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THE ADAPTIVE OBSERVER

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CHAPTER I

THE BEGINNING

1.1 Techniques of System Control

The underlying notion of modern control system theory is the acceptable control of a process that has the inherent capability of being influenced. The process is mathematically described in a way so as to potentially define its behavior completely for any choice of control stimulus; the collection of all variables for defining the process behavior is called the *state*. The primary objective of control system theory is to define the evolution of a control stimulus in order to cause the evolution of the process state to behave in a desired manner.

In some cases, the control may not be dependent upon the system state. However, for most instances the control does depend upon the system state and consequently at least some of the variables in the state evolution must be measured in order to construct the evolution of the control: such a situation is called *feedback control*.

For a significant percentage of applications, it is either impossible or undesirable - perhaps from a cost standpoint - to measure all the state variables called for in the control law. In this instance, the choice is either to alter the criteria defining the acceptability of the state behavior so as to eliminate the necessity of measuring the unavailable state variables, or to find a simple way to make the unavailable measurements that are needed. Moreover, with the presently known criteria of system control, the conditions for which the desired behavior of the system state can be made to depend only upon available measurements are not always clearly understood.

1.2 *Purpose of Work Herein Reported*

This dissertation adopts the second choice noted above, namely the simple generation of state from the available measurements, for use in systems for which the criteria defining the acceptable state behavior mandates a control that is dependent upon unavailable measurement. (It is assumed that the system structure allows such a generation of state.)

To be sure, this approach has been actively investigated previously [1]-[4] for systems in which the parameters are known. This dissertation proposes an adaptive means for determining the state of a linear time-invariant differential system having unknown parameters by using only available measurements. This procedure is called an *adaptive observer.*

The adaptive observer not only generates the state of a linear time-invariant dynamical system having unknown parameters but also simultaneously *identifies* all or some - dependent upon the observer structure - of the system parameters.

The adaptive observer possesses some noise suppression qualities, dependent upon a free choice of observer eigenvalues. In the absence of noise the adaptive algorithm employed is guaranteed to converge to the proper values regardless of the magnitude of parameter ignorance or of the magnitude of the adaptive gain constants selected, and requires no derivatives of output or of other measurements for implementation. There is inherent freedom in adjusting the adaptive rate of convergence. Some restriction, however, is placed upon the system input to insure parameter identification.

The adaptive observer appears in either a full-order version (that is, of same dynamic order as the system being observed, discounting the adaptive algorithm) for single-output systems or a reduced order observer for multiple output systems. The full-order adaptive observer completely identifies the system parameters and generates as well the entire state of the single-output system. The reduced-order observer generates the remainder of system state information needed to completely construct, along with the output information, the state of the system, and partially identifies the parameters of the system.

1.3 Organization of the Dissertation

The major contribution of this dissertation appears in Chapters IV and VI. Chapter IV has appeared substantially in [71] and Chapter VI in [72]. In these two chapters, the single-input single-output adaptive observer and the reduced adaptive observer is developed. Chapter V surveys the investigations into the adaptive observer by other authors undertaken subsequent to the initial report by this author of the material in Chapter IV. Chapter II examines the basic ideas for the non-adaptive (Luenberger) observer for linear time-invariant dynamical systems with known parameters, and in doing so lays a foundation for the construction of Chapters IV and VI. Chapter III is a survey of the Lyapunov synthesis technique. Lyapunov synthesis is employed in the adaptive algorithm for the adaptive observer. This survey has appeared as [69] and contains 40 references. Chapter VII outlines a considerable amount of work left for future research.

CHAPTER II

THE NON-ADAPTIVE OBSERVER

2.1 The Worth of The Observer

Unlike the Kalman filter [1], the Luenberger observer [2,3,4] for generating the state of a system using input and output measurements is theoretically postulated in a noise-free environment. The worth of the observer, however, rests ultimately upon the likelihood that noise is inherent in the available measurements.

To understand the motivation for the observer, consider the three following means of generating the state of the system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix} r \tag{II.1}$$

$$y = [1 \quad 0 \quad 0]\, x = x_1$$

when given the input r, the output y, the structure of the system matrix, input matrix, and output matrix, and the values of the parameters in each of these matrices. Although these three methods are discussed here with respect to (II.1) for illustration, the comments apply to a general linear system.

One possibility is to determine x_2 and x_3 by differentiators, illustrated in Figure 2.1(a). This method simply recognizes that

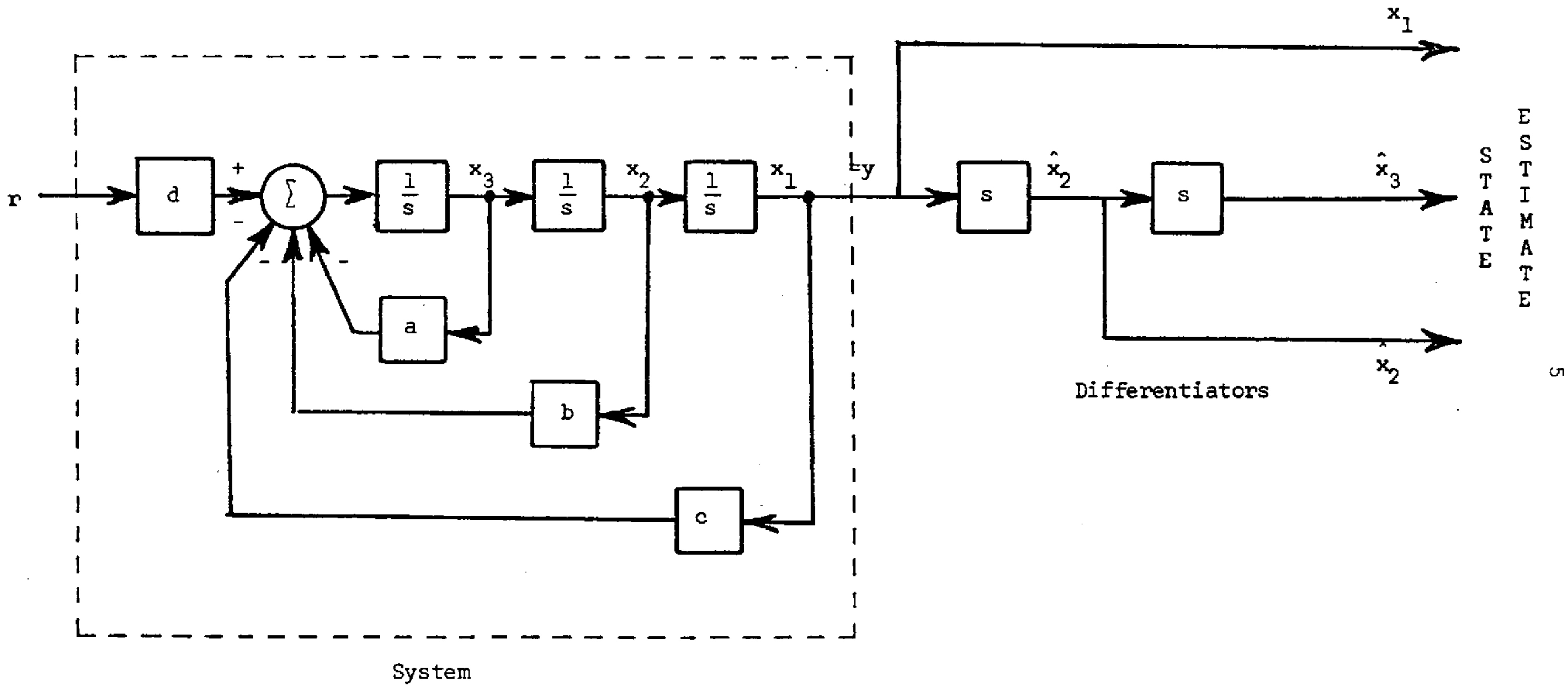


Figure 2.1 (a)

Generation of State by Differentiation

$$x_2 = \dot{x}_1 = \dot{y}$$
$$x_3 = \dot{x}_2 = \ddot{x}_1 = \ddot{y}$$

from (II.1). Thus one may ascertain x_2 by differentiating y and x_3 by differentiating y twice. In the absence of measurement noise introduced at any point within or without the system, and by assuming the availability of a perfect differentiation algorithm, this differentiation technique is successful. However, in the presence of broad band or high-frequency noise (even assuming the feasibility of a perfect differentiator), the magnitude Bode-plot characteristic of a differentiator prohibits a meaningful result to emerge. Since noise within a system, or in the differentiation itself, is very common, the differentiation technique has little value.

The parallel-model scheme, illustrated in Figure 2.1(b), is based upon the certain knowledge of the system parameters and structure. A model is constructed identical to the system, but so as to allow access to the model state (unlike the corresponding system, in which only the output is accessible). The system state is identical to the model state when both the system and the model are in a noiseless environment, when the same input is applied to each, and when the initial conditions are the same in each. The initial condition requirement may be waived if the system is stable, since then the effect of the initial condition eventually vanishes.

The state of the model is therefore used in place of the system state to implement a control law. This arrangement is an open loop process in

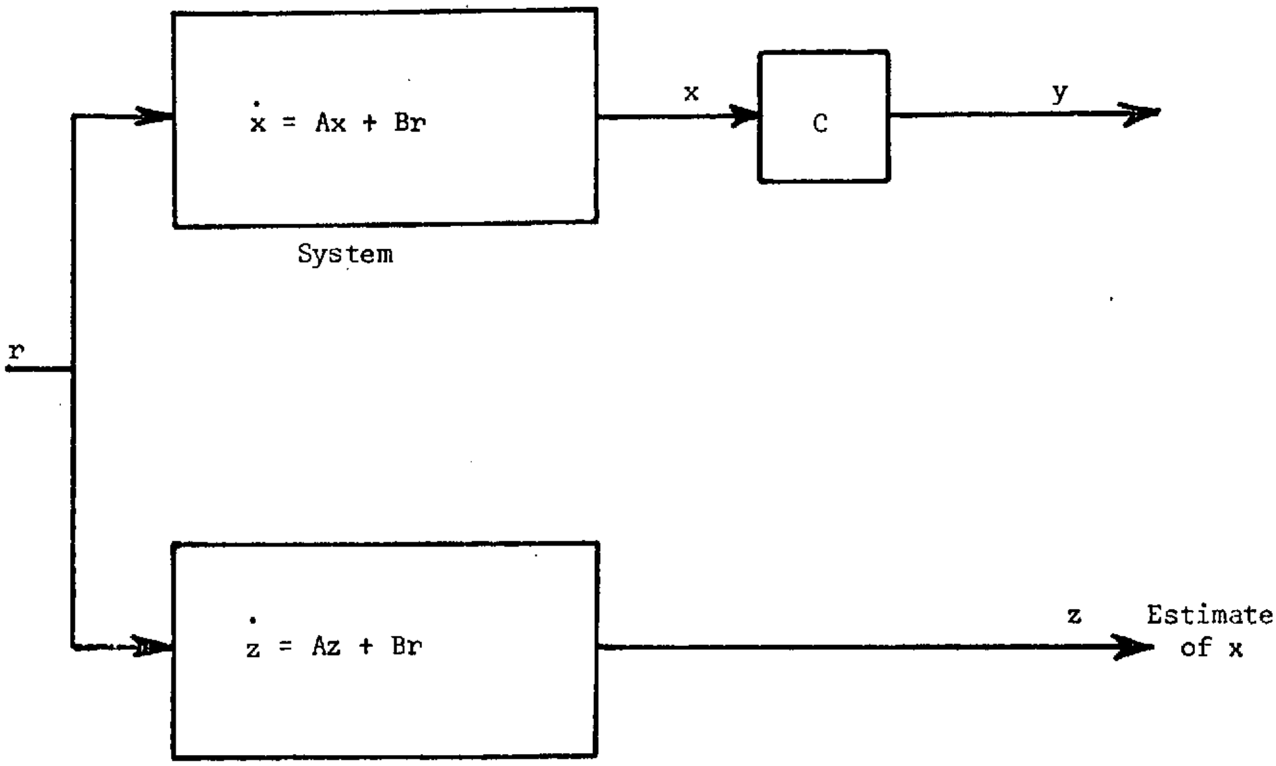


Figure 2.1 (b)

Generation of State By Parallel-Model Scheme that the output of the system is not connected to the input.

The lack of feedback in the system control law does not allow the decreased sensitivity to disturbances - occurring at the system input, within the system, or in the output measurements - that is a beneficial property of feedback control. Rather, noise originating within the system or (if possible) within the model seriously affects the behavior of the system, especially since the system may well receive disturbances unequal to that received by the model. Consequently, while the parallel-model scheme is more valid than the differentiation scheme, it has limited usefulness in a noisy environment.

The third scheme is the model-following scheme of Figure 2.1.c). The observer is characterized by this scheme.

When the model estimate of the system state is used to control the system, the result is feedback control. Moreover, it will be later seen that the model-following scheme allows an inherent capability of noise suppression not present in the other two schemes, an additional advantage.

From the preceeding it may be readily perceived that, although the observer is postulated theoretically to be implemented in a disturbance-free environment, its worth rests in the inherent ability of the observer both to be less sensitive to the origination of measurement noise and to possess the capability to suppress the adverse effects of noise.

However, the observer is not necessarily optimal with regard to noise suppression as is the Kalman filter. Consequently, the usefulness of the observer is mainly limited to those cases for which the signal-

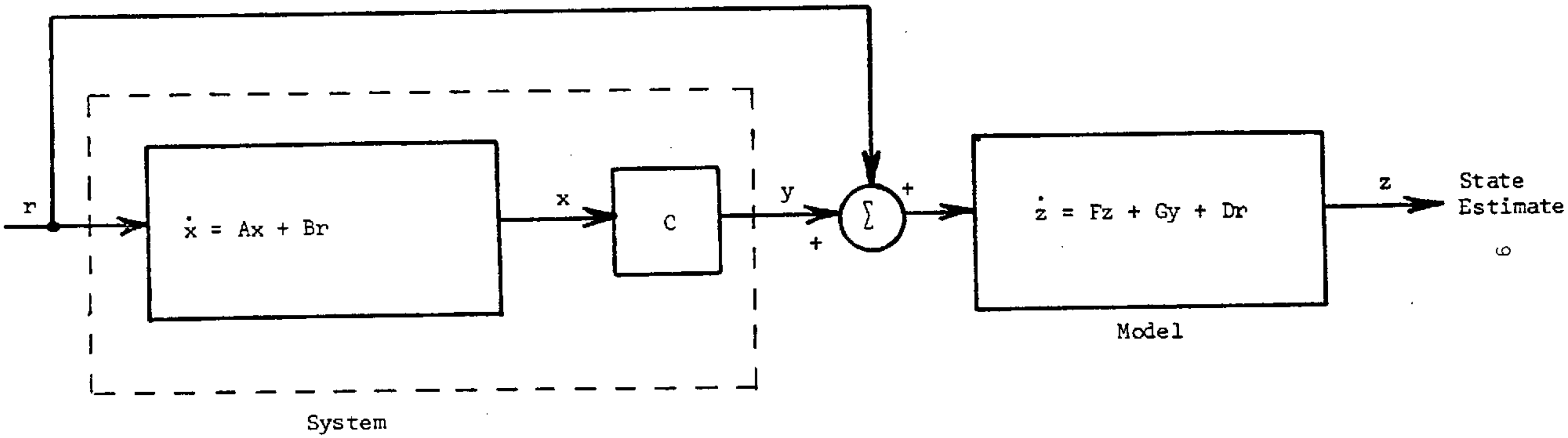


Figure 2.1 (c)

Generation of State By Model Following Process to-noise ratio is high enough so that it is not considered worthwhile to implement the optimal Kalman filter. Since the various forms of the observer allow considerable reduction in complexity, when compared with the Kalman filter, the use of the observer rather than the Kalman filter, when noise considerations allow, presents an attractive alternative.

2.2 *The Full-Order Observer*

The basic notion of the model-following scheme illustrated in Figure 2.1(c) is that the model, which is fed by the output of the system whose state is to be observed, tends to track a linear transformation of the system state if the model is stable. It is this basic tendency which allowed Luenberger [2] to formulate the particular dynamics of the model so that it eventually perfectly tracks a transformation of the system state. When the model is designed so, it is called an observer.

The full-order observer is of the same dynamic order as that of the system. The Luenberger observer [2,3,4], which is discussed in this section, is defined for a linear time-invariant differential system in the absence of a disturbance vector. This system may be described by the equations

$$\dot{x} = Ax + Br \qquad \text{(II.2)}$$
$$y = Cx$$

in which $x \in \mathcal{E}^n$ is the system state, $r \in \mathcal{E}^m$ is the system command input vector, $y \in \mathcal{E}^p$ is the system output, and A, B, and C are appropriately-dimensioned matrices which are known and constant with time. It is assumed that the pair (C,A) is completely observable. Moreover, it is assumed without restriction that C is of rank p.

The observer is hypothesized as

$$\dot{z} = Fz + Gy + Dr \tag{II.3}$$

for which $z \in \mathcal{E}^n$ is the estimate of the system state x. By constructing the observer (II.3) so that the observer state z is available for measurement and by defining F, G, and D so that $z \to x$, then the state of the system x can be determined by measuring z rather than x. This is accomplished, it is noted, by utilizing only the available system measurements y and r in the observer (II.3).

Rather than make $z \to x$ as just described, the system (II.2) can first be transformed by the non-singular square matrix T by defining $x = T\bar{x}$; then (II.2) becomes

$$\begin{aligned} \dot{\bar{x}} &= T^{-1}AT\bar{x} + T^{-1}Br \\ y &= CT\bar{x} \end{aligned} \tag{II.4}$$

Now the observer may be built so that $z \to \bar{x}$ or that $z \to T^{-1}x$.* Consequently a transformation of the state of the system (II.2) may be observed. This additional freedom possibly allows greater flexibility in actually constructing the observer.

An error vector $e \in \mathcal{E}^n$ may be defined as

$$e \equiv z - \bar{x} = z - T^{-1}x \tag{II.5}$$

in which e is a comparison between the observer state and the system state. The desired condition that $z \to \bar{x}$ or, equivalently, $z \to T^{-1}x$ is the same as requiring $e \to 0$.

To find conditions on G, F, and D in (II.3) so that $e \to 0$, a differential equation in the dependent variable e is derived and conditions are imposed so that the equilibrium $e = 0$ is asymptotically

* This unusual notation is employed in anticipation of later chapters.

stable.

