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Technical Memorandum 33-648

A Preliminary Design and Implementation of the Low-Thrust Simulation and Trajectory Search Program (LOWTRAJ)

Chen-wan L. Yen

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PREFACE

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CONTENTS

ABSTRACT

This work represents the results of one phase of research conducted for the JPL Solar Electric Propulsion (SEP) Navigation Software System development program. It deals only with the problem of designing the flight quality trajectory program, which is a major subset of the entire navigation software system.

In this phase of research (breadboard development phase), attempts were made to assess the SEP trajectory software functional requirements, to investigate the program design method satisfying these requirements, to identify the primary anticipated problem areas, and to provide solutions to these problem areas. These efforts culminated in the development of a compact breadboard program, "LOWTRAJ." A functional description and the mathematical formulation of the program are presented.

The results of tests performed using LOWTRAJ indicate that the primary requirements of the flight quality trajectory program can be met with this type of design. Future extensions of the program, further refined to support flight operation, should be straightforward.

I. INTRODUCTION

In the past decade, various trajectory programs have been developed to study the applications of solar electric propulsion (SEP) in interplanetary explorations. These programs were of a first-generation type, oriented primarily to support the assessment of SEP mission feasibility, payload performance, and reliability. In this early conceptual stage of SEP technology, the hardware performance characteristics were not well defined. Simplified approximations and idealizations were used to represent low-thrust propulsion, which made the mathematical formulation and implementation of optimal trajectory search feasible. This resulted in a large quantity of preliminary information stressing the upper limit of SEP spacecraft performance capabilities. It appears that with currently available technology a SEP spacecraft can be an attractive candidate for some selected missions.

To prepare for an actual SEP flight, second-generation SEP trajectory software is being considered. It is intended to be a low-cost program, serving primarily as a test tool (breadboard) for the development of a third-generation flight program. It is also intended to become the main core of a larger breadboard SEP Navigation Software System (SEPNSS), which will include the trajectory, orbit determination (OD), and guidance control software.

For the support of a SEP flight, particularly in performing reliable navigation and guidance functions, it is critical to have a very accurate "theoretical prediction" of the spacecraft state. An accurate simulation of SEP spacecraft controls and their net effect is required to achieve this goal. Optimum payload performance to the last few percent, the characteristic of the first-generation software, is now considered to be of secondary importance. To design a practical mission, optimality must be violated anyway to accommodate specific restrictions required in hardware design, and to satisfy other mission constraints imposed for the sake of mission reliability, scientific experiments, and communications. The development of elegant mathematical approaches using the calculus of variations to generate suboptimal controls satisfying many and varied types of constraints appear to be rather difficult, and the potential performance margin to be gained is expected to be slight. Therefore, a direct parametric trajectory search procedure that may or may not explicitly optimize the payload is proposed. In addition to satisfying the requirements of establishing a valid reference trajectory, the trajectory program is expected to interact intimately with the OD and guidance programs. The use of the direct parametric search procedure is most natural, and in harmony with the needs of these user programs.

In Section II, a general description of a trajectory program is given. Details of specific problems and solutions required for the low-thrust application will be discussed in Section III. Section IV will be devoted entirely to the mathematical formulations and solutions of these specific problems.

II. GENERAL TRAJECTORY SOFTWARE FUNCTIONS AND REQUIREMENTS

A simplified diagram of a SEPNSS is shown in Fig. 1. This illustrates typical (ballistic or SEP) OD and guidance software functions, and their relationships to one another. The subprograms PATH, VARY, and SEARCH are the major structural constituents of the LOWTRAJ. As one can infer from the diagram, the design philosophy of the OD and guidance programs must strongly influence the formulation of PATH and VARY. Since it is impractical to expect synchronized progress in the development of the OD, guidance, and trajectory programs, the major linkages of PATH-VARY with the OD or guidance programs will not be attempted until later in the SEPNSS development phase.

Descriptions of the general functions and requirements for each of the three major subprograms of LOWTRAJ are given in the following. Requirements unique to the low-thrust applications, demanding special study, are emphasized.

A. Trajectory Simulation Program (PATH)

The function of this program is to theoretically predict the spacecraft path by numerical integration of the equations of motion. It must accurately account for and model all forces acting on the spacecraft, particularly the low-thrust force, and must integrate, maintaining high numerical accuracy. This is an open loop path predictor that does not perform a targeting function.

B. Variational Equations Integrator (VARY)

This program generates variations of the spacecraft trajectory as induced by small perturbations of the trajectory parameters $\{\vec{P}\}$ (i.e., $\partial\vec{X}(t)/\partial\vec{P}$). This is accomplished by the integration of the variational equations. These are the basic data required by the user programs, namely trajectory search, OD, and the guidance programs. The major objective in the construction of this subprogram is to identify the important parameters with respect to which partial derivatives are required by the search program. At the same time the potential needs of the OD and guidance programs must be considered. Then the derivation and integration of the necessary variational equations must be performed, and the efficient transfer of this information to the user programs must be executed.

C. Trajectory Search Program (SEARCH)

This program performs the deterministic targeting function. It drives PATH and VARY, and iteratively searches for the trajectory shaping parameters that will satisfy the required boundary conditions and various mission constraints. A large number of search parameters (perhaps as many as 100 or more) can be involved in this process. This program must handle searches with many degrees of freedom, preferably with a capability to optimize the payload. It must have the ability to handle many constraints of different types, and it must attain convergence with high reliability and speed.

III. SPECIFIC PROBLEMS AND PROPOSED SOLUTIONS

A. Accurate Low-Thrust Modeling in PATH

An accurate modeling of the low-thrust force depends on a good understanding of the performance characteristics of the thrust-producing hardware subsystems, the policies of their operation, and the many modes in which they may be operated. Mathematical simulations of these control processes by means of simple equations and parameterization leads to correct representation of the net force exerted on the spacecraft. The key to successful modeling lies in the manner in which these parameters are introduced. The versatility of the program, its compatibility to the search, OD, and guidance programs are established at this point.

In the following, a brief discussion of the three major thrust-producing hardware systems, and the assumptions being made on their operating policies and modes will be presented. The resulting model and the meaning of some important parameters will be analyzed.

1. Simulation of the power subsystem function. Power subsystem components pertinent to low-thrust control include the solar array and switching and control function subsystems for power management and distribution. Theoretically, solar array maximum output power minus housekeeping power should be available to the thrust subsystem. However, a closed loop maximum power operating policy for the thrust subsystem may not be acceptable. Unpredictable fluctuations in solar array output power, compounded by the already noisy thruster performance at known operating levels, would make the overall low-thrust noise level too large to be tolerable for accurate navigation and guidance. Therefore, regardless of the maximum power available for propulsion from the solar array, the power input to the thrust subsystem will be programmed to be less than the maximum point. The devices for the solar array power regulation have as yet to be specified by the power subsystem specialists. Still, the regulation generally consists of triggers to command the level changes, and automatic maintenance of the set level between the triggers. The on-board command and control timer triggers these changes, and the interval between the triggers is estimated to be on the order of 1 to 10 days depending on the power profile. Basically, the power regulation can be achieved in two ways.

First, it is assumed that the regulation device automatically controls both the current and the voltage output of the solar array at a fixed value between regulation switching times (for convenience call it a "power stage"). This implies a piecewise constant power operation. Second, if the regulation device controls only the current (or the voltage) and lets the voltage (or the current) operate at the natural solar array output, the program must model the solar panel current-voltage (I-E) characteristic curves as a function of solar distance. This is not only more complex in mathematical formulation, but it also adds greater uncertainty in the thrust magnitude, because the accuracy of the given (I-E) curves as a function of solar distance is doubtful. If there are any uncertainties, these must be fed to a navigation program and their impact measured. The present trajectory software package includes only the first policy and models power input to the thrust subsystem as being piecewise constant.

