq M_Y = y_o^2 z_o Y/6$$

As the moment M exceeds M_Y, however, we need to account for the stress-

strain relationship. Suppose that

$$\varepsilon_o^{(p)} = \varepsilon^*(\tau_o/\tau_\ell - 1)$$

as before, except that the exponent is by implication set to unity.
Then

$$\delta\sigma = E\{1 - 1/[1 + Y/(\sqrt{2}E\varepsilon)]\}\delta\varepsilon$$

and, since $\delta\varepsilon = y\delta\kappa$, we have

$$\delta M = 2z_o \int_0^{y_p/2} Ey^2\delta\kappa\,dy + 2z_o \int_{y_p/2}^{y_o/2} E\{1 - 1/[1 + Y/(\sqrt{2}E\varepsilon^*)]\}y^2\delta\kappa\,dy$$

$$= (Ey_o^3 z_o/12)\left((y_p/y_o)^3 + \{1 - 1/[1 + Y/(\sqrt{2}E\varepsilon^*)]\}\,[1 - (y_p/y_o)^3]\right)\delta\kappa$$

where $y_p = 2Y/E\kappa$. Observing that y_p depends upon κ, we integrate this
last expression to obtain the final relationship. Setting

$$\kappa^* = Ey_o\kappa/2Y$$

$$\beta^* = 1/[1 + Y/(\sqrt{2}E\varepsilon^*)]$$

we have

$$M/M_Y = (1-\beta^*)\kappa^* - \beta^*/2\kappa^{*2} + 3\beta^*/2$$

the additive constant of integration being adjusted to connect this result with that for purely elastic behavior. At $\kappa^* = 1$, $M/M_Y = 1$; as κ^* becomes large $M/M_Y \to (1-\beta^*)\kappa^* + 3\beta^*/2$. In addition, perfect plasticity is characterized by $\beta^* \to 1$ so that we observe the limiting situation of $M/M_Y \to 3/2$. For perfect. plasticity, it is easy to show that

$$v = \frac{Yx^2}{Ey_o \sqrt{(3-2M/M_Y)}}$$

so long as $1 < M/M_Y < 3/2$.

Thus we have solution to some elementary problems. Slightly more complicated problems may of course be devised, but the algebra becomes far more difficult and it is not in our interest here to pursue the matter. The point remains, however, that a variety of approaches is available to solve problems and that, while we may develop explicit formulae for simple cases, we are always in a position to observe the process of response from purely elastic to combined elasto-plastic flow. We may also allow excitation to reverse and examine the process whereby residual stresses develop.

SOLVING MORE COMPLEX PROBLEMS

To this point in the development, it should be clear that the theory of elasto-plastic flow is fully described by a set of one or more differential equations, and boundary and initial conditions. The governing differential equations are typically* written in terms of displacement increments; they are quasilinear with variable coefficients which depend upon the current stress state and given material properties and thereby reflect the nonlinear aspect of the process. The boundary conditions are in terms of displacement increments or traction increments which depend linearly on the gradients of displacement increments. Initial conditions perforce are to be given in terms of stress and strain fields.

Problem solving will usually entail numerical procedures and use of a large, high-speed digital computer. The issue then becomes identifying an appropriate *method*. If we look at the kind of problem to be treated, as summarized above, we observe that the equations for incremental quantities are similar to those for ordinary elasticity, but with two important exceptions. The first is that the apparent material properties, as described by the coefficients, are inhomogeneous and effectively anisotropic. The second is that distinction must be

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*Alternate formulation in terms of stress function(s) is permissible but, on the whole, has not proven to be particularly useful. Other than the one example shown above, we do not pursue this form further.

made between loading and therefore continuing elasto-plastic flow, and
unloading and thus only elastic behavior.

These exceptions give no especial impediment to problem solving
when a computer is used, for these devices are easily programmed to
handle what are, after all, merely complex details. Accordingly, the
question of selecting a method appropriate to elasto-plastic flow is
no more difficult than settling on a numerical technique for ordinary
elasticity.

