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FINITE ELEMENT METHOD FOR THERMAL ANALYSIS

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16. Abstract

A two- and three-dimensional, finite-element thermal-analysis program which handles conduction with internal heat generation, convection, radiation, specified flux, and specified temperature boundary conditions is presented. Elements used in the program are the triangle and tetrahedron for two- and three-dimensional analysis, respectively.

The theory used in the program is developed, and several sample problems demonstrating the capability and reliability of the program are presented. A guide to using the program, description of the input cards, and program listing are included.

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CONTENTS

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NOTATION

Unless stated otherwise, the following notation is used.

α	Absorptivity
Δ	Area
ϵ	Emissivity
ρc	Heat Capacity
σ	Stefan-Boltzmann Constant ($5.66961\ (10^{-8})\ W\,m^{-2}K^4$)
τ	Time
B	Boundary region (B_e represents an element boundary.)
C	Heat Capacity Matrix
F	Force Matrix
$F_{m\text{-}n}$	Form Factor between m, n
h	Heat Transfer (convective) Coefficient
i,j,k,l,m,n	Nodes of an element (subscripts)
k	Conductivity coefficient
K	Conductivity matrix
L	Boundary length
q_n	Flux Normal to Surface
Q	Internal Heat Generation
S	Surface (S_e represents an element surface.)
t	Thickness
T	Temperature
T_∞	Ambient Fluid Temperature in convection
$\dot{T}$	$\partial T/\partial \tau$
V	Volume

Standard units of the International System have been used in all sample problems of this document.

FINITE ELEMENT METHOD FOR THERMAL ANALYSIS

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INTRODUCTION

The finite element method (FEM) was applied to structural analysis by Turner et al. about 17 years ago. As use of the method in the structural field has become widespread and its basic ideas are now more clearly defined, the approach is recognized as applicable to any problem that can be formulated in variational form. This recognition is significant because it stimulates extension of the technique into other fields, such as thermal analysis. Interest in this extension has been reinforced by the possibility of coordinating the two areas of analysis. Using the FEM approach, structural and thermal data are more easily exchanged; using the same computer programs, the two analyses are more closely interrelated and more truly representative of things as they are in the real world.

Conceptually, the FEM involves dividing the continuum into a finite number of elements—triangles, for example. The temperature field within each triangle is described in terms of the temperatures at the vertices. The heat equation with boundary conditions, written as an integral equation in variational form, is minimized. The result is a system of algebraic equations which may be solved for the desired solution matrix of predicted temperatures.

Work has been done to develop and verify the FEM approach for thermal analysis. Computer programs exist to handle two-dimensional transient or steady-state heat conduction with internal heat generation and the following boundary conditions: convection or a specified temperature along the boundary B, and radiation or a constant heat flux normal to B. The program described in this paper can handle the two-dimensional case, as well as the three-dimensional case subject to the same boundary conditions, or the two-dimensional case with radiation, convection, or normal flux over the surface (rather than along the boundary) of the two-dimensional continuum. This paper presents the theoretical basis of the program, several illustrative problems, and computer program listings.

THEORETICAL DEVELOPMENT

Two-dimensional Case

Consider the equation for two-dimensional (x,y) transient heat flow in a uniformly thick plate of thickness t

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + Q - \rho c \frac{\partial T}{\partial \tau} = 0 \qquad (1)$$

where the desired temperature T must satisfy the following boundary conditions:

- Specified temperature T along boundary segment B
- Fixed flux q_n normal to B or surface S, $q_n = -k(\frac{\partial T}{\partial n})$
- Convection between the region and its surroundings along B or over the surface S, $h(T_\infty - T) = k\,(\frac{\partial T}{\partial n})$
- Radiation between the region and its surroundings along B or over the surface S, or radiation between two boundary or surface segments i and j, $\sigma F_{i\text{-}j}(\epsilon_i T_i^4 - a_i T_j^4) = -k\,(\frac{\partial T}{\partial n})$

Equation (1) together with its boundary conditions specifies the problem in a unique manner. Using the calculus of variations, an alternate formulation is possible by which the entire problem may be expressed in one equation—an integral equation in variational form:

$$
\begin{aligned}
X = & \int_0^t \iint_S \left[\frac{1}{2} k_x \left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{2} k_y \left(\frac{\partial T}{\partial y}\right)^2 - QT + T\rho c \frac{\partial T}{\partial \tau}\right] dS\, dz \\
& + \int_0^t \int_B q_n T\, dB\, dz + \iint_S q_n T\, dS + \int_0^t \int_B \left(\frac{1}{2} hT^2 - hT_\infty T\right) dB\, dz \\
& + \iint_S \left(\frac{1}{2} hT^2 - hT_\infty T\right) dS + \int_0^t \int_B q_r T\, dB\, dz \\
& + \iint_S q_r T\, dS
\end{aligned}
\qquad (2)^*
$$

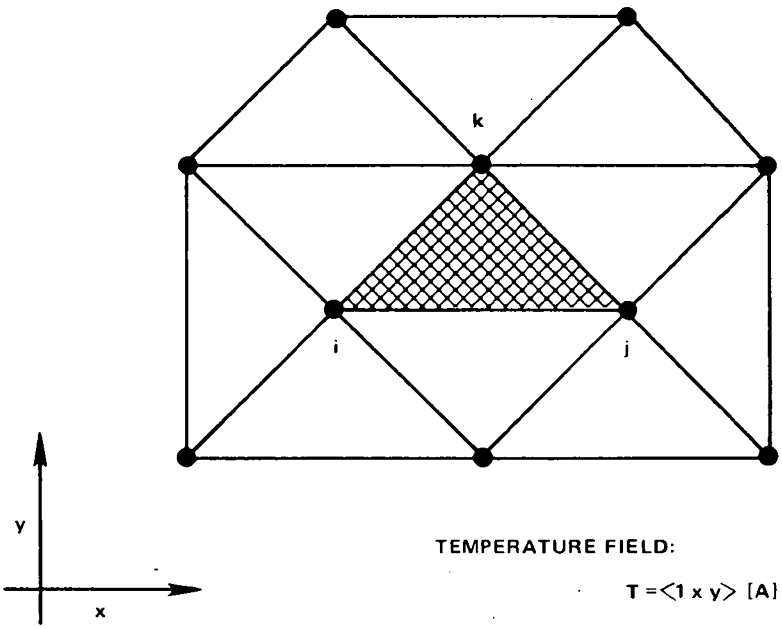
By Euler's theorem, this function has a minimum represented by Equation (1) with its boundary conditions.

Minimization of X requires an explicit formulation of the temperature T. To find this, the region is subdivided into a finite number of triangular elements (Figure 1), and Equation (2) may be restated as: $X = \sum X_e$, where the summation is taken over all elements e, and X_e represents the functional applied to each triangular element. Assuming the temperature field is linearly distributed throughout each element,

$$
T = a + bx + cy = [1\ x\ y]\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \qquad (3)
$$

within an element.

*Note: q_r represents radiative flux. The actual definition and treatment of this begins on page 15.



$$T = \langle 1\ x\ y \rangle [A] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

Figure 1. Division of continuum into a finite number of elements.

To determine the unknowns, a, b, and c, the known temperatures at each of the vertices, or nodes, of the triangular element are used to set up a system of equations:

$$T_i = a + bx_i + cy_i$$
$$T_j = a + bx_j + cy_j$$
$$T_k = a + bx_k + cy_k$$

In matrix form, this becomes

$$\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

Solving the system explicitly by Cramer's rule yields

$$a = \frac{(x_j y_k - x_k y_j)T_i + (x_k y_i - x_i y_k)T_j + (x_i y_j - x_j y_i)T_k}{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}$$

$$b = \frac{(y_j - y_k)T_i + (y_k - y_i)T_j + (y_i - y_j)T_k}{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}$$

$$c = \frac{(x_k - x_j)T_i + (x_i - x_k)T_j + (x_j - x_i)T_k}{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}$$

so that

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} x_j y_k - x_k y_j & x_k y_i - x_i y_k & x_i y_j - x_j y_i \\ y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \tag{4}$$

where

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

is equal to twice the area of the triangle defined by points i, j, and k.

Substituting Equation (4) into Equation (3) yields

$$T = [1 \; x \; y] \, \frac{1}{2\Delta} \begin{bmatrix} x_j y_k - x_k y_j & x_k y_i - x_i y_k & x_i y_j - x_j y_i \\ y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

This result can be simplified by letting $[N] = [1 \; x \; y]$ and $[A] = \frac{1}{2\Delta}$ times the square matrix:

$$T = [N] \; [A] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \tag{5}$$

The element temperature, T, as described by Equation (5) is an explicit and unique linear function of the nodal temperatures T_i, T_j, T_k and the spatial coordinates x and y. This equation provides the desired explicit formulation of T needed to minimize Equation (2).

Evaluation of $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial \tau}$ with T defined by Equation (5) yields the following explicit expressions for each of these terms:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left([1\ x\ y]\ [A]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}\right) = [0\ 1\ 0]\ [A]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

$$\frac{\partial T}{\partial y} = [0\ 0\ 1]\ [A]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \qquad \frac{\partial T}{\partial \tau} = [1\ x\ y]\ [A]\begin{Bmatrix} \dot{T}_i \\ \dot{T}_j \\ \dot{T}_k \end{Bmatrix} \tag{6}$$

The functional $X=\sum X_e$ (Equation 2) can be minimized with respect to the temperature $\bar{T}$ over the entire surface S by the equation

$$\left\{\frac{\partial X}{\partial T}\right\} = \begin{Bmatrix} \frac{\partial X}{\partial T_1} \\ \frac{\partial X}{\partial T_2} \\ \vdots \\ \frac{\partial X}{\partial T_n} \end{Bmatrix} = \begin{Bmatrix} \sum \frac{\partial X_e}{\partial T_1} \\ \sum \frac{\partial X_e}{\partial T_2} \\ \vdots \\ \sum \frac{\partial X_e}{\partial T_n} \end{Bmatrix} = \sum_e \frac{\partial X_e}{\partial T}$$

where $T_1, T_2, \cdots T_n$ are the temperatures at the nodal points $1, 2, \cdots n$ in the surface S.

For each triangular element e,

$$\left\{ \frac{\partial X_e}{\partial T} \right\} = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \\ \dfrac{\partial X_e}{\partial T_i} \\ 0 \\ \vdots \\ 0 \\ \dfrac{\partial X_e}{\partial T_j} \\ 0 \\ \vdots \\ 0 \\ \dfrac{\partial X_e}{\partial T_k} \\ 0 \\ \vdots \\ 0 \end{array} \right\}$$

where i, j, and k are the nodal points of e.

Now, consider the expression $\frac{\partial X_e}{\partial T_i}$ defined over the element surface S_e with boundary B_e. (Analogous equations will hold for $\frac{\partial X_e}{\partial T_j}$ and $\frac{\partial X_e}{\partial T_k}$.)

$$\begin{aligned}
\frac{\partial X_e}{\partial T_i} = & \int_0^t \left[\iint_{S_e} \left[k_x \frac{\partial T}{\partial x} \frac{\partial}{\partial T_i} \frac{\partial T}{\partial x} + k_y \frac{\partial T}{\partial y} \frac{\partial}{\partial T_i} \frac{\partial T}{\partial y} \right] dS - \iint_{S_e} Q \frac{\partial T}{\partial T_i} dS \right] dz \\
& + \int_0^t \iint_{S_e} \rho c \frac{\partial T}{\partial t} \frac{\partial T}{\partial T_i} dS \, dz + \int_0^t \int_{B_e} q_n \frac{\partial T}{\partial T_i} dB \, dz + \iint_{S_e} q_n \frac{\partial T}{\partial T_i} dS \\
& + \int_0^t \int_{B_e} \left(hT \frac{\partial T}{\partial T_i} - hT_\infty \frac{\partial T}{\partial T_i} \right) dB \, dz + \iint_{S_e} \left(hT \frac{\partial T}{\partial T_i} - hT_\infty \frac{\partial T}{\partial T_i} \right) dS \\
& + \int_0^t \int_{B_e} q_r \frac{\partial T}{\partial T_i} dB \, dz + \iint_{S_e} q_r \frac{\partial T}{\partial T_i} dS
\end{aligned} \tag{7}$$

To evaluate $\frac{\partial X_e}{\partial T_i}$, consider each portion of the integral equation separately. First, consider the double integral over S_e dealing with conduction.

Conduction

Note that by defining $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$ explicitly as in Equation (6)

$$\frac{\partial}{\partial T_i}\frac{\partial T}{\partial x} = [0\ 1\ 0]\ [A] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = a_{21} = a_{ji}$$

$$\frac{\partial}{\partial T_i}\frac{\partial T}{\partial y} = [0\ 0\ 1]\ [A] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = a_{31} = a_{ki}$$

Thus

$$\int_0^t\iint_{S_e} \left[k_x \frac{\partial T}{\partial x}\frac{\partial}{\partial T_i}\frac{\partial T}{\partial x} + k_y \frac{\partial T}{\partial y}\frac{\partial}{\partial T_i}\frac{\partial T}{\partial y}\right] dS\ dz$$

$$= \int_0^t\iint k_x \left[[0\ 1\ 0]\ [A]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}\right] a_{ji} + k_y \left[[0\ 0\ 1]\ [A]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}\right] a_{ki}\ dx\,dy\,dz$$

$$= \int_0^t\iint \left[k_x\ (a_{ji}T_i + a_{jj}T_j + a_{jk}T_k)a_{ji} + k_y\ (a_{ki}T_i + a_{kj}T_j + a_{kk}T_k)a_{ki}\right] dx\ dy\ dz$$

Assuming constant k_x and k_y over the element

$$= \int_0^t\iint [a_{ji}\ a_{ki}] \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} dx\ dy\ dz$$

$$= [a_{ji}\ a_{ki}] \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \iint dx\ dy \int_0^t dz$$

$$= \frac{1}{2\Delta}[\,y_j - y_k \quad x_k - x_j\,] \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \frac{1}{2\Delta} \begin{bmatrix} y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} t\ \Delta$$

$$\left(\frac{\partial X_{COND}}{\partial T_i}\right) = \frac{t}{4\Delta}\,[y_j - y_k \quad x_k - x_j]\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\begin{bmatrix} y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

Combining this with similar expressions for $\dfrac{\partial X_{COND}}{\partial T_j}$ and $\dfrac{\partial X_{COND}}{\partial T_k}$ results in

$$\left\{\frac{\partial X_{COND}}{\partial T}\right\}_e = \frac{t}{4\Delta}\begin{bmatrix} y_j - y_k & x_k - x_j \\ y_k - y_i & x_i - x_k \\ y_i - y_j & x_j - x_i \end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$$

$$\bullet \begin{bmatrix} y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \qquad (8)$$

Performing these operations for each element and summing the resulting matrices yields the total n-by-n conductivity matrix K for the continuum (where n = the number of individual nodes). The above portion of the K matrix will contain nonzero terms to be added to K_{ii}, K_{ij}, K_{ik}, K_{ji}, K_{jj}, K_{jk}, K_{ki}, K_{kj}, K_{kk}, and zeros elsewhere.

