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AN ANALYTIC STUDY ON BOUNDS FOR THE
ASSOCIATED LEGENDRE FUNCTIONS

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ABSTRACT

The conjecture is made that the largest maximum of the normalized associated Legendre function $P_{nm}(\xi)$ lies in the interval (ξ_1, ξ_2) with

$$\xi_1 = \frac{\left[\frac{m(m+4)}{4}\right] - 1}{n(n+1)}$$

$$\xi_2 = \frac{\left[\frac{m(m+4)}{4}\right]}{n(n+1)}$$

where $[\ ]$ indicates the greatest integer function. A procedure is developed for verifying this conjecture. An on-line algebraic manipulator, IAM, is used to implement the procedure and the verification is carried out for all $n \geq 2m$, for $m = 1$ through 6. A rigorous proof of the conjecture is not yet available.

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I. INTRODUCTION

In this report, we describe the results of an analytic study on bounds for the normalized associated Legendre functions P_{nm}. The motivation for the study is to provide a rational basis for the truncation of the geopotential series in spherical harmonics in various orbital analyses. One would require, in addition, bounds on the geopotential coefficients, each of which is defined as an integral involving the product of the Earth's density function and the corresponding spherical harmonic[1]. Thus, bounds on the associated Legendre functions may be expected to enter the truncation problem in two ways: directly as the functions occur explicitly in the geopotential series and indirectly as they occur in the geopotential coefficients.

The goal of the study is to find a realistic upper bound for the normalized function $P_{nm}(\cos\theta)$ over the range 0 to π of the polar angle θ. The normalization factor used in this study is

$$A_{nm} = \sqrt{(2-\delta_{mo})(2n+1)\frac{(n-m)!}{(n+m)!}} \quad , \tag{1.1}$$

where δ_{mo} is the Kronecker δ which vanishes for $m \neq 0$ and is unity for $m = 0$. The unnormalized Legendre polynomials $P_{n0}(\cos\theta)$ $(m = 0)$ are known to be bounded between ± 1 and take on their extreme values at $\theta = 0$ and π. The normalized Legendre polynomials are thus bounded by $\pm\sqrt{2n+1}$. Realistic bounds of P_{nm} for $m \geq 1$ appear not to be known for the full range of variation of θ from 0 to π.

Making use of the well known integral for the unnormalized functions,

$$\int_{-1}^{1}\left[P_{nm}(x)\right]^2 dx = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!} < 2\left[\mathrm{LUB}\,|P_{nm}|\right]^2 \tag{1.2}$$

we obtain that for the normalized function

$$\text{LUB}\,|P_{nm}(\xi)| \geq \sqrt{2-\delta_{mo}} = \sqrt{2} \tag{1.3}$$

for $m > 0$. (It will be assumed that $m > 0$ throughout the rest of the report). This is a rather weak result. Another weak result may be obtained from consideration of the integral representation of the unnormalized function:

$$P_{nm}(\cos\theta) = e^{i m \pi/2} \frac{(n+m)!}{n!\,\pi} \int_0^{\pi} [\cos\theta + i \sin\theta \cos\varphi)^n \cos m\varphi \, d\varphi \tag{1.4}$$

(Hobson[2], p. 98, Eq. (20)). Noting that the absolute value of the integrand has LUB of 1, we obtain for the normalized function

$$|P_{nm}(\cos\theta)| \leq A_{nm} \frac{(n+m)!}{n! \cdot \pi} \cdot \pi$$

$$= \sqrt{\frac{2(2n+1)(n-m)!\,(n+m)!}{(n!)^2}} \tag{1.5}$$

for all θ. This bound must be conservative since the fact that the integrand is negative over roughly half of the interval of integration is not taken into account.

Using an asymptotic formula from Jahnke-Emde[3], (p. 117), again for unnormalized functions,

$$P_{nm}(\cos\theta) = (-n)^m \sqrt{\frac{2}{n\pi \sin\theta}} \sin\left[(n+\frac{1}{2})\theta + \frac{\pi}{4} + m\frac{\pi}{2}\right] \tag{1.6}$$

valid for

$$n >> m, \quad \epsilon < \theta < \pi - \epsilon, \quad 0 < \epsilon < \pi/6 \tag{1.7}$$

we may conclude that in the interval $(\epsilon, \pi-\epsilon)$ the normalized functions satisfy

$$\text{LUB}\,|P_{nm}(\cos\theta)| \leq n^m \sqrt{\frac{4(2n+1)}{n\pi \sin\epsilon} \frac{(n-m)!}{(n+m)!}} \tag{1.8}$$

This LUB, of course, becomes unbounded at the endpoints of the interval $(0, \pi)$. Now we know that P_{nm} contains a factor $\sin^m\theta$ which vanishes at $\theta = 0$ and

$\theta = \pi$, so that the bounds for θ near either ϵ or $\pi - \epsilon$ are likely to be conservative. Hobson[2], (p. 302) derives and discusses a more elaborate asymptotic expression in which $\sin\theta$ appears in the denominator. This fact suggests that a realistic LUB corresponds to a value of θ in the neighborhood of the endpoints of the interval $(0, \pi)$ – just where the bound (1.8) is weakest.

The associated Legendre function $P_{nm}(\cos\theta)$ is proportional to the hypergeometric function

$$F(m-n, n+2m+1, m+1; \sin^2\theta) \tag{1.9}$$

and it is known (see Lanczos[4], p. 453 and p. 369) that the hypergeometries functions

$$F(-r, r+2\gamma-1, \gamma; \frac{1-x}{2}) \tag{1.10}$$

are oscillatory with the amplitude increasing as x values from 0 to 1. Hence if we make the identification

$$r = n-m, \gamma = m+1, x = \cos\theta \tag{1.11}$$

we can conclude that the largest maximum of $|P_{nm}(\cos\theta)|$ and hence the LUB is that peak of $|P_{nm}|$ closest to $\theta = 0$ (or π, since P_{nm} is either symmetric or antisymmetric about $\theta = \pi/2$, according as n-m is even or odd).

In the early stages of this investigation, we examined with some care Jahnke-Emde's[3] plots of their normalized associated Legendre functions (pp. 112-113). Their normalization factor is just twice our A_{nm}. From these plots it appears that for given n, $|P_{n1}|$ has a larger LUB than P_{nm} for $m = 2, 3, \sim n$, an observation which contrasts with bounds such as (1.5) and (1.8) which increase as m increases.

A powerful on-line algebraic manipulator, IAM, was made available by Applied Data Research, Inc.* for this investigation. A great many experiments

*Princeton, N.J.

were tried, using IAM, and the method finally adopted was to make a systematic search for the smallest θ for which the derivative of P_{nm} vanishes for various values of n and m, with the hope of ultimately completing the analysis by a proof by induction. Such a value of θ corresponds, as noted earlier, to the largest maximum of $|P_{nm}|$, and the final step would be to estimate this maximum value, again using IAM. The details of the procedure developed, the listings of the IAM programs used, and their output appear in the following sections.

The principal result of the study is the conjecture that the largest maximum of $P_{nm}(\cos\theta)$ occurs for

$$\frac{1}{n(n+1)}\left[\frac{m(m+4)-1}{4}\right]^{*} < \sin^2\frac{\theta}{2} < \frac{1}{n(n+1)}\left[\frac{m(m+4)}{4}\right] \tag{1.12}$$

for all $m \geq 2$ and all n sufficiently large - indications are that sufficiently large is not a stringent limitation. The conjecture has been verified for $m = 1, 2, 3, 4, 5, 6$; a rigorous proof is, so far, elusive.

The verification of this conjecture for $m = 2, 3, 4, 5, 6$ involves a rather intricate line of reasoning. In Section II, we introduce some notations, discuss some pertinent properties of the associated Legendre functions, and prove the three theorems basic to the verification procedure. In Section III, we treat P_{n1} as a special case, partly because it can be more completely analyzed than the general case, partly because some of the formalism developed for $m \geq 2$ is not well suited to the case $m = 1$, and partly because its relative simplicity can serve as an introduction to the more complex cases. Thus both Sections II and III may be regarded as background for Section IV. In Section IV, we describe the IAM program used in the verification of the conjecture (1.12), and present the output for $m = 2$. The quantity of output increases rapidly with m so the output for $m = 3, 4, 5$, and 6 are contained in Appendix B. (Appendix A contains the IAM program and its output for the cumbersome proof of one of the results of Section II). In Section V, estimates for bounds on the largest

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*[ ] indicates the "greatest integer" function.

maximum of P_{nm} are calculated for $m = 2, 3, 4, 5, 6$. Once more, output from an IAM program is utilized. The program is straightforward and the output fairly lengthy; both appear in Appendix C. Finally, some comparisons are made of these bounds with double precision calculations[5] of the largest maximum of P_{nm}, for selected values of n and m.

II. SOME USEFUL PROPERTIES OF THE ASSOCIATED LEGENDRE FUNCTIONS

We have seen in the previous section that the bounds on the Legendre polynomials P_{no} are known. In our analysis of bounds for the associated Legendre functions, we therefore omit the case $m = 0$. For all $m > 0$, $P_{nm}(\cos 0)$ and $P_{nm}(\cos \pi)$ vanish and the largest of the maxima of P_{nm} is that one closest to $\theta = 0$; this largest maximum is, of course, the LUB $|P_{nm}|$. It is convenient to introduce

$$\xi = \sin^2 \frac{\theta}{2} \tag{2.1}$$

as the independent variable, in terms of which the normalized associated Legendre functions are

$$P_{nm}(\xi) = B_{nm}(\xi(1-\xi))^{m/2} \sum_{k=0}^{n-m} C_k(-\xi)^k \tag{2.2}$$

where the coefficients C_k are given by

$$C_k = \frac{(n-m)!\,(n+m+k)!\,m!}{k!\,(m+k)!\,(n-m-k)!\,(n+m)!} \tag{2.3}$$

and satisfy the recursion relationship

$$\frac{C_{k+1}}{C_k} = \frac{x-(m+k)(m+k+1)}{(k+1)(m+k+1)} \tag{2.4}$$

with

$$C_0 = 1, \quad x = n(n+1) \tag{2.5}$$

The coefficient B_{nm} is

$$B_{nm} = \frac{(n+m)!}{m!\,(n-m)!} A_{nm} = \frac{1}{m} \sqrt{2(2n+1)\frac{(n+m)!}{(n-m)!}} \tag{2.6}$$

We denote the polynomial factor of P_{nm} by P_{nm}^*:

$$P_{nm}^* = \sum_{k=0}^{n-m} C_k(-\xi)^k \tag{2.7}$$

This factor defines n-m real zeroes for P_{nm} and also determines the sign of P_{nm} for any specified value of ξ. In addition to these zeroes, the function $P_{nm}(\xi)$ possesses zeroes at $\xi = 0$ and $\xi = 1$.

It is rather easily shown by direct differentiation of Eq. (2.2) that

$$P_{nm}'(\xi) = \bar{B}_{nm} (\xi(-\xi))^{m/2-1} \; \bar{P}_{nm}^{*}(\xi) \tag{2.8}$$

with

$$\bar{B}_{nm} = \frac{m}{2} B_{nm}$$

$$\bar{P}_{nm}^{*} = \sum_{k=0}^{n-m+1} \bar{C}_k (-\xi)^k \tag{2.9}$$

where

$$\bar{C}_0 = 1$$

$$\bar{C}_k = \frac{(n-m)!\,(n+m+k-1)!\,(m-1)!\,[(m+2k)y-m(m+k)(m+k-1)]}{k!\,(m+k)!\,(n-m-k+1)!\,(n+m)!} \tag{2.10}$$

for

$$1 \leq k \leq n-m+1$$

The polynomial part $\bar{P}_{nm}^{*}$ of P_{nm}' defines n-m+1 zeroes for P_{nm}', and also determines its size for a given value of ξ. We note that for m=1, P_{nm}' becomes infinite at $\xi = 0$ and $\xi = 1$; for $m > 1$ these points are zeroes as for P_{nm}.

Now we are interested in finding the smallest zero of $\bar{P}_{nm}^{*}$ and then estimating the value of P_{nm} at this point. Since P_{nm}^{*} and $\bar{P}_{nm}^{*}$ are of degree n-m and n-m+1, respectively, a complete analysis can be carried out analytically for $n-m+1 \leq 4$. In fact, a complete analysis can be carried out analytically for $n-m+1 \leq 8$, since use of the expressions for P_{nm} and P_{nm}' in terms of $\cos\theta$ would lead to the analysis of polynomials of the fourth degree (or less) in $\cos^2\theta$. In the present study, therefore, we could assume that we were seeking information in the neighborhood of the smallest zero of a polynomial containing a fairly large number of zeroes (> 8) in the interval $(0,1)$. We would thus presumably be concerned with relatively small values of the argument and might reasonably hope to gain some information from examination of the first few terms, rather than the entire polynomial.