Accordingly, we may choose from a more or less standardized list:
finite differences, finite elements, and so on. In the specific problems
we have solved and released for publication [15-19], we have preferred
finite elements of the simplest kind, largely for reasons of convenience
and economy. These considerations have proven sensible in dealing with
more complex situations of bending [4] and finite deformations [13].
We have also formulated an approach using the boundary-integral equation
method [5], and Riccardella has implemented this technique for the
case of plane strain [20]. The overall algorithm has been described
as well, in a context that transcends specific method [21,22,10]. In
most methods of practical interest, differential equations of the
form (27), but reduced to a specific situation, are then reduced to
algebraic equations which we represent as

$$[A] \ \{\delta u\} \ = \ \{\delta t\} \tag{48}$$

with boundary conditions being specified in terms of certain elements of $\{\delta u\}$ and/or $\{\delta t\}$. In (48), $[A]$ is a matrix of coefficients that replace the differential operator in (27), $\{\delta u\}$ is a vector of displacement increments at points distributed throughout the domain of interest, and $\{\delta t\}$ is a vector of traction or force increments at these points such that, except for specified boundary locations $\{\delta t\}$ is largely null. Note that (48) obtains for both finite differences and finite elements, and that a parallel form is sufficient to represent the equations for the boundary integral technique.

Program Outline*: Typically, a computer program for elasto-plastic flow in two spatial variables comprises an executive routine followed by one that sets up data and initializes various arrays, and then by several operational routines. The initializing routine reads in geometric information, loading data, and material properties. The load data may consist of tractions and/or displacements, in either of two orthogonal directions. The material data include two elastic constants and the relation between $\varepsilon_o^{(p)}$ and τ_o. This is read in as a table of coordinates of a sequence of points along the octahedral stress - plastic octahedral strain curve; spacing between the points

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*This discussion is taken largely from [10]; we appreciate the cooperation of that publication's editor in releasing the text for presentation here .

is roughly proportional to the curve's radius of curvature. It is also useful to provide an artificial extension to the curve, and we assume indefinite continuation of the slope on logarithmic scales, as evaluated near the end of the input data. Alternatively, one could employ a simple straight line extension. Use of this technique precludes inadvertant loss of a run because the stress-strain curve was out-run; the analyst is enabled to decide where to cut off his results after they are in hand.

The first load step is necessarily elastic, and no difficulty is encountered in performing it for an arbitrary level of loading. The matrix [A] is assembled, the vector {δt} is computed together with whatever modifications in [A] are required due to imposition of boundary conditions, and the resulting equations are solved for {δu}. Deformation and stress increments are determined. Excitation may then be scaled so that the most highly stressed point in the domain just exceeds the proportional limit, e.g., $\tau_o/\tau_\ell = 1.00010$.

Before beginning the next load step, certain operations are necessary, so that the initiation of yield may be taken into account. The table of points describing the stress-strain curve is scanned, beginning with that at the proportional limit (i.e., $(0, \tau_\ell)$ in the table described above). The scan continues until a point is found whose coordinates exceed that of the point in the material; the next highest point is then selected for further computation. The slope of

the curve $(2\mu_0^{(p)})$ is then taken to be that of a straight line connecting this point on the curve and the material point.

This procedure accomplishes several things. It is clear that the value of $\mu_0^{(p)}$ thus determined is a reasonable approximation to an average over the increment of loading to come. Proximity of the material point to the point in the curve will not exert undue influence on the value of $\mu_0^{(p)}$ which, incidentally, is necessarily positive. If the material point happens to be off the curve, it will be steered back on; both overshoot and undershoot are corrected. Furthermore, extensive experience has demonstrated that this procedure is inherently self-correcting.

At the position where yield has just begun, no significant plastic strain or strain energy has occurred, and no values of these quantities are stored. The calculation proceeds with whatever further data reduction and output preparation are appropriate. In our finite element programs, for example, we find strains, stresses, and several of their invariants. As is evident from (37), for example, at least the stresses are needed to prepare for the next load increment.