Internal Heat Generation

The internal heat generation portion of this equation is

$$\left(\frac{\partial X_Q}{\partial T_i}\right)_e = -\int_0^t\!\!\iint_{S_e} Q\left(\frac{\partial T}{\partial T_i}\right) dS\, dz = -\int_0^t\!\!\iint Q\ [1\ x\ y]\ [A]\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} dx\, dy\, dz$$

$$= -\iint Q\ (a_{ii} + a_{ji}x + a_{ki}y)\, dx\, dy \int_0^t dz$$

Assuming constant Q over the element

$$= -Q\ \Delta\ (a_{ii} + a_{ji}\bar{x} + a_{ki}\bar{y})\ t$$

since

$$\iint dx\, dy = \Delta, \quad \bar{x} = \frac{1}{\Delta}\iint x\, dx\, dy, \quad \bar{y} = \frac{1}{\Delta}\iint y\, dx\, dy$$

where $\bar{x}$ and $\bar{y}$ are coordinates of the centroid of the element. Now, $\bar{x} = (1/3)(x_i + x_j + x_k)$ and $\bar{y} = (1/3)(y_i + y_j + y_k)$. Substituting these into the equation yields

$$\left(\frac{\partial X_Q}{\partial T_i}\right)_e = -Q\Delta t\left[a_{ii} + \frac{a_{ji}}{3}(x_i + x_j + x_k) + \frac{a_{ki}}{3}(y_i + y_j + y_k)\right]$$

$$= -\frac{Q\Delta t}{2\Delta}\left(x_j y_k - x_k y_j + \frac{(y_j - y_k)}{3}(x_i + x_j + x_k) + \frac{(x_k - x_j)}{3}(y_i + y_j + y_k)\right)$$

$$= -\frac{Q\Delta t}{3}\left(\frac{1}{2\Delta}\right)\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = -\frac{Q\Delta t}{3}\left(\frac{1}{2\Delta}\right)(2\Delta) = -\frac{Q\Delta t}{3}$$

Similarly, $\dfrac{\partial X_Q}{\partial T_j} = \dfrac{\partial X_Q}{\partial T_k} = -\dfrac{Q\Delta t}{3}$. Thus if $F = \sum F$ for each element, the contribution to the force matrix is given by

$$\left\{\frac{\partial X_Q}{\partial T}\right\}_e = \begin{Bmatrix} -Q\Delta t/3 \\ -Q\Delta t/3 \\ -Q\Delta t/3 \end{Bmatrix} \qquad (9)$$

where $-Q\Delta t/3$ is added to the terms F_i, F_j, and F_k.

Heat Capacity

The contribution due to the heat capacity term is

$$\left(\frac{\partial X_{\rho c}}{\partial T_i}\right)_e = \int_0^t\iint_{S_e} \rho c\,\frac{\partial T}{\partial \tau}\,\frac{\partial T}{\partial T_i}\,dS\,dz = \int_0^t\iint \rho c\,[1\ x\ y]\,[A]\begin{Bmatrix}\dot{T}_i \\ \dot{T}_j \\ \dot{T}_k\end{Bmatrix}[1\ x\ y]\,[A]\begin{Bmatrix}1 \\ 0 \\ 0\end{Bmatrix}dx\,dy\,dz$$

Assuming constant ρc over e

$$= \rho c\iint (a_{ii} + a_{ji}x + a_{ki}y)\,[1\ x\ y]\,[A]\begin{Bmatrix}T_i \\ T_j \\ T_k\end{Bmatrix}dx\,dy\int_0^t dz$$

$$= \rho c\iint\left[a_{ii} + a_{ji}x + a_{ki}y \quad a_{ii}x + a_{ji}x^2 + a_{ki}xy \quad a_{ii}y + a_{ji}xy + a_{ki}y^2\right]dx\,dy\,[A]\begin{Bmatrix}\dot{T}_i \\ \dot{T}_j \\ \dot{T}_k\end{Bmatrix}t$$

Using the identities $\iint x^2\, dx\, dy = \frac{\Delta}{6}(x_ix_j + x_ix_k + x_jx_k + \sum x_m^2)$ and $\iint xy\, dx\, dy = \frac{\Delta}{12}(\sum_{m,n} x_m y_n + \sum_m x_m y_m)$, where m = i, j, k and n = i, j, k, this becomes

$$\left(\frac{\partial X_{\rho c}}{\partial T_i}\right)_e = \rho c t \frac{\Delta}{12}\Big[12a_{ii} + 4a_{ji}\sum_m x_m + 4a_{ki}\sum_m y_m,\ 4a_{ii}\sum_m x_m + 2a_{ji}$$

$$(x_ix_j + x_ix_k + x_jx_k + \sum x_m^2)\ a_{ki}\sum_m x_m y_n + a_{ki}\sum_m x_m y_m,\ 4a_{ii}\sum_m y_m$$

$$+ a_{ji}\sum_{m,n} x_m y_n + a_{ji}\sum_m x_m y_m + 2a_{ki}(y_iy_j + y_iy_k + y_jy_k + \sum y_m)\Big]\,[A]\,\{T\}$$

Defining $b_m^n = a_{im} + a_{jm}x_n + a_{km}y_n$, this may be put into the form

$$\left(\frac{\partial X_{\rho c}}{\partial T_i}\right)_e = \frac{\rho c t}{12}\begin{bmatrix} b_i^i & b_i^j & b_i^k \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} b_i^i & b_j^i & b_k^i \\ b_i^j & b_j^j & b_k^j \\ b_i^k & b_j^k & b_k^k \end{bmatrix}\{\dot{T}\} = \frac{\rho c t \Delta}{12}\,[2\ \ 1\ \ 1]$$

for the two-dimensional triangle. Consideration of $\frac{\partial X_{\rho c}}{\partial T_j}$ and $\frac{\partial X_{\rho c}}{\partial T_k}$ shows that the additions to the total heat capacity matrix [C] due to this element are

$$\left\{\frac{\partial X_{\rho c}}{\partial T}\right\}_e = \frac{\rho c t \Delta}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \tag{10}$$

Thus far, three matrices (K, F, and C) have been defined. The conductivity matrix, K, is multiplied by the vector

$$\{T\} = \begin{Bmatrix} T_1 \\ \vdots \\ T_n \end{Bmatrix};$$

the force matrix, F, is multiplied by one; and the heat capacity matrix, C, is multiplied by

$$\frac{\partial T}{\partial \tau} = \begin{Bmatrix} \dot{T}_1 \\ \vdots \\ \dot{T}_n \end{Bmatrix}.$$

In the evaluation of the remaining integrals of X, if a term is to be multiplied by $\{T\}$, it will be placed in the appropriate row and column of $[K]$. If it is neither a coefficient of $\{T\}$ nor of $\{\dot{T}\}$, it will be placed in $\{F\}$.

Convection

Case 1: Convection along boundary of thickness t — For convection along an element boundary (i, j), one again assumes that T varies linearly so that the temperature T at any point on the boundary of length L is given by

$$T = T_i + (T_j - T_i)\frac{x}{L}$$

Now

$$\left(\frac{\partial X_h}{\partial T_i}\right)_e = \int_0^t\int_{B_e}\left(hT\frac{\partial T}{\partial T_i} - hT_\infty\frac{\partial T}{\partial T_i}\right) dB\, dz$$

$$= \int_0^L h\left[T_i + (T_j - T_i)\frac{x}{L}\right]\left[(1 - \frac{x}{L}) - hT_\infty(1 - \frac{x}{L})\right] dx \int_0^t dz$$

If h and T_∞ are constant over B_e

$$= h\left[(T_i - T_\infty)L + \frac{(T_j - 2T_i + T_\infty)L^2}{2L} + \frac{(T_i - T_j)L^3}{3L^2}\right] t$$

$$= hLt\left(T_i - T_\infty + \frac{T_j}{2} - T_i + \frac{T_\infty}{2} + \frac{T_i}{3} - \frac{T_j}{3}\right)$$

$$= hLt\left(\frac{1}{3}T_i + \frac{1}{6}T_j - \frac{1}{2}T_\infty\right)$$

$$= \begin{bmatrix}\frac{hLt}{3} & \frac{hLt}{6} & 0\end{bmatrix}\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix} - \begin{Bmatrix}hLT_\infty t/2\\ hLT_\infty t/2\\ 0\end{Bmatrix}$$

The expression for $\frac{\partial X_h}{\partial T_j}$ is similar, and $\frac{\partial X_h}{\partial T_k} = 0$ because T_k is not a part of the convective boundary. Thus

$$\left\{\frac{\partial X_h}{\partial T}\right\}_e = \begin{bmatrix}hLt/3 & hLt/6\\ hLt/6 & hLt/3\end{bmatrix}\begin{Bmatrix}T_i\\ T_j\end{Bmatrix} - \begin{Bmatrix}hLT_\infty t/2\\ hLT_\infty t/2\end{Bmatrix} \tag{11}$$

where the terms of the square matrix are added into $[K]$, increasing K_{ii}, K_{ij}, K_{ji}, and K_{jj}, and the terms of the column matrix are added to F_i and F_j.

<u>Case 2: Convection over surface</u> – Now assume that convection exists not only along an outside boundary but over the entire element surface (i, j, k). Then

$$\left(\frac{\partial X_h}{\partial T_i}\right)_e = \iint_{S_e} \left(hT \frac{\partial T}{\partial T_i} - hT_\infty \frac{\partial T}{\partial T_i} \right) dS$$

Assuming h and T_∞ constant over the element surface

$$= \left(h \iint [1\ x\ y]\ [A] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} [1\ x\ y]\ [A] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} dx\, dy \right.$$

$$\left. - hT_\infty \iint [1\ x\ y]\ [A] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} dx\, dy \right)$$

$$= \left(h \iint [1\ x\ y]\ [A] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} [1\ x\ y] \begin{Bmatrix} a_{ii} \\ a_{ji} \\ a_{ki} \end{Bmatrix} dx\, dy \right.$$

$$\left. - hT_\infty \iint [1\ x\ y] \begin{Bmatrix} a_{ii} \\ a_{ji} \\ a_{ki} \end{Bmatrix} dx\, dy \right)$$

$$= h \frac{\Delta}{12} (2T_i + T_j + T_k) - h \frac{\Delta}{3} T_\infty$$

$$= \frac{h\Delta}{12} [2\ 1\ 1] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} - \frac{h\Delta}{3} T_\infty$$

The terms $\frac{\partial X_h}{\partial T_j}$ and $\frac{\partial X_h}{\partial T_k}$ are found in a similar manner, yielding a final convective matrix of

$$\left\{\frac{\partial X_h}{\partial T}\right\}_e = \frac{h\Delta}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} - \begin{Bmatrix} h\Delta T_\infty/3 \\ h\Delta T_\infty/3 \\ h\Delta T_\infty/3 \end{Bmatrix} \tag{12}$$

Each element of the 3-by-3 matrix (each term multiplied by $h\Delta/12$) is added to its corresponding (i, j, k) component of the larger matrix [K]. Likewise, $-hT_\infty\Delta/3$ is added to F_i, F_j, and F_k.

Specified Flux Normal to Boundary or Surface

Case 1: Normal to boundary of thickness t — In this situation

$$\left(\frac{\partial X_q}{\partial T_i}\right)_e = \int_0^t \int_{B_e} q_n \frac{\partial T}{\partial T_i} \, dB \, dz$$

Since T is assumed to be linear over the element, it also is linear along a boundary (i, j) and thus may be written as $T = T_i + (T_j - T_i)\, x/L$, where L is the length of (i, j). Using this expression

$$\frac{\partial T}{\partial T_i} = 1 - \frac{x}{L}$$

$$\frac{\partial T}{\partial T_j} = \frac{x}{L}$$

$$\frac{\partial T}{\partial T_k} = 0$$

Therefore, assuming that the flux is normal to boundary (i, j)

$$\left(\frac{\partial X_q}{\partial T_i}\right)_e = \int_0^L q_n \left(1 - \frac{x}{L}\right) dx \int_0^t dz$$

If q_n is constant on (i, j)

$$= \left(q_n x - q_n \frac{x^2}{2L}\right)\Bigg|_0^L t = q_n \left(L - \frac{L}{2}\right)t = \frac{q_n L t}{2}$$

Similarly

$$\frac{\partial X_q}{\partial T_j} = \int_0^L q_n \frac{x}{L} \, dx \int_0^t dz = \left(q_n \frac{x^2}{2L}\right) t = \frac{q_n L t}{2}$$

$$\frac{\partial X_q}{\partial T_k} = 0$$

The flux matrix for this element is, therefore

$$\left\{\frac{\partial X_q}{\partial T}\right\}_e = \left\{\begin{matrix} tq_n L/2 \\ tq_n L/2 \\ 0 \end{matrix}\right\} \begin{matrix} \leftarrow i \\ \leftarrow j \\ \leftarrow k \end{matrix} \tag{13}$$

Since this matrix is multiplied by neither $\{T\}$ nor $\{\dot{T}\}$, it is added to $\{F\}$, increasing the terms F_i and F_j by $(tq_n L/2)$.

Case 2: Normal to surface — If the flux is normal to the surface of element (i, j, k), then

$$\left(\frac{\partial X_q}{\partial T_i}\right)_e = \iint_{S_e} q_n \frac{\partial T}{\partial T_i}\, dS = q_n \iint [1\ x\ y]\,[A] \left\{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right\} dS$$

$$= q_n \iint_{S_e} [1\ x\ y] \left\{\begin{matrix} a_{ii} \\ a_{ji} \\ a_{ki} \end{matrix}\right\} dS$$

where q_n is constant over (i, j, k)

$$= q_n \iint a_{ii} + a_{ji}x + a_{ki}y \; dS$$

$$= \frac{q_n \Delta}{3} \left(3a_{ii} + a_{ji}(x_i + x_j + x_k) + a_{ki}(y_i + y_j + y_k)\right)$$

$$= \frac{q_n \Delta}{3} \left(\frac{1}{2\Delta}\right)(2\Delta) = \frac{q_n \Delta}{3}$$

Likewise, $\partial X_q/\partial T_j = \partial X_q/\partial T_k = q_n \Delta/3$, and the flux addition to the $\{F\}$ matrix becomes

$$\left\{\frac{\partial X_q}{\partial T}\right\}_e = \left\{\begin{matrix} q_n \Delta/3 \\ q_n \Delta/3 \\ q_n \Delta/3 \end{matrix}\right\} \tag{14}$$

where $q_n \Delta/3$ is added to F_i, F_j, and F_k.