With such an approach in mind, we introduce some useful notation and set

$$
\begin{aligned}
Pr_k(\xi) &= C_{k-1}\,\xi^{k-1} - C_k\,\xi^k \\
\bar{P}r_k(\xi) &= \bar{C}_{k-1}\,\xi^{k-1} - \bar{C}_k\,\xi^k \\
S_L(\xi) &= \sum_{k=0}^{L} C_k \cdot (-\xi)^k \\
\bar{S}_L(\xi) &= \sum_{k=0}^{L} \bar{C}_k \cdot (-\xi)^k
\end{aligned}
\tag{2.11}
$$

The functions Pr and $\bar{P}r$ are pairs of terms from the polynomial parts $P_{nm}{}^*$ and $\bar{P}_{nm}{}^*$, respectively; S and $\bar{S}$ are partial sums for these two polynomials. Using this notation, the following properties of $P_{nm}{}^*$ and $\bar{P}_{nm}{}^*$ may be established:

Property 1: The single non-vanishing zeroes ζ_k and $\bar{\zeta}_k$ of Pr_k and $\bar{P}r_k$, respectively, are increasing functions of k. Since

$$
\begin{aligned}
\zeta_k &= C_{k-1}/C_k \\
\bar{\zeta}_k &= \bar{C}_{k-1}/\bar{C}_k
\end{aligned}
\tag{2.12}
$$

this statement is proved if we can show that

$$
\frac{C_k}{C_{k+1}} - \frac{C_{k-1}}{C_k} = \frac{C_k^{\,2} - C_{k-1}\,C_{k+1}}{C_k\,C_{k+1}} > 0
$$

$$\frac{\bar{C}_k^{\,2} - \bar{C}_{k-1}\bar{C}_{k+1}}{\bar{C}_k \bar{C}_{k+1}} > 0 \tag{2.13}$$

The proof is carried out by factoring, and discarding, the numerous common (positive) factors from the numerators, discarding the (positive) denominators, and evaluating what is left subject to the condition that

$$\begin{aligned} &n-m \geq k+1 \text{ for } Pr \\ &n-m+1 \geq k+1 \text{ for } \bar{P}r \end{aligned} \tag{2.14}$$

which is necessary in order for these pairs to appear at all. The calculation for Pr is quite easy; that for $\bar{P}r$ is not, and was carried out using IAM. Both are given in Appendix A.

Property 2:

$$\begin{aligned} &S_L(\xi) > 0 \text{ for } \xi > \zeta_1, \; n-m \geq L \\ &\bar{S}_L(\xi) > 0 \text{ for } \xi < \bar{\zeta}_1, \; n-m+1 \geq L \end{aligned} \tag{2.15}$$

This property follows immediately from Property 1, since all these partial sums are unity for $\xi = 0$ and cannot change sign until after the first pair becomes negative.

Property 3: If $S_{2L}(\xi) < 0$ for some L and some $\xi < \zeta_{2L+2}$, then $P_{nm}{}^*(\xi) < 0$. Similarly if $\bar{S}_{2L}(\bar{\xi}) < 0$ for some L and some $\bar{\xi} < \bar{\zeta}_{2L+2}$, then $\bar{P}_{nm}{}^*(\bar{\xi}) < 0$. We prove only the first statement; only minor modifications are required for the proof of the second.

Proof:

$$P_{nm}{}^*(\xi) = S_{2L}(\xi) - \sum_{k=1}^{\left[\frac{n-m}{2}\right]-L} Pr_{2L+2k}(\xi) - \left(n-m-2\left[\frac{n-m}{2}\right]\right) C_{n-m}\, \xi^{n-m} \tag{2.16}$$

where $[\;]$ indicates the greatest integer function; and $n-m \geq 2L$. The partial sum S_{2L} terminates with an even power of ξ, and is followed by pairs of even index which appear preceded by minus signs in P_{nm}*. If n-m is even, there will be an integral number of such pairs; if it is odd, there will be one term left over which is an odd power of ξ and hence has a minus sign. Now since $\xi < \zeta_{2L+2}$

$$Pr_{2L+2k}(\xi) = C_{2L+2k}\left(\frac{C_{2L+2k-1}}{C_{2L+2k}} - \xi\right)\xi^{(2L+2k-1)}$$

$$= C_{2L+2k}(\zeta_{2L+2k} - \xi)\,\xi^{(2L+2k-1)} > 0 \tag{2.17}$$

Hence every term on the right hand side of Eq. (2.15) is negative and $P_{nm}{}^*(\xi) < 0$ as claimed.

Property 4: If $S_{2L+1}(\xi) > 0$ and $\xi < \zeta_{2L+3}$, then $P_{nm}{}^*(\xi) > 0$ and correspondingly for $\bar{P}_{nm}{}^*(\xi)$. The proofs are minor variations on those for Property 3.

In applying these properties, we must always keep in mind that for $P_{nm}{}^*(\xi)$, n-m, the highest power of ξ appearing in the polynomial, must not be less than the highest power of ξ occurring in a partial sum, or pair, explicitly required in the analysis. For $\bar{P}_{nm}{}^*(\xi)$, n-m+1 must not be less than the highest power used.

We are now in a position to prove the main theorems on which we base our calculation of bounds for the associated Legendre functions:

Theorem I: If for a given m, there exist L, $\bar{L}$, ξ_1, ξ_2 satisfying the following conditions:

$$\xi_1 < \bar{\zeta}_{2\bar{L}+3} \tag{2.18}$$

$$\xi_2 < \frac{2\bar{L}+1}{2\bar{L}+2}\,\zeta_{2\bar{L}+2} < \bar{\zeta}_{2\bar{L}+2} \tag{2.19}$$

$$\xi_2 < \zeta_{2L+2} \tag{2.20}$$

$$S_{2L}(\xi_2) > 0 \tag{2.21}$$

$$\bar{S}_{2\bar{L}+1}(\xi_1) > 0 \tag{2.22}$$

$$\bar{S}_{2\bar{L}}(\xi_2) < 0 \tag{2.23}$$

$$\bar{S}_{2\bar{L}}(\xi) \text{ has exactly one zero to the left of } \xi_2 \tag{2.24}$$

$$\bar{S}_{2\bar{L}}'(\xi) < 0 \text{ for } 0 < \xi < \xi_2 \tag{2.25}$$

then ξ_1 and ξ_2 bracket the largest maximum of $P_{nm}(\xi)$ for all n such that

$$n > m+2\bar{L} + 2 \tag{2.26}$$

and this largest maximum is the only extremum of P_{nm} contained in the interval (ξ_1, ξ_2).

Proof: The conditions (2.22) and (2.23) guarantee at least one zero in the interval (ξ_1, ξ_2) and this zero must therefore be the one specified by condition (2.24). Conditions (2.18), (2.19), (2.22) and (2.23), together with Properties 3 and 4 of P_{nm}', imply that

$$P_{nm}'(\xi_1) > 0 \text{ and } P_{nm}'(\xi_2) < 0 \tag{2.27}$$

and hence P_{nm} has at least one maximum point in the interval (ξ_1, ξ_2).

Now $\bar{S}_{2\bar{L}}(\xi)$ differs from $\bar{P}_{nm}{}^*(\xi)$ by a sequence of negative $\bar{P}r$ functions and, if n-m is even, the single negative term $(-\bar{C}_{n-m+1}\,\xi^{n-m+1})$:

$$\bar{P}_{nm}{}^*(\xi) = \bar{S}_{2\bar{L}}(\xi) - \sum_{k=1}^{\left[\frac{n-m+1}{2}\right]-2\bar{L}} \bar{P}r_{2\bar{L}+2k}(\xi)$$

$$-(n-m+1-2\left[\frac{n-m+1}{2}\right])\,\bar{C}_{n-m+1}\,\xi^{n-m+1} \tag{2.28}$$

Consider for a moment

$$\bar{P}r_{2J}(\xi) = \bar{C}_{2J-1}\,\xi^{2J-1} - \bar{C}_{2J}\,\xi^{2J} \tag{2.29}$$

This function possesses two zeroes; one at $\xi = 0$ and the other at $\xi = \bar{\zeta}_{2J}$ and possesses a maximum at

$$\mu_{2J} = \frac{2J-1}{2J}\,\bar{\zeta}_{2J} \tag{2.30}$$

Since both $\bar{\zeta}_{2J}$ and the fraction $(2J-1)/2J$ increase with J, μ_{2J} also increases with J. To match the parameters of the theorem, we identify J with $\bar{L}+1$, and we see that condition (2.19) guarantees that ξ_2 lies to the left of the maximum points of all $\bar{P}r$ functions in Eq. (2.27). Thus, for all ξ in the interval $(0, \xi_2)$, $\bar{P}_{nm}{}^*(\xi) - \bar{S}_{2\bar{L}}(\xi)$ is a decreasing function of ξ. Further, condition (2.25) says that $\bar{S}_{2\bar{L}}(\xi)$ is decreasing in this interval, and hence, finally, $\bar{P}_{nm}{}^*$ is also decreasing. Since $\bar{P}_{nm}{}^*(0)$ is unity and $\bar{P}_{nm}{}^*(\xi_2)$ is negative, it follows that $P_{nm}{}'(\xi)$ has exactly one zero in the interval $0 < \xi \le \xi_2$, so that $P_{nm}(\xi)$ has exactly one maximum, its largest, and no minima to the left of ξ_2. Conditions (2.20) and (2.21), together with the last statement, imply that $P_{nm}(\xi)$ has no zeroes in the interval $0 < \xi \le \xi_2$. Thus, not only do ξ_1 and ξ_2 bracket the largest maximum of $P_{nm}(\xi)$, but do so in a way that makes the following theorem for bounding this maximum useful:

Theorem II: If the conditions of Theorem I are satisfied, then $P_{nm_{max}}$, the largest maximum of $P_{nm}(\xi)$ satisfies the inequalities

$$A < P_{nm_{max}}(\xi) < B \tag{2.31}$$

where

$$A = \sqrt{2(2n+1)x}\;(\xi_1(1-\xi_1))^{m/2}\,P_{nm}{}^*(\xi_2) \tag{2.32}$$

$$B = \sqrt{2(2n+1)x}\;(\xi_2(1-\xi_2))^{m/2}\,P_{nm}{}^*(\xi_1)$$

with

$$x = n(n+1), \quad n > m + 2\bar{L} + 2 \tag{2.33}$$

Proof: We know that the polynomial part, $P_{nm}^*(\xi)$, of $P_{nm}(\xi)$ is an oscillatory function with n-m real zeroes in the interval (0,1), and hence n-m-1 extrema interlaced with the zeroes in this interval. Since $P_{nm}^*(0) = 1$, and Theorem I guarantees that no zeroes of P_{nm}^* lie to the left of ξ_2, it follows that $P_{nm}^*(\xi)$ is decreasing in the interval $(0, \xi_2)$, and hence also in the interval (ξ_1, ξ_2), so that

$$P_{nm}^*(\xi_1) > P_{nm}^*(\xi_2) \tag{2.34}$$

Further, since the function $(\xi(1-\xi))^{m/2}$ vanishes at $\xi = 0$ and $\xi = 1$, and has its only maximum at $\xi = 1/2$, it is an increasing function in the interval (ξ_1, ξ_2), so long as $\xi_2 < 1/2$, a condition implied by the conditions of Theorem I. Thus,

$$(\xi_1(1-\xi_1))^{m/2} < (\xi_2(1-\xi_2))^{m/2} \tag{2.35}$$

The inequalities (2.34) and (2.35) establish the conclusion of the theorem.

We need one more theorem to make conditions (2.24) and (2.25) of Theorem I more explicit:

Theorem III: Given (1) a polynomial of even degree 2K

$$R_{2K}(r) = \sum_{k=0}^{K} a_k r^k \tag{2.36}$$

with coefficients such that

$$\begin{aligned} a_k &> 0 \quad k \text{ even} \\ a_k &< 0 \quad k \text{ odd} \end{aligned} \tag{2.37}$$

and (2) $\bar{r}$ such that

$$b_0 = \sum_{j=0}^{2K} a_j \bar{r}^j < 0$$

$$b_1 = \sum_{j=0}^{2K-1} a_{j+1} \bar{r}^j (j+1) < 0$$

$$b_2 = \sum_{j=0}^{2K-2} a_{j+2} \bar{r}^j \frac{(j+1)(j+2)}{2!} > 0 \tag{2.38}$$

$$\vdots$$

$$b_\ell = \sum_{j=0}^{2K-\ell} a_{j+\ell} \bar{r}^j \frac{(j+\ell)!}{\ell!j!} \begin{matrix} > 0 & \ell \text{ even} \\ < 0 & \ell \text{ odd} \end{matrix}$$

$$\vdots$$

$$b_{2K} = a_{2K} > 0$$

then, in the interval $(0, \bar{r})$, $R_{2K}(r)$ is decreasing and has exactly one zero to the left of $\bar{r}$.

