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THE SECOND - ORDER
GRAVITATIONAL RED SHIFT

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I. INTRODUCTION

Gravitation has always been one of the most fascinating phenomena of nature. Recently, a resurgence of interest in this field has occurred, spurred on by many new astronomical observations. Because the force of gravity is weak compared to nuclear and electromagnetic forces, it is virtually impossible to perform laboratory experiments involving relativistic gravitation. However, with advances in space technology, we have now enlarged our laboratory for controlled experiments to the size of the solar system.

New tests of gravitation are presently being made, using new technology: Light bending by a gravitational field is measured, using very long baseline interferometry to observe the microwave signals from quasi stellar objects as the signals pass close to the sun. Powerful radars are being used to send pulses across the solar system to measure the time delay produced by the sun's gravity. Space tracking of probes flying past Mars and Venus have yielded valuable orbital data on the general relativistic corrections to the motion of objects in a gravitational field.

It is significant that atomic clocks have been involved in nearly all of these experiments. Indeed, today, atomic clocks offer a measuring capability within a precision of better than 1 part in 10^{14}. As no other instruments of comparable capability exist, it seems logical to further explore space-time with clocks, and to make measurements in terms of data expressed as time intervals.

It appears that a direct measurement of the non-linear term of the gravitational field equations can now be made by the use of very stable clocks. A vehicle containing such a clock could be put into a highly eccentric orbit around the sun, or allowed to fall into the sun, and its frequency would be compared with an identical clock on earth. This is a measure of the so-called "second-order gravitational redshift".

The only other presently feasible test for the second-order term is that which measures the perhelion advance of a planet or satellite; all the other current experimental tests of relativity can only measure first-order effects. This frequency shift experiment would be a distinct, additional test for the nonlinear term. A great advantage of this experiment is that it does not depend at all on the integrated characteristics of the orbit. As we will show, the only requirements are the knowledge of the instantaneous gravitational potential and the velocities of the clocks.

At present, a highly accurate test of the first-order gravitational redshift is in development, using a terrestrial rocket probe. Some of the techniques perfected for that experiment will be useful in a heliocentric second-order redshift experiment. The terrestrial probe experiment uses atomic hydrogen maser clocks in the probe and on the ground that are connected by microwave signals. The nongravitational effects due to the relative motion of the probe and ground clocks are accounted for by continuously measuring the path length in terms of the phase of coherently returned signals originally transmitted to the probe. The phase variations due to path length changes are then automatically removed from the clock signal.

The terrestrial experiment is expected to test the so-called principle of equivalence. This principle, first stated by Einstein in 1907, asserts that all freely falling, nonrotating, "sufficiently small" laboratories are equivalent; i.e., all the laws of physics will appear the same in all freely falling, infinitesimal laboratories. At present, the best tests of the principle of equivalence are the elegant experiments performed by R. V. Pound and co-workers, using Mössbauer emission and absorption from Fe^{57}. These experiments, performed over a vertical distance of 75 feet, have verified Einstein's postulate of equivalence to 1%. The terrestrial probe experiment seeks to extend the distance to 10,000 miles, and the accuracy to 20 ppm. The successful completion of this test will do much to put the equivalence principle on a sound experimental basis, and establish the validity of the cornerstone of Einstein's General Theory of Relativity.

However, the terrestrial probe experiment will not be sufficiently sensitive to permit distinguishing between the several theories of gravitation now in contention. Other presently viable theories of gravitation have the principle of equivalence as a basis. The <u>first-order</u> gravitational redshift can be derived immediately from the principle of equivalence alone; a knowledge of the field equations of the theory is not necessary. Thus, it is important to emphasize that a measurement of the <u>second-order</u> redshift, the nonlinear term, is a test of the actual field equations of a theory of gravitation, and could help establish the validity of such a theory.

II. FREQUENCY SHIFT IN A GRAVITATIONAL FIELD

We derive the exact expression for the gravitational frequency shift of light emitted by a source at one position and received by an observer elsewhere. The rigorous, covariant approach is used in order to avoid any ambiguities in the definition or interpretation of any terms appearing in the final expression.

The ratio of the frequency of light emitted by a source(s) and received by an observer (o) can be given as:

$$\frac{\nu_s}{\nu_o} = \frac{(\vec{p} \cdot \vec{v})_s}{(\vec{p} \cdot \vec{v})_o} = \frac{(p_\alpha v^\alpha)_s}{(p_\alpha v^\alpha)_o} , \tag{1}$$

where $p_\alpha \equiv \frac{dx_\alpha}{ds}$ is the photon "four-momentum", $v^\alpha \equiv \frac{dx^\alpha}{ds}$ is the "four-velocity" of the source or observer, and ds is the invariant infinitesimal element of length in four-space.[1] $ds = c d\tau$, where τ is the so-called "proper time" for a particle; for light, $ds = 0$, and, in that case, ds is simply an arbitrary variable. The symbol α represents numbers from 1 to 4. Similar subscripts and superscripts imply summation over that index; this is the usual convention.

The proof of Eq. (1) is quite direct. $(\vec{p} \cdot \vec{v})$ is the product of two four-vectors; it is a scalar--an <u>invariant</u> quantity. Being invariant, it has the same value in any coordinate system. We can easily find the invariant value of $(\vec{p} \cdot \vec{v})$ by considering an inertial system in which the particle is at rest. (By the principle of equivalence, one can always make a transformation into an inertial system at any point in a gravitational field.) In this system, from special relativity, we have, with c = 1, $p_{\alpha_{Light}} = (p_1, p_2, p_3, E = h\nu)$ and $v^{\alpha}_{Particle} = (0, 0, 0, 1)$, so that $(\vec{p} \cdot \vec{v}) \equiv (p_{\alpha} v^{\alpha}) = h\nu$. Thus,

$$\frac{(\vec{p} \cdot \vec{v})_s}{(\vec{p} \cdot \vec{v})_o} = \frac{h\nu_s}{h\nu_o} = \frac{\nu_s}{\nu_o} \quad .$$

Eq. (1) is quite general; we now apply it to the special case of the spherically symmetric gravitational field. The general form for this field, in the (r, θ, φ, t) coordinate system, is given as[1]

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\phi^2 + \sin^2\theta d\theta^2) \quad . \qquad (2)$$

Upon solving the geodesic equations of motion, for either light or a particle, it is found that the motion can be confined to a plane, ϕ = constant (chosen as $= \pi/2$), just as in the Newtonian case of central force motion. Eq. (2) can then be taken as

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 \quad . \qquad (3)$$

<u>Evaluation of $\vec{p}_{Light}$ and $\vec{v}_{Particle}$</u>

The geodesic equations for θ and t give

$$r^2 \frac{d\theta}{ds} = q = \text{constant} \qquad (4)$$

$$B \frac{dt}{ds} = k = \text{constant} \quad . \qquad (5)$$

If, for a moment, we consider a test particle at infinity, of finite mass m, with velocity v as measured at infinity, Eqs. (4) and (5) can be written as

$$r^2 \frac{d\theta}{ds} = q = \frac{\ell v}{\sqrt{1 - v^2}} , \tag{6}$$

$$B \frac{dt}{ds} \to \frac{dt}{ds} = k = \frac{1}{\sqrt{1 - v^2}} , \tag{7}$$

where ℓ is the classical impact parameter of the particle as measured at infinity. Therefore, for light, where v = 1, we have

$$\frac{q}{k} = \ell \quad . \tag{8}$$

Using Eqs. (3), (4), (5), (8) and $ds^2 = o$ for light, we have for $\vec{p}_{Light} \equiv \frac{dx^\alpha}{ds}$:

$$p^0 = \frac{dt}{ds} = \frac{k}{B}$$

$$p^1 = \frac{dr}{ds} = \frac{k}{\sqrt{AB}} \sqrt{1 - \frac{B\ell^2}{r^2}}$$

$$p^2 = \frac{d\theta}{ds} = \frac{k\ell}{r^2} ,$$

and, finally, for $p_\beta = g_{\beta\alpha} \frac{dx^\alpha}{ds}$:

$$p_0 = g_{00}\, p^0 = k \tag{9a}$$

$$p_1 = g_{11}\, p^1 = - k \sqrt{\frac{A}{B}} \sqrt{1 - \frac{B\ell^2}{r^2}} \tag{9b}$$

$$p^2 = g_{22}\, p^2 = - k\ell \quad . \tag{9c}$$

For a particle, $ds^2 \neq o$. Using Eq. (3), we find for $v^\alpha_{Particle} \equiv \frac{dx^\alpha}{ds}$:

$$v^0 = \frac{dt}{ds} = \frac{1}{\sqrt{B - A\,v_r^{\,2} - r^2 v_\theta^2}} \tag{10a}$$

$$v^1 = \frac{dr}{ds} = \frac{dr}{dt}\,\frac{dt}{ds} = v_r\,v^0 \tag{10b}$$

$$v^2 = \frac{d\theta}{ds} = \frac{d\theta}{dt}\,\frac{dt}{ds} = v_\theta\,v^0\;, \tag{10c}$$

where $v_r \equiv \frac{dr}{dt}$ and $v_\theta \equiv \frac{d\theta}{dt}$. Therefore,

$$\vec{p}\cdot\vec{v} = p_\alpha\,v^\alpha$$

$$= k\left(1, -\sqrt{\frac{A}{B}}\sqrt{1 - \frac{B\ell^2}{r^2}}\,, -\ell\right)v^0\,(1, v_r, v_\theta)$$

$$= k\,\frac{1}{\sqrt{B - Av_r^{\,2} - r^2 v_\theta^{\,2}}}\left[1 - \sqrt{\frac{A}{B}}\sqrt{1 - \frac{B\ell^2}{r^2}}\,v_r - \ell v_\theta\right]. \tag{11}$$

In the absence of gravity, A = B = 1, and the ratio of the frequency sent by a source, moving with velocity (v_r, v_θ) at a point r, versus the frequency received by an observer at rest is

$$\frac{(p\cdot v)_s}{(p\cdot v)_0} = \frac{k\left[1 - \sqrt{1 - \frac{\ell^2}{r^2}}\,v_r - \frac{\ell}{r}\,(rv_\theta)\right]}{\sqrt{1 - v_r^{\,2} - r^2 v_\theta^{\,2}}}\cdot\frac{1}{k}$$

$$= \frac{\left[1 - \sqrt{1 - \frac{\ell^2}{r^2}}\;v_r - \frac{\ell}{r}\,(rv_\theta)\right]}{\sqrt{1 - v_r^{\,2} - r^2 v_\theta^{\,2}}}\;.$$

This is just the special relativistic doppler equation, as we should expect: The numerator is $(1 = v_{Particle}\cos\theta)$, as is verified by a simple geometrical construction. The denominator is $\sqrt{1 - v^2}$, the inverse Lorentz factor (c = 1).

Eq. (11) shows that the presence of a gravitational field "modifies" the well-known doppler-shift equation through the functions A(r) and B(r). The parameter ℓ is the single constant of the motion for light. In the absence of gravity, it is clearly the impact parameter of the photon; in the presence of a gravitational field, this unambiguous physical interpretation of ℓ can only be made at infinity, as will be discussed below (Section VII). No <u>physical</u> interpretation can be made for $r \neq \infty$, even though ℓ is still a valid mathematical constant of the motion for the photon.

We can write Eq. (11) in a more compact form, if we define

$$\gamma \equiv \frac{1}{\sqrt{B - A\,v_r^{\,2} - r^2 v_\theta^{\,2}}} \quad , \tag{12}$$

$$\vec{\beta} \equiv (v_r,\ r v_\theta,\ v_\phi = 0) \quad , \tag{13}$$

(c will always be taken = 1, unless otherwise specified), and

$$\vec{\varepsilon} = \left(\sqrt{\frac{A}{B}} \sqrt{1 - \frac{B\ell^2}{r^2}}\ ,\ \frac{\ell}{r}\ ,\ 0 \right) \quad . \tag{14}$$

Eq. (11) becomes

$$\vec{p} \cdot \vec{v} = k\,\gamma\,(1 - \vec{\beta} \cdot \vec{\varepsilon}) \quad . \tag{15}$$

In the absence of gravitation, γ reduces to the Lorentz factor, and $\vec{\varepsilon}$ is the unit direction vector of the photon at the point r. We emphasize that, when A and B $\neq 1$, γ and $\vec{\varepsilon}$ are modified by these functions, and such a simple interpretation cannot be made.