The lower bound of solar array maximum output power during t_i to t_{i+1} (i-th power stage) is estimated based on the spacecraft state $(r, \dot{r})$ at time t_i and the conventional power curve formula. This power minus the housekeeping power is the available input power to the thrust subsystem at the i-th stage $[P_a(i)]$. Actual input power to the thrust subsystem will be $[\nu_i P_a(i)]$, which comprises the basis for panel power regulation. Here ν_i is a power utilization factor for the i-th power stage. $\nu_i = 1$ represents full utilization, $\nu_i = 0$ represents a coast period, and $0 < \nu_i < 1$ represents partial utilization. Even though a bang-bang-type control, where $\nu_i = 0$, or 1, is the optimal control policy, one may want to design the nominal path with ν_i's slightly less than 1 to provide some guidance reserve. In addition ν_i's can be used to simulate expected or unpredictable solar array degradations caused by solar flares or meteorite impacts. It may even be assigned a value larger than 1 to simulate conditions where the actual output of the panel in space indeed exceeds the theoretical prediction.

2. Simulation of the thrust subsystem function. The two major thrust subsystem components considered are power conditioner (PC) units and thruster (THR) units. A combined operation of one PC unit and a thruster constitutes a thrust unit.

The characteristic numbers of the thrust subsystem are the maximum and minimum power ratings of a thrust unit, the number of thrust units available, the efficiency of power conversion, and the specific impulse (I_{sp}). The efficiencies of the PC and THR depend on the respective input power levels. The specific impulse depends on the operating power (throttling level) of the THR. Currently available data on these (Ref. 1) can be represented adequately by quadratic functions of the input power to the thrust unit.

The operating policy of the thrust subsystem assumes that the total input power to the thrust subsystem $[\nu_i P_a(i)]$ is nominally distributed equally among the minimum number ($N_{min}(i)$) of thrust units required to match the power. This is a maximum efficiency policy for a given power level. If the number of available thrust units are less than the $N_{min}(i)$, then all units will be operated at maximum level. As an option one may specify the minimum number of units to be operated. This could, for example, be the case if one wants to maintain at least two thrusters operating, so that three-axis attitude control can be maintained using low thrust.

3. <u>Simulation of the thrust vector control system function.</u> Thrust vector control is assumed to be implemented with the aid of sun and star sensors for attitude reference. Gross reorientation of the thrust vector is achieved by gimballing the sensors by a desired amount, thus offsetting the tracking, then applying torque to the spacecraft to reacquire the stars. This maneuver will be performed at specified intervals (call it angular stages), which may range from a few to hundreds of days depending on the mission. Between gross reorientations, the autonomous attitude control command system maintains the thrust vector cone and clock angles within a specified tolerance band.

B. Setup for Variational Equations in VARY

The dimensions of the partial derivatives to be given by VARY depend on the needs of the user programs. SEARCH requires partial derivatives with respect to injection state, solar array power at 1 AU (P_0), thrust vector directions (two angles) for all angular stages, power utilization factors ν_i for all power stages, the arrival time, and the arrival velocity bias of the spacecraft with respect to the target.

Further expansion of VARY to meet the needs of OD and guidance functions is as yet to be determined.

C. Design of Program SEARCH

1. Input requirements. As discussed previously, this program is of a second-generation type emphasizing the accuracy in the trajectory through accurate hardware function modeling. The capability to search for optimal gross mission parameters such as P_0, I_{sp}, launch dates, and flight time is considered to be outside the scope of this program. Nevertheless optimal or at least near-optimal mission profile is preferred. Therefore, the first guess of the general thrust profile is obtained from the first-generation trajectory optimizing programs, such as CHEBYTOP or EPITOP (Refs. 2, 3). This is mandatory not only for the sake of performance, but also for easier convergence. This program does not have the capability to self-start, nor is it meant to generate the "ballistic conic path" equivalent of a low-thrust trajectory. Once the crude profile is given, it will readjust all the free search variables by means of a modified Newton Raphson method to satisfy the required boundary conditions, one of which may include the final mass with a given tolerance of, say, plus or minus 5 to 10 kg. It can, as an option, perform a limited optimal search for the maximum final mass.

2. Versatility. The flexibility of the program is specifically geared for the needs of flight project analysis and design. It must accurately simulate the controls of specific form required for mission implementation. It must also be able to simulate various types of control malfunctions so that impacts of these uncertain hardware functions can be analyzed and a reliable mission designed.

These goals can be achieved if one allows, by option, all the low-thrust control parameters to be included in the search or to be fixed. The only parameters that cannot be given this freedom are the stage times, both for power stages or angle stages. However, the user will have the freedom to assign almost any stage pattern as long as it does not exceed the designated dimension of the stages in the program, which currently is 200 for power stages and 50 for angle stages.

Some examples of the desirable features are:

(1) The user can design the trajectory with thrust on-off time specified. In first generation trajectory software, this was not possible because the optimality condition controlled and internally generated the switching time. Coast phases could not be arbitrarily specified. This capability is crucial; it will be needed to satisfy the science experiment requirements, navigational needs, and for reliable mission design.

(2) Since all the power utilization parameters can be searched or fixed, any throttling levels may be commanded.

(3) Thrust vector and/or spacecraft attitude can be constrained for any desired period of time. Such constraints are imposed usually by the limited structural flexibility of the spacecraft, thrust-subsystem thermal control requirements, the science experiments or the communications requirements.

(4) Thruster arcing or failures can be simulated and updates of trajectory can be made.

(5) Solar panel degradations, minor or major, can be simulated and their impacts can be measured.

3. Organization of search variables.

(1) Independent variables: To attain maximum flexibility, a large dimension in independent-variable space is introduced. This high degree of freedom consists mainly of thrust angles modeled in multistage fashion. Since SEP is a power-limited propulsion device, these angles are the main source of trajectory shaping capability, particularly after spacecraft initial injection.

Other important degrees of freedom that can be used for control are the thrust duration and the time of encounter. The search on thrust duration is performed on ν_i allowing only quantum jumps, that is 0 to 1, or 1 to 0.

The degrees of freedom of 0, 2, and 3 can be assigned, by option, to departure and arrival velocity biases ($\vec{V}_B$).

These correspond to cases where it does not search on $\vec{V}_B$ (rendezvous), $\vec{V}_B$ is constrained in magnitude but has 2 degrees of freedom in the choice of direction (flyby with given relative speed), and, thirdly, $\vec{V}_B$ is unconstrained.

Other variables such as injection mass, injection time, and P_0 are included to meet the primary needs of flight quality mission design.

(2) Dependent variables: The program is organized in a manner such that dependent variables always include the spacecraft final position and velocity minus position and velocity biases. These are always searched to coincide with the state of the target body. Analytic ephemerides of the major planets, asteroids, and comets are internally linked to the program.

An additional dependent variable included is the final mass of the spacecraft. This is included so that limited optimal control of the final mass can be accomplished.

4. Search procedure. As it has been stressed in Section III, the many degrees of freedom of search are due to the many angle variables. Since these are characteristically the same controls appearing consecutively and progressively in stages, it is likely that high correlations exist among the partial derivatives $[\partial\vec{Y}/\partial(\alpha_I, \beta_I)]$ (where $\vec{Y}$ is the vector of dependent variables, and α_I and β_I are the thrust cone and clock angles for stages $I = 1, \cdots, N$, etc., to be searched). It is unwise to perform standard iterative linear analysis of the form $\mathbf{M}\Delta\mathbf{X} = \Delta\mathbf{Y}$ without fully analyzing the singularity of matrix $\mathbf{M}$. The search algorithm makes extensive utilization of the information obtained in performing the "Singular-Value Decomposition Analysis" (Ref. 4) of matrix $\mathbf{M}$. Methods of obtaining solutions within the framework of linear algebra is discussed in greater detail in Section IV.