The second load step follows the same pattern used for the first, with two exceptions. Assembly of [A] is modified to account for yield through introduction of suitable material parameters where yield has begun: Also, after finding {δu}, the same considerations obtain in determining the corresponding stress increments.

We have preferred to reassemble [A] fully for each load step, rather than to modify a retained and originally elastic version of this array. While there is no unanimity over this point among practitioners in this sort of analysis, it is pertinent to note certain bases of our preference. Fresh reassembly of [A] for each load step is not, in our experience, costly in time; the core storage allocated to [A] may be overwritten for temporary purposes once {δu} is determined; the more complex bookkeeping needed to modify [A] from step to step is avoided and the overall code is thereby more compact; and this choice directly admits multi-cycle loading in that different types of boundary conditions may be imposed from one step to the next.

Subsequent steps follow the pattern outlined above, and the computation nominally continues to whatever termination point has been set. Because, however, unloading either locally or globally may occur during any load step, we insert an additional procedure prior to finalization of the results for each step. Once increments in displacements, strains, and stresses are found, the solution is regarded as having only a candidate status. The program checks to determine whether the candidate solution implies an increase in τ_o at all points currently yielding. If so, the candidate solution is accepted and the program moves on to completion of the load step.

If not, then [A] is reassembled with elastic constants in those positions corresponding to material points where elastic unloading seems to be in process. The degree of yield at which unloading is

(tentatively) first detected is stored, and an iteration within the increment is begun until there is internal consistency between the presumed and actual behavior at each material point. Once such consistency is achieved, whether or not iteration is required, the increment in plastic octahedral strain is computed via

$$\delta\epsilon_o^{(p)} = 2\mu_o^{(p)} \delta\tau_o$$

and the plastic work is obtained through simple trapezoidal quadrature at those points where yield continues.

Where unloading has occurred, τ_o is traced so that when it again reaches the level where unloading began, the elasto-plastic flow rule is again used for computation of [A], and plastic strain continues to accumulate.

With this additional procedure, unloading on any scale may be incorporated into the analysis. Experience has shown that the iteration rarely requires more than one cycle for internal consistency to be achieved. We may also remark that the present computer programs exclude Bauschinger effects largely as a matter of convenience. Had we the requisite data, we should be able to program not only this effect but cyclic stress-strain behavior as well. We recognize, however, that obtaining such data is itself a difficult task.

Automatic Loading: The procedure outlined above works well with smooth stress-strain curves, i.e., those with a gradually changing slope all the way from initial yield to high strain levels. In other

situations such as that of a bilinear curve, additional steps are needed. The reason, of course, is that there is every chance for overshooting the curve at its knee when load progression is externally specified.

We thus introduce the option of what is termed automatic loading. The analyst merely prescribes unit load steps, and the actual load increment level is internally scaled, once the solution is beyond the stage of candidacy. The scheme, operative for displacement, force, or mixed loading in any order, works as follows.

Once the candidate solution, based on unit loading, is found acceptable, two potential scale factors are determined. One is found so that just one material point yields (or re-yields), and the other so that the increment in total octahedral strain nowhere exceeds a small fraction of its current value. The lesser of these two factors is selected and all incremental quantities are scaled by its value. In this manner, the stress-strain curve is tracked with no little precision. It may be noted, however, that the cost of solving a given problem is greater using automatically rather than externally determined load step size, since many increments are very small compared to what the analyst might otherwise have set.

This method is especially useful in multi-cycle excitation in which progression of one system of loads is interrupted and another takes over in such a fashion as to force local unloading and perhaps

reloading. That the procedure is admissible is a direct result of the quasilinear nature of our original formulation; the programs in hand merely exploit this feature.