Differentiating (II.5) gives

$$\dot{e} = \dot{z} - \dot{\bar{x}}$$

$$= Fz + Gy + Dr - T^{-1}AT\bar{x} - T^{-1}Br$$

which may be rewritten as

$$\dot{e} = Fe + (F-T^{-1}AT + GCT)\bar{x} + (D - T^{-1}B)r \qquad \text{(II.6)}$$

In (II.6), suppose that F is an asymptotically stable matrix [5]; then e is bounded whenever the inputs $(F - T^{-1}AT + GCT)\bar{x}$ and $(D - T^{-1}B)r$ are bounded. To make $e \to 0$ it is sufficient to make $F - T^{-1}AT + GCT = D - T^{-1}B = 0$. Doing this, F, G, and D are defined so that

$$FT^{-1} = T^{-1}A - GC \qquad \text{(II.7)}$$

$$D = T^{-1}B$$

Then (II.6) becomes

$$\dot{e} = (T^{-1}AT - GCT)e \qquad \text{(II.8)}$$

and the equilibrium of e is asymptotically stable if the eigenvalues of $T^{-1}AT - GCT$ all have negative real parts. This implies that the initial condition of e vanishes at a rate dependent upon the values of the eigenvalues of $T^{-1}AT - GCT$. Thus $e \to 0$ asymptotically.

The question remains, *is it possible to make $F = T^{-1}AT-GCT$ have eigenvalues all with negative real parts for any matrices A and C?*

Luenberger showed [2] for a single-output system, and Wonham showed [6] for a multi-output system, that the answer is affirmative whenever the pair (C,A) is completely observable.

In addition, Luenberger showed that if the eigenvalues of F and A are not identical, then the non-singular matrix T exists satisfying

(II.7). This result is somewhat intuitive, since $G = 0$ in (II.7) is equivalent to saying that F and A have the same eigenvalues, and implies that the output information is omitted from the observer (II.3).

Thus with $F = T^{-1}AT - GCT$, the solution to (II.8) is

$$e(t) = \exp[F(t-t_0)]e(t_0) \qquad \text{(II.9)}$$

in which the behavior of e is explicitly obtained. The more negative the real parts of the eigenvalues of F are, the more rapidly the error between the system state and the observer estimate of that state vanishes.

However, eigenvalues of F with highly negative real parts tend to make the observer behave like a set of differentiators. For effective noise suppression F should be chosen with time constants roughly equal to the time constants of the system (II.2) [4]. It is seen that a tradeoff exists between rapid estimation of system state and effective noise suppression in the estimate.

At present, criteria for the optimal location for placing the eigenvalues of F for the various kinds of Luenberger observers has not been reported, although the Kalman filter does so for the full order observer.

If the transformation matrix T is chosen as the identity matrix, then the resulting observer configuration is known as the identity observer. For $T = I$, the equations (II.7) are

$$F = A - GC$$
$$\text{(II.10)}$$
$$D = B.$$

Then the observer estimate z approaches x.

2.3 *The Reduced-Order Observer*

In (II.2) it is seen that the output y represents information about the state x directly without necessity of generating that information. This can be illustrated explicitly by introducing the transformation.

$$\bar{x} = \begin{bmatrix} C \\ \cdots \\ M \end{bmatrix} x \tag{II.11}$$

in which M is any matrix selected so that the transformation is non-singular. Then in the transformed system, analogous to (II.4), the output equation is

$$y = [I_p \;\vdots\; 0]\,\bar{x} \tag{II.12}$$

where I_p is an identity matrix of rank p.

Using the transformation (II.11) it is seen that there is no necessity of generating the first p state variables of $\bar{x}$ by an observer since they are directly measurable in the output y according to (II.12). The question naturally arises, *is there an observer structure which generates only the lower n-p state variables of the state $\bar{x}$, and if so, is there an advantage in employing it rather than the full order observer?*

The answer to both questions is affirmative. The advantage of the observer structure which generates only the missing state variables in (II.12) is that its dynamic order is n-p rather than the full-order n dimension.

The reduced-order observer is due to Luenberger [3], but the development given below follows Gopinath [7].

It is assumed that transformation (II.11) has been made (if necessary) of system (II.2) so that the output equation is as (II.12).

Then (II.2) is written as

$$\dot{x} = Ax + Br$$

$$y = [I_p \vdots 0]x \tag{II.13}$$

in which it is understood that A and B have been transformed by (II.11) and the notation $\bar{x}$ has been dropped in favor of x for simplicity.

Now (II.13) is partitioned. The first p state variables of x equals y, and the remaining are denoted as w so that

$$x = \begin{bmatrix} y \\ w \end{bmatrix}$$

Then accordingly (II.13) is written as

$$\dot{y} = A_{11}y + A_{12}w + B_1r$$

$$\dot{w} = A_{21}y + A_{22}w + B_2r \tag{II.14}$$

for which y is measurable and $w \in \mathcal{E}^{n-p}$ is not.

The reduced observer is hypothesized as

$$\dot{z} = Fz + Ly + Dr \tag{II.15}$$

in which $z \in \mathcal{E}^{n-p}$ is an estimate of a function of w as illustrated in Figure 2.2.

An error vector is defined as

$$e = z + Gy - w$$

The object is to make $e \to 0$; then $z + Gy = w$. To insure $e \to 0$, the matrices L, F and D are to be chosen so that the error equation is asymptotically stable about its equilibrium $e = 0$. The underlying motivation in accomplishing this is to construct an identity observer for the second equation in (II.14) using y and r as multiple inputs.

From above,

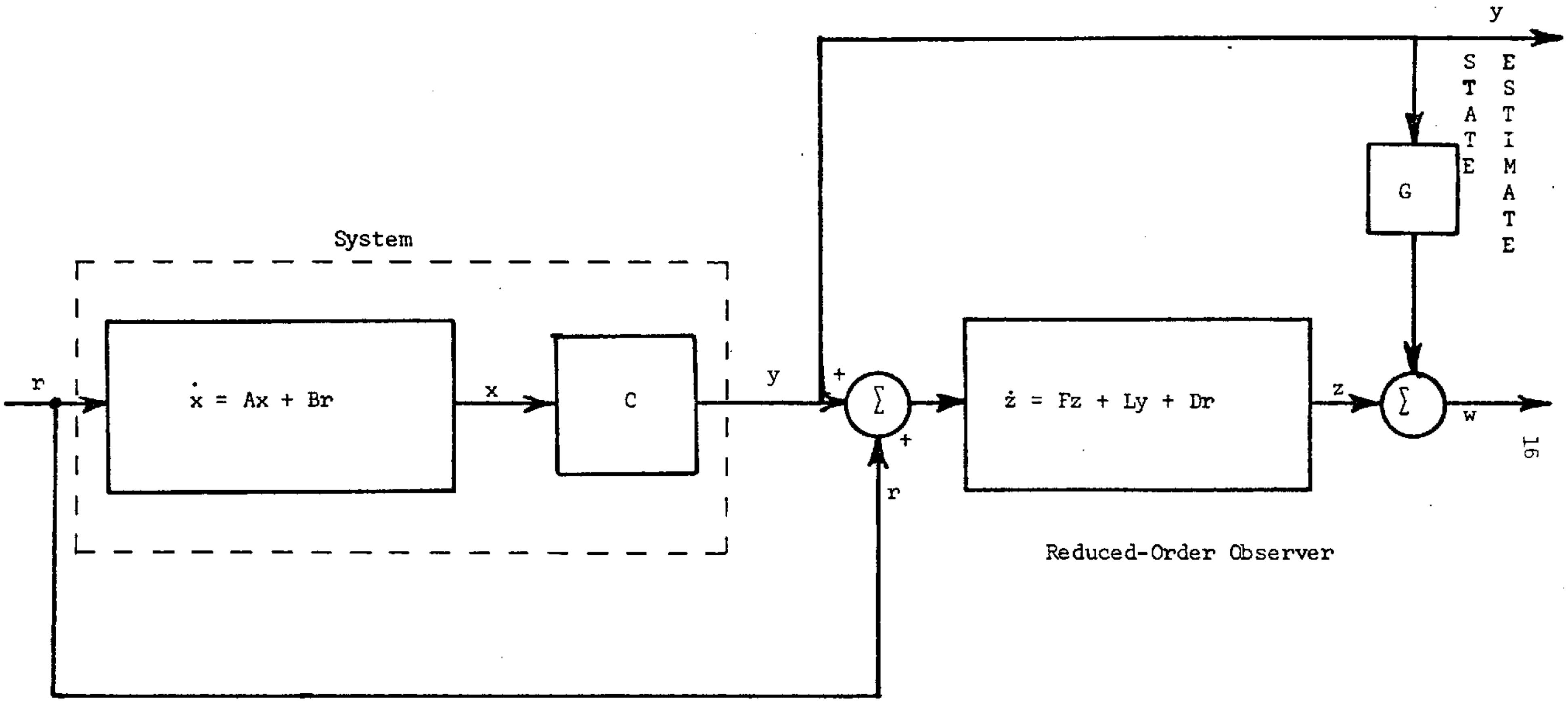


Figure 2.2

Reduced-Order Observer

$$\dot{e} = \dot{z} + G\dot{y} - \dot{w}$$

$$= [Fz + Ly + Dr] + G\dot{y} - [A_{21}y + A_{22}w + B_2r]$$

$$= Fe + (F - A_{22})w + (L - FG - A_{21})y + G\dot{y} + (D-B_2)r$$

Suppose that $L = FG + A_{21} - GA_{11}$. The above becomes

$$\dot{e} = Fe + (F-A_{22})w + G(\dot{y} - A_{11}y) + (D - B_2)r$$

Since, from (II.14),

$$\dot{y} - A_{11}y = A_{12}w + B_1r$$

the error equation is equivalent to

$$\dot{e} = Fe + (F - A_{22} + GA_{12})w + (\dot{D} + GB_1 - B_2)r \qquad \text{(IV.16)}$$

Equation (II.16) is analogous to (II.6), which is the full-order error equation. As in the full-order equation, if F and D are defined as

$$F = A_{22} - GA_{12}$$
$$\qquad \text{(II.17)}$$
$$D = B_2 - GB_1$$

then (II.16) becomes

$$\dot{e} = Fe \qquad \text{(II.18)}$$

for which the discussion following (II.8) applies here.

The n-p eigenvalues of F may be arbitrarily chosen if and only if the pair (A_{22}, A_{12}) is completely observable [6]. It has been shown [7] that if (C,A) is completely observable, then so is (A_{22}, A_{12}). (The simple proof is omitted here.) Consequently the eigenvalues of F may be arbitrarily chosen when observing the state of system (II.2).

The observer for the system is therefore

$$\dot{z} = (A_{22} - GA_{12})z + (A_{22} - GA_{12})Gy + (A_{21} - GA_{11})y$$
$$+ (B_2 - GB_1)r \qquad \text{(II.19)}$$

and the estimate of **x** is

$$\begin{bmatrix} y \\ \cdots \\ z+Gy \end{bmatrix} \equiv \hat{x}$$

for the state $x \in \mathcal{E}^n$ of system (II.13).

One important point regarding this reduced-order observer will be made here. The particular form (II.19) of the reduced observer derived here is but one of many possible. That is to say, the system (II.13) may be transformed by a non-singular matrix T, analogous to (II.4), to allow many different forms of the observer equation (II.15).

2.4 Closed-Loop Properties of the Observer

An important feature of the observer is the ability to not adversely affect the stability of a closed-loop system employing the observer within the loop. An example of this application, illustrated in Figure 2.3, is the pole placement problem. The state estimate generated by the observer is used in a linear time-invariant feedback to adjust the system poles. Since the observer estimate of the system state is used rather than the (unmeasurable) system state, it would be undesirable - yet conceivable - that the system becomes unstable in the closed loop due to the presence of the observer. However, Luenberger showed [2] that the introduction to the observer within a closed-loop employing a linear time-invarient feedback does not affect the stability of the system. In fact, the poles of the composite system employing the observer are exactly the poles of the system, assuming the control law enjoyed state feedback, plus the poles of the observer. Since the observer poles may always be chosen stable, the overall stability of a closed-loop system is not affected when generating the state by means of an observer.

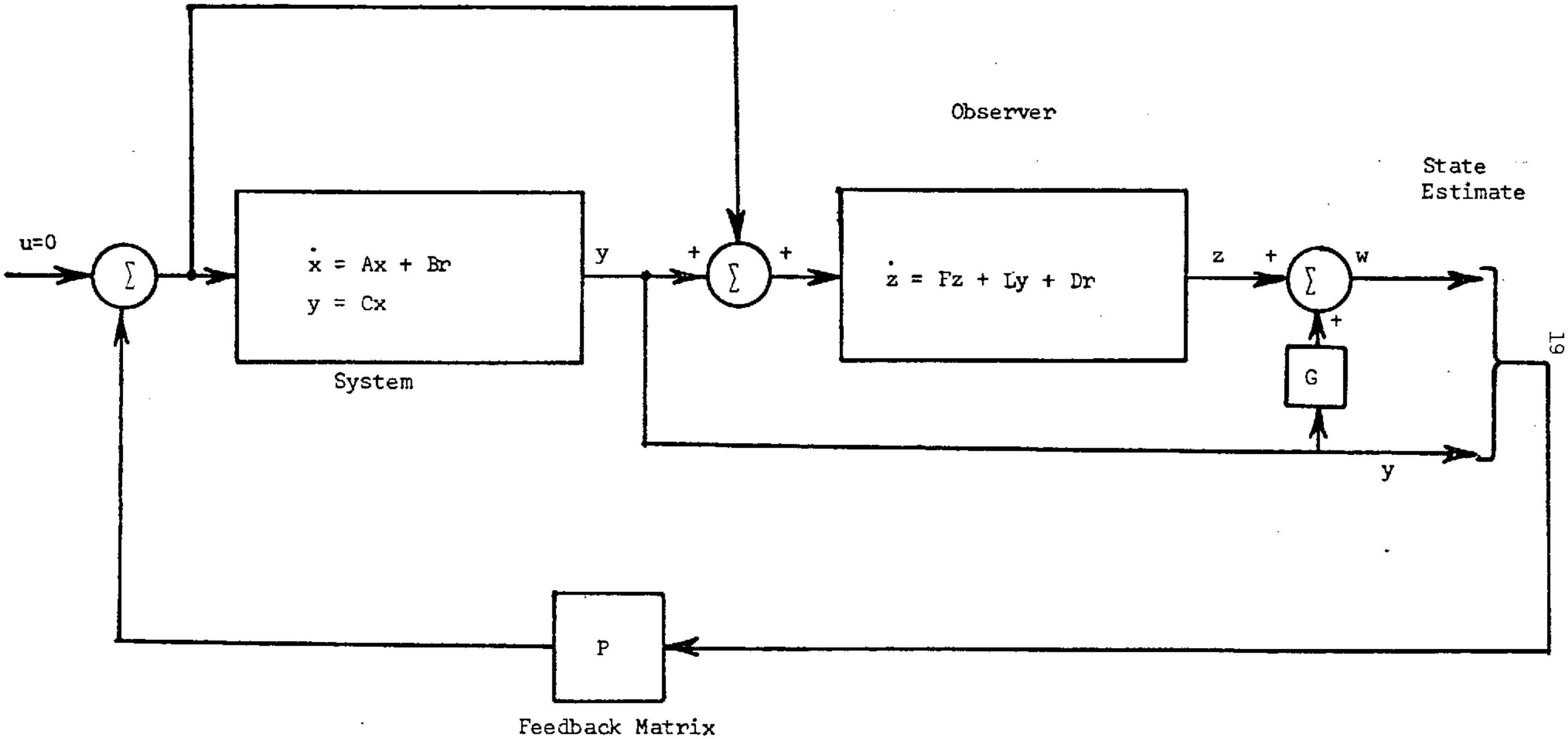


Figure 2.3

Reduced-Order Observer in Constant Feedback Loop

To see this, suppose the system

$$\dot{x} = Ax + Br \qquad (II.2)$$

$$y = Cx$$

has the control law

$$r = Kx \qquad (II.20)$$

Then supposing that the state is available for measurement (or that $K = MC$ for some M), the closed-loop system is

$$\dot{x} = (A + BK)x \qquad (II.21)$$

and its poles are the eigenvalues of the matrix $A + BK$.

However, if an observer is employed to generate the state of (II.2) for implementing (II.20), then using the equation (II.15) the control law becomes

$$r = K\hat{x} = Ez + Hy \qquad (II.22)$$

where z is the reduced-observer estimate and

$$K = ET^{-1} + HC \qquad (II.23)$$

Then the composite system is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BHC & BE \\ GC + T^{-1}BHC & F + T^{-1}BE \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \qquad (II.24)$$

Introducing the coordinate change $\zeta = z - T^{-1}x$, the above becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A + BK & BE \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \qquad (II.25)$$

The eigenvalues of (II.25), and therefore (II.24), are the eigenvalues of $A + BK$ and of F.

Consequently, an n^{th} order system that is completely controllable and completely observable with m independent output variables may, by use of a linear feedback law and an observer of order n-p, have the 2n-p eigenvalues of the closed-loop system take on any preassigned value.

2.5 Other Kinds of Observers

Other kinds of observers include observers which generate a linear function of system state variables, and observers which generate the state of time-varying systems, discrete systems, and stochastic systems.

The linear-function observer generates a function ε of the form

$$\varepsilon = a^T x$$

where a^T is any desired row vector and x is the system state. The utility of such a function ε is that many linear feedback laws merely require this function for control of pole placement within a time-invarient system. An observer which generates a linear function may be considerably lower in dynamical order than the multivariable observer of Section 2.3 which generates the state x. The linear-function observer was developed by Luenberger [3].

The major result of the linear-function observer is that the function $a^T x$ can be generated by an observer of order ν-1 where ν is the observability index [3] of system (II.2). The observability index ν is defined as the least positive integer for which the matrix

$$[C \vdots CA \vdots CA^2 \vdots \dots \vdots CA^{\nu-1}]^T$$

has rank n (equal to system order). For any completely observable system, $(n/p)-1 \leq \nu-1 \leq n-p$, and in many cases ν-1 is much less than n-p. Therefore there is often a considerable savings in dynamic order by generating $a^T x$ directly rather than generating x and then forming $a^T x$.

The estimate of $\varepsilon = a^T x$ is

$$\hat{\varepsilon} = b^T y + c^T z \tag{II.26}$$

$$\dot{z} = Fz + Ly + T^{-1} Br$$

where

$$T^{-1}A - FT^{-1} = L \tag{II.27}$$

$$b^T C + c^T T^{-1} = a^T$$

and

$$z \in \mathcal{E}^{\nu-1}$$

The underlying notion for this capability is that the matrix noted by T^{-1} need not be square if it is understood that its inverse does not exist. The design procedure is to choose T^{-1} so that the second equation in (II.27) is satisfied; then the remaining values of T^{-1} and the values of L may now be determined by the first of these equations. Other techniques can also be used in the design.

Observers for time-varying systems have been reported in [8] and [9]. Discrete time observers have been investigated in [10] - [12] and stochastic observers in [13] - [15].

Other papers of note are [16] - [21].