IV. MATHEMATICAL FORMULATION OF LOWTRAJ

A. Spacecraft Trajectory

Generally, the sources of acceleration of the spacecraft relative to the center of integration to be considered include the following perturbations:

(1) The Newtonian point-mass acceleration relative to the center of integration.

(2) The acceleration due to low thrust.

(3) The acceleration due to chemical motor burns.

(4) The acceleration due to solar radiation pressure.

(5) The accelerations due to other smaller order gravitational interactions, including N-body effects, planet oblateness effects, mascon effects, and relativistic effects.

(6) The accelerations due to small perturbations originating in the spacecraft, attitude controls (especially the low-thrust type), and due to gas leaks. Nonavailability of solar power for low thrust during solar occultation must also be included.

However, due to the experimental nature of the program and to maintain low cost and efficiency of program operation, numerical integration is performed in single precision. Inclusion of perturbations (4) to (6) in the single precision algorithm is not meaningful, thus they are excluded in LOWTRAJ but will be required in the flight quality program. In the current scheme of the trajectory search program, the inclusion of these small forces is not expected to influence the basic algorithms.

B. Glossary of Notations

Unless otherwise stated, the following notations will be used consistently without explanation.

$\vec{F}$	low-thrust acceleration
f	low-thrust magnitude
$\vec{G}$	gravitational acceleration
H	mass flow rate
m	spacecraft mass at time t
m_{0i}	spacecraft mass at the beginning of i-th power stage
$\vec{r}$	spacecraft position vector at time t
$\vec{r}_{0i}$	spacecraft position vector at the beginning of i-th power stage
$\vec{s}$	unit vector of specified reference star position
t_i	initial time of power stage i
$\vec{v}_{0i}$	spacecraft velocity vector at the beginning of i-th power stage
$\vec{v}$	spacecraft velocity vector at time t
$\vec{X}$	spacecraft state vector at time t where $\vec{X} = (\vec{r}, \vec{v}, m)$
$\vec{X}_{0i}$	spacecraft state vector at the beginning of i-th power stage
α, β	thrust cone and clock angles with respect to sun and a star
Δt_i	i-th power stage interval
Δt_I	I-th angular stage interval
μ	gravitational constant of the sun
ν	power utilization factor
$\vec{\xi}$	low-thrust unit vector

Subscripts

i	power-stage
I	angular-stage

C. Derivation of Equations of Motion

1. Mathematical force model. Mathematical expressions of the spacecraft accelerations and mass flow rate are given as follows.

a. Gravitational acceleration

$$\vec{G}(\vec{r}) = -\frac{\mu\vec{r}}{r^3} \tag{1}$$

b. Low-thrust acceleration

$$\vec{F}_{Ii}\ (\vec{r}_{0i},\ \vec{v}_{0i},\ \alpha_I,\ \beta_I,\ \nu_i) = \frac{f_i\ (\vec{r}_{0i},\ \vec{v}_{0i},\ \nu_i)\ \vec{\xi}\ (\alpha_I,\ \beta_I,\ \vec{r})}{m} \tag{2}$$

where $f_i\ (\vec{r}_{0i},\ \vec{v}_{0i},\ \nu_i)$ is the thrust magnitude for the i-th power stage, and $\vec{\xi}\ (\alpha_I,\ \beta_I,\ \vec{r})$ is the thrust unit vector during I-th angle stage.

Note: ν_i is constant during the i-th power stage and α_I, β_I are constants during the I-th angle stage.

c. Mass flow rate

$$H_i = f_i\ (r_{0i},\ v_{0i},\ \nu_i)/c$$

where c is the thruster exhaust velocity.

Derivation of $f_i\,(\vec{r}_{0i}, \vec{v}_{0i}, \nu_i)$

Let

$$r_p = r_{0i} + \frac{1}{2}\left[\frac{\vec{r}_{0i} \cdot \vec{v}_{0i}}{|\vec{r}_{0i} \cdot \vec{v}_{0i}|} + 1\right]\left[\frac{\vec{r}_{0i} \cdot \vec{v}_{0i}}{|\vec{r}_{0i}|} \cdot \Delta t_i\right] \tag{3}$$

where r_p is the estimated upper bound of the spacecraft-sun distance, and this is used to estimate the lower bound of the solar-panel maximum output power during Δt_i. To maintain the validity of such a power estimate, Δt_i must not be too large. Then,

$$p_{max} = p_0 \sum_{i=1}^{5} a_i r_p^{-(i+3)/2} \tag{4}$$

where p_{max} = estimated lower bound of panel maximum output power during the i-th power stage, a_i = solar panel maximum power curve coefficients and p_0 = solar panel output power at 1 AU. Let

$$p_a = p_{max} - p_h$$

where p_a = available input power to the thrust unit and p_h = housekeeping power. Let

$$N_{min} = \frac{p_a \nu_i}{p_{r_1}} + 1 \text{ (integer operation)} \tag{5}$$

$$N_{max} = \frac{p_a \nu_i}{p_{r_2}} \text{ (integer operation)} \tag{6}$$

Then

$$N_o = N_{min} \text{ or } \begin{cases} \text{If } N_{min} > N_a \text{ then } N_o = N_a \\ \text{If } N_{min} < N_d \text{ then } N_o = N_d \\ \text{If } N_o > N_{max} \text{ then } N_o = N_{max} \end{cases}$$

where

N_{min} = minimum number of thrust units required to be in operation

N_{max} = maximum number of thrust units one may be operating without throttling below the minimum rated power

p_{r_1} = maximum power rating of a thrust unit

p_{r_2} = minimum power rating of a thrust unit

N_o = number of thrusters actually operated

N_a = available number of thrust units

N_d = desired lower limit of thrust units operating

Let

$$p_{op} = \frac{p_a \nu_i}{N_o} \tag{7}$$

$$\left.\begin{aligned} c &= c_0 + c_1\, p_{op} + c_2\, p_{op}^2 \\ \eta &= \eta_0 + \eta_1\, p_{op} + \eta_2\, p_{op}^2 \end{aligned}\right\} \tag{8}$$

where

p_{op} = the operating level of each thrust unit

c = thruster exhaust velocity ($I_{sp}g$)

η = thrust unit power conversion efficiency

c_0, c_1, c_2 = polynomial coefficients used to express exhaust velocity as a function of operating power

η_0, η_1, η_2 = efficiency coefficients

Then

$$f_i = \frac{2\eta \nu_i p_a}{c} \tag{9}$$

<u>Spacecraft mass flow rate</u>

$$H_i = \frac{2\eta \nu_i p_a}{c^2} \tag{10}$$

where H_i = mass flow rate during i-th power stage.

<u>Derivation of $\vec{\xi}(\alpha_I, \beta_I, \vec{r})$.</u> Let

$$\vec{k}' = \frac{\vec{r}}{|\vec{r}|}, \; \vec{j}' = \vec{k}' \times \vec{s}, \; \vec{i}' = \vec{k}' \times \vec{j}' \tag{11}$$

where $\vec{s}$ is the reference star unit vector and $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ are unit vectors of sun-star reference frame. Let

$$\vec{\xi}' = (\sin \alpha_I \cos \beta_I, \sin \alpha_I \sin \beta_I, \cos \alpha_I) \tag{12}$$

where $\vec{\xi}'$ is a thrust unit vector in the sun-star reference frame and α_I, β_I are the thrust cone and clock angles. Then,

$$\vec{\xi} = \mathbf{T}(\vec{r})\vec{\xi}' \qquad (13)$$

where $\mathbf{T}(\vec{r})$ is a coordinate transformation matrix with the following components:

$$\mathbf{T}(r) = \begin{pmatrix} i'_x & j'_x & k'_x \\ i'_y & j'_y & k'_y \\ i'_z & j'_z & k'_z \end{pmatrix} \qquad (14)$$

where subscripts x, y, and z denote the x, y, and z components of the vector.

