N74-11116

CR 132353

OCTOBER 1972

ANALYSIS AND DESIGN OF THREE DIMENSIONAL SUPERSONIC NOZZLES

ATL TR 166 - VOLUME IV
SIMILARITY LAWS FOR NOZZLE FLOWS

By

A. Ferri and G. Roffe

PREPARED FOR

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
HAMPTON, VIRGINIA

UNDER

CONTRACT NO. NAS1-10327

ADVANCED TECHNOLOGY LABORATORIES, INC.
Merrick and Stewart Avenues
Westbury, New York 11590

INDEX

LIST OF FIGURES

I. INTRODUCTION

The development of nozzles for hypersonic aircraft requires the ability to analyze the behavior of high temperature gases, often chemically reactive, flowing through ducts of complicated geometry. For this purpose, analyses such as those presented in the preceeding volumes of this report have been and will continue to be developed. However, the extremely complicated nature of these flow fields necessitates the adoption of certain simplifying assumptions in the construction of the analytical models. It is therefore both necessary and desirable to carry out the parallel development of an experimental program aimed at assessing the accuracy and areas of applicability of the analytical results as well as at the acquisition of data for those physically interesting configurations for which suitable analytical tools may not yet be available.

Since the exact duplication of all gas conditions at the nozzle entrance station, especially the extremely high temperature levels, makes experiments both difficult and expensive, it is logical to investigate the possibility of simulating the actual nozzle flows with low temperature nonreactive gases. To pursue this goal, it is necessary to develop the similarity laws for nozzle flows and determine the parameters which must be duplicated. This effort has been carried out and the results of the analysis are presented here. In addition to the development of these similarity rules, a number of cool gas flows have been considered to determine their suitability as substitutes for actual high temperature gases in an experimental program.

II. SIMILARITY REQUIREMENTS

The proper simulation of a flow field requires that at any geometrically similar point the Mach number and flow direction be the same as those which exist at the corresponding point in the alternate flow and that the ratio of static pressure from one point to another in each of the flows be identical. The necessity to match the variation of p with θ requires that the value of Γ used in the characteristics equations and defined for equilibrium chemistry as

$$\Gamma = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_S$$

be duplicated at corresponding points. The requirement that the Mach number and flow deflection be identical at corresponding points requires that the initial conditions be similar in p and identical in M and θ. In addition, the variation of V/V_i and a/a_i with p must be the same, where V_i and a_i are the gas velocity and speed of sound at the reference initial conditions.

It should be noted that although the ratio V_i/a_i must be the same for the nozzle and the model, the actual values V_i and a_i can be different since only the local Mach number must be the same.

The assumption is made that the chemistry is in equilibrium. Therefore, the conditions of similarity require that the variation of pressure ratio (p/p_i), Γ and c_p with temperature ratio (T/T_i) be the same at geometrically similar points. This will assure that the variation of velocity ratio, Mach number and pressure ratio, as functions of temperature ratio are also the same. These conditions imply that the variation of three thermodynamic parameters with temperature must be closely simulated in the model, within the range of temperature ratio existing from the throat to the nozzle exit.

III. THERMODYNAMIC RELATIONS

Consider a volume containing a unit weight of gas in thermodynamic equilibrium consisting of n moles and with a static enthalpy H per mole. In this case, the following thermodynamic relations are valid if the gas is a mixture of species i and n_i is the number of moles of each specie:

$$\Sigma n_i = n \tag{1}$$

$$\left(\frac{\partial H}{\partial T}\right)_p = n\, c_p = T \left(\frac{\partial S}{\partial T}\right)_p \tag{2}$$

where c_p is the specific heat per mole at constant pressure. The flow is isentropic between discontinuities and is in chemical equilibrium. Therefore,

$$\left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial p}\right)_T dp = 0 \text{ (isentropic flow)} \tag{3}$$

$$\left(\frac{\partial n}{\partial T}\right)_p dT + \left(\frac{\partial n}{\partial p}\right)_T dp = 0 \text{ (equilibrium chemistry)} \tag{4}$$

The Gibbs free energy is given by

$$G = H - TS \tag{4}$$

and

$$dH = T\, dS - Vdp \tag{5}$$

Then

$$\left(\frac{\partial V}{\partial T}\right)_p = \left(\frac{\partial S}{\partial p}\right)_T \qquad (6)$$

because

$$\left(\frac{\partial n}{\partial p}\right)_T = V, \quad \left(\frac{\partial n}{\partial T}\right)_p = -S \qquad (7)$$

but Equation (3) gives

$$\left(\frac{\partial S}{\partial p}\right)_T = - \left(\frac{\partial S}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_S \qquad (8)$$

Therefore,

$$\left(\frac{\partial T}{\partial p}\right)_S = \frac{T \left(\frac{\partial V}{\partial T}\right)_p}{n \; c_p} \qquad (9)$$

The equation of state gives

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{R_o T}{p} \left(\frac{\partial n}{}\right)_p + \frac{n R_o}{p} \qquad (10)$$

Therefore

$$\left(\frac{\partial \ \ln T}{\partial \ \ln p}\right)_S = \frac{R_o}{c_p} \left[\left(\frac{\partial \ \ln n}{\partial \ \ln T}\right)_p + 1 \right] \qquad (11)$$

If we define

$$D_T = \left(\frac{\partial \ln n}{\partial \ln T}\right)_p \tag{12}$$

and

$$\alpha_T = \left(\frac{\partial \ln p}{\partial \ln T}\right) \frac{c_p}{R_o} \left[\frac{1}{1+D_T} \right] \tag{13}$$

α_T must be the same at corresponding points or at the same value of temperature ratio $(\frac{T}{T_i})$. In addition, since

$$nH = \Sigma n_i H_i \tag{14}$$

then

$$n\, c_p = \left(\frac{\partial nH}{\partial T}\right)_p = \sum_i n_i\, c_{p_i} + \sum_i \frac{n_i H_i}{T} D_{T_i} \tag{15}$$

where D_{T_i} is defined as

$$D_{T_i} = \left(\frac{\partial \ln n_i}{\partial \ln T}\right)_p \tag{16}$$

Thus

$$c_p = \Sigma \frac{n_i\, c_{p_i}}{n} + \Sigma \frac{n_i H_i D_{T_i}}{nT} \tag{17}$$

where

$$\Sigma\, n_i\, D_{T_i} = n\, D_T \tag{18}$$

The relation between p and ρ is given by

$$\left(\frac{\partial p}{\partial \rho}\right)_S = a^2 = \frac{\left(\frac{\partial p}{\partial T}\right)_S}{\left(\frac{\partial \rho}{\partial T}\right)_S} \tag{19}$$

But

$$\rho = \frac{pW}{R_o T} \tag{20}$$

where W is the molecular weight and nW = 1.