CHAPTER III

THE LYAPUNOV SYNTHESIS TECHNIQUE

3.1 *Lyapunov Stability*

It is not the purpose of this thesis to delve deeply into the general stability problem or to survey the stability literature. However, within this section the basic concepts in stability which are used in the subsequent sections for synthesizing an adaptive algorithm are briefly described.

The concept of Lyapunov stability deals with the family of motions defined by the differential equation

$$\dot{x} = f(x, t) \quad t \geq t_0 \tag{III.1}$$

in which it is assumed that the trivial solution $x = 0$ is a member of that family in the sense that

$$0 = f(0,t), t \geq t_0$$

Moreover, $x = 0$ is also called the equilibrium of (III.1). Each member of the family of motions defined by (III.1) is designated by

$$p\ (t,\ x_0,\ t_0)\ ,\quad t \geq t_0$$

where $p(t_0, x_0, t_0) = x_0$. It is assumed that each $p(t,x_0,t_0)$ is continuous in all its arguments whenever x_0 is in the neighborhood of the equilibrium; in other words, $p(t, x_0, t_0)$ satisfies a Lipschitz condition [22].

The notion of Lyapunov stability is that the stability of an equilibrium of (III.1) can be determined by examining the behavior of the entire solution, for all $t \geq t_0$, with respect to the initial value

x_0. In particular, if the initial value is close to the equilibrium, then the subsequent motion $p(t, x_0, t_0)$ remains correspondingly close to the equilibrium.

The precise definition [23] is that an equilibrium of the differential equation (III.1) is called *stable in the sense of Lyapunov* if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$||p(t, x_0, t_0)|| \leq \varepsilon \qquad \text{for all } t \geq t_0$$

whenever $||x_0|| \leq \delta$.

Consequently it is the *relationship* between the initial value x_0 and the ultimate bound on the motion $p(t, x_0, t_0)$ that denotes stability in the sense of Lyapunov. Indeed, it is easy to visualize trajectories which are bounded for all $t \geq t_0$ but not stable due to a lack in relationship between that bound and the closeness of x_0 from the equilibrium. It should be pointed out that the definition of stability applies only in the neighborhood of the equilibrium.

An equilibrium of (III.1) is said to be *attractive* [5] if there exists a real number $\eta > 0$ for which

$$\lim_{t \to \infty} p(t, x_0, t_0) = 0$$

whenever $||x_0|| \leq \eta$, assuming there is no other initial value x' such that $f(x', t) = 0$ for $t \geq t_0$ and $||x'|| \leq \eta$.

From the preceeding discussion it is readily seen that the attractiveness of the equilibrium bears no relation to the concept of stability in that a motion which is stable need not be attractive and vice versa.

Attractiveness of the equilibrium is a concept which applies only in the neighborhood of the equilibrium defined by $||x_0|| \leq \eta$.

If in addition, however, the motion $p(t, x_0, t_0)$ is attractive for every $\eta > 0$ then the equilibrium is said to be globally attractive.

The concept of stability and attractiveness may be combined to form the concept of asymptotic stability. The equilibrium of (III.1) is said to be *asymptotically stable in the sense of Lyapunov* if it is both attractive and stable in the sense of Lyapunov.

Moreover, if the equilibrium of (III.1) is both stable and globally attractive then the equilibrium is said to be *globally asymptotically stable*.

Given the preceeding definitions (which by no means are the totality of definitions currently in use in the theory of Lyapunov stability [see 23]),it is desired, for purposes of this thesis, to state certain methods for determining the stability of the equilibrium of a differential equation with the ultimate aim of constructing a control to influence the stability of the equation. For this purpose, the so-called Direct Method of Lyapunov will be briefly examined.

3.2 Lyapunov's Direct Method

The Direct Method is a means of determining stability and asymptotic stability (either global or nonglobal). Unfortunately, the ability to use this method usually depends upon the ingenuity of the user.

A simple explanation [5] of the Direct Method can be had with reference to Figure 3.1. Consider a differential equation of the form

$$\dot{x} = f(x) \tag{III.2}$$

which, for purposes of illustration here, is taken as a vector equation with $x \in \mathcal{E}^2$. Consider also a quadratic function

$$V = x^T P x$$

in which P is a symmetric matrix having its two eigenvalues positive;

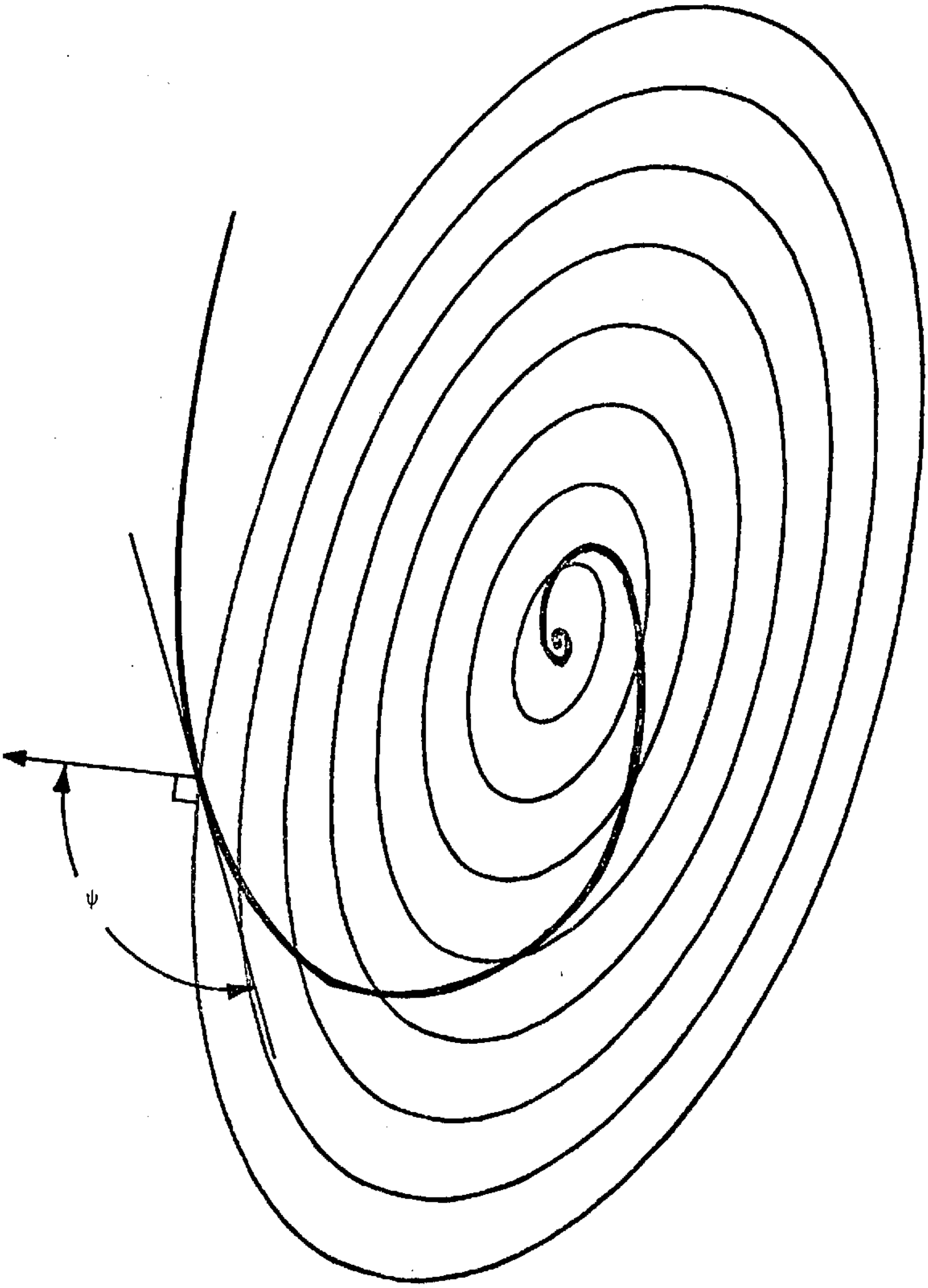


Figure 3.1

Criteria for an Asymptotically Stable Trajectory such a matrix is a member of a class of matrices called *positive definite* [24]. The ellipses in Figure 3.1 represent the locus of the equations

$$V = C_i$$

for positive constants C_i.

Suppose the solution $p(t, x_0)$ of (III.2) spirals toward the origin as indicated in Figure 3.1. Such a motion represents an asymptotically stable equilibrium with respect to the initial value. It is seen that any motion which is always directed inward to the ellipses is an asymptotically stable motion. This property is taken as an easy test to determine whether the equilibrium of (III.2) is asymptotically stable. In order to gain a relationship in terms of V to insure that the motion spirals inward with respect to the ellipses, the angle between the trajectory (in the direction of motion) and the ellipse is required always to be negative.

Taking the outer normal to an ellipse as

$$\text{grad } V = \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix}$$

it is desired to compute the angle ψ between a trajectory and the outer normal. Then

$$\cos \psi = \frac{(\text{grad } V)^T \dot{x}}{||(\text{grad } V)^T \dot{x}||}$$

The trajectory is always inward with respect to the function V whenever $\cos \psi < 0$. Since the denominator of $\cos \psi$ is always positive, the condition is that

$$(\text{grad } V)^T \dot{x} < 0$$

since in this case

$$(\text{grad } V)^T\dot{x} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \frac{d}{dt} V$$

the requirement that $\dot{V}$ be negative along the trajectory insures that the solutions of (III.2) are asymptotically stable.

This simple viewpoint can be expanded into a rigorous test for stability known as the Direct Method.

One theorem for stability is the following [5]: let $V(x)$ be a function with continuous first order partial derivatives. Suppose there exists a region G such that $0 < V(x) < a$ in which $\dot{V} \leq 0$. Let M be the largest invariant subset of the set $\dot{V} = 0$. Then every motion of (III.2) which begins in G tends toward M.

As a corollary to the above theorem, the more familiar theorem is given:

If additionally $V(x)$ is positive definite with derivative $\dot{V}$ for (III.2) negative semi-definite (definite) or identically zero, then the equilibrium of (III.2) is (asymptotically) stable. If in addition there exists a monotone increasing function $w(||x||)$ such that $V(x) \geq w(||x||)$ and if $w(||x||) \to \infty$ as $||x|| \to \infty$, then the equilibrium of (III.2) is globally (asymptotically) stable.

In analyzing an equation of the form (III.1) or (III.2) to determine the stability of an associated equilibrium, a function V is sought which satisfies the condition of the theorems here given (or the numerous other theorems omitted here). Failure for the V function to be as described does not imply instability of the equilibrium however, only that the choice of V was inappropriate. Although there exist several methods of finding appropriate Lyapunov functions, in the general case, the ingenuity of the investigator is the sole means of constructing an appropriate function.

In *synthesizing* a control r to make the equilibrium of the equation

$$\dot{x} = f(x, t, r) \tag{III.3}$$

stable, the situation is somewhat altered. A Lyapunov-function candidate is first selected, its derivative found, and a control is selected to make the derivative negative semi-definite or definite. With this procedure, ingenuity is required additionally in the choice of control.

The next section deals specifically with the question of synthesis. Suffice it to say here that with the synthesis technique as presently developed the derivative of V is negative definite in some of the parameters over which V is defined but only semi-definite over all the parameters. This involves the stability of *non-compact manifolds*. One of the few results in the study of stability on a non-compact manifold is the following [25].

Assume for the system

$$\dot{x} = f(t, x, \phi)$$
$$\dot{\phi} = g(t, x, \phi)$$

that f is bounded for bounded x and r and all $t \geq t_0$, and that there exists a function $V(x,\phi)$ satisfying

1. $V(x,\phi)$ positive definite with continuous first partial derivatives
2. $V(x,\phi) \to \infty$ as $||x||^2 + ||\phi||^2 \to \infty$
3. $\dot{V}(x,\phi) \leq - W(x) + h_1(t)q(x,\phi) + h_2(t)\, V(x,\phi)$

where

i) $W(x)$ is continuous and positive definite

ii) $q(x,\phi)$ is continuous

iii) $\int_0^{\infty} |h_i(t)|\, dt < \infty$ for $i = 1,2$

then the state $x = 0$, $\phi=0$ is eventually stable and, corresponding to each $\tau > 0$, there is a T_τ such that $||x(t_0)||^2 + ||y(t_0)||^2 < \tau^2$ for some $t_1 \geq T_\tau$ implies $\phi(t)$ is bounded and $x(t) \to 0$ as $t \to \infty$.

If, in addition,

4. for some $K > 0$ and some $0 < \alpha < 1$

$$|q(x,\phi)| \leq KV^{\alpha}(x,\phi)$$

then all solutions $\phi(t)$ are bounded, and all $x(t) \to 0$ as $t \to \infty$.

*3.3 Synthesis Techniques**

This section surveys the literature devoted to the synthesis of model-tracking adaptive systems based on application of Lyapunov's second method. In the early work [26] - [34], the model tracking problem was approached by using the sensitivity or gradient approach, without assurance of global asymptotic stability. Rang [35], Shackcloth and Butchart [36, 37] and Parks [38], were first to employ the Lyapunov design in finding an adaptive control law which guaranteed global stability. It is the purpose here to introduce the basic synthesis procedure, and to critically review extensions to the theory which have appeared since 1966, relating to: design for relative stability, reduction of order techniques, design with disturbance, design with time variable parameters, multivariable systems, identification, and an adaptive observer.

* This section exclusively used underscoring to indicate vector for purposes of clarity.

Synthesis Using Lyapunov's Second Method

The basic problem to be considered in this section is that of designing a model tracking system for stability without specifying exact values of the plant parameters. Lyapunov's stability theorems offer a means of *synthesizing* various control laws which offer possible solutions to this design problem, the particular solution depending in part upon the form of the Lyapunov function selected. In this section the rudimentary ideas involved will be introduced in a somewhat limited context. In subsequent sections elaborations on the elementary theory will be discussed, indicating some practical design considerations as well as defects in the method.

The concept which is central to adaptive schemes to be discussed in this section can be explained with reference to the model tracking system in Figure 3.2, for which the state equation of the stable model is given by

$$\dot{\underline{y}} = A\underline{y} + B\underline{r} \qquad \text{(model)} \quad \underline{y}(0) = \underline{y}^o \tag{III.4}$$

and that of the time-invariant plant by

$$\dot{\underline{x}} = A^*\underline{x} + B^*\underline{u}. \qquad \text{(plant)} \quad \underline{x}(0) = \underline{x}^o \tag{III.5}$$

Here $\underline{y} = \{y_i\}$, $\underline{x} = \{x_i\}$ are n dimensional state vectors, and $\underline{r}$, $\underline{u}$ are m dimensional control inputs. A^*, B^* contain unknown coefficients. If the differential equation of the tracking error ($\underline{e} = \underline{y} - \underline{x}$) is now written in the form

$$\dot{\underline{e}} = A\underline{e} + \underline{f} \tag{III.6}$$

where $\underline{f} = (A - A^*)\underline{x} + B\underline{r} - B^*\underline{u}$, then the control objective is to manipulate $\underline{f}$ in some way so that $\lim_{t\to\infty} \underline{e}(t) = \underline{0}$. To this end we introduce

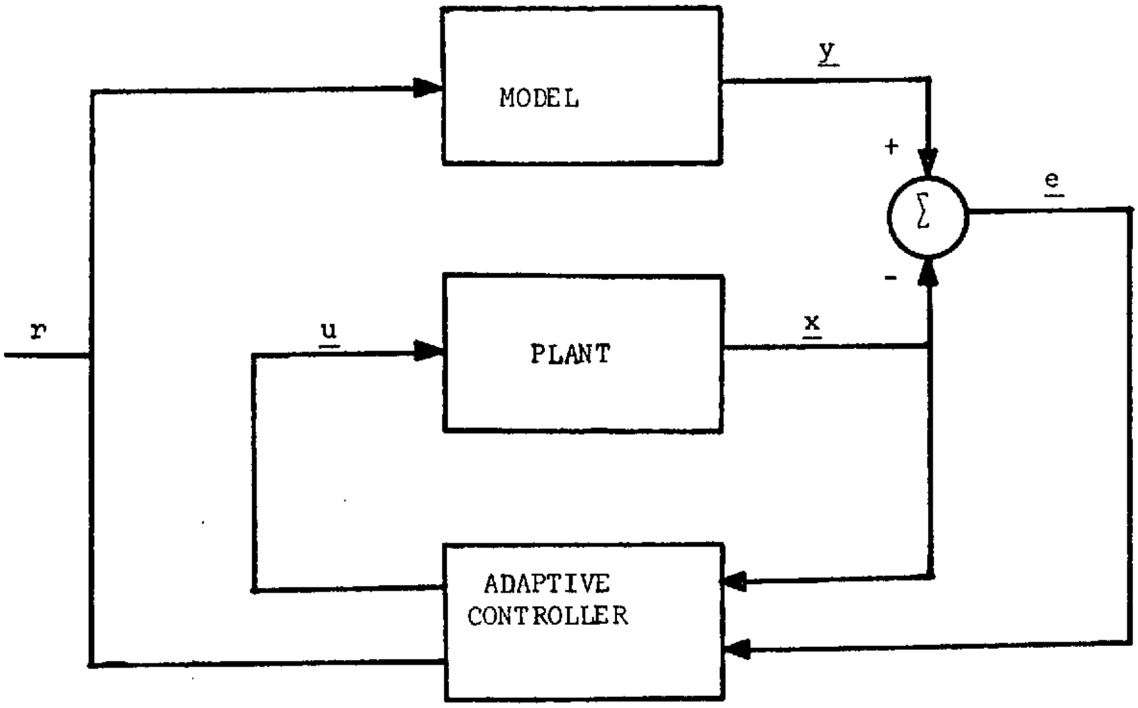


Figure 3.2

Model-Reference Adaptive Scheme

the positive definite function

$$V = \underline{e}^T P \underline{e} + \underline{h}(\Phi,\Psi) \qquad \text{(III.7)}$$

where Φ, Ψ are matrices of parameter vectors $\underline{\phi}_i$, $\underline{\psi}_i$ to be defined. Then along the trajectory of (III.6) we obtain for the time derivative of V

$$\dot{V} = -\underline{e}^T Q \underline{e} + 2\underline{e}^T P \underline{f} + \dot{\underline{h}} \qquad \text{(III.8)}$$

where

$$-Q = A^T P + PA. \qquad \text{(III.9)}$$

By a theorem of Lyapunov [39], with any $Q^T = Q > 0$, it follows that $P = P^T > 0$ is a unique solution to (III.9) iff A is a stability matrix, as assumed.

Using the classification suggested by Phillipson [40], we will introduce two methods which have been reported for causing $\underline{e} \to \underline{0}$.

Input Modification

This scheme uses the solution obtained with $\underline{h} \equiv \underline{0}$. Although it is not in itself useful in synthesizing an *adaptive* control law, i. e. one in which a set of parameters are automatically adjusted so as to reduce the tracking error, it has been used to advantage in solving certain of the design problems to be discussed in the survey.

