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APPROXIMATIONS IN THE MINIMUM TIME-TO-CLIMB PROBLEM

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SYMBOLS

a	speed of sound, ft/sec.
C_{D_o}	zero lift drag coefficient
C_{L_α}	lift coefficient
D	drag divided by weight
D_L	drag due to lift divided by weight
D_o	zero lift drag divided by weight
D_{L_1}	drag due to lift divided by weight evaluated at lift equal to weight
D'	drag, lbs.
E	specific energy, sec^2.
F	thrust less zero lift drag, divided by weight
g	acceleration due to gravity
g_o	acceleration due to gravity at sealevel
h	altitude divided by g_o, sec^2.
h'	altitude, ft.
H	Hamiltonian function
$\tilde{H}$	part of Hamiltonian containing control
L	lift divided by weight
L'	lift, lbs.
M	Mach number
r_o	radius of earth, ft.
S	reference area, ft^2.
t	time, sec.
T	thrust divided by weight
T'	thrust, lbs.

u sine of the flight path angle

V velocity divided by g_o, sec.

V' velocity, ft/sec.

W airplane total weight, lbs.

x' component of position vector measured along fundamental parallel, ft.

y' component of position vector measured along fundamental meridian, ft.

α angle of attack, deg.

β weight flowrate, lbs/sec.

γ flight path angle, deg.

η drag due to lift factor

θ bank angle, deg.

λ_i adjoint variable associated with state variable i

ρ atmospheric density, slugs/ft^3.

τ thrust off-set angle, deg.

ϕ time-to-climb, sec.

ϕ^* minimum time-to-climb, sec.

χ heading angle, deg.

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SUMMARY

The minimum time-to-climb problem is formulated as a third order system and three approximate solutions based on reduced order systems are presented. The first of these is the often used energy state, the second is the less frequently used two state and the third is a slightly altered form of the second, herein called the modified two state. These three approximations are discussed and compared both qualitatively and, by using a numerical example, quantitatively. The numerical example is also solved by the steepest descent method to provide a basis for comparison. It is concluded that the modified two-state approximation is significantly better than the other two. This approximation is used to assess the sensitivity of climb performance to various vehicle parameters and it is found that, as expected, thrust and weight influence the time-to-climb most strongly.

INTRODUCTION

In the early years of flight, when aircraft speeds were relatively low, performance optimization of aircraft was studied on a steady-state basis. With the advent of high-speed aircraft, however, dynamic effects could no longer be neglected. Consideration of all dynamic effects leads to problems of such computational complexity that the effort expended in their solution is often not warranted for the purposes of preliminary performance estimation, and thus approximate solutions have been sought.

The earliest and most widely used of these approximations is that of energy state, sometimes called energy climb. This approximation was proposed by Lush[1] and applied to the minimum time-to-climb problem by Rutowski[2]. Systematic application of this approximation to several aircraft performance optimization problems, including minimum time-to-climb, is presented in reference 3. In this approximation, only the energy is treated as a state variable and velocity (or altitude) plays the role of a control variable. Boundary conditions are met by adjoining constant energy paths to the optimal path.

Another approximation, called the two state in this paper, has also been used, but to a much lesser extent. In this approximation, drag due to lift is ignored. Altitude and energy (or velocity) are state variables and flight path angle is the control variable. Thus boundary conditions on altitude and velocity may be satisfied; however, the flight path angle will be discontinuous.

In this paper, the minimum time-to-climb program is first precisely stated and the necessary conditions for optimal control are determined by employing Pontryagin's maximum principle.[4,5,6] The concept of a singular approximation is then introduced and it is shown that the two possible singular approximations are just the two previously mentioned. These two approximations are discussed and compared both qualitatively and, using a numerical example, quantitatively.

A third approximation is proposed called the modified two state, which is based on the two state but includes drag due to lift and accounts for the time required to change flight path angle. Comparison with the two-state approximation shows that inclusion of drag due to lift has a negligible effect but that the time to change the flight path angle is significant, particularly at high speeds. The modified two-state approximation is used to assess the sensitivity of climb performcance to various vehicle parameters for a specific numerical example.

1. MINIMUM TIME-TO-CLIMB PROBLEM

The most general system of equations of motion commonly employed in aircraft trajectory computations is the following seventh order system which describes a variable weight point mass moving over a spherical non-rotating earth (c.f., ref. 7):

$$\begin{aligned}
\dot{x}' &= \frac{V' r_o}{r_o + h'} \frac{\cos\gamma\cos\chi}{\cos y'/r_o} \\
\dot{y}' &= \frac{V' r_o}{r_o + h'} \cos\gamma\sin\chi \\
\dot{h}' &= V'\sin\gamma \\
\dot{V}' &= \frac{g_o T'}{W}\cos(\alpha + t) - \frac{g_o D'}{W} - g\sin\gamma \qquad (1.1) \\
\dot{\chi} &= \frac{g_o}{WV'}[L' + T'\sin(\alpha + \tau)]\frac{\sin\theta}{\cos\gamma} \\
\dot{\gamma} &= \frac{g_o}{WV'}[L' + T'\sin(\alpha + \tau)]\cos\theta - \frac{g}{V'}\cos\gamma + \frac{V'}{r_o + h'}\cos\gamma \\
\dot{W} &= -\beta
\end{aligned}$$

where $T' = T'(h',V',\beta,\alpha)$, $L' = L'(h',V',\alpha)$, $D' = D'(h',V',\alpha)$, $g = g(h')$ and τ and g_o are given constants. In these equations, the state vector is $(x',y',h',V',\chi,\gamma,W)$ and the control vector is (α,β,θ).

For the purposes of the present paper, the following simplifying assumptions are made*: 1) $r_o \to \infty$ ("flat earth" assumption), 2) W is constant, 3) $\tau = 0$, 4) $g(h') = g_o$, 5) T' is not a function of α, 6) $T'\cos\alpha = T'$ and $L'+T'\sin\alpha = L'$. In addition, it is clear that for the minimum time-to-climb problem that β should be set to its maximum value (this will be considered further later) and that if boundary conditions on x', y', and χ are not specified then only planar motion, say x'=constant, need be considered. Using these assumptions and noting that if range is to be neither constrained nor optimized then it is uncoupled from the rest of the system, the following system is obtained:

$$\dot{h}' = V' \sin\gamma$$

$$\dot{V}' = \frac{g_o}{W}(T' - D') - g_o \sin\gamma \qquad (1.2)$$

$$\dot{\gamma} = \frac{g_o}{WV'} L' - \frac{g_o}{V'} \cos\gamma$$

The decrease in system order which results from the uncoupling of the range equation makes the minimum time-to-climb problem relatively easy to solve compared with other performance optimization problems. Making the change of variables

$$V = V'/g_o,\ h = h'/g_o,\ T = T'/W,\ D = D'/W,\ L = L'/W \qquad (1.3)$$

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*These assumptions have been found to be well-founded for many problems of interest. They have been made here for simplicity and do not affect the subsequent developments.

in (1.2) results in

$$\begin{aligned} \dot{h} &= V \sin \gamma \\ \dot{V} &= T - D - \sin \gamma \\ \dot{\gamma} &= \frac{1}{V} (L - \cos \gamma) \end{aligned} \tag{1.4}$$

where T(h,V) and D(h,V,L) are known functions. Let

$$\begin{aligned} D(h,V,L) &= D_o(h,V) + D_L(h,V,L) \\ F(h,V) &= T(h,V) - D_o(h,V) \end{aligned} \tag{1.5}$$

Then (1.4) becomes

$$\begin{aligned} \dot{h} &= V \sin \gamma \\ \dot{V} &= F - D_L - \sin \gamma \\ \dot{\gamma} &= \frac{1}{V} (L - \cos \gamma) \end{aligned} \tag{1.6}$$

For the purpose of discussing approximations, it is advantageous to introduce the energy variable E, defined by

$$E = \frac{1}{2} V^2 + h \tag{1.7}$$

and use it to replace V in the equations of motion. Differentiating (1.7) and using (1.6) leads to the system

$$\begin{aligned} \dot{h} &= V \sin \gamma \\ \dot{E} &= V (F - D_L) \\ \dot{\gamma} &= \frac{1}{V} (L - \cos \gamma) \end{aligned} \tag{1.8}$$

where

$$V = V(h,E) = \sqrt{2(E - h)} \tag{1.9}$$

and $F = F(h,E)$, $D_L = D_L(h,E,L)$.

The minimum time-to-climb problem is now stated as follows: Suppose that there exists a region Ω of (h,E,L) space such that in Ω we have 1) D_L as a function of L is even, $D_L = 0$ for $L = 0$, and

$\frac{\partial^2 D_L}{\partial L^2} > 0$, 2) $F - D_L > 0$, and 3) $E-h > 0$. It is desired to find the control history $L(t)$ such that among all control histories which transfer the system (1.8) from the values h_o, E_o at time $t = 0$ to the values h_f, E_f at the time $t = \phi$ (the case of specified γ_o and γ_f will be discussed later) such that the trajectories are entirely contained in Ω, $L(t)$ minimizes the transfer time ϕ. The existence of such an optimal control is assumed. The maximum principle of Pontryagin[4,5,6] will now be used to obtain the extremal controls, one of which must be the optimum control. If the extremal control is unique it will be the optimal control.

The H function for the problem (1.8) is

$$H = \lambda_o + \lambda_h V \sin\gamma + \lambda_E V(F - D_L) + \lambda_\gamma \frac{1}{V}(L - \cos\gamma) \quad (1.10)$$

The adjoint variables satisfy

$$\begin{aligned}
\dot{\lambda}_h &= -\lambda_h\left(\frac{\partial V}{\partial h}\right)_E \sin\gamma - \lambda_E\left(\frac{\partial V}{\partial h}\right)_E (F - D_L) - \lambda_E V\left(\frac{\partial F}{\partial h} - \frac{\partial D_L}{\partial h}\right)_E \\
&\quad + \lambda_\gamma \frac{1}{V^2}\left(\frac{\partial V}{\partial h}\right)_E (L - \cos\gamma) \\
\dot{\lambda}_E &= -\lambda_h\left(\frac{\partial V}{\partial E}\right)_h \sin\gamma - \lambda_E\left(\frac{\partial V}{\partial E}\right)_h (F - D_L) - \lambda_E V\left(\frac{\partial F}{\partial E} - \frac{\partial D_L}{\partial E}\right)_h \\
&\quad + \lambda_\gamma \frac{1}{V^2}\left(\frac{\partial V}{\partial E}\right)_h (L - \cos\gamma) \qquad (1.11) \\
\dot{\lambda}_\gamma &= -\lambda_h V \cos\gamma - \lambda_\gamma \frac{1}{V}\sin\gamma
\end{aligned}$$

The transversality conditions give the boundary values for (1.8) and (1.11) as

$$\begin{aligned}
h(0) &= h_o & h(\phi) &= h_f \\
E(0) &= E_o & E(\phi) &= E_f \qquad (1.12) \\
\lambda_\gamma(0) &= 0 & \lambda_\gamma(\phi) &= 0
\end{aligned}$$

The part of H which contains the control L explicitly is

$$\tilde{H} = -\lambda_E V D_L + \lambda_\gamma \frac{1}{V} L \tag{1.13}$$

and the optimality condition is

$$L = \arg\operatorname{Max} \tilde{H} \tag{1.14}$$

The various cases which may arise are discussed in ref. 8. If the energy of the final state is greater than that of the initial state it is reasonable to assume that $\lambda_E > 0$ since λ_E may be viewed in most cases* as the negative sensitivity of trajectory time ϕ to energy level E. Therefore in the sequel it will be assumed that $\lambda_E > 0$; this condition must be verified for any candidate solution. Under this assumption, (1.14) becomes

$$\frac{\partial\tilde{H}}{\partial L} = -\lambda_E V \frac{\partial D_L}{\partial L} + \lambda_\gamma \frac{1}{V} = 0 \tag{1.15}$$

provided this gives trajectories in region Ω.