Equations of motion. To maintain the symmetry in the expression, the equations of motion are expressed in seven first order differential equations.

$$\dot{\vec{X}} = \begin{bmatrix} \dot{\vec{r}} \\ \dot{\vec{v}} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \vec{v} \\ \vec{G} + \vec{F}_{Ii} \\ H_i \end{bmatrix} \qquad (15)$$

Since the first derivatives $\dot{\vec{X}}$ are discontinuous at the bounds of every power stage and angular stage (which are designed to coincide with one of the power-stage times), numerical integrations are performed piecewise by power stage increments with a restart procedure for each discontinuity.

D. Derivation of Variational Equations

Partial derivatives of the spacecraft state with respect to search parameters are obtained by numerical integrations of the variational equations. In functional form, equations of motion are given by

$$\dot{\vec{X}} = \vec{\mathscr{F}}(\vec{X}, \vec{Q}) \tag{16}$$

where $\vec{Q}$ is the parameter set with respect to which partial derivatives are required. For search purposes, the parameter set $\vec{Q}$ includes $(\vec{X}_{0i}, \vec{q})$, where $\vec{q} = (\alpha_I, \beta_I, \nu_i, p_o)$ for all i and I desired.

The variational equations are of the following general form:

$$\frac{d}{dt}\left(\frac{\partial\vec{X}}{\partial\vec{Q}}\right) = \frac{\partial\vec{\mathscr{F}}(\vec{X}, \vec{Q})}{\partial\vec{X}} \cdot \frac{\partial\vec{X}}{\partial\vec{Q}} + \frac{\partial\vec{\mathscr{F}}(\vec{X}, \vec{Q})}{\partial\vec{Q}} \tag{17}$$

where $\partial\vec{X}/\partial\vec{Q}$ is a 7 by 11 matrix with initial conditions given by

$$\frac{\partial\vec{X}}{\partial\vec{X}_{0i}} = \mathbf{I}, \quad \text{and} \quad \frac{\partial\vec{X}}{\partial\vec{q}} = 0 \tag{18}$$

where $\mathbf{I}$ is a 7 by 7 unit matrix. For convenience, the following matrix notations are introduced: Let

$$\frac{\partial\vec{X}}{\partial\vec{Q}} = \left[\frac{\mathbf{U}}{\mathbf{V}}\right], \quad \mathbf{A} = \frac{\partial\vec{\mathscr{F}}}{\partial\vec{X}}, \quad \mathbf{C} = \frac{\partial\vec{\mathscr{F}}}{\partial\vec{Q}} = \left(\frac{\partial\vec{\mathscr{F}}}{\partial\vec{X}_{0i}}, \frac{\partial\vec{\mathscr{F}}}{\partial\vec{q}}\right) \tag{19}$$

where

$$\mathbf{U} = \frac{\partial \vec{X}}{\partial \vec{X}_{0i}} = \text{7 by 7 state transition matrix}$$

$$\mathbf{V} = \frac{\partial \vec{X}}{\partial \vec{q}} = \text{7 by 4 control transition matrix}$$

$\mathbf{A}$ = 7 by 7 matrix

$\mathbf{C}$ = 7 by 11 matrix

1. <u>Computation of the **A** matrix.</u> From Eqs. (15), (1), and (2),

$$\mathbf{A} = \begin{array}{c} \left[\begin{array}{c:c:c} 0 & I & 0 \\ \hdashline \dfrac{3\mu\vec{r}\vec{r}^T}{r^3} - I\dfrac{\mu}{r^3} + \dfrac{\partial\vec{\xi}}{\partial\vec{r}} f_{i/m} & 0 & -\dfrac{1}{m}\vec{F}_{Ii} \\ \hdashline 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\,3 \\ \}\,3 \\ \}\,1 \end{array} \\ \begin{array}{ccc} \underbrace{\qquad\qquad}_{3} & \underbrace{\quad}_{3} & \underbrace{\quad}_{1} \end{array} \end{array} \tag{20}$$

where the dimensions of submatrices are as indicated. $(\partial\vec{\xi}/\partial\vec{r})$ is obtained by differentiation of Eq. (13), and from Eqs. (11), (12), and (14).

2. Computation of the $\mathbf{C}$ matrix

a. Derivation of $(\partial\vec{\mathscr{F}}/\partial\vec{X}_{0i})$. From Eq. (15),

$$\frac{\partial\vec{\mathscr{F}}}{\partial\vec{X}_{0i}} = \begin{array}{c} \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \dfrac{\partial\vec{F}_{Ii}}{\partial\vec{r}_{0i}} & \dfrac{\partial\vec{F}_{Ii}}{\partial\vec{v}_{0i}} & 0 \\ \hdashline \dfrac{\partial H_i}{\partial\vec{r}_{0i}} & \dfrac{\partial H_i}{\partial\vec{v}_{0i}} & 0 \end{array}\right] \begin{array}{l} \}\,3 \\ \}\,3 \\ \}\,1 \end{array} \\ \begin{array}{ccc} \underbrace{\qquad}_{3} & \underbrace{\qquad}_{3} & \underbrace{\quad}_{1} \end{array} \end{array} \tag{21}$$

where

$$\frac{\partial F_{Ii}}{\partial(\vec{r}_{0i}, \vec{v}_{0i})} = \frac{\partial f_i}{\partial(\vec{r}_{0i}, \vec{v}_{0i})} \cdot \frac{\vec{\xi}}{m}$$

From Eqs. (9), (7), (8), (5), and (4),

$$\frac{\partial f_i}{\partial(\vec{r}_{0i}, \vec{v}_{0i})} = f_i \left[\left\{\frac{1}{\eta} \cdot \frac{\partial\eta}{\partial p_{op}} - \frac{1}{c} \cdot \frac{\partial c}{\partial p_{op}}\right\} \frac{\partial p_{op}}{\partial p_a} + \frac{1}{p_a}\right] \left[\frac{\partial p_a}{\partial(\vec{r}_{0i}, \vec{v}_{0i})}\right]$$

where, from Eqs. (7) and (8), one can show that

$$\frac{\partial p_{op}}{\partial p_a} = \frac{\nu_i}{N_o}, \quad \frac{\partial\eta}{\partial p_{op}} = (\eta_1 + 2\eta_2 p_{op}), \quad \frac{\partial c}{\partial p_{op}} = (c_1 + 2c_2 p_{op})$$

from Eq. (4),

$$\frac{\partial p_a}{\partial(\vec{r}_{0i}, \vec{v}_{0i})} = p_o \left\{ \sum_i \left(-\frac{i+3}{2}\right) a_i r_p^{-(i+5)/2} \right\} \left(\frac{\partial r_p}{\partial(\vec{r}_{0i}, \vec{v}_{0i})} \right)$$

from Eq. (3),

$$\frac{\partial r_p}{\partial \vec{r}_{0i}} = \frac{r_{0i}}{|\vec{r}_{0i}|} + \frac{1}{2}\left(\frac{\vec{r}_{0i} \cdot \vec{v}_{0i}}{|\vec{r}_{0i} \cdot \vec{v}_{0i}|} + 1 \right) \left(\frac{\vec{v}_{0i}}{|\vec{r}_{0i}|} - \frac{\vec{r}_{0i} \cdot \vec{v}_{0i}}{|\vec{r}_{0i}|^3} \vec{r}_{0i} \right) \Delta t_i$$

$$\frac{\partial r_p}{\partial \vec{v}_{0i}} = \frac{1}{2}\left(\frac{\vec{r}_{0i} \cdot \vec{v}_{0i}}{|\vec{r}_{0i} \cdot \vec{v}_{0i}|} + 1 \right) \frac{\vec{r}_{0i}}{|\vec{r}_{0i}|} \Delta t_i$$