In addition,

$$\left(\frac{\partial \rho}{\partial T}\right)_S = \frac{W}{R_o T}\left(\frac{\partial p}{\partial T}\right)_S + \frac{p}{R_o T}\left(\frac{\partial W}{\partial T}\right)_S - \frac{pW}{R_o T^2} \tag{21}$$

and

$$\left(\frac{\partial W}{\partial T}\right)_S = \left(\frac{\partial W}{\partial T}\right)_p + \left(\frac{\partial W}{\partial P}\right)_T \left(\frac{\partial p}{\partial T}\right)_S \tag{22}$$

Then, if we define

$$D_p = \left(\frac{\partial \ln n}{\partial \ln p}\right)_T \tag{23}$$

$$a^2 = \frac{c_p \frac{p}{\rho}}{(1-D_p)c_p-(1+D_T)^2 R_o} \tag{24}$$

Then if Γ is defined as

$$\Gamma = \frac{c_p \quad R_o}{(1-D_p)c_p - (1+D_T)^2 R_o} \tag{25}$$

$$a^2 = \Gamma \left(\frac{p}{\rho}\right) \tag{26}$$

If we transform c_p into specific heat per unit mass c_{p_m}

then

$$\Gamma = \frac{c_{p_m}}{c_{p_m}(1-D_p) - (1+D_T)^2 \frac{R_o}{W}} \tag{27}$$

$$c_{p_m} = \Sigma \alpha_i \, c_{p_{m_i}} + \alpha_i \, h_i \, D_{T_i} \tag{28}$$

$$\alpha_T = \frac{c_{p_m}}{R_o W} \left(\frac{1}{1+D_T}\right) \tag{29}$$

where α_i is the mass fraction of species i and D_{T_i} is the coefficient defined in Equation (16).

From the preceding we conclude that the three thermodynamic parameters which must be duplicated are Γ, α_T and c_p or equivalently D_p, D_T and c_p.

Unfortunately, the gases entering the nozzle of an actual vehicle undergo a substantial change in chemical composition as they flow through the duct and this change has a pronounced effect on the values of D_T and D_p. Since any cold gas simulation will, of neccessity, preclude these chemical reactions, it does not appear possible to obtain perfect simulation over the entire range of temperatures existing in the nozzle. However, gases are available which permit close simulation of Γ, c_p and α_T for some conditions of interest. Some of these gases are investigated and defined in the next section where their variation of critical thermodynamic properties with temperature ratio is presented within the temperature range of interest.

IV. COLD FLOW SIMULATION

In order to assess the relative merits of various gases with regard to their ability to simulate the hot exhaust flow through a hypersonic nozzle, it is first necessary to define the properties of a typical hot flow. To do this, we select the case of a scramjet propelled vehicle flying at a Mach number of 8 at an altitude of 100,000 feet. The scramjet engine operation is characterized by an inlet static pressure ratio of 140 with a total pressure recovery of 60% and constant pressure combustion. For hydrogen-air combustion at equivalence ratios of 0.6 and 1.0 these operational assumptions result in the following set of gas stream characteristics at the nozzle entrance.

ϕ	p(pa)	h(j/kg)	$\rho(kg/m^3)$	$T(^oK)$
0.6	1.56×10^5	6×10^6	0.162	2821
1.0	1.56×10^5	4.2×10^6	0.204	2381

Assuming equilibrium chemistry, the temperature-pressure history through an isentropic expansion from the nozzle entrance conditions to a static pressure 200 times lower than the initial value is obtained from the tables of Reference (1) and is shown in Figure (1). It should be noted that the actual degree of expansion would be expected to be significantly less than a factor