An Alternate Approach: The foregoing discussion has all been in the context of converting differential equations to algebraic increment equations such as (48). While this time-honored progression is certainly viable, one might inquire whether an alternate procedure might be used whereby a functional is to be extremized. Again, the quasilinear nature of the formulation proves useful, for we may utilize such an approach in conjunction with any of the four integral theorems outlined above. Indeed, Jones [4] has shown that, with a suitable choice of minimization method, such an approach proves efficacious in solving elasto-plastic problems. Incorporation of the procedures described above, e.g., checking for local unloading, gives a method for problem solving that is especially useful in treating cases that involve plates and shells which perforce include both an elastic core and a variable position of the neutral surface. Much of the advantage that is realized follows from the fact that the results of the kth increment are an excellent estimate for the behavior in the (k+1)th increment, and convergence of the minimization algorithm is quite rapid.

Remark on Infinite Domains: From a research standpoint, one of the limitations on numerical methods for solving problems is that the analyst is constrained to treat problems in a finite domain. It is almost truistic, however, that analytical results against which he or she would compare numerical data are for infinite domains. As a result, such comparisons are not always on a equal basis and evaluation of numerical findings is sometimes impeded.

Riccardella has recently suggested a means for overcoming this limitation. Recapitulating (19)-(21) we have the (incremental) stress-strain relations and differential equations in the form

$$\delta\sigma_{ij}/2\mu = E_{ijk\ell}\delta u_{k,\ell} \tag{19}$$

$$(E_{ijk\ell}\delta u_{k,\ell})_{,j} = 0 \tag{20}$$

Subject to boundary conditions on δu_i and/or

$$\delta t_i = 2\mu E_{ijk\ell}\delta u_{k,\ell} n_j \tag{21a}$$

Let us suppose now that the problem at hand involves an infinite domain with some sort of localized feature such as a cut-out or perforation. We then decompose the original problem into two parts. The first is the infinite domain with no local feature but subject to remote loading. The solution to such a problem is normally described in a simple manner, and we denote its solution in terms of the quantities

δu_i^∞ and $\delta \sigma_{ij}^\infty$. In many cases of interest, the first problem is fully

elastic and may thereby be represented analytically.

The second problem is the residual problem for which (20) becomes

$$[E_{ijk\ell}(\delta u_{k,\ell} - \delta u_{k,\ell}^\infty)]_{,j} = 0$$

or, if

$$\delta u_i^r = \delta u_i - \delta u_i^\infty$$

then

$$(E_{ijk\ell}\delta u_{k,\ell}^r)_{,j} = 0 \qquad (49)$$

In (49), it is important to bear in mind that $E_{ijk\ell}$ depends in a

complicated and nonlinear fashion on the stress field for the

full problem; $E_{ijk\ell}$ is not therefore subject to simple decomposition.

Recognizing, however, that there is a value of $E_{ijk\ell}$ pertinent to the

first problem, we may rewrite (19) as

$$(\delta \sigma_{ij}^\infty + \delta \sigma_{ij}^r)/2\mu = (E_{ijk\ell}^\infty + E_{ijk\ell}^r)\,\delta u_{k,\ell}^\infty + E_{ijk\ell}\,\delta u_{k,\ell}^r$$

or, since we define $E_{ijk\ell}^\infty$ by

$$\delta \sigma_{ij}^\infty/2\mu = E_{ijk\ell}^\infty \delta u_{k,\ell}^\infty$$

we have

$$\delta \sigma_{ij}^r/2\mu = E_{ijk\ell}^r \delta u_{k,\ell}^\infty + E_{ijk\ell}\delta u_{k,\ell}^r$$

It follows that (21) leads to

$$\delta t_i^r = 2\mu[E_{ijk\ell}^r \delta u_{k,\ell}^\infty + E_{ijk\ell}\delta u_{k,\ell}^r]n_j \qquad (50)$$

The second problem is thus described by the differential equation (49) subject to local boundary conditions such that δu_i^r and/or δt_i^r annihilate the residuals of δu_i^∞ and/or δt_i^∞. The second problem nominally involves an infinite domain; inasmuch, however, as the residual stresses and displacements die off quickly with distance from the local feature, a large but finite domain may be adequate for analysis.