III. THE DOPPLER-CANCELLING TECHNIQUE

Because of the motion of the space vehicle relative to earth, the first-order (in v/c) doppler effect is clearly the largest contribution to the signal sent by the vehicle clock. The lowest-order gravitational contribution is of order v^2/c^2 ($\approx \frac{GM}{c^2 r}$). It is thus essential to set up an experiment that eliminates the first-order terms.[10,11,12] This can be done by placing both a clock and a transponder on the probe. The transponder serves as a reflector that allows us to measure the two-way distance in terms of signal phase. By combining the received signals appropriately, we can remove the first-order doppler terms, as follows:

We first consider the situation where the earth station is equipped with a clock and the probe is equipped with both a clock and a transponder. The earth station sends out a signal of proper frequency ν at proper time t_1 (with the station at position 1, with velocity $\vec{\beta}_1$) to the probe's transponder, which receives it at time t_2 (position 2, velocity $\vec{\beta}_2$). The probe transponder re-transmits the signal to the earth, along with the probe's clock signal; both signals are received at the earth station at time t_3 (position 3, velocity $\vec{\beta}_3$). (Figure 1)

Eqs. (1) and (15) give the frequency shift of the clock signal received from the probe as

$$\left.\frac{\nu_3}{\nu_2}\right|_{\text{Clock}} = \frac{\gamma_3}{\gamma_2}\,\frac{(1-\vec{\beta}_3\cdot\vec{\varepsilon}_{23})}{(1-\vec{\beta}_2\cdot\vec{\varepsilon}_{23})} \quad , \tag{16}$$

where $\vec{\varepsilon}_{ij}$ is the "direction vector" of the light signal from position i to j. ($\vec{\beta}_2 \cdot \vec{\varepsilon}_{23}$ is evaluated at position 2, $\vec{\beta}_3 \cdot \vec{\varepsilon}_{23}$ is evaluated at position 3, etc.).

In a similar manner, the frequency shift of the earth-probe-earth transponder signal, received on the earth at time t_3 is

$$\left.\frac{\nu_3}{\nu_1}\right|_{\text{Transponder}} = \left(\frac{\nu_2}{\nu_1}\right)\left(\frac{\nu_3}{\nu_2}\right)$$

$$= \frac{\gamma_3}{\gamma_1}\,\frac{(1-\vec{\beta}_2\cdot\vec{\varepsilon}_{12})}{(1-\vec{\beta}_1\cdot\vec{\varepsilon}_{12})}\,\frac{(1-\vec{\beta}_3\cdot\vec{\varepsilon}_{23})}{(1-\vec{\beta}_2\cdot\vec{\varepsilon}_{23})} \;. \tag{17}$$

The doppler-cancelling technique consists of combining these two received signals in the form:

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = \left.\frac{\nu_3}{\nu_2}\right|_{\text{Clock}} - \frac{1}{2}\left.\frac{\nu_3}{\nu_1}\right|_{\text{Transponder}} - \frac{1}{2} \;. \tag{18}$$

The observed $\frac{\Delta\nu}{\nu}$ is very small, and contains the information we seek; it contains only terms in β^2 and higher order.

IV. THE DUAL TRANSPONDER SYSTEM

It is also possible to equip the earth station with a transponder, and use the doppler-cancelling scheme for earth-based clock and transponder signals received by the <u>probe</u>. (This information is subsequently transmitted to the earth station for analysis.) This <u>dual</u> transponder-clock arrangement provides twice the number of observables for a self-consistent analysis of the experiment; the added information gained from the second system permits a more direct and tractable means for explicitly evaluating the various parameters in the experiment. The dual system would also aid in eliminating possible nonreciprocal propagation effects.[12]

For the probe-based system, with a signal sent from the probe at time t_0 (position 0, velocity $\vec{\beta}_0$), received at earth at time t_1, and received back at the probe at time t_2 along with the earth-clock signal, the doppler-cancelling technique gives

$$\frac{\Delta\nu}{\nu}\bigg|_{\text{Probe}} = \frac{\nu_2}{\nu_1}\bigg|_{\text{Clock}} - \frac{1}{2}\,\frac{\nu_2}{\nu_0}\bigg|_{\text{Transponder}} - \frac{1}{2}\ . \qquad (19)$$

V. THE SECOND-ORDER REDSHIFT IN A SPHERICALLY SYMMETRIC GRAVITATIONAL FIELD

We will consider the gravitational field of the sun as spherically symmetric in the first approximation. A possibly significant correction to this model will be considered later. The solution of the field equations of general relativity for the spherically symmetric case was first given by Schwarzschild[2] shortly after Einstein's publication of the basic theory. In a plane ($\phi = \pi/2$), the exact solution is given as

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 \quad , \tag{3}$$

with

$$B(r) = 1 - \frac{2GM}{r} \tag{20}$$

$$A(r) = 1/B(r) \quad . \tag{21}$$

Eddington[3] and Robertson[4] put the spherically symmetric solution in a more general, parameterized form in order to show more explicitly the contribution of the various-order terms in (GM/r). In the "standard" (r, θ, t) coordinate system, we have

$$ds^2 = B\,dt^2 - A\,dr^2 - r^2d\theta^2 \quad ,$$

with

$$A = 1 - 2\alpha \frac{GM}{r} + 2(\beta - \alpha\gamma) \frac{G^2M^2}{r^2} + \ldots \tag{22}$$

$$B = 1 + 2\gamma \frac{GM}{r} + \ldots \quad , \tag{23}$$

where α, β, and γ are unknown, dimensionless parameters.

If the Einstein field equations give the physically correct description of the spherically symmetric gravitational field, then the solution would be the exact Schwarzschild solution, and in the parameterized representation we would have $\alpha = \beta = \gamma = 1$. This, of course, is what is to be tested. In contrast, the Brans-Dicke theory[5] would have $\alpha = \beta = 1$, and $\gamma = \frac{\omega+1}{\omega+2}$, where ω is the unknown dimensionless parameter of that theory, which governs the admixture of the scalar field to the tensor field.

Actually, $\alpha = 1$ simply follows from the empirical definition of the mass, M. We will thus set $\alpha = 1$ in all further calculations.

Earth-Based Station

Combining Eqs. (16) - (18), we have for the doppler-cancelled signal at the earth station:

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = \left.\frac{\nu_3}{\nu_2}\right|_{\text{Clock}} - \frac{1}{2}\left.\frac{\nu_3}{\nu_1}\right|_{\text{Transponder}} - \frac{1}{2}$$

$$= \frac{\gamma_3}{\gamma_2}\frac{(1-Ae')}{(1-Ap)} - \frac{1}{2}\frac{\gamma_3}{\gamma_1}\frac{(1-Ap')}{(1-Ae)}\frac{(1-Ae')}{(1-Ap)} - \frac{1}{2}$$

$$= \frac{(1-Ae')}{(1-Ap)}\left[\frac{\gamma_3}{\gamma_2} - \frac{1}{2}\frac{\gamma_3}{\gamma_1}\frac{(1-Ap')}{(1-Ae)}\right] - \frac{1}{2} \quad , \tag{24}$$

where $Ae' \equiv \vec{\beta}_3 \cdot \vec{\varepsilon}_{23}$, $Ap \equiv \vec{\beta}_2 \cdot \vec{\varepsilon}_{23}$, $Ae = \vec{\beta}_1 \cdot \vec{\varepsilon}_{12}$, and $Ap' = \vec{\beta}_2 \cdot \vec{\varepsilon}_{12}$.

We will initially assume that the directions and vector velocities do not change during the signal propagation time (i.e., infinite propagation velocity). In this case, $\vec{\beta}_3 = \vec{\beta}_1$, $\gamma_3 = \gamma_1$, $\vec{\varepsilon}_{23} = -\vec{\varepsilon}_{12}$, which gives $Ae' = -Ae$ and $Ap' = -Ap$. Eq. (24) becomes

$$\frac{\Delta\nu}{\nu}\bigg|_{\text{Earth}} = \frac{(1+Ae)}{(1+Ap')}\left[\frac{\gamma_1}{\gamma_2} - \frac{1}{2}\frac{(1-Ap')}{(1-Ae)}\right] - \frac{1}{2} \quad . \tag{25}$$

Expanding to fourth order in β (or, equivalently, second order in GM/r), and using the Eddington-Robertson parameterized form for the gravitational field, we have (Appendix A):

$$\frac{\Delta\nu}{\nu}\bigg|_{\text{Earth}} = \left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}(\beta_1^2-\beta_2^2)\right] + Ae(Ap'-Ae) \tag{26a}$$

$$- (Ap'-Ae)\left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}(\beta_1^2-\beta_2^2) + Ae(Ap'-Ae)\right] \tag{26b}$$

$$+ m\left[\gamma\left(\frac{\beta_{r_1}^2}{r_1} - \frac{\beta_{r_2}^2}{r_2}\right) + \frac{3}{2}\frac{\beta_1^2}{r_1}\frac{1}{2}\frac{\beta_2^2}{r_2} - \frac{1}{2}\left(\frac{\beta_2^2}{r_1} + \frac{\beta_1^2}{r_2}\right)\right] \tag{26c}$$

$$+ \frac{1}{8}\left[3\beta_1^4 - \beta_2^4 - 2\beta_1^2\beta_2^2\right] \tag{26d}$$

$$+ m^2\left[(\beta-\gamma)\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) + \frac{3}{2}\frac{1}{r_1^2} - \frac{1}{2}\frac{1}{r_2^2} - \frac{1}{r_1 r_2}\right] \tag{26e}$$

$$+ (Ap'-Ae)\left\{Ap'\left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}(\beta_1^2-\beta_2^2) + Ae(Ap'-Ae)\right] + Ae^3\right\} \tag{26f}$$

where G has been set = 1 ($GM/c^2 \rightarrow m$) and $\beta_i^{\,2} = \beta_{r_i}^{\,2} \quad \beta_{\theta_i}^{\,2}$. Eq. (26a) is the second-order term, (26b) is the third-order term, and the rest are all fourth-order in $\vec{\beta}$ (or second order in m/r). The nonlinear Eddington-Robertson parameter, β, characterizing the second-order (in m/r) purely gravitational redshift, appears only in the term (26e).

Probe-Based Station

A similar analysis can be made for the probe-based doppler-cancelling system. Eq. (19) becomes

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Probe}} = \frac{(1-Ap')}{(1-Ae)} \left[\frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_0}\frac{(1-Ae'')}{(1-Ap'')}\right] - \frac{1}{2} \ , \tag{27}$$

where $Ae'' \equiv \vec{\beta}_1 \cdot \vec{\varepsilon}_{01}$ and $Ap'' \equiv \vec{\beta}_0 \cdot \vec{\varepsilon}_{01}$. For an infinite signal-propagation velocity, $\vec{\beta}_2 = \vec{\beta}_0$, $\gamma_2 = \gamma_0$, $\vec{\varepsilon}_{01} = -\vec{\varepsilon}_{12}$, which gives $Ae'' = -Ae$ and $Ap'' = -Ap'$. Eq. (27) then becomes

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Probe}} = \frac{(1-Ap')}{(1-Ae)} \left[\frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{(1+Ae)}{(1+Ap')}\right] - \frac{1}{2} \ . \tag{28}$$

Eq. (28) has the same form as the earth station relation, Eq. (25), with the substitutions $Ae \rightarrow (-Ap')$, $Ap' \rightarrow (-Ae)$ and the (1, 2) indices reversed. The fourth-order expansion of Eq. (28) can then be found from Eq. (26) using these substitutions:

$$\frac{\Delta\nu}{\nu}\Bigg|_{Probe} = -\left[\left(\frac{m}{r_1}-\frac{m}{r_2}\right)+\frac{1}{2}\left(\beta_1^2-\beta_2^2\right)\right] - Ap'(Ap'-Ae) \qquad (29a)$$

$$+ (Ap'-Ae)\left[\left(\frac{m}{r_1}-\frac{m}{r_2}\right)+\frac{1}{2}\left(\beta_1^2-\beta_2^2\right) + Ap'(Ap'-Ae)\right] \qquad (29b)$$

$$+ m\left[\gamma\left(\frac{\beta_{r_2}^{\,2}}{r_2}-\frac{\beta_{r_1}^{\,2}}{r_1}\right)+\frac{3}{2}\frac{\beta_2^2}{r_2}-\frac{1}{2}\frac{\beta_1^2}{r_1}-\frac{1}{2}\left(\frac{\beta_1^2}{r_2}+\frac{\beta_2^2}{r_1}\right)\right] \qquad (29c)$$

$$+ \frac{1}{8}\left[3\beta_2^4-\beta_1^4-2\beta_1^2\beta_2^2\right] \qquad (29d)$$

$$+ m^2\left[(\beta-\gamma)\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)+\frac{3}{2}\frac{1}{r_2^2}-\frac{1}{2}\frac{1}{r_1^2}-\frac{1}{r_1r_2}\right] \qquad (29e)$$

$$+ (Ap'-Ae)\left\{Ae\left[\left(\frac{m}{r_1}-\frac{m}{r_2}\right)+\frac{1}{2}\left(\beta_1^2-\beta_2^2\right) + Ap'(Ap'-Ae)\right] - Ap'^3\right\} . \qquad (29f)$$

VI. FINITE SIGNAL TRANSIT TIME

We will, in fact, have a finite go-return time for the propagation of the signal. For a heliocentric probe, there would only be a relatively small displacement of the earth or probe during this time, as compared with the total displacement over a whole period of its motion (approximately 3 parts in 10^{-5} for any of the vector positions and velocities, at best). Therefore, r, $\vec{\beta}$, and $\vec{\varepsilon}_{ij}$ would not change appreciably in direction during the signal transit time. We will use a perturbation analysis to investigate the additional contributions due to the finite displacement of the earth or probe during the signal transit time.