In the same manner, one obtains from Eqs. (10), (7), (8), (5), and (4)

$$\frac{\partial H_i}{\partial(\vec{r}_{0i}, \vec{v}_{0i})} = H_i \left[\left\{ \frac{1}{\eta} \frac{\partial \eta}{\partial p_{op}} - \frac{2}{c} \frac{\partial c}{\partial p_{op}} \right\} \frac{\partial p_{op}}{\partial p_a} + \frac{1}{p_a} \right] \frac{\partial p_a}{\partial(\vec{r}_{0i}, \vec{v}_{0i})}$$

b. <u>Derivations of $(\partial \vec{\mathscr{F}} / \partial \vec{q})$.</u> From Eqs. (15), (2), (13), and (12)

$$\frac{\partial \vec{\mathscr{F}}}{\partial(\alpha_I, \beta_I)} = \left[\begin{array}{c} 0 \\ \hdashline \dfrac{\partial \vec{F}_{Ii}}{\partial(\alpha_I, \beta_I)} \\ \hdashline 0 \end{array} \right] \begin{array}{l} \}3 \\ \}3 \\ \}1 \end{array} \qquad (22)$$

where

$$\frac{\partial \vec{F}_{Ii}}{\partial(\alpha_I, \beta_I)} = \frac{f_i}{m}\left[\frac{\partial \vec{\xi}}{\partial(\alpha_I, \beta_I)}\right] = \left[\frac{f_i}{m}\right]\left[\mathbf{T}\right]\left[\frac{\partial \vec{\xi}}{\partial(\alpha_I, \beta_I)}\right]$$

$$\frac{\partial \vec{\xi}'}{\partial \alpha_I} = \begin{bmatrix} \cos \alpha_I \cos \beta_I \\ \cos \alpha_I \sin \beta_I \\ 0 \end{bmatrix} \qquad \frac{\partial \vec{\xi}'}{\partial \beta_I} = \begin{bmatrix} -\sin \alpha_I \sin \beta_I \\ \sin \alpha_I \cos \beta_I \\ -\sin \beta_I \end{bmatrix}$$

From Eqs. (15), (2), (9), (8), and (7)

$$\frac{\partial \vec{\mathscr{F}}}{\partial \nu_i} = \begin{bmatrix} 0 \\ \dfrac{\partial \vec{F}_{Ii}}{\partial \nu_i} \\ \dfrac{\partial H_i}{\partial \nu_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{f_i}{\partial \nu_i} \cdot \dfrac{\vec{\xi}}{m} \\ \dfrac{\partial H_i}{\partial \nu_i} \end{bmatrix} \tag{23}$$

where

$$\frac{\partial f_i}{\partial \nu_i} = \frac{2\eta p_a}{c} + f_i \left\{\frac{1}{\eta}\frac{\partial \eta}{\partial p_{op}} - \frac{1}{c}\frac{\partial c}{\partial p_{op}}\right\}\frac{\partial p_{op}}{\partial \nu_i}$$

$$\frac{\partial p_{op}}{\partial \nu_i} = \frac{p_a}{N_o}$$

$$\frac{\partial H_i}{\partial \nu_i} = \frac{2\eta p_a}{c^2} + H_i \left\{ \frac{1}{\eta} \frac{\partial \eta}{\partial p_{op}} - \frac{2}{c} \cdot \frac{\partial c}{\partial p_{op}} \right\} \frac{\partial p_{op}}{\partial \nu_i}$$

From Eqs. (15), (2), (9), (8), (7), and (4)

$$\frac{\partial \vec{\mathscr{F}}}{\partial p_0} = \begin{bmatrix} 0 \\ \\ \dfrac{\partial \vec{F}_{Ii}}{\partial p_0} \\ \\ \dfrac{\partial H_i}{\partial p_0} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \dfrac{\partial f_i}{p_0} \cdot \dfrac{\vec{\xi}}{m} \\ \\ \dfrac{\partial H_i}{\partial p_0} \end{bmatrix} \tag{24}$$

where

$$\frac{\partial f_i}{\partial p_0} = f_i \left\{ \frac{1}{p_a} + \frac{1}{\eta} \cdot \frac{\partial \eta}{\partial p_{op}} \cdot \frac{\partial p_{op}}{\partial p_a} - \frac{1}{c} \cdot \frac{\partial c}{\partial p_{op}} \cdot \frac{\partial p_{op}}{\partial p_a} \right\} \frac{\partial p_a}{\partial p_0}$$

$$\frac{\partial p_a}{\partial p_0} = \sum_{i=1}^{5} a_i \, r_p^{-(i+3)/2}$$

E. Search Algorithms

1. Problem statement. Let the dependent variables be

$$\vec{Y}(t_F) = \vec{X}(t_F) - \vec{X}_B \tag{25}$$

where $\vec{X}_B = (\vec{x}_B, \vec{v}_B, 0)$ is a seven vector representing spacecraft position and velocity bias to the target planet. Let the target state vector be $\vec{Y}_w$ where $\vec{Y}_w = [\vec{x}_p(t_F), \vec{v}_p(t_F), m(t_F)]$ is the target planet position and velocity vectors and the desired spacecraft final mass, if any, at final time, t_F. Since $\vec{Y}(t_F)$ is a function of independent search variables $\vec{Q}$, where $\vec{Q}$ is a subset of all available independent variables, $[\vec{r}(t_0), \vec{v}(t_0), m(t_0), \alpha_I, \beta_I, \nu_i, \vec{V}_B, t_F]$, in a linear approximation, the solution to the following equations gives the required corrections $\delta\vec{Q}$ to the independent variables.

$$\frac{\partial \vec{Y}}{\partial \vec{Q}} \cdot \delta\vec{Q} = (\vec{Y}_w - \vec{Y}) \tag{26}$$

In actual nonlinear problems, procedure Eq. (26) is performed iteratively many times, until a satisfactory solution, $(\vec{Y} - \vec{Y}_w) \simeq 0$, is attained.

2. <u>Derivation of $(\partial\vec{Y}/\partial\vec{Q})$.</u> Due to the discontinuities arising in the thrust controls, the equations of motion and the variational equations are integrated piecewise with reinitialization performed at each discontinuity. Propagation and accumulation of the partial derivatives to the final time is required to obtain $(\partial\vec{Y}/\partial\vec{Q})$.

a. <u>Propagation of the state transition matrix.</u> To obtain $[\partial\vec{X}(t_F)/\partial\vec{X}(t_i)]$ one must propagate stagewise information using the following chain rule:

$$\frac{\partial\vec{X}(t_F)}{\partial\vec{X}(t_i)} = \frac{\partial\vec{X}(t_F)}{\partial\vec{X}(t_{F-1})} \; \frac{\partial\vec{X}(t_{F-1})}{\partial\vec{X}(t_{F-2})} \cdots \frac{\partial\vec{X}(t_{i+1})}{\partial\vec{X}(t_i)} \tag{27}$$

where

$$\left[\frac{\partial \vec{X}(t_{i+1})}{\partial \vec{X}(t_i)} = \frac{\partial \vec{X}(t_{i+1})}{\partial \vec{X}_{0i}}\right]$$

is obtained in the process of piecewise integration of the variational equations. Therefore, $[\partial \dot{Y}/\partial(\vec{x}_0, \vec{v}_0, \vec{m}_0)]$ is obtained in this manner with $t_i = t_0$.

b. Propagation of control matrix for control component ν_i. Since ν_i represents control applied only during t_i to t_{i+1}, $[\partial \vec{X}(t)/\partial \nu_i] \neq 0$ only for $t_i \leq t \leq t_{i+1}$. To propagate this control effect to the final state, the following computation is required:

$$\frac{\partial \vec{X}(t_F)}{\partial \nu_i} = \frac{\partial \vec{X}(t_F)}{\partial \vec{X}(t_{i+1})} \frac{\partial \vec{X}(t_{i+1})}{\partial \nu_i} \tag{28}$$

where $[\partial \vec{X}(t_{i+1})/\partial \nu_i]$ is available at the end of the variational equation integration for the i-th power stage.

c. Propagation and accumulation of control matrix for control components α_I and β_I. Since α_I and β_I represent controls applied during t_I to t_{I+1}, $[\partial \vec{X}(t)/\partial(\alpha_I, \beta_I)] \neq 0$ only for $t_I < t \leq t_{I+1}$.