We have not in fact used this procedure to date. It may be observed, however, that the formulation is altogether straightforward and follows that often used in elasticity for problems of this type, i.e., where there is a local disturbance in an otherwise uniform field. The one complexity for elasto-plastic flow — as opposed to elasticity — is the manner in which the generalized modulus $E_{ijk\ell}$ is decomposed.

Results: A variety of results has been obtained with this approach, most dealing with planar crack problems. Early work in plane stress [23] and subsequent effort in plane strain [18] was directed toward overall characterization of elasto-plastic response of cracked bodies. Concern since has focussed on deformed crack shapes in stretching [15,19] and bending [4]. Considerable effort has been directed toward the question of physical fidelity of computed results, and the finding is in the main satisfactory [16,17]. There

is significant sensitivity to the accuracy with which material data
are represented [10], and some work has been directed toward reconciling
experimental measurements with material response [13]. This issue is
far from resolved and, indeed, remains as one of the primary technological
questions of mechanics.

CONCLUDING REMARKS

We have sought in this report to provide an overview of the theory of elasto-plastic flow without becoming deeply involved in the great array of detail that perforce accompanies its implementation and use. Our emphasis has been largely on three main aspects which deserve restatement at this point. First, the theory has an extensive scope. It can accommodate a broad variety of flow rules which reflect work-hardening, plastic compressibility, anisotropy, small or large deformations, and so on; the choice of all these resides with the analyst. Next, the theory is mathematically complete and coherent, and the analyst need have no concern for, say, the conditions under which it will 'work' or not. Finally, the theory is straightforward to use and gives results that are physically realistic insofar as we have been able to judge.

It should also be mentioned that the theory is economical to use. The codes now in hand, although refined over a period of some years, consume only modest amounts of computer storage and time, as a few examples will show. Using a Univac 1108, our planar codes for finite elements will accommodate up to 600 degrees of freedom but need less than 45,000 words of storage. Running times for 242 degrees of freedom are about 4 seconds per load step and for 436 degrees of freedom, under 13 seconds per load step. Costs for up to 10^2 load steps are thereby within reach for many purposes. Indeed we have on occasion run problems consisting of 35 to 40 steps for classroom use.

The approach then is attractive in a variety of respects, and we look forward to its use by more people. At the same time, there are several items worthy of attention that should be mentioned. We believe that solution methods other than simple finite elements and the boundary-integral technique should be explored as potential vehicles for solving problems of elasto-plastic flow. We have looked at finite differences, but the context of that effort made statement of boundary conditions awkward. Yet there is reason to suppose that some classes of problem would be tractable in finite differences. More complex finite elements, including special elements as might be used to examine locally singular behavior, merit study and possible development.

We suggest that the breadth of the theory is in need of exploration. Such features as anisotropy and plastic compressibility are natural extensions of work done to date. Any effort along these lines, however, should be coupled with careful experimentation so that extraction of material properties procedes in parallel with development of a capability for solving problems.

In addition to the features noted above, we believe that the economies attainable with this approach make study of cyclic loading especially attractive. Two important technological processes, fatigue response of structures and the Bauschinger effect, are thereby accessible to study, and we look for renewed analytical efforts along these lines.

The theory is also amenable to purely analytical treatments such as the important results reported recently by Budiansky [24]. The availability of solutions to non-trival problems provides both insight into the process of plastic flow and 'check problems' for developers of computer programs. Such work is usually inexpensive to perform but has far-reaching value.

Finally, we urge the technical community to use this method. Even without further develppment and extension, it offers considerable advantage at reasonable cost and it provides information that may in some circumstances prove valuable to have. Thus incorporation of the theory of elasto-plastic flow into more general structural programs is warranted and we urge such effort upon the technical community.

REFERENCES

1. R. Hill, *The Mathematical Theory of Plasticity*, Oxford University Press (1950).

2. D. C. Drucker, A More Fundamental Approach to Plastic Stress-Strain Relations, *Proceedings, First U. S. National Congress of Applied Mechanics* (1951) 487-491.