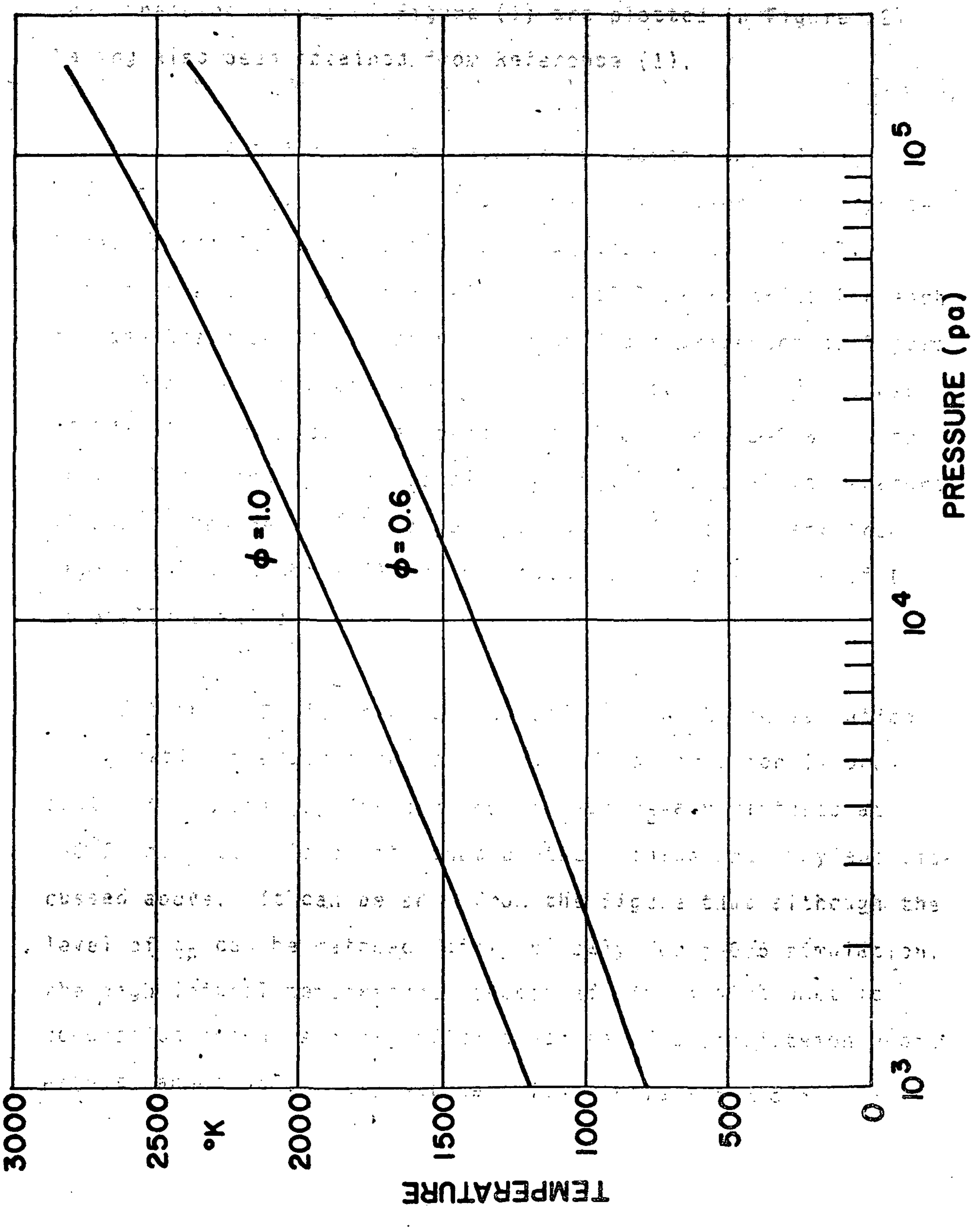

FIGURE 1. PRESSURE-TEMPERATURE PATH THROUGH EQUILIBRIUM ISENTROPIC EXPANSION.

of 200 but this wider range provides an adequate margin for the purposes of comparison. The values of Γ corresponding to the expansion curves of Figure (1) are plotted in Figure (2), having also been obtained from Reference (1).

The values of Γ for several hydrocarbon gases (CH_4, C_2H_4, C_2H_6, C_4H_8-2) are shown in Figure (3) as functions of temperature. Since it is necessary to match the function $\Gamma(T/T_i)$ through the expansion, several values of T_i were tried for each gas and the resulting Γ compared with the requirement of Figure (2). The best simulation was found to be obtained by the use of CH_4 with an initial temperature of 550°K when ϕ=0.6 was to be simulated and by the use of C_2H_4 with an initial temperature of 500°K when ϕ=1.0 was to be simulated. It is evident from Figure (4) that the ability to provide the proper value of Γ is easily attained.

In addition to Γ, it is also desirable to match the variation of c_p with temperature ratio (T/T_i). This variation is presented in Figure (5) for the equilibrium H_2-air mixtures at ϕ=0.6 and ϕ=1.0 along with that of the methane and ethylene discussed above. It can be seen from the figure that although the level of c_p can be matched fairly closely for ϕ=0.6 simulation, the high initial temperature associated with stoichiometric combustion produces a significant degree of dissociateion whose effect cannot be closely matched with a relatively cool gas.