With $\underline{h} \equiv 0$, asymptotic stability in $\underline{e}$ will be satisfied if [41]

$$\begin{aligned} \underline{e}^T P \underline{f} &= 0 \text{ for } A - A^* = 0 \\ &\leq \text{otherwise.} \end{aligned} \qquad \text{(III.10)}$$

This inequality cannot be satisfied except in special cases depending on the system structure. In some restricted cases, such as if (III.4), (III.5) are in phase variable form, a solution exists. In this case $\underline{u}$ becomes a scalar, $B = \underline{b}$, and b_n, f_n are the only nonzero elements of $\underline{b}$, $\underline{f}$ respectively. We then have the simplification

$$\underline{e}^T P \underline{f} = (\underline{e}^T \underline{p}_n)\ f_n$$

where $\underline{p}_n$ is the nth column of P. Writing

$$f_n = f'_n + b_n u,$$

and assuming that $b_n > 0$, we see that conditions (2.7) are fullfilled if

$$|u| \geq |f'_n / b_n|$$

$$\text{sgn } u = \text{sgn } \underline{e}^T \underline{p}_n, \quad \underline{e} \neq \underline{0},$$

as might be realized by a switching function. Solutions of this form have been reported, together with design considerations [42,43].

Feedback Synthesis

This scheme differs from input modification in that parameters in the system are adjusted continuously so that in the simplified case treated here $\underline{e} \to \underline{0}$.

In feedback synthesis, we write for (III.7)

$$V = \underline{e}^T P \underline{e} + \sum_{i=1}^{n} \underline{\phi}_i^T \underline{\phi}_i + \sum_{i=1}^{m} \underline{\psi}_i^T \underline{\psi}_i \qquad \text{(III.11)}$$

wherein $\underline{\phi}_i$, $\underline{\psi}_i$ are misallignment parameter vectors to be defined in terms of the elements of the matrices A-A* and B-B* which express the misallignment between the model and the plant. In this case (III.8) becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} + 2(\underline{e}^T P \underline{F} + \sum_{i=1}^{n} \dot{\underline{\phi}}_i^T \underline{\phi}_i + \sum_{i=1}^{m} \dot{\underline{\psi}}_i^T \underline{\psi}_i). \qquad \text{(III.12)}$$

Let E define the whole state space with $\underline{\xi} \epsilon E$, where $\underline{\xi}$ is defined by $\underline{\xi}^T = [\underline{e}_1^T, \underline{\phi}_1^T, \ldots, \underline{\phi}_n^T, \underline{\psi}_1^T, \ldots, \underline{\psi}_m^T]$. Let $E_1 \epsilon E$, where E_1 is the n dimensional subspace with $\underline{e} \epsilon E_1$. Clearly V in (III.11) is positive definite in E.

The basic idea in feedback synthesis is to specify $(\dot{\underline{\phi}}_i, \dot{\underline{\psi}}_i)$ in (III.12) so that

$$\underline{e}^T P \underline{f} + \sum_{i=1}^{n} \dot{\underline{\phi}}_i^T \underline{\phi}_i + \sum_{i=1}^{m} \dot{\underline{\psi}}_i^T \underline{\psi}_i \equiv 0 \qquad \text{(III.13)}$$

and consequently

$$\dot{V} = -\underline{e}^T Q \underline{e} \qquad \text{(III.14)}$$

Since $\dot{V}$ is only negative *semidefinite* in E, but negative definite in E_1, we may conclude from the theorems of Lyapunov [39] that the equilibrium at $\underline{e} = \underline{0}$ is asymptotically stable, and the equilibrium at $\underline{\xi} = \underline{0}$ is stable. It follows in the present formulation that $\underline{e} \to \underline{0}$, and that the missallignment parameter vectors are bounded. It will be shown in certain cases that $\underline{\xi} \to \underline{0}$ if the frequency content of $\underline{r}$ is rich enough [44, 59].

In implementing the controls to satisfy (III.13), there are two schemes which will be described in this survey as *direct* and *indirect adaptation*. Direct adaptation assumes that plant parameters are adjustable. Indirect adaptation requires that adjustment take place external to the plant.

Direct Adaptation

In this scheme $\underline{u} = \underline{r}$ in (III.5). Φ and Ψ are in turn defined by $\Phi = A-A^*$, $\psi = B-B^*$, with columns $\underline{\phi}_i$ and $\underline{\psi}_i$ respectively. Then (III.6) becomes

$$\dot{\underline{e}} = A\underline{e} + \underline{f} \qquad \text{(III.15)}$$

wherein

$$\underline{f} = \Phi\underline{x} + \Psi\underline{r}.$$

It is seen that (III.13) can be satisfied if

$$\dot{\underline{\phi}}_i^T = - e^T P x_i, \quad (i=1,\ldots n) \qquad \text{(III.16)}$$

$$\dot{\underline{\psi}}_i^T = - e^T P r_i, \quad (i=1,\ldots m).$$

If adaptive control (Figure 3.2) is the objective, then elements of A*, B* should be adjusted, in which case (III.16) becomes

$$\dot{\underline{a}}_i^{*T} = \underline{e}^T P x_i, \quad i=1,\ldots n$$
$$\text{(III.17)}$$
$$\dot{\underline{b}}_i^{*T} = \underline{e}^T P r_i, \quad i=1,\ldots m.$$

Here it is assumed that A, B are constant. In the identification section control is applied to matrices A, B instead of A*, B*.

Indirect Adaptation

Whereas the adaptive control law in (III.17) can be implemented for the multivariable system if direct adaptation is possible, restrictions must be placed on the form of the state equations in order to apply indirect adaptation [34, 45]*.

Assuming that the elements of A*, B* are not directly adjustable it becomes necessary to modify the system. Consider the single-input single-output plant defined by

$$\frac{x_1}{u} = \frac{k^*}{s^n + a_n^* s^{n-1} + \ldots + a_1^*}$$

By introducing k_r, k_1, k_2, $\ldots k_n$ as adjustable parameters as in Figure 3.3, it is seen that the compensated system has been cast into the form

$$\dot{\underline{x}} = A^* \underline{x} + \underline{b}^* r$$

* As discussed in the section on reduction of order, if it is desired to require convergence of fewer than all the states, the restriction on form can be relaxed if there are l.h.p. zeros in the plant transfer function.

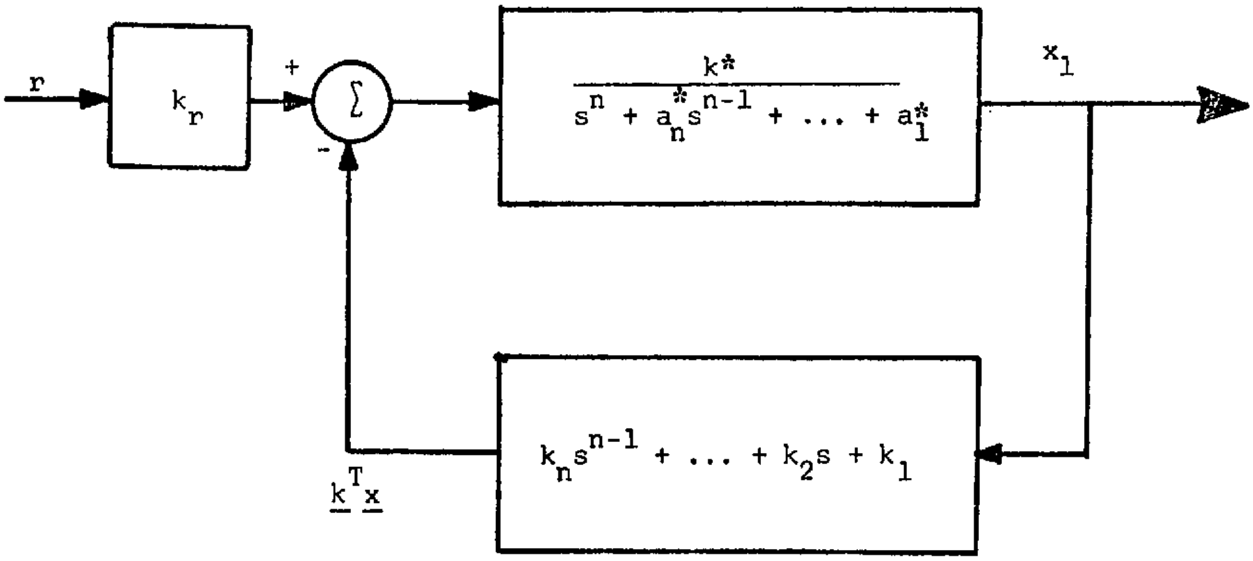


Figure 3.3

Model-Reference Adaptation by Feedback

where

$$A^* = \begin{bmatrix} 0 & \vdots & 1 & & \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ a^*_{n1} & & \cdots & a^*_{nn} & \end{bmatrix}, \quad \underline{b}^{*T} = [0,\ldots,0,b^*_n],$$

with $b^*_n = k_r k^*$, and $a^*_{ni} = -a^*_i + k_i$, $i = 1,\ldots n$. Thereby the problem is amenable to direct adaptation by application of (III.17). For the single-input case it is noted that only n coefficients in A^* can be adjusted. The multivariable problem suffers restrictions also [45].

Degree of Stability - Improved Speed of Response

The *adaptive step response* [36] has been helpful in analyzing the relative stability of adaptive systems. With the aid of this concept Phillipson [40] showed that it is possible in case of an adaptive gain to improve the relative stability by appropriately modifying the adaptive control law. Gilbart and Monopoli [46] formalized the synthesis procedure by redefining the V function in (III.11).

In the ensuing discussion let the error equation be defined by

$$\dot{\underline{e}} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 \\ (k - k^*) \end{bmatrix} r \qquad \text{(III.18)}$$

where k^* is the only adjustable parameter. If (III.11) is written as

$$V = \underline{e}^T P \underline{e} + \frac{\psi^2}{h_1}, \; h_1 = \text{const.} \qquad \text{(III.19)}$$

then (III.8) becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} + 2(\underline{e}^T \underline{p}_2 (k-k^*) r + \frac{\dot{\psi}\psi}{h_1}) \qquad \text{(III.20)}$$

wherein $\underline{p}_2^T = [p_{12} p_{22}]$, and P is found according to (III.9). In this case, with $\psi = k - k^*$, it is seen that (III.13) is satisfied if

$$\dot{k}^* = (p_{12}e_1 + p_{22}e_2)\, h_1\, r. \qquad \text{(III.21)}$$

If r is assumed to be a step of magnitude R, then differentiating (III.18) and using (III.21) leads to the characteristic equation

$$S^3 + a_1S^2 + a_0S + (p_{12} + p_{22}S)h_1R^2 = 0 \qquad \text{(III.22)}$$

The roots of this equation characterize the adaptive step response. A typical root locus, as shown in Figure 3.4, demonstrates that the relative stability is degraded with increasing R^2, even though the system by design cannot be unstable.

As a means of improving the relative stability with increasing R^2 Phillipson modified the adaptive rule in (III.21) by setting $u = k^*r + u_1$, and using u_1 to insert input modification. Thus, with $u_1 = \gamma\dot{k}^*r$, (III.8) becomes

$$\dot{V} = -\underline{e}^TQ\underline{e} - 2(\underline{e}^T\underline{p}_2)^2\, h_1r^2. \qquad \text{(III.23)}$$

The effect has been to make $\dot{V}$ more negative with r^2. It is also instructive to examine the adaptive step response. The equation corresponding to (III.22) is

$$S^3 + a_1S^2 + a_0S + (p_{12} + p_{22}S)\,(1 + \gamma S)h_1R^2 = 0 \qquad \text{(III.24)}$$

The root locus indicates an improved relative stability for large R^2 (see Figure 3.5).

More in the spirit of involving the Lyapunov function in the synthesis procedure, Gilbart and Monopoli [46] have proposed modifying the V function in (III.11) so that the desired result is obtained more routinely. The concept will be illustrated using the system equation

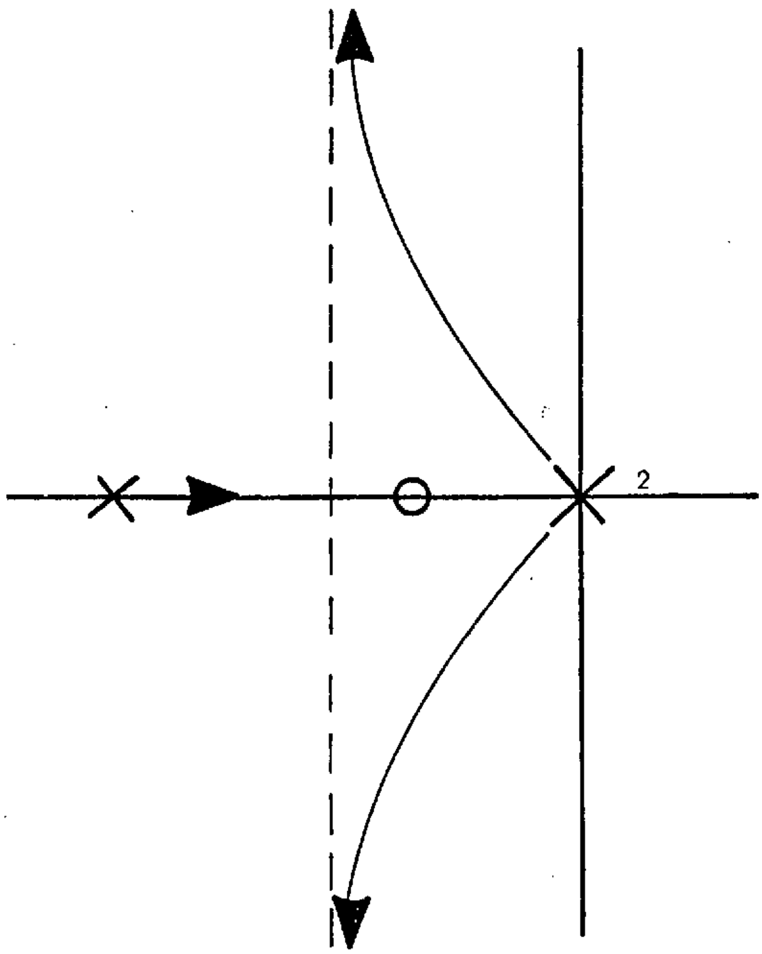


Figure 3.4

Root Locus for Change in Adaptive Gain Parameter

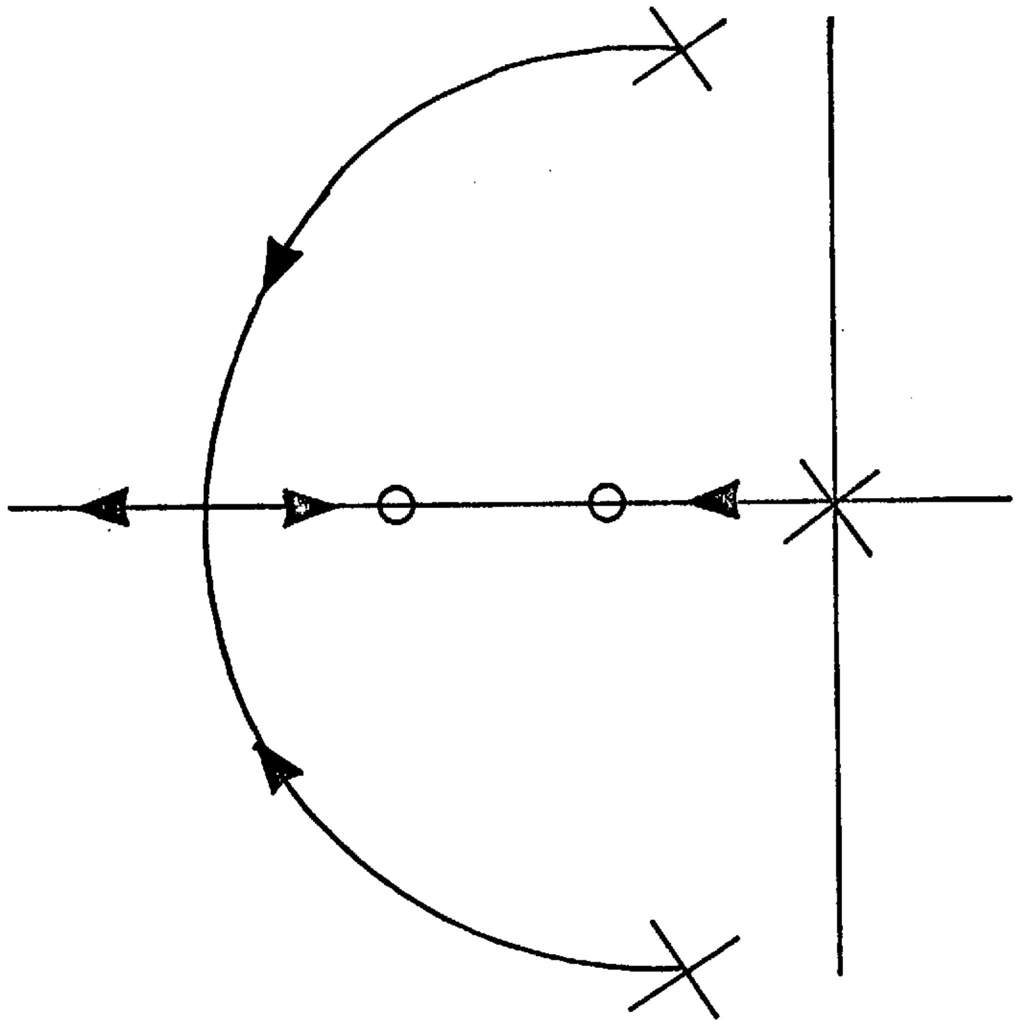


Figure 3.5

Improved Degree of Stability

(III.18). According to [46] we introduce the modified V function

$$V = \underline{e}^T p \underline{e} + \frac{(\psi + h_1 z)^2}{h_2}. \tag{III.28}$$

(III.8) now becomes

$$\dot{V} = -\underline{e}^T Q\underline{e} + 2\underline{e}^T \underline{P}_2 (k-k^*)r + \frac{2}{h_2}(\psi\dot{\psi} + h_1\psi\dot{z} + h_1 z\dot{\psi} + h_1^2 z\dot{z}). \tag{III.29}$$

If we equate for this example

$$\begin{aligned} z &= \underline{e}^T \underline{p}_2 r \\ \psi &= k-k^* \\ \dot{\psi} &= -h_2 z - h_1\dot{z} \end{aligned} \tag{III.30}$$

then (III.29) becomes

$$\dot{V} = -\underline{e}^T Q\underline{e} - 2h_1(\underline{e}^T \underline{p}_2 r)^2. \tag{III.31}$$

This is seen to be identical to (III.23). For the adaptive control, we have from (III.30)

$$k^* = h_1\underline{e}^T \underline{p}_2 r + h_2 \int_{t_0}^{t} \underline{e}^T \underline{p}_2 r dt + k(t_0). \tag{III.32}$$

The general matix formulation of this scheme for the entire parameter set Φ,Ψ is given in [47], together with simulation results showing that improvement in convergence time can be obtained.