Proof: It is readily verified that the polynomial $\bar{R}_{2K}(\rho)$ defined by

$$\bar{R}_{2K}(\rho) = R_{2K}(\bar{r} + \rho) \tag{2.39}$$

has coefficients b_j; that is,

$$\bar{R}_{2K}(\rho) = \sum_{j=0}^{2K} b_j \rho^j \tag{2.40}$$

From Descartes' rule of signs, there is exactly one negative value of ρ for which $\bar{R}_{2K}$ vanishes and hence exactly one value of $r < \bar{r}$ for which $R_{2K}(r)$ vanishes. This establishes the second conclusion of the theorem.

The derivative of $\bar{R}_{2K}(\rho)$ is given by:

$$\bar{R}_{2K}'(\rho) = \sum_{j=0}^{2K} j b_j \rho^j \tag{2.41}$$

and from the hypotheses on the b's, the coefficients of $\bar{R}_{2K}'(\rho)$ alternate in sign (b_0 does not appear). Hence again using Descartes' rule of signs, $\bar{R}_{2K}'(\rho)$ has no negative zeroes and is therefore always negative: $\bar{R}_{2K}'(0) = b_1 < 0$. This implies that in the interval $(0, \bar{r})$, $R_{2K}(r)$ decreases from $R_{2K}(0) = a_0 > 0$ to $R_{2K}(\bar{r}) = b_0 < 0$. This completes the proof of Theorem III.

In the application of Theorem III to Theorem I, we identify $R_{2K}(r)$ with $\bar{S}_{2\bar{L}}(\xi)$ and $\bar{r}$ with ξ_2. It now remains to discuss how we evaluate the parameters L, $\bar{L}$, ξ_1, ξ_2, on which Theorem I depends. As mentioned in the introduction, extensive experimentation with IAM led to the conjecture that

$$\xi_1 = \frac{\left[\frac{m(m+4)}{4} - 1\right]}{n(n+1)} \tag{2.42}$$

$$\xi_2 = \frac{\left[\frac{m(m+4)}{4}\right]}{n(n+1)}$$

$$[\ ] = \text{greatest integer function}$$

will satisfy all the required conditions. To arrive at values of L and $\bar{L}$, one simply starts with a small value of $\bar{L}$, say 1, and checks to see if ξ_1 and ξ_2, defined by (2.42) satisfy all the conditions of Theorem I; if not, increment $\bar{L}$ and check again. In all cases tested ($m = 2, 3, 4, 5, 6$), an $\bar{L}$ has been found and appears to be of the order of $2m$. The same trial and error technique is then used to find L. What is lacking is a rigorous proof for the existence of $\bar{L}$ and L for ξ_1 and ξ_2 defined by Eq. (2.42)

III. CALCULATION OF BOUNDS FOR P_{n1}

Using the notation $x = n(n+1)$ and Eqs. (2.1) to (2.9), expressions for $P_{n1}(\xi)$ may be obtained as

$$P_{n1}(\xi) = \sqrt{2(2n+1)\cdot x}\left[1 - \frac{x-2}{2}\xi + \frac{(x-2)(x-6)}{2!\cdot 3!}\xi^2 \ldots\right] \text{times} \sqrt{\xi(1-\xi)} \tag{3.1}$$

$$P_{n1}'(\xi) = \frac{1}{2}\sqrt{2(2n+1)x}\left[1 - \frac{3x-2}{2}\xi + \frac{(x-2)(5x-6)}{2!\cdot 3!}\xi^2\right]\frac{1}{\sqrt{\xi(1-\xi)}} \tag{3.2}$$

We note at once from Property 4 of the last section that

$$P_{n1}(\xi) > 0 \text{ for } \xi < \frac{2}{x-2} = \zeta_1$$

$$P_{n1}'(\xi) > 0 \text{ for } \xi < \frac{2}{3x-2} = \bar{\zeta}_1 < \zeta_1 \tag{3.3}$$

so that we may identify ξ_1 of Theorem I with ζ_1:

$$\xi_1 = \frac{2}{3x-2} \tag{3.4}$$

For this case (m=1), it is quite easy to find a value for ξ_2 with $\bar{L} = 1$, so that the results will hold for $n \geq 2$. A near-minimum value for ξ_2 is obtained by seeking the largest α for which

$$\bar{S}_2(\xi_2) < 0 \text{ with } \xi_2 = \frac{1}{x+\alpha} \tag{3.5}$$

Substitution of ξ_2 in the first three terms of Eq. (3.2) for $P_{n1}'(\xi)$ and clearing fractions yields

$$12(x+\alpha)^2 - 6(3x-2)(x+\alpha) + (x-2)(5x-6)$$
$$= -x^2 + x(6\alpha - 4) + 12(\alpha^2 + \alpha + 1) < 0 \tag{3.6}$$

as the inequality to be satisfied by α for all $n \geq 2$, which implies $x \geq 6$. Trial and error shows that $\alpha \leq 3/4$ satisfies the condition, while $\alpha = 7/8$ is too large.

We next verify that for $\alpha \le \frac{3}{4}$, $\xi_2 > \xi_1$:

$$\xi_2 - \xi_1 = \frac{1}{x+\alpha} - \frac{2}{3x-2} = \frac{x-2(\alpha+1)}{(x+\alpha)(3x-2)} > 0 \tag{3.7}$$

provided $x > 2(\alpha+1)$, which is so since $x > 6$ and $2(\alpha+1) < 2(1+\frac{7}{8}) = \frac{15}{4}$. Note that $\alpha = 6$ corresponds to the value of ξ_2 given by the conjecture of the last section; for $\alpha = 6$

$$\xi_2 = \frac{1}{x} = \frac{1}{x}\left[\frac{m(m+4)}{4}\right]_{m=1} \tag{3.8}$$

We are now ready to estimate bounds: The polynomial part of P_{n1} is decreasing in the interval $(0, \xi_2)$ since $\xi_2 < \zeta_1$. Thus, in the interval (ξ_1, ξ_2) $P_{n1}{}^*$ has its maximum value at ξ_1 and its minimum value at ξ_2. The other factor of P_{n1}, $\sqrt{\xi(1-\xi)}$, increases in the interval $(0, 1/2)$ and hence also in the interval (ξ_1, ξ_2). Finally, therefore, the lower and upper bounds for the largest maximum of P_{n1} are given by:

$$A = \sqrt{2(2n+1)x}\ \sqrt{\xi_1(1-\xi_1)}\ P_{n1}{}^*(\xi_2)$$

$$B = \sqrt{2(2n+1)x}\ \sqrt{\xi_2(1-\xi_2)}\ P_{n1}{}^*(\xi_1) \tag{3.9}$$

respectively. We now expand the various factors in Eq. (3.9) in powers of $1/x$:

$$\sqrt{x\xi_1(1-\xi_1)} = \sqrt{\frac{2x(3x-4)}{(3x-2)^2}} = \frac{\sqrt{6}}{3}\left(1-\frac{2}{x}\right)\left(1+\frac{2}{x}\right) + O\left(\frac{1}{x^2}\right) = \frac{\sqrt{6}}{3} + O\left(\frac{1}{x^2}\right) \tag{3.10}$$

$$\sqrt{x\xi_2(1-\xi_2)} = \sqrt{\frac{x(x+\alpha-1)}{(x+\alpha)^2}} = \left(1+\frac{\alpha-1}{x}\right)\left(1-\frac{\alpha}{x}\right) + O\left(\frac{1}{x^2}\right) = 1 - \frac{1+\alpha}{x} + O\left(\frac{1}{x^2}\right) \tag{3.11}$$

$$P_{n1}^*(\xi_1) = 1 - \frac{x-2}{2}\,\frac{2}{3x-2} + \frac{(x-2)(x-6)}{12}\left(\frac{2}{3x-2}\right)^2 = \frac{19}{27} + \frac{16}{81x} + O\left(\frac{1}{x^2}\right) \tag{3.12}$$

$$P_{n1}^*(\xi_2) = \frac{7}{12} + \frac{1+\alpha}{3x} + O\left(\frac{1}{x^2}\right) \tag{3.13}$$

Using these results, A and B become

$$A = \sqrt{2(2n+1)}\,\frac{\sqrt{6}}{3}\left(\frac{7}{12} + \frac{1+\alpha}{3x}\right) + O\left(\frac{1}{x^2}\right) \sim \sqrt{n+\frac{1}{2}}\,\frac{7\sqrt{6}}{18}$$

$$B = \sqrt{2(2n+1)}\,\frac{19}{27}\left(1 - \frac{1+\alpha}{2x} + \frac{16}{57x}\right) + O\left(\frac{1}{x^2}\right) \sim \frac{38}{27}\sqrt{n+\frac{1}{2}} \tag{3.14}$$

In order to see that A and B not only bound the first maximum of P_{n1}, but also provide reasonably realistic bounds, we tabulate A, B, and the largest maximum of P_{n1} calculated in double precision [5] for selected values of n:

n	A	$P_{n1_{max}}$	B
2	1.51	1.94	2.23
6	2.43	2.99	3.59
10	3.09	3.78	4.56
14	3.63	4.44	5.36
18	4.10	5.01	6.05

(Only the first three digits of the double precision values are used for this table). The bounds, for this range of values for n, deviate some 15-20% from the exact value, which does not seem excessive for bounds intended to be valid for <u>all</u> n. Note, however, that the deviations of the bounds from the exact value appear to increase with n; there is, at present, nothing in the theory on which to base an estimate of a rate of change in the deviations.

IV. LOCATING THE LARGEST MAXIMUM OF P_{nm}

In this section, we first describe the procedure developed for verifying the conjecture (2.42), then present the IAM program implementing the procedure, and then the output for m = 2. Because of its bulk, the output for m = 3, 4, 5, 6 is placed in Appendix C.

The problem is to verify, for a sequence of values of m, that the largest maximum of $P_{nm}(\xi)$ lies at a point $\bar{\xi}$ such that

$$\xi_1 = \frac{z_1}{x} < \bar{\xi} < \xi_2 = \frac{z}{x} \tag{4.1}$$

with

$$x = n(n+1)$$
$$z = \left[\frac{m(m+4)}{4}\right] \tag{4.2}$$
$$z_1 = z - 1$$

The steps of the procedure are as follows:

1. Specify m, and hence z and z_1.
2. Find $\bar{L}_1$ the minimum value of $\bar{L}$ for which

$$\bar{S}_{2\bar{L}} < 0; \text{ note } n \geq m + \bar{L} \tag{4.3}$$

3. Calculate the coefficients of u in the polynomial $\bar{T}_{2\bar{L}_1}(u)$ defined by

$$\bar{T}_{2\bar{L}_1}(u) = \bar{S}_{2\bar{L}_1}\left(\frac{z}{x} + u\right) \tag{4.4}$$

using Eqs. (2.9), for the coefficients $\bar{C}$ of $\bar{S}$, and Theorem III. Check to see if conditions (2.28) of Theorem III are satisfied. If not, increase $\bar{L}_1$ by 1 and repeat.

4. Verify that $\bar{S}_{2\bar{L}_1+1}(\xi_1) > 0$.

5. Verify that ξ_1 and ξ_2 satisfy the various other conditions imposed by Theorem I for $n \geq m + \bar{L}_1$.

6. Verify that an L exists for which condition (2.23) is satisfied.

The IAM program below, called WORK1, implements the first four steps of this procedure; the last two are done by "hand". IAM utilizes an ALGOL-like language. It executes PARTS of a program and/or STEPS within the parts. The individual command strings are identified in a "decimal" notation: PART.STEP, eg. 4.3, 5.11, etc. Note that STEP 5.11 would be executed between STEPS labeled 5.1 and 5.2. The commands are almost self-explanatory; $\uparrow$ denotes exponentiation, $\leftarrow$ is a replacement operator, FOR generates a loop, WITH indicates substitution. All arithmetic is carried out in integer form to full precision, using as many "words" as necessary to store the result. There are, of course, limits to the storage capability, and when the address space of the system is exhausted, a message to this effect is returned and calculation stops. WORK1 successfully carried out the first four steps of the procedure outlined above for m = 2, 3, 4, 5, and 6. Time did not permit further trials, and most likely storage would be exhausted for m = 7 or 8; considerable care in conserving storage had to be exercised to carry through m = 6, for which 14 terms of the P_{nm}' series were required.