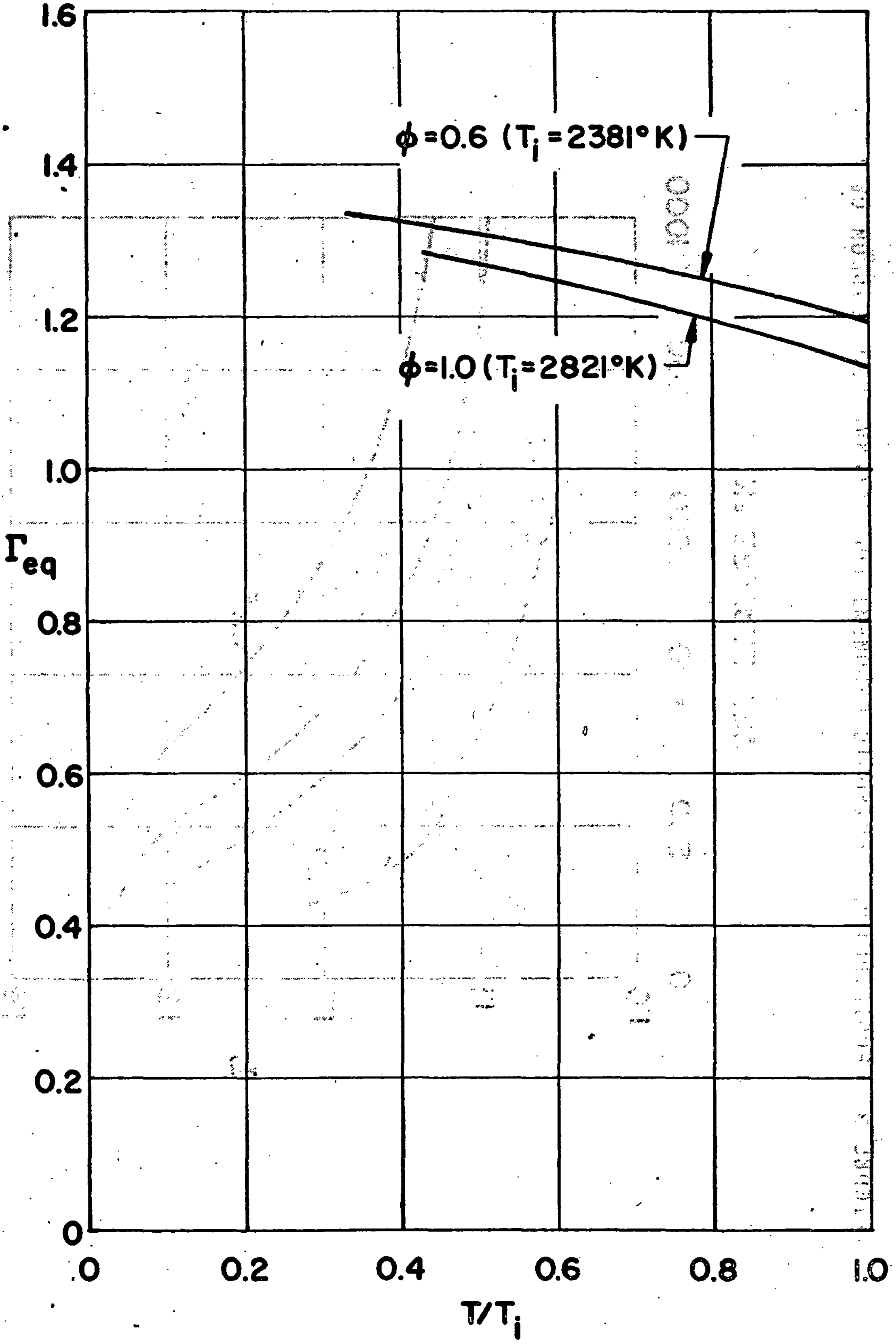

FIGURE 2. VARIATION OF EQUILIBRIUM Γ THROUGH NOZZLE EXPANSION.

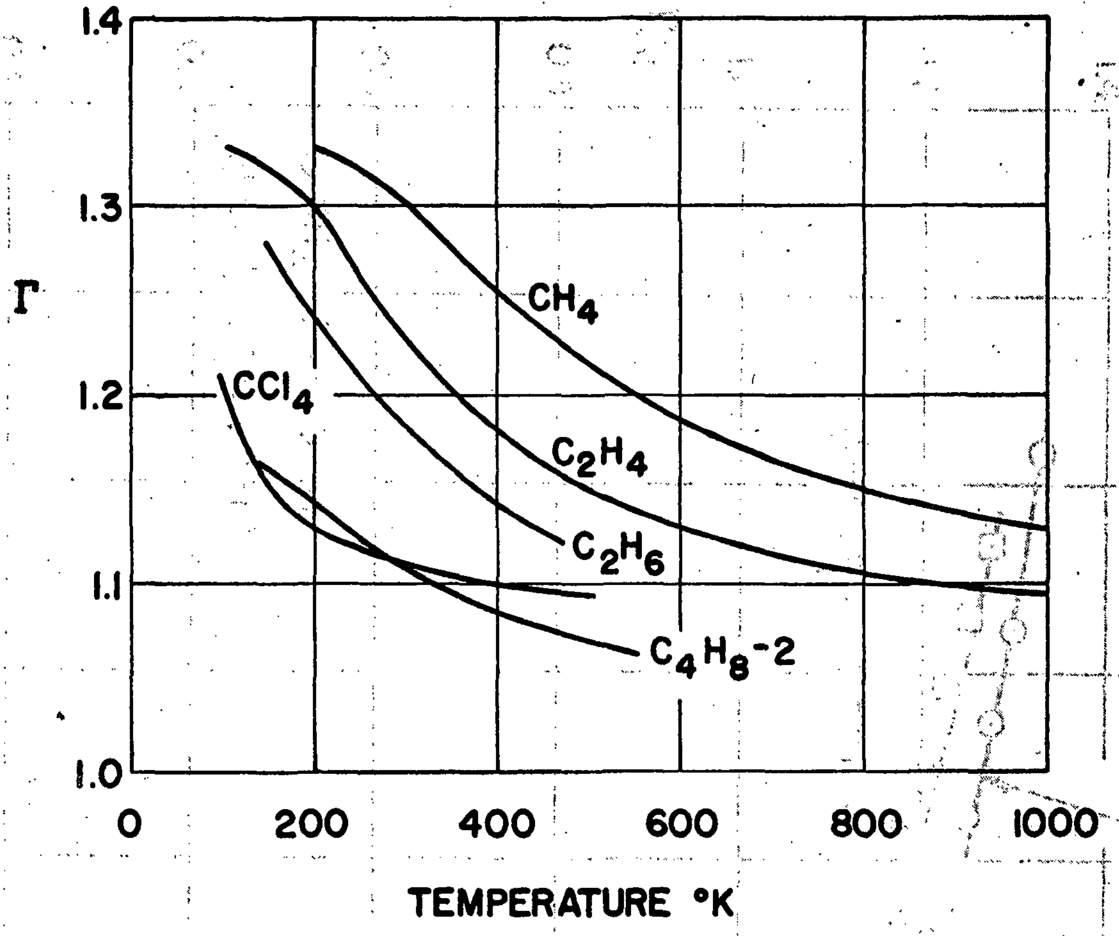

FIGURE 3. EQUILIBRIUM ISENTROPIC EXPONENT FOR SEVERAL HYDROCARBON GASES. (REFERENCE 2)