Radiation

Radiation between element e and other radiative elements R or between boundary e and other radiative boundaries R may be described for the transient situation by

$$q_r = \sum_R (\sigma\epsilon_e \Delta_e F_{e\text{-}R}\ T_e^4 - \sigma a_e \Delta_e F_{e\text{-}R}\ T_R^4)$$

where ϵ, a, and $F_{e\text{-}R}$ are assumed constant over individual elements

$$= \sigma\epsilon_e \Delta_e T_e^4 \sum F_{e\text{-}R} - \sigma a_e \Delta_e \sum F_{e\text{-}R} T_R^4$$

$$= \sigma\epsilon_e \Delta_e T_e^4 - \sigma a_e \Delta_e \sum F_{e\text{-}R} T_R^4$$

$$\approx \sigma\epsilon_e \Delta_e T_{\textcircled{e}}^4 - \sigma a_e \Delta_e \sum F_{e\text{-}R} T_{\textcircled{R}}^4$$

where $T_{\textcircled{e}}$ is the temperature of boundary e found in the previous time step of the computer-iterative algorithm. This approximation is numerically stable and accurate if $\delta\tau$ is chosen small enough to ensure small values of $|\ T_e^4 - T_{\textcircled{e}}^4 |$ and $|\ T_R^4 - T_{\textcircled{R}}^4 |$. Treating T^4 in this manner leads to the explicit evaluation of a q_r at the beginning of each time step. Hence, during that time step, q_r may be treated as a specified constant flux:

$$\begin{aligned} \left(X_r\right)_e &= \iint_{S_e} (\sigma\epsilon_e T_{\textcircled{e}}^4 - \sigma a_e \sum_R F_{e\text{-}R} T_{\textcircled{R}}^4)\ T_e dS \\ \left(\frac{\partial X_r}{\partial T_i}\right)_e &= (\sigma\epsilon_1 T_{\textcircled{e}}^4 - \sigma a_e \sum F_{e\text{-}R} T_{\textcircled{R}}^4) \iint_{S_e} \frac{\partial T}{\partial T_i} dS \end{aligned} \tag{15}$$

<u>*Case 1: Radiation between two boundaries or between a boundary and a constant temperature heat source*</u> – Letting $q_n = (\sigma\epsilon_e T_e^4 - \sigma a_e \sum_R F_{e\text{-}R} T_{\textcircled{R}}^4)$, this case is handled in the same way as Case 1 of "Flux Normal to Boundary or Surface," the previous section. The expression corresponding to Equation (13) is

$$\left\{\frac{\partial X_r}{\partial T}\right\}_e = \begin{Bmatrix} tq_n L/2 \\ tq_n L/2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{tL}{2}(\sigma\epsilon_e T_{\textcircled{e}}^4 - \sigma a_e \sum F_{e\text{-}R} T_{\textcircled{R}}^4) \\ \frac{tL}{2}(\sigma\epsilon_e T_{\textcircled{e}}^4 - \sigma a_e \sum F_{e\text{-}R} T_{\textcircled{R}}^4) \\ 0 \end{Bmatrix} \begin{matrix} \leftarrow i \\ \leftarrow j \\ \leftarrow k \end{matrix} \tag{16}$$

where $T_{e} = (T_{i} + T_{j})/2$ when i and j are nodes of the radiating boundary e. Also, $T_{R} = (T_{p} + T_{m})/2$ when T_{R} is on an element boundary, (p, m) and $T_{R} = T_R$ when T_R is a specified-temperature heat source.

Case 2: Radiation between two surfaces or between a surface and a constant-temperature heat source – Following the procedure in Case 2 of the previous section, the following addition to the F matrix results

$$\left\{\frac{\partial X_r}{\partial T}\right\}_e = \begin{Bmatrix} q_n/3 \\ q_n/3 \\ q_n/3 \end{Bmatrix} = \begin{array}{l} \left\{\begin{array}{l} \Delta_e/3 \ (\sigma\epsilon_e T_{e}^4 - \sigma a_e \sum F_{e\text{-}R} T_{R}^4) \\ \Delta_e/3 \ (\sigma\epsilon_1 T_{e}^4 - \sigma a_e \sum F_{e\text{-}R} T_{R}^4) \\ \Delta_e/3 \ (\sigma\epsilon_e T_{e} - \sigma a_e \sum F_{e\text{-}R} T_{R}^4) \end{array}\right\} \end{array} \begin{array}{l} \leftarrow i \\ \leftarrow j \\ \leftarrow k \end{array} \tag{17}$$

where $T_{e} = (T_{i} + T_{j} + T_{k})/3$ and i, j, and k are nodes of the radiating element e. Also, $T_{R} = (T_{p} + T_{m} + T_{n})/3$ if T_R is a part of the modeled continuum, and $T_{R} = T_R$ if T_R is a specified-temperature heat source.

Three-dimensional Case

The previous development for two-dimensional heat transfer may be expanded to handle the three-dimensional heat equation

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + Q - \rho c \frac{\partial T}{\partial \tau} = 0 \tag{18}$$

where the desired T must satisfy the same boundary conditions given for the two-dimensional case.

In variational form this problem becomes

$$\begin{aligned} X = \iiint_V \left[\frac{k_x}{2}\left(\frac{\partial T}{\partial x}\right)^2 + \frac{k_y}{2}\left(\frac{\partial T}{\partial y}\right)^2 + \frac{k_z}{2}\left(\frac{\partial T}{\partial z}\right)^2 - QT + \rho c \frac{\partial T}{\partial t}\right] dV + \iint_S q_n T \, dS \\ + \iint_S hT^2 - hT_\infty T \ dS + \iint_S q_r T \, dS \end{aligned} \tag{19}$$

which, as before, may be minimized to yield $\{T\}$. Assuming that T is linear over the three-dimensional tetrahedron and proceeding as for two dimensions, one may derive

$$T = a + bx + cy + dz = [1\ x\ y\ z] \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$$= [1\ x\ y\ z] \frac{1}{6V} \begin{bmatrix} a_i & a_j & a_k & a_l \\ b_i & b_j & b_k & b_l \\ c_i & c_j & c_k & c_l \\ d_i & d_j & d_k & d_l \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_l \end{Bmatrix} \tag{20}$$

$$= [1\ x\ y\ z] \frac{1}{6V} [A] \{T\} \quad \text{for the tetrahedron}$$

where

$$V = \frac{\begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \end{vmatrix}}{6} \quad \text{is the volume of the tetrahedron.}$$

$$a_i = \begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} \qquad b_i = \begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix} \tag{21}$$

$$c_i = \begin{vmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_l & 1 & z_l \end{vmatrix} \qquad d_i = \begin{vmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_l & y_l & 1 \end{vmatrix}$$

Other constants may be defined by cyclic interchange of the subscripts in the order i,j,k,l. Each time a cyclic interchange is made, it is necessary to multiply the newly defined constant by −1.

The process of minimizing X_e with respect to T_i leads to

$$\frac{\partial X_e}{\partial T_i} = \iiint_{V_e} \left[k_x \frac{\partial T}{\partial x} \frac{\partial}{\partial T_i} \frac{\partial T}{\partial x} + k_y \frac{\partial T}{\partial y} \frac{\partial}{\partial T_i} \frac{\partial T}{\partial y} \right] dV$$

$$- \iiint_{V_e} Q \frac{\partial T}{\partial T_i} dV + \iiint_{V_e} \rho c \frac{\partial T}{\partial t} \frac{\partial T}{\partial T_i} dV + \iint_{S_e} q_n \frac{\partial T}{\partial T_i} dS \qquad (22)$$

$$+ \iint_{S_e} (hT \frac{\partial T}{\partial T_i} - hT_\infty \frac{\partial T}{\partial T_i}) dS + \iint_{S_e} q_r \frac{\partial T}{\partial T_i} dS$$

Note that the radiative, flux, and convective terms in this equation are identical to those for the surface conditions in the two-dimensional analysis. Hence, their evaluation has been completed. Now, consider the evaluation of the conductive, internal heat generation, and heat capacity integrals.

Conduction

Following the same procedure as for two-dimensional conduction, but including k_z and redefining T, the addition to the [K] matrix for element (i, j, k, l) becomes

$$\left\{ \frac{\partial X_{COND}}{\partial T} \right\}_e = \frac{1}{36V} \begin{bmatrix} b_i & c_i & d_i \\ b_j & c_j & d_j \\ b_k & c_k & d_k \\ b_l & c_l & d_l \end{bmatrix} \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{bmatrix} b_i & b_j & b_k & b_l \\ c_i & c_j & c_k & c_l \\ d_i & d_j & d_k & d_l \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_l \end{Bmatrix} \qquad (22)$$

Nonzero terms are added to $[K_{mn}]$ where m = i,j,k,l and n = i,j,k,l.

Internal Heat Generation

Using

$$\int dx\, dy\, dz = V$$

and

$$\bar{x} = \frac{1}{V} \int x\, dV \text{ (tetrahedron)}$$

$$\bar{y} = \frac{1}{V} \int y\, dV$$

$$\bar{z} = \frac{1}{V} \int z\, dV$$

where $(\bar{x}, \bar{y}, \bar{z})$ are coordinates of the centroid of the tetrahedron, and following the method used for internal heat generation in two dimensions yields

$$\left\{\frac{\partial X_Q}{\partial T}\right\}_e = \left\{\begin{matrix} \frac{-QV}{4} \\ \frac{-QV}{4} \\ \frac{-QV}{4} \\ \frac{-QV}{4} \end{matrix}\right\} \tag{23}$$

where $\frac{-QV}{4}$ is added to F_i, F_j, F_k, and F_l.

Heat Capacity

The contribution due to the heat capacity term is evaluated by using the identities

$$\int x^2\, dV = \frac{V}{10}\left[\left(\sum_m x_m^2\right) + x_i x_j + x_i x_k + x_i x_l + x_j x_k + x_j x_l + x_k x_l\right]$$

$$\int xy\, dV = \frac{V}{20}\left[\sum_{m,n} x_m y_n + \sum_m x_m y_m\right]$$

where m = i, j, k, l; n = i, j, k, l.

Again, by letting $b_m^n = \frac{1}{6V}(a_m + b_m x_n + c_m y_n + d_m z_n)$, the three-dimensional equivalent of Equation (10) becomes

$$\left\{\frac{\partial X_{\rho c}}{\partial T}\right\}_e = \left(\frac{\rho c V}{20}\right) \begin{bmatrix} b_i^i & b_i^j & b_i^k & b_i^l \\ b_j^i & b_j^j & b_j^k & b_j^l \\ b_k^i & b_k^j & b_k^k & b_k^l \\ b_l^i & b_l^j & b_l^k & b_l^l \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_i^i & b_j^i & b_k^i & b_l^i \\ b_i^j & b_j^j & b_k^j & b_l^j \\ b_i^k & b_j^k & b_k^k & b_l^k \\ b_i^l & b_j^l & b_k^l & b_l^l \end{bmatrix} \left\{\begin{matrix} T_i \\ T_j \\ T_k \\ T_l \end{matrix}\right\} \tag{24}$$

$$= \left(\frac{\rho c V}{20}\right) \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

General Matrix Equation

Performing the above operations for each element and summing the resulting matrices results in a system of simultaneous linear equations which may be written in matrix form as

$$[K]\{T\} + [C]\{\dot{T}\} + \{F\} = 0. \tag{25}$$

To facilitate computer solution, time derivatives are replaced by finite difference approximations, yielding

$$([K_\tau] + \frac{2}{\delta\tau}[C])\{T_\tau\} = (\frac{2}{\delta\tau}[C] - [K_{\tau-\delta\tau}])\{T_{\tau-\delta\tau}\} - \{F_\tau + F_{\tau-\delta\tau}\} \tag{26}$$

From Equation (25)

$$[K_\tau]\{T_\tau\} + [C]\{\dot{T}_\tau\} + \{F_\tau\} = 0 \tag{25a}$$

$$[K_{\tau-\delta\tau}]\{T_{\tau-\delta\tau}\} + [C]\{\dot{T}_{\tau-\delta\tau}\} + \{F_{\tau-\delta\tau}\} = 0 \tag{25b}$$

Then

$$\{T_\tau\} = \{T_{\tau-\delta\tau}\} + \Delta T = \{T_{\tau-\delta\tau}\} + \frac{(\delta\tau)\{\dot{T}_\tau\}}{2} + \frac{\{\dot{T}_{\tau-\delta\tau}\}}{2}$$

or

$$\{\dot{T}_\tau\} = \frac{2}{\delta\tau}(\{T_\tau\} - \{T_{\tau-\delta\tau}\}) - \{\dot{T}_{\tau-\delta\tau}\}$$

Substituting this result into Equation (25a) gives

$$[K_\tau]\{T_\tau\} + [C](\frac{2}{\delta\tau}(\{T_\tau\} - \{T_{\tau-\delta\tau}\}) - \{\dot{T}_{\tau-\delta\tau}\}) + \{F_\tau\} = 0$$

or

$$([K_\tau] + \frac{2}{\delta\tau}[C])\{T_\tau\} = [C](\{\dot{T}_{\tau-\delta\tau}\} + \frac{2}{\delta\tau}\{T_{\tau-\delta\tau}\}) - \{F_\tau\} \tag{27}$$

Now, from Equation (25b)

$$\{\dot{T}_{\tau-\delta\tau}\} = [C]^{-1}\left(-[K_{\tau-\delta\tau}]\{T_{\tau-\delta\tau}\} - \{F_{\tau-\delta\tau}\}\right)$$

Then, substituting this into Equation (27) yields

$$([K_\tau] + \frac{2}{\delta\tau}[C])\left\{T_\tau\right\} = [C]\,(-[C]^{-1}\,[K_{\tau-\delta\tau}]\left\{T_{\tau-\delta\tau}\right\} - [C]^{-1}\left\{F_{\tau-\delta\tau}\right\})$$

$$+\ [C]\,\frac{2}{\delta\tau}\left\{T_{\tau-\delta\tau}\right\} - \left\{F_\tau\right\}$$

which can be reduced to Equation (26). To find only the steady state distribution for a problem with no radiative boundaries, the matrix equation $[K]\left\{T\right\} + \left\{F\right\} = 0$ is used. Solution of this linear system yields the steady state results in one step. Using Equation (26) instead would require a number of $\delta\tau$ iterations before reaching the steady state temperatures. For problems including radiation, it is necessary to use Equation (26) with an appropriate $\delta\tau$ and iterate to the steady state temperature distribution. (This iteration is necessitated by the use of the approximation $T_e^4 = T_{(e)}^4$.)

A computer program has been written to solve the matrix Equation (26). The program also includes consideration of the boundary condition that holds specified boundary nodes at constant temperatures. To consider such a boundary condition, let

$$A \;=\; [K_\tau] + \frac{2}{\delta\tau}[C]$$

$$B \;=\; \frac{2}{\delta\tau}[C] - [K_{\tau-\delta\tau}]$$

$$C \;=\; -(\left\{F_\tau\right\} + \left\{F_{\tau-\delta\tau}\right\}).$$

Then

$$A\left\{T_\tau\right\} = B\left\{T_{\tau-\delta\tau}\right\} + C. \tag{28}$$

Suppose that of the n nodes representing the thermal model, the last k nodes are held fixed. Thus the first m = n − k nodes are variable. The matrix Equation (28) can be partitioned into m-by-m and k-by-k submatrices as follows.

$$\begin{bmatrix} A_{m\text{ by }m} & A_{m\text{ by }k} \\ A_{k\text{ by }m} & A_{k\text{ by }k} \end{bmatrix} \begin{Bmatrix} T_m \\ T_k \end{Bmatrix}_\tau = \begin{bmatrix} B_{m\text{ by }m} & B_{m\text{ by }k} \\ B_{k\text{ by }m} & B_{k\text{ by }k} \end{bmatrix} \begin{Bmatrix} T_{(m)} \\ T_{(k)} \end{Bmatrix}_{\tau-\delta\tau} + \begin{Bmatrix} C_m \\ C_k \end{Bmatrix} \tag{29}$$

Matrix multiplication yields two matrix equations

$$A_{mm} T_m + A_{mk} T_k = B_{mm} T_{(m)} + B_{mk} T_{(k)} + C_m \tag{29a}$$

$$A_{km} T_m + A_{kk} T_k = B_{km} T_{(m)} + B_{kk} T_{(k)} + C_k \tag{29b}$$

Only the first equation need be solved for the unknown temperatures $\{T_m\}$. If the specified nodes are not necessarily the last k, a similarity transformation may be applied to cast the system into this form. After solution is obtained, again it may be applied to return the answers to their original order.