Since the angle stages are designed to be larger than or equal to the power stages, each Δt_I contain several Δt_i's. Therefore, accumulations and propagations of the piecewise control matrix must be performed.

Let $t_I = t_k$ and $t_{I+\ell} = t_{k+\ell}$, implying Δt_I contains ℓ-power stages of equal duration. Then

$$\frac{\partial \vec{X}(t_{I+1})}{\partial \alpha_I} = \sum_{n=1}^{\ell} \frac{\partial \vec{X}(t_{I+1})}{\partial \vec{X}(t_{k+n})} \frac{\partial \vec{X}(t_{k+n})}{\partial \alpha_I} \tag{29}$$

Propagation of Eq. (29) to the final time as in Eq. (28) leads to

$$\frac{\partial \vec{Y}}{\partial \alpha_I} = \frac{\partial \vec{X}(t_F)}{\partial \vec{X}(t_{I+1})} \cdot \frac{\partial \vec{X}(t_{I+1})}{\partial \alpha_I} \tag{30}$$

The same procedure applies to obtain $(\partial \vec{Y}/\partial \beta_I)$.

d. <u>Accumulation of control matrix for control component p_0.</u> Since p_0 is a control applied all through the flight duration, procedure (29) is used to obtain $(\partial \vec{Y}/\partial p_0)$. Let k be the total number of power stages. Then

$$\frac{\partial \vec{Y}}{\partial p_0} = \sum_{n=2}^{k} \frac{\partial \vec{X}(t_F)}{\partial \vec{X}(t_n)} \cdot \frac{\partial \vec{X}(t_n)}{\partial p_0} \tag{31}$$

e. <u>Derivation of $(\partial \vec{Y}/\partial \vec{V}_B)$.</u> From Eq. (25),

$$\frac{\partial \vec{Y}}{\partial \vec{V}_B} = \begin{array}{c} \left[\begin{array}{c} 0 \\ \cdots \\ -I \\ \cdots \\ 0 \end{array} \right] \begin{array}{l} \}3 \\ \\ \}3 \\ \\ \}1 \end{array} \end{array} \tag{32}$$

f. Derivation of $(\partial \vec{Y}/\partial t_F)$. From Eq. (25),

$$\frac{\partial \vec{Y}}{\partial t_F} = \frac{\partial \vec{X}(t_F)}{\partial t_F} - \frac{\partial \vec{X}_B}{\partial t_F} = \frac{\partial \vec{X}(t_F)}{\partial t_F} + \frac{\partial \vec{Y}_w(t_F)}{\partial t_F} \tag{33}$$

3. Numerical search technique. In this section, a linear analysis technique used to solve for $\delta\vec{Q}$ of Eq. (26) is given. Before proceeding to the detailed discussion of the analysis, row and column scalings will be performed to normalize Eq. (26). Let the dimensions of dependent variable space be n_Y and that of independent variables be n_Q. Generally $n_Q > n_Y$. Scaling factors for the independent variables (S_i, $i = 1, \ldots, n_Q$) are the conjectured largest step sizes within which the linear approximation $[\vec{Y}(\vec{Q} + \delta\vec{Q}) = \vec{Y}(\vec{Q}) + (\partial\vec{Y}/\partial\vec{Q})\ \delta\vec{Q}]$ holds. Scaling factors for the dependent variables (T_i, $i = 1, \ldots, n_Y$) are the accepted tolerance of the dependent variable deviations from the desired value. With these scalings, Eq. (26) can be transformed into a normalized form

$$\mathbf{mx} = \mathbf{y} \tag{34}$$

where

$$\mathbf{x}_i = \delta Q_i/S_i,\ i = 1, 2, \cdots, n_Q$$

$$\mathbf{y}_j = (Y_{wj} - Y_j)/T_j,\ j = 1, 2, \cdots, n_Y$$

$$\mathbf{m}_{ij} = \partial \mathbf{y}_i/\partial \mathbf{x}_j$$

Here, bold face letters are used exclusively to denote matrices, including raw and column vectors. In this normalized expression, convergence is considered to be attained if $\|\mathbf{y}\| \leq 1$. The linear neighborhood constraint on independent variables is implied by $\| \mathbf{x} \| \le 1$, where $\| \mathbf{x} \| = \sum_{i}^{n_Q} x_i^2$.

Solution of Eq. (34) is obtained first by performing a singular value decomposition of $\mathbf{m}$. The information gained in this singular value analysis is used further to control the selection of a particular solution. These policies of solution selection are of two types. The first type, called "minimal control policy," attempts to solve Eq. (34) using minimum $\| \mathbf{x} \|$. The second type, called "final mass optimizing policy," attempts to solve Eq. (34) while maximizing the final mass.

a. <u>Singular value decomposition of matrix $\mathbf{m}$.</u> Let $\mathbf{m}$ be a real n_Y by n_Q matrix; there then exist matrices $\mathbf{u}$, $\mathbf{s}$, and $\mathbf{v}$ such that

$$\mathbf{m} = \mathbf{u}\mathbf{s}\mathbf{v}^T \tag{35}$$

where $\mathbf{u}$ and $\mathbf{v}$ are square orthonormal matrices of orders n_Y and n_Q respectively (i.e., $\mathbf{u}\mathbf{u}^T = \mathbf{u}^T\mathbf{u} = \mathbf{v}\mathbf{v}^T = \mathbf{v}^T\mathbf{v} = \mathbf{I}$).

$\mathbf{s}$ is a n_Y by n_Q matrix where only nonzero elements are on the principal diagonal (singular values) (i.e., $\mathbf{s}_{ij} = 0$ for $i \neq j$) and

$$\mathbf{s}_{11} > \mathbf{s}_{22} > \mathbf{s}_{33} > \cdots > \mathbf{s}_{n_Y n_Y} \tag{36}$$

The basic algorithms to compute $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{s}$ were given in Ref. 5 and the program that performs this computation exists in the JPL computer library.