We will assume here that the earth moves in a circular orbit around the sun and ignore the earth-moon motion and earth rotation. We write

$$\vec{\beta}_3 = \vec{\beta}_1 + \vec{\delta} \quad , \tag{30}$$

and $|\beta_3|^2 = |\beta_1|^2$ in this case. Defining $\vec{\varepsilon}$ by

$$\vec{\varepsilon}_{23} = - \vec{\varepsilon}_{12} + \vec{\varepsilon} \quad , \tag{31}$$

we have

$$
\begin{aligned}
Ae' &\equiv \vec{\beta}_3 \cdot \vec{\varepsilon}_{23} \\
&= (\vec{\beta}_1 + \vec{\delta}) \cdot (-\vec{\varepsilon}_{12} + \vec{\varepsilon}) \\
&= - \vec{\beta}_1 \cdot \vec{\varepsilon}_{12} + \vec{\beta}_1 \cdot \vec{\varepsilon} - \vec{\varepsilon}_{12} \cdot \vec{\delta} + \vec{\varepsilon} \cdot \vec{\delta} \\
&= - Ae + \vec{\beta}_1 \cdot \vec{\varepsilon} - \vec{\varepsilon}_{12} \cdot \vec{\delta} + \vec{\varepsilon} \cdot \vec{\delta} \quad , \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
Ap &\equiv \vec{\beta}_2 \cdot \vec{\varepsilon}_{23} \\
&= \vec{\beta}_2 \cdot (-\vec{\varepsilon}_{12} + \vec{\varepsilon}) \\
&= - \vec{\beta}_2 \cdot \vec{\varepsilon}_{12} + \vec{\beta}_2 \cdot \vec{\varepsilon} \\
&= - Ap' + \vec{\beta}_2 \cdot \vec{\varepsilon} \quad . \qquad (33)
\end{aligned}
$$

All the terms have been retained in these expansions, since we are carrying our results through to fourth order in β.

We substitute Eqs. (32) and (33) into the general relation for the earth station, Eq. (24). Further calculations (Appendix B) ultimately yield

$$
\begin{aligned}
\frac{\Delta\nu}{\nu}\bigg|_{\text{Earth}} &= \left(\frac{\Delta\nu}{\nu}\right)_0 \quad [\ = \text{Eq. } (26)] \qquad (34a) \\
&+ \frac{1}{2} [(\vec{\varepsilon}_{12} \cdot \vec{\delta}) + (\vec{\beta}_2 - \vec{\beta}_1) \cdot \vec{\varepsilon}] \qquad (34b) \\
&- \frac{1}{2} [(\vec{\varepsilon}_{12} \cdot \vec{\delta}) Ae + (\vec{\beta}_2 \cdot \vec{\varepsilon}) \ Ap' - (\vec{\beta}_1 \cdot \vec{\varepsilon}) Ae + (\vec{\varepsilon} \cdot \vec{\delta})] \qquad (34c) \\
&+ [(\vec{\varepsilon}_{12} \cdot \vec{\delta}) + (\vec{\beta}_2 - \vec{\beta}_1) \cdot \vec{\varepsilon}] [(\frac{m}{r_1} - \frac{m}{r_2}) + \frac{1}{2} (\beta_1{}^2 - \beta_2{}^2)] \qquad (34d) \\
&+ \frac{1}{2} \{ - (\vec{\beta}_1 \cdot \vec{\varepsilon}) \ [Ae(2 \ Ap' - Ae) +
\end{aligned}
$$

$$+ (\vec{\beta}_2 \cdot \vec{\varepsilon}) [2\ Ae\ (Ap' - Ae) + Ap'^2]$$

$$+ (\vec{\varepsilon}_{12} \cdot \vec{\delta})\ [Ae\ (2\ Ap' - Ae) + (\vec{\varepsilon} \cdot \vec{\delta})\ Ae$$

$$+ (\vec{\beta}_2 \cdot \vec{\varepsilon})\ (\vec{\varepsilon}_{12} \cdot \vec{\delta}) + (\vec{\beta}_2 \cdot \vec{\varepsilon})\ ((\vec{\beta}_2 - \vec{\beta}_1) \cdot \vec{\varepsilon}\} . \qquad (34e)$$

Eq. (34b) is of lowest order β^2; (34c) is of lowest order β^3; (34d) and (34e) are of order β^4. This is relatively simple to verify:

To find the order of δ, we can rewrite $\vec{\beta}_3 = \vec{\beta}_1 + \vec{\delta}$ as $\vec{\beta}_3 \approx \vec{\beta}_1 + \frac{\partial \vec{\beta}_1}{\partial t} \Delta T$, where ΔT is the signal go-return time. For the earth, moving with velocity v in a circular orbit around the sun, with "radius" R, $\frac{\partial \vec{\beta}_1}{\partial t} = \frac{1}{c} \frac{v^2}{R}$. With $\Delta T \propto \frac{R}{c}$, we have $\vec{\delta} \approx \frac{\partial \vec{\beta}_1}{\partial t} \cdot \Delta T \propto \frac{1}{c} \frac{v^2}{R} \left(\frac{R}{C} \right) = \frac{v^2}{c^2} \equiv \beta^2$. This simple analysis shows that the lowest order dependence for δ is $\mathcal{O}(\beta^2)$.

To see the dependence of $\vec{\varepsilon} = \vec{\varepsilon}_{12} + \vec{\varepsilon}_{23}$, we will use a plane geometry argument[6]; this is not rigorously correct (as discussed in Section VII), but it will at least give the lowest-order dependence. We have

$$\vec{\varepsilon} \equiv \vec{\varepsilon}_{12} + \vec{\varepsilon}_{23}$$

$$\approx \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} + \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|} .$$

With $\vec{r}_3 \approx \vec{r}_1 + c\Delta T \vec{\beta}_1$,

$$\vec{\varepsilon} \approx \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|} - \frac{(\vec{r}_2-\vec{r}_1) - c\Delta T\vec{\beta}_1}{|\vec{r}_2-\vec{r}_1| \; [1 - \frac{c\Delta T\vec{\beta}_1 \cdot \vec{\varepsilon}_{12}}{|\vec{r}_2-\vec{r}_1|}]} .$$

Expanding the denominator of the second term yields

$$\vec{\varepsilon} \approx \frac{c\Delta T\vec{\beta}_1}{|\vec{r}_2-\vec{r}_1|} - \vec{\varepsilon}_{12} \left(\frac{c\Delta T\vec{\beta}_1 \cdot \vec{\varepsilon}_{12}}{|\vec{r}_2-\vec{r}_1|}\right) .$$

With $\Delta T \approx 2 \; |\vec{r}_2 - \vec{r}_1| \; /c$, we finally have

$$\vec{\varepsilon} \approx 2\vec{\beta}_1 - 2 \; \vec{\varepsilon}_{12} \; (\vec{\beta}_1 \cdot \vec{\varepsilon}_{12}) .$$

$\vec{\varepsilon}$ is at least of order β .

We see that the inclusion of the finite signal-propagation time introduces further terms of all orders in the doppler-cancelling data; these terms must, of course, be carefully treated on the same footing as the pure doppler and relativistic effects calculated in the preceding section.

A similar analysis has been made for the probe-based station. In this case, no restriction to circular motion can be assumed for the probe. With $\vec{r}_2 \equiv \vec{r}_0 + \vec{\rho}$, $\vec{\beta}_2 = \vec{\beta}_0 + \vec{\delta}'$, $\vec{\varepsilon}_{12} = -\vec{\varepsilon}_{01} + \vec{\varepsilon}'$ Appendix C gives

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Probe}} = \left(\frac{\Delta\nu}{\nu}\right)_0 \qquad [= \text{Eq. } (29)] \tag{35a}$$

$$- \frac{1}{2} \left[(\vec{\varepsilon}_{12} \cdot \vec{\delta}') + (\vec{\beta}_2 - \beta_1) \cdot \vec{\varepsilon}'\right] \tag{35b}$$

$$+ \frac{1}{2} \Big\{ [3 \text{ Ap}' - 2 \text{ Ae}]\left[(\vec{\varepsilon}_{12} \cdot \vec{\delta}') + (\vec{\beta}_2 \cdot \vec{\varepsilon}')\right] + (\text{Ae} - 2 \text{ Ap}')(\vec{\beta}_1 \cdot \vec{\varepsilon}') - (\vec{\varepsilon}' \cdot \delta') + \frac{m}{r_0} \left(\frac{\vec{r}_0 \cdot \vec{\rho}}{r_0^2}\right) - (\vec{\beta}_0 \cdot \vec{\delta}') \Big\} \tag{35c}$$

$$- \frac{1}{2} \{ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')[5\ Ap'(Ap'-Ae) + 2\ Ae^2 + (2\vec{\beta}_2-\vec{\beta}_1)\cdot\vec{\varepsilon}'$$

$$+ \vec{\varepsilon}_{12}\cdot\vec{\delta}'] + (\vec{\varepsilon}'\cdot\vec{\delta}')\ Ap' + (\vec{\beta}_2\cdot\vec{\varepsilon}')[5\ Ap'(Ap'-Ae)$$

$$+ 2\ Ae^2 + (\vec{\beta}_2-\vec{\beta}_1)\cdot\vec{\varepsilon}'] - (\vec{\beta}_1\cdot\vec{\varepsilon}')[Ae(Ae-Ap') + 2\ Ap'^2]$$

$$+ \frac{m}{r_0}\ [(\frac{\rho^2}{2r_0{}^2}) - \frac{3}{2}\ (\frac{\vec{r}_0\cdot\vec{\rho}}{r_0})^2 + \frac{1}{2}\ \delta'^2 - 2(Ap'-Ae)\ (\frac{\vec{r}_0\cdot\vec{\rho}}{r_0{}^2})]$$

$$- 2(Ap'-Ae)(\vec{\beta}_0\cdot\vec{\delta}')\ \} \tag{35d}$$

Eq. (35b) is of lowest-order β^2; (35c) is of lowest-order β^3; Eq. (35d) is of order β^4. (ρ is of the order of $r\beta$, for $\vec{r}_2 = r_0 + \vec{\rho} \approx \vec{r}_0 + c\Delta T\ \vec{\beta}_0 \approx \vec{r}_0 + 2\ |\vec{r}_0 - \vec{r}_1|\ \vec{\beta}_0$).

VII. THE REALITY AND INTERPRETATION OF COORDINATES IN THE SECOND-ORDER REDSHIFT EXPERIMENT

Throughout this work, we have used the so-called "standard (r, θ, ϕ, t) coordinate system". While this or any other coordinate system is quite valid to work in, within the framework of general relativity, there is the question of whether the (r, θ, ϕ, t) system physically corresponds with our familiar Euclidean concept of a spherical coordinate system. In fact, it does not, and the lack of correspondence with our familiar geometrical concepts must be accounted for carefully in the analysis of the second-order redshift experiment. These questions must be carefully considered when we analyze the tracking of the probe from the earth. Positions, velocities, angles, etc., will all be affected.

We can illustrate the nature of the problem. Let us consider, for a moment, the exact Schwarzschild solution in a plane,

$$ds^2 = (1 - \frac{2m}{r})\, dt^2 - (1 - \frac{2m}{r})^{-1}\, dr^2 - r^2 d\theta^2 \quad .$$

The infinitesimal elements of spatial distance in the r and direction are given by[1] .

$$d\ell_r = (1 - \frac{2m}{r})^{-\frac{1}{2}}\, dr \tag{36}$$

$$d\ell_\theta = r\, d\theta \quad . \tag{37}$$

These expressions are found by setting up an operational definition for the measurement of spatial displacement at a point in a gravitational field, using light signals.[1] For the static

Schwarzschild solution, these $d\ell$ happen to be simply the spatial components of Eq. (3), but this is not generally true for an arbitrary gravitational field.