Sample Problems

Following are some sample problems in one, two, and three dimensions to which the preceding method was applied. Each sample includes a description of the problem and the temperature distributions predicted at certain times as well as at steady state. For the first three problems, exact or finite difference temperature predictions also are given for comparison.

Problem 1: One Dimension—Finite Element and Exact Solutions

One-dimensional convection occurs at the surface of a semi-infinite solid with constant initial temperature T_0 into a medium at 0°K. The exact solution is given by

$$\frac{T}{T_0} = \operatorname{erf} \frac{X}{2a\tau} + e^{hx+h^2 a\tau} \operatorname{erfc}\left[\frac{X}{2a\tau} + ha\tau\right]$$

where $a = k/\rho c$ is the thermal diffusivity.

Letting $k = 1\ \mathrm{W m^{-1} K^{-1}}$, $\rho c = 1\ \mathrm{J m^{-3} K^{-1}}$ $h = 1\ \mathrm{W m^{-2} K^{-1}}$, $T_0 = 100°\mathrm{K}$ and $T_\infty = 0°\mathrm{K}$, the temperature distributions (exact and finite element) for various times are plotted in Figure 2. As can be seen, agreement of results is excellent.[1]

[1]Problem and analytical solution graphs were obtained from the work by R.V.S. Yalamanchili and S.C. Chu (see Sources).

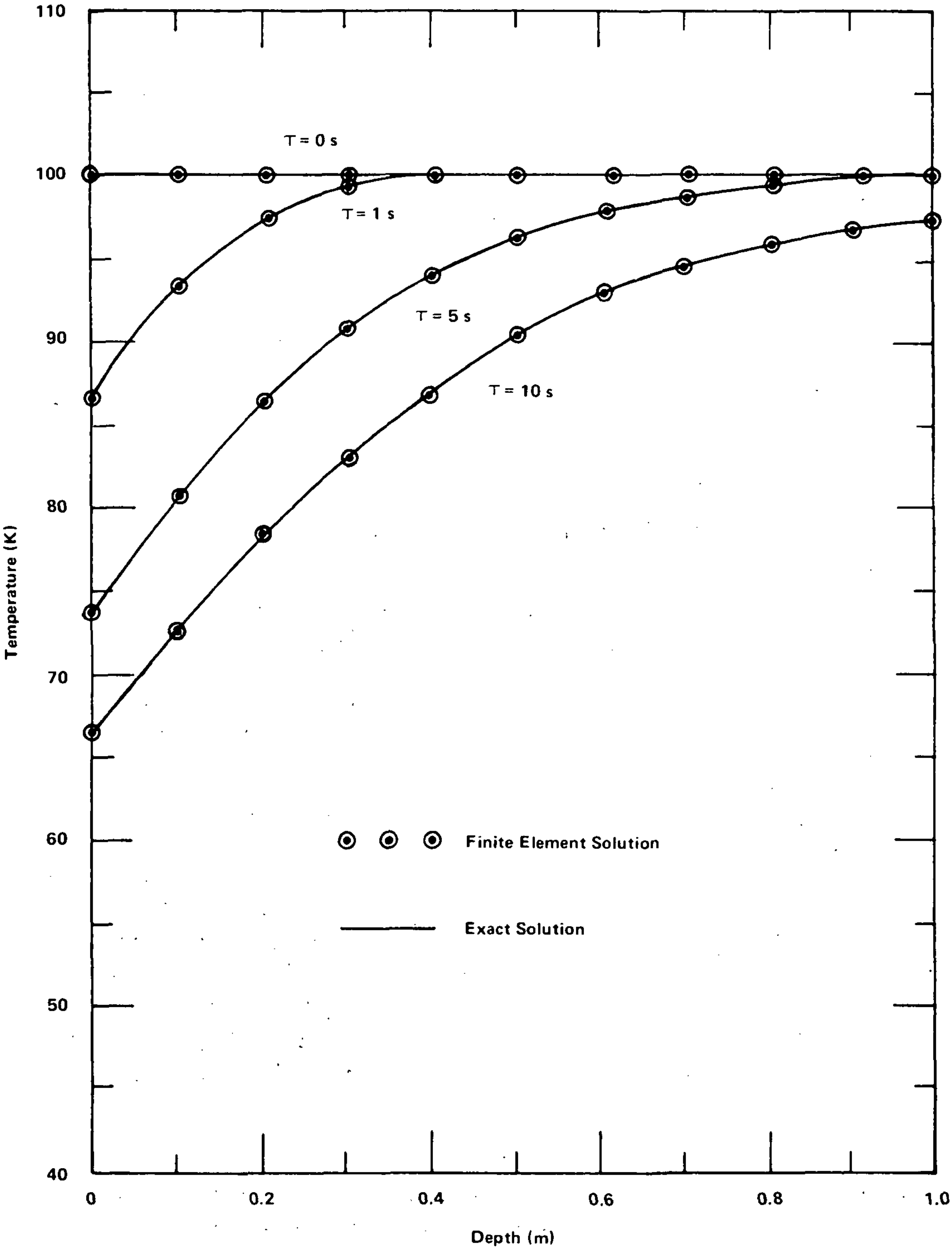


Figure 2. Comparison of finite element and analytical solutions for heat flow in a semi-infinite solid, where $h = k = \rho c = 1$; $T_0 = 100$ K, and $T_\infty = 0$ K.

The following thermal properties are used in sample problems 2 to 6.

Material: aluminum

$k_x = k_y = k_z = 62.99 \text{ W m}^{-1}\text{K}^{-1}$

$\rho c = 72.76 \text{ J m}^{-3}\text{K}^{-1}$

$h = 42.03 \text{ W m}^{-3}\text{K}^{-1}$

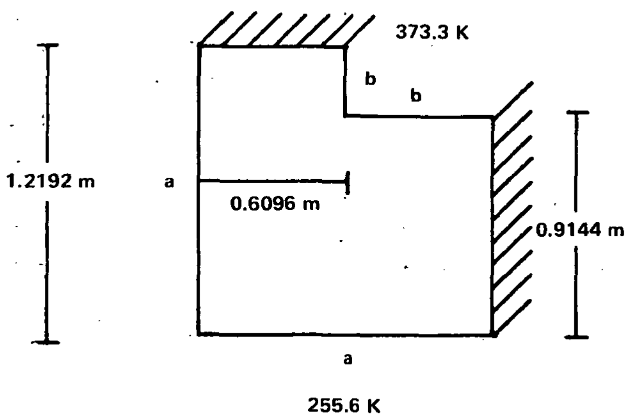
$\epsilon = 0.1 = a$; absorptivity

$T_o = 297.2\text{K}$

$t = 0.0254 \text{ m}$

Problem 2: Two Dimensions – With Radiation and Convection (Finite Element and Finite Difference Solutions)

Consider one quarter of a symmetric pipe where the external surface, a, radiates to an environment at 255.6K and the internal surface, b, is a convective boundary with $T_\infty = 373.3\text{K}$.



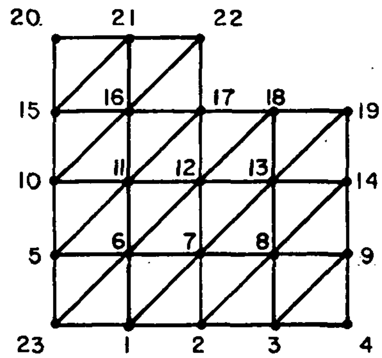


Table 1 shows transient and steady state temperature predictions for nodes 1 to 23, found by the finite element program. Also shown are predictions for corresponding nodes of an implicit finite difference program using a 30-node model.

Table 1

Comparison of Finite Element and Finite Difference Results for Problem 2.

Node	T at 1800s		T at 3600s		T at 7200s	
	Finite Element	Finite Difference	Finite Element	Finite Difference	Finite Element	Finite Difference
1	297.4K	297.9K	300.6K	301.0K	308.6K	308.7K
2	298.0	298.4	301.7	301.9	309.7	309.7
3	298.6	298.9	302.7	302.9	310.8	310.7
4	299.0	299.1	303.3	302.3	311.3	311.1
5	297.6	298.1	300.9	301.3	308.9	308.9
6	298.1	298.4	301.6	301.8	309.6	309.6
7	299.1	299.3	303.0	303.2	310.9	310.9
8	299.9	300.2	304.3	304.4	312.2	312.2
9	300.2	300.4	304.8	304.9	312.7	312.7
10	298.8	299.2	303.1	303.1	311.0	310.9
11	299.9	300.1	304.2	304.2	312.1	311.9
12	302.5	302.5	306.9	306.9	314.6	314.4
13	304.4	304.5	309.1	309.2	316.6	316.6
14	304.9	305.0	309.8	309.9	317.2	317.3
15	300.7	300.7	305.6	305.3	313.4	313.1
16	302.7	302.3	307.5	307.2	315.2	314.8
17	309.6	309.2	313.9	313.6	320.9	320.1
18	313.4	313.6	317.8	317.8	324.3	324.4
19	314.1	314.4	318.6	318.9	325.1	325.3
20	301.7	301.6	306.8	306.6	314.6	314.3
21	303.9	304.1	308.9	308.9	316.6	316.4
22	311.9	312.4	316.5	316.8	323.2	323.4
23	297.1	297.7	299.9	300.6	307.9	308.2

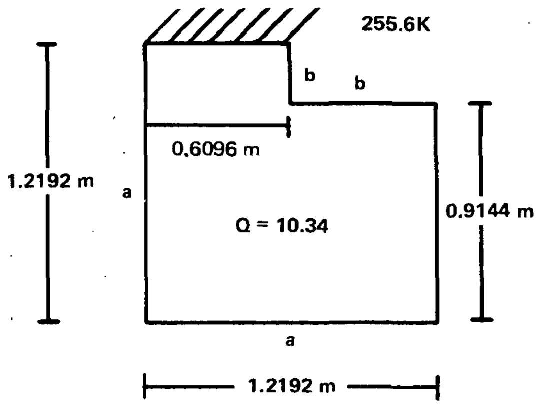
Table 1 (Continued)

Comparison of Finite Element and Finite Difference Results for Problem 2.

Node	T at 18,000s		T at 36,000s		T for Steady State	
	Finite Element	Finite Difference	Finite Element	Finite Difference	Finite Element	Finite Difference
1	328.1K	327.8K	347.9K	347.7K	370.9K	370.9K
2	328.9	328.7	348.4	348.7	371.1	371.1
3	329.1	329.4	348.8	348.6	371.1	371.1
4	330.1	329.7	349.1	348.7	371.2	371.2
5	328.3	328.1	348.1	347.8	370.9	370.9
6	328.8	328.5	348.4	348.1	371.1	371.1
7	329.8	329.5	348.9	348.7	371.2	371.2
8	330.7	330.4	349.4	349.2	371.3	371.3
9	331.0	330.8	349.7	349.4	371.3	371.3
10	329.8	329.4	348.9	348.6	371.1	371.1
11	330.6	330.2	349.4	349.1	371.2	371.2
12	332.3	331.9	350.4	350.1	371.4	371.4
13	333.7	333.6	351.2	351.0	371.4	371.5
14	334.2	334.1	351.4	351.3	371.5	371.5
15	331.6	331.0	349.9	349.5	371.2	371.2
16	332.8	332.2	350.7	350.2	371.3	371.3
17	336.8	336.1	352.9	352.4	371.6	371.6
18	339.2	339.1	354.2	354.1	371.7	371.8
19	339.7	339.6	354.6	354.4	371.8	371.8
20	332.3	331.9	350.3	350.0	371.2	371.2
21	333.7	333.4	351.2	350.9	371.4	371.4
22	338.4	338.3	353.8	353.7	371.7	371.7
23	327.6	327.5	347.6	347.4	370.8	370.8

Problem 3: Two Dimensions, With Internal Heat Generation, Specified Flux, and Convection (Finite Element and Finite Difference Solutions)

Consider one quarter of a symmetric pipe with the external boundary, a, exposed to the normal flux described by the curve below; with the internal convective boundary, b, exposed to a fluid at 255.6K; and with internal heat generation throughout the pipe, ($Q = 10.34\ \mathrm{W\,m^{-3}}$). The model used is the same as that for problem 2, illustrated previously. Results are shown in Table 2.



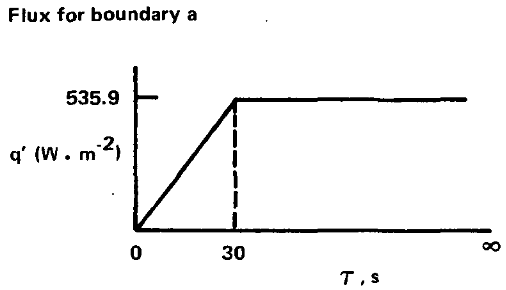


Table 2

Comparison of Finite Element and Finite Difference Results for Problem 3.

Node	T at 1800s		T at 3600s		T at 7200s	
	Finite Element	Finite Difference	Finite Element	Finite Difference	Finite Element	Finite Difference
1	298.8K	298.2K	298.0K	297.5K	295.2K	294.8K
2	298.2	297.8	297.0	296.7	294.1	293.9
3	297.7	297.4	296.2	296.0	293.3	293.1
4	297.5	297.3	295.8	295.8	292.9	292.8
5	298.7	298.2	297.8	297.3	295.0	294.6
6	297.8	297.4	296.8	296.5	294.0	293.7
7	296.9	296.7	295.6	295.4	292.8	292.6
8	296.3	296.2	294.7	294.6	291.8	291.6
9	296.1	296.0	294.3	294.2	291.4	291.3
10	297.7	297.4	296.3	296.1	293.4	293.3
11	296.5	296.3	295.0	294.9	292.2	292.1
12	294.7	294.7	293.1	293.1	290.3	290.2
13	293.6	293.5	291.6	291.5	288.9	288.7
14	293.2	293.2	291.2	291.1	288.4	288.2
15	296.6	296.5	294.7	294.8	291.8	291.9
16	294.8	294.9	292.9	293.1	290.1	290.3
17	290.8	291.1	289.0	289.3	286.4	286.7
18	288.5	288.4	286.7	286.4	284.2	284.1
19	288.1	288.0	286.1	285.9	283.7	283.5
20	296.0	295.9	294.0	294.0	291.1	291.1
21	294.1	294.1	292.1	292.1	289.3	289.3
22	289.4	289.2	287.6	287.4	285.1	284.9
23	299.6	298.6	299.0	298.0	296.2	295.3

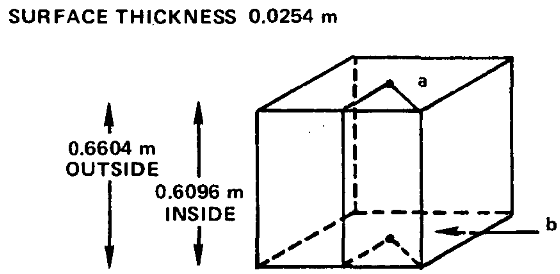
Table 2 (Continued)

Comparison of Finite Element and Finite Difference Results for Problem 3.