Consider the following orthogonal coordinate transformations of vector spaces $\mathbf{x}$ and $\mathbf{y}$ into $\mathbf{x}'$ and $\mathbf{y}'$:

$$\mathbf{x}' = \mathbf{v}^T\mathbf{x}, \qquad \mathbf{y}' = \mathbf{u}^T\mathbf{y} \tag{37}$$

Matrix $\mathbf{m}$, in the representation of $\mathbf{x}'$, $\mathbf{y}'$ coordinate systems, would be $\mathbf{m}' = \mathbf{u}^T\mathbf{m}\mathbf{v} = \mathbf{u}^T\mathbf{u}\mathbf{s}\mathbf{v}^T\mathbf{v} = \mathbf{s}$, and Eq. (34) is reduced to the following simple form:

$$\mathbf{s}\mathbf{x}' = \mathbf{y}' \tag{38}$$

Since $\mathbf{s}$ is nonzero only along the principal diagonal, i.e.,

$$\mathbf{s} = \left[\begin{array}{c:c} \underbrace{\begin{matrix} s_{11} & & & 0 \\ & s_{22} & & \\ & & \ddots & \\ 0 & & & s_{n_Y n_Y} \end{matrix}}_{n_Y} & \underbrace{\qquad 0 \qquad}_{n_Q - n_Y} \end{array}\right] = \left.\left[\begin{array}{c:c} \underbrace{\;\mathbf{s}_I\;}_{n_Y} & \underbrace{\;\mathbf{s}_{II}\;}_{n_Q - n_Y} \end{array}\right]\right\} n_Y \tag{39}$$

If one lets

$$\mathbf{x}'_I = \left.\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_{n_Y} \end{bmatrix}\right\} n_Y \qquad \mathbf{x}'_{II} = \left.\begin{bmatrix} x'_{n_Y+1} \\ x'_{n_Y+2} \\ \vdots \\ x'_{n_Q} \end{bmatrix}\right\} n_Q - n_Y \tag{40}$$

then Eq. (38), can be written as two separate equations

$$\mathbf{s}_I\mathbf{x}'_I = \mathbf{y}' \quad \text{and} \quad \mathbf{s}_{II}\mathbf{x}'_{II} = 0$$

The most general solution of Eq. (38) can be expressed by

$$\mathbf{x}' = \begin{bmatrix} \mathbf{x}'_I \\ \text{-----} \\ 0 \end{bmatrix} \begin{matrix} \}\, n_Y \\ \\ \}\, n_Q - n_Y \end{matrix} + \begin{bmatrix} 0 \\ \text{-----} \\ \mathbf{x}'_{II} \end{bmatrix} \begin{matrix} \}\, n_Y \\ \\ \}\, n_Q - n_Y \end{matrix} \tag{41}$$

where

$$\begin{cases} \mathbf{x}'_I = \mathbf{s}_I^{-1}\, \mathbf{y}' \\ \\ \mathbf{x}'_{II} = \mathbf{c} \end{cases}$$

Since $\mathbf{s}_{II} = [0]$, $\mathbf{c}$ can be any arbitrary vector of dimension $(n_Q - n_Y)$, orthogonal to first m components of $\mathbf{x}'$ space.

As one must be aware, $n_Q > n_Y$ implies that the problem posed is underdeterministic. The multitude of possible solutions given in Eq. (41) merely indicates this fact.

b. Method of solution selection for minimal control policy.

By definition, minimal control policy implies that Eq. (34) is satisfied while minimizing $\|\mathbf{x}\|$ (call it minimal length). Note that $\|\mathbf{x}\| = \|\mathbf{x}'\|$ holds due to the orthonormality of $\mathbf{v}$ used in Eq. (37).

According to Eq. (41), $\| x' \| = \| x'_I \| + \| x'_{II} \|$, since x'_I is constrained by Eq. (41), minimum length $\| x \|$ is obtained if one set $x_{II} = c = 0$. Therefore, the basic choice of solution for Eq. (38) is:

$$x' = \begin{bmatrix} s_I^{-1} y' \\ ----- \\ 0 \end{bmatrix} \begin{matrix} \} \; n_Y \\ \\ \} \; n_Q \end{matrix} \tag{42}$$

If one performs inverse transformation of Eq. (37),

$$x = vx' = v \begin{bmatrix} s_I^{-1} y' \\ ----- \\ 0 \end{bmatrix} = v \begin{bmatrix} s_I^{-1} u y \\ ----- \\ 0 \end{bmatrix}$$

i.e.,

$$\boxed{x_i = \sum_{j=1}^{n_Y} v_{ij} \cdot \frac{1}{s_{jj}} \sum_{\ell=1}^{n_Y} u_{j\ell} y_\ell, \qquad i = 1, 2, \cdots, n_Q} \tag{43}$$

Let $\mathbf{z} = \partial m(t_f)/\partial \mathbf{x}$ be a 1 by n_Q row vector, and $m(t_f)$ is the spacecraft final mass. In the $\mathbf{x}'$ representation $\mathbf{z}$ is transformed to

$$\mathbf{z}' = \frac{\partial m(t_f)}{\partial \mathbf{x}'} = \mathbf{z}\mathbf{V} \tag{45}$$

Then $m(t_f) = m_0(t_f) + \mathbf{z}'\mathbf{x}'$, where $m_0(t_f)$ is the current mass, and $m(t_f)$ is the linear estimate of the mass after corrections $\mathbf{x}'$ are applied to the independent variables. Further, let

$$\mathbf{z}' = (\underbrace{\mathbf{z}'_I}_{n_Y}, \quad \underbrace{\mathbf{z}'_{II}}_{n_Q - n_Y})$$

Then

$$m(t_f) = m_0(t_f) + \mathbf{z}'_I\mathbf{x}'_I + \mathbf{z}'_{II}\mathbf{x}'_{II} \tag{46}$$

As was shown in Eq. (41), the addition of an arbitrary vector $\mathbf{x}'_{II}$, which is orthogonal to $\mathbf{x}'_I$, does not disturb the constraint $\mathbf{S}'\mathbf{x}' = \mathbf{y}'$ or equivalently $\mathrm{m}\mathbf{x} = \mathbf{y}$. Therefore, it is possible to construct a vector that will modify $m(t_f)$ by making $\mathbf{z}'_{II}\mathbf{x}'_{II}$ as large as possible. Since $\mathbf{z}'_{II}\mathbf{x}'_{II}$ is linear and unbounded, a notion of maximum is not valid. However, since $\|\mathbf{x}'_{II}\| < 1$ is also necessary in this type of linear iteration procedure, we can assert that for $\|\mathbf{x}'_{II}\| = 1$, the maximum expected $\mathbf{z}'_{II}\mathbf{x}'_{II}$ can be obtained if one chooses $\mathbf{x}'_{II} \,/\!/\, (\mathbf{z}'_{II})^T$, with length 1; i.e., The basic solution given in Eq. (43) is generally not directly applicable for highly nonlinear problems. One often would encounter the cases where even the minimal length solution of Eq. (42) has length $\| \mathbf{x} \|$ exceeding 1. This violates the nominal linear domain constraint and is undesirable. To resolve this dilemma, a careful inspection of singular values $\mathbf{s}_{11}$, $\mathbf{s}_{22}, \ldots, \mathbf{s}_{n_Y n_Y}$ is in order. Since $\mathbf{x}' = \mathbf{s}_I^{-1}\mathbf{y}'$, $\mathbf{x}'$ can become large if some components of $\mathbf{s}_I$ becomes rather small. As $\mathbf{s}_{ii}$ are given in the order of decreasing magnitude, one can readily examine such situations. When the ratios of $\mathbf{s}_{11}$ to $\mathbf{s}_{ii}$ (defined as condition number in Ref. 4) becomes larger than some number (e.g., $\sim 10^5$) for $i \geq k + 1$, one may consider that the given matrix $\mathbf{m}$ of Eq. (35) actually is ill-conditioned. If this situation is encountered, it is likely that one is dealing with a nearly correlated matrix $\mathbf{m}$ whose effective rank is k ($< n_Y$) instead of n_Y. Then, the proposed solution is obtained by replacing Eq. (43) by

$$\boxed{\mathbf{x}_i = \sum_{j=1}^{k} v_{ij} \cdot \frac{1}{\mathbf{s}_{jj}} \sum_{\ell=1}^{n_Y} u_{j\ell} y_\ell, \qquad k \leq n_Y, \qquad \|\mathbf{x}\| < 1} \tag{44}$$

In the event that the condition number of matrix $\mathbf{m}$ is not large, uniform scaling of the solution given by Eq. (43) is recommended to restrict $\|\mathbf{x}\| < 1$.

c. Method of solution selection for the final mass optimization.