Using Eqs. (36) and (37), we can find the ratio of the circumference of a "circle" at coordinate r from the origin, to the radial distance. This is

$$\int_0^2 d\ell_\theta \;/\; \int_0^r d\ell_r \;=\; r \cdot \int_0^{2\pi} d\theta \;/\; \int_0^r \frac{dr}{(1-\frac{2m}{r})^{\frac{1}{2}}}$$

$$= 2\pi r \;/ \int_0 \frac{dr}{(1-\frac{2m}{r})^{\frac{1}{2}}}$$

The integral in the denominator is obviously not going to be equal to r, but some value greater than r (since $2m < r$). Therefore, the ratio of the circumference to the radial distance will be less than 2π, in disagreement with classical Euclidean geometry. The "radial coordinate, r" cannot be interpreted simply as the straight line distance from the origin to a point r.

Further study[7] shows that there is indeed no way physically to interpret the coordinates in terms of our familiar geometrical concepts. The coordinates are a valid means of cataloging the points and events in space-time, but they can only be given a real physical meaning if they are somehow eventually written completely in terms of observables (proper-clock time intervals and proper frequencies, in this case). This is the self-consistent analysis of the redshift experiment that ultimately must be made.

We now consider the specific equations of the second-order redshift experiment. In the compact notation defined by Eqs. (12) - (14), the frequency was written as

$$\nu \equiv \vec{p}\cdot\vec{v} = k\ \gamma\ (1 - \vec{\beta}\cdot\vec{\varepsilon}\ ) \quad . \tag{15}$$

However, it is incorrect to say that $\vec{\varepsilon}$ is simply the plane direction cosine of the photon, since its explicit form in the (r, θ, φ, t) "standard" coordinate system is

$$\vec{\varepsilon} \equiv (\sqrt{\frac{A}{B}}\ \sqrt{1 - \frac{B\ell^2}{r^2}}\ ,\ \frac{\ell}{r}\ ,\ 0) \quad . \tag{14}$$

Only when A = B = 1 will $\vec{\varepsilon}$ be the actual classical direction cosine for the photon. In the presence of a gravitational field, where A and B $\neq$ 1, the form of $\vec{\varepsilon}$ is modified, and no unambiguous geometric association can be made.

Indeed, even ℓ cannot be associated with the classical impact parameter of the photon, when a gravitational field is present. By combining Eqs. (3), (8), (20), and (21), we find the $r - \theta$ equation for light ($ds^2 = o$) in the Schwarzschild field to be

$$\frac{1}{r^4}\ (\frac{dr}{d\theta})^2 = \frac{2m}{r^3} - \frac{1}{r^2} + \frac{1}{\ell^2} \quad . \tag{38}$$

The impact parameter $\bar{r}$, is the point where $\frac{dr}{d\theta} = 0$. From Eq. (38), we have

$$0 = \frac{2m}{\bar{r}^3} - \frac{1}{\bar{r}^2} + \frac{1}{\ell^2} \quad ,$$

or

$$\ell^2 = \frac{\bar{r}^2}{1 - \frac{2m}{\bar{r}}} \quad . \tag{39}$$

$\ell \neq \bar{r}$ except as $\bar{r} \to \infty$. We see that ℓ cannot be considered the "impact parameter", except at infinity; it is merely a constant of the motion for the photon, and no deeper meaning can be attributed to it.

All the parameters appearing in the redshift equations (ℓ, r_i, β_i, etc.) must eventually be evaluated rigorously in terms of the observables, using the redshift and light propagation equations in a self-consistent scheme.

We note that in the above discussions of the various-order terms appearing in the doppler-cancelled data, we have assumed the $\vec{\beta}_i \cdot \vec{\varepsilon}_{ij}$ terms to be only of order β_i. There are, of course, higher order corrections to $\vec{\beta}_i \cdot \vec{\varepsilon}_{ij}$, involving terms in m/r, due to the presence of A (r) and B (r); we have not explicitly expanded them out in this present work, in order to avoid further complicating the equations. These questions will be addressed in the tracking study phase of this work.

VIII. FURTHER CORRECTIONS TO THE REDSHIFT: SOLAR QUADRUPOLE MOMENT

There are many corrections that should be included in any realistic analysis of the second-order redshift experiment. Of particular importance is the possibility of a solar quadrupole moment, as suggested by Dicke and Goldenberg[8]. We will now consider this correction, reserving such effects as the earth's rotation, etc., for a later analysis. We will use the "weak-field" approach to find the quadrupole contribution.

In a"weak-field", the general metric components for the gravitational field can be written as the classical, special relativistic term ($-\delta_{\mu\nu}$, the Kronecker delta function), plus a small perturbation, $\gamma_{\mu\nu}$; i.e., $g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$. It was shown by Einstein[9], that, if we define

$$\gamma_{\mu\nu} \equiv \gamma_{\mu\nu}' - \frac{1}{2}\delta_{\mu\nu}\gamma'_{\rho\rho} , \tag{40}$$

then

$$\gamma_{\mu\nu}'(x,y,z,t) = -\frac{4G}{c^2}\int \frac{T_{\mu\nu}(x',y',z',t-R/c)}{R}\,dv' , \tag{41}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, with the primes denoting the source coordinates, and $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{\frac{1}{2}}$. To lowest order in v/c, we have

$$T_{\mu\nu} \simeq T^{\mu\nu} = \rho \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} , \tag{42}$$

where $\rho(r')$ is the density. For a nonrotating mass distribution, we have

$$T_{44} = -\rho(r') \tag{43}$$

as the only nonzero component. In this case, Eq. (52) shows that the only nonzero component of $\gamma_{\mu\nu}'$ will be γ_{44}.

We combine Eqs. (41) and (42), and expand $^1/R$ in spherical coordinates:

$$\gamma_{44}' = \frac{4G}{c^2}\int \rho(r')\left[\frac{1}{r}\sum_{\ell=o}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos\chi)\right] dv' \quad , \tag{44}$$

where χ is the angle between a source point (r', θ', ϕ') and the observer point (r, θ, ϕ), with

$$\cos\chi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \ . \tag{45}$$

Expanding Eq. (44), we have

$$\gamma_{44}' = \frac{4G}{c^2 r}\int \rho(r')\left[1 + \left(\frac{1}{r}\right) r'\cos\gamma + \left(\frac{1}{r^3}\right)\frac{1}{2} r'^2(3\cos^2\gamma-1) + \ldots\right] r'^2\sin\theta'\, dr' d\theta' d\phi' \ . \tag{46}$$

Inserting Eq. (45) into (46) and integrating over θ $(o \to \pi)$, $(o \longrightarrow 2\pi)$, and r, we find

$$\gamma_{44}' = \frac{4GM}{c^2 r} - \frac{2G}{c^2 r^3}\left[I_x\left(1-\frac{3x^2}{r^2}\right) + I_y\left(1-\frac{3y^2}{r^2}\right) + I_z\left(1-\frac{3z^2}{r^2}\right)\right] + \frac{12G}{c^2 r^5}\left[xy\,P_{xy} + xz\,P_{xz} + yz\,P_{yz}\right] , \tag{47}$$

where $M \equiv \int \rho(r')\, dv'$, $I_i \equiv \int \rho(r')\, x_i'^2 dv'$, and $P_{ij} \equiv \int \rho(r') x_i' x_j' dv'$.

If we consider the sun to be an oblate spheroid around the z axis, then $I_x = I_y$ and $P_{ij} = 0$. The nonzero components of g_{ij} are then, to lowest order,

$$g_{11} = g_{22} = g_{33} = -1 - \frac{2GM}{c^2 r} \tag{48}$$

$$g_{44} = 1 - \frac{2GM}{c^2 r} - \frac{G}{c^2} (I_x - I_z) \frac{1}{r^3} \left(1 - \frac{3z^2}{r^2}\right) . \tag{49}$$

Confining our attention to the plane $z = 0$ ($\phi = \pi/2$), g_{44} becomes

$$g_{44} = 1 - \frac{2GM}{c^2 r} - \frac{G}{c^2} (I_x - I_z) \frac{1}{r^3} . \tag{50}$$

According to Dicke and Goldenberg[8], $I_x - I_z \approx 4 \times 10^{-5} M_\odot R_\odot^2$, and Eq. (50) can be written as

$$g_{44} \approx 1 - \frac{2m}{r} - (4 \times 10^{-5}) \frac{m}{r} \left(\frac{R_\odot}{r}\right)^2 , \tag{51}$$

where our previous notation ($G = C = 1$) has now been used.

We have carried out this analysis only to first-order in m/r, as this is all that is necessary; even if Dicke and Goldenberg are correct, the quadrupole term in Eq. (51) would only be of the order of $\sim 10^{-15} - 10^{-16}$; i.e., of order β^4 . The possibility of such a quadrupole contribution should be included in any analysis of the measured data.

IX. MODEL ORBIT CALCULATIONS

We have used a computer to calculate the time dependence and magnitude of all the terms in the second-order redshift experiment (with $\gamma = \beta = 1$), for various orbits of a heliocentric probe, such as the proposed NASA-ESRO heliocentric satellite mission (Figures 2 - 4). These calculations have been made for an earth-based station that moves in a circular orbit of radius 1 AU around the sun. Even though this model is idealized, as are the assumed probe orbits, we can still obtain a reasonable estimate of the magnitude of the various terms as the mission progresses.

These calculations show that the combination of sufficiently accurate and stable clocks, and judiciously chosen orbits, make a second-order redshift experiment quite feasible.

APPENDIX A

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = \frac{(1+Ae)}{(1+Ap')}\left[\frac{\gamma_1}{\gamma_2} - \frac{1}{2}\frac{(1-Ap')}{(1-Ae)}\right] - \frac{1}{2} \quad . \tag{A1}$$

Expanding $(1 + Ap')^{-1}$ and $(1 - Ae)^{-1}$ to fourth-order in β , we have

$$= (1+Ae)(1-Ap'+Ap'^2-Ap'^3+Ap'^4) \times$$

$$\left[\frac{\gamma_1}{\gamma_2} - \frac{1}{2}(1-Ap')(1+Ae+Ae^2+Ae^3+Ae^4)\right] - \frac{1}{2}$$

$$= (1-Ap'+Ap'^2-Ap'^3+Ap'^4+Ae-AeAp'+AeAp'^2$$

$$-AeAp'^3) \times$$

$$[(\gamma_1/\gamma_2-1) - \frac{1}{2}(-1+Ae+Ae^2+Ae^3+Ae^4$$

$$-Ap'-Ap'Ae-Ap'Ae^2-Ap'Ae^3)] - \frac{1}{2}$$

$$= (1-Ap'+Ap'^2-Ap'^3+Ap'^4+Ae-AeAp'+AeAp'^2-AeAp'^3) \times$$

$$(\gamma_1/\gamma_2-1)$$

$$+Ae(Ap'-Ae)-Ae(Ap'-Ae)^2+Ap'Ae(Ap'-Ae)^2$$

$$+Ae^3(Ap'-Ae) \quad . \tag{A2}$$

Explicitly, from Eqs. (12), (22), and (23), γ_i is given as

$$\gamma_i = [1 - \frac{2m}{r_i} + (\beta-\gamma)\frac{2m^2}{r_i{}^2} - \beta_{r_i}{}^2 - \gamma\frac{2m}{r_i}\beta_{r_i}{}^2 - r^2\beta_{\theta_i}{}^2]^{-\frac{1}{2}}$$

$$= [1 - \frac{2m}{r_i} + (\beta-\gamma)\frac{2m^2}{r_i{}^2} - \beta_i{}^2 - \gamma\frac{2m}{r_i}v_{r_i}{}^2]^{-\frac{1}{2}} , \qquad \text{(A3)}$$

where $\beta_i{}^2 \equiv \beta_{r_i}{}^2 + \beta_{\theta_i}{}^2$. (The unsubscripted β and γ are the Eddington-Robertson coefficients.)