Reduction of Order

The basic model-reference Lyapunov adaptive law requires measurement of the *entire* error vector for its implementation. If all the state variables of the plant under control are not available, then the basic adaptive law is inadequate since ignorance of parameters and inherent system noise may prohibit their generation by an observer or by differentiators.

To overcome this practical problem investigators have attempted to find adaptive laws that require a minimum of state variable measurements. The first such attempt by Parks [38] was for the restrictive system shown in Figure 3.6 in which only the constant plant input gain K_V is mismatched to model gain K. The adaptive law adjusts K_c so that the error vanishes. Parks use of Kalman's lemma [48] subsequently extended by Monopoli [49] using the Kalman-Meyer lemma [50], has shown that the adaptive law $\dot{K}_c = \lambda^{-1} e_1 r$, in which only the output error e_1 appears, is sufficient for asymptotic stability of $\underline{e}$ if

1) $\frac{N(s)}{D(s)}$ is a positive real function (III.33)

2) $[1 \;\; 0 \;\; 0 \;\; .. \;\; 0] \, \underline{Ab} \neq \underline{0}$

where $\frac{N(s)}{D(s)}$ is the transfer function representation of the plant equation $\underline{\dot{x}} = A\underline{x} + \underline{b}r$. If 2) does not hold but 1) does, the set $\dot{V} = 0$ must be examined to insure asymptotic stability of $\underline{e}$.

Monopoli [49] extended the criteria to nonpositive real transfer functions that can be made positive real by multiplying by a polynomial in s with roots of negative real parts. In doing so, the resulting adaptive law requires n-m-2 derivatives of the output error e_1 where n is the number of plant poles and m is the number of plant zeroes.

In the more general case where adaptation of both plant poles and zeros are desired, the basic indirect adaptive scheme fails (but not direct adaptation) even with full state measurement available. This is seen since, referring to Figure 3.7 with D(s) = 1, the error equation

$$N(s)e_1 = [Q(s)-N(s) + P(s)\, H(s)]x_1 + [M(s) - P(s)\, G(s)]r \quad \text{(III.34)}$$

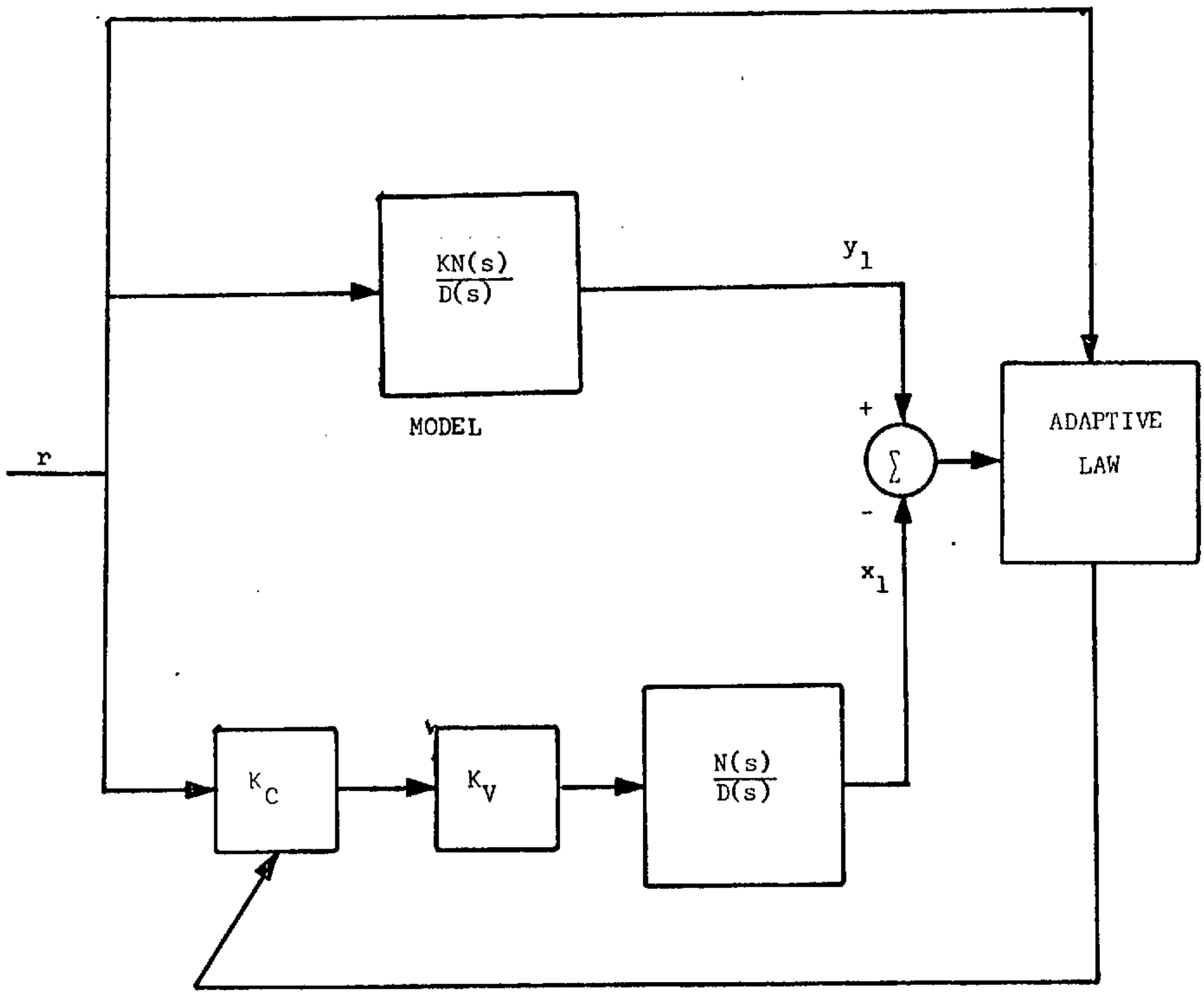


Figure 3.6

Input Adaptation

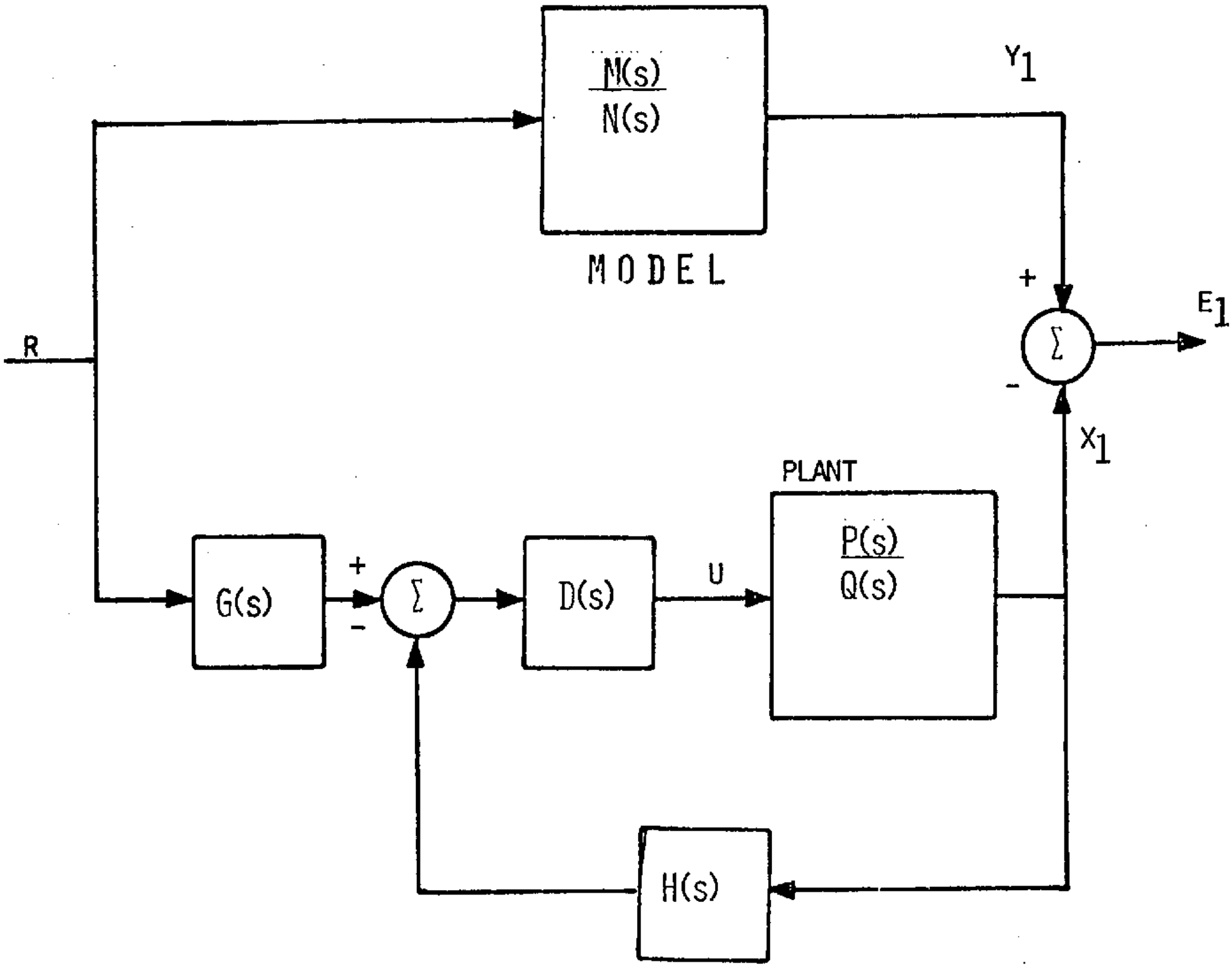


Figure 3.7

Reduction of Order Technique

reveals H(s), a polynomial, may have at most only n-m free parameters so as not to increase the order of the nth order polynomial Q-N+PH; consequently there exists no steady-state solution to the problem of adjusting n parameters in Q-N+PH state feedback.

Monopoli and Gilbart [51], however, have employed dynamic feedback on a reduced state of order n-m-1 to accomplish adaptation with plant zeroes. Moreover, they have shown that the plant matrix A* need not be of any particular form; nor must A* be stable nor $C^*(sI-A^*)^{-1}B^*$ be positive real; and the output matrix C* need not be known.

Their basic idea is for D(s), after adaptation, to cancel P(s), with the model zeroes placed in cascade with the plant by G(s). Due to the cancellation, the zeroes of the plant necessarily must be in the LHP.

The vector error equation (III.6) is "collapsed" to yield a scalar error equation in the output error e_1 from which the synthesis proceeds:

$$e_1^{(n)} + \sum_{i=0}^{n-1} a_i e_1^{(i)} = \sum_{i=0}^{n-1} \Delta a_i x_1^{(i)} + \sum_{i=0}^{m} \Delta b_i r^{(i)} + \sum_{i=0}^{m} \hat{b}_i u^{(i)} \quad \text{(III.35)}$$

Note that the output error, defined as the difference between the scalar model output and the scalar plant output, is always available for measurement by definition.

For clarity in illustrating the synthesis procedure, a 2nd order system with one zero will be treated here. For this system, (III.35) becomes

$$(s^2+a_1s+a_0)e_1 = (\Delta a_1 s+\Delta a_0)x_1 + \Delta b_0 r + (s+\hat{b}_0)u \quad \text{(III.36)}$$

in which initial conditions have been ignored and $\Delta b_1=0$ and $\hat{b}_1=1$ for simplicity in this treatment. Dividing (III.36) by s+c, $0<c<a_1$, and taking the Laplace inverse transformation yields

$$\dot{e}_1 + (a_1-c)e_1 = [c(a_1-c)-a_2]\mathcal{L}^{-1}\left(\frac{e_1}{s+c}\right) + \Delta a_1 x_1$$

$$+ [\Delta a_0 - c\Delta a_1]\mathcal{L}^{-1}\left(\frac{x_1}{s+c}\right) + \Delta b_0 \mathcal{L}^{-1}\left(\frac{r}{s+c}\right)$$

$$+ u + [\hat{b}_0 - c]\mathcal{L}^{-1}\left(\frac{u}{s+c}\right). \qquad \text{(III.37)}$$

Now if u is chosen as $u = \Sigma k_i v_i$ where v_i are each of the terms on the right side of (III.37) (i.e. $v_1 = \mathcal{L}^{-1}\left(\frac{e_1}{s+c}\right)$, $v_2 = x_1$, etc.) then (III.37) has the form for which the indirect adaptation scheme can be applied with the important difference that the resulting adaptive law contains only e_1 and not its derivatives.

In general, a division polynomial, such as that which produced (III.37), should be of order m with zeroes chosen so that the left side of (III.37) is stable. Then the left side contains n-m terms, so the resulting adaptive law contains n-m-1 derivatives of e_1.

It should be pointed out that since vector state information is destroyed in collapsing the error equation to (III.35), convergence of e_1 (output error) does *not* in general imply that the state variables converge. If the output matrix is known and A is in companion form, then convergence of e_1 implies convergence of the first n-m state variables.

At present, indirect adaptation cannot take place when there is a right half-plane zero in the system.

Effects of Disturbance in Adaptive Control

As noted in the synthesis section, if disturbance is neglected then, according to (III.14), $\dot{V}$ becomes negative definite in E_1 and negative semi-definite in E. However, it has been demonstrated, and in some cases proved, that the input to the system can be so chosen that the solution $\dot{V} \equiv 0$ can be satisfied only at the origin of the entire E space.

Recently Lindorff [52] and Narendra et al. [53] have considered the effect of disturbance upon stability. In [52] it is shown that, even though $\underline{e}$ remains bounded (theoretically), disturbance can cause the adaptive gains to be unstable (unbounded). In [53] a modified scheme is derived such that $\dot{V}$ is strictly negative in E outside of some bounded region about the origin, thereby guaranteeing boundedness in E, without placing special requirements on the input signal.

In [52] the effect of disturbance $\underline{d}$ and incomplete adaptation has been examined with reference to the single input plant described by the equation

$$\underline{\dot{x}} = A^*\underline{x} + \underline{b}^*(r + u_1) + \underline{d}. \quad \text{(III.38)}$$

The tracking error in this case defined by

$$\underline{\dot{e}} = A\underline{e} + \underline{f}_1 + \underline{f}_2, \; \underline{f}_1 + \underline{f}_2 \equiv \underline{f}$$

where $\underline{f}_1$ includes all adjustable parameters Φ, Ψ (although these need not be the entire set of unknown parameters), and $\underline{f}_2$ contains the remaining terms in $\underline{f}$, including $\underline{d}$. The adaptive controls when applied to this problem cause $\dot{V}$ to be reduced in the form

$$\dot{V} = -\underline{e}^T Q\underline{e} + \underline{e}^T P\underline{f}_2. \quad \text{(III.39)}$$

Since $\dot{V}$ in (III.39) is strictly negative outside of some bounded region about $\underline{e} = \underline{0}$ in E_1, and indefinite elsewhere, $\underline{e}$ will ultimately lie inside a calculable region about the origin. However, due to the presence of $\underline{f}_2$, $\dot{V}$ is indefinite in E, and stability in E can no longer be guaranteed. This has been demonstrated for the case in which (III.38) is in phase variable form [52], and $\underline{d}$ and r are constant. It is shown that parameter errors can be unbounded if the disturbance is opposite in sign to r.

Narendra et al. [53] have considered the problem of synthesizing an adaptive control law which guarantees boundedness of the errors when disturbance is present. Since their method also allows for time varying plant parameters, we shall include this degree of generality in the discussion, although the time variable case is treated in more detail later. For simplicity of exposition consider the first order differential equations

$$\text{(stable model)}\quad \dot{y} = -ay + r \tag{III.40}$$

$$\text{(plant)}\quad \dot{x} = -(a^*+k)x + r + d,\ a^* = a^*(t)$$

in which k is an adjustable parameter, d is a disturbance entering the plant, and r is the common input. The tracking error ($e \equiv y - x$) is governed by

$$\dot{e} = -ae + \phi x - d \tag{III.41}$$

with $\phi = -a + a^* + k$. If we now choose the positive definite form

$$V = \frac{1}{2}(\gamma e^2 + \phi^2),\ \gamma > 0 \tag{III.42}$$

and introduce the modified adaptive control

$$\dot{k} = -\beta k - \gamma e x \tag{III.43}$$

the equation for $\dot{V}$ becomes

$$\dot{V} = -\gamma a e^2 - \beta\phi^2 + (\beta(a^* - a) + \dot{a}^*)\phi - \gamma d e. \tag{III.44}$$

From this result it follows that boundedness in e, ϕ is guaranteed since $\dot{V}$ contains a negative definite part in e and ϕ. These quadratic terms control the sign of $\dot{V}$ for large enough values of $|e|$, $|\phi|$, if $(a^* - a)$, d and $\dot{a}^*$ are bounded. This result in turn depends upon the introduction of β in the modified adaptive control law (III.43).

This design has been generalized [53] for the single-input n^{th} order plant. Simulation results indicate that, in the absence of noise disturbance, best results are obtained with $\beta = 0$ and with the input rich enough in frequency content to assume asymptotic stability in the entire E space.

Time-Varying Parameters

For the time-varying plant

$$\dot{\underline{x}} = A^*(t)\ \underline{x} + B^*(t)(\underline{r}-\underline{u}) \qquad \text{(III.45)}$$

to behave as a time-invariant model requires an indirect adaptive law that depends upon the time derivative of an unavailable quantity if the synthesis is followed. Porter and Tatnall [54] have pointed out that this term may be ignored under some conditions on $A^*(t)$ and $B^*(t)$ for eventual asymptotic stability of $\underline{e}$. These conditions are restrictive, however.

In the more general case, use of a new adaptive law due to Narendra et al. [53] leads to the determination of Lagrange stability bounds whenever the time varying plant parameters satisfy certain restrictions. Reference is also made here to the next section.