3. Y. C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, Inc. (1965).

4.* D. P. Jones, Elasto-Plastic Bending of Cracked Plates, Including the Effects of Crack Closure, Report SM-83A, Department of Mechanical Engineering, Carnegie-Mellon University, (1972); Ph.D. Dissertation.

5. J. L. Swedlow and T. A. Cruse, Formulation of Boundary-Integral Equations for Three-Dimensional Elasto-Plastic Flow, *International Journal of Solids and Structures* 7 (1971) 1673-1683.

6. J. L. Swedlow, Character of the Equations of Elasto-Plastic Flow in Three Independent Variables, *International Journal of Non-Linear Mechanics* 3 (1968) 325-336; 4 (1969) 77.

7. W. T. Koiter, Stress-Strain Relations, Uniqueness and Variational Theorems for Elastic-Plastic Materials with a Singular Yield Surface, *Quarterly of Applied Mathematics* 11 (1953) 350-354.

*Also issued as CR-112268, National Aeronautics and Space Administration (1973).

8. W. R. Osgood, Combined-Stress Tests on 24ST Aluminum Alloy Tubes, *Journal of Applied Mechanics* 14 (1947) 147-153.

9. D. C. Drucker, Relation of Experiments to Mathematical Theories of Plasticity, *Journal of Applied Mechanics* 16 (1949) 349-357.

10. J. L. Swedlow, A Procedure for Solving Problems of Elasto-Plastic Flow, *Computers and Structures* 3 (1973).

11. J. R. Rice, A Note on the 'Small Strain' Formulation for Elastic-Plastic Problems, Division of Engineering, Brown University (1970).

12. R. Hill, *op. cit.* 53-57.

13.* J. R. Osias, Finite Deformation of Elasto-Plastic Solids, Report SM-72-22, Department of Mechanical Engineering, Carnegie-Mellon University (1972); Ph.D. Dissertation.

14. W. Prager, *Introduction to Mechanics of Continua,* Ginn & Co. (1961).

15. J. L. Swedlow, Further Comment on the Association Between Crack Opening and G_I, *International Journal of Fracture Mechanics,* (1967) 75-79.

16. J. L. Swedlow, Initial Comparison Between Experiment and Theory of the Strain Fields in a Cracked Copper Plate, *International Journal of Fracture Mechanics* 5 (1969) 25-31.

17. J. H. Underwood, J. L. Swedlow, and D. P. Kendall, Experimental and Analytical Strains in an Edge-Cracked Sheet, *Engineering Fracture Mechanics* 2 (1971) 183-196.

---

*Also issued as CR-2199, National Aeronautics and Space Administration (1973).

18. J. L. Swedlow, Elasto-Plastic Cracked Plates in Plane Strain, *International Journal of Fracture Mechanics* 5 (1969) 33-44.

19. J. E. Srawley, J. L. Swedlow, and E. Roberts, Jr., On the Sharpness of Cracks Compared with Wells's COD, *International Journal of Fracture Mechanics* 6 (1970) 441-444; 7 (1971) 242-246.

20. P. C. Riccardella, Implementation of the Boundary Integral Technique for Problems of Planar Elasticity and Plasticity, Report SM-73-10, Department of Mechanical Engineering, Carnegie-Mellon University (1973).

21. J. L. Swedlow, Analysis of Cracks and Notches, *Transactions of the American Society for Metals Quarterly* 60 (1967) 551.

22. J. L. Swedlow, Computer Solution to Nonlinear Problems: The Example of Plasticity, *Computational Approaches in Applied Mechanics,* E. Sevin, editor, American Society of Mechanical Engineers (1969) 191-199.

23. J. L. Swedlow, M. L. Williams, and W. H. Yang, Elasto-Plastic Stresses and Strains in Cracked Plates, *Proceedings, First International Conference on Fracture* 1 (1966) 259-282.

24. B. Budiansky, An Exact Solution to an Elastic-Plastic Stress-Concentration Problem, *PMM* 35 (1971) 18-25.

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