With these preliminaries, here is WORK1:

```
 TYPE WORK1
1.1:I2←I1-1;Z1←Z-1;TYPE M,I1,I2,Z,Z1
1.2:MSQ←M↑2
1.3:TWOM←2*M
1.4:MM(1)←MSQ+M
1.5:FOR I←2 TO I2,MM(I)←MM(I-1)+TWOM+2*(I-1)
1.6:MSQ←2*MSQ
1.7:FOR I←1 TO I2,MMM(I)←(1+I)*M+(I+1)*(I+1)
4.1:Q(0)←M;FOR I←1 TO I1,Q(I)←Q(I-1)*I*(M+I);&
CP(0)←Q(I1);DELETE Q
4.2:CP(1)←CP(0)/MM(1)
4.3:FOR I←2 TO I1,(CP(I)←(CP(I-1)/MMM(I-1))*(X-MM(I-1))&
;DELETE MMM(I-1))
```

```
5.1:MMM(1)←M*MM(1);MMM(2)←MSQ+TWOM
5.15:DELETE MM
5.2:FOR I←3 TO I1,MMM(I)←MMM(I-1)+TWOM
5.3:Y(1)←(M+2)*X-MMM(1);DELETE MMM(1)
5.4:FOR I←2 TO I1,(Y(I)←Y(I-1)+2*X-MMM(I)&
;DELETE MMM(I))
5.6:FOR I←1 TO I1,(CP(I)←CP(I)*Y(I)*(-1)↑I;&
DELETE Y(I))
6.1:D(0)←1;D(1)←X
6.2:FOR I←2 TO I1,D(I)←D(1)*D(I-1)
6.3:FOR I←0 TO I1,CP(I)←D(I1-I)*CP(I)
6.4:DELETE D
8.1:$DIST←15
8.3:X2←U+(I2+M)*(I2+M-1)
8.4:X1←U+(I1+M)*(I2+M)
8.5:AI1←SUM(I←0 TO I1:CP(I)*Z1↑I)
8.6:ODD←AI1 WITH [X=X1];TYPE ODD
8.65:DELETE AI1,ODD
8.7:AI2←SUM(I←0 TO I2:CP(I)*Z↑I)
8.8:TEST(0)←AI2 WITH [X=X2];TYPE TEST(0)
8.9:DELETE TEST(0),AI2
9.1:FOR J←1 TO I2,IP(J)←J
9.2:T←SUM(I←1 TO I2:(IP(I)*Z↑(I-1))*CP(I))
9.3:TEST(1)←T WITH [X=X2]
9.4:TYPE TEST(1);DELETE TEST(1)
9.5:FOR K←2 TO (I2-1),(FOR J←K TO I2,IP(J)←IP(J)*&
(J-K+1)/K;T←SUM(I←K TO I2:(IP(I)*Z↑(I-K))*CP(I));&
TEST(K)←T WITH [X=X2];TYPE TEST(K);DELETE TEST(K),T)
9.6:DELETE ALL VALUES
10.1:C(0)←CP(0)/X↑I1
10.11:FOR I←1 TO I2,C(I)←COEFF(CP(I),X,I1)
10.12:TST1(0)←SUM(I←0 TO I2:C(I)*Z↑I);TYPE TST1(0)
10.13:IF TST1(0).GT.0,(I1←I1+2;DELETE TST1;TO STEP 21.1)
10.15:FOR J←1 TO I2,IP(J)←J
10.2:T←SUM(I←1 TO I2:(IP(I)*Z↑(I-1))*C(I))
10.3:TST1(1)←T
10.4:TYPE TST1(1)
10.5:FOR K←2 TO (I2-1),(FOR J←K TO I2,IP(J)←IP(J)*&
(J-K+1)/K;T←SUM(I←K TO I2:(IP(I)*Z↑(I-K))*C(I));&
TST1(K)←T;TYPE TST1(K);IF (TST1(K)*(-1)↑K).LT.0,(I1←I1+2;&
DELETE TST1;TO STEP 21.1))
10.8:DELETE C,T,TST1
10.9:DO PART 8;DO PART 9
21.1:I2←I1-1
21.3:DO PART 1;DO PART 4
21.4:DO PART 5;DO PART 6;TO STEP 10.1
```

Execution of the program is accomplished by entering the command DO PART 21 from the terminal; this must be preceded by entry of m, z, and an initial value for an odd integer I1. The even integer I2 = I1 - 1 plays the role of 2L in step 2 of the procedure outlined above. Note that PART 21 executes PARTS 1, 4, 5, and 6 and then skips to STEP 10.1. PARTS 1, 4, 5, and 6 evaluate an array CP of (I1+1) coefficients which are proportional to the first (I1+1) terms of the polynomial part of $P_{nm}'(\xi)$. The proportionality factor is the least common denominator of these coefficients, including the power of x which will occur on substitution of ξ_1 or ξ_2 for the argument. IAM works more efficiently if it is not asked to carry fractions, since its first step is to get the common denominator, which requires storage and is carried throughout the calculation.

Referring to Eqs. (2.8), (2.9) and (4.1), we note first that

$$\bar{C}_0 = 1$$

$$\bar{C}_1 = \frac{(m+2)x - m^2(m+1)}{m(m+1)} \tag{4.5}$$

$$\bar{C}_2 = \frac{[x-m(m+1)]\cdot[(m+4)x-m(m+1)(m+2)]}{2!\, m(m+1)(m+2)}$$

$$\vdots$$

$$\bar{C}_{I1} = \frac{\{[x-m(m+1)][x-(m+1)(m+2)]\cdots[x-(m+I1-2)(m+I1-1)]\cdot[(m+2I1)x-m(m+I1-1)(m+I1)]\}}{I1!\, m(m+1)\cdots(m+I1)}$$

and the corresponding terms in $\bar{P}_{nm}{}^*$ are these coefficients multiplied by the corresponding powers of z/x or z_1/x for evaluation at ξ_1 and ξ_2 respectively. In either case, the lowest common denominator of the partial sum $\bar{S}_{I1}$, which serves also as a common denominator for $\bar{S}_{I2}$, will be $I1!\, m(m+1)\cdots(m+I1)x^{I1}$ and CP(I) at the end of PART 6 of WORK1 is this factor times the corresponding $\bar{C}_I$ of Eq. 4.5:

$$CP(0)=I1!\, m(m+1)\cdots(m+I1)\, x^{I1} \tag{4.6}$$

$$CP(1)=-I1!\,(m+2)(m+3)\cdots(m+I1)x^{I1}[(m+2)x-m^{2}(m+1)]$$

$$\vdots$$

$$CP(I1) = (-1)^{I1}(x-m(m+1))(x-(m+1)(m+2))\cdots$$

$$(x-(m+I1-2)(m+I1-1))((m+2I1)x-m(m+I1-1)(m+I1))$$

PART 1 and STEPS 4.1 and 4.2 set up the factors independent of x in these expressions. STEP 4.3 calculates all but the power of x and the factor involving (m+2I) 4. The array Y is set up in STEPS 5.3 and 5.4 to generate this last factor and STEP 5.6 inserts it along with the proper sign into the CP's. Finally, in PART 6 the array D generates the powers of x which are inserted into the CP's by STEP 6.3.

Having generated expressions for the terms of the partial sums, we next want to find $\bar{L}_1$ of Eq. (4.3). The search is done by successively calculating

$$\bar{S}_{I2} \sum_{k=0}^{I2} CP(k)\, z^{k} \tag{4.7}$$

for I2 = 2, 4, 6, ... until a negative value results. In practice, this is a very lengthy calculation which strains the storage capacity of IAM, so a similar preliminary search is carried out using only the leading terms in x. Note from Eq. (4.6) that each CP has leading term x^{I1} and that x = n(n+1) can be arbitrarily large. Thus, in effect, we first seek an $\bar{L}_1$ for very large x. This is accomplished in STEP 10.11 by constructing an auxiliary array C(I) from the coefficients of x^{I1} in CP(I). The rest of PART 10 calculates and prints out

$$TST1(0) = \sum_{k=0}^{I2} C(k)\, z^{k} \tag{4.8}$$

incrementing I2 by 2 and repeating the calculation until a negative value results. At this point the test is undertaken for zeroes of this sum to the left of z, using Theorem III. If more than one zero is detected, I1, and hence also I2, is incremented by 2. The sequence of tests for Theorem III requires

$$\begin{aligned}
&TST\,1(0) < 0 \\
&TST\,1(1) = \sum_{k=1}^{I2} k\,C(k)\,z^{k} < 0 \\
&TST\,1(2) = \sum_{k=2}^{I2} \frac{k(k-1)}{1.2}\,C(k)\,z^{k} > 0 \\
&\vdots \\
&TST\,1(I2-1) = -\,C(I2-1)\,z^{I2-1} + I2\,C(I2)\,z^{I2} < 0 \\
&TST\,1(I2) = C(I2)\,z^{I2} > 0
\end{aligned} \tag{4.9}$$

The last of these tests does not appear in the output. Once these tests are passed for the leading terms in x, the value of I2 so obtained is used as the initial value for the corresponding sequence of tests using the complete coefficients of Eq. (4.6), with the following modification:

First, note that use of I2 for $\bar{P}_{nm}{}^*$ requires

$$n-m+1 \geq I2 \tag{4.10}$$

and hence

$$x \geq (m+I2-1)(m+I2) \tag{4.11}$$

WORK 1 defines

$$\begin{aligned}
x2 &= (m+I2)(m+I2-1) + u \\
x1 &= (m+I1)(m+I2) + u
\end{aligned} \tag{4.12}$$

with $u \geq 0$ to incorporate the fact that $u = 0$ corresponds to the minimum value of $n(n+1)$ consistent with the existence of $\bar{S}_{I2}$ and $\bar{S}_{I1}$, with I2 corresponding to $2\bar{L}$ and I1 to $2\bar{L}+1$ of Theorem I. In STEPS 8.5 and 8.6, IAM first evaluates $\bar{S}_{I1}(z_1)$, labeled ODD in the output, with $x = x_1$. If the coefficients of all powers of u are positive, then $\bar{S}_{I1}(z_1)$ will be positive for all $n \geq m+I1-1$, as required by Theorem I. In STEP 8.8, $\bar{S}_{I2}$ (z), labeled TEST(0) in the output, is evaluated with $x = x_2$. If the coefficients of all powers of u are negative, $\bar{S}_{I2}$ will be negative for all $n \geq m+I2-1$, again as required for the application of Theorem I. In PART 9, the calculations corresponding to Eq. (4.9) are carried out using the full coefficients CP, $x=x_2$, and arranging the output in powers of u. The results of the calculation are returned in the array TEST(J), $J = 0, 1, \ldots I2$. The inequalities corresponding to those of (4.9) will be valid for all $n \geq m+I2-1$ if, and only if, the coefficient of each power of u satisfies the inequality individually. This test is made by "human" inspection of the output. In all cases run, the value of I2 found by the preliminary tests led to an array TEST(J) satisfying the above conditions.

At this point we display the output of WORK1 for m=2. For m=2

$$z = \left[\frac{2(2+4)}{4}\right] = 3 \tag{4.13}$$

and we initialized I1 to 3:

```
.IAM
WELCOME TO IAM(72321)
*LOAD FROM "WORK1"
*M←2;I1←3;Z←3;DO PART 21

 M: 2


 I1: 3


 I2: 2


 Z: 3


 Z1: 2


 TST1(Ø): 9Ø


 M: 2
```



```
 I1: 5
```

```
I2: 4

Z: 3

Z1: 2

TST1(0): - 77490

TST1(1): - 93240

TST1(2): 34020

TST1(3): - 2520

ODD:
52256*U↑5 + 11387840*U↑4
+ 992819328*U↑3 + 43288291584*U↑2 + 943994926080*U + 8237259878400


TEST(0):
- 77490*U↑5 - 11872980*U↑4
- 721700280*U↑3 - 21810222000*U↑2 - 328224960000*U - 1969349760000


TEST(1):
- 93240*U↑5 - 14802480*U↑4
- 930696480*U↑3 - 29091182400*U↑2 - 453309696000*U - 2821754880000


TEST(2):
34020*U↑5 + 4906440*U↑4
+ 287083440*U↑3 + 8473701600*U↑2 + 125574624000*U + 744629760000


TEST(3):
- 2520*U↑5 - 408240*U↑4
- 24625440*U↑3 - 706708800*U↑2 - 9761472000*U - 52254720000
```

We see that Eq. (4.13) led to a value of 90 for TST1(0), so WORK1 incremented I1 (and hence also I2) by 2 and started over. This time the array TST1 produced the acceptable pattern −, −, +, −; as noted above. TST1(I2) must be positive, unless the last coefficient CP(I2) is incorrectly computed. Next ODD is calculated and we see that it is positive for all non-negative u. Finally, TEST(0), TEST(1), TEST(2) and TEST(3) are calculated, the output is inspected, and it is verified that for $u \geq 0$, the sign pattern −, −, +, − is correct.