Expanding to second order in m/r yields

$$\begin{aligned}
\frac{\gamma_1}{\gamma_2} - 1 = \frac{1}{2}\,[&\frac{2m}{r_1} - (\beta-\gamma)\frac{2m^2}{r_1{}^2} + \beta_1{}^2 + \gamma\frac{2m}{r_1}\beta_{r_1}{}^2 \\
&- \frac{2m}{r_2} + (\beta-\gamma)\frac{2m^2}{r_2{}^2} - \beta_2{}^2 - \gamma\frac{2m}{r_2}\beta_{r_2}{}^2] \\
&+ \frac{1}{8}\,[\frac{12m^2}{r_1{}^2} + \frac{12m}{r_1}\beta_1{}^2 + 3\beta_1{}^4 - \frac{4m^2}{r_2{}^2} - \frac{4m}{r_2}\beta_2{}^2 - \beta_2{}^4] \\
&- \frac{1}{4}\,[\frac{4m^2}{r_1r_2} + \frac{2m}{r_2}\beta_1{}^2 + \frac{2m}{r_1}\beta_2{}^2 + \beta_1{}^2\beta_2{}^2] \\
= (&\frac{m}{r_1} - \frac{m}{r_2}) + \frac{1}{2}(\beta_1{}^2-\beta_2{}^2) \\
&+ m\,[\gamma(\frac{\beta_{r_1}{}^2}{r_1} - \frac{\beta_{r_2}{}^2}{r_2}) + \frac{3}{2}\frac{\beta_1{}^2}{r_1} - \frac{1}{2}\frac{\beta_2{}^2}{r_2} - \frac{1}{2}(\frac{\beta_2{}^2}{r_1} + \frac{\beta_1{}^2}{r_2})] \\
&+ \frac{1}{8}\,[3\beta_1{}^4 - \beta_2{}^4 - 2\beta_1{}^2\beta_2{}^2] \\
&+ m^2\,[(\beta-\gamma)(\frac{1}{r_2{}^2} - \frac{1}{r_1{}^2}) + \frac{3}{2}\frac{1}{r_1{}^2} - \frac{1}{2}\frac{1}{r_2{}^2} - \frac{1}{r_1r_2}]. \qquad \text{(A4)}
\end{aligned}$$

The first line of Eq. (A4) is of second-order in β ; the other lines are all fourth-order, with the second line being a mixed doppler-gravitational term, the third line a pure doppler term, and the fourth line being the purely second-order (in m/r) redshift term.

Combining Eqs. (A4) and (A2), we finally obtain

$$
\begin{aligned}
\frac{\Delta\nu}{\nu}\bigg|_{\text{Earth}} = {} & \left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\beta_1{}^2 - \beta_2{}^2\right) + Ae\ (Ap' - Ae) \\
& \qquad + \\
& - (Ap' - Ae)\ \left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\beta_1{}^2 - \beta_2{}^2\right) + Ae\ (Ap' - Ae)\right] \\
& + m\ \left[\gamma\left(\frac{\beta_{r_1}{}^2}{r_1} - \frac{\beta_{r_2}{}^2}{r_2}\right) + \frac{3}{2}\frac{\beta_1{}^2}{r_1} - \frac{1}{2}\frac{\beta_2{}^2}{r_2} - \frac{1}{2}\left(\frac{\beta_2{}^2}{r_1} + \frac{\beta_1{}^2}{r_2}\right)\right] \\
& + \frac{1}{8}\ \left[3\beta_1{}^4 - \beta_2{}^4 - 2\beta_1{}^2\beta_2{}^2\right] \\
& + m^2\left[(\beta-\gamma)\left(\frac{1}{r_2{}^2} - \frac{1}{r_1{}^2}\right) + \frac{3}{2}\frac{1}{r_1{}^2} - \frac{1}{2}\frac{1}{r_2{}^2} - \frac{1}{r_1 r_2}\right] \\
& + (Ap' - Ae)\{\ Ap'\ \left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\beta_1{}^2 - \beta_2{}^2\right) + Ae(Ap' - Ae)\right] \\
& \qquad + Ae^3\}\ .
\end{aligned}
$$

APPENDIX B

The general relation for the doppler-cancelled signals received at the earth station is

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = \frac{(1-Ae')}{(1-Ap)}\left[\frac{\gamma_3}{\gamma_2} - \frac{1}{2}\frac{\gamma_3}{\gamma_1}\frac{(1-Ap')}{(1-Ae)}\right] - \frac{1}{2} \quad . \tag{B1}$$

From Eqs. (32) and (33),

$$Ae' = - Ae + \vec{\beta}_1\cdot\vec{\varepsilon} - \vec{\varepsilon}_{12}\cdot\vec{\delta} + \vec{\varepsilon}\cdot\vec{\delta} \quad , \tag{B2}$$

$$Ap = - Ap' + \vec{\beta}_2\cdot\vec{\varepsilon} \quad . \tag{B3}$$

Substituting Eqs. (B2) and (B3) into Eq. (B1), we have

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = \frac{(1+Ae) - (\vec{\beta}_1\cdot\vec{\varepsilon}-\vec{\varepsilon}_{12}\cdot\vec{\delta}+\vec{\varepsilon}\cdot\vec{\delta})}{(1+Ap') - (\vec{\beta}_2\cdot\vec{\varepsilon})}\left[\frac{\gamma_3}{\gamma_2} - \frac{1}{2}\frac{(1-Ap')}{(1-Ae)}\right] - \frac{1}{2} \quad . \tag{B4}$$

$\gamma_3 = \gamma_1$, since we have assumed that the earth moves in a circle with a constant velocity. Expanding the denominator to fourth-order yields

$$\begin{aligned}\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = {} & [(1-Ap'+Ae+Ap'^2-Ap'^3+Ap'^4-AeAp'+AeAp'^2-AeAp'^3) \\ & - (\vec{\beta}_1\cdot\vec{\varepsilon}) + (\vec{\varepsilon}_{12}\cdot\vec{\delta}) + (\vec{\beta}_1\cdot\vec{\varepsilon})Ap' - (\vec{\varepsilon}_{12}\cdot\vec{\delta})\, Ap' \\ & - (\vec{\beta}_1\cdot\vec{\varepsilon})\, Ap'^2 + (\vec{\varepsilon}_{12}\cdot\vec{\delta})\, Ap'^2 - (\vec{\varepsilon}\cdot\vec{\delta}) + (\vec{\varepsilon}\cdot\vec{\delta})\, Ap' \\ & + (\vec{\beta}_2\cdot\vec{\varepsilon}) - 2\,(\vec{\beta}_2\cdot\vec{\varepsilon})\, Ap' + (\vec{\beta}_2\cdot\vec{\varepsilon})\, Ae + 3\,(\vec{\beta}_2\cdot\vec{\varepsilon})\, Ap'^2 \\ & - 2\,(\vec{\beta}_2\cdot\vec{\varepsilon})\, AeAp' - (\vec{\beta}_2\cdot\vec{\varepsilon})\,(\vec{\beta}_1\cdot\vec{\varepsilon}) + (\vec{\beta}_2\cdot\vec{\varepsilon})\,(\vec{\varepsilon}_{12}\cdot\vec{\delta}) \\ & + (\beta_2\cdot\varepsilon)^2] \times \frac{1}{2}\left[2\frac{\gamma_3}{\gamma_2} - 1 + Ap' - Ae + AeAp'\right. \\ & \left. - Ae^2 - Ae^3 - Ae^4 + Ap'Ae^2 + Ap'Ae^3\right] - \frac{1}{2} \quad . \end{aligned} \tag{B5}$$

The explicit functional form of γ_3/γ_2 can be taken from Eq. (A4) of Appendix A; inserting this into Eq. (B5), multiplying out, and cancelling and combining terms, finally yields.

$$
\begin{aligned}
\left.\frac{\Delta\nu}{\nu}\right|_{\text{Earth}} = \; & \left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\beta_1{}^2 - \beta_2{}^2\right) + Ae(Ap'-Ae) \\
& + \frac{1}{2}\left[\, (\vec{\varepsilon}_{12}\cdot\vec{\delta}) + (\vec{\beta}_2 - \vec{\beta}_1)\cdot\vec{\varepsilon} \,\right] \\
& \qquad\qquad + \\
& - (Ap'-Ae)\left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\beta_1{}^2 - \beta_2{}^2\right) + Ae\,(Ap'-Ae)\right] \\
& - \frac{1}{2}\left[\, (\vec{\varepsilon}_{12}\cdot\vec{\delta})\, Ae + (\vec{\beta}_2\cdot\vec{\varepsilon})\, Ap' - (\vec{\beta}_1\cdot\vec{\varepsilon})\, Ae + (\vec{\varepsilon}\cdot\vec{\delta}) \,\right] \\
& + m\left[\gamma\left(\frac{\beta_{r_1}{}^2}{r_1} - \frac{\beta_{r_2}{}^2}{r_2}\right) + \frac{3}{2}\frac{\beta_1{}^2}{r_1} - \frac{1}{2}\frac{\beta_2{}^2}{r_2} - \frac{1}{2}\left(\frac{\beta_2{}^2}{r_1} + \frac{\beta_1{}^2}{r_2}\right)\right] \\
& + \frac{1}{8}\left[\, 3\beta_1{}^4 - \beta_2{}^4 - 2\beta_1{}^2\beta_2{}^2 \,\right] \\
& + m^2\left[(\beta - \gamma)\left(\frac{1}{r_2{}^2} - \frac{1}{r_1{}^2}\right) + \frac{3}{2}\frac{1}{r_1{}^2} - \frac{1}{2}\frac{1}{r_2{}^2} - \frac{1}{r_1 r_2}\right] \\
& + (Ap'-Ae)\left\{ Ap'\left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\beta_1{}^2 - \beta_2{}^2\right)\right.\right. \\
& \qquad\qquad \left.\left. + Ae\,(Ap'-Ae) + Ae^3 \right.\right\} \\
& + \left[\, (\vec{\varepsilon}_{12}\cdot\vec{\delta}) + (\vec{\beta}_2 - \vec{\beta}_1)\cdot\vec{\varepsilon}\,\right]\left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\left(\vec{\beta}_1{}^2 - \vec{\beta}_2{}^2\right)\right] \\
& + \frac{1}{2}\Big\{ - (\vec{\beta}_1\cdot\vec{\varepsilon})\left[Ae\,(2\,Ap'-Ae)\right] + (\vec{\beta}_2\cdot\vec{\varepsilon})\big[\, 2\,Ae(Ap'-Ae) \\
& \qquad\qquad + Ap'^2\big] + (\vec{\varepsilon}_{12}\cdot\vec{\delta})\left[Ae(2\,Ap'-Ae)\right] \\
& \qquad + (\vec{\varepsilon}\cdot\vec{\delta})\, Ae + (\vec{\beta}_2\cdot\vec{\varepsilon})\,(\vec{\varepsilon}_{12}\cdot\vec{\delta}) + (\vec{\beta}_2\cdot\vec{\varepsilon})\,(\vec{\beta}_2 - \vec{\beta}_1)\cdot\vec{\varepsilon} \,\Big\}
\end{aligned}
$$

APPENDIX C

The general relation for the doppler-cancelled signals received at the probe-based station is

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Probe}} = \frac{(1-Ap')}{(1-Ae)}\left[\frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_0}\frac{(1-Ae'')}{(1-Ap'')}\right] - \frac{1}{2} \quad . \tag{C1}$$

With $\vec{\beta}_2 \equiv \vec{\beta}_0 + \vec{\delta}'$ and $\vec{\varepsilon}_{12} = -\vec{\varepsilon}_{01} + \vec{\varepsilon}'$, we have

$$\begin{aligned}
Ap'' &\equiv \vec{\beta}_0 \cdot \vec{\varepsilon}_{01} \\
&= (\vec{\beta}_2 - \vec{\delta}') \cdot (-\vec{\varepsilon}_{12} + \vec{\varepsilon}') \\
&= -\vec{\beta}_2 \cdot \vec{\varepsilon}_{12} + \vec{\beta}_2 \cdot \vec{\varepsilon}' + \vec{\varepsilon}_{12} \cdot \vec{\delta}' - \vec{\varepsilon}' \cdot \vec{\delta}' \\
&= -Ap' + \vec{\beta}_2 \cdot \vec{\varepsilon}' + \vec{\varepsilon}_{12} \cdot \vec{\delta}' - \vec{\varepsilon}' \cdot \vec{\delta}' \quad .
\end{aligned} \tag{C2}$$

$$\begin{aligned}
Ae'' &\equiv \vec{\beta}_1 \cdot \vec{\varepsilon}_{01}' \\
&= -\vec{\beta}_1 \cdot \vec{\varepsilon}_{12} + \vec{\beta}_1 \cdot \vec{\varepsilon}' \\
&= -Ae + \vec{\beta}_1 \cdot \vec{\varepsilon}' \quad .
\end{aligned} \tag{C3}$$