To illustrate their adaptive scheme, consider a first-order time varying plant (III.45) with $B^*(t) = b^*(t)$. The corresponding error equation with $u = k_1x + k_2r$ is

$$\begin{aligned} \dot{e} - ae &= (a-a^*(t) + b^*k_1)x + (b-b^*(t) + b^*(t)k_2)r \\ &= \phi_1 x + \phi_2 r \end{aligned} \qquad \text{(III.46)}$$

$$a < 0$$

Using the adaptive law [53]

$$\dot{k}_1 = -\beta k_1 - xe$$

$$\dot{k}_2 = -\beta k_2 - re \qquad \text{(III.47)}$$

the Lyapunov function may be shown to be (with b*(t) chosen as in (III.49)).

$$V = b^*(t)\, e^2 + \phi_1^2 + \phi_2^2$$

$$\dot{V} = +b^*\left(\frac{\dot{b}^*}{b^*} + 2a\right) e^2 + 2\left(\frac{\dot{b}^*}{b^*} - \beta\right) \phi_1^2 + 2\left(\frac{\dot{b}^*}{b^*} - \beta\right)\phi_2^2$$

$$+ 2(a^* - a)\left(\frac{\dot{b}^*}{b^*} - \beta\right)\phi_1 - 2b\left(\frac{\dot{b}^*}{b^*} - \beta\right)\phi_2 - 2\dot{a}^*\phi_1 - 2\beta b^*\phi_2$$

(III.48)

Now if the following restrictions are placed upon the parameters

(i) $0 < b_1 \leq b^*(t) \leq b_2 < \infty$

(ii) $-\infty < a_1 \leq a^*(t) \leq a_2 < \infty$

(iii) $-\infty < b_3 \leq \frac{\dot{b}^*}{b^*} \leq b_4 < a_2$ (III.49)

(iv) $\beta > b_4$

(v) $|\dot{a}^*| \leq a_3$

The function $\dot{V}$ can be shown to be negative outside of a region in the space (e, ϕ_1, ϕ). Bounds on the region of attraction can therefore be found by determining the smallest ellipse V = C that encloses the region of indefinite $\dot{V}$.

It is noted that (III.47) does not produce an asymptotically stable error even in time-invariant systems unless $\beta = 0$. This follows from the observation that the adaptive parameters k cannot converge to a non-zero value whenever the error e vanishes simultaneously.

Narendra [55] extended the above technique along these lines for nth order systems.

Monopoli, Gilbart and Thayer [49] produced a *"practically Asymptotically* *stable"* system of Figure 3.4 when only the time-varying plant input gain $K_v(t)$ is adapted. The plant must have the same general attributes as in the section entitled Degree of Stability. They found, using a Lyapunov function $V = 1/2(\lambda_1 \underline{w}^T P \underline{w} + \lambda(t)\ \phi^2)$, that practical asymptotic stability of $\underline{e}$ is attained with an adaptive law $\dot{k}_c = \lambda_1 \underline{w}^T P b \underline{r} + \delta K(K-K_c K_v)|r|$ where $\underline{w}$ is related to n-m-1 derivatives of e_1, n & m defined as in the Degree on Stability section. The region of attraction is inversely proportional to the magnitude of δ and r.

The reduction of order with time-varying parameters was extended by Gilbart and Monopoli [56] to include adaptation of time-varying plant input coefficients and characteristic polynomial coefficients. A scalar error equation in e_1 analogous to (III.35) but with time-varying coefficients on the right side is assumed. For clarity, a second order example is demonstrated here. Suppose first that the second order plant is without input derivatives. After dividing by plant input coefficient $b^*(t)$, the scalar error equation is

$$\frac{1}{b^*(t)}(\ddot{e}_1+a_1\dot{e}_1+a_0e_1) = \frac{\Delta b}{b^*(t)} r + \frac{\Delta a_1}{b^*(t)} \dot{x} + \frac{\Delta a_0}{b^*(t)} x - u, b^*(t)>0 \ \forall t>0.$$

This can in turn be put in the form

$$\frac{1}{b^*(t)}(\ddot{e}_1+a_1\dot{e}_1+a_0e_1) = [f_1+g_1(t)]r + [f_2+g_2(t)]\dot{x} + [f_3+g_3(t)]x - u \qquad \text{(III.50)}$$

in which the coefficients have been broken into time-invarient parts f and time-varying part g. Choosing $u = (k_1^f+k_1^g)r + (k_2^f+k_2^g)\dot{x} + (k_3^f+k_3^g)x$ and

$$\dot{k}_1^f = \alpha_1 e^T P_2 r$$

$$\dot{k}_1^f = -a_i e^T P_2 x^{(2-i)} \quad i = 2,3 \qquad \text{(III.51)}$$

$$\dot{k}_1^g = M_1 \ \text{sat}\left[\frac{\beta_1}{M_1} e^T P_2 r \right]$$

$$\dot{k}_i^g = M_i \ \text{sat} \left[\frac{\beta_i}{M_i} e^T P_2 x^{(2-i)} \right] \quad i = 2,3$$

then the function $V = e^t Pe + \sum_{i=1}^{3} (f_i + k_i^f)$ has a negative derivative outside a region whose boundary is directly proportional to the bound on $|g_i(t)|$ and $b(t)$ and inversely proportional to β_i. Consequently (III.51) delivers strong practical stability of $\underline{e}$.

It is shown also by Gilbart and Monopoli [56] that a reduction of order technique similar to that in the Reduction of Order section, but modified to account for the time-varying nature of the scalar error equation (III.35), can be applied so that the resulting equation is in a form similar to (III.50) in order that adaptive laws like (III.51) may be used to produce a strong practically stable system. The reduction of order allows only n-m-1 derivatives to be used in forming the adaptive laws.

Adaptive Control of Multivariable Systems

It has been noted by Winsor and Roy [34] that the adaptive control law in the form of (III.17) can be implemented for the general multivariable plant if the plant parameters are directly adjustable (direct adaptation). The practical case in which feedback control signals are used to implement compensation (indirect control) warrants attention, however, particularly for the multi-input systems.

Starting with (III.4) and (III.5),(III.6), Lindorff [45] has shown that certain conditions are imposed on the form of (III.4) (III.5) in order that the Lyapunov design may yield a unique set of controls. This may be clarified by considering the term in (III.8)

$$\underline{e}^T P \underline{f} = \sum_{i=1}^{n} \underline{e}^T \underline{p}_i f_i \tag{III.52}$$

in which

$$f_i = \sum_{j=1}^{n} \phi_{ij} x_j + \sum_{j=1}^{m} \psi_{ij} r + \sum_{j=1}^{m} b^*_{ij} u_j.$$

Since P is positive definite, $\underline{p}_j$ and $\underline{p}_i$ are linearly independent for all $i \neq j$. Therefore a different u_j must be identified with each f_i in generating a particular component of the adaptive control law. It follows that not more than m components of $\underline{f}$ can be nonzero, and that a stable adaptive control law can be realized if (1) there are no more outputs than inputs, (2) the state equation is written in partitioned phase variable form, (3) the matrices B*, B are in triangular form. Extension of the reduction of order technique [51] to the multivariable problem has not been reported.

Identification

The identification problem can be approached so that the process is inherently stable. Lion [59] has derived a very practical solution to the problem of identification for single-input, single-output plants, with guaranteed asymptotic stability. Kudva and Narendra [57] have applied the Lyapunov synthesis method to the identification of time variable systems, illustrating an application of direct adaptation.

Lion uses the so called Generalized Equation Error System of Figure 3.8. In this scheme the parameters of $\tilde{N}$, $\tilde{D}$ are adjusted so as to minimize e^2. A significant feature of the method is that no derivatives of u, y are required (G is a low pass filter).

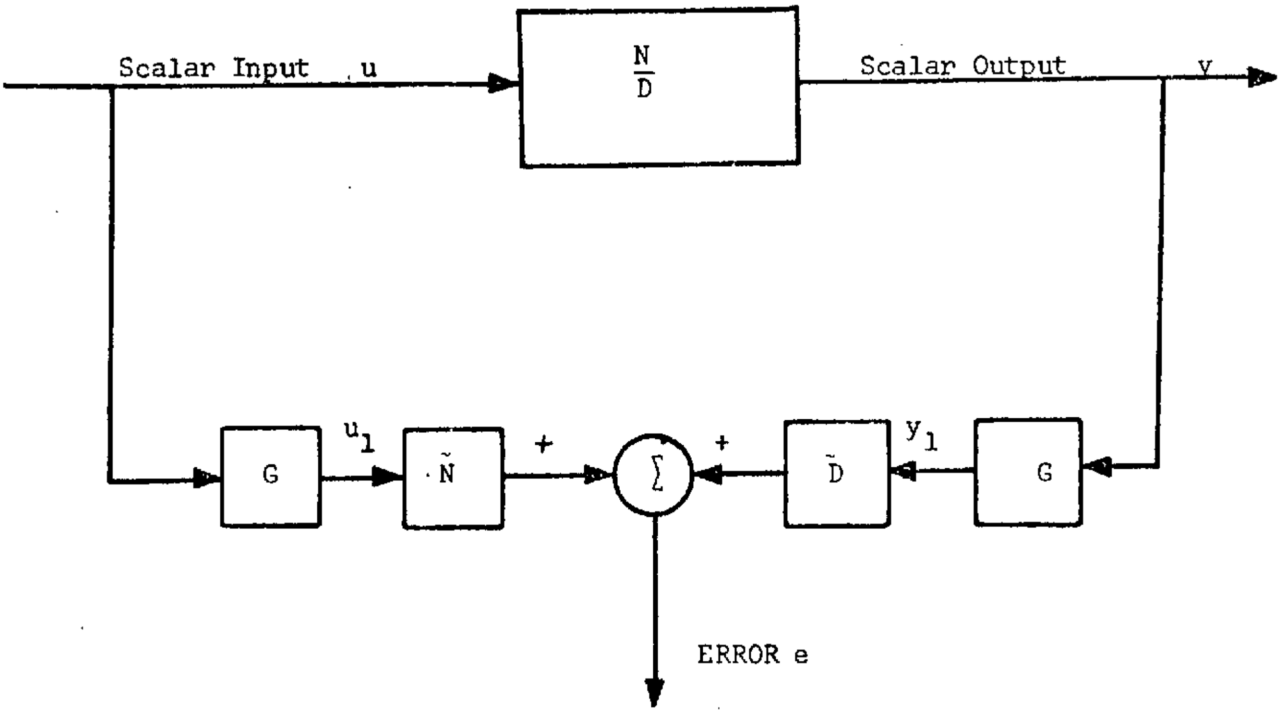


Figure 3.8

Lion's Identification Scheme

The scheme is based on the scalar error equation

$$e = (\tilde{D} - D)\, y_1 + (\tilde{N} - N)\, u_1. \qquad \text{(III.53)}$$

in which

$$\tilde{D}-D = \sum_{k=0}^{n} \Delta\alpha_k p^k$$

$$\tilde{N}-N = \sum_{k=0}^{m} \Delta\beta_k p^k$$

with $p \equiv d/dt$, and u_1, y_1 are filtered values of u, y. G(s) may have the simple form $1/s^{n-1}$.

If the parameter adjustment law is defined as

$$\Delta\dot{\alpha}_j = -ke(p^j y_1) \quad p\equiv d/dt$$

$$\Delta\dot{\beta}_i = -ke(p^i u_1) \qquad \text{(III.54)}$$

and the parameter misalignment vector is defined by

$$\underline{\phi}^T = [\Delta\alpha_0, \Delta\alpha_1, \ldots, \Delta\alpha_{n-1}, \nabla\beta_0, \ldots \nabla\beta_m],$$

it is possible to show that the function

$$V = \frac{1}{2}\, \underline{\phi}^T \underline{\phi} \text{ is a Lyapunov function with}$$

$$\dot{V} = \underline{\phi}^T \underline{\dot{\phi}} = -ke^2 \leq 0. \qquad \text{(III.55)}$$

However $\dot{V}$ in (III.55) is only negative semi-definite, i. e., e may be zero for $\underline{\phi} \neq \underline{0}$, and $\underline{\phi} = \underline{0}$ is required for identification. By application of a theorem due to Lasalle [58], Lion has shown that global asymptotic stability will be achieved if u is periodic, and meets certain conditions as to frequency content.

Kudva and Narendra [57] have used direct adjustment of the model to solve the identification problem for the multivariable time-variable plant. In constrast to Lion, all plant states must be known. No restrictions are placed on the form of the state equations.

Given the state equations

$$\text{Plant } \dot{\underline{x}} = A^*\underline{x} + B^*\underline{r}$$

$$\text{Model } \dot{\underline{y}} = C\underline{y} + (A-C)\underline{x} + B\underline{r} \tag{III.56}$$

where C is a stability matrix, and A, B represent the model, then the error equation ($\underline{e} = \underline{y}-\underline{x}$) becomes

$$\dot{\underline{e}} = C\underline{e} + \Phi\underline{x} + \Psi\underline{u} \tag{III.57}$$

in which $\Phi \equiv A-A^*$, $\Psi \equiv B-B^*$. It is seen that $\underline{e} \to \underline{0}$ if the parameter misalignment matrices approach zero.

Following the scheme outlined in the Time-Varying Parameters section, for a single-input plant, the direct adjustment scheme for a time-variable multi-input plant becomes

$$\dot{\underline{a}}_i = -R_i\underline{a}_i - P\underline{e}x_i$$

$$\dot{\underline{b}}_i - -S_i\underline{b}_i - P\underline{e}r_i \tag{III.58}$$

where it is observed that (III.57) is a modification of (III.17). R_i, S_i are any positive definite diagonal matrices, and are introduced so that V will contain/a negative definite component in E(See the Time-varying Parameter section). It is noted that the model parameters ($\underline{a}_i$, $\underline{b}_i$), rather than the plant parameters, are adjusted in this case. Computer simulation of/a fourth-order two-input plant is shown to yield good results.

CHAPTER IV

THE SINGLE-INPUT SINGLE-OUTPUT ADAPTIVE OBSERVER - PART I

4.1 Motivation for an Adaptive Observer

In this chapter the first adaptive observer, which was developed by this author, is described. In the next chapter, the subsequent development by other investigators based upon the formulation described herein is detailed.

The motivation of the adaptive observer rests upon the necessity of certain knowledge of the system parameters for formulating a non-adaptive observer. This may be seen with reference to equation (II.6), copied below

$$\dot{e} = Fe + (F-T^{-1}AT + GCT)x + (D-T^{-1}B)r \qquad \text{(II. 6)}$$

As shown in section 2.2, the above equation is made asymptotically stable by defining

$$FT^{-1} = T^{-1}A-GC, \text{ i.e. } T^{-1}AT-GCT=F \qquad \text{(II. 7)}$$

$$D = T^{-1}B$$

and choosing the eigenvalues of F to all have negative real parts by choice of G.

If the elements of the matrices A, B, or C are unknown, then the equations (II. 6) cannot be assuredly satisfied. In this case, (II.6) becomes

$$\dot{e} = Fe + \Phi x + \Psi r \qquad \text{(IV. 1)}$$

for which $\Phi \neq 0$ and $\Psi \neq 0$. Thus, even if F may be chosen to be an asymptotically stable matrix, the error e in general does not vanish whenever the system is excited by a non-vanishing r.

Since e is defined as

$$e = z - x,$$

the condition that e does not vanish implies that the observer state z does not eventually reproduce the system state x.

It is therefore proposed that the matrices Φ and Ψ be adaptively adjusted to zero during system operation. Since (IV.1) is then asymptotically stable (for suitable choice of F) when $\Phi = \Psi = 0$, the observer state becomes equal to the system state.

Moreover, since Φ and Ψ are defined here as the ignorance in equations (II. 6) when A, B, and C are unknown, adaptively reducing Φ and Ψ to zero allows the values of A and B to be ascertained if C is known. Thus in addition to adaptively generating the state of an unknown system, the full order adaptive observer identifies the parameters of the unknown system.

As will be seen, the adaptive observer converges to a Luenberger observer, implying the noise suppression characteristic of the Luenberger observer is retained in some measure in the adaptive observer.

4.2 Development of the Adaptive Observer

The Luenberger observer [2]-[4] allows extraction of the state of an observable linear system when given (1) the system input, (2) the system output, (3) the form of the system, and (4) the parameter values of the system. In those cases for which the system parameters are unknown, the state observation is subject to error. Some previous investigators of parameter ignorance in observers [60], [61] alleviate to some degree the observation error, but they are unable to guarantee that the error vanishes or that their computational algorithm converges when the magnitude of parameter ignorange is large. The basics of a full order adaptive observer which negates these disadvantages have been previously reported [44]. The present chapter, following [71], considerably simplifies the exposition of the previous paper and extends, both computationally and theoretically, the topic of that paper. Briefly, the full order adaptive observer for single-input single-output observable continuous linear differential systems in the absence of a deterministic or random disturbance vector guarantees the vanishing of observation error regardless of the size of the constant or slowly varying parameter ignorance. The observer parameters are directly changed in a Lyapunov adaptive way so as to eventually yield the unknown full order Luenberger observer. The observer poles may throughout be placed freely in the stable region, and no derivatives are required in the adaptive law.

The Problem

A differential system is assumed of the form

$$\dot{w} = \tilde{A}w + Br, \quad w(0) = w^0$$
$$y = [1 \ \ 0 \ \ 0 \ \ \dots \ \ 0]w \equiv Cw$$
$$\tilde{A} \ n \times n$$
$$B \ n \times 1 \qquad \text{(IV. 2)}$$

for which only the single-output $y = Cw = w_1$ is available for measurement. It is assumed that a similarity transformation has been made, if necessary, so that the single-input single-output system has the form of (IV. 2). It is assumed that some or all of the elements of matrices $\tilde{A}$ and B are unknown, w^0 may be unknown, and the pair $(C,\tilde{A})$ is completely observable. The observer is of the form

$$\dot{z} = Kz + GCw + Dr + Hu, \quad z(0) = z^0$$

$$F\ n \times n \quad G\ n \times 1$$

$$D\ n \times 1 \quad H\ n \times n \text{ and diagonal} \qquad \text{(IV. 3)}$$

where K is arbitrary and u is a control vector yet to be defined, but with the property that $u \to 0$ as $t \to \infty$. The problem is to adaptively form a triple (G,D,T) so that the error vector defined as $e = z - T^{-1}w$ vanishes as the system adapts. T is a nonsingular square matrix with the property that CT = C. Figure 4.1 illustrates the adaptive strategy. It is noted that T^{-1} between w and x in Figure 4.1 is not physically realized.

The Adaptive Strategy

The overall strategy for solving the problem is to determine an adaptive law for progressively forming the triple (G,D,T). The adaptive law shall require that the error between the system output and the corresponding observer state variable be asymptotically stable in the sense of Lyapunov. This requirement alone does not guarantee that the error between the system state and the observer state vanishes, since the system output remains unchanged under a range of similarity transformations of the system state. In the section entitled "The Transformation" it is therefore shown that the adaptive observer generates but one of these similar states and that, by introducing the transformation matrix T, the system state can always be adaptively constructed by the constrained matrix $\hat{T}(t)$ whenever the system is observable, since $\lim_{t \to \infty} \hat{T}(t) = T$.