At this point we may conclude the following: For m=2

$$\bar{S}_4\left(\frac{3}{x}\right) < 0 \qquad n \geq 5$$

$$\bar{S}_5\left(\frac{3}{x}\right) > 0 \qquad n \geq 6 \tag{4.14}$$

Referring to Theorem I, we identify

$$\xi_1 = \frac{2}{x}, \quad \xi_2 = \frac{3}{x}, \quad \bar{L}_1 = 2 \tag{4.15}$$

We must next verify that conditions (2.18) and (2.19) are satisfied. Before doing this we discuss three special cases, corresponding to $n = m+2\bar{L}_1-1$, $n = m+2\bar{L}_1$, $n = m+2\bar{L}_1+1$, for which these conditions may not be relevant. The existence of the partial sum $\bar{S}_{2\bar{L}_1}$ implies that the first of the special cases represents the smallest value of n consistent with given m and $\bar{L}_1$.

1. $n = m+2\bar{L}_1-1$: $\bar{S}_{2\bar{L}_1} = \bar{P}_{nm}{}^*$ and therefore neither $\bar{S}_{2\bar{L}_1+1}$ nor the $\bar{P}r$'s associated with conditions (2.18) and (2.19) appear. These conditions must be replaced, in this case, by the condition

$$\bar{S}_{2\bar{L}_1}(z_1) > 0 \tag{4.16}$$

2. $n = m+2\bar{L}_1$: $\bar{S}_{2\bar{L}_1+1} = \bar{P}_{nm}{}^*$. In this case, conditions (4.14) replace conditions (2.18) and (2.19): $\bar{P}_{nm}{}^*(z) = \bar{S}_{2\bar{L}_1}(z) - \bar{C}_{n-m+1}\, z^{(n-m+1)} < 0$ if $\bar{S}_{2\bar{L}_1}(z) < 0$.

3. $n = m+2\bar{L}_1+1$: In this case

$$\bar{P}_{nm}^{*}(z) = \bar{S}_{2\bar{L}_1} - Pr_{2\bar{L}_1+2} = \bar{S}_{2\bar{L}_1+1} + \bar{C}_{2\bar{L}_1+2}\, z^{(2\bar{L}_1+2)} \tag{4.17}$$

Hence condition (2.19) is required, but (2.18) is not, since if $\bar{S}_{2\bar{L}_1+1} > 0$, then also $\bar{P}_{nm}^{*} > 0$.

Verification that the modified conditions hold for these special cases is easy. We proceed to the verification of conditions (2.18) and (2.19) for the general case for $m = 2$; $n \geq m+2\bar{L}_1+2$:

$$\bar{\zeta}_{2\bar{L}_1+3} = \bar{\zeta}_7 = \frac{\bar{C}_7}{\bar{C}_8} = \frac{8\cdot 10\,(16x-2\cdot 8\cdot 9)}{(n-8)(n+9)(18x-2\cdot 9\cdot 10)} = \frac{640}{9x}\;\frac{1-9/x}{(1-72/x)(1-10/x)} > \frac{2}{x} \tag{4.18}$$

$$\bar{\zeta}_{2\bar{L}_1+2} = \frac{\bar{C}_6}{\bar{C}_7} = \frac{7\cdot 9}{(n+8)(n-7)}\;\frac{14x-2\cdot 7\cdot 8}{16x-2\cdot 8\cdot 9} = \frac{441}{8x}\;\frac{1-8/x}{(1-50/x)(1-9/x)} > \frac{3}{x}$$

for all $n \geq 8$, which implies $x \geq 72$.

We have now satisfied all conditions of Theorem I, except that on $S_{2L}(\xi_2)$. This last test is carried out by another IAM program EVAL, which is used, not only for this test, but also in the next section, for the calculation of bounds using Theorem II. Here we shall merely state that EVAL does indeed verify that L exists for which conditions (2.20) and (2.21) of Theorem I are satisfied.

The output of IAM for $m = 3, 4, 5,$ and 6 appear in Appendix B. Because of prior experimentation, $\bar{L}_1$ was already known for these values of m, and to reduce the bulk of output, I1 was initialized to $2\bar{L}_1+1$. The output exhibits all the desired "sign" patterns for the array TST1, ODD, and the array TEST for the values input for m, I1, z, z_1. Verification of conditions (2.18) and (2.19) for the general case, or their substitutes for the special cases, is routine, though tedious; the details are omitted from the report.

The table below summarizes the information given by the output of WORK1 (plus the additional verifications), showing $I2 = 2\bar{L}_1$, ξ_1, ξ_2 and the lower bounds on n and $x = n(n+1)$ for which ξ_1 and ξ_2 bound the largest maximum of P_{nm}:

n	$I2=2\bar{L}_1$	ξ_1	ξ_2	lower bound for n	lower bound for x
2	4	2/x	3/x	6	42
3	4	4/x	5/x	7	56
4	6	7/x	8/x	10	110
5	8	10/x	11/x	13	182
6	12	14/x	15/x	18	342

V. BOUNDS FOR THE LARGEST MAXIMUM OF P_{nm}

The case $m = 1$ is discussed in Section III. For $m = 2, 3, 4, 5, 6$, ξ_1 and ξ_2 given in the table at the end of the last section can be used, together with Theorem II (Section II) to obtain the desired bounds. For this purpose, we require approximate values of $P_{nm}{}^*(\xi_1)$ and $P_{nm}{}^*(\xi_2)$ for $m = 2, 3, 4, 5, 6$. These values of the polynomial parts of P_{nm} were calculated using the IAM program EVAL, reproduced in Appendix C. The code is fairly obvious. The output arrays SUMX(I) and SUMIN(I) refer to $(2m)!\ y^m$ times the first I terms of $P_{nm}{}^*$ evaluated at ξ_1 and ξ_2, respectively. The parameter y refers to n(n+1), denoted by x in the rest of the report. It will be noticed that for $m = 2$, the numerator of ξ_2, z, was inadvertently set equal to 2, instead of 3 in the first command. We shall therefore outline here the calculation for $m = 2$, and this will serve as an illustration of how EVAL calculates SUMX and SUMIN for $m = 3, 4, 5, 6$.

From Eq. (2.3)

$$P_{n2}{}^*(\xi) = 1 - \frac{(n-2)(n+3)}{3}\,\xi + \frac{(n-2)(n-3)(n+3)(n+4)}{2!\cdot 3\cdot 4} \cdots \tag{5.1}$$

from which we readily obtain

$$P_{n2}{}^*(\xi) = 1 - \frac{x-6}{3}\,\xi + \frac{(x-6)(x-12)}{24}\,\xi^2 - \frac{(x-6)(x-12)(x-20)}{5\cdot 72}\,\xi^3 \ldots \tag{5.2}$$

Substituting $\xi_1 = 2/x$ and $\xi_2 = 3/x$ and arranging in inverse powers of x, we obtain

$$\begin{aligned} \mathrm{SUMX}(3) &= \frac{43}{90} + \frac{83}{45\cdot x} + \frac{12}{5x^2} + \frac{160}{3x^3} \\ \mathrm{SUMIN}(3) &= \frac{3}{10} + \frac{21}{10x} - \frac{27}{5x^2} + \frac{108}{x^3} \end{aligned} \tag{5.3}$$

We now observe that, while powers up to ξ^3 were retained in calculating SUMX and SUMIN, Theorem I requires $n \geq 6$ and hence $x \geq 42$, so that only the first two terms need be retained for approximate values of $P_{n2}{}^*$:

$$P_{n2}{}^*(\xi_2) \sim \frac{3}{10} - \frac{21}{10x}$$

$$P_{n2}{}^*(\xi_1) \sim \frac{43}{90} + \frac{83}{45x} \tag{5.4}$$

To implement Theorem II, we next calculate B_{n2} in powers of $1/x$. From Eq. (2.6), it is readily verified that

$$\begin{aligned} B_{nm} &= \frac{1}{m!}\sqrt{2(2n+1)x(x-2)(x-6)\ldots(x-m(m-1))} \\ &= \frac{2x^{m/2}}{m!}\sqrt{(n+1/2)(1-2/x)(1-6/x)\ldots(1-m(m-1)/x)} \end{aligned} \tag{5.5}$$

and for m = 2

$$\begin{aligned} B_{n2} &= x\sqrt{(n+1/2)(1-2/x)} = \sqrt{n+1/2}\; x\left[1 - \frac{1}{x} - \frac{1}{2x^2}\ \ldots\right] \\ &\sim \sqrt{n+1/2}\; x\ (1-1/x) \end{aligned} \tag{5.6}$$

We also need to evaluate $(\xi(1-\xi))^{2/2}$ at ξ_1 and ξ_2:

$$\xi_1(1-\xi_1) = \frac{2}{x}\left(1 - \frac{2}{x}\right)$$

$$\xi_2(1-\xi_2) = \frac{3}{x}\left(1 - \frac{3}{x}\right) \tag{5.7}$$

We now insert these various expressions into Eqs. (2.32) for A and B, the bounds on the largest maximum of P_{n2}, we obtain

$$\begin{aligned} A &= \sqrt{n+1/2}\; x\left(1 - \frac{1}{x}\right)\cdot\frac{2}{x}\left(1 - \frac{2}{x}\right)\left(\frac{3}{10} - \frac{21}{10x}\right) \\ &\sim \frac{3}{5}\sqrt{n+1/2}\;\left(1 - \frac{10}{x}\right) \end{aligned} \tag{5.8}$$

$$\begin{aligned} B &= \sqrt{n+1/2}\; x\left(1 - \frac{1}{x}\right)\cdot\frac{3}{x}\left(1 - \frac{3}{x}\right)\left(\frac{43}{90} + \frac{83}{45x}\right) \\ &\sim \frac{1}{30}\sqrt{n+1/2}\;(43-6/x) \end{aligned}$$

as approximate values. The table below tabulates the values as well as $P_{n2_{max}}$, the largest maximum of P_{n2} obtained by a double precision calculation[5] for selected values of n:

n	A	$P_{n2_{max}}$	B
4	.64	2.16	3.02
5	.94	2.35	3.35
6	1.16	2.53	3.64
10	1.77	3.18	4.64
15	2.26	3.84	5.64
20	2.65	4.41	6.49

Note that we have included the values $n = 4, 5$, even though they are below the bound imposed by Theorem I. We do the same for $m = 3, 4, 5$, and 6 below, since it appears that Theorem I is conservative in this respect; in fact, some preliminary analysis to generalize Theorem I indicates that the bracketing of $P_{nm_{max}}$ by ξ_1 and ξ_2 holds for all $n \geq 2m$.

The output of the arrays SUMX and SUMIN from EVAL is included in Appendix C, along with the listing for EVAL, for $m = 3, 4, 5$, and 6. Tables similar to that for $m = 2$, above, were obtained by the following sequence of steps:

1. Replace y in SUMX and SUMIN by x.
2. Approximate $P_{nm}*(\xi_1)$ and $P_{nm}*(\xi_2)$ by truncating

$$P_{nm}*(\xi_1) \sim \frac{\text{SUMIN(m)}}{(2m)!\ x^m}$$

$$P_{nm}*(\xi_2) \sim \frac{\text{SUMX(m)}}{(2m)!\ x^m} \qquad (5.9)$$

to a reasonable number of terms in $1/x$, recalling the lower bound for x given at the end of Section IV.

3. Expand $(\xi_1(1-\xi_1))^{m/2}$ and $(\xi_2(1-\xi_2))^{m/2}$, including all powers of $1/x$ retained in 2.

4. Expand B_{nm}, as given by Eq. (5.5) similarly.

5. Calculate A and B from Eq. (2.32), again in truncated form.

6. Evaluate A and B for the desired values of n and $x = n(n+1)$.

The tables so constructed are:

m=3:

n	A	$P_{n3_{max}}$	B
6	2.10	2.30	3.34
9	2.05	2.72	3.79
12	2.15	3.10	4.22
15	2.29	3.44	4.65
18	2.51	3.75	5.18
20	2.53	3.95	5.30

m=4:

n	A	$P_{n4_{max}}$	B
8	2.25	2.42	3.30
10	2.18	2.65	3.48
12	2.28	2.87	3.76
15	2.51	3.18	4.18
17	2.69	3.37	4.46
20	2.90	3.64	4.84

m=5:

n	A	$P_{n5_{max}}$	B
10	2.13	2.51	2.78
11	2.28	2.61	3.03
13	2.37	2.81	3.41
15	2.60	2.99	3.69
18	2.74	3.25	4.05
20	2.83	3.42	4.26

m=6:

n	A	$P_{n6_{max}}$	B
12	1.98	2.59	3.26
14	2.28	2.76	4.43
16	2.44	2.93	4.23
17	2.51	3.01	4.21
18	2.57	3.09	4.22
19	2.63	3.17	4.24
20	2.69	3.24	4.28

The principal conclusion to be drawn from these tables is that the values provided for ξ_1 and ξ_2, from the conjecture of Section II, appear to provide realistic bounds on the largest maximum of P_{nm}.