Substituting Eqs. (C2) and (C3) into (C1), and expanding to fourth-order, yields

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{Probe}} = (1-Ap')\,(1+Ae+Ae^2+Ae^3+Ae^4)\ \times$$

$$\left[\frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_0}\,(1-Ap'+Ap'^2-Ap'^3+Ap'^4+Ae\right.$$

$$-\ AeAp'+AeAp'^2-AeAp'^3$$

$$+ (\vec{\varepsilon}_{12}\cdot\vec{\delta}') + (\vec{\beta}_2-\vec{\beta}_1)\cdot\vec{\varepsilon}'$$

$$- (\vec{\varepsilon}'\cdot\vec{\delta}') - 2\ Ap'\ (\vec{\beta}_2\cdot\vec{\varepsilon}') - 2\ Ap'\ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')$$

$$+ Ae\ (\vec{\beta}_2\cdot\vec{\varepsilon}') + Ae\ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')$$

$$+ Ap'\ (\vec{\beta}_1\cdot\vec{\varepsilon}')$$

$$+ 2\ Ap'\ (\vec{\varepsilon}'\cdot\vec{\delta}') + (\vec{\beta}_2\cdot\vec{\varepsilon}')^2 + 2\ (\vec{\beta}_2\cdot\vec{\varepsilon}')\ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')$$

$$+ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')^2 + 3\ Ap'^2\ (\vec{\beta}_2\cdot\vec{\varepsilon}')$$

$$+ 3\ Ap'^2\ (\vec{\varepsilon}_{12}\cdot\vec{\delta}') - Ae\ (\vec{\varepsilon}'\cdot\vec{\delta}')$$

$$- 2\ AeAp'\ (\vec{\beta}_2\cdot\vec{\varepsilon}') - 2\ AeAp'\ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')$$

$$- (\vec{\beta}_1\cdot\vec{\varepsilon}')\ (\vec{\beta}_2\cdot\vec{\varepsilon}') - (\vec{\beta}_1\cdot\vec{\varepsilon}')\ (\vec{\varepsilon}_{12}\cdot\vec{\delta}')$$

$$- Ap'^2\ (\vec{\beta}_1\cdot\vec{\varepsilon}')\ )\ ] - \frac{1}{2} \quad . \tag{C4}$$

The explicit form for γ_2/γ_1, can be immediately picked up from Eq. (A4) of Appendix A. To find γ_2/γ_0 explicitly, we use $\vec{r}_2 = \vec{r}_0 + \vec{\rho}$:

$$r_2^{\,2} = \vec{r}_2 \cdot \vec{r}_2 = r_0^{\,2} + 2\,\vec{r}_0 \cdot \vec{\rho} + \rho^2$$

$$= r_0^{\,2}\,\left(1 + \frac{2\,\vec{r}_0\cdot\vec{\rho}}{r_0^{\,2}} + \frac{\rho^2}{r_0^{\,2}}\right) \quad .$$

Therefore, similarly to Eq. (A4), we have

$$\begin{aligned}
\frac{\gamma_2}{\gamma_0} = {} & 1 + \left(\frac{m}{r_2} - \frac{m}{r_0}\right) + \frac{1}{2}\,(\beta_2^{\,2}-\beta_0^{\,2}) \\
& + m\,\left[\,\gamma\,\left(\frac{\beta_{r_2}^{\,2}}{r_2} - \frac{\beta_{r_0}^{\,2}}{r_0}\right) + \frac{3}{2}\,\frac{\beta_2^{\,2}}{r_2} - \frac{1}{2}\,\frac{\beta_0^{\,2}}{r_0} - \frac{1}{2}\,\left(\frac{\beta_0^{\,2}}{r_2} + \frac{\beta_2^{\,2}}{r_0}\right)\right] \\
& + \frac{1}{8}\,[\,3\beta_2^{\,4} - \beta_0^{\,4} - 2\beta_2^{\,2}\beta_0^{\,2}] \\
& + m^2\,\left[\,(\beta-\gamma)\,\left(\frac{1}{r_0^{\,2}} - \frac{1}{r_2^{\,2}}\right) + \frac{3}{2}\,\frac{1}{r_2^{\,2}} - \frac{1}{2}\,\frac{1}{r_0^{\,2}} - \frac{1}{r_2 r_0}\right]
\end{aligned} \qquad (C5)$$

$$\begin{aligned}
\approx {} & 1 - \frac{m}{r_0}\,\left[\,\frac{\vec{r}_0\cdot\vec{\rho}}{r_0^{\,2}} - \frac{\rho^2}{2r_0^{\,2}} + \frac{3}{2}\,\left(\frac{\vec{r}_0\cdot\vec{\rho}}{r_0^{\,2}}\right)^2\right] \\
& + (\vec{\beta}_0\cdot\vec{\delta}\,') + \frac{1}{2}\,(\delta'^{\,2}) \quad .
\end{aligned} \qquad (C6)$$

The fourth-order terms in Eq. (C5) do not give any additional corrections in the perturbation expansion.

Inserting Eq. (C6) and the explicit form for γ_2/γ_1 into Eq. (C4) finally yields

$$\begin{aligned}
\left.\frac{\Delta\nu}{\nu}\right|_{\text{Probe}} = {} & -\left[\,\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\,(\beta_1^{\,2}-\beta_2^{\,2})\right] - Ap'(Ap'-Ae) \\
& - \frac{1}{2}\,[(\vec{\varepsilon}_{12}\cdot\vec{\delta}\,') + (\vec{\beta}_2-\vec{\beta}_1)\cdot\vec{\varepsilon}\,'] \\
& + (Ap'-Ae)\,\left[\left(\frac{m}{r_1} - \frac{m}{r_2}\right) + \frac{1}{2}\,(\beta_1^{\,2}-\beta_2^{\,2}) + Ap'(Ap'-Ae)\right]
\end{aligned}$$

$$+ \frac{1}{2} \{ [3\, Ap'-2\, Ae] \, [(\vec{\varepsilon}_{12}\cdot\vec{\delta}') + (\vec{\beta}_2\cdot\vec{\varepsilon}')]$$

$$+ (Ae-2\, Ap') \, (\vec{\beta}_1\cdot\vec{\varepsilon}') - (\vec{\varepsilon}'\cdot\vec{\delta}')$$

$$+ \frac{m}{r_0} \, (\frac{\vec{r}_0\cdot\vec{\rho}}{r_0{}^2}) - (\vec{\beta}_0\cdot\vec{\delta}') \}$$

$$+ m \, [\, \gamma \, (\frac{\beta_{r_2}{}^2}{r_2} - \frac{\beta_{r_1}{}^2}{r_1}) + \frac{3}{2} \frac{\beta_2{}^2}{r_2} - \frac{1}{2} \frac{\beta_1{}^2}{r_1} - \frac{1}{2} \, (\frac{\beta_1{}^2}{r_2} + \frac{\beta_2{}^2}{r_1})]$$

$$+ \frac{1}{8} \, [\, 3\, \beta_2{}^4 - \beta_1{}^4 - 2\beta_1{}^2\beta_2{}^2 \,]$$

$$+ m^2 \, [(\beta-\gamma) \, (\frac{1}{r_1{}^2} - \frac{1}{r_2{}^2}) + \frac{3}{2} \frac{1}{r_2{}^2} - \frac{1}{2} \frac{1}{r_1{}^2} - \frac{1}{r_1 r_2}]$$

$$+ (Ap'-Ae) \, \{ \, Ae \, [(\frac{m}{r_1} - \frac{m}{r_2}) + \frac{1}{2} \, (\beta_1{}^2-\beta_2{}^2)$$

$$+ Ap' \, (Ap'-Ae) - Ap'^3 \, \}$$

$$- \frac{1}{2} \{ \, (\vec{\varepsilon}_{12}\cdot\vec{\delta}') \, [\, 5\, Ap' \, (Ap'-Ae) + 2\, Ae^2$$

$$+ (\, 2\, \vec{\beta}_2-\vec{\beta}_1)\cdot\vec{\varepsilon}' + \vec{\varepsilon}_{12}\cdot\vec{\delta}'] + (\vec{\varepsilon}'\cdot\vec{\delta}') \, Ap'$$

$$+ (\vec{\beta}_2\cdot\vec{\varepsilon}') \, [\, 5\, Ap'(Ap'-Ae) + 2\, Ae^2 + (\vec{\beta}_2-\vec{\beta}_1)\cdot\vec{\varepsilon}']$$

$$- (\beta_1\cdot\varepsilon') \, [\, Ae \, (Ae - Ap') + 2\, Ap'^2]$$

$$+ \frac{m}{r_0} \, [(\frac{\rho^2}{2r_0{}^2}) - \frac{3}{2} \, (\frac{\vec{r}_0\cdot\vec{\rho}}{r_0})^2 + \frac{1}{2}\delta'^2 - 2\, (Ap'-Ae) \, (\frac{\vec{r}_0\cdot\vec{\rho}}{r_0{}^2})]$$

$$- 2\, (Ap'-Ae) \, (\vec{\beta}_0\cdot\vec{\delta}') \, \}$$

CAPTIONS

Figure 2a. Distance from the sun (in AU), versus period, for an earth-launched heliocentric probe, that has a perhelion of its motion at 0.29 AU. (Period = 0.52 years)

Figure 2b. Curve a is the total redshift for an earth-launched heliocentric probe that has a perhelion of its motion at 0.29 AU, plotted as a function of the probe's period. (Actually $-\Delta\nu/\nu$ is plotted.) The infinite signal-propagation-velocity case is shown here. Curve b is the third-order(in β) terms that contribute to the total redshift. Curve c is the (negative of the) purely gravitational fourth-order contribution (i.e., the m^2 term). Some of the points at the beginning and end of the orbit have been omitted for simplicity; they either drop below the scale of the figure, or have opposite sign to the general curve.

Figure 3a. Same as Figure 2a, with perhelion = 0.5 AU. (Period = 0.65 years)

Figure 3b. Same as Figure 2b, with perhelion = 0.5 AU.

Figure 4a. Same as Figure 2a, with perhelion = 0.1 AU. (Period = 0.41 years)

Figure 4b. Same as Figure 2b, with perhelion = 0.1 AU.