The required adaptive law may be explicitly obtained by the use of Lyapunov's direct method. In the section entitled "The Adaptive Law"

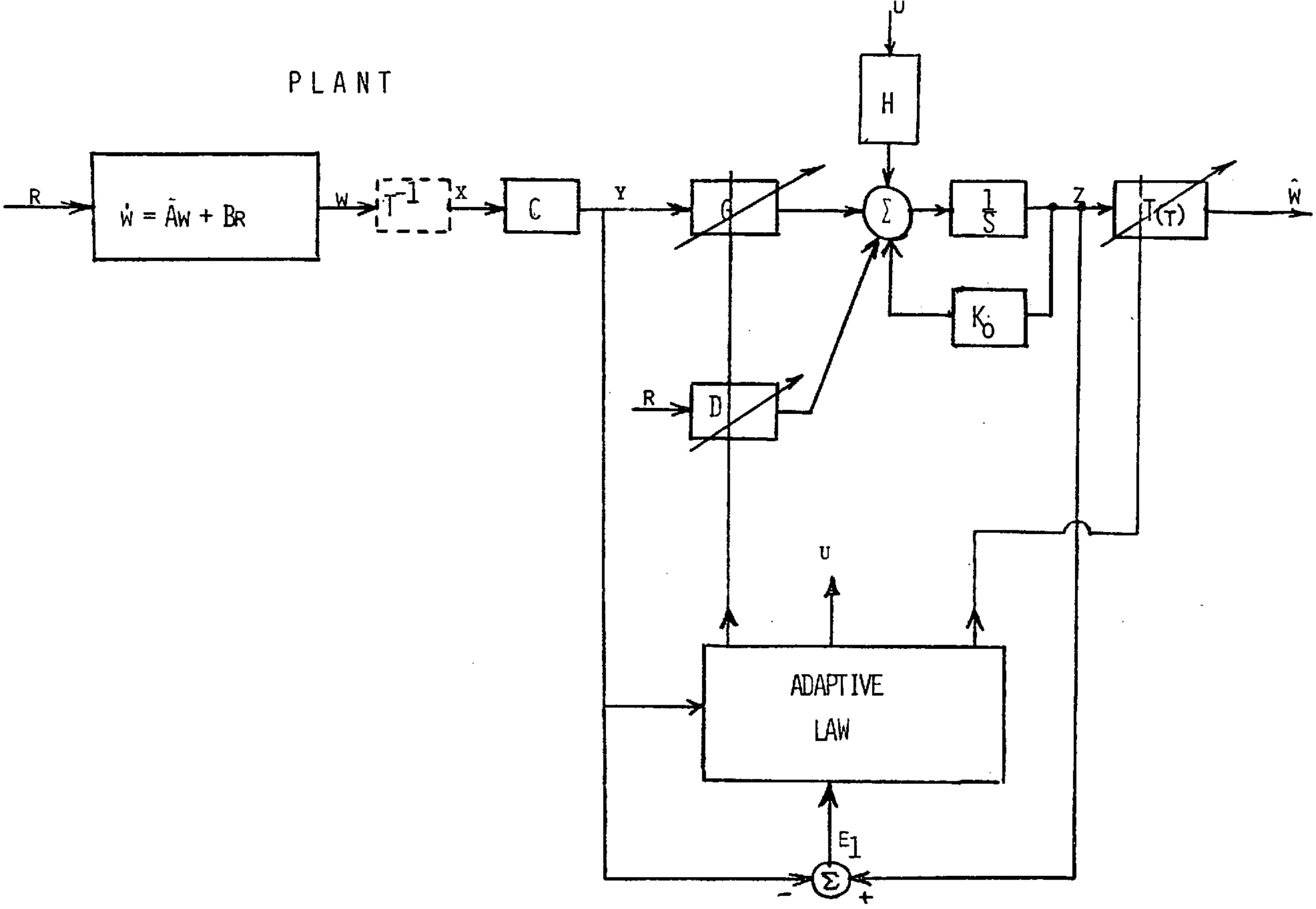


Figure 4.1
Single-Output Adaptive Observer

the equation of error is manipulated in such a way as to obtain it in terms of only those variables which can be directly measured without recourse to differentiation; the resultant equation is therefore necessarily defined on a noncompact manifold of the error space. A Lyapunov function is introduced on this noncompact manifold, and the adaptive law is accordingly synthesized. Theorem 2 expands the validity of the law to the compact manifold.

The Transformation

Define a transformation $x = T^{-1}w$ so that $e = z - x$. Then (IV. 2) becomes

$$\dot{x} = \tilde{A}_0 x + T^{-1}Br, \quad x(0) = T^{-1}w^0$$
$$y = CTx = Cx, \qquad \tilde{A}_0 = T^{-1}\tilde{A}T \tag{IV.2A}$$

and (IV.3) becomes

$$\dot{z} = K_0 z + GCx + Dr + Hu, \quad z(0) = z^0. \tag{IV.3A}$$

It is desired for subsequent development that $\tilde{A}_0 = T^{-1}\tilde{A}T$ be in the "output" form

$$\tilde{A}_0 = \begin{bmatrix} -a_{11} & 1 & 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & 1 & 0 & \cdots & 0 \\ -a_{31} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_{n1} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

wherein the first column contains the system parameters, and all other elements are zero, save the super-diagonal elements which are unity. The following theorem defines the restriction that must be placed upon $\tilde{A}$ so that both $\tilde{A}_0 = T^{-1}\tilde{A}T$ and $CT = C$.

THEOREM 4.1 [CARROLL]: Let $\tilde{A}$ be an $n \times n$ matrix, $C = [1\ 0\ 0\ \dots\ 0]$ a $1 \times n$ matrix, $\tilde{A}_0$ an $n \times n$ matrix in output form. Then there exists a matrix T such that $\tilde{A} = T\tilde{A}_0T^{-1}$ and $CT = C$ iff the pair $(C,\tilde{A})$ is completely observable.

PROOF: $(\tilde{A},C)$ is observable and by duality $(\tilde{A}^T,C^T)$ is controllable iff there exists a transformation Q such that

$$Q\tilde{A}^TQ^{-1} = \begin{bmatrix} 0 & & & \\ 0 & \cdot & & \\ \vdots & \vdots & I & \\ \cdot & & \cdots & \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}, \quad QC^T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \cdot \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$Q^{-T}\tilde{A}Q^T = \begin{bmatrix} 0 & 0 & \cdots & a_1 \\ & \cdots & & a_2 \\ & & \cdot & \\ & I & & a_3 \\ & & \cdot & \vdots \\ & & \cdot & \cdot \\ & & & a_n \end{bmatrix}, \quad CQ^T = [0\ \ 0\ \ \dots\ 0\ \ 1].$$

Let

$$M \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & 1 & 0 \\ \cdot & \cdot & \cdot & & \vdots \\ 0 & 1 & & & \cdot \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} = M^T = M^{-1}$$

and let

$$T = MQ^{-T}, \; T^{-1} = Q^T M^{-1}.$$

Then

$$T\tilde{A}T^{-1} = M[Q^{-T}\tilde{A}Q^{-T}]M^{-1} = \begin{bmatrix} a_n & 1 & 0 & \cdots & 0 \\ a_{n-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ a_2 & \vdots & & & 1 \\ a_1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Also,

$$C(MQ^{-T}) = C. \qquad \text{Q.E.D.}$$

As a result of the theorem, any observable system (IV. 2) may be placed by similarity transformation into system (IV.2A) with $CT = C$. The elements of T may be unknown since $\tilde{A}$ is unknown. The problem will be considered as defined by (IV.2A) and (IV.3A), so that $e = x - z$ must vanish. Eventually the problem of contructing w from x will be solved by introducing the adaptive matrix $\hat{T}(t)$. It will be shown that $\lim_{t\to\infty} \hat{T}(t) = T$ so that $\hat{w} = \lim_{t\to\infty} \hat{T}(t)z = w$ (in which $\hat{w}$ is the observer estimate of w) since $\lim_{t\to\infty} z = x$.

After the adaptation of the observer (IV.3) to the unknown system (IV.2) has been essentially completed, the values of G and D are then

$$D = T^{-1}B$$

$$G = \frac{(T^{-1}\tilde{A}T-K_0)C^T}{CC^T}.$$

Consequently, the adaptive observer converges to a Luenberger identity observer of the transformed system (IV.2A) with knowledge of parameters.

It is noted that there is no physical realization of matrix T^{-1} between w and x in Figure 4.1.

The Adaptive Law

It is now assumed, more for explanatory purpose than actual practical need, that some stable "nominal" plant matrix is either known or is chosen so that $\tilde{A}_0 = A_0 + \Delta A_0$, where A_0 has all known elements and is in output form. Consequently, ΔA_0 contains all zero elements, except for the left column which has elements that are to be adapted. Letting $e = z - x$, the vector error equation is

$$\dot{e} = Ke + (K + GC - A_0 - \Delta A_0)x + \Delta Br + Hu, \quad e(0) = e^0$$

where $\Delta B = D - T^{-1}B$. A theorem of Luenberger [2] allows the eigenvalues of A_0 - GC to be arbitrarily placed by selection of G (with the sole exception that A_0 - GC cannot have the same eigenvalues as A_0). For the above error equation, let $G = G_1 + G_2$ and $K = A_0 - G_2C$. Then as a result of the theorem of Luenberger and of the special forms of A_0 and C, the vector error equation is

$$\dot{e} = K_0e + (G_1C - \Delta A_0)x + \Delta Br + Hu \qquad \text{(IV. 4)}$$

where K_0 is an arbitrary stable constant matrix in output form with eigenvalues differing from A_0. The adaptive strategy is to change G_1 and ΔB to eliminate the influence of x and r in (IV.4); since by assuming K_0 is a constant matrix, changing G_1 is equivalent to changing G and will be considered as such in the ensuing.

For notational convenience in the next sections, the following definitions are made

$$K_0 = \begin{bmatrix} -k_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ -k_{n-3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -k_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$GC - \Delta A_0 = \begin{bmatrix} \alpha_{n-1} & 0 & 0 & \cdots & 0 \\ \alpha_{n-2} & 0 & 0 & \cdots & 0 \\ \alpha_{n-3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\Delta B = [0 \quad 0 \quad \cdots \quad 0 \quad \beta_m \quad \beta_{m-1} \quad \beta_{m-2} \cdots \quad \beta_0]^T. \qquad \text{(IV.5)}$$

$$G = [g_{n-1} \quad g_{n-2} \quad g_{n-3} \quad \cdots \; g_0]^T.$$

H = diag $[0, h_{n-2}, h_{n-3}, \ldots, h_o]$ and $n \times n$.

n = order of plant.

m = number of zeros in system transfer function.

The error between plant state x and observer state z may be measured only by the scalar state variable $e_1 = z_1 - y = z_1 - x_1$. To insure that only available measurements are called for in the adaptive laws, a differential equation of first order in terms of the error variable e_1 is derived by using the so-called reduction of order technique similar to that of Gilbart and Monopoli [51]. To this end, (IV.4) is first "collapsed" to yield the equivalent scalar form of the differential equation given by

$$\sum_{i=0}^{n} k_i e_1^{(i)} = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1-j} \binom{i+j}{i} \alpha_{i+j}^{(i)} x_1^{(j)}$$

$$+ \sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{i+j}{i} \beta_{i+j}^{(i)} r^{(j)}$$

$$+ \sum_{i=0}^{n-2} h_i u_i^{(i)} \tag{IV.6}$$

where $(\cdot)^{(i)}$ denotes the ith time derivative of $(\cdot)$.

Equation (IV.6) can now be reduced in order by first altering its form and then appropriately defining the n-1 elements of the control vector u. Thus, if λ_1 is taken to be a real and arbitrary characteristic value of K_0 and $p \triangleq d/dt$, then (IV.6) can be written in the form

$$(p + \lambda_1) \left(\sum_{i=0}^{n-1} a_i p^i\right) e_1 = \left(\sum_{i=0}^{n-1} a_i p^i\right) \left[\sum_{i=0}^{n+m} \phi_i v_i\right] + f_x$$

$$+ f_r + \sum_{j=0}^{n-2} h_j u_j^{(j)} \tag{IV.7}$$

where it is understood that the left side of (IV.7) is a partially factored form of the left side of (IV.6). The right side of (IV.7) has likewise been constructed to be equivalent to the right side of (IV.6). The underlying purpose is to form f_x and f_r from measurable quantities so that a u can be realized which satisfies the equation

$$f_x + f_r + \sum_{j=0}^{n-2} h_j u_j^{(j)} \equiv 0$$

subject to the condition that $\lim_{t\to\infty} u(t) = 0$. It follows, disregarding initial conditions, that (IV.7) can be reduced to the form

$$(p + \lambda_1)e_1 = \sum_{i=0}^{n+m} \phi_i v_i$$

where ϕ_i and v_i are appropriately defined to ensure the desired asymptotic behavior of u.

The defining equations of v_i are given by

$$\sum_{j=0}^{n-1} a_j v_i^{(j)} = x_1^{(i)}, \qquad i = 0, 1, 2, \ldots, n-2 \qquad \text{(IV.8)}$$

$$v_{n-1} = x_1$$

$$\sum_{j=0}^{n-1} a_j v_1^{(i)} = r^{(i-n)}, \qquad i = n, n+1, n+2, \ldots, 2(n-1) \qquad \text{(IV.8A)}$$

$$v_{2n-1} = r$$

where it is seen that each v_i is obtained from x_1 or r by low-pass filtering, and hence is realizable.

The defining equations for ϕ_i are given by

$$\phi_i = \begin{cases} \alpha_i - a_i\alpha_{n-1}, & i = 0, 1, 2, \ldots, n-2 \\ \alpha_{n-1}, & i = n - 1 \\ \beta_{i-n}, & i = n, n+1, n+2, \ldots, m+n < 2n-1. \end{cases} \qquad \text{(IV.9)}$$

However, should m=n-1, then (IV.9) should be corrected to the extent that

$$\phi_i = \begin{cases} \beta_{i-n} - a_{i-n}\,\beta_{n-1}, & i = n, n+1, n+2, \ldots, 2n-2 \\ \beta_{n-1}, & i = 2n-1. \end{cases} \qquad \text{(IV.9A)}$$

These have been chosen so that the terms in f_x and f_r contain only derivatives of ϕ_i as required, so that f_x and f_r may be realizable. Although ϕ_i is not directly measurable, it will be seen that $\dot{\phi}_i$ is a measurable quantity to be defined by the adaptive algorithm.

The resulting expressions for f_x and f_r become

$$f_x = \sum_{k=1}^{n-2} \sum_{j=0}^{k-1} \frac{d^j}{dt^j} [x^{(k-j-1)} \dot{\phi}_k]$$

$$- \sum_{k=0}^{n-2} \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-j} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k] \qquad \text{(IV.10)}$$

$$f_r = \sum_{k=1}^{m} \sum_{j=0}^{k-1} \frac{d^j}{dt^j} [\dot{\phi}_{n+k} r^{(k-j-1)}]$$

$$- \sum_{k=n}^{n+m} \sum_{j=0}^{n-2} \sum_{i=0}^{n-j-2} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k]. \qquad \text{(IV.11)}$$

However, if m=n-1, then (IV.11) should be changed to

$$f_r = \sum_{k=1}^{n-2} \sum_{j=0}^{k-1} \frac{d^j}{dt^j} [\dot{\phi}_{n+k} r^{(k-j-1)}]$$

$$- \sum_{k=n}^{2n-2} \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-j} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k]. \qquad \text{(IV.11A)}$$

It is further noted that, since $\dot{\phi}_k$ is the change in parameters due to adaptation, as adaptation is completed $\dot{\phi}_k \to 0, 0 \le k \le n + m$, and consequently, $\lim_{t\to\infty} u_j = 0$.

As noted, the implementation of u_j reduced (IV.7) to

$$(p + \lambda_1)(\sum_{i=0}^{n-1} a_i p^i) e_1 = (\sum_{i=0}^{n-1} a_i p^i) \left[\sum_{i=0}^{n+m} \phi_i v_i \right]. \tag{IV.12}$$

Taking the Laplace transform of (IV.12) and dividing by $\sum_{i=0}^{n-1} a_i s^i$ yields

$$(s + \lambda_1) s_1 = \left[\sum_{i=0}^{n+m} \phi_i v_i \right] + \frac{\mathcal{L}(\text{initial conditions})}{\sum_{i=0}^{n-1} a_i s^i} \tag{IV.13}$$

for which follows

$$\dot{e}_1 + \lambda_1 e_1 = \sum_{i=0}^{n+m} \phi_i v_i + \sum_{i=2}^{n-1} \psi_i \exp \ [-\lambda_i t] \tag{IV.14}$$

where ψ_i are unknown constants or time-dependent functions depending upon the initial conditions and $\{\lambda_i\}$, the set of characteristic values of $\sum_{i=0}^{n-1} a_i s^i$.

A Lyapunov function is now to be formed so that stability of the adaptive observer may be assured. To this end, a positive definite function of the measured error e_1 and the unknown parameter errors ϕ_i is defined as

$$V = \frac{1}{2} (m_s e_1^2 + \sum_{i=0}^{n+m} m_i \phi_i^2). \tag{IV.15}$$

Following Shackcloth [37], $\dot{V}$ can be made to be

$$\dot{V} = -m_s \lambda_1 e_1^2 + e_1 \sum_{i=0}^{n-1} \psi_i \exp[-\lambda_i t] \tag{IV.16}$$

when

$$\dot{\phi}_i = - \frac{m_s}{m_i} v_i e_1, \quad 0 \leq i \leq n + m. \tag{IV.17}$$

Other adaptive laws can easily be chosen instead [57].

Implementation of the adaptive law in (IV.17) can be accomplished by reference to (IV.9) and to the definitions of the variables α_i and β_i. For example,

$$\dot{\phi}_{n-1} = \dot{\alpha}_{n-1} = \dot{g}_{n-1} = -\frac{m_s}{m_{n-1}} x_1 e_1$$

$$\dot{\phi}_{n-2} = \dot{\alpha}_{n-2} - a_n - 2\dot{\alpha}_{n-1} = \dot{g}_{n-2} + a_{n-2} \frac{m_s}{m_{n-1}} x_1 e_1$$

$$= -\frac{m_s}{m_{n-2}} v_{n-2} e_1, \text{ etc.}$$

in which $\dot{g}_1$ may be ascertained.

From the form of $\dot{V}$, e_1 is stable in the sense of Lagrange with the region of attraction determined by the unknown constants ψ_i and the decaying exponential time function. Clearly, the region of attraction shrinks exponentially with time and eventually vanishes; consequently, e_1 is eventually asymptotically stable and $\lim_{t\to\infty} e_1 = 0$. All derivatives of e_1 must vanish in the limit as well since the scalar error equation (IV.14) is linear and of first order and possesses finite frequencies.

However, the Lyapunov function (IV.15) is defined on a noncompact manifold. Consequently, $\{\phi_i\}$ is shown to be (eventually) stable but not necessarily asymptotically stable [5]. It is evident from (IV. 4) that each ϕ_i must vanish by adaptation in order to observe the correct plant state. Theorem 2 defines the restriction placed upon r(t) in order to guarantee that each $\phi_i \to 0$ as $t \to 0$.