If $T(h,V,\beta)$ is a monotonically increasing function of β subject to $0 \leq \beta \leq \beta_M$, the condition $\lambda_E > 0$ implies from (1.10) that $\beta = \beta_M$ on optimal trajectories.

For later reference, the classes of trajectories called zoom climbs ($\gamma = +90°$) and dives ($\gamma = -90°$) will be investigated with regard to the necessary conditions. From (1.8c) L=0 on such trajectories and from (1.11c)

$$\dot{\lambda}_\gamma = \mp \lambda_\gamma \frac{1}{V}$$

Thus, using (1.12),

$$\lambda_\gamma(t) = \lambda_\gamma(0) e^{\mp \int_0^t \frac{1}{V} dt} = 0$$

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*Conditions for which this is true are discussed in ref. 4.

This solution satisfies the condition (1.14) if $\lambda_E > 0$. The state equations are now

$$\dot{h} = \pm V$$

$$\dot{E} = VF$$

the adjoint equations are

$$\dot{\lambda}_h = \mp \lambda_h \left(\frac{\partial V}{\partial h}\right)_E - \lambda_E \left(\frac{\partial V}{\partial h}\right)_E F - \lambda_E V \left(\frac{\partial F}{\partial h}\right)_E$$

$$\dot{\lambda}_E = \mp \lambda_h \left(\frac{\partial V}{\partial E}\right)_h - \lambda_E \left(\frac{\partial V}{\partial E}\right)_h F - \lambda_E V \left(\frac{\partial F}{\partial E}\right)_h$$

and H is

$$H = \lambda_o \pm \lambda_h V + \lambda_E VF$$

Noting that

$$\dot{V} = F \mp 1$$

$$\left(\frac{\partial V}{\partial h}\right)_E = - \frac{1}{\sqrt{2\ (E - h)}} = - \left(\frac{\partial V}{\partial E}\right)_h$$

$$\dot{F} = \frac{\partial F}{\partial h}\dot{h} + \frac{\partial F}{\partial E}\dot{E}$$

we form $\dot{H} = 0$ to get

$$0 = \pm\dot{\lambda}_h V \pm \lambda_h \dot{V} + \dot{\lambda}_E VF + \lambda_E \dot{V} F + \lambda_E V\dot{F}$$

$$0 = \lambda_h \pm F\lambda_E$$

Thus

$$H = \lambda_o = 0$$

so that zoom climbs and dives are "abnormal" arcs. This also shows that $\lambda_E \neq 0$. It may be concluded that in the fortuitous event that both the initial and final points lie in the right sense on one of the directed arcs

$$\frac{dE}{dh} = \pm F \qquad (1.16)$$

that the optimal control is given by $L = 0$ and the optimal trajectory by (1.16) together with $\gamma = \pm 90$.

Solution of the nonlinear two-point boundary value system (1.8) + (1.11) + (1.12) + (1.14) has proved to be a formidable computational problem. Therefore, there have been many attempts to obtain approximate solutions based on simplified equations. Many of these approximations are critically discussed in reference 8. All of them are found to exhibit undesirable features. The approximate methods most often employed are based on neglecting certain of the terms in (1.8) (and hence also in (1.11)). If right-hand side terms are neglected (e.g., the aerodynamic forces) the order of the system remains the same; this may be termed regular approximation. If, however, terms on the left-hand side are neglected, the order of the system is reduced and thus not all of the boundary conditions can be met. This loss in boundary conditions is usually accounted for by arbitrarily saying that the functions instantaneously jump, usually at the boundaries, in just such a way as to satisfy the boundary conditions. Hopefully, the approximation will be accurate everywhere but in small neighborhoods of such jumps. It is natural to call such approximations singular. Singular approximations are a double-edged sword; the considerable simplification resulting from decrease in system order is accompanied by a radical change in system behavior.

If one takes the view that control variables are variables which may be changed instanteously, then it is intuitively clear how singular approximations are to be made. A relatively "fast" state variable (i.e., a variable capable of changing across its range relatively rapidly compared with the other state variables) should be relegated to the role of a control variable by neglecting its derivative term. Thus control variables may be viewed as limiting or degenerate forms of state variables. Such a variable will lose its boundary conditions and it must be assumed that these variables jump to meet their boundary conditions.

The reason E has been substituted for V in the state equations for the minimum time-to-climb problem is that this results in variables of more widely separated "speed". It has been found in practical problems for supersonic aircraft that γ is relatively fast as compared with h and that h is relatively fast as compared with E (V is about the same speed as h). Thus there are two natural singular approximations for this problem: In the "energy state" approximation, only E is a state variable; in the "two state" approximation both E and h are state variables. These two singular approximations in the minimum time-to-climb problem are discussed and compared in the remainder of this paper.

There are two possible procedures for solving singular approximation problems in optimal control. In the first of these, the necessary conditions are formulated for the full system of equations and then the appropriate terms on the left-hand side are neglected. In the second procedure, the left-hand side terms are neglected previous to the formulation of the necessary conditions. It can be shown that for the energy state and two state approximations these two procedures are equivalent.

In the first procedure, used for the energy state approximation in this paper, the left hand sides of adjoint equations corresponding to state equations with left hand sides neglected will be neglected. This step may be formalized by inserting a parameter into the state equations prior to formulation of the necessary conditions in such a way that the required approximation is obtained when the parameter is set to zero. This procedure results in a problem which falls under the domain of singular perturbation theory. This theory is presented in detail in ref. 9 for linear systems.

The second procedure, used for the two state approximation herein, results in a problem with state dependent control constraints. Such problems are treated in section 3.6 of ref. 4. Comparing the two procedures shows that the adjoint variables associated with the state variables whose derivatives are neglected in the first procedure may be viewed formally as the ordinary Lagrange multipliers arising in the maximization of the Hamiltonian in the second procedure.

2. THE ENERGY STATE APPROXIMATION

Consistent with remarks in the previous section, the energy state approximation is obtained as a limiting case of (1.8) + (1.11) + (1.12) + (1.14) by letting $\dot{h}, \dot{\gamma}, \dot{\lambda}_h, \dot{\lambda}_\gamma \to 0$. For (1.8),

$$0 = V \sin \gamma$$

$$\dot{E} = V\,(F - D_L)$$

$$0 = \frac{1}{V}\,(L - \cos \gamma)$$

so that $\gamma = 0$, $L = 1$ and

$$\dot{E} = V\,(F - D_{L_1}) \tag{2.1}$$

where we now regard $F = F\ (E,V)$ and $D_{L_1} = D_L\ (E,V,1)$. From (1.11)

$$0 = \lambda_E \frac{\partial}{\partial h}\left[V\ (F - D_{L_1})\right]_E$$

$$\dot{\lambda}_E = -\ \lambda_E \frac{\partial}{\partial E}\left[V\ (F - D_{L_1})\right]_h \tag{2.2}$$

$$0 = -\ \lambda_h V$$

The last of these gives $\lambda_h = o$.

The appropriate boundary conditions are

$$E(o) = E_o;\ E(\phi) = E_f \tag{2.3}$$

Equation (1.10) becomes

$$H = \lambda_o + \lambda_E\ V\ (F - D_{L_1}) \tag{2.4}$$

From $H = 0$, $\lambda_o < 0$ (note that $\lambda_o = 0$ is not possible) and our assumption that $V(F - D_{L_1}) > 0$, (2.4) implies that

$$\lambda_E > 0 \tag{2.5}$$

Thus the first of (2.2) implies

$$\frac{\partial}{\partial h}\left[V(F - D_{L_1})\right]_E = 0 \tag{2.6}$$

The optimality condition (1.15) becomes an equation for λ_γ

$$\lambda_\gamma = \lambda_E\ V^2\ \frac{\partial D_{L_1}}{\partial L} \tag{2.7}$$

once (2.4) with $H = 0$ and $\lambda_o = -1$ has been solved for λ_E. The second of (2.2) is then satisfied identically.

It may happen that (2.6) has multiple roots. In this case the root which maximizes H as given by (2.4) must be selected. If a "jump" between roots occurs, V will be discontinuous (it is now, in effect, a control variable) but the state variable E will be continuous. From (1.9),

$$\left(\frac{\partial}{\partial V}\right)_E = \left(\frac{\partial}{\partial h}\right)_E \left(\frac{\partial h}{\partial V}\right)_E = -V\left(\frac{\partial}{\partial h}\right)_E \qquad (2.8)$$

so that (2.6) may also be written

$$\frac{\partial}{\partial V}\left[V\left(F - D_{L_1}\right)\right]_E = 0 \qquad (2.9)$$

It is of interest to note that treating V as a control and E as a state variable, and maximizing H as given by (2.4) with respect to V results in a control law identical to (2.9). This control law gives V in terms of E and has been called the "energy climb path". The total elapsed time is obtained from (2.1) as

$$\phi = \int_{E_o}^{E_f} \frac{dE}{V\left(F - D_{L_1}\right)} \qquad (2.10)$$

where V = V(E) is obtained from (2.9).

Returning now to the original problem (1.8), it is seen that the energy climb trajectory will not meet the boundary conditions on the fast variable h. This was to be expected since the energy state approximation is singular. Since h (or equivalently V) is now regarded, in a limiting sense, as a control variable, its boundary conditions can be met by instanteous changes during which the state E remains constant.

Consider the resulting trajectories in the (h,V) plane. "Control law" (2.9) may be written in terms of V and h as

$$\frac{\partial}{\partial V}\left[V\left(F - D_{L_1}\right)\right]_E = \frac{\partial}{\partial V}\left[V\left(F - D_{L_1}\right)\right]_h + \frac{\partial}{\partial h}\left[V\left(F - D_{L_1}\right)\right]_V \left(\frac{\partial h}{\partial V}\right)_E$$

$$= F - D_{L_1} + V\left(\frac{\partial F}{\partial V} - \frac{\partial D_{L_1}}{\partial V}\right)_h - V^2\left(\frac{\partial F}{\partial h} - \frac{\partial D_{L_1}}{\partial h}\right)_V = 0 \qquad (2.11)$$

Denote this curve by $f_E\ (h,V) = 0$; as discussed earlier, this function may be discontinuous in both arguments if multiple roots occur. The initial arc lies on the constant energy contour through the initial point

$$f_o(h,V) = 1/2\ (V_o^2 - V^2) + (h_o - h) = 0 \tag{2.12}$$

Similarly, the final arc lies on

$$f_f(h,V) = 1/2\ (V_f^2 - V^2) + (h_f - h) = 0 \tag{2.13}$$

Under the assumption that $f_E(h,V) = 0$ as described above exists in Ω and intersects both $f_o(h,V) = 0$ and $f_f(h,V) = 0$, an energy state approximate minimum time-to-climb trajectory exists. Such a trajectory is shown schematically in Figure 1 for the case of one discontinuity in $f_E(h,V)$.

The following observations are now made:

(1) The energy state approximation (2.1) may be obtained from (1.8) by setting $\gamma = 0$ in (1.8); this provides an alternate way of looking at this approximation.