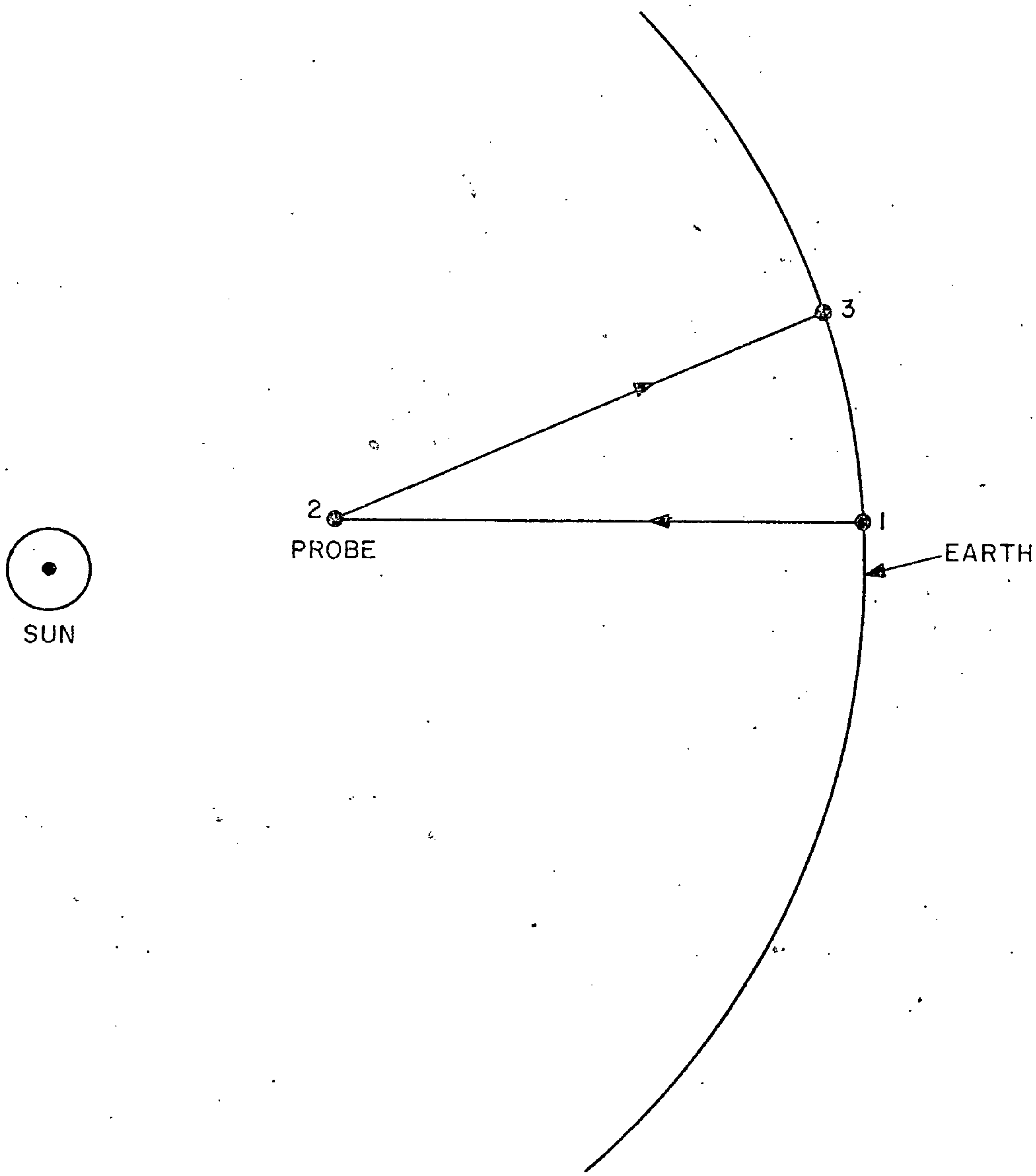


Figure 1

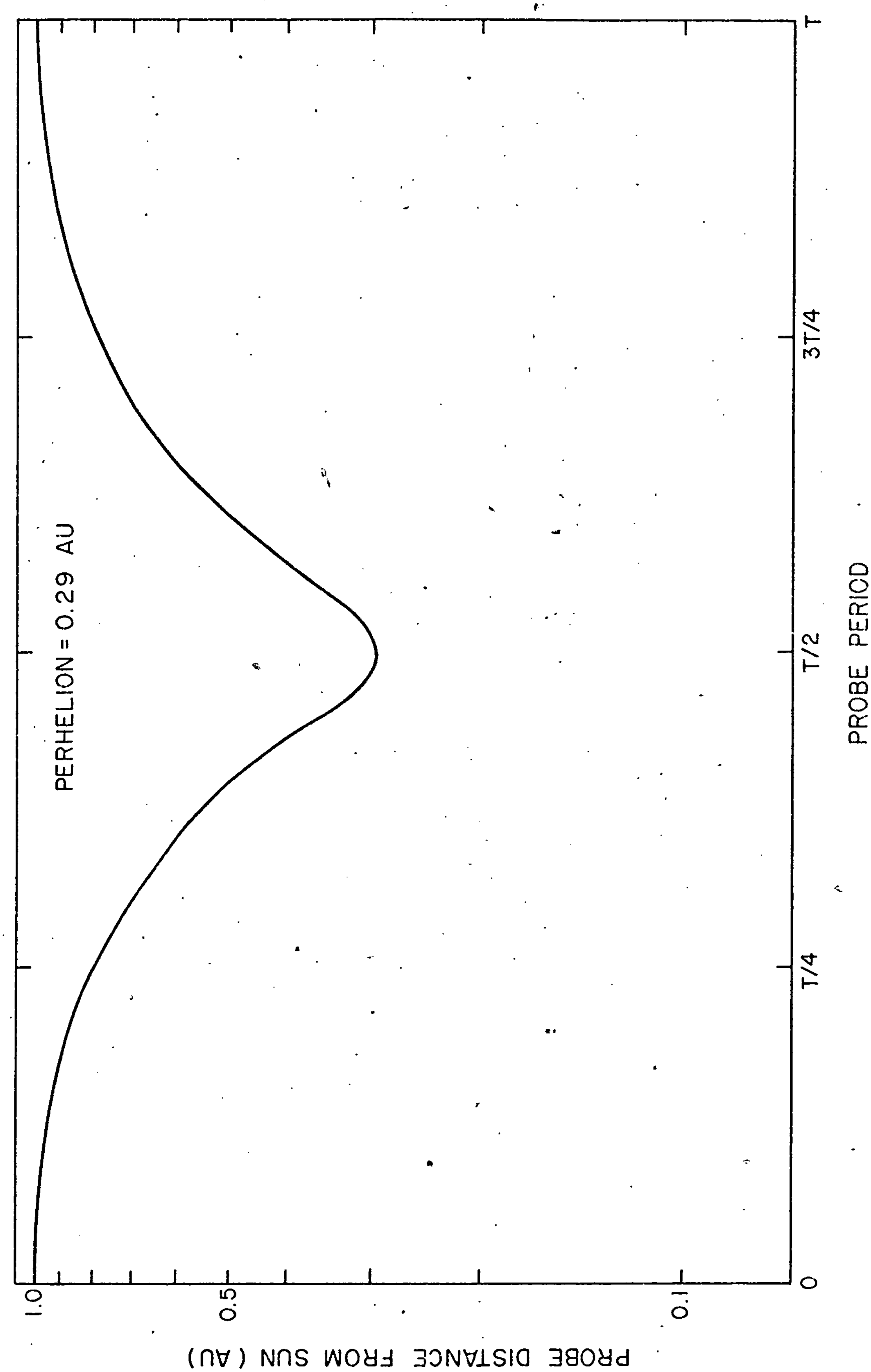

PERHELION = 0.29 AU
PROBE DISTANCE FROM SUN (AU)
1.0
0.5
0.1
PROBE PERIOD
0
T/4
T/2
3T/4
T

Figure 2a

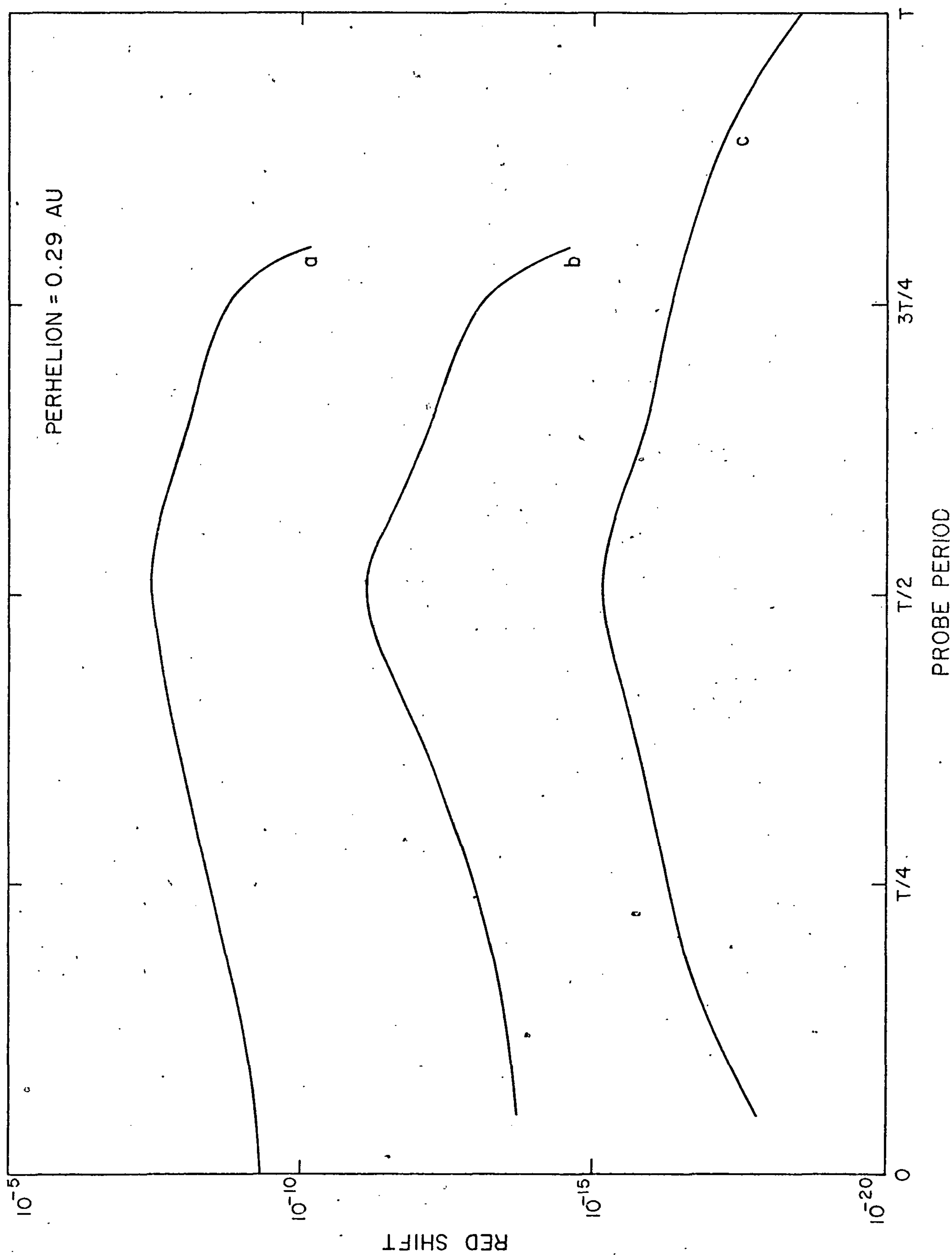

PERHELION = 0.29 AU
a
b
c
RED SHIFT
10^{-5}
10^{-10}
10^{-15}
10^{-20}
PROBE PERIOD
0
T/4
T/2
3T/4
T

Figure 2b

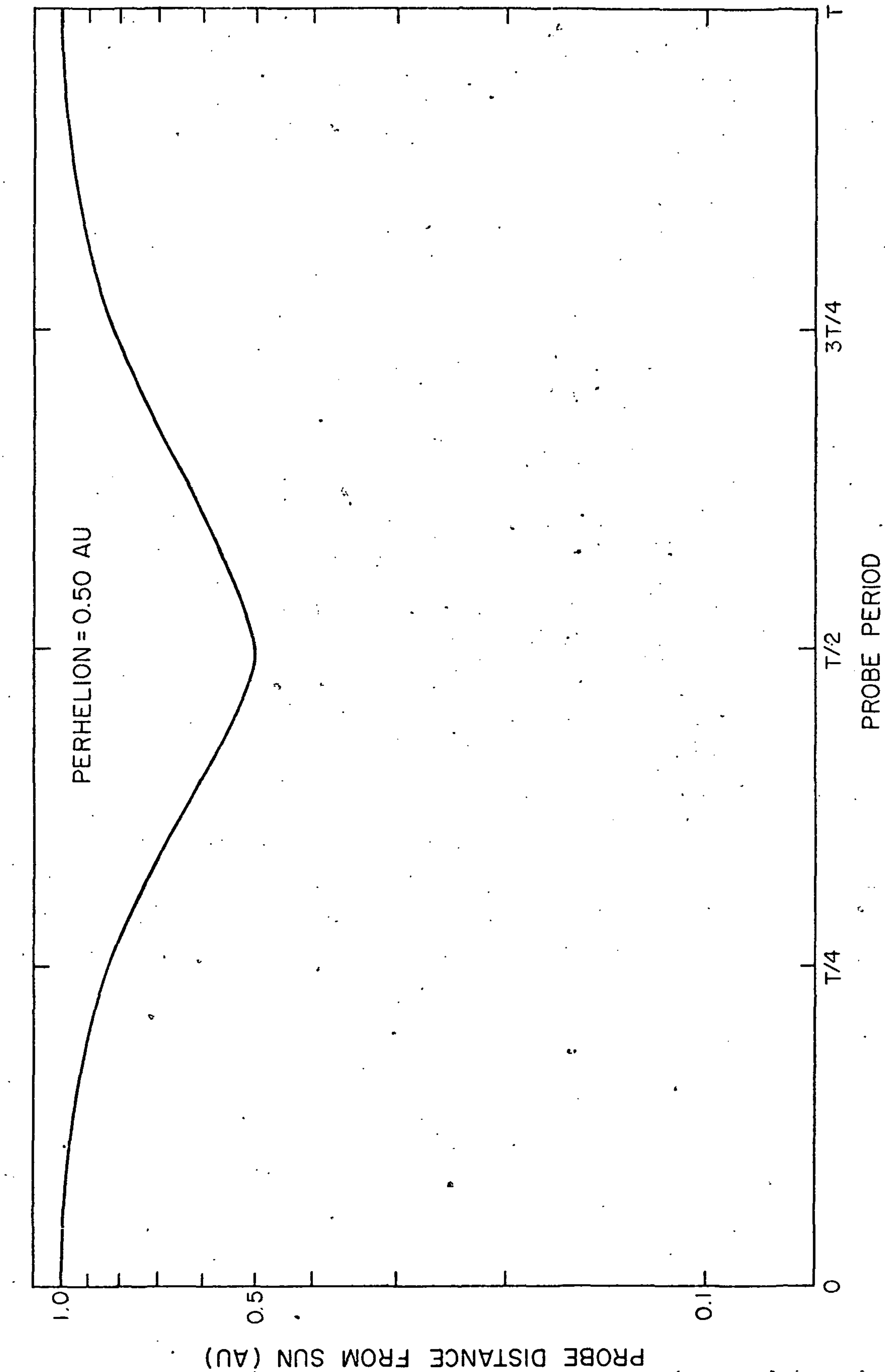

PERHELION = 0.50 AU
PROBE DISTANCE FROM SUN (AU)
1.0
0.5
0.1
PROBE PERIOD
0
T/4
T/2
3T/4
T

Figure 3a