If one were concerned with the outcome of the final mass, and wished to maximize the mass while satisfying the constraint $\mathbf{mx} = \mathbf{y}$, the following scheme is suggested.

$$x'_{II} = \frac{(z'_{II})^T}{\sqrt{\|z'_{II}\|}} \tag{47}$$

Then

$$z'_{II}x'_{II} = \frac{z'_{II}z'_{II}}{\sqrt{\|z'_{II}\|}} = \sqrt{\|z'_{II}\|} \tag{48}$$

This is the mass gain one can expect if one modifies the minimal length solution by the addition of component x'_{II}.

Therefore the best mass optimizing solution is

$$x = vx' = v\begin{bmatrix} x'_I \\ ----- \\ x'_{II} \end{bmatrix} = v\begin{bmatrix} s_I^{-1}y' \\ --------- \\ \dfrac{(z'_{II})^T}{\sqrt{\|z'_{II}\|}} \end{bmatrix} = v\begin{bmatrix} s_I^{-1}uy \\ --------- \\ \dfrac{(zv)_{II}^T}{\sqrt{\|(zv)_{II}^T\|}} \end{bmatrix}$$

i. e.,

$$x_i = \sum_{j=1}^{n_Y} v_{ij} \cdot \frac{1}{s_{jj}} \sum_{\ell=1}^{n_Y} u_{j\ell}y_\ell + \left(\sum_{j=n_Y+1}^{n_Q} v_{ij} \sum_{\ell=1}^{n_Q} v_{j\ell}^T z_\ell^T\right) \Bigg/ \sqrt{\sum_{j=n_Y+1}^{n_Q} \left(\sum_{\ell=1}^{n_Q} v_{j\ell}^T z_\ell\right)^2} \tag{50}$$

Note here that the index n_Y can take values $k < n_Y$, when the matrix given is ill-conditioned.

d. Comments on the search techniques. In principle, if the search variables are updated iteratively using Eq. (50) for corrections, it eventually will satisfy the boundary conditions where $\|y\| < 1$ is reached. Further, if the mass increment indicator $\sqrt{\|z'_{II}\|}$ of Eq. (48) becomes smaller than the pre-assigned number, e.g., 1 to 2 kg, one may consider that an optimal final mass is attained. To date, extensive tests using minimal control policy solutions given in Subsection E-3-b have been performed. The results are very satisfactory in most cases. The algorithm described in Subsection E-3-c is still under investigation, however it appears promising.

V. FUTURE DEVELOPMENT

It is intended that future work will include:

(1) A thorough test of the optimizing algorithm.

(2) Further refinements and verification of the representations of low-thrust subsystem characteristics.

(3) A more detailed development of the requirements in the interface with the OD and guidance programs, particularly the input/output specifications.

(4) Investigations into the modeling of the lower order perturbations and verification of current findings that these perturbations will not be a major problem.

The final product from these studies will be a set of software requirements specifications for a flight quality trajectory program.

REFERENCES

1. Finke, R. C., IOM Reply to Attn. of: 3330: Ion Thruster System Parameters. NASA Lewis Research Center, Cleveland, Ohio, Nov. 17, 1972.

2. Johnson, F. T. "Approximate Finite-Thrust Trajectory Optimization," AIAA Journal, Vol. 7, No. 6, June 1969.

3. Sauer, C. G., Jr., "Optimization of a Solar-Electric-Propulsion Planetary Orbiter Spacecraft," AAS Paper 68-104, presented at the AAS/AIAA Astrodynamics Specialists Conference, Jackson, Wyoming, September 3, 1968.

4. Lawson, C. L., Application of Singular Value Analysis. Mathematical Software, 1971, Academic Press Inc., New York and London.

5. Golub, G., and Businger, P., Least Squares, Singular Values and Matrix Approximations, An ALGOL procedure for Computing the Singular Value Decomposition, Technical Report CS73, Stanford University, Palo Alto, Calif., 1967.

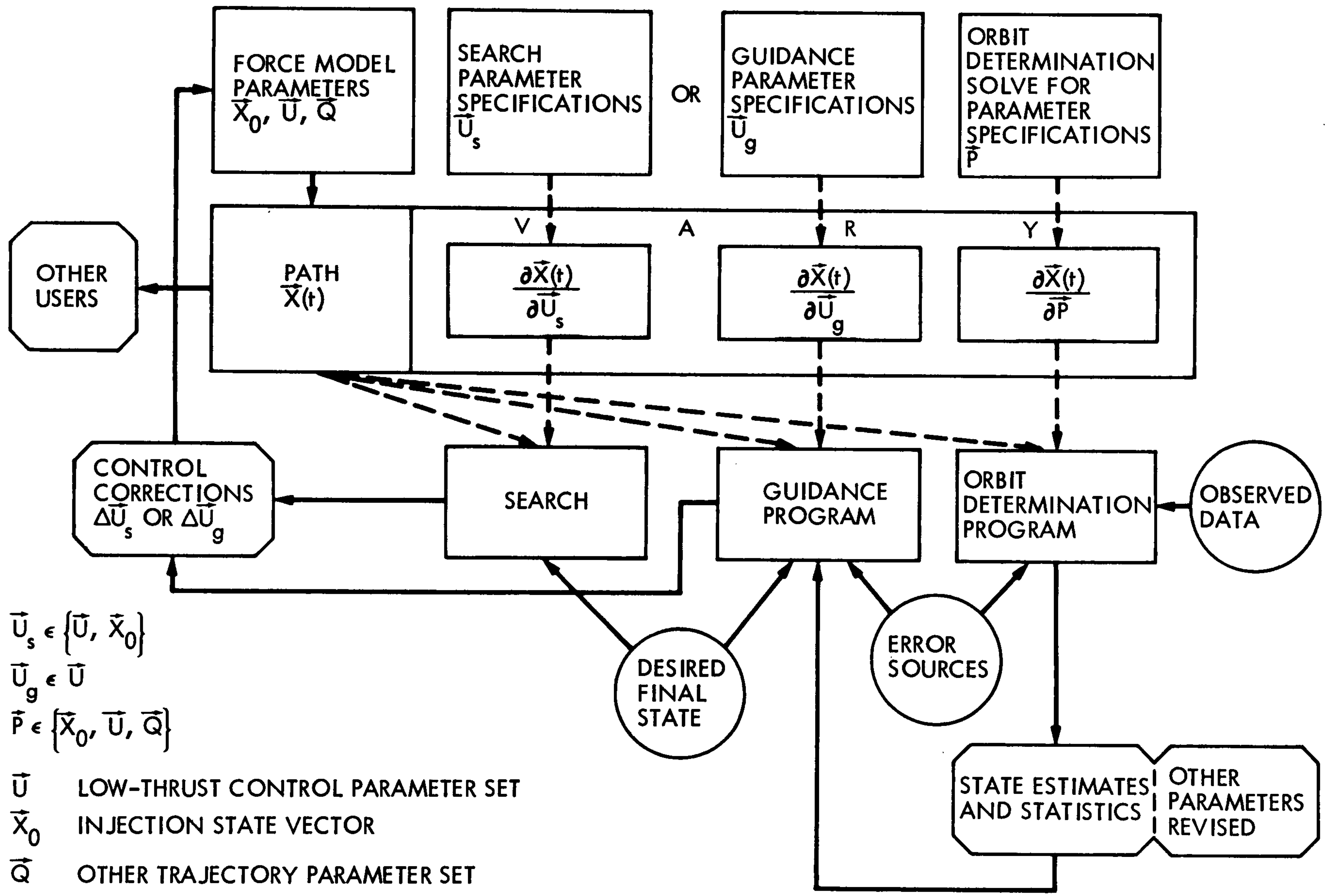

Fig. 1. Trajectory software functions and interfaces

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