(2) The control law (2.9) may be determined directly without recourse to the maximum principle. The time-to-climb is obtained from (2.1) as

$$\phi = \int_{E_o}^{E_f} \frac{dE}{V\left(F - D_{L_1}\right)}$$

This integral will be minimized when $V\left(F - D_{L_1}\right)$ is maximized with respect to V for each E, giving (2.9). This may be interpreted as maximizing "excess power" $V\left(F - D_{L_1}\right)$ while holding E constant and was the argument originally used by Rutowski.[2]

(3) The portions of the trajectory on which E = constant (such as on arcs $f_o(h,E) = 0$ and $f_f(h,E) = 0$) have undesirable features. In the first place, such arcs are traversed in zero elapsed time. It is often

said that in energy state approximations, one may "trade h for V with no penalty t." Further, the implicit assumption that γ is small is violated; in fact, the condition $|\sin\gamma| \le 1$ is violated. To see this, use (1.6) to compute the slope

$$\frac{dh}{dV} = \frac{V \sin\gamma}{F - D_{L_1} - \sin\gamma} = -\frac{V}{1 - \frac{F - D_{L_1}}{\sin\gamma}} \tag{2.14}$$

Thus, starting at a given point (h,E(h,V),1) in Ω there is an admissible region for trajectories in the (h,V) plane obtained by letting γ range from -90° to +90°. On the other hand, from (1.7),

$$\left(\frac{dh}{dV}\right)_E = -V$$

which does not lie in the admissible region for $F - D_L > 0$. The situation is illustrated in Figure 2.

(4) For low-speed aircraft, a "quasi-steady-state" approximation is often employed in which changes in velocity and flight path angle are ignored in comparison to changes in altitude. Setting $\dot{V} = 0$ and $\dot{\gamma} = 0$ in (1.6) leads to

$$\phi = \int_{h_o}^{h_f} \frac{dh}{V\left(F - D_{L_1}\right)}$$

Thus for minimum time-to-climb ϕ^*,

$$\frac{\partial}{\partial V}\left[V\left(F - D_{L_1}\right)\right]_h = 0$$

$$F - D_{L_1} + V\left(\frac{\partial F}{\partial V} - \frac{\partial D_{L_1}}{\partial V}\right)_h = 0 \tag{2.15}$$

Comparing (2.15) with (2.11), it is seen that the former neglects a term which may be expected to be small for small velocity.

(5) Since F and D_L are usually given in terms of h' and Mach number M,

$$M = \frac{V'}{a(h')} \tag{2.16}$$

it is useful to write the energy climb path (2.9) in terms of h' and M. The result is

$$F - D_{L_1} + \frac{aM}{g_o}\left[\left(\frac{g_o}{a} + M^2 \frac{\partial a}{\partial h'}\right)\left(\frac{\partial F}{\partial M} - \frac{\partial D_{L_1}}{\partial M}\right)_{h'} - aM\left(\frac{\partial F}{\partial h'} - \frac{\partial D_{L_1}}{\partial h'}\right)_M\right] = 0 \tag{2.17}$$

(6) From (2.4) and (2.5) it follows that $\beta = \beta_M$ on the energy climb path.

(7) Since the energy state formulation is independent of γ it follows that the solution of the problem with specified boundary values of γ is the same as that for γ free.

(8) From (2.11) it is seen that if F and D_{L_1} are the same exponential function of h then (2.11) is no longer a function of h and the energy climb path is a vertical line in the (h,V) plane.

(9) A graphical interpretation of the energy state solution is presented in reference 10. Referring to (2.4), let

$$\zeta\ (E,V) = \frac{1}{V}\ ;\ \psi\ (E,V) = F - D_{L_1}$$

Then

$$H = -1 + \lambda_E \frac{\psi}{\zeta}$$

with $\lambda_E > 0$ so that the optimality condition is

$$V = \arg \mathrm{Max}\ \frac{\psi}{\zeta}$$

Thus the optimal value of V is that which maximizes the slope of $\psi = \psi\ (E,\zeta)$ with E fixed as shown on Figure 3.

3. THE TWO-STATE APPROXIMATION

In the two state approximation, it is assumed that drag due to lift D_L is negligibly small and that γ is a fast variable as compared with h and E. Let $\dot{\gamma}$, $D_L \to 0$ in (1.8) to get

$$\dot{h} = V \sin \gamma$$

$$\dot{E} = VF$$

$$0 = \frac{1}{V} (L - \cos \gamma)$$

so that L is uncoupled from the state equations. It is convenient to let

$$u = \sin \gamma, \ |u| \leq 1 \tag{3.1}$$

Then the state equations are

$$\begin{aligned} \dot{h} &= Vu \\ \dot{E} &= VF \end{aligned} \tag{3.2}$$

and the lift is given by

$$L = \sqrt{1 - u^2} \tag{3.3}$$

The adjoint equations associated with (3.2) are

$$\begin{aligned} \dot{\lambda}_h &= - \lambda_h \left(\frac{\partial V}{\partial h}\right)_E u - \lambda_E \left(\frac{\partial (VF)}{\partial h}\right)_E \\ \dot{\lambda}_E &= - \lambda_h \left(\frac{\partial V}{\partial E}\right)_h u - \lambda_E \left(\frac{\partial (VF)}{\partial E}\right)_h \end{aligned} \tag{3.4}$$

From (1.12) the boundary conditions are

$$\begin{aligned} h(0) &= h_o \qquad h(\phi) = h_f \\ E(0) &= E_o \qquad E(\phi) = E_f \end{aligned} \tag{3.5}$$

so that all state variable boundary conditions have been retained. The H function is

$$H = \lambda_o + \lambda_h Vu + \lambda_E VF \tag{3.6}$$

We have a system with state (h,E) and control u; the part of H explicity a function of u is

$$\tilde{H} = \lambda_h Vu \tag{3.7}$$

Since V > 0, the optimality condition is

$$u = \begin{Bmatrix} -1 \text{ if } \lambda_h < 0 \\ ? \text{ if } \lambda_h \equiv 0 \\ +1 \text{ if } \lambda_h > 0 \end{Bmatrix} \tag{3.8}$$

so that λ_h is a "switching function."

The possibility of a singular arc, on which

$$\lambda_h \equiv 0 \tag{3.9}$$

is now investigated. If (3.9) holds, (3.4) become

$$\begin{aligned} 0 &= -\lambda_E \frac{\partial (VF)}{\partial h} \\ \dot{\lambda}_E &= -\lambda_E \frac{\partial (VF)}{\partial E} \end{aligned} \tag{3.10}$$

Equation (3.6) is

$$H = \lambda_o + \lambda_E VF \tag{3.11}$$

Since $H = 0$, $\lambda_o < 0$ ($\lambda_o = 0$ is not possible in this case), this gives

$$\lambda_E = \frac{1}{VF} \tag{3.12}$$

so that $\lambda_E > 0$. Thus the first of (3.10) implies that

$$\frac{\partial}{\partial h} [VF]_E = 0 \tag{3.13}$$

on the singular arc, if it exists. To find u, differentiate (3.9) twice:

$$\dot{\lambda}_h = -\lambda_h \frac{\partial V}{\partial h} u - \lambda_E \frac{\partial (VF)}{\partial h} = 0$$

$$\ddot{\lambda}_h = - \dot{\lambda}_h \frac{\partial V}{\partial h} u - \lambda_h \left[\frac{\partial^2 V}{\partial h^2} \dot{h} + \frac{\partial^2 V}{\partial h \partial E} \dot{E} \right] u$$

$$- \lambda_h \frac{\partial V}{\partial h} \dot{u} - \dot{\lambda}_E \frac{\partial (VF)}{\partial h} - \lambda_E \left[\frac{\partial^2 (VF)}{\partial h^2} + \frac{\partial^2 (VF)}{\partial h \partial E} \dot{E} \right] = 0$$

$$\ddot{\lambda}_h = - \lambda_E \left[\frac{\partial^2 (VF)}{\partial h^2} Vu + \frac{\partial^2 (FV)}{\partial h \partial E} VF \right] = 0 \qquad (3.14)$$

Since $\lambda_E \neq 0$, this implies

$$u = - \frac{F \; [\partial^2 (FV)/\partial h \partial E]}{\partial^2 (FV)/\partial h^2} \qquad (3.15)$$

provided this does not violate $|u| \leq 1$. The singular arc must satisfy an additional necessary condition[5,11] called the Kelley condition or the strengthened convexity condition. Since u appears linearly in $\ddot{\lambda}_h = \frac{d^2}{dt^2} \quad \frac{\partial H}{\partial u}$ the singular arc is of order m = 2. The Kelley condition is

$$\frac{\partial \ddot{\lambda}_h}{\partial u} \leq 0 \qquad (3.16)$$

which, from (3.14), leads to

$$\frac{\partial^2 (FV)}{\partial h^2} > 0 \qquad (3.17)$$

Note that equality is not allowed in view of (3.15). It may be concluded that the arc given by (3.13) is extremal if (3.17) and the condition

$$-1 \leq - \frac{F \; [\partial^2 (FV)/\partial h \partial E]}{\partial^2 (FV)/\partial h^2} \leq 1 \qquad (3.18)$$

are satisfied.

Two-state approximate solutions are formed by joining arcs u = +1, u = -1, and $\frac{\partial}{\partial h} [VF]_E = 0$ in such a way that minimum time results and the boundary conditions are satisfied. If (3.13) has multiple roots, this process may become quite complicated since switching between roots occurs along paths $u = \pm 1$ and takes a non-zero elapsed time. If (3.13) has only a single root (defines a single valued function $h = h(E)$), then a solution might appear in the (h,V) plane as shown in Figure 4. Since $m/2 = 1$ is odd, the control u will in general be discontinuous at the junctions of arcs $u = \pm 1$ with the singular arc.[5]

The following observations may be made.

(1) The critical assumption is the neglecting of D_L; if this is done in (1.8) the first two equations become uncoupled from the third, the latter being now regarded as an equation for L. However, L will now have to be regarded as unbounded since γ will be discontinuous.

(2) Setting D_L, $\dot{\gamma} \to 0$ in (1.6) leads to the two state approximation as would be expected.

(3) The singular arc (3.13) in terms of E and V is

$$\frac{\partial}{\partial V} [VF]_E = 0 \tag{3.19}$$

in terms of h and V is

$$F + V\left(\frac{\partial F}{\partial V}\right)_h - V^2\left(\frac{\partial F}{\partial h}\right)_V = 0 \tag{3.20}$$

and in terms of h' and M is

$$F + \frac{aM}{g_o}\left[\left(\frac{g_o}{a} + M^2 \frac{\partial a}{\partial h'}\right)\left(\frac{\partial F}{\partial M}\right)_{h'} - aM\left(\frac{\partial F}{\partial h'}\right)_M\right] = 0 \tag{3.21}$$

(4) The equation for the singular arc may be determined by a Green's theorem argument.[12] To see this, set $D_L = 0$ in the first two of (1.6), eliminate $\sin\gamma$, and evaluate the resulting line integral over a closed curve in the (h,V) plane.

$$t = \oint \left[\frac{1}{VF}\, dh + \frac{1}{F}\, dV \right]$$

By Green's theorem, this is equivalent to the surface integral over the enclosed area:

$$t = \oint \left[\frac{\partial}{\partial V} \left(\frac{1}{VF}\right)_h - \frac{\partial}{\partial h} \left(\frac{1}{F}\right)_V \right] dhdV$$

By considering various closed curves, optimal control may be deduced; in particular, the "critical arc" is

$$\frac{\partial}{\partial V} \left(\frac{1}{VF}\right)_h - \frac{\partial}{\partial h} \left(\frac{1}{F}\right)_V = 0$$

or

$$F + V \left(\frac{\partial F}{\partial V}\right)_h - V^2 \left(\frac{\partial F}{\partial h}\right)_V = 0$$

which is the singular arc (3.20).

(5) On the arcs $u = \pm 1$, the full equations of motion (1.8) and the complete necessary conditions are satisfied.

(6) Since u is discontinuous at the junctions of $u = \pm 1$ with the singular arc, the jumps in the fast variable γ occur in the interior of the trajectory and not at the boundaries $t = 0$ and $t = \phi$.

(7) Since $\lambda_E > 0$ on the singular arc, it follows that $\beta = \beta_M$.