Node	T at 18,000s		T at 36,000s		T for Steady State	
	Finite Element	Finite Difference	Finite Element	Finite Difference	Finite Element	Finite Difference
1	288.1K	287.7K	280.7K	280.3K	272.2K	272.4K
2	287.1	286.9	279.9	279.6	271.4	271.8
3	286.3	286.2	279.2	279.0	271.0	271.3
4	286.1	285.9	279.0	278.8	270.8	271.2
5	287.9	287.6	280.6	280.2	272.1	272.4
6	286.9	286.7	279.7	279.4	271.3	271.7
7	285.9	285.7	278.8	278.6	270.6	270.9
8	285.1	284.9	278.1	277.9	270.0	270.3
9	284.7	284.6	277.8	277.6	269.8	270.1
10	286.5	286.4	279.4	279.2	271.2	271.6
11	285.4	285.3	278.4	278.2	270.3	270.7
12	283.8	283.8	277.1	277.2	269.3	269.7
13	282.6	282.4	276.1	275.9	268.6	268.9
14	282.2	282.1	275.8	275.6	268.4	268.7
15	285.2	285.2	278.4	278.3	270.5	270.3
16	283.7	283.8	277.1	276.9	269.4	269.8
17	280.7	280.9	274.7	274.7	267.8	268.2
18	278.8	278.7	273.2	273.1	266.7	266.9
19	278.3	278.2	272.8	272.7	266.4	266.7
20	284.6	284.6	277.9	277.8	270.2	270.6
21	283.1	283.0	276.6	276.4	274.7	269.4
22	279.6	279.6	273.9	273.7	267.2	267.3
23	289.0	288.2	281.6	280.7	272.9	273.3

Problem 4: Three Dimensions – Specified Temperature and Convective Surfaces

Consider a hollow cube where the outside surface temperature for side a is held constant at 422.2K and the convective surface of side b is exposed to a fluid with T_∞ = 255.6K. Tetrahedral elements have been used to model an eighth of this symmetric cube, resulting in the nodal points shown in the sketch. Results of the thermal analysis are shown in Table 3.



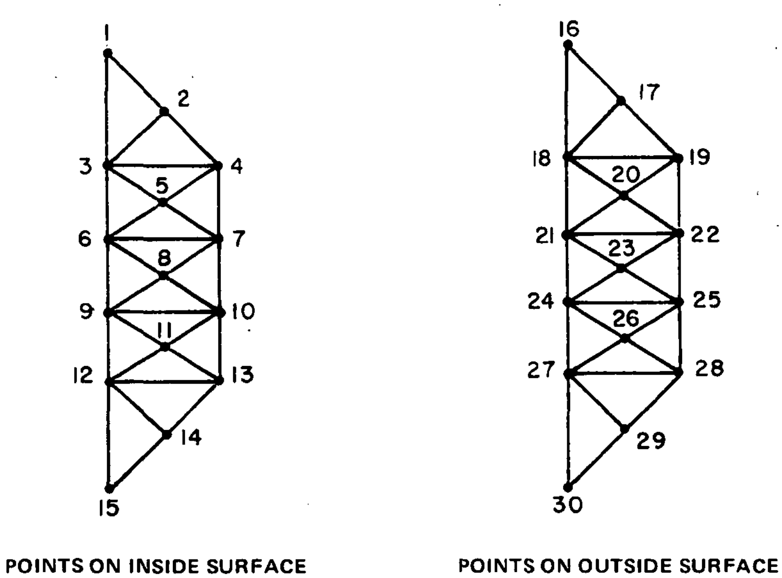


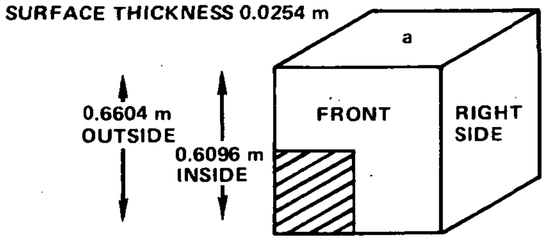
Table 3

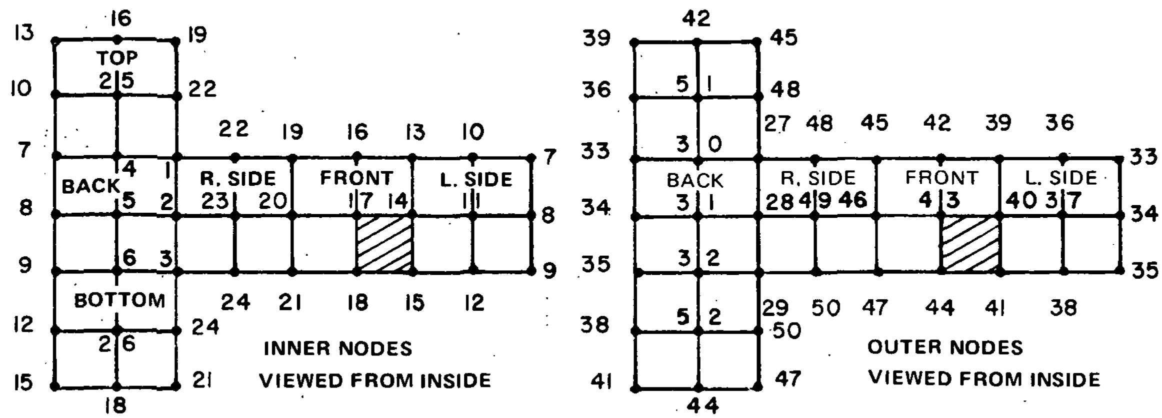
Finite Element Results for Problem 4.

Node	T at 120s	T at 300s	T at 900s	T at 1800s	T at 3600s	T for Steady State
1	422.1K	422.2K	422.2K	422.2K	422.2K	422.2K
2	422.1	422.1	422.2	422.2	422.2	422.2
3	416.9	418.5	419.7	420.1	420.3	420.3
4	413.9	416.5	418.4	419.1	419.3	419.4
5	366.2	383.9	396.4	400.8	402.9	403.3
6	323.8	350.3	371.1	379.2	383.1	383.8
7	321.8	348.1	369.9	378.4	382.4	383.2
8	303.1	323.7	346.3	357.1	362.3	363.3
9	296.2	305.2	323.1	334.6	340.6	341.7
10	296.5	305.8	325.2	337.2	343.2	344.4
11	294.4	293.9	305.2	316.3	322.2	323.3
12	290.3	284.8	289.3	298.3	303.3	304.2
13	292.4	288.1	295.6	305.9	311.5	312.6
14	287.6	281.1	283.1	290.8	295.3	296.1
15	285.6	277.3	275.6	281.7	285.4	286.2
16	422.2	422.2	422.2	422.2	422.2	422.2
17	422.2	422.2	422.2	422.2	422.2	422.2
18	422.2	422.2	422.2	422.2	422.2	422.2
19	422.2	422.2	422.2	422.2	422.2	422.2
20	365.9	383.8	396.3	400.7	402.8	403.2
21	323.6	350.3	371.1	379.3	383.1	328.3
22	322.1	348.1	369.7	378.2	382.2	383.0
23	303.1	323.7	346.3	357.1	362.3	363.3
24	296.2	305.3	323.1	334.7	340.6	341.7
25	296.7	306.0	325.5	337.5	288.0	344.7
26	294.4	294.0	305.4	316.4	322.3	323.4
27	290.1	284.2	288.3	297.2	302.2	303.2
28	292.3	287.4	294.6	304.9	310.6	311.6
29	287.5	281.0	282.9	290.7	295.1	295.9
30	285.2	277.1	275.2	281.3	285.0	285.7

Problem 5: Three Dimensions – Specified Temperature, Flux, and Radiative Surfaces

The outside surface temperature for side a of a hollow cube is specified as T_a = 233.3K; the shaded portion of the cube is exposed to normal flux, an external q = 1398.1 W m^{-2}; and radiation exists between all internal surfaces. Tetrahedral elements have been used to model the entire cube. The grid point locations of the models are shown in the "flattened diagrams."





Results of the finite element analysis are shown in Table 4. The temperatures for nodes 27,30, 33, 36, 39, 42, 45, 48, and 51 are constant (T = 233.3K) and hence are not shown in this table.

Table 4

Finite Element Results for Problem 5.

Node	T at 180s	T at 300s	T at 900s	T at 3600s	T at 7200s	T for Steady State
1	236.1K	235.1K	234.5K	233.9K	233.6K	233.4K
2	286.6	266.6	256.4	245.3	237.5	235.3
3	298.3	284.5	269.4	252.2	239.9	236.4
4	235.6	234.8	234.3	233.8	233.5	233.4
5	285.9	266.7	256.7	245.6	237.7	235.4
6	297.6	285.5	270.4	252.8	240.3	236.8
7	236.7	235.4	234.8	234.1	233.6	233.4
8	286.7	266.9	257.0	245.8	237.9	235.7
9	298.1	284.6	269.9	252.7	240.4	236.9
10	235.6	234.8	234.3	233.9	233.6	233.4
11	285.2	267.1	257.6	246.6	238.8	236.6
12	297.5	286.7	272.2	254.7	242.3	238.7
13	235.9	235.0	234.6	234.0	233.7	233.6
14	287.2	269.3	259.9	249.0	241.2	239.0
15	300.5	289.4	275.2	258.1	245.8	242.3
16	235.6	234.9	234.4	234.0	233.7	233.6
17	287.3	269.4	259.9	248.9	241.2	239.0
18	298.4	288.2	273.8	256.4	244.0	240.5
19	236.7	235.5	234.9	234.2	233.8	233.6
20	286.6	267.4	257.6	246.6	238.8	236.5
21	297.9	285.3	270.8	253.8	241.7	238.3
22	235.6	234.8	234.3	233.9	233.4	233.4
23	285.8	266.5	256.7	245.7	237.9	235.7
24	297.4	285.4	270.5	252.9	240.6	237.0
25	233.6	233.5	233.4	233.4	233.3	233.3
26	295.8	231.8	272.4	254.3	241.6	237.9
28	286.6	266.6	256.4	245.3	237.5	235.3
29	299.1	285.5	270.2	252.5	240.1	236.5
31	285.9	266.8	256.7	245.6	237.7	235.4
32	298.1	286.2	270.9	253.1	240.4	236.8
34	286.8	266.9	257.0	245.8	237.9	235.7
35	298.7	285.3	270.4	252.9	240.5	236.9
37	285.3	267.2	257.6	246.7	238.8	236.6
38	298.1	287.4	272.8	255.1	242.4	238.9
40	287.3	269.4	260.0	249.1	241.4	239.2
41	301.5	290.8	276.3	258.8	246.4	242.8

Table 4 (Continued)

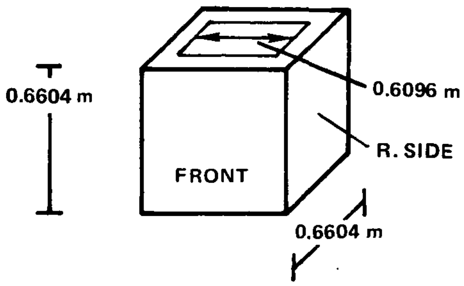
Finite Element Results for Problem 5.

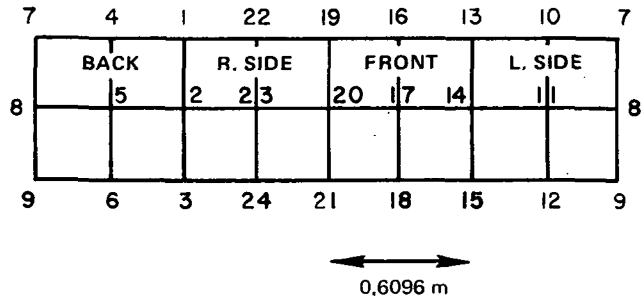
Node	T at 180s	T at 300s	T at 900s	T at 3600s	T at 7200s	T for Steady State
43	287.3K	269.4K	259.9K	249.0K	241.2K	239.0K
44	298.9	288.9	274.3	256.8	244.3	240.7
46	286.7	267.3	257.5	246.5	238.7	236.4
47	298.6	286.1	271.4	254.1	241.8	238.3
49	285.9	266.6	256.7	245.7	237.9	233.7
50	297.8	285.9	270.8	253.2	240.7	237.1
52	295.8	287.3	272.4	254.4	241.6	237.9

Problem 6: Three Dimensions – Figure With Radiative and Flux Surfaces

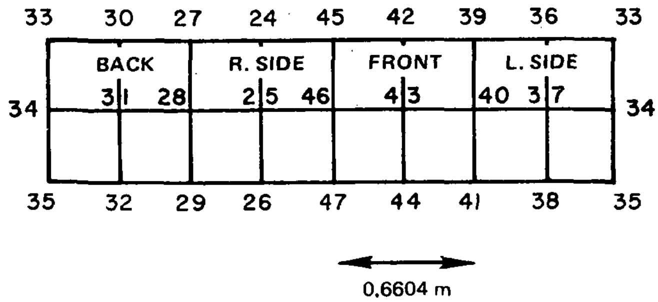
A figure with front, back, and two sides has all internal and one external (right side) surfaces radiating with an environment of T_e = 255.6K; the outer surface on the left side is exposed to a specified normal flux q = 1576.2 W m^{-2}; and all inner surfaces radiate with each other. Tetrahedral elements with grid points as shown have been used to model the figure. Results of the analysis are shown in Table 5.

SURFACE THICKNESS 0.0254 m





INNER NODES



OUTER NODES

Table 5

Finite Element Results for Problem 6.