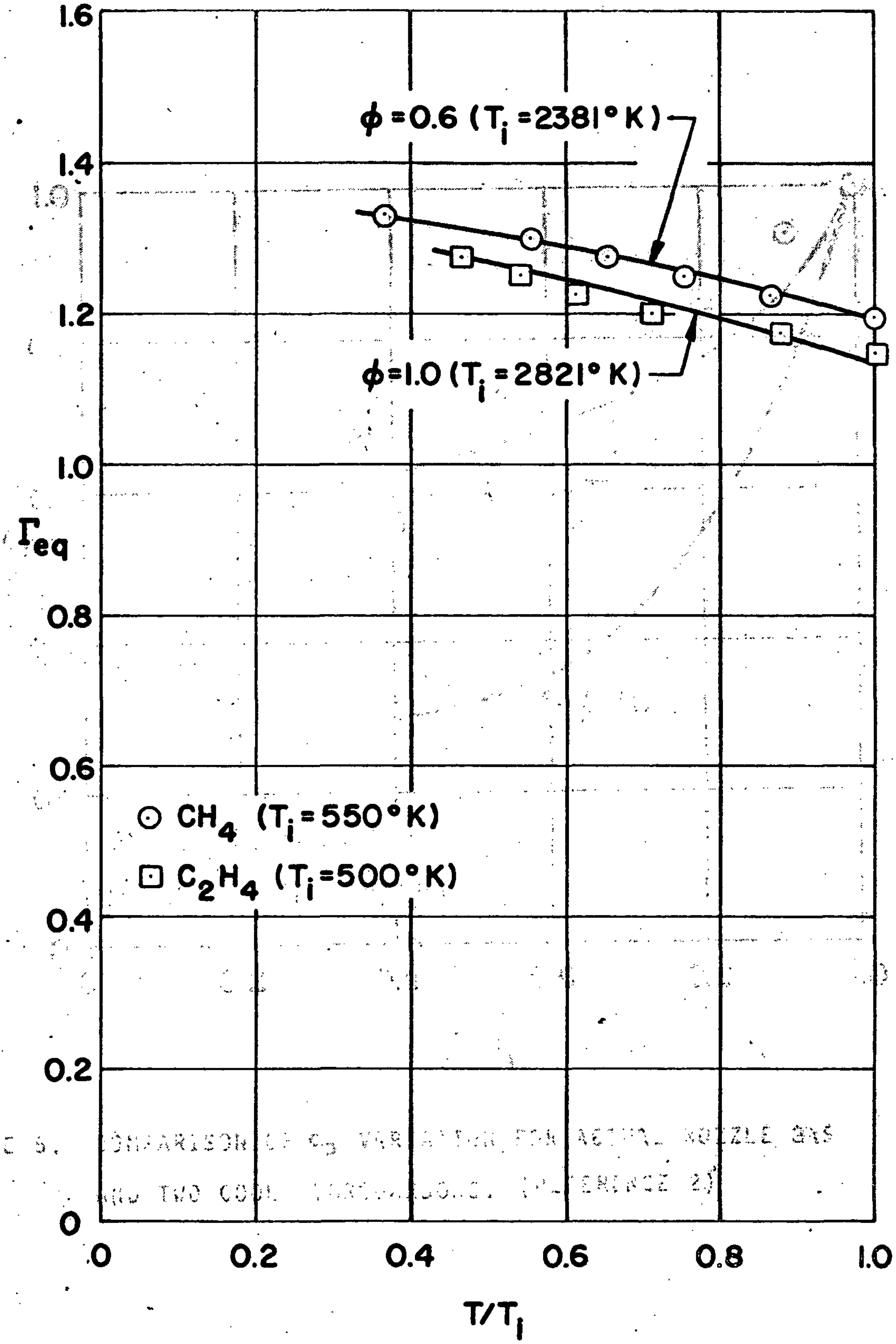

FIGURE 4. COMPARISON OF Γ VARIATION FOR ACTUAL NOZZLE GAS AND TWO COOL HYDROCARBONS.

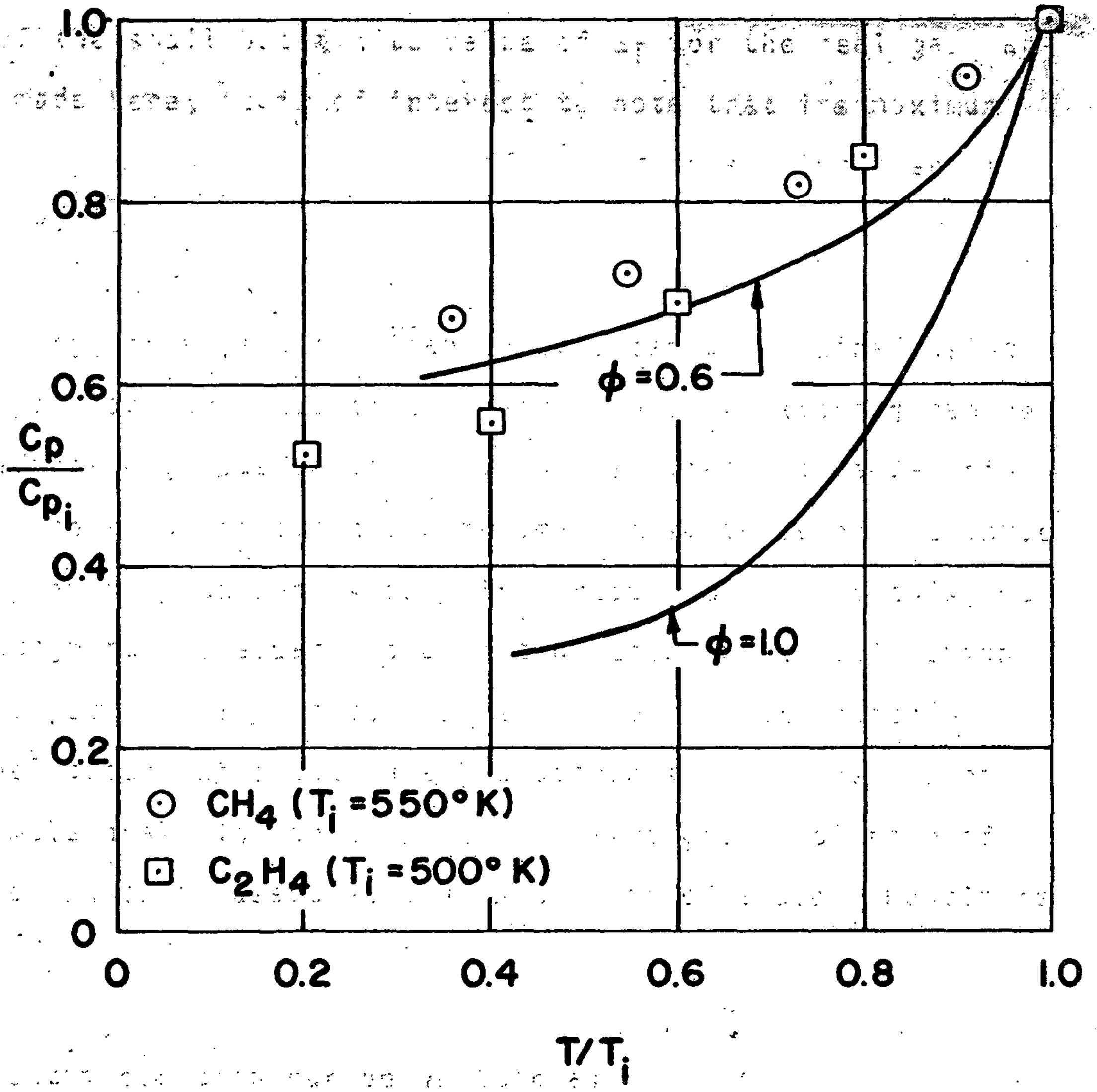

FIGURE 5. COMPARISON OF c_p VARIATION FOR ACTUAL NOZZLE GAS AND TWO COOL HYDROCARBONS. (REFERENCE 2)

Finally, the value of α_T is zero for the nonreactive substitute gases while it varies from zero to a small negative number for the high temperature gases. Although a quantitative evaluation of the small but finite value of α_T for the real gas cannot be made here, it is of interest to note that its maximum magnitude for ϕ=0.6 is an order of magnitude smaller then it is for ϕ=1.0.