(8) Since h and u do not appear in the second of (3.2), it is tempting to treat E as a single state variable and V as the control. This leads directly to (3.13) but does not give the arcs $u = \pm 1$ necessary to meet the boundary conditions. This happens of course because h must be treated as a state variable due to (3.5).

(9) Although this method technically neglects drag due to lift, D_L, in practice it is desirable to include an average amount of drag due to lift, say D_L at $L = 1$.

(10) Since γ is treated as a control variable, boundary conditions on γ are met by instantaneous changes and there is no difference between the cases of free and specified boundary values of γ in the two-state approximation.

(11) From (3.20) it is seen that if F is an exponential function of h then (3.20) is no longer a function of h and the singular arc is a vertical line in the (h,V) plane.

(12) From (1.8) the assumption that γ is a fast variable worsens as V increases so that the two-state approximation is of doubtful value for hypersonic or higher speeds.

4. THE MODIFIED TWO-STATE APPROXIMATION

Both the energy state and two-state approximations have been shown to exhibit undesirable features. The two-state approximation employs one more state variable than the energy state and thus more accuractely models the system dynamical behavior. The disadvantage of the two-state approximation is that integrations (1.16) are required to obtain the connecting arcs to the singular arc, whereas the connecting arcs in the energy state approximation are algebraic relations. These integrations however are quite simple and there appears to be no reason to use the energy state approximation as compared with the two state.

There are two undesirable features of the two-state approximation. First is the absence of drag due to lift. This may be at least partially compensated for by adding the drag due to lift at L = 1 to the singular arc equation. From (2.11) and (3.20) we see that if this is done the energy climb path is identical to the singular arc.

The second undesirable feature is the jumps in γ at the junction points between the connecting arcs $\sin\gamma = \pm 1$ and the singular arc and, for the case of fixed initial and final γ, at the endpoints. A time correction for these jumps may be derived as follows. At a junction point, h and E will to a first approximation remain constant while γ changes rapidly, say from γ_1 to γ_2. Referring to (1.8), we have a single state variable system with control L

$$\dot{\gamma} = \frac{1}{V}(L - \cos\gamma)$$

Solving for the elapsed time $t_2 - t_1$

$$t_2 - t_1 = V\int_{\gamma_1}^{\gamma_2} \frac{d\gamma}{L - \cos\gamma}$$

If L is bounded, say $L_m \leq L \leq L_M$, then, for minimum $t_2 - t_1$

$$L^* = \begin{Bmatrix} L_M \text{ if } \gamma_2 > \gamma_1 \\ L_m \text{ if } \gamma_2 < \gamma_1 \end{Bmatrix}$$

so that

$$t_2 - t_1 = V\int_{\gamma_1}^{\gamma_2} \frac{d\gamma}{L^* - \cos\gamma}$$

$$t_2 - t_1 = \frac{2V}{\sqrt{L^{*2} - 1}}\left[\tan^{-1}\frac{\sqrt{L^{*2} - 1}\,\tan\frac{\gamma_2}{2}}{L^* - 1} - \tan^{-1}\frac{\sqrt{L^{*2} - 1}\,\tan\frac{\gamma_1}{2}}{L^* - 1}\right] \tag{4.1}$$

provided that $|L^*| > 1$. Note that if $|L_M|$, $|L_m| >> 1$ then this is approximately

$$t_2 - t_1 \approx \left|\frac{V(\gamma_2 - \gamma_1)}{L^*}\right|$$

In summary, the modified two-state approximation is formed by "patching together" the following: a) the energy climb path (2.11), b) the arcs $\sin \gamma = \pm 1$, and c) the arcs characterized by h = constant, E = constant, $L = L^*$ which take time $t_2 - t_1$ as given by (4.1) and which are needed at the junctions of arcs a) and b) and, if γ_0 and γ_f are specified, at the endpoints. This approximation may be expected to be a good one provided that the time spent on arcs c) is small compared with that spent on a) and b). From (4.1) the approximation will be poor when V is large or when $(L^* - 1)$ is small.

5. COMPARISON OF APPROXIMATIONS - A NUMERICAL EXAMPLE

The three approximations previously discussed have been programmed for a digital computer. The computer program computes the path in the (M,h') plane and the minimum time-to-climb ϕ^*. The most general case is shown in Figure 5. It may happen that one or more of the arcs shown will be absent in a given example. Note that the condition h' = constant > 0 has been imposed for physical reasons.

To gain insight into the nature of the paths which result from the approximations and to compare these approximations with each other, a numerical example is now considered. The data is that of "airplane 2" of ref. 3 for which the aerodynamics are represented by

$$L' = C_{L_\alpha} \alpha \, 1/2 \, \rho V'^2 S$$

$$D' = (C_{D_o} + \eta \, C_{L_\alpha} \alpha^2) \, 1/2 \, \rho V'^2 S$$

Figure 6 shows the singular arc (3.21) and the energy climb path (2.17), both of which are single valued for this airplane. Since the difference in these two curves is that $D_L = 0$ in the former and $D_L = D_{L_1}$ in the latter, it may be concluded that drag due to lift has a very small effect.

Both the energy state and two-state trajectories for the minimum time-to-climb between $h_o' = 20{,}000$ ft, $M_o = 0.4$ and $h_f' = 80{,}000$ ft, $M_f = 1.0$ are shown in Figure 7 where the elapsed times on the various arcs are indicated in parentheses. The fact that the energy state trajectory traverses the arcs E = constant in zero elapsed time is offset to some extent by the fact that this trajectory remains on the arc h' = constant and the energy climb path longer than the two-state trajectory remains on h' = constant and the singular arc. However ϕ^* as predicted by the energy state approximation is significantly lower than that predicted by the two state. The closer to the energy climb path that the end points lie the less this discrepancy will be. The modified two-state path is the same as the two-state path except that the time-to-climb is 157 seconds instead of 132 seconds.

Figure 8 shows the energy time histories of the energy and two-state trajectories. Since E is a state variable in both cases, E(t) is continuous. These histories agree very closely except at the terminal point where additional time must be spent by the two-state trajectory in meeting the terminal condition on h'.

The minimum time-to-climb ϕ^* as computed by the various approximations is presented in Table I. The energy state and two-state cases are as discussed before. The third column of the table shows the results of including D_L at L = 1 in the singular arc computation (i.e., using the energy climb path). Comparison shows that inclusion of drag due to lift results in a negligible change in ϕ^*. The elapsed times of the modified two-state approximation are shown in the last three columns. For this approximation, the cases of free and fixed boundary values of γ must be distinguished. In the first case, it is seen that accounting for the time to change γ at points 2, 3 and 4 has added 22 seconds to ϕ^*, a significant amount. The requirement for level flight at the beginning and end points adds another 24 seconds. Note that the elapsed time in changing γ is higher at the higher velocity points. It may be concluded that neglecting the time required to rotate a vehicle is not generally a good approximation at high speeds.

Figure 9 illustrates alternate types of trajectories which may occur. Type I is that previously discussed. In Type II, arc 2-3 has vanished and in Type III arc 1-2 has vanished. Also shown is ϕ^* for the modified two-state approximation for each trajectory.

The time histories of energy, flight path angle, altitude, and Mach number are shown in Figure 10 parts a, b, c, and d respectively for the Type I modified two-state trajectory of Figure 9. Figure 10 illustrates the nature of minimum time-to-climb trajectories for supersonic aircraft: The initial portion of the trajectory (about two thirds of the total time) is essentially a low, constant altitude acceleration during which speed and energy are steadily increased. During the final portion of the trajectory, a "zoom" maneuver is performed in which speed is traded for altitude with energy remaining approximately constant.

The steepest descent trajectory optimization program of reference 13 was applied to the example discussed above to provide a standard solution for comparative purposes. For the present application, the steepest descent solution used equations describing motion of a constant weight vehicle flying in a great circle path over a spherical, non-rotating earth; using these assumptions in (1.1) gives

$$\dot{h}' = V' \sin \gamma$$

$$\dot{V}' = \frac{g_o}{W} (T' \cos \alpha - D') - g \sin \gamma$$

$$\dot{\gamma} = \frac{g_o}{WV'} (L' + T' \sin \alpha) - \left(\frac{g}{V'} + \frac{V'}{r_o + h'} \right) \cos \gamma$$

The difference between this system of equations of motion and the system (1.2) is relatively small. For the steepest descent solution, γ_o was fixed at zero and γ_f was free.

The steepest descent path is shown in Figure 11 along with the approximate paths. The agreement is quite good, the major difference being the relative smoothness of the steepest descent path. The times-to-climb are compared in Table II where it is seen that the modified two-state method is in extremely good agreement with the steepest descent value. The other two approximations underestimate the time by a significant amount.

One of the most important aspects of a preliminary design study is a sensitivity analysis, i.e., a determination of those parameters which have strong effects on performance. Use of the modified two-state approximation in a sensitivity analysis is illustrated in Figure 12 for the Type III trajectory of Figure 9. It is apparent that the climb performance as measured by ϕ^* is very sensitive to thrust T', slightly less sensitive to weight W and final altitude h_f', and not sensitive to the aerodynamic parameters C_{L_α} and η.

CONCLUDING REMARKS

The minimum time-to-climb problem has been formulated as a third order system. Various approximate solutions based on reduced order systems have been developed, discussed, and compared. It was found that the energy state approximation has undesirable features and may significantly underestimate the minimum time-to-climb. The two-state approximation is an improvement over the energy state but has the undesirable feature of resulting in a discontinuous flight path angle history. The modified two-state approximation is developed to overcome this deficiency and is thought to be the superior approximation. Consideration of a numerical example showed good agreement between the values of minimum time-to-climb as predicted by the modified two-state approximation and by a steepest descent solution. A sensitivity analysis indicated thrust and weight had a large effect on climb performance.

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TABLE 1.— MINIMUM TIME-TO-CLIMB BY VARIOUS APPROXIMATIONS.

Time, sec	Energy state	Two state ($D_L = 0$)	Two state ($D_L = D_{L_1}$)	Mod two state $L_M = 7$, $L_m = -2$ γ_0, γ_f free	Mod two state $L_M = 7$, $L_m = -2$ $\gamma_0 = 0$, γ_f free	Mod two state $L_M = 7$, $L_m = -2$ $\gamma_0 = \gamma_f = 0$
t_1	0	0	0	0	8	8
t_{12}	0	14	14	14	14	14
t_2	0	0	0	8	8	8
t_{23}	46	36	36	36	36	36
t_3	0	0	0	2	2	2
t_{34}	58	44	44	44	44	44
t_4	0	0	0	14	14	14
t_{45}	0	38	39	39	39	39
t_5	0	0	0	0	0	18
Minimum time-to climb, ϕ^*	104	132	133	157	165	183

t_{NM} = Elapsed time on arc NM

t_N = Elapsed time at point N

TABLE 2.— COMPARISON OF MINIMUM TIME-TO-CLIMB BY VARIOUS APPROXIMATIONS WITH STEEPEST DESCENT SOLUTION, $\gamma_o = 0$, γ_f FREE.