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APPENDIX A

PROOF OF THE INEQUALITIES (2.13)

Proof of the Inequalities (2.13)

The first of these inequalities is

$$\frac{C_k}{C_{k+1}} - \frac{C_{k-1}}{C_k} > 0 \tag{A.1}$$

where the C's are coefficients in the polynomial part of P_{nm}. For all three of these coefficients to appear in P_{nm}, k must exceed zero. Since the C's are all positive

$$\frac{C_k}{C_{k-1}} - \frac{C_{k+1}}{C_k} > 0 \tag{A.2}$$

is an inequality equivalent to (A.1), but somewhat simpler to manipulate. From Eq. (2.4) we have

$$\frac{C_{k+1}}{C_k} = \frac{x-(m+k)(m+k+1)}{(k+1)(m+k+1)} \tag{A.3}$$

and hence

$$\frac{C_k}{C_{k-1}} = \frac{x-(m+k-1)(m+k)}{k(m+k-1)} \tag{A.4}$$

with

$$x = n(n+1) \geq (m+k+1)(m+k+2) \tag{A.5}$$

from the first of inequalities (2.14). Substitution of Eqs. (A.3) and (A.4) into Eq. (A.2), and clearing fractions, yields the following inequality to be verified;

$$(m+1)x-2m(m+1)(m+2) + (m+1)^2(m+2) = (m+1)\,[x-(m+2)(m-1)] > 0 \tag{A.6}$$

From inequality (A.5)

$$x-(m+2)(m-1) \geq m^2 + (2k+3)m+(k+1)(k+2)-(m^2+m-2) = 2(k+1)m+(k+1)(k+2)+2 > 0 \tag{A.7}$$

which establishes (A.6) and hence (A.1).

The second inequality to be verified is

$$\frac{\bar{C}_k^{\,2} - \bar{C}_{k+1}\,\bar{C}_{k-1}}{\bar{C}_k\,\bar{C}_{k-1}} > 0 \tag{A.8}$$

where the $\bar{C}$'s are coefficients in the polynomial part of $P_{nm}{}'$, and, as above, $k \geq 1$ is required if all three are present. Also, as above, all the $\bar{C}$'s are positive and hence the denominator in inequality (A.8) may be discarded. The case $k=1$ is a special case for which (A.8) reduces to

$$\bar{C}_1^{\,2} - \bar{C}_2 > 0 \tag{A.9}$$

since, according to Eq. (2.10), $\bar{C}_0 = 1$. $\bar{C}_1$ and $\bar{C}_2$ are given by

$$\bar{C}_1 = \frac{(m+2)x-m^2(m+1)}{m(m+1)}$$

$$\bar{C}_2 = \frac{(x-m(m+1))((m+4)x-m(m+1)(m+2))}{2!\,m(m+1)(m+2)} \tag{A.10}$$

The verification of inequality (A.9) is laborious, but has been carried out by hand since it was inadvertently omitted from the IAM program for the general case. The details of the proof are omitted; the key step is to make use of the second inequality of (2.14)

$$n \geq m+k = m+1, \text{ which implies } n = m+L+1 \text{ with } L \geq 0 \tag{A.11}$$

so that we may set

$$x = n(n+1) = (m+L+1)(m+L+2) \tag{A.12}$$

where L is any integer ≥ 0. Then the sequence of steps

1. Substitute (A.10) into (A.9)
2. Clear fractions
3. Collect terms in ascending powers of x

4. Substitute for x from (A.12)

5. Arrange in ascending powers of L

6. Observe that the resulting expression is non-negative for non-negative values of L.

For the general case, $k > 1$, the three coefficients involved in (A.6) are, from Eq. (2.10)

$$\bar{C}_{k-1} = \frac{(n-m)!\,(n+m+k-2)!\,(m-1)!\,[(m+2k-2)x-m(m+k-1)(m+k-2)]}{(k-1)!\,(m+k-1)!\,(n-m-k+2)!\,(n+m)!} \tag{A.13}$$

$$\bar{C}_{k} = \frac{(n-m)!\,(n+m+k-1)!\,(m-1)!\,[(m+2k)x-m(m+k)(m+k-1)]}{k!\,(m+k)!\,(n-m-k+1)!\,(n+m)!}$$

$$\bar{C}_{k+1} = \frac{(n-m)!\,(n+m+k)!\,(m-1)!\,[(m+2k+2)x-m(m+k)(m+k+1)]}{(k+1)!\,(m+k+1)!\,(n-m-k)!\,(n+m)!}$$

We discard the denominator and the positive factor

$$\frac{(n-m)!\,(m-1)!}{(n+m)!} \tag{A.14}$$

common to all three $\bar{C}$'s. We next clear fractions, discard the positive common denominator, and a number of other common positive factors in the numerator to obtain, as the inequality to be verified

$$\begin{aligned}&(k+1)(m+k+1)(n-m-k+2)(n+m+k-1)[(m+2k)x-m(m+k)(m+k-1)]^2\\ &\quad -k(m+k)(n-m-k+1)(n+m+k)[(m+2k-2)x-m(m+k-1)(m+k-2)]\\ &\quad \text{times } [(m+2k+2)x-m(m+k+1)(m+k)] > 0\end{aligned} \tag{A.15}$$

The proof of this inequality is <u>very</u> laborious and IAM was used to carry out most of the remaining calculations. The program written for this purpose follows:

```
TYPE PROOF
30.1:N←M+K+J;$DIST←15
30.2:F1←K*(M+K)*(N-M-K+1)*(N+M+K)
30.3:F2←(M+2*K-2)*(N+1)*N-M*(M+K-1)*(M+K-2)
30.4:F3←(M+2*K+2)*N*(N+1)-M*(M+K+1)*(M+K)
30.5:F4←(K+1)*(M+K+1)*(N-M-K+2)*(N+M+K-1)
30.6:F5←((M+2*K)*(N+1)*N-M*(M+K)*(M+K-1))↑2
30.7:TST←F4*F5-F1*F2*F3
30.8:TSTJ(0)←TST WITH [J=0];TYPE TSTJ(0)
30.9:FOR I←1 TO 6,(TST←(TST-TSTJ(I-1))/J;&
TSTJ(I)←TST WITH [J=0];TYPE TSTJ(I);&
DELETE TSTJ(I-1))
30.95:DELETE TST,TSTJ,N,F1,F2,F3,F4,F5
```

The first instruction makes use of condition (2.14):

$$n \geq m+k \tag{A.16}$$

so that J in the program must satisfy

$$J \geq 0 \tag{A.17}$$

The command $DIST ← 15 requires the system to expand all powers of multinomials up to and including the 15th. In terms of the parameters F1, F2, F3, F4, F5 generated in STEPS 30.2 to 30.6, the left hand side of (A.15) is given by

$$F4*F5-F1*F2*F3 \tag{A.18}$$

as indicated in STEP 30.7. This expression, written symbolically as a polynomial in J, K, and M turned out to be too complex to be contained in its entirety in the system. STEPS 30.9 and 30.95 calculate in turn the coefficients of the powers of J from the 0th through the 6th, and these coefficients are listed in the array TSTJ(I), I=0 to 6 as output:

```
IAM
WELCOME TO IAM(72321)
*LOAD FROM "PROOF"
*


DO PART 30


TSTJ(0):

8*K↑9 + K↑8*(48*M + 40) + K↑7*(120*M↑2 + 232*M + 64)

+ K↑6*(160*M↑3 + 560*M↑2 + 368*M + 32)

+ K↑5*(120*M↑4 + 720*M↑3 + 872*M↑2 + 200*M - 8)

+ K↑4*(48*M↑5 + 520*M↑4 + 1088*M↑3 + 488*M↑2 - 16*M - 8)

+ K↑3*(8*M↑6 + 200*M↑5 + 752*M↑4 + 600*M↑3 + 8*M↑2 - 32*M)

+ K↑2*(32*M↑6 + 272*M↑5 + 392*M↑4 + 40*M↑3 - 48*M↑2)

+ K*(40*M↑6 + 128*M↑5 + 32*M↑4 - 32*M↑3) + 16*M↑6 + 8*M↑5 - 8*M↑4


TSTJ(1):

48*K↑8 + K↑7*(264*M + 200) + K↑6*(600*M↑2 + 1036*M + 264)

+ K↑5*(720*M↑3 + 2208*M↑2 + 1296*M + 104)

+ K↑4*(480*M↑4 + 2472*M↑3 + 2560*M↑2 + 516*M - 24)

+ K↑3*(168*M↑5 + 1528*M↑4 + 2584*M↑3 + 952*M↑2 - 48*M - 16)

+ K↑2*(24*M↑6 + 492*M↑5 + 1392*M↑4 + 816*M↑3 - 28*M↑2 - 40*M)

+ K*(64*M↑6 + 376*M↑5 + 320*M↑4 - 8*M↑3 - 32*M↑2)

+ 40*M↑6 + 44*M↑5 - 4*M↑4 - 8*M↑3
```

```
TSTJ(2):

120*K↑7 + K↑6*(588*M + 408) + K↑5*(1176*M↑2 + 1816*M + 432)

+ K↑4*(1224*M↑3 + 3248*M↑2 + 1724*M + 128)

+ K↑3*(696*M↑4 + 2960*M↑3 + 2638*M↑2 + 472*M - 24)

+ K↑2*(204*M↑5 + 1432*M↑4 + 1908*M↑3 + 588*M↑2 - 40*M - 8)

+ K*(24*M↑6 + 344*M↑5 + 638*M↑4 + 280*M↑3 - 28*M↑2 - 8*M)

+ 32*M↑6 + 76*M↑5 + 36*M↑4 - 10*M↑3 - 2*M↑2


TSTJ(3):

160*K↑6 + K↑5*(672*M + 432) + K↑4*(1128*M↑2 + 1568*M + 352)

+ K↑3*(960*M↑3 + 2188*M↑2 + 1064*M + 72)

+ K↑2*(432*M↑4 + 1452*M↑3 + 1128*M↑2 + 172*M - 8)

+ K*(96*M↑5 + 452*M↑4 + 474*M↑3 + 108*M↑2 - 8*M)

+ 8*M↑6 + 52*M↑5 + 58*M↑4 + 11*M↑3 - 3*M↑2


TSTJ(4):

120*K↑5 + K↑4*(408*M + 248) + K↑3*(534*M↑2 + 680*M + 144)

+ K↑2*(336*M↑3 + 660*M↑2 + 292*M + 16)

+ K*(102*M↑4 + 260*M↑3 + 170*M↑2 + 16*M)

+ 12*M↑5 + 32*M↑4 + 21*M↑3 + M↑2


TSTJ(5):

48*K↑4 + K↑3*(120*M + 72) + K↑2*(108*M↑2 + 132*M + 24)

+ K*(42*M↑3 + 72*M↑2 + 24*M) + 6*M↑4 + 9*M↑3 + 3*M↑2


             3    2                    2             3    2
TSTJ(6): 8*K  + K *(12*M + 8) + K*(6*M  + 8*M) + M  + M
```

We can conclude that the left hand side of (A.15) (or (A.18)) is positive if TSTJ(I) > 0 for I = 0 to 6. Recalling that the case m=0 is excluded from this theory, it is easily seen by inspection that the coefficients of the various powers of k in each TSTJ(I) are positive. Since $k > 1$, it follows that TSTJ(I) > 0 for I = 0 to 6.

It might be remarked that carrying out this calculation by hand, without errors, would indeed be a formidable task - as will be obvious to the reader who elected to carry out the omitted proof of the special case (A.9).