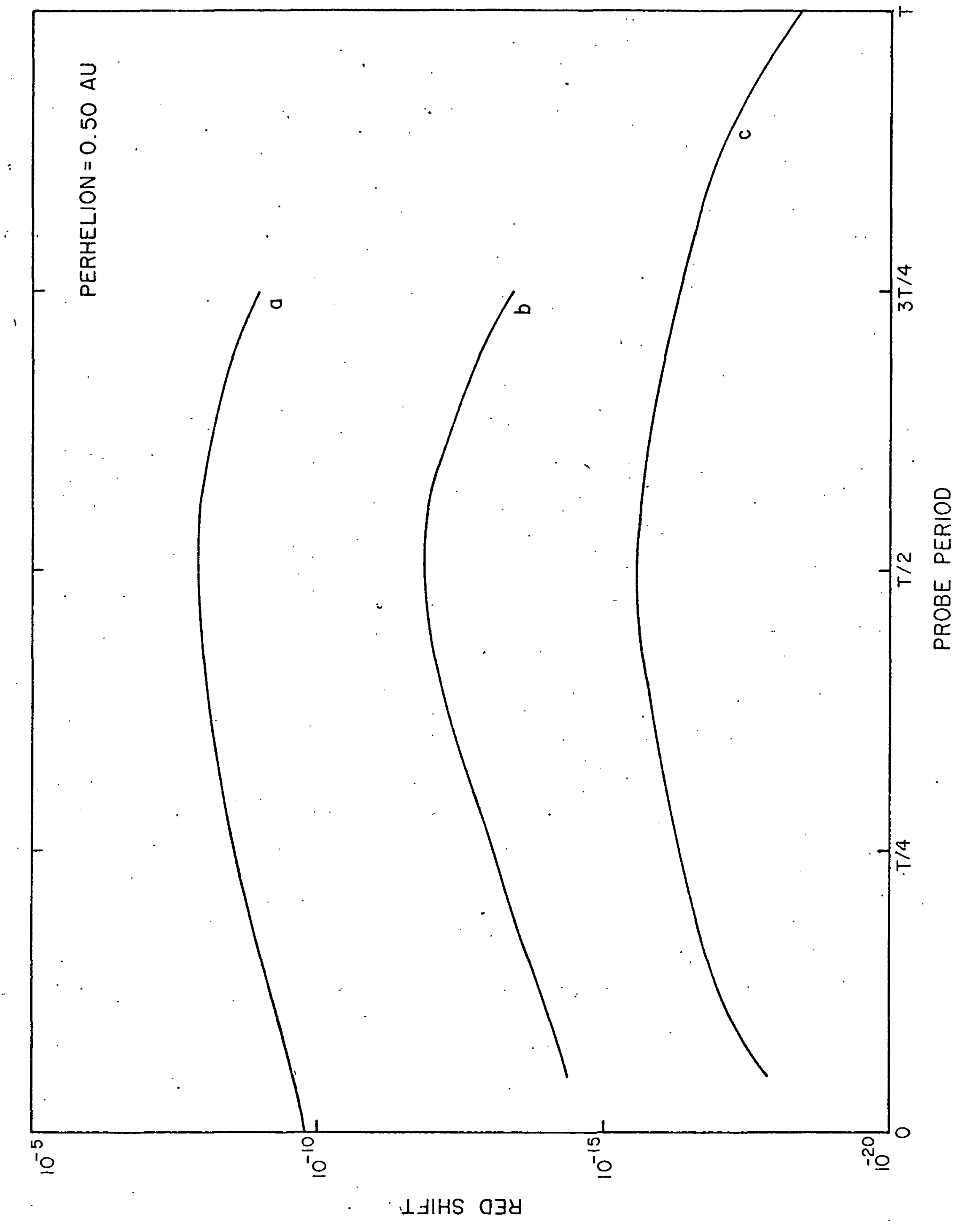

PERHELION = 0.50 AU
a
b
c
0
T/4
T/2
3T/4
T
PROBE PERIOD
10^{-5}
10^{-10}
10^{-15}
10^{-20}
RED SHIFT

Figure 3b

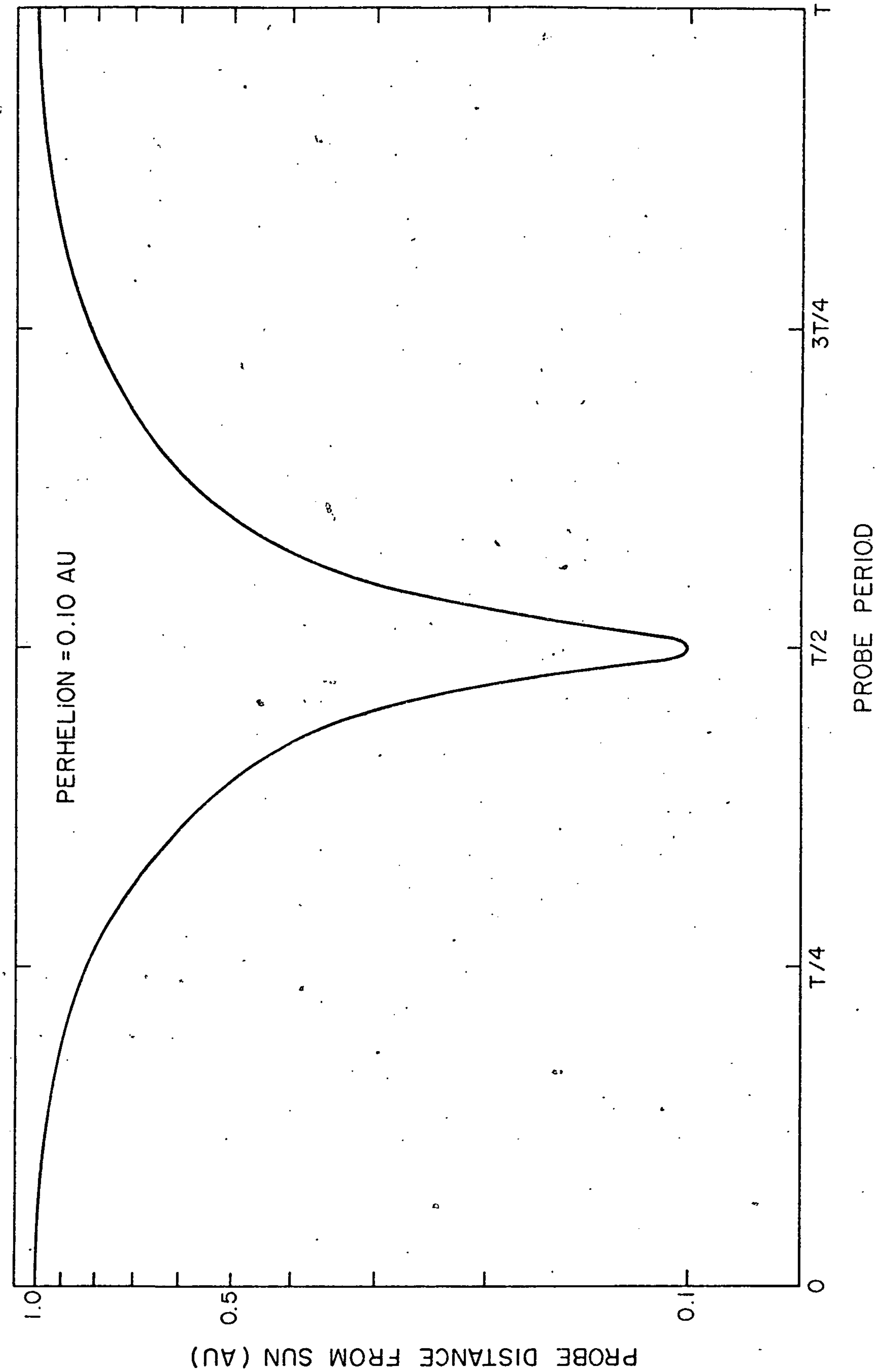
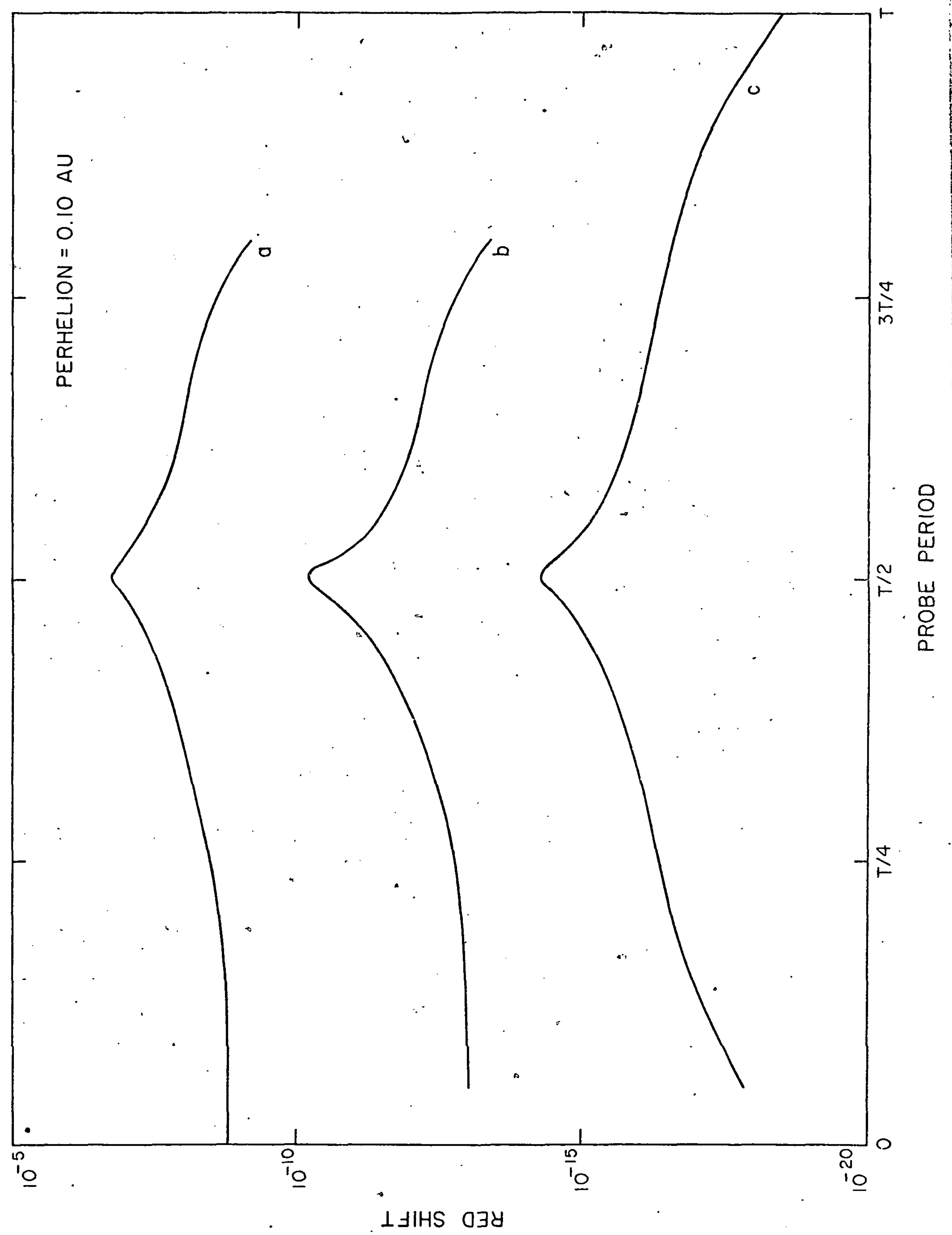

PERHELION = 0.10 AU
PROBE PERIOD
0
T/4
T/2
3T/4
T
PROBE DISTANCE FROM SUN (AU)
1.0
0.5
0.1

Figure 4a



PERHELION = 0.10 AU
a
b
c
0
T/4
T/2
3T/4
T
PROBE PERIOD
10^{-5}
10^{-10}
10^{-15}
10^{-20}
RED SHIFT

Figure 4b

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