	Minimum time-to-climb, ϕ^*, sec
Energy state†	104
Two state†	132
Mod. two state	165
Steepest descent	162

† Not influenced by boundary conditions on γ

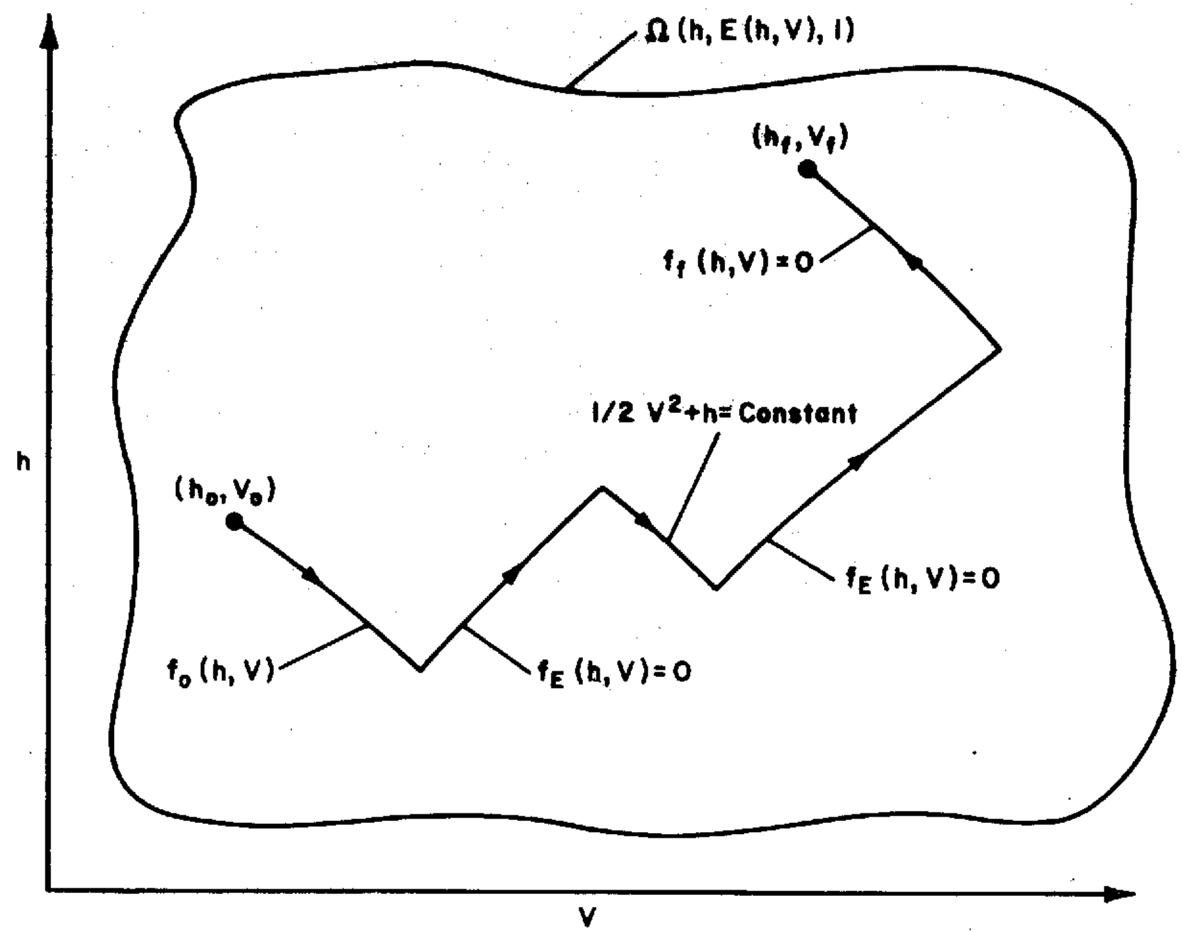

FIGURE 1.– Sketch of a typical energy-state minimum time-to-climb path.

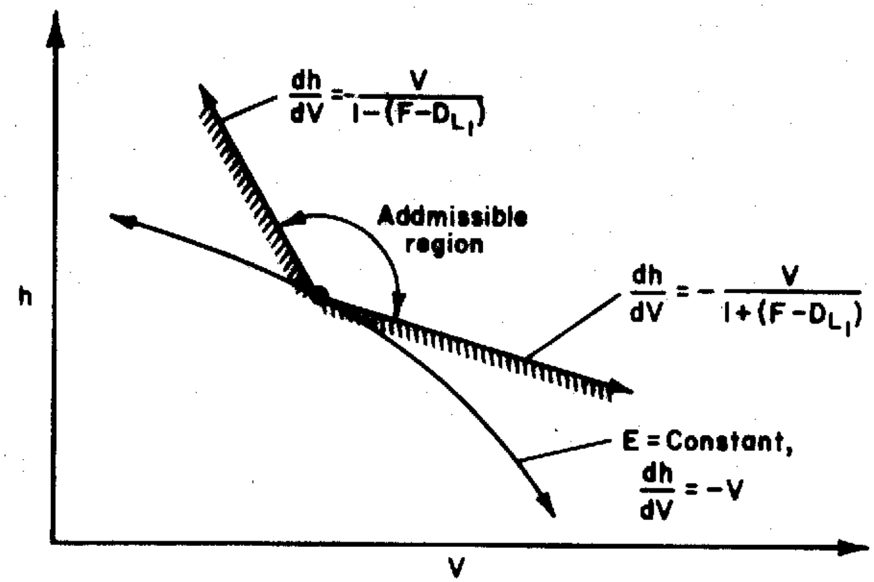

FIGURE 2.— Sketch showing relation of constant energy paths to admissible region.

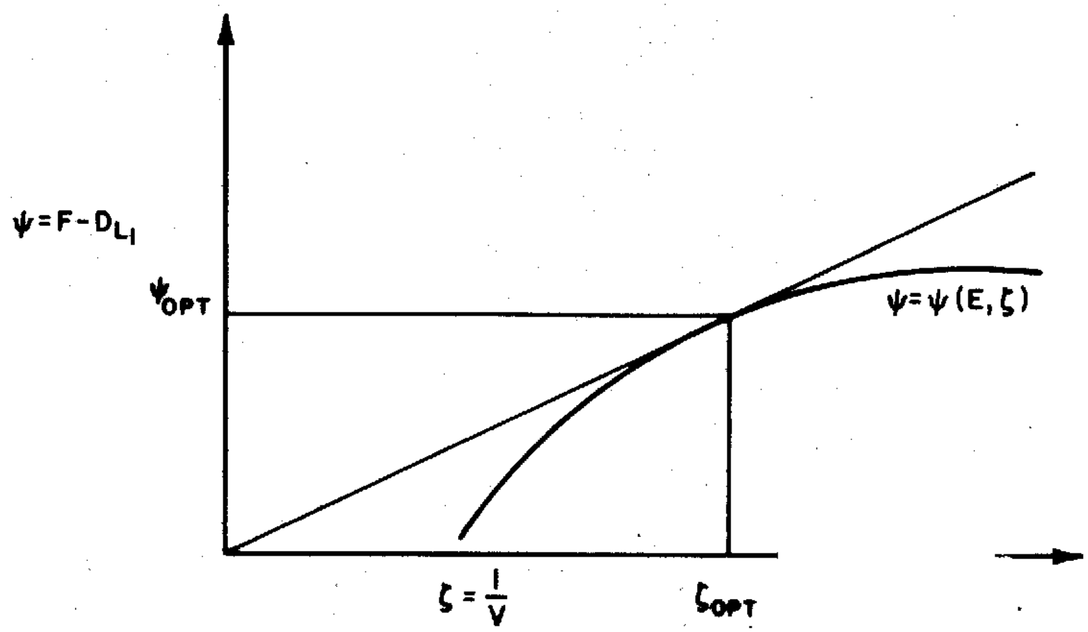

FIGURE 3.— Graphical construction of energy-state approximation.

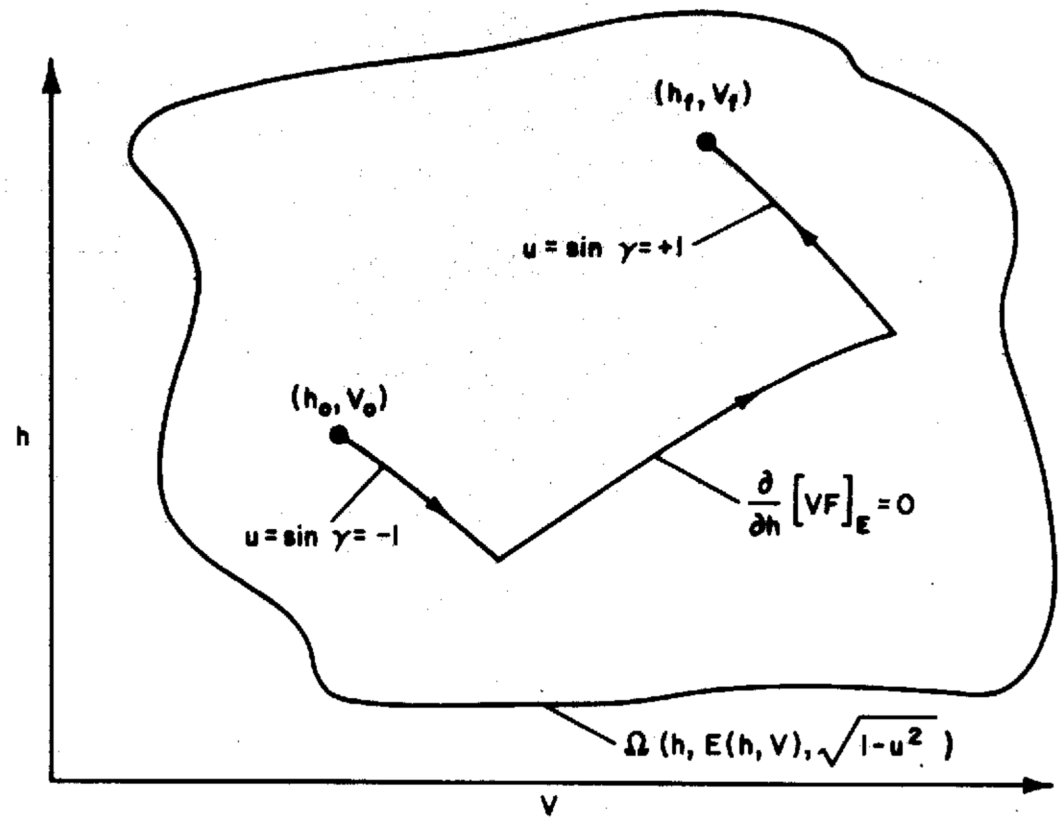

FIGURE 4.— Sketch of a typical two-state minimum time-to-climb path.

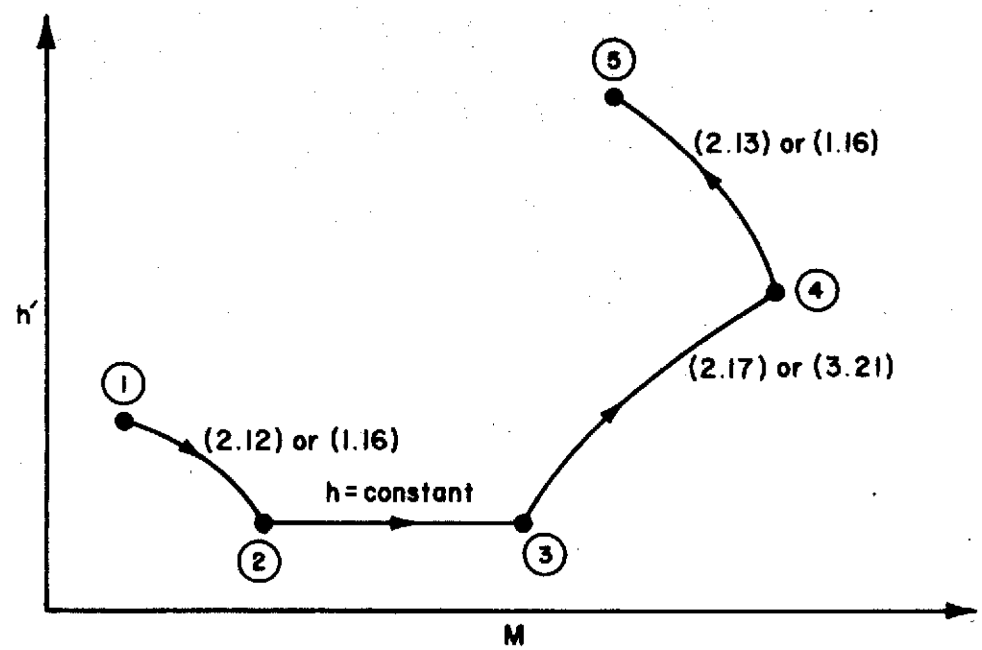

FIGURE 5.— Sketch of minimum time-to-climb path in (M,n') plane.