Summarizing, excellant duplication of Γ can be obtained using a cool gas for any vlaue of ϕ. Good c_p and α_T matching can be obtained for moderate values of ϕ, but a quantitative definition of the error associated with small mismatches of the three thermodynamic functions is certainly required. To do this, we will integrate the equations of motion used in an equilibrium characteristics analysis to analyze a one wave expansion of both real gases and substitute hydrocarbons. The results of this simple analysis can then be used as a good indicator of the error which is associated with the use of a cool substitute gas.

The momentum equation can be written as

$$\frac{dp}{\rho} + \tfrac{1}{2}\, dV^2 = 0 \qquad (30)$$

and the isentropic law in the form

$$\frac{dp}{p} = \Gamma \frac{d\rho}{\rho} \tag{31}$$

For a one wave flow, a convenient coordinate system can be defined by the local Mach line and a normal to it. Defining the velocity component along the characteristic as V_t, we have for one wave flow

$$\frac{d\,V_t}{d\delta} = a \tag{32}$$

where δ is the angle defining the wave orientation from the y-axis and a is the local sonic velocity, given by Equation (26). From the definition of our coordinate system, we have

$$V^2 = V_t^2 + a^2 \tag{33}$$

Integrating (30) and (31) over a small step in Δp through which Γ may be assumed constant, we obtain

$$\left(\frac{\Gamma+1}{\Gamma-1}\right) a^2 + V_t^2 = c^2 \tag{34}$$

Substituting into (32), we have

$$\frac{dV_t}{d\delta} = \sqrt{\frac{\Gamma-1}{\Gamma+1}}\ \sqrt{c^2-V_t^2} \tag{35}$$

or

$$\int_{V_{t_1}}^{V_{t_2}} \frac{dV_t}{\sqrt{c^2-V_t^2}} = \int_{\delta_1}^{\delta_2} \sqrt{\frac{\Gamma-1}{\Gamma+1}} \, d\delta \tag{36}$$

which yields

$$\beta_2 = \beta_1 - \sqrt{\frac{\Gamma-1}{\Gamma+1}} \, (k_2-k_1) \tag{37}$$

where

$$k \equiv \tan^{-1} \frac{V_{t/c}}{\sqrt{1-(V_t/c)^2}} \tag{38}$$

and

$$\beta = \theta + \mu = \pi/2 - \delta \tag{39}$$

Equations (33), (34), (37) and (38) can be used to evaluate the expansion of a gas through a series of steps in Δp, the value of Γ being a constant in each step. This analysis was programmed for numerical solution using the curvefits for $h(T,p,\phi)$ and $\Gamma(T,p,\phi)$ described in Volume I of this report. The program listing is presented in the Appendix. The program was used to compute the variation of pressure and impulse function* with flow turning angle through a one wave expansion for the $\phi=0.6$ and 1.0 nozzle entrance conditions described earlier. The program was also used with the curvefits of Reference (1) for the functions $h(T)$ and $[c_p/R](T)$ for the gases CH_4 and C_2H_4. Since the temperature range for the hydrocarbons was fairly low, the equilibrium isentropic exponent was defined by the expression

$$\Gamma = \frac{c_p/R}{(c_p/R)-1} \tag{40}$$

The results of these calculations are presented in Figures (6) and (7) where it is clear that fairly good representation of pressure distribution and impulse can be achieved with the use of cool substitute gases.

---

*Impulse function is defined here as

$$F/F^* = \frac{1+\Gamma m^2}{m\sqrt{2(\Gamma+1)[1+(\frac{\Gamma-1}{2})m^2]}}$$

following the notation of Reference (3).