APPENDIX B

OUTPUT OF WORK1 TO VERIFY CONJECTURE

FOR m = 3, 4, 5, 6

m = 3

```
•IAM
WELCOME TO IAM(72321)
*LOAD FROM "WORK1"
*M←3;I1←5;Z←5;DO PART 21

 M: 3

 I1: 5

 I2: 4

 Z: 5

 Z1: 4

 TST1(Ø): - 778ØØ

 TST1(1): - 1328ØØ

 TST1(2): 5592Ø

 TST1(3): - 128Ø

 ODD:

 99328*U↑5 + 307712ØØ*U↑4 + 3795431424*U↑3

 + 23368761344Ø*U↑2 + 7192405147648*U + 886Ø468248576Ø


 TEST(Ø):

 - 778ØØ*U↑5 - 199Ø6ØØØ*U↑4 - 1789236ØØØ*U↑3

 - 75135Ø96ØØØ*U↑2 - 15Ø876432ØØØØ*U - 1161706613760Ø
```

```
TEST(1):

- 132800*U↑5 - 34068800*U↑4 - 3238934400*U↑3

- 148704019200*U↑2 - 3362743641600*U - 30231561830400


TEST(2):

55920*U↑5 + 10342560*U↑4

+ 807036480*U↑3 + 32631431040*U↑2 + 670654494720*U + 5522714265600


TEST(3):

- 1280*U↑5 - 491840*U↑4

- 48885120*U↑3 - 2062344960*U↑2 - 39654074880*U - 286123622400
```

m = 4

```
.IAM
WELCOME TO IAM(72321)
*LOAD FROM "WORK1"
*M←4;I1←7;Z←8;DO PART 21

 M: 4

 I1: 7

 I2: 6

 Z: 8

 Z1: 7

 TST1(0): - 1051359232

 TST1(1): - 922758144

 TST1(2): 223534080

 TST1(3): - 15473920
```

```
TST1(4): 1090320

TST1(5): - 5544

ODD:

96680234*U↑7

+ 110962352200*U↑6 + 49573318410672*U↑5 + 11655729860711104*U↑4

+ 1588276197961984160*U↑3 + 126774443025818448000*U↑2

+ 5524042423811804928000*U + 101804347438976286720000


TEST(0):

- 1051359232*U↑7 - 669522022400*U↑6

- 180675309101056*U↑5 - 26869411366060032*U↑4

- 2381557828659609600*U↑3 - 125825553687601152000*U↑2

- 3666290213009571840000*U - 45377077416694087680000


TEST(1):

- 922758144*U↑7 - 620726937600*U↑6

- 177442608279552*U↑5 - 28065436646252544*U↑4

- 2658741078789734400*U↑3 - 151035821063221248000*U↑2

- 4766493826307727360000*U - 64479590990030684160000


TEST(2):

223534080*U↑7

+ 139815244800*U↑6 + 37912103792640*U↑5 + 5744277409812480*U↑4

+ 523366306670592000*U↑3 + 28613846697891840000*U↑2

+ 868115487981772800000*U + 11265426294738739200000
```

```
TEST(3):

- 15473920*U↑7 - 10220672000*U↑6

- 2785324917760*U↑5 - 412389054443520*U↑4

- 36115102324224000*U↑3 - 1878210682951680000*U↑2

- 53800686603417600000*U - 655112749932748800000


TEST(4):

1090320*U↑7 + 560128800*U↑6 + 127579119360*U↑5

+ 16490351243520*U↑4 + 1290102660000000*U↑3 + 60479909582400000*U↑2

+ 1562069124000000000*U + 17068047504076800000


TEST(5):

- 5544*U↑7 - 6930000*U↑6

- 2116743552*U↑5 - 298627426944*U↑4 - 22862606659200*U↑3

- 980891692704000*U↑2 - 22165403863680000*U - 204999081154560000
```

<u>m = 5</u>

```
 IAM
WELCOME TO IAM(72321)
*LOAD FROM "WORK1"
*M←5;I1←9;Z←11;DO PART 21

 M: 5


 I1: 9


 I2: 8


 Z: 11


 Z1: 10
```

```
TST1(0): - 17172780156378

TST1(1): - 21271465952352

TST1(2): 3954506831352

TST1(3): - 284857239744

TST1(4): 12362967540

TST1(5): - 290128608

TST1(6): 8506008

TST1(7): - 16128

ODD:

8327711488000*U↑9 + 15546080348544000*U↑8

+ 12790995456325632000*U↑7 + 6096970461109113856000*U↑6

+ 1857536542517692121088000*U↑5 + 375451584679455299106816000*U↑4

+ 50381858806734421935505408000*U↑3

+ 4330703236252216782857601024000*U↑2

+ 216481739496505058686151688192000*U

+ 4796706165203789542456364630016000


TEST(0):

- 17172780156378*U↑9 - 23484678338339064*U↑8

- 14196045908490093648*U↑7 - 4975579169060728536000*U↑6

- 1113270555467463550930080*U↑5 - 164679126124474266208219008*U↑4

- 16074108546949298820042883584*U↑3

- 995646861630436023278967932928*U↑2

- 35374636306257631070763277762560*U

- 546064091011089811207696416768000
```

TEST(1):

- 21271465952352*U↑9 - 31045455262350144*U↑8

- 20130982577094445056*U↑7 - 7613871508307544657408*U↑6

- 185128838744339307467360*U↑5 - 3001185570566210626446605952*U↑4

- 32439505717657970137046421504*U↑3

- 225438334573462911687841741 2096*U↑2

- 914017916286475760754065849057 28*U

- 1647219167649918753516544484966400

TEST(2):

3954506831352*U↑9 + 5636770118334432*U↑8 + 3572809189314805440*U↑7

+ 1321041222583083783936*U↑6 + 313920061401055331376000*U↑5

+ 49708567953334297395666432*U↑4 + 5244410274527423501752707072*U↑3

+ 355447938422681072123787878400*U↑2

+ 14042035794681283349632891355136*U

+ 246329946269902355895574777036800

TEST(3):

- 284857239744*U↑9 - 398664188733888*U↑8 - 246955192953460992*U↑7

- 88965822613739613696*U↑6 - 20551856335936104453120*U↑5

- 3157832077547419582700544*U↑4 - 322737552640985564622532608*U↑3

- 21154547401374726928722051072*U↑2

- 806849173438954709568979009536*U

- 13640756861286678178141883596800

TEST(4):

12362967540*U↑9 + 16341408079920*U↑8

+ 9637024742785440*U↑7 + 3316981360543804800*U↑6

+ 732742703695618862400*U↑5 + 107586607508423198749440*U↑4

+ 10489669087745431395947520*U↑3 + 654462594523265976730275840*U↑2

+ 23698088026617327436482969600*U + 379288231763912321349058560000

```
TEST(5):

- 290128608*U↑9 - 392029412544*U↑8

- 228209615682048*U↑7 - 75895162363722240*U↑6

- 15979784172107374080*U↑5 - 2215530833349518834688*U↑4

- 202563176790072654065664*U↑3 - 11783442334732507395145728*U↑2

- 395775539228071052815564800*U - 5846699438719102726963200000


TEST(6):

8506008*U↑9

+ 9142642656*U↑8 + 4468528419264*U↑7 + 1291838248173312*U↑6

+ 241319265675949440*U↑5 + 29986137704387870208*U↑4

+ 2464712123717211088896*U↑3 + 128702577173599879962624*U↑2

+ 3863275813280916627259392*U + 50686994823099331195699200


TEST(7):

- 16128*U↑9 - 46928448*U↑8 - 32039804160*U↑7

- 10503523183104*U↑6 - 1997014366771200*U↑5 - 236327888658613248*U↑4

- 17687909887524753408*U↑3 - 813955648588624281600*U↑2

- 20981827261430073851904*U - 231191682528193177190400
```

<u>m = 6</u>

```
*M←6;I1←13;Z←15;DO PART 21

 M: 6

 I1: 13

 I2: 12

 Z: 15

 Z1: 14
```

```
TST1(0): - 562104262949070000138750

TST1(1): - 516418426215008734950000

TST1(2): 868314567408449411250 0

TST1(3): - 5788701112358542950000

TST1(4): 220896029759569807500

TST1(5): - 55849405858491960 0

TST1(6): 10167863351045400

TST1(7): - 139563590166000

TST1(8): 1549199029950

TST1(9): - 12385933560

TST1(10): 122949684

TST1(11): - 160056

ODD:

4716065119064668823552*U↑13 + 29064030974986852042784768*U↑12

+ 7691458685955905433487844966 4*U↑11

+ 1189753200988085595664968779694 08*U↑10

+ 1217472441251154951754147109073059 84*U↑9

+ 8776320525120482347916603774497062912 0*U↑8

+ 460957462519083111685301975148655033712 64*U↑7

+ 1792503673658594228181918680688833888727859 2*U↑6

+ 5173855177833040446246865359397708258395291648*U↑5

+ 1096827992605914313180514072387945676009986064384*U↑4

+ 16623065625667005574996802827549606144804248354816 0*U↑3

+ 1707426091800008934076680142299550108298235692974080 0*U↑2

+ 1066244625604993951709649877832114256625735160561664000*U

+ 3059459088107487551257207068609979541745436377219072000 0
```

```
TEST(0):

- 56210426294907000138750*U↑13 - 220746525969335467402372500*U↑12

- 399602795314544695887653835000*U↑11

- 441453435712931353101066074970000*U↑10

- 332014522159078502918842412187300000*U↑9

- 179492948298305379000218562641906040000*U↑8

- 71749592490147209417478299131263067920000*U↑7

- 21467819729152815797537427966014298129120000*U↑6

- 4806847577218157250866613654598253985546240000*U↑5

- 795306046968069227671919031800032952407564800000*U↑4

- 94482315816834569417725037946351092663443845120000*U↑3

- 7629521341287663782655844929133720858744844943360000*U↑2

- 375189154283111111570869532697714420796712882012160000*U

- 8481755985513114910006183194681202778692427207147520000

TEST(1):

- 51641842621500873495000*U↑13 - 210251855147405586499530000*U↑12

- 395092944083706239336226300000*U↑11

- 453724639678621565578881053640000*U↑10

- 355277334932971273330168445413200000*U↑9

- 200303511848556393026592559797855840000*U↑8

- 83654480522660340415111811001857252160000*U↑7

- 26203804603012706811103808181316134460800000*U↑6

- 6156216072931246350912329696671411969505280000*U↑5

- 1071377440756971375965412321016173206778931200000*U↑4

- 134250626870099829994861077147985234815552798720000*U↑3

- 11470204007227641899185448593243324499553997946880000*U↑2

- 598905418879477976116794197604847459623099039744000000*U

- 14433277820600385587234154940034794127874994093424640000
```

TEST(2):

```
868314567408449412500*U↑13 + 3496036071997454601973500 0*U↑12

+ 64959563281965599467355850000*U↑11

+ 737545421534738811222839153400 00*U↑10

+ 570898754975869876930645734606 00000*U↑9

+ 318138090072475759744872183280 74000000*U↑8

+ 131307581583850314006562349393 9464752 0000*U↑7

+ 406418980710949164508764054262 80260880 00000*U↑6

+ 943334095733240173855032747432 9259511884 80000*U↑5

+ 162168537359528540623500873717 9786172598016 00000*U↑4

+ 200696814201663896347717471508 3105697527685120 0000*U↑3

+ 169324622172255624470640429681 2730403747606364160 000*U↑2

+ 872878511505006301203120538136 1846700322484387840 0000*U

+ 207647049341465650236276663895 6276795005175741808 640000
```