Node	T at 3600s	T at 7200s	T at 18,000s	T at 36,000s	T at 72,000s	T for Steady State
1,19	310.3K	334.0K	404.9K	509.8K	645.3K	715.7K
2,20	310.3	334.1	404.9	509.8	645.3	715.6
3,21	310.4	334.2	405.1	509.9	645.3	715.6
4,16	318.9	343.6	415.0	521.6	661.3	735.1
5,17	318.8	343.5	414.9	521.5	661.2	735.0
6,18	319.1	343.7	415.1	521.7	661.2	734.9
7,13	330.1	358.5	430.2	537.6	679.5	755.1
8,14	332.9	358.4	430.2	537.6	679.4	754.9
9,15	333.0	358.4	430.2	537.5	679.3	754.9
10	339.6	365.2	437.0	544.4	686.5	762.1
11	339.5	365.1	436.9	544.4	686.4	762.0
12	283.6	364.7	436.5	543.9	685.9	761.4
22	308.7	332.3	402.8	507.0	640.4	709.1
23	308.7	332.3	402.8	507.0	640.4	709.1
24	309.1	332.7	403.3	507.4	640.8	709.5
25	308.7	332.3	402.8	506.9	640.3	709.0
26	309.1	332.7	403.2	507.3	640.7	709.3
27,45	309.9	333.8	404.6	509.4	644.8	715.1
28,46	310.0	333.8	404.6	509.5	644.8	715.1
29,47	310.1	333.9	404.7	509.6	644.8	715.0
30,42	318.9	343.6	415.0	521.6	661.3	735.1
31,43	318.8	343.5	414.9	521.5	661.2	735.0
32,44	319.1	343.7	415.1	521.7	661.2	734.9
33,39	333.4	358.8	430.6	538.1	680.1	755.6
34,40	333.3	358.8	430.6	538.0	679.9	755.5
35,41	333.4	358.8	430.6	538.0	679.9	755.2
36	339.7	365.3	437.1	544.5	686.6	762.2
37	339.6	365.2	437.0	544.5	686.5	762.1
38	283.7	364.9	436.6	544.1	686.0	761.6
48	308.7	332.3	402.8	506.9	640.3	708.9

FINITE ELEMENT COMPUTER PROGRAM

Input Cards and Data Required

The following is a description of data necessary for the finite element thermal analysis program written for an IBM 360-95 computer. Each data deck must first include card types 1 to 5. If radiation boundary conditions exist, these are described by card types 6 and 7. When time dependent data are used, card type 7 is followed by sets of card types 8 to 13. (One card type 8, plus combinations of types 9 to 13 must be included for each increment where there is a change.) Finally, card types 14 and 15 must be included.

Card Type	Data Description	Data Type	Field Location
1 Title Card	Title of thermal analysis problem	Alphanumeric	1-80
2 General Information (one card)	Number of nodes	Integer (I)	1-4
	Number of elements	I	5-8
	Number of materials	I	9-12
	Dimension of elements (2 = triangles, 3 = tetrahedrons)	I	13-16
	Blank spaces		17-20
	*Number of increments	I	21-24
	*Time increment (hr)	Real (R)	25-32
	Specified boundary code (0 = no, 1 = yes)	I	33-36
	Radiation code (0 = no, 1 = boundary radiation, 2 = surface radiation)	I	37-40
	Convection code (0 = no, 1 = boundary convection, 2 = surface convection)	I	41-44
	Normal flux code (0 = no, 1 = boundary flux, 2 = surface flux)	I	45-48
	Internal heat generation code (0 = no, 1 = yes)	I	49-52

*Note: For steady state results only, place a 0 in field 21-24 (number of increments) and a negative real number in field 25-32 (time increment size).

Card Type	Data Description	Data Type	Field Location
3 Nodal Information (one card per node)	Node number	I	1-8
	x-coordinate (ft)	R	9-16
	y-coordinate (ft)	R	17-24
	z-coordinate (ft)	R	25-32
	Initial temperature of node (°F or °R; °R must be used if radiative boundary conditions exist.)*	R	33-40

*Note: If this node temperature is specified (fixed), precede temperature value by a negative sign (−).

Card Type	Data Description	Data Type	Field Location
4 Element Information (one card per element)	Element number	I	1-4
	*Node i of element	I	5-8
	*Node j of element	I	9-12
	*Node k of element	I	13-16
	*Node l of element	I	17-20
	Material of element	I	21-24
	Thickness of element (ft)	R	25-32
	Blank spaces		33-36
	Boundary condition code (0 = no boundary conditions, 1 = boundary conditions)	I	37-40
	**Internal heat generation value (Btu/hr ft^3)	R	41-48
	**Convective coefficient h (Btu/hr ft^2 F)	R	49-56
	**Ambient fluid temperature (°F or °R)	R	57-64
	†Value of normal flux (Btu/hr ft^2)	R	65-72

*Note: For triangular elements, describe i,j,k in a counterclockwise direction, leaving l blank. When a boundary of the triangle is subject to boundary conditions, denote the boundary nodes by i and j. For tetrahedral elements, again let any surface subject to a boundary condition have nodes i,j,k; and number the nodes such that, viewed from point l, the i,j, and k are counterclockwise.

**Note: These values all are boundary condition variables. Leave the data location blank (or use 0) if the information requested is unapplicable to the element described by columns 1-32.

†Note: The flux in is negative; flux out is positive.

Card Type	Data Description	Data Type	Field Location
5 Material Information (one card per material)	Material number	I	1-8
	Conductivity coefficient, k_x, of material (Btu/hr ft F)	R	9-16
	Conductivity coefficient, k_y, of material	R	17-24
	*Conductivity coefficient K_z, of material	R	25-32
	Heat capacity, ρc, of material	R	33-40
	*Material internal heat generation, Q (Btu/hr ft^3)	R	41-48
	*Emissivity of material	R	49-56
	*Absorptivity of material	R	57-64

*Note: Leave blank if inapplicable to material. If an element or boundary of this material participates in radiation, 49-64 must be specified.

If no radiative boundaries exist, omit card types 6 and 7.

Card Type	Data Description	Data Type	Field Location
6 Radiation Information (one card)	*Temperature of environment to which continuum is radiating	R	1-8
	**Maximum number of iterations to be performed to attain steady state temperatures	I	9-16
	† Γ , Acceptance criterion for steady state answers (If left blank, = 0.00001 is used.)	R	17-24

*Note: Leave field 1-8 blank if there is radiation but none with the environment (or with any fixed temperature source).

**Note: This number, N6, should be $\geq$ number of increments specified on card type 2, N2, denoting the number of increments for which output is desired. However, N6 denotes the maximum number of desired iterations (increments) to try to obtain a steady state distribution. Printed results will include temperatures at increments 1 through N2 and at steady state (or at N6 if steady state has not been attained.) This parameter is required for radiative problems only.

†Note: Answers are assumed to be at steady state, when for each node $|1 - T_\tau/T_{\tau+d\tau}| \leq \Gamma$.
This parameter is required for radiative problems only.

Card Type	Data Description	Data Type	Field Location
7	*Radiating element 1	I	1-4
Radiation	*Radiating element 2	I	5-8
Information	*F_{1-2}	R	9-16
(one card for	Radiating element 1	I	17-20
five radiating	Radiating element 3	I	21-24
conditions)	F_{1-3}	R	25-32
	**.		. . .
	Radiating element n	I	
	Radiating element m	I	
	†F_{n-m}	R	

*Note: *If 1 and 2 are radiating elements, separate entries must be made for* F_{1-2} and F_{2-1}, one entry containing "element 1, element 2, F_{1-2}" and the other containing "element 2, element 1, F_{2-1}."

**Note: Begin new card if needed and proceed as above, using two integer fields of four for identification of the two radiating elements, followed by one real field of eight for the form factor value.

†Note: After all radiative form factors have been read in, place a negative integer in the next integer field of four. (I.e., place this negative number where the next element title would be if there were another form factor to be described.) This number is a code to denote the end of radiative data.

If no time dependent data exist, omit card types 8–13.

8	Increment of first or next change in any input data	I	1-4
	Number of elements with new ambient fluid temperature at this increment	I	5-8
	Number of elements with new normal flux value at this increment	I	9-12
	Number of elements with new internal heat generation value at this increment	I	13-16
	Number of materials with new internal heat generation value at this increment	I	17-20
	Number of nodes with new specified temperatures at this increment	I	21-24
	*New increment value (if it changes)	R	25-32

*Note: For no change in increment size, leave field 25-32 blank.

To describe time changing data for the first (or next) increment, follow card type 8 with the appropriate combination of 9, 10, 11, 12, and 13. (I.e., if only the ambient fluid temperature is changing at this increment, then only card type 9 is needed to describe it. This type, therefore, will follow 8. If, however, there is no change in ambient fluid temperature, but flux or specified nodal temperature change, card 8 is followed immediately by 10, then by 13). To describe a change in increment size, only card 8 is needed.

Card Type	Data Description	Data Type	Field Location
9 Time Changing Data–Ambient Fluid Temperature	Number of first element subject to new ambient fluid temperature	I	1-8
	New ambient fluid temperature for element	R	9-16
	Number of second element subject to new ambient fluid temperature	I	17-24
	New ambient fluid temperature for element	R	25-32
	*.		. . .
	Number of last element subject to new ambient fluid temperature	I	
	New ambient fluid temperature for element	R	

*Note: Begin new card if needed and proceed as above, each element having an integer field of 8 for its identification and a real field of 8 for its new value.

Card Type	Data Description	Data Type	Field Location
10 Time Changing Flux Data	Number of first element with change in flux value	I	1-8
	New value of flux	R	9-16
	Number of second element with change in flux value	I	17-24
	New value of flux	R	25-32
	*.		. . .
	Number of last element with change in flux value	I	
	New value of flux	R	

*Note: Begin new card if needed and proceed as before, each element having an integer field of 8 for its identification and a real field of 8 for its new value.

Card Type	Data Description	Data Type	Field Location
11 Time Changing Data–Internal Heat Generation (Element)	Number of first element with internal heat generation change	I	1-8
	New value for internal heat generation	R	9-16
	Number of second element with internal heat generation change	I	17-24
	New value for internal heat generation	R	25-32
	*.		. . .
	Same information for last element with internal heat generation change		
*Note: Begin new card if needed and proceed as before, each element having an integer field of 8 for its identification and a real field of 8 for its new value.			
12 Time Changing Data–Internal Heat Generation (Material)	Number of first material with internal heat generation change	I	1-8
	New value of internal heat generation	R	9-16
	Number of second material with internal heat generation change	I	17-24
	New value of internal heat generation	R	25-32
	*.		. . .
	Same information for last material with internal heat generation change		
*Note: Begin new card if needed and proceed as before, each element having an integer field of 8 for its identification and a real field of 8 for its new value.			

Card Type	Data Description	Data Type	Field Location
13 Time Changing Data--Specified Nodal Temperature	Number of first node with new specified temperature value change	I	1-8
	New value of nodal temperature	R	9-16
	Number of second node with new specified temperature value change	I	17-24
	New value of nodal temperature	R	25-32
	*.		. . .
	Same information for last node with specified temperature value change		

*Note: Begin new card if needed and proceed as before, each node having an integer field of 8 for its identification and a real field of 8 for its new value.

If data change at a later time increment, return to card type 8. If not, use type 14.

Card Type	Data Description	Data Type	Field Location
14 End of Time Dependent Data Code Card	Integer greater than the maximum increment specified on card 2, field 21-24	I	1-4
15	*Terminator card (i.e., /* for 360-95)		1-2