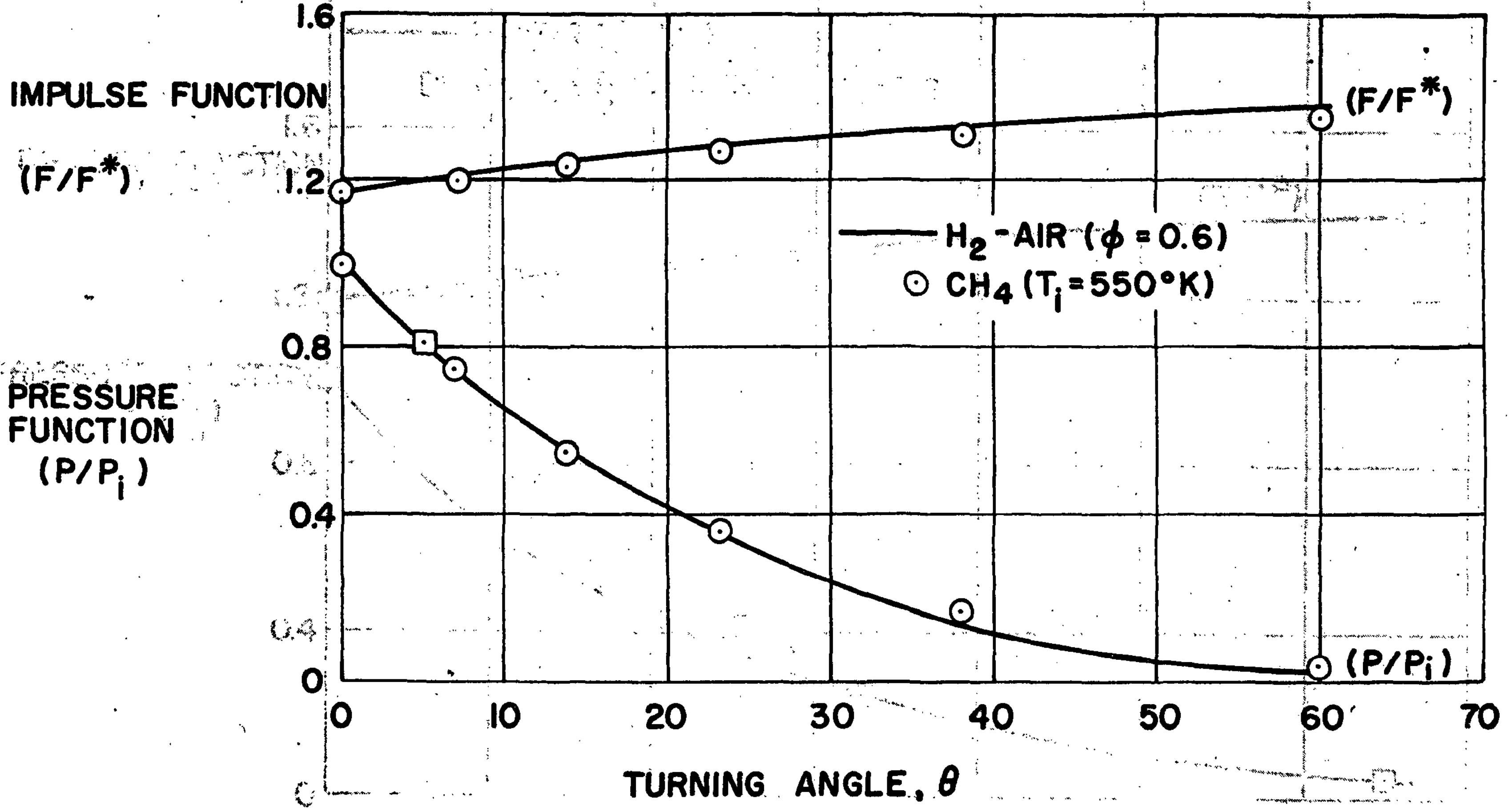

FIGURE 6. COMPARISON OF STATIC PRESSURE DISTRIBUTION AND IMPULSE FUNCTION FOR EQUILIBRIUM H2-AIR (ϕ=0.6) AND CH4 (T_i=550^{0}K).

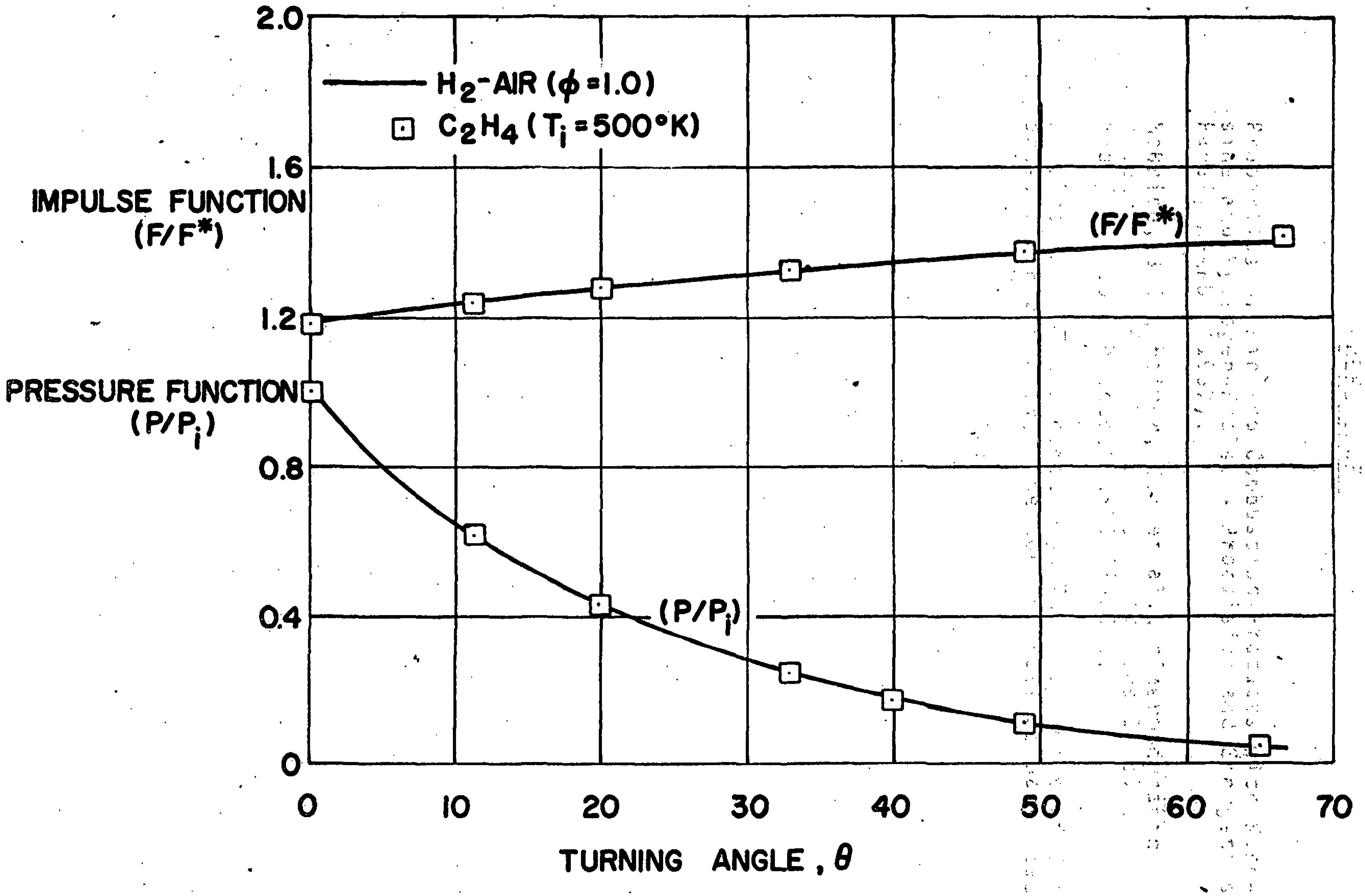

FIGURE 7. COMPARISON OF STATIC PRESSURE DISTRIBUTION AND IMPULSE FUNCTION FOR EQUILIBRIUM STOICHIOMETRIC H_2-AIR AND C_2H_4 (T_i=500°K).