TEST(3):

```
- 578870111235854295000*U↑13 - 230260240254275218845 0000*U↑12

- 42259477809921918680791800 00*U↑11

- 473809695944249690209512900 0000*U↑10

- 362077402856407787802055631 4000000*U↑9

- 199147451535606974666976194 2230240000*U↑8

- 811057487518939996202970872 012443200000*U↑7

- 247639987217527162365996073 239629477760000*U↑6

- 566859627293878332311041965 27440393856000000*U↑5

- 960757840258652851034829371 8127805148416000000*U↑4

- 117191208202875008591499854 28826703309493043200 00*U↑3

- 974198240421605174490877580 85503529734946816000 000*U↑2

- 494668883061472168194510423 6353592014353024942080 000*U

- 115871411764609165297427780 1967995802369602355200 00000
```

```
TEST(4):

22089602975956980750*U↑13

+ 86491002543982696588500*U↑12 + 156195689415213130744347000*U↑11

+ 172259143093717771535408250000*U↑10

+ 129433596450385062619125316260000*U↑9

+ 69970518596628628170780919761144000*U↑8

+ 27996770504240767719968332553302032000*U↑7

+ 8394753607311485555936963301017405664000*U↑6

+ 1886262462178359470453628348106255603200000*U↑5

+ 313676486921320634295943928563523576578560000*U↑4

+ 37523199314813738015423264308248900527788032000*U↑3

+ 3057578038452175234913906048078748584921235456000*U↑2

+ 152107928208571885762889443855329650933273198592000*U

+ 3488954088079574088445883089691182162429083648000000
```

```
TEST(5):

- 558494058584919600*U↑13 - 2142662019449632480800*U↑12

- 3789225074163506961297600*U↑11 - 4089859378859733960658320000*U↑10

- 3005799008077653995763043488000*U↑9

- 1588366179425304301573043591155200*U↑8

- 620860711883269178016664389017625600*U↑7

- 181745794690693486784804862937239091200*U↑6

- 39841805042504877515374549581687521280000*U↑5

- 6459517168573458186339920338039023587328000*U↑4

- 752818207186657559705506897698968496183705600*U↑3

- 59720164266934734335749087866430652508851404800*U↑2

- 2890144210404008986942354078884334953293257113600*U

- 64438906132691514550825884716384821390501478400000
```

```
TEST(6):

10167863351045400*U↑13 + 37986041396482779600*U↑12

+ 65376576000357477055200*U↑11 + 68623918054871623541697600*U↑10

+ 49009670433545317134028790400*U↑9

+ 25145499320697226640444737862400*U↑8

+ 9534620661328195038640811314137600*U↑7

+ 2704996636231675642131202569113472000*U↑6

+ 574131509156234848399952829359879577600*U↑5

+ 90033413450909494587661434986732891750400*U↑4

+ 10138443021477329246579563453409089093632000*U↑3

+ 776267072302493023717104601261862514720768000*U↑2

+ 36218949047079636618827533040605194990963916800*U

+ 777666996997867121179205957383950265692979200000


TEST(7):

- 139563590166000*U↑13 - 505607814851911200*U↑12

- 842165940795966072000*U↑11

- 854085193896421169654400*U↑10 - 588426788430571912028064000*U↑9

- 290820781641508750890761817600*U↑8

- 106073025275055925463091829171200*U↑7

- 28905790970250481770947691284428800*U↑6

- 5884591266676637266881562326802022400*U↑5

- 883793210920424262309414701266292736000*U↑4

- 95169485783506928321046440862841149849600*U↑3

- 6957214922863672669709835661615899082752000*U↑2

- 309425635647526623842881281451065862835404800*U

- 6322495954953233870393362639809739436851200000
```

```
TEST(8):

1549199029950*U↑13 + 534085013752290 0*U↑12

+ 848050805093906700 0*U↑11 + 8202719839548302125200*U↑10

+ 5387717233171136613578400*U↑9 + 25360516597892305838229936 00*U↑8

+ 87972426757027437078409750 0800*U↑7

+ 2276157682146137051982105397440 00*U↑6

+ 43912655642784039470723912658739200*U↑5

+ 623727782688294460170480575798691840 0*U↑4

+ 6338358636100343351541768403833053184 00*U↑3

+ 43628392811718179373374938138204038758400*U↑2

+ 18227437197746423626149068244720195993600 00*U

+ 349009314542003231833476983829980774400000 00

TEST(9):

- 12385933560*U↑13 - 42094444700880*U↑12 - 64886449373871840*U↑11

- 60267171904427442240*U↑10 - 377091694709436794630 40*U↑9

- 1680560690292937501386624 0*U↑8 - 5492129463360493402666314240*U↑7

- 13330623763628017192227254630 40*U↑6

- 2403406265715841678533231081062 40*U↑5

- 31786941713626239114479086450851840*U↑4

- 2997132167593005999711581491595182080*U↑3

- 190732988639646079303914212051527925760*U↑2

- 7340530929376193890802461232939925504000*U

- 128987675480883909365489172549206016000000
```

```
TEST(10):

122949684*U↑13

+ 356079304152*U↑12 + 479443248185616*U↑11 + 395851656158820576*U↑10

+ 222779168227694460096*U↑9 + 899489463800236478357776*U↑8

+ 2673089306519585273537817б*U↑7 + 59062807939094568388944529б*U↑6

+ 96843705717134727028279144857б*U↑5

+ 1162003290381608255470129506140 16*U↑4

+ 990400137234006654785408007418б752*U↑3

+ 5671670993391014961409378813964451 84*U↑2

+ 1953868168902203463492358604964495 3600*U

+ 30549468104235989008546574830141 4400000


TEST(11):

- 160056*U↑13 - 787582224*U↑12 - 1356448350816*U↑11

- 1261252019117376*U↑10 - 741950691782990976*U↑9

- 298016918468024461056*U↑8 - 85039302554076938236416*U↑7

- 17564318265935066256423936*U↑6 - 2635242297457649636343914496*U↑5

- 284193567614790517765936398336*U↑4

- 21429294530725134119954975293440*U↑3

- 1069776715869965366362488488067072*U↑2

- 31663623306759717874343575604428800*U

- 419022432774206248141796356915200000
```

APPENDIX C

IAM PROGRAM EVAL AND OUTPUT FOR PARTIAL SUMS OF THE POLYNOMIAL PARTS OF P_{nm} EVALUATED AT ξ_1 AND ξ_2 FOR m = 3, 4, 5, 6

EVAL Listing

```
 TYPE EVAL
15.1:INZ(2)←2;INZ(3)←5;INZ(4)←8;INZ(5)←11;INZ(6)←15
15.2:$DIST←15
15.3:FOR L←2 TO 6,(M←L;Z/INZ(L);K←M;Z1←Z-1;&
DELETE INZ(L);TYPE M,K,Z,Z1;A(Ø)←1;B(Ø)←1;&
FOR I←1 TO K,(C←M+I;B(I)←-(Y-C*(C-1))*B(I=1);&
C←K-I+1;A(I)←C*(M+C)*A(I-1));DELETE C;&
FOR I←Ø TO K,(A1(I)←A(I)*Z↑(K-I);&
A2(I)←A(I)*Z1↑(K-I));DELETE A;&
FOR I←Ø TO K,B(I)←B(I)*Y↑(K-I);&
SUMX(1)←A1(K)*B(Ø)+A1(K-1)*B(1);TYPE SUMX(1);&
DELETE A1(K),A1(K-1);&
SUMIN(1)←A2(K)*B(Ø)+A2(K-1)*B(1);TYPE SUMIN(1);&
DELETE A2(K),A2(K-1),B(Ø),B(1);&
FOR I←2 TO K,(SUMX(I)←SUMX(I-1)+A1(K-I)*B(I);&
DELETE SUMX(I-1),A1(K-I);TYPE SUMX(I);&
SUMIN(I)←SUMIN(I-1)+A2(K-I)*B(I);&
DELETE SUMIN(I-1),A2(K-I),B(I);TYPE SUMIN(I));&
DELETE SUMX(K),SUMIN(K),M,Z,K,Z1)

•
```

EVAL Output

```
 IAM
WELCOME TO IAM(72321)
*LOAD FROM "EVAL"
*DO PART 15

 M: 2


 K: 2


 Z: 2


 Z1: 1
```

$$SUMX(1): 8*Y^2 + 96*Y$$

$$SUMIN(1): 16*Y^2 + 48*Y$$

$$SUMX(2): 12*Y^2 + 24*Y + 288$$

$$SUMIN(2): 17*Y^2 + 30*Y + 72$$

M: 3

K: 3

Z: 5

Z1: 4

$$SUMX(1): - 180*Y^3 + 10800*Y^2$$

$$SUMIN(1): 8640*Y^2$$

$$SUMX(2): 270*Y^3 - 3600*Y^2 + 108000*Y$$

$$SUMIN(2): 288*Y^3 - 576*Y^2 + 69120*Y$$

$$SUMX(3): 145*Y^3 + 4150*Y^2 - 42000*Y + 900000$$

$$SUMIN(3): 224*Y^3 + 3392*Y^2 - 7680*Y + 460800$$

M: 4

K: 4

Z: 8

Z1: 7

SUMX(1): $- 24192 Y^{4} + 1290240 Y^{3}$

SUMIN(1): $- 16128 Y^{4} + 1128960 Y^{3}$

SUMX(2): $18816 Y^{4} - 860160 Y^{3} + 25804800 Y^{2}$

SUMIN(2): $16800 Y^{4} - 517440 Y^{3} + 19756800 Y^{2}$

SUMX(3): $2432 Y^{4} + 647168 Y^{3} - 18432000 Y^{2} + 412876800 Y$

SUMIN(3): $5824 Y^{4} + 492352 Y^{3} - 9878400 Y^{2} + 276595200 Y$

SUMX(4): $6528 Y^{4} + 40960 Y^{3} + 13729792 Y^{2} - 309657600 Y + 5780275200$

SUMIN(4): $8225 Y^{4} + 137004 Y^{3} + 8974252 Y^{2} - 146941200 Y + 3388291200$

M: 5

K: 5

Z: 11

Z1: 10

SUMX(1): $-3024000 \cdot Y^{5} + 199584000 \cdot Y^{4}$

SUMIN(1): $-2419200 \cdot Y^{5} + 181440000 \cdot Y^{4}$

SUMX(2): $2203200 \cdot Y^{5} - 176774400 \cdot Y^{4} + 6586272000 \cdot Y^{3}$

SUMIN(2): $1900800 \cdot Y^{5} - 129600000 \cdot Y^{4} + 5443200000 \cdot Y^{3}$

SUMX(3): $-192600 \cdot Y^{5} + 129888000 \cdot Y^{4} - 6092301600 \cdot Y^{3} + 169047648000 \cdot Y^{2}$

SUMIN(3): $100800 \cdot Y^{5} + 100800000 \cdot Y^{4} - 4082400000 \cdot Y^{3} + 127008000000 \cdot Y^{2}$

SUMX(4):

$539450 \cdot Y^{5}$

$- 16522000 \cdot Y^{4} + 4528279800 \cdot Y^{3} - 161534419200 \cdot Y^{2} + 3719048256000 \cdot Y$

SUMIN(4):

$600800 \cdot Y^{5}$

$+ 800000 \cdot Y^{4} + 3171600000 \cdot Y^{3} - 98784000000 \cdot Y^{2} + 2540160000000 \cdot Y$

SUMX(5):

$378399 \cdot Y^{5} + 30182790 \cdot Y^{4} - 707166108 \cdot Y^{3}$

$+ 121481147304 \cdot Y^{2} - 3644667290880 \cdot Y + 73637155468800$

SUMIN(5):

$500800 \cdot Y^{5} + 29800000 \cdot Y^{4}$

$- 79200000 \cdot Y^{3} + 7694640000 \cdot Y^{2} - 2032128000000 \cdot Y + 45722880000000$

M: 6

K: 6

Z: 15

Z1: 14

SUMX(1): $- 547430400*Y^{6} + 43110144000*Y^{5}$

SUMIN(1): $- 479001600*Y^{6} + 40236134400*Y^{5}$

SUMX(2): $414849600*Y^{6} - 51193296000*Y^{5} + 2263282560000*Y^{4}$

SUMIN(2): $359251200*Y^{6} - 41912640000*Y^{5} + 1971570585600*Y^{4}$

SUMX(3):

$$- 119750400*Y^{6} + 39688704000*Y^{5} - 2766234240000*Y^{4} + 90531302400000*Y^{3}$$

SUMIN(3):

$$- 75398400*Y^{6} + 31977792000*Y^{5} - 2117612851200*Y^{4} + 73605301862400*Y^{3}$$

SUMX(4):

$$80724600*Y^{6} - 12434796000*Y^{5}$$

$$+ 218710206000*Y^{4} - 11316412800000*Y^{3} + 305543145600000*Y^{2}$$

SUMIN(4):

$$76728960*Y^{6} - 7575321600*Y^{5}$$

$$+ 164114995968 0*Y^{4} - 8096583204864 0*Y^{3} + 2318567008665600*Y^{2}$$

```
SUMX(5):

26049600*Y↑6 + 7794954000*Y↑5 - 727512840000*Y↑4

+ 90989260200000*Y↑3 - 3888730944000000*Y↑2 + 91662943680000000*Y


SUMIN(5):

38005632*Y↑6 + 6752309760*Y↑5 - 423113209344*Y↑4

+ 63624906196992*Y↑3 - 2599605433958400*Y↑2 + 64919876242636800*Y


SUMX(6):

37440225*Y↑6 + 2076860250*Y↑5 + 436016722500*Y↑4 - 31694605425000*Y↑3

+ 3172187731500000*Y↑2 - 118397968920000000*Y + 2520730951200000000


SUMIN(6):

45535168*Y↑6 + 2972482688*Y↑5 + 346013833984*Y↑4 - 17472714244608*Y↑3

+ 2067868155967488*Y↑2 - 73936525720780800*Y + 1666276823561011200

*
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