*Note: To submit more than one data deck per run, assemble each deck as above (excluding card type 15); stack the decks one behind the other; follow the final deck with card 15.

```
      COMMON    XIYJZK(12),X(100,3),H(100,100),P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      DIMENSION S(100,100),TO(100),NELEM(5),MELEM(5),FOR(5),IDENT(20)
      INTEGER*4 STEADY
      REAL * 8 DFORCE, DC, DB, DDIF,DDT,DCHK
C
      NORDER = 100
10000 CONTINUE
      TIME = 0.
      INCR = 0
      READ(60,99,END=10001) IDENT
   99 FORMAT(20A4)
      READ (60,100,END=10001) NODES,NEL,NMAT,NDIM,MAXINC,DELTA,NCONST,
     2    NRAD,NCONV,NFLUX,NHEAT
  100 FORMAT(4I4,I8,F8.0,5I4)
10002 CONTINUE
      WRITE(61,101) IDENT,NDIM,NEL,NODES,NMAT,NCONST,NRAD,NCONV,NFLUX,
     1  NHEAT, MAXINC, DELTA
  101 FORMAT(1H1/1H0, 45X,20A4/1H0,
     1 /1H0,55X,11HDIMENSION =,I3,
     1 /1H0,55X,20HNUMBER OF ELEMENTS =, I3,
     1 /1H0,55X,17HNUMBER OF NODES =, I3,
     1 /1H0,55X,21HNUMBER OF MATERIALS =, I3,
     1  /1H0,55X,43HCONSTANT TEMPERATURE INPUT (0=NO, 1=YES) = ,I1,
     1  /1H0,55X,68HRADIATION INPUT (0=NO, 1=BOUNDARY RADIATION, 2=SURFA
     1CE RADIATION) = ,I1,
     1  /1H0,55X,71HCONVECTION INPUT (0=NO, 1=BOUNDARY CONVECTION, 2=SUR
     1FACE CONVECTION) = ,I1,
     1  /1H0,55X,53HFLUX INPUT (0=NO, 1=BOUNDARY FLUX, 2=SURFACE FLUX) =
     1  ,I1,
     1  /1H0,55X,47HINTERNAL HEAT GENERATION INPUT (0=NO, 1=YES) = ,I1,
     1 /1H0,55X,35HMAXIMUM NUMBER OF TIME INCREMENTS =,I3,
     1/1H0,55X,44HTIME INCREMENT (+TRANSIENT, -STEADY STATE) =,F15.5,
     1 /1H0)
C     SET SWITCH FOR STEADY STATE (-DELTA) OR TRANSIENT (+DELTA)
      STEADY = 2
      IF (DELTA) 11000,11000,12000
11000 STEADY = 1
      DELTA = -DELTA
12000 CONTINUE
```

```
C
C     N O D E   I N F O R M A T I O N
C
      WRITE (61,104)
  104 FORMAT(1H0,50X, 16HNODE INFORMATION/1H0,8X,1HN,17X,1HX,14X,1HY,
     2   14X,1HZ,14X,1HT/)
      DO 105 K = 1,NODES
      READ(60,102) I,X(I,1),X(I,2),X(I,3), T(I)
  102 FORMAT(I8,4E8.0)
      WRITE(61,103) I,X(I,1),X(I,2),X(I,3),T(I)
  103 FORMAT(5X,5HNODES,I3,5X,4F15.5)
  105 CONTINUE
C     TABULAR PRINT REGULATOR
      LINE = 60
      LINBLK = (NODES+5)/5
C
C
C     E L E M E N T   I N F O R M A T I O N
C
      WRITE (61,113)
  113 FORMAT(1H0/1H0,50X,19HELEMENT INFORMATION/1H0,5X,1HE,5X,1HI,5X,
     1   1HJ,5X,1HK,5X,1HL,3X,8HMATERIAL,7X,9HTHICKNESS,6X,
     1   1HQ,8X,7HBC CODE,10X,1HH,11X,1HT,10X,4HFLUX/)
      DO 112 N = 1,NEL
      READ(60,110) I,(INDEX(I,J),J=1,4),MATERL(I),TH(I),
     1   NBND1(I),HEAT(I),HCONV(I),AMB(I),FLUX(I)
  110 FORMAT(6I4,F8.0,I8,4F8.0)
      WRITE(61,111) I, (INDEX(I,J),J=1,4),MATERL(I),TH(I),HEAT(I),
     1   NBND1(I),HCONV(I),AMB(I),FLUX(I)
  111 FORMAT(4X,I3,4I6,4X,I4,5X,2F12.3,5X,I5,5X,3F12.3)
  112 CONTINUE
C
C     M A T E R I A L   I N F O R M A T I O N
C
      WRITE (61,114)
  114 FORMAT(1H0/1H0,50X,20HMATERIAL INFORMATION  /1H0,  8X,1HM, 8X,2HKX
     1,10X,2HKY,10X,2HKZ,10X,2HPC,10X,1HQ,11X,1HE,11X,1HA/)
      DO 120 M = 1,NMAT
      READ(60,115) I,COND(I,1),COND(I,2),COND(I,3),PC(I),Q(I),E(I),A(I)
  115 FORMAT(I8,7F8.0)
      WRITE(61,116) I,COND(I,1),COND(I,2),COND(I,3),PC(I),Q(I),E(I),A(I)
  116 FORMAT (5X,I5,7F12.3)
  120 CONTINUE
```

```
C
C     R A D I A T I O N   I N F O R M A T I O N
C
      IF(NRAD) 130,130,131
  131 READ(60,139) TWALL,MXSSIN ,RADTOL
  139 FORMAT (F8.0,I8,F8.0)
      IF (RADTOL .LE. 0.) RADTOL = .00001
      WRITE (61,135) TWALL, MXSSIN , RADTOL
  135 FORMAT(1H0/1H0,50X,17HRADIATION FACTORS/1H0,6HTWALL=,F12.4,5X,13HM
     2AX. SS INC.=   , I4, 5X, 10HTOLERANCE= , F12.6/1H0)
      STEADY = 2
      DO 1720 I=1,NODES
 1720 TO(I) = ABS(T(I))
      DO 137 I = 1,NEL
      DO 137 J = 1,NEL
  137 FORM(I,J) = 0.
  132 READ (60,133) (NELEM(I),MELEM(I),FOR(I),I=1,5)
  133 FORMAT (5(2I4,F8.0))
      DO 145 I=1,5
      ITEM=I
      IF(NELEM(I)) 144,144,136
  136 N=NELEM(I)
      M=MELEM(I)
      FORM(N,M) = FOR(I)
      IF(FORM(N,N)) 145,138,145
  138 FORM(N,N) = -1.
  145 CONTINUE
      GO TO 146
  144 ITEM=ITEM-1
      IF (ITEM) 130,130,146
  146 WRITE(61,134)(NELEM(I),MELEM(I),FOR(I),I=1,ITEM)
  134 FORMAT(2X,5(2HF(,I2,1H,,I2,3H) =,F7.5, 8X))
      IF (ITEM .EQ. 5) GO TO 132
  130 CONTINUE
      READ (60,140) INC, NAMBS, NFLUXS, NHEATS, NQS, NODS, DELTAN
  140 FORMAT (6I4,F8.0)
C
C     HOW MANY FIXED NODES
C
      NCONST = 0
      DO 150 N = 1,NODES
      IF(T(N)) 151,150,150
  151 NCONST = NCONST + 1
  150 CONTINUE
      NVAR = NODES - NCONST
      WRITE (61,154) NVAR,NCONST,NODES
  154 FORMAT(1H0,10X, 5HNVAR=,I3,5X,7HNCONST=,I3,5X,6HNODES=,I3)
```

```
C
C     CONSTRUCT SIMILARITY TRANSFORMATION MATRIX FOR CONSTANT NODES
C
      IF (NCONST .EQ. 0) GO TO 992
      DO 155 I = 1,NODES
      DO 156 J = 1,NODES
  156 S(I,J) = 0.
  155 S(I,I) = 1.0
C
      NLAST = NODES
      DO 160 N = 1,NVAR
      IF(T(N)) 161,160,160
  161 T(N) = - T(N)
      IF (T(N) .LE. 1.E-75)  T(N) = 0.
  162 IF(T(NLAST)) 163,164,164
  163 T(NLAST) = -T(NLAST)
      IF ( T(NLAST) .LE. 1.E-75)  T(NLAST) = 0.
      NLAST = NLAST - 1
      GO TO 162
  164 S(N,N) = 0.
      S(NLAST,NLAST) = 0.
      S(NLAST,N) = 1.
      S(N,NLAST) = 1.
      NLAST = NLAST - 1
  160 CONTINUE
      NVAR1 = NVAR + 1
      IF (NODES - NVAR1) 992, 167,167
  167 DO 165 N = NVAR1,NODES
      IF(T(N)) 166,165,165
  166 T(N) = -T(N)
      IF (T(N) .LE. 1.E-75)  T(N) = 0.
  165 CONTINUE
C
  992 CONTINUE
C
      DO 200 I = 1,NODES
      FO(I) = 0.
      DO 200 J = 1,NODES
      H(I,J) = 0.
      P(I,J) = 0.
  200 CONTINUE
C     INITIALIZE THE FORCE VECTOR AND H MATRIX
      NTCD = 0
15010 DO 199 I=1,NODES
  199 F(I) = 0.
```

```
C
C     C A L C U L A T I O N S
C
      DO 2000 N=1,NEL
      IF (NTCD .EQ. 2) GO TO 15011
      CALL CONDUC (N,MATERL(N))
C     OMIT SPECIFIC HEAT IF STEADY-STATE
      GO TO(15011,15012),STEADY
15012 CALL SPECIF (N,MATERL(N))
15011 NN=MATERL(N)
      TOTAL=HEAT(N) + Q(NN)
      IF (TOTAL) 220,221,220
  220 CALL  HETGEN(N,TOTAL)
  221 IF (NBND1(N)) 2000,2000,224
  224 IF (FLUX(N)) 226,227,226
  226 CALL FLUXN(N,NFLUX)
  227 IF (HCONV(N)) 228,229,228
  228 CALL CONVEC(N,NCONV,NTCD)
  229 IF (NRAD) 2000,2000,233
  233 IF (FORM(N,N)) 230,2000,230
  230 DO 232 J = 1,NEL
      IF (FORM(N,J)) 231,232,231
  231 CALL RADIAN(N,J,FORM(N,J),NN,TWALL,NRAD)
  232 CONTINUE
 2000 CONTINUE
      IF (NTCD .NE. 2) GO TO 234
      NTCD = 0
      GO TO 15113
  234 GO TO (15101,15103),STEADY
15103 DO 201 I=1,NODES
  201 FO(I) = F(I)
      WRITE (61,11119) (I,VOLUME(I),I=1,NEL)
11119 FORMAT (1H0/1H0,50X,26HAREA OR VOLUME OF ELEMENTS   /(/1X,
     2   6(2HA(,I3,2H)=, F8.3,5X)))
```

```
C
C     COMPUTE COEFFICIENT MATRICES   (H+2/DELT*P) AND (H-2/DELT*P)
C
C     SKIP IF STEADY-STATE
15102 FACTOR = 2./DELTA
      DO 250 I = 1,NODES
      DO 250 J = I,NODES
      HIJ = H(I,J)
      PIJ = P(I,J)*FACTOR
      H(I,J) = HIJ + PIJ
      H(J,I) = H(I,J)
      P(I,J) = HIJ - PIJ
  250 P(J,I) = P(I,J)
15101 IF (NCONST .EQ. 0) GO TO 329
      DO 321 J=1,NODES
      DO 320 I=1,NODES
      V(I) = 0.
      DO 320 K=1,NODES
  320 V(I) = V(I) + S(I,K)*H(K,J)
      DO 321 I=1,NODES
  321 H(I,J) = V(I)
      DO 326 I=1,NODES
      DO 325 J=1,NODES
      V(J) = 0.
      DO 325 K=1,NODES
  325 V(J) = V(J) + H(I,K) * S(K,J)
      DO 326 J=1,NODES
  326 H(I,J) = V(J)
  329 DO 332 I=1,NODES
      DO 332 J=I,NODES
      HH(I,J) = H(I,J)
  332 HH(J,I) = H(J,I)
      CALL FACTR (H,PER,NVAR,NORDER,IER)
      IF (IER .NE. 0) WRITE(61,1005) IER
 1005 FORMAT (1H0/20H ***** ERROR CODE =    ,I2, 50H.  SEE WRITE UP FOR
     2EXPLANATION.  *****                    /1H0)
      IF (NTCD .EQ. 1) GO TO 419
C
    1 CONTINUE
```

```
C
      INCR = INCR + 1
      NTCD = 0
      IF (INCR - INC) 420,410,420
  410 WRITE (61,500) INCR,NAMBS,NFLUXS,NHEATS,NQS,NODS,DELTAN
  500 FORMAT (1H0/22H CHANGES FOR INCREMENT   , I5/1H0, I5,
     1  21H  AMBIENT FLUID TEMPS, 5X, I5, 24H  NORMAL FLUX BOUNDARIES,
     2  5X, I5, 24H  INTERNAL HEAT ELEMENTS  /1H0, I5, 25H  INTERNAL HEA
     2T MATERIALS, 5X, I5, 23H  SPECIFIED NODAL TEMPS  /1H0,
     4F10.5, 20H  NEW INCREMENT SIZE  /1H0)
      IF (DELTAN) 419, 419, 411
  411 NTCD = 1
      DO 216 I=1,NODES
      DO 216 J=1,NODES
      H(I,J) = 0.
  216 P(I,J) = 0.
      DO 217 N=1,NEL
      CALL CONDUC (N,MATERL(N))
      CALL SPECIF(N,MATERL(N))
      IF (HCONV(N)) 401,217,401
  401 CALL CONVEC(N,NCONV,NTCD)
  217 CONTINUE
      DELTA = DELTAN
      GO TO 15102
  419 NTCD = 0
      IF (NAMBS) 413,413,412
  412 READ (60,510) (N,AMB(N), I=1,NAMBS)
  510 FORMAT (5(I8,F8.0))
      WRITE (61,520) N,AMB(N)
  520 FORMAT     (1H0,4HAMB(,I3,2H)=,F12.5)
      NTCD = 2
  413 IF (NFLUXS) 415,415,414
  414 READ (60,510) (N,FLUX(N),I=1,NFLUXS)
      WRITE (61,530)  N,FLUX(N)
  530 FORMAT     (1H0,5HFLUX(,I3,2H)=,F12.5)
      NTCD = 2
  415 IF (NHEATS) 417,417,416
  416 READ (60,510) (N,HEAT(N),I=1,NHEATS)
      WRITE (61,540)  N,HEAT(N)
  540 FORMAT     (1H0,5HHEAT(,I3,2H)=, F12.5)
      NTCD = 2
  417 IF (NQS) 420,420,418
  418 READ (60,510) (N,Q(N),I=1,NQS)
      WRITE (61,550)  N,Q(N)
  550 FORMAT     (1H0,2HQ(,I3,2H)=,F12.5)
      NTCD = 2
```

```
C
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
  420 CONTINUE
      TIME = TIME + DELTA
      IF (NRAD .NE. 0) NTCD = 2
      IF (NTCD .EQ. 2) GO TO 15010
15113 DO 270 I=1,NODES
      FORCE(I) = -F(I) -FO(I)
      GO TO (270,15112),STEADY
15112 DO 270 J = 1,NODES
      FACTOR = P(I,J)*T(J)
      FORCE(I) = FORCE(I) - FACTOR
  270 CONTINUE
C
      GO TO (15211,15212),STEADY
15212 IF (INCR - INC) 440,430,440
  430 IF (NODS) 432,432,431
  431 READ (60,510) (N,T(N),I=1,NODS)
      WRITE (61,560)  N,T(N)
  560 FORMAT    (1H0,2HT(,I3,2H)=,F12.5)
  432 READ (60,140) INC,NAMBS,NFLUXS,NHEATS,NQS,NODS, DELTAN
  440 CONTINUE
C
C     COMPENSATE FOR CONSTANT TEMPERATURES
C
15211 IF (NCONST .EQ. 0) GO TO 302
C     PERFORM SIMILARITY TRANSFORMATION
      DO 327 I=1,NODES
      FO(I) = 0.
      V(I) = 0.
      DO 327 J=1,NODES
      FO(I) = FO(I) + S(I,J) * FORCE(J)
  327 V(I) = V(I) + S(I,J)*T(J)
      DO 328 I=1,NODES
      FORCE(I) = FO(I)
  328 T(I) = V(I)
      K1 = NVAR + 1
C
      DO 304 I=1,NVAR
      DFORCE = FORCE(I)
      DC = 0.
      DO 300 J=K1,NODES
      DB = HH(I,J) * T(J)
  300 DC = DC + DB
      DDIF = DFORCE - DC
  304 FORCE(I) = DDIF
  302 CONTINUE
C
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
```

```
C
      TOTAL = 0.
      DO 305 I= 1,NVAR
  305 TOTAL = TOTAL + (FORCE(I) * FORCE (I))
      TOTAL = SQRT(TOTAL)
      IF (TOTAL .GT. 1.E-75) GO TO 310
      TOTAL = 0.
      GO TO 309
  310 DO 306 I=1,NVAR
  306 FORCE(I) = FORCE(I) / TOTAL
      EPSI = 2.**(-23)
      CALL RSLMC(HH,H,FORCE,T,NVAR,EPSI,IER,NORDER,V,PER)
      IF (IER .GT. 2) WRITE(61,1005) IER
  309 DO 307 I=1,NVAR
  307 T(I) = T(I) * TOTAL
      DO 308 I=1,NODES
  308 FO(I) = F(I)
C
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