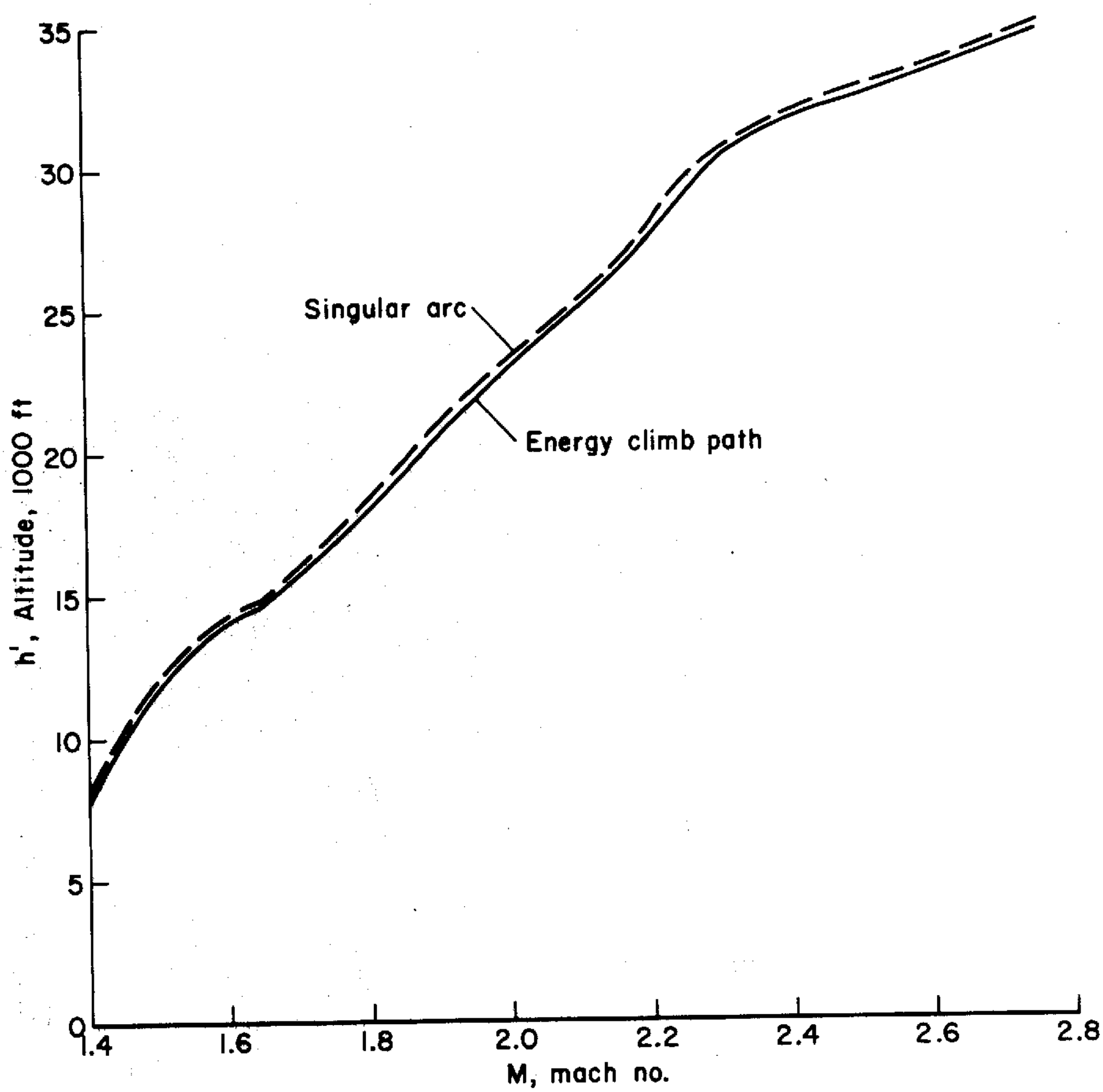

FIGURE 6.– Singular arc and energy climb path.

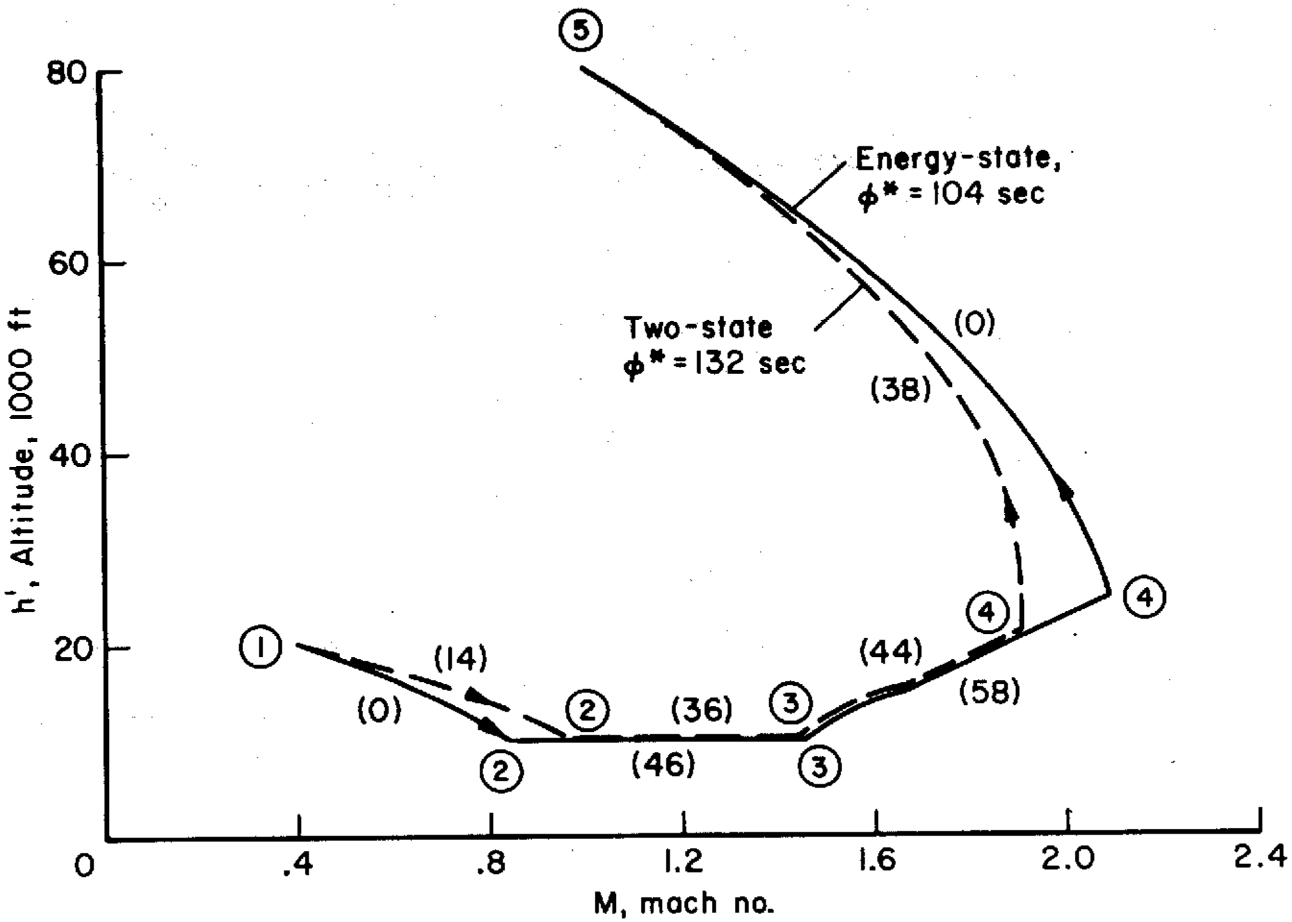

FIGURE 7.— Comparison of energy state and two-state trajectories.

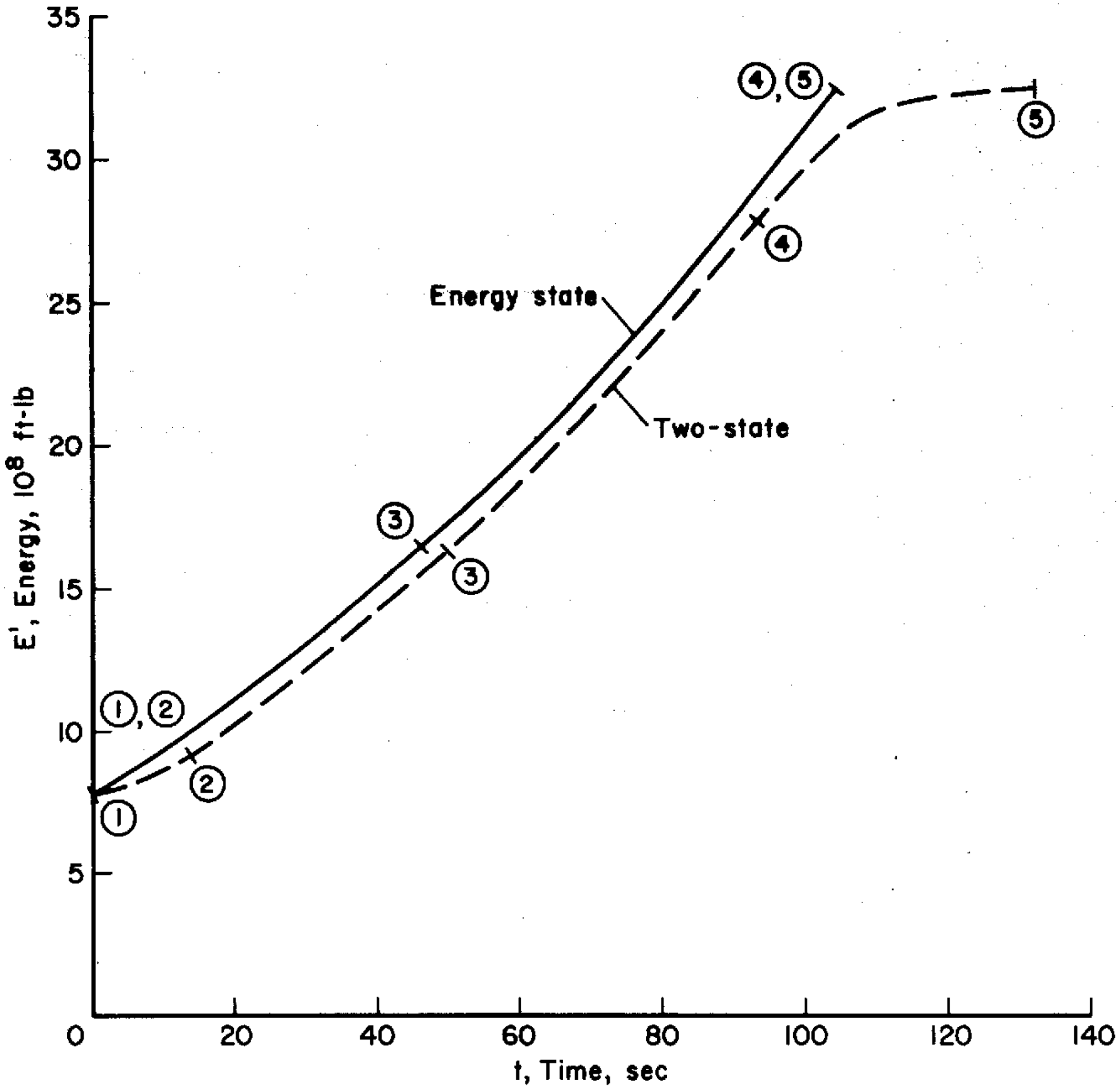

FIGURE 8.— Energy time histories.