REFERENCES

1. Properties of Air and Combustion Products with Kerosine and Hydrogen Fuels. - Propulsion and Energetics Panel, AGARD (1967).

2. McBride, B.J., Heimer, S.; et al., "Thermodynamic Properties to 6000^{o}K for 210 Substances Involving the First 18 Elements," NSAS SP-3001.

3. Shapiro, A., "The Dynamics and Thermodynamics of Compressible Fluid Flow," The Ronald Press Company (1954).

APPENDIX

PROGRAM LISTING IN BASIC LANGUAGE

```
5  PRINT "I","P","T","RHO","H"
6  PRINT "GAMMA","MACH","THETA"
7  PRINT
10  DIM J[1000],K[1000]
15  DIM B[1000],Z[1000]
20  GOTO 512
30  LET T=T0=1500
40  GOSUB 180
50  LET E0=(H-H1)/H
60  LET H0=H1
70  LET T1=T0*1.1
80  LET T=T1
90  GOSUB 180
100  LET E1=(H-H1)/H
110  IF ABS(E1)<.0001 THEN 390
120  LET T9=T0-E0*(T1-T0)/(E1-E0)
130  LET E0=E1
140  LET T0=T1
150  LET H0=H1
160  LET T1=T9
170  GOTO 80
180  LET A=1.E-07*(-.1042*F↑2+.8242*F+.987)
190  LET B=.001*(.01167*F↑2+.1503*F+.938)
200  LET C=-.0284*F↑2+.6731*F+.4293
210  IF F <= 1 THEN 250
220  LET A=1.E-07*(1.787*F↑2-5.48*F+5.4)
230  LET B=.001*(-.1867*F↑2+1.11*F+.176)
240  LET C=-.0933*F↑2+3.975*F-2.808
250  IF T <= 2000 THEN 330
260  LET A=.000001*(1.792*F↑2+.3983*F+.31)
270  LET B=.001*(-9.05*F↑2-.07917*F+.245)
280  LET C=10.86*F↑2-.1183*F+.97
290  IF F <= 1 THEN 330
300  LET A=.000001*(4.81*F↑2-13.9*F+11.59)
310  LET B=.001*(-23.08*F↑2+66.82*F-52.61)
320  LET C=27.05*F↑2-73.73*F+58.39
330  LET H1=A*T↑2+B*T+C
340  IF T <= 2000 THEN 370
350  LET Z9=.125*(LOG(P)/2.3-5)↑2-.275*(LOG(P)/2.3-5)
360  LET H1=H1*(1+(1+F)*(T/2000-1)*Z9)
370  LET H1=H1*1.E+06
380  RETURN
390  LET M=0
400  IF T <= 1000 THEN 420
410  LET M-2.15E-08*T↑2+.000091*T-.0695
420  LET N=4.E-09*T↑2-.00002*T-.019
430  IF F <= 1 THEN 450
440  LET N=.0339*SQR(T)-.000391*T-.681
450  LET G=-1.833E-07*T↑2+.000075*T+1.367
460  IF T<500 THEN 500
470  LET G=2.E-08*T↑2-.000138*T+1.423
480  IF T<2000 THEN 500
490  LET G=7.267E-08*T↑2-.000457*T+1.85
500  LET G=G+M*(LOG(P)/2.3-5)+N*(F-1)
510  RETURN
512  READ P,P9,R1,D1,H,V,F
514  DATA 1.56E+06,15600,1.62,5000,6.E+06,2075,1
516  GOTO 600
520  PRINT "INITIAL PRESSURE";
530  INPUT P
540  PRINT "FINAL PRESSURE";
550  INPUT P9
```

```
560  PRINT "RHO";
570  INPUT R1
580  PRINT "DELTA PRESSURE";
590  INPUT D1
600  LET I1=1000
610  LET J[1]=P
620  LET K[1]=R1
622  GOTO 690
630  PRINT "ENTHALPY";
640  INPUT H
650  PRINT "VELOCITY";
660  INPUT V
670  PRINT "FUEL TO AIR RATIO";
680  INPUT F
690  LET H2=H+.5*V*V
700  GOSUB 30
710  LET C8=P/R1↑G
720  LET R=R1
722  GOSUB 2000
724  LET Z[1]=ATN(V1/C1/SQR(1-(V1/C1)↑2))
726  LET B[1]=ATN(1/SQR((V/A1)↑2-1))
728  LET T1=0
730  FOR I=2 TO I1
740  LET C8=P/R↑G
744  IF I<2 THEN 750
746  : BETA(###)=######.     MU=######     THETA=#######
750  IF INT(I/20)*20 <> I THEN 770
760  PRINT I,P,T,R,H
765  PRINT G,V/A1,57.3*T1
766  PRINT
770  LET J[I]=J[I-1]-D1
780  LET K[I]=(J[I]/C8)↑(1/G)
790  LET V=SQR(V*V+2*G/(G-1)*(J[I-1]/K[I-1]-J[I]/K[I]))
800  LET H=H2-.5*V*V
810  LET P=J[I]
820  LET R=K[I]
830  GOSUB 30
835  GOSUB 950
840  LET I2=I
850  IF J[I] <= P9 THEN 890
860  IF J[I]-D1>P9 THEN 880
870  LET D1=J[I]-P9
880  NEXT I
890  LET I=K[I2]*V*V+J[I2]
900  LET R=K[I2]
910  LET P=J[I2]
920  GOSUB 30
925  GOSUB 950
930  PRINT I2,J[I2],T,K[I2],H,G,V,I
940  STOP
950  GOSUB 2000
990  LET Z[I]=ATN(V1/C1/SQR(1-(V1/C1)↑2))
1000  LET B[I]=B[I-1]-SQR(G1)*(Z[I]-Z[I-1])
1010  LET U=ATN(1/SQR((V/A1)↑2-1))
1020  LET T1=B[I]-U
1024  GOTO 1030
1025  PRINT USING 746,I,B[I],57.3*U,57.3*T1
1030  RETURN
1990  STOP
2000  LET A1=SQR(G*P/R)
2010  LET G1=(G+1)/(G-1)
2020  LET C1=SQR(G1*A1↑2+V↑2-A1↑2)
2030  LET V1=SQR(V↑2-A1↑2)
2040  RETURN
3000  END
```