C     O U T P U T
C
C     INVERT THE SIMILARITY TRANSFORMATION
      IF (NCONST .EQ. 0) GO TO 888
      DO 330 I = 1,NODES
      V(I) = 0.
      DO 330 J = 1,NODES
      V(I) = V(I) + S(I,J)*T(J)
  330 CONTINUE
      DO 331 I = 1,NODES
  331 T(I) = V(I)
C
C     O U T P U T
C
  888 IF (NRAD .EQ. 0) GO TO 889
      IF (INCR .LE. MAXINC) GO TO 889
      DO 1700 I=1,NODES
      DDT = T(I) / TO(I)
      DCHK = DABS(1. - DDT)
      CHK = DCHK
      IF (CHK .GT. RADTOL) GO TO 1701
 1700 CONTINUE
      STEADY = 1
      GO TO 889
```

```
C VALUES DID NOT MEET TOLERANCE.  THEREFORE PRINTOUT WITH
C      ACCEPTABLE TOLERANCE.
 1701 IF (INCR - MXSSIN) 1711,1702,1702
 1711 DO 1712 I=1,NODES
 1712 TO(I) = T(I)
      GO TO 1
 1702 LINE = LINE + LINBLK + 3
      IF (LINE - 60) 1703,1703,1704
 1704 WRITE (61,997) IDENT
      LINE = 10 + LINBLK
 1703 TOL = CHK
      DO 1706 J=I,NODES
      DDT = T(J) / TO(J)
      DCHK = DABS(1.-DDT)
      CHK = DCHK
      IF (CHK .GT. TOL) TOL = CHK
 1706 CONTINUE
      WRITE (61,1705) TOL
 1705 FORMAT (1H0/3X,78HSTEADY STATE TOLERANCE NOT MET...AT MAXIMUM SS I
     CNCREMENT, ABS(1-T(N)/T(N-1)) =,  E12.6,1H./1H0)
      STEADY = 1
      GO TO 15122
  889 LINE = LINE + LINBLK + 3
      IF(LINE - 60) 901,901,900
  900 WRITE (61,997) IDENT
  997 FORMAT(1H1/1H0,25X,20A4/1H0)
      LINE =10 + LINBLK
  901 GO TO (15121,15122),STEADY
15121 WRITE(61,998) INCR
  998 FORMAT(1H0/30X,45HSTEADY-STATE NODAL TEMPERATURES---INCREMENT #,
     1I4/1H0)
      GO TO 15123
15122 WRITE(61,999) INCR,TIME
  999 FORMAT(10H0INCREMENT,I5,5X,9HAT TIME =,F12.5,5X,50(1H-)/)
15123 WRITE (61,1000) (I,T(I),I=1,NODES)
 1000 FORMAT(5(6X,2HT(,I2,3H) =,F12.4))
      GO TO (10000,3),STEADY
    3 IF(INCR-MAXINC) 1,2,2
C      CHANGE TRANSIENT TO STEADY-STATE
    2 IF (NRAD .LE. 0) GO TO 1710
      DO 1709 I=1,NODES
 1709 TO(I) = T(I)
      GO TO 1
 1710 STEADY = 1
      GO TO 992
10001 STOP
      END
```

```
      SUBROUTINE CONDUC(N,M)
      COMMON XI,XJ,XK,XL,YI,YJ,YK,YL,ZI,ZJ,ZK,ZL,X(100,3),H(100,100),
     1  P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      INTEGER * 4 STEADY
C
C     CONDUCTIVITY MATRIX
C
      I = INDEX(N,1)
      J = INDEX(N,2)
      K = INDEX(N,3)
      XI = X(I,1)
      XJ = X(J,1)
      XK = X(K,1)
      YI = X(I,2)
      YJ = X(J,2)
      YK = X(K,2)
      IF (NDIM .EQ. 3) GO TO 100
      AREA(N) = ABS(XJ*YK - XK*YJ - XI*(YK-YJ) + YI*(XK-XJ))/2.
      VOLUME (N) = AREA(N)
      BI = YJ-YK
      BJ = YK-YI
      BK = YI-YJ
      CI = XK-XJ
      CJ = XI-XK
      CK = XJ-XI
      CX = COND(M,1)*TH(N)/(4.*AREA(N))
      CY = COND(M,2)*TH(N)/(4.*AREA(N))
      H(I,I) = H(I,I) + BI*CX*BI + CI*CY*CI
      H(I,J) = H(I,J) + BJ*CX*BI + CJ*CY*CI
      H(I,K) = H(I,K) + BK*CX*BI + CK*CY*CI
      H(J,J) = H(J,J) + BJ*CX*BJ + CJ*CY*CJ
      H(J,K) = H(J,K) + BK*CX*BJ + CK*CY*CJ
      H(K,K) = H(K,K) + BK*CX*BK + CK*CY*CK
      H(J,I) = H(I,J)
      H(K,I) = H(I,K)
      H(K,J) = H(J,K)
      RETURN
  100 CONTINUE
```

```
C
C     CONDUCTIVITY MATRIX FOR THE 3 DIMENSIONAL TETRAHEDRON
C
      L=INDEX(N,4)
      XL = X(L,1)
      YL = X(L,2)
      ZI = X(I,3)
      ZJ = X(J,3)
      ZK = X(K,3)
      ZL = X(L,3)
      DET(1,1) = XJ*(YK*ZL-YL*ZK) - YJ*(XK*ZL-XL*ZK) + ZJ*(XK*YL-XL*YK)
      DET(2,1) = -YK*ZL + YL*ZK + YJ*(ZL - ZK) - ZJ*(YL - YK)
      DET(2,2)=-(-YL*ZI + YI*ZL + YK*(ZI-ZL) - ZK*(YI-YL))
      DET(2,3) = -YI*ZJ + YJ*ZI + YL*(ZJ - ZI) - ZL*(YJ - YI)
      DET(2,4)=-(-YJ*ZK + YK*ZJ + YI*(ZK - ZJ) - ZI*(YK - YJ))
      DET(3,1)=-(XJ*(ZL - ZK) -(XK*ZL - XL*ZK) + ZJ*(XK - XL))
      DET(3,2)=  XK*(ZI - ZL) - (XL*ZI - XI*ZL) + ZK*(XL - XI)
      DET(3,3)=-(XL*(ZJ - ZI) - (XI*ZJ - XJ*ZI) + ZL*(XI - XJ))
      DET(3,4)=  XI*(ZK - ZJ) - (XJ*ZK - XK*ZJ) +ZI*(XJ - XK)
      DET(4,1) = -XJ*(YK-YL) + YJ*(XK - XL) - XK*YL + XL*YK
      DET(4,2)=-(-XK*(YL - YI) + YK*(XL - XI) - XL*YI + XI*YL)
      DET(4,3) = -XL*(YI -YJ) + YL*(XI - XJ) - XI*YJ + XJ*YI
      DET(4,4)=-(-XI*(YJ - YK) + YI*(XJ - XK) - XJ*YK + XK*YJ )
      IF (NBND1(N) .EQ. 0)  GO TO 200
      AREA(N) = SQRT(.25*(DET(2,4)**2 + DET(3,4)**2 + DET(4,4)**2))
  200 VOLUME(N) = ABS((DET(1,1)+DET(2,1)*XI+DET(3,1)*YI+DET(4,1)*ZI)/6.)
      CW = 36.*VOLUME(N)
      CX = COND(M,1)/CW
      CY = COND(M,2)/CW
      CZ = COND(M,3)/CW
```

```
      H(I,I) = H(I,I) + DET(2,1)*CX*DET(2,1) + DET(3,1)*CY*DET(3,1)
     2      + DET(4,1)*CZ*DET(4,1)
      H(I,J) = H(I,J) + DET(2,2)*CX*DET(2,1) + DET(3,2)*CY*DET(3,1)
     2      + DET(4,2)*CZ*DET(4,1)
      H(I,K) = H(I,K) + DET(2,3)*CX*DET(2,1) + DET(3,3)*CY*DET(3,1)
     2      + DET(4,3)*CZ*DET(4,1)
      H(I,L) = H(I,L) +  DET(2,1) *CX*DET(2,4) + DET(3,1)*CY*DET(3,4)
     2     + DET(4,1)*CZ*DET(4,4)
      H(J,J) = H(J,J) + DET(2,2)*CX*DET(2,2) + DET(3,2)*CY*DET(3,2)
     2      + DET(4,2)*CZ*DET(4,2)
      H(J,K) = H(J,K) + DET(2,3)*CX*DET(2,2) + DET(3,3)*CY*DET(3,2)
     2      + DET(4,3)*CZ*DET(4,2)
      H(J,L) = H(J,L) + DET(2,2)*CX*DET(2,4) + DET(3,2)*CY*DET(3,4)
     2      + DET(4,2)*CZ*DET(4,4)
      H(K,K) = H(K,K) + DET(2,3)*CX*DET(2,3) + DET(3,3)*CY*DET(3,3)
     2      + DET(4,3)*CZ*DET(4,3)
      H(K,L) = H(K,L) + DET(2,3)*CX*DET(2,4) + DET(3,3)*CY*DET(3,4)
     2      + DET(4,3)*CZ*DET(4,4)
      H(L,L) = H(L,L) + DET(2,4)*CX*DET(2,4) + DET(3,4)*CY*DET(3,4)
     2      + DET(4,4)*CZ*DET(4,4)
      H(J,I) = H(I,J)
      H(K,I) = H(I,K)
      H(L,I) = H(I,L)
      H(K,J) = H(J,K)
      H(L,J) = H(J,L)
      H(L,K) = H(K,L)
      RETURN
      END
```

```
      SUBROUTINE SPECIF(N,M)
C
C     SPECIFIC HEAT MATRIX
C
      COMMON XX(4),Y(4),Z(4),X(100,3),H(100,100),P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      I = INDEX(N,1)
      J = INDEX(N,2)
      K = INDEX(N,3)
      IF (NDIM .EQ. 3) GO TO 100
      FACTOR = PC(M)*AREA(N)*TH(N) / 12.
      FACTO2 = FACTOR + FACTOR
   10 P(I,I) = P(I,I) + FACTO2
      P(I,J) = P(I,J) + FACTOR
      P(I,K) = P(I,K) + FACTOR
      P(J,J) = P(J,J) + FACTO2
      P(J,K) = P(J,K) + FACTOR
      P(K,K) = P(K,K) + FACTO2
      P(J,I) = P(I,J)
      P(K,I) = P(I,K)
      P(K,J) = P(J,K)
      RETURN
  100 CONTINUE
      L=INDEX(N,4)
      FACTOR = PC(M)*VOLUME(N)*.05
      FACTO2 = FACTOR + FACTOR
      P(I,L) = P(I,L)     +        FACTOR
      P(J,L) = P(J,L)      +       FACTOR
      P(K,L) = P(K,L)    +         FACTOR
      P(L,L) = P(L,L) +            FACTO2
      P(L,I) = P(I,L)
      P(L,J) = P(J,L)
      P(L,K) = P(K,L)
      GO TO 10
      END
```

```
      SUBROUTINE RADIAN (N,M,FNM,MAT,TWALL,NRAD)
C
CRADIATION NORMAL TO BOUNDARY
C
      COMMON    XIYJZK(12),X(100,3),H(100,100),P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      REAL * 8 DF1,DF2,DF3,DT4
      I = INDEX(N,1)
      J = INDEX(N,2)
      IF (NRAD .EQ. 2) GO TO 100
      XL1 = SQRT((X(I,1)-X(J,1))**2 + (X(I,2)-X(J,2))**2) * TH(N)
      IF (N.NE.M) GO TO 20
      T41 = (.005*(T(I) + T(J)))**4 * .0857 * XL1 * E(MAT)
      IF (FNM) 5,6,6
    5 NOWALL = 0
      GO TO 10
    6 NOWALL = 1
   10 DF1 = F(I)
      DF2 = F(J)
      DT4 = T41
      DF1 = DF1 + DT4
      DF2 = DF2 + DT4
      F(I) = DF1
      F(J) = DF2
      IF (NOWALL .EQ. 0) RETURN
      T41 = (TWALL * .01) ** 4
      GO TO 30
   20 K=INDEX(M,1)
      L=INDEX(M,2)
      T1 = .005 * (T(K) + T(L))
      T41 = T1 ** 4
   30 T41 = -T41 * .0857 * XL1 * A(MAT) * FNM
      NOWALL = 0
      GO TO 10
  100 CONTINUE
      K= INDEX(N,3 )
      A1 = AREA(N)
      T1 = (T(I) + T(J) + T(K)) / 3.
      IF (N .NE. M) GO TO 220
```

```
C
CIF N EQUALS M, CALCULATE RADIATION EMITTED ONLY PLUS THAT ABSORBED
CFROM KNOWN SOURCE.
C
      T41 = ((.01*T1)**4)* .1714 * A1 * E(MAT) /  3.
      IF (FNM) 205,206,206
  205 NOWALL = 0
      GO TO 210
  206 NOWALL = 1
  210 DF1 = F(I)
      DF2 = F(J)
      DF3 = F(K)
      DT4 = T41
      DF1 = DF1 + DT4
      DF2 = DF2 + DT4
      DF3 = DF3 + DT4
      F(I)  = DF1
      F(J)  = DF2
      F(K)  = DF3
      IF (NOWALL .EQ. 0) RETURN
      T41 = (TWALL * .01) ** 4
      GO TO 230
  220 II = INDEX(M,1)
      JJ = INDEX(M,2)
      KK = INDEX(M,3)
      T1 = .01 * (T(II) + T(JJ) + T(KK)) / 3.
      T41 = T1 ** 4
  230 T41 = -T41 * .1714 * A1 * A(MAT) * FNM          /  3.
      NOWALL = 0
      GO TO 210
      END
```

```
      SUBROUTINE HETGEN (N,TOTAL)
C
C     INTERNAL HEAT GENERATION - FORCE
C
      COMMON    XIYJZK(12),X(100,3),H(100,100),P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      I = INDEX(N,1)
      J = INDEX(N,2)
      K = INDEX(N,3)
      IF (NDIM .EQ. 3) GO TO 100
      FACTOR = -TOTAL * AREA(N) * TH(N) /3.
   10 F(I) = F(I) + FACTOR
      F(J) = F(J) + FACTOR
      F(K) = F(K) + FACTOR
      RETURN
  100 CONTINUE
      L = INDEX(N,4)
      FACTOR = -TOTAL*VOLUME(N)*.25
      F(L) = F(L) + FACTOR
      GO TO 10
      END
```

```
      SUBROUTINE FLUXN (N,NFLUX)
C
C     FLUX NORMAL TO SURFACE - FORCE
C
      COMMON    XIYJZK(12),X(100,3),H(100,100),P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      NR = INDEX(N,1)
      NS = INDEX(N,2)
      IF (NFLUX .EQ. 2) GO TO 200
      D= SQRT((X(NR,1)-X(NS,1))**2 + (X(NR,2)-X(NS,2))**2) * TH(N)
      FACTOR = FLUX(N) * D * .5
      F(NR) = F(NR) + FACTOR
      F(NS) = F(NS) + FACTOR
      RETURN
  200 NT = INDEX(N,3)
      FACTOR = (FLUX(N) * AREA(N)          ) / 3.
      F(NR) = F(NR) + FACTOR
      F(NS) = F(NS) + FACTOR
      F(NT) = F(NT) + FACTOR
      RETURN
      END
```

```
      SUBROUTINE CONVEC (N,NCONV,NTCD)
C
C
C     CONVECTION ACROSS BOUNDARY       + AMBIENT
      COMMON   XIYJZK(12),X(100,3),H(100,100),P(100,100),FORCE(100),
     1  AREA(150),HEAT(150),HCONV(150),COND(5,3),Q(5),PC(5),T(100),
     2  FO(100),F(100),AMB(150),FLUX(150),FORM(150,150),E(5),A(5),
     3  HH(100,100),TH(150),V(100),PER(100),DET(4,4),VOLUME(150),
     4  STEADY,INDEX(150,4),NBND1(150),MATERL(150),INCR,NEL,NODES,NMAT,
     5  NDIM
      I = INDEX(N,1)
      J = INDEX(N,2)
      IF (NCONV .EQ. 2) GO TO 100
      H1 = SQRT((X(I,1) - X(J,1))**2 + (X(I,2) - X(J,2))**2)*TH(N)
     2   *HCONV(N)/6.
      IF (NTCD .EQ. 2) GO TO 2
      H2 = 2. * H1
      H(I,I) = H(I,I) + H2
      H(J,J) = H(J,J) + H2
      H(I,J) = H(I,J) + H1
      H(J,I) = H(I,J)
      IF (NTCD .EQ. 1) RETURN
    2 H4 = -H1*AMB(N)*3.
      F(I) = F(I) + H4
      F(J) = F(J) + H4
      RETURN
  100 CONTINUE
C
CONVECTION OVER SURFACE
C
      K = INDEX(N,3)
      H1 = HCONV(N) * AREA(N)          / 12.
      IF (NTCD .EQ. 2) GO TO 200
      H2 = H1 * 2.
      H(I,I) = H(I,I) + H2
      H(J,J) = H(J,J) + H2
      H(K,K) = H(K,K) + H2
      H(I,J) = H(I,J) + H1
      H(I,K) = H(I,K) + H1
      H(J,K) = H(J,K) + H1
      H(K,J) = H(J,K)
      H(K,I) = H(I,K)
      H(J,I) = H(I,J)
      IF (NTCD .EQ. 1) RETURN
  200 H4 = H1 * 4. * AMB(N)
      F(I) = F(I)  - H4
      F(J) = F(J) -H4
      F(K) = F(K) - H4
      RETURN
      END
```

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