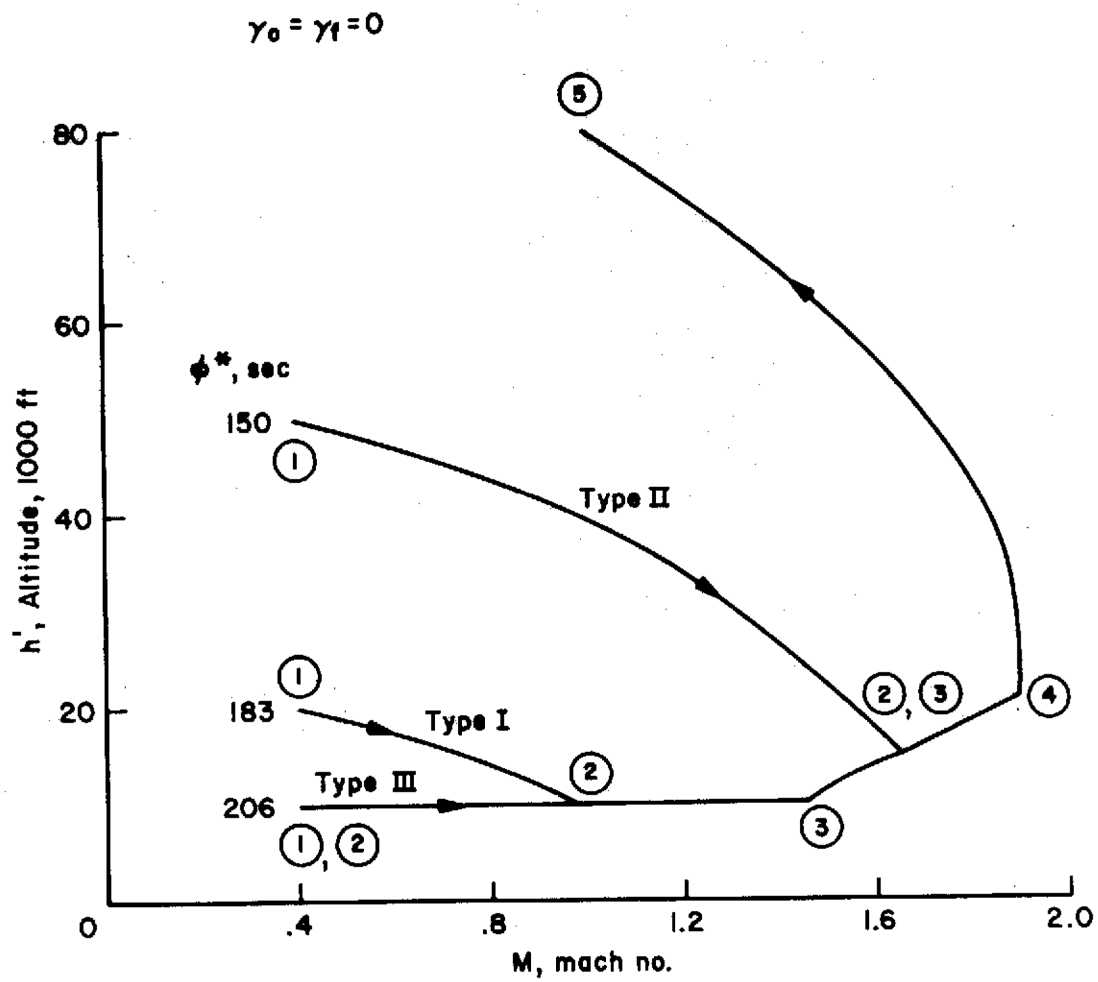

FIGURE 9.— Modified two-state trajectories.

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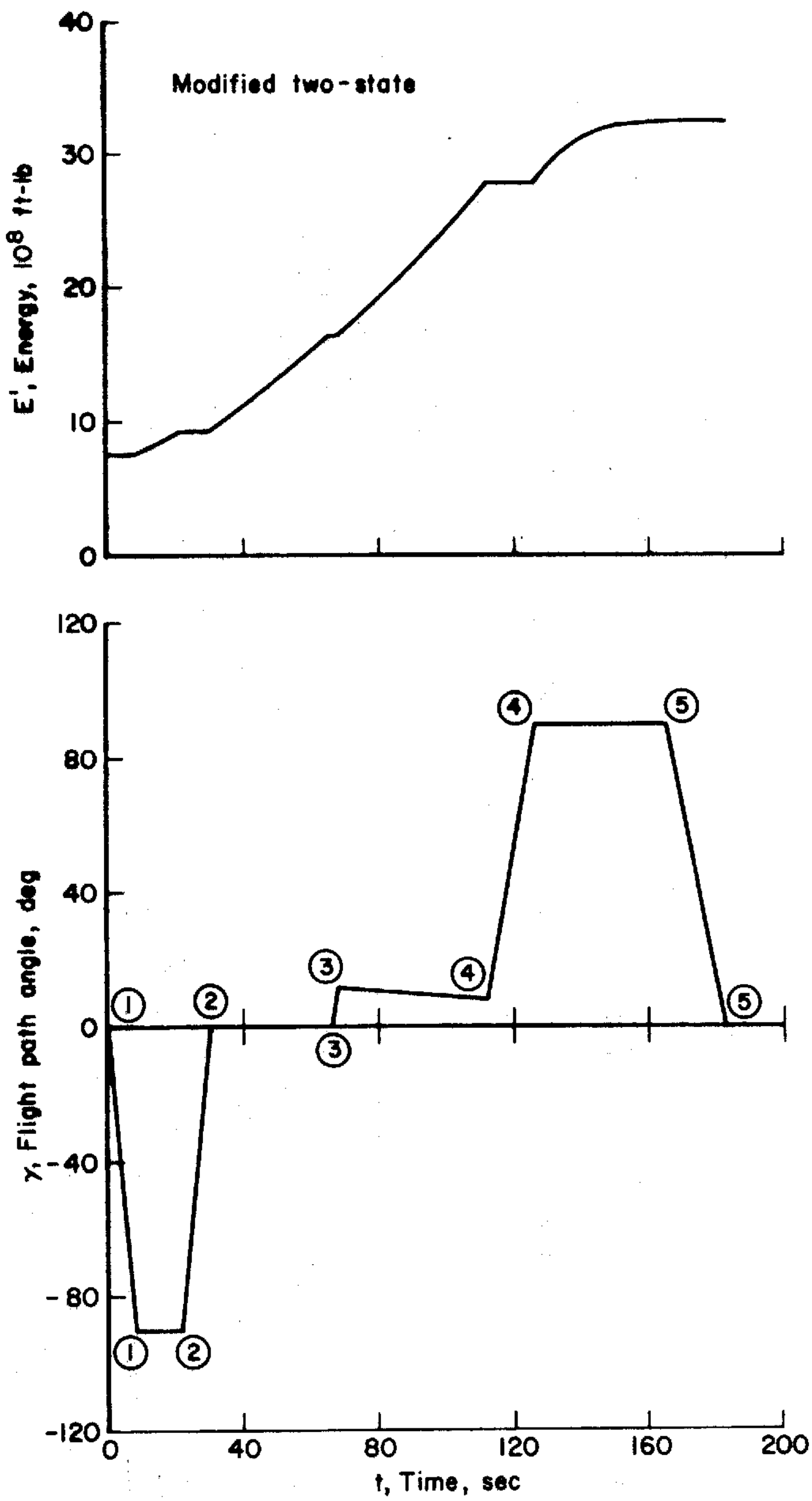

FIGURE 10.— Time histories.

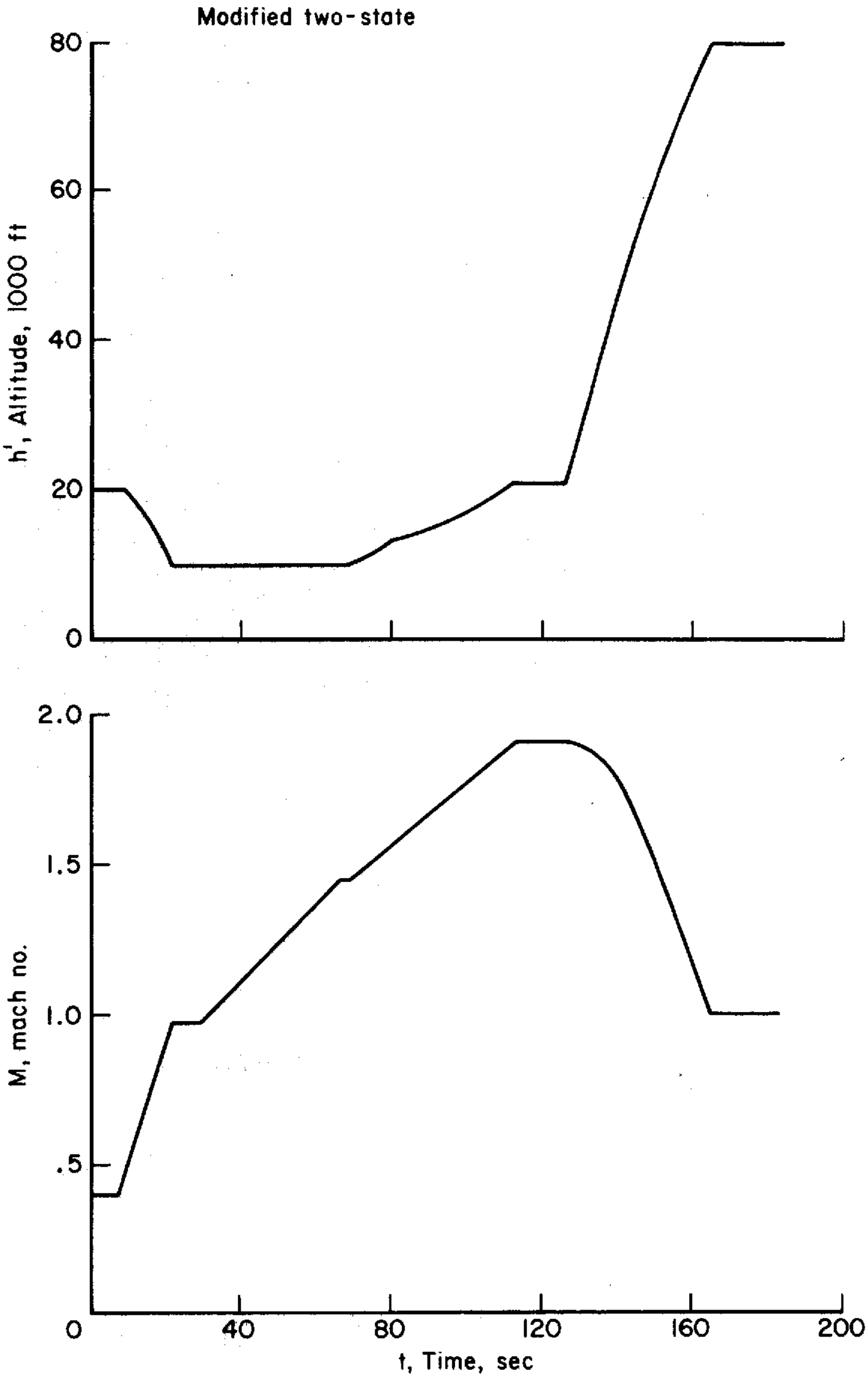

FIGURE 10.— Time histories (concluded).

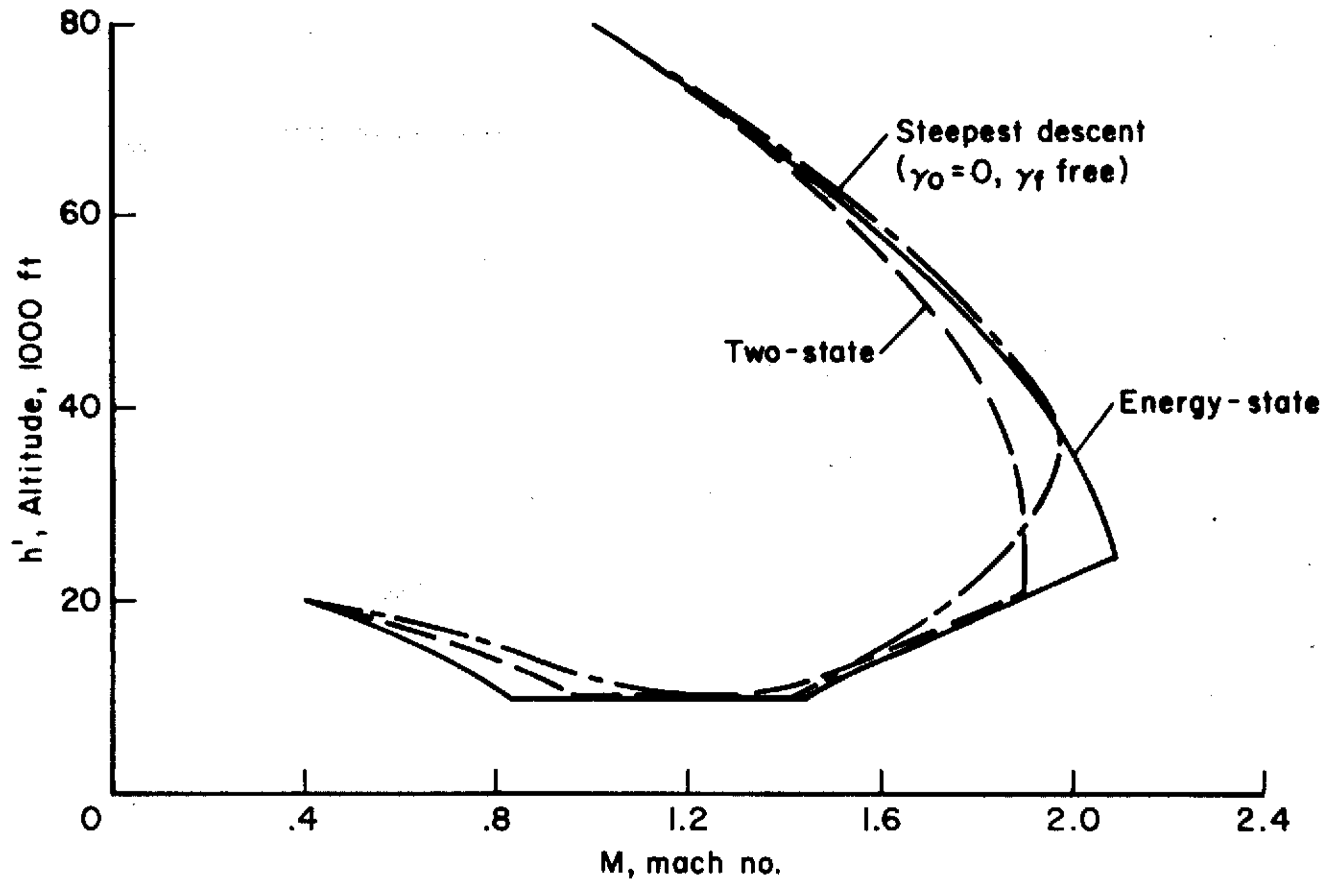

FIGURE 11.— Comparisons of approximations with steepest descent.

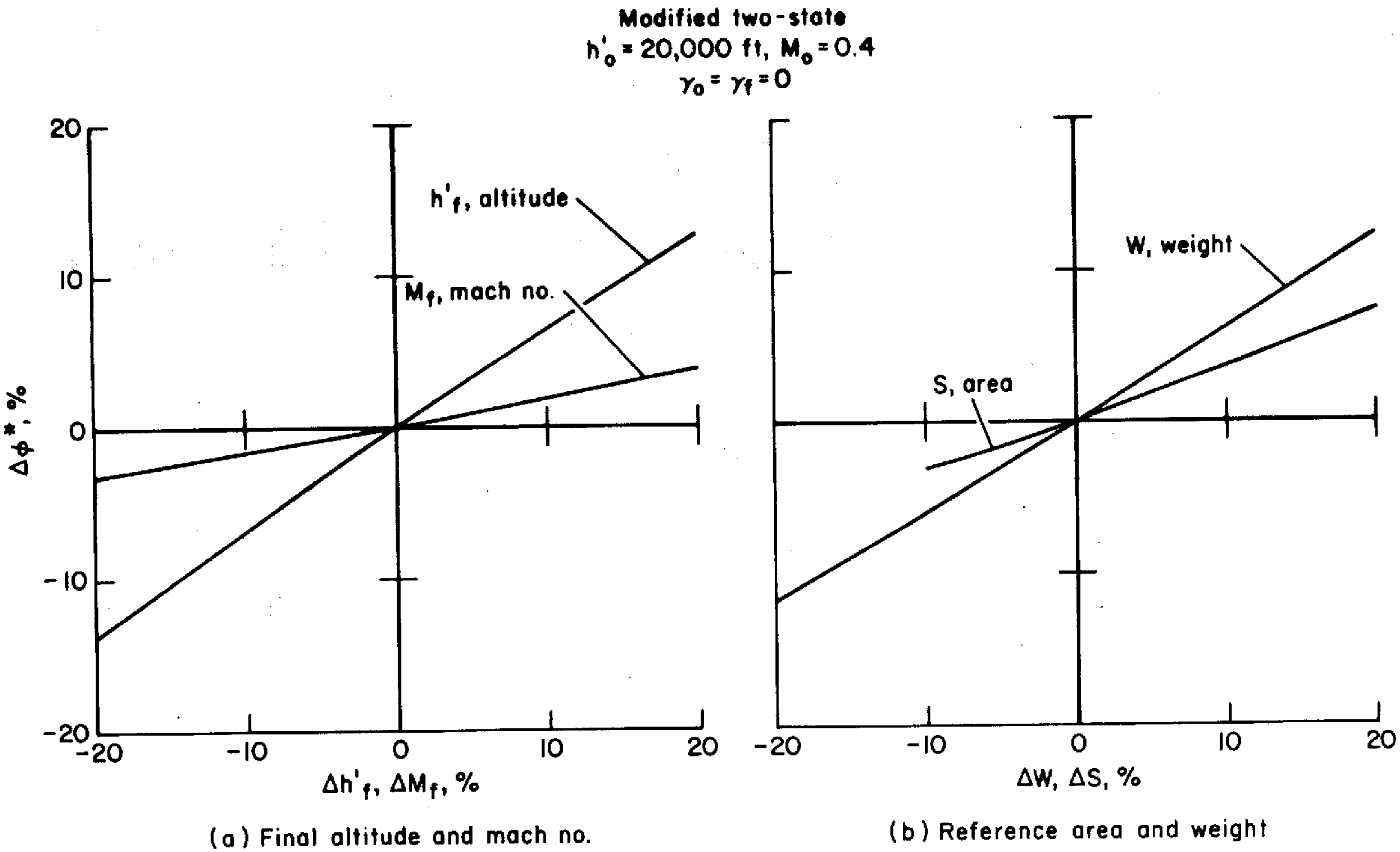

FIGURE 12.— Sensitivities.

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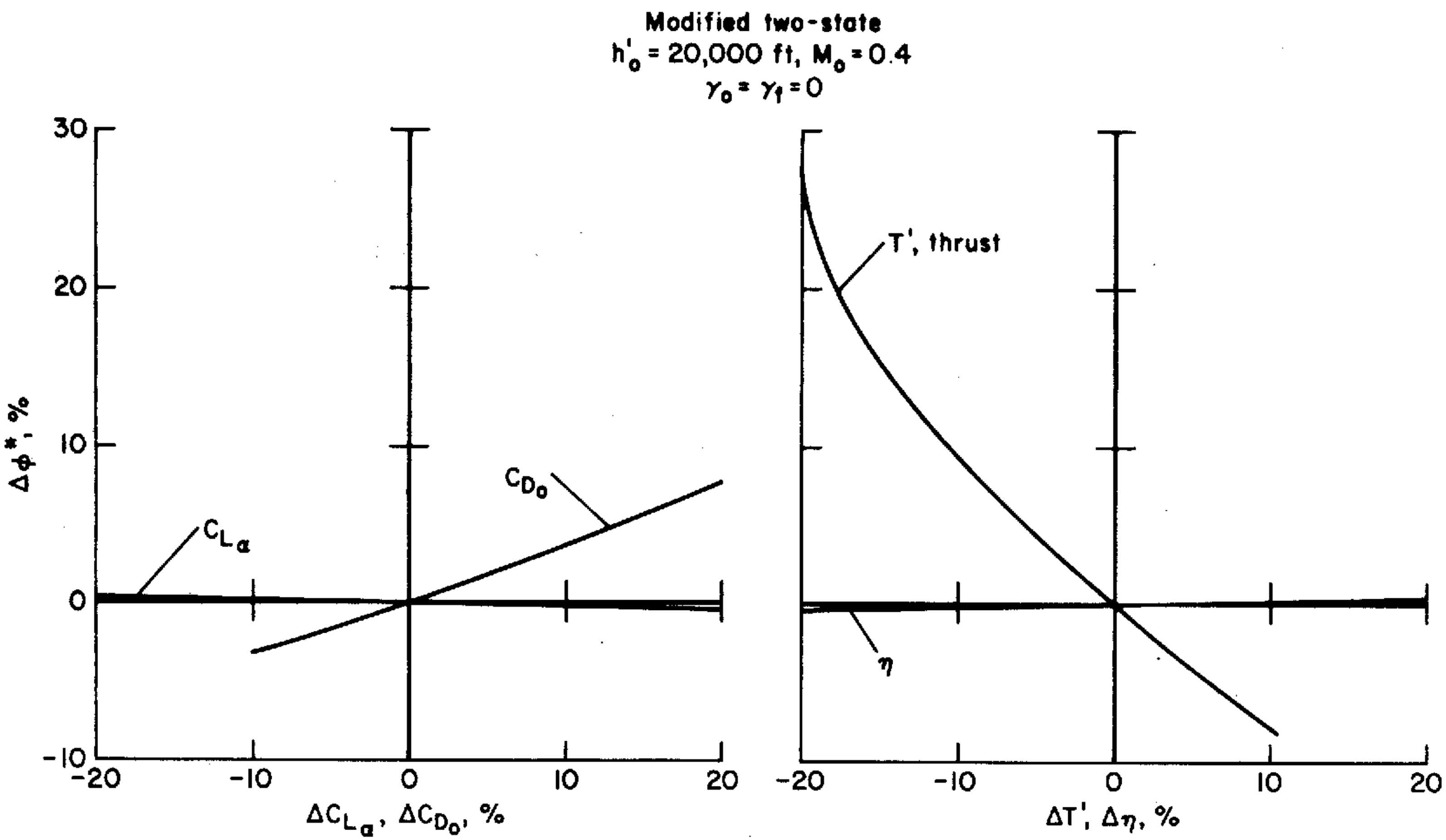

(c) Lift coefficient and zero lift drag

(d) Thrust and drag due to lift

FIGURE 12.— Sensitivities (concluded).