THEOREM 4.2 [*CARROLL*]: *Suppose there exists no set of real constants* $\{q_i\}$, $i = 0, 1, 2, \ldots, n + m$ *for which the command input* $r(t)$ *of the* *observable and controllable system in its steady-state condition is a solution of the homogeneous differential equation*

$$\sum_{i=0}^{n+m} q_i \frac{d^i}{dt^i} r = 0$$

where n and m are defined in (IV.5). Then $\lim_{t\to\infty} \phi_i(t) = 0$, $i = 0, 1, 2, \ldots, n + m$, *and* $\lim_{t\to\infty} e(t) = 0$ *is assured.*

PROOF: It has been shown that

$$\lim_{t\to\infty} e_1(t) = 0$$

$$\lim_{t\to\infty} u_j(t) = 0 \text{ for each } j$$

$$\lim_{t\to\infty} \phi_i(t) = \text{constant for each } i. \qquad \text{(IV.18)}$$

Therefore from (IV.8) and (IV.8A)

$$\lim_{t\to\infty} \alpha_i(t) = \text{constant for each } i$$

$$\lim_{t\to\infty} \beta_i(t) = \text{constant for each } i \qquad \text{(IV.19)}$$

Referring to n equations (IV.4), each equation may be differentiated in a manner so as to form the vector $e_s(t) = [e_1^{(n)}, e_2^{(n-1)}, e_3^{(n-2)}, \ldots, \dot{e}_n]^T$. Employing (IV.18) and (IV.19) in determining $\lim_{t\to\infty} e_s(t)$ and letting $\beta_i \equiv 0$ for $i > m$, (IV.20) results.

$$0 = e_2^{(n-1)} + \alpha_{n-1} y^{(n-1)} + \beta_{n-1} r^{(n-1)}$$

$$e_i^{(n+1-i)} = e_{i+1}^{(n-i)} + \alpha_{n-i} y^{(n-i)} + \beta_{n-i} r^{(n-i)}, \quad i = 2,3,4,\ldots n - 1$$

$$\dot{e}_n = \alpha_0 y + \beta_0 r. \qquad \text{(IV-20)}$$

All e's may be easily eliminated from (IV.20), yielding

$$0 = \left(\sum_{i=0}^{n-1} \alpha_i s^i\right)y + \left(\sum_{i=0}^{m} \beta_i s^i\right)r. \qquad \text{(IV.21)}$$

Let the stable (but unknown) plant transfer function be

$$\left(\sum_{i=0}^{n} a_i s^i\right)y = \left(\sum_{i=0}^{m} b_i s^i\right)r, \quad a_n \equiv 1. \qquad \text{(IV.22)}$$

Combining (IV.21) and (IV.22) yields

$$0 = \left[\left(\sum_{i=0}^{n-1} \alpha_i s^i\right)\left(\sum_{i=0}^{m} b_i s^i\right) + \left(\sum_{i=0}^{m} \beta_i s^i\right)\left(\sum_{i=0}^{n} a_i s^i\right)\right]r. \qquad \text{(IV.23)}$$

Equation (IV.23) represents a condition upon $r(t)$ which is assured in the limit by (IV.18) and (IV.19), that is to say, after adaptation has forced e_1 to vanish. Two distinct possibilities exist regarding the solution of the $(n + m)$th-order linear homogeneous differential equation (IV.23): a) either the steady-state system command input $r(t)$ obeys (IV.23) for some values α_i and β_i, or b) the $n + m + 1$ coefficients of polynomial in brackets are in the limit each zero. By supposition of the theorem, a) cannot occur; consequently b) must be true.

Using the assumption of observability and controllability to insure that (IV.22) is relatively prime, it is easy to show by mathematical induction that condition b) implies that the constants α_i and β_i are each zero, which was to be proved.

COROLLARY: If the steady-state command input $r(t)$ is periodic, a sufficient condition in order for $\lim_{t\to\infty} e(t) = 0$ in (IV.4) is that $r(t)$ contain at least $[n + m + 1]/2$ distinct frequencies in its steady-state condition.

It is emphasized that r is the command input to the system and not necessarily a special identification signal. If the steady-state r is periodic, then it must possess the required number of frequencies. However, an r which is steady-state aperiodic must meet no restrictions to satisfy Theorem 2.

Reconstruction of T

Using the "nominal" matrix A_0 as initial condition, the actual value of the system parameters may be determined by integrating the change in parameters $\{\phi_i\}$, defined in (IV.17), until adaptation is complete, and combining appropriately in the form of the matrix T. Thus $\hat{T}(t)$ "drifts" toward T as adaptation progresses and $\lim_{t\to\infty} \hat{T}(t) = T$. The example makes this technique clear.

$\hat{w}$, the estimate of w, is constructed from the observer output z by forming $\hat{T}(t)z$. Consequently, $\lim_{t\to\infty} \hat{w} = w$.

Example

A third-order plant with one zero is considered for illustration. Let the plant be described by

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + \alpha_0) & -(a_1 + \alpha_1) & -(a_2 + \alpha_2) \end{bmatrix} w + \begin{bmatrix} 0 \\ c_1 \\ c_0 \end{bmatrix} r$$

$$y = w_1 \qquad \text{(IV.2*)}$$

in which α_0, α_1, α_2, c_0, and c_1 are unknown. a_0, a_1, a_2 are the nominal values. In output form, (IV.1*) is

$$\dot{x} = \begin{bmatrix} -(a_2 + \alpha_2) & 1 & 0 \\ -(a_1 + \alpha_1) & 0 & 1 \\ -(a_0 + \alpha_0) & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_1 - \beta_1 \\ b_0 - \beta_0 \end{bmatrix} r \qquad \text{(IV.2A*)}$$

$$y = x_1 = w_1.$$

The error equation (IV.4) is now

$$\dot{e} = \begin{bmatrix} -k_2 & 1 & 0 \\ -k_1 & 0 & 1 \\ -k_0 & 0 & 0 \end{bmatrix} e + \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ \beta_1 \\ \beta_0 \end{bmatrix} r + \begin{bmatrix} 0 \\ u_1 \\ u_0 \end{bmatrix} \qquad \text{(IV-4*)}$$

and the scalar equation (IV.6) is now

$$\dddot{e}_1 + k_2\ddot{e}_1 + k_1\dot{e}_1 + k_0e_1 = (\alpha_0 + \dot{\alpha}_1 + \ddot{\alpha}_2)x_1$$
$$+ (\alpha_1 + 2\dot{\alpha}_2)\dot{x}_1 + \alpha_2\ddot{x}_1 + \beta_1\dot{r} + (\beta_0 + \dot{\beta}_1)r + \dot{u}_1 + u_0. \qquad \text{(IV.6*)}$$

The scalar error equation (IV.6*) is equivalent to

$$(p + \lambda_1)(p^2 + a_1p + a_0)e_1 = (p^2 + a_1p + a_0)\left(\sum_{i=0}^{4} \phi_i v_i\right)$$
$$+ x\dot{\phi}_1 - \sum_{k=0}^{1} \sum_{j=0}^{1} \sum_{i=0}^{1-j} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k] + \dot{\phi}_4 r$$
$$- \sum_{k=3}^{4} \sum_{j=0}^{1} \sum_{i=0}^{1-j} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k] + \dot{u}_1 + u_0. \qquad \text{(IV.7*)}$$

It is seen from the right side of (IV.7*) that v_k and $\dot{v}_k$ must be generated in order to allow removal by u_0 and u_1. v_k, which is defined in (IV.8), may be generated by state-space means.

To illustrate generation of v_k, the following notation is adopted:

$$v_k(1) = v_k$$
$$v_k(2) = \frac{d}{dt} v_k$$
$$\vdots$$
$$v_k(i) = \frac{d^i}{dt^i} v_k.$$

Then the generation of v_1 and $\dot{v}_1$ defined by

$$\ddot{v}_1 + a_1\dot{v}_1 + a_0 v_1 = \dot{x}_1$$

results in

$$\begin{bmatrix} \dot{v}_1(1) \\ \dot{v}_1(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} v_1(1) \\ v_1(2) \end{bmatrix} + \begin{bmatrix} 1 \\ -a_1 \end{bmatrix} x_1.$$

Thus both v_1 and $\dot{v}_1$ are available measurements without the benefit of differentiation of v_1 or x_1. Other v_k may be generated similarly.

Using this notation, the implementation of inputs as

$$u_0 = \dot{\phi}_0(v_0(2) + a_1 v_0(1)) + \dot{\phi}_1(v_1(2) + a_1 v_1(1))$$

$$+ \dot{\phi}_3(v_3(2) + a_1 v_3(1)) + \dot{\phi}_4(v_4(2) + a_1 v_4(1))$$

$$u_1 = \dot{\phi}_0 v_0(1) + \dot{\phi}_1 v_1(1) + \dot{\phi}_4 v_4(1) + \dot{\phi}_3 \; v_3(1)$$

reduces (IV.7*) to

$$(p + \lambda_1)(p^2 + a_1 p + a_0)e_1 = (p^2 + a_1 p + a_0) \left(\sum_{i=0}^{4} \phi_i v_i \right) \qquad \text{(IV.12*)}$$

Defining $\dot{\phi}_1$ as in (IV.17), the observer has the form

$$\dot{z} = \begin{bmatrix} -k_2 & 1 & 0 \\ -k_1 & 0 & 1 \\ -k_0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} g_2 \\ g_1 \\ g_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b_1 \\ b_0 \end{bmatrix} r + \begin{bmatrix} 0 \\ u_1 \\ u_0 \end{bmatrix}$$

where

$$\dot{b}_1 = -\frac{m_s}{m_4} e_1 v_4$$

$$\dot{b}_0 = -\frac{m_s}{m_3} e_1 v_3$$

$$\dot{g}_2 = -\frac{m_s}{m_2} e_1 x_1$$

$$\dot{g}_1 = -e_1 \left(\frac{m_s}{m_1} v_1 + a_1 \frac{m_s}{m_2} x_1\right)$$

$$\dot{g}_0 = -e_1 \left(\frac{m_s}{m_0} v_0 + a_0 \frac{m_s}{m_2} x_1\right)$$

and

$$\hat{w} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \int_0^t \dot{g}_2 dt - a_2 & 1 & 0 \\ \left[\int_0^t \dot{g}_2 dt - a_2\right]^2 + \int_0^t g_1 dt - a_1 & \int_0^t \dot{g}_2 dt - a_2 & 1 \end{array}\right] z$$

$$z = \hat{T}(t) z.$$

$\hat{w}$ is the estimate of plant state w, and $\lim_{t\to\infty} \hat{w} = w$.

Note that CT=C.

A. *Simulation*

The third-order system of the example was simulated on a digital computer using the following parameters:

$a_0 = 24$	$\alpha_0 = 0$	$c_1 = 30$	$k_0 = 24$	$m_0/m_3 = 8000$
$a_1 = 26$	$\alpha_1 = 74$	$c_2 = 195$	$k_1 = 26$	$m_0/m_5 = 2000$
$a_2 = 9$	$\alpha_2 = 0$	$b_1 = 30$	$k_2 = 9$	$g_0 = g_2 = 0$

The eigenvalues of the observer (determined by $\{k_i\}$)were $\lambda_1 = -4$, $\lambda_2 = -2$, $\lambda_3 = -3$. The input to the plant was a square wave of magnitude 1 and frequency 6t. Two parameters, b_0 and g_1 were adjusted by the adaptive law. These were initially set at $b_0 = 73$, $g_1 = -5$ corresponding to a correct value of $b_0 = 75$, $g_1 = -74$. Figure 4.2 illustrates the behavior of b_0, g_1, e_2 and e_3 as a function of time.

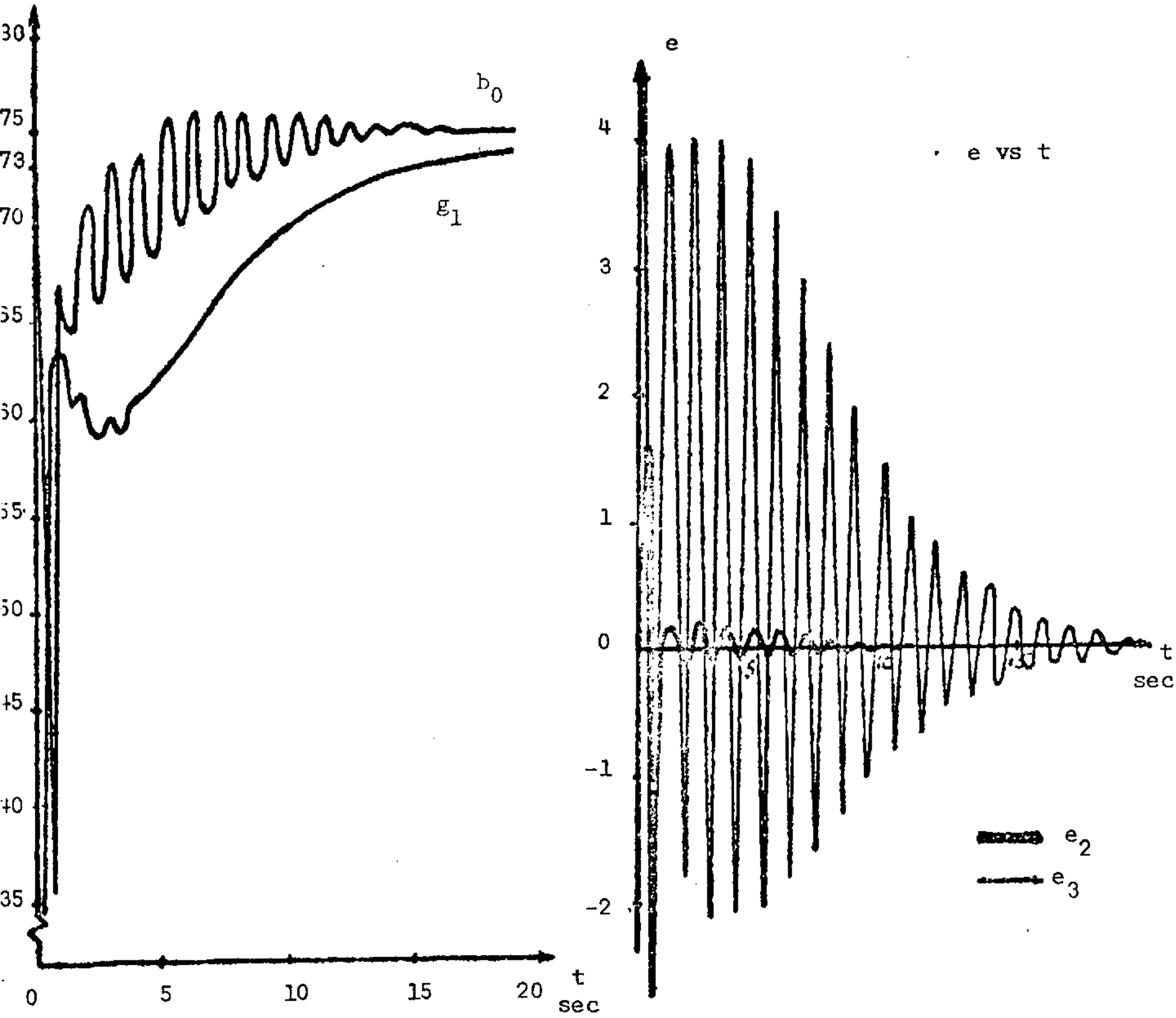


Figure 4.2

Adaptive Response

It is noted that the behavior is somewhat slow and oscillatory.

Remark

As has been previously stated, $\hat{w}$ and $\hat{T}z$ and $\lim_{t\to\infty} \hat{w} = w$. In the general case of an artibrary plant matrix $\tilde{A}$, the determinant of $\hat{T}$ may vanish for some instances of time. These momentary occurrences, of course, have no detrimental effect on $\hat{w}$ since convergence of $\hat{w}$ to w is guaranteed. It need not be feared, either, that the elements of $\hat{T}$ may become impossibly large at any time during operation, since the elements of $\hat{T}$ are guaranteed by the Lyapunov function (IV.15) and (IV.16) to be stable in the sense of Lyapunov and therefore bounded.

The following theorem summarizes the results of this section.

THEOREM 4.3 [*CARROLL*] *A full-order adaptive observer (IV.3A) can be constructed to observe the state of and to identify the parameters of system (IV.2) having unknown parameters in matrices $\tilde{A}$ and B iff $(\tilde{A},C)$ is a completely observable pair and if the command input r is periodic, possesses more steady state frequencies than half the number of parameters being adapted.*

The adaptation is accomplished by algorithm (IV.17), dependent upon definitions (IV.9) or (IV.9A), when the control u is implemented according to (IV.7) subject to definitions (IV.8), (IV.8A), (IV.10), (IV.11), (IV.11A).

4.3 An Alternate Adaptive Algorithm for the Observer

In Section 4.2 the single-output adaptive observer was synthesized. The adaptive algorithm depends upon the filtered-output and filtered-input variables v_i, defined in eqs. (IV.8) and (IV.8A). The transfer function of these filters is

$$\frac{s^i}{s^{n-1}+a_{n-2}s^{n-2}+\ldots a_1 s+a_0} \qquad \text{(IV.24)}$$

for i=0, 1, 2, ..., n-2. This configuration apparently allows effective noise suppression since high frequency signals contained in the input of the filters are attenuated.

However, according to (IV.9) and (IV.17), the adaptive laws for α_i and β_i are given by

$$\dot{\alpha}_i = -\left(\frac{m_s}{m_i} v_i + \frac{m_s}{m_{n-1}} a_i x_1\right)e_1 \qquad i = 0, 1, \ldots, n-2$$

$$\dot{\beta}_{i-n} = -\left(\frac{m_s}{m_i} v_i + \frac{m_s}{m_{2n-1}} a_{i-n}\, r\right)e_1 \qquad i = n, n+1, \ldots, 2n-2$$

except for i = n-1 and i = 2n-1, and for which it is assumed that $\alpha_{n-1}\neq 0$, $\beta_{n-1}\neq 0$. Therefore for the case in which $\alpha_{n-1}\neq 0$, $\beta_{n-1}\neq 0$, the adaptive algorithm depends upon the combination of v_i and x_1 or r. The resulting transfer function is, for the first of the above equations,

$$\frac{m_s}{m_i}\;\frac{s^i}{s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_1 s+a_0} + \frac{m_s}{m_{n-1}} a_i$$

which equals

$$\frac{\frac{m_s}{m_{n-1}} a_i \left(s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_0\right)+ \frac{m_s}{m_i} s^i}{s^{n-1} + a_{n-2}s^{n-2}+\ldots+ a_1 s + a_0} \qquad \text{(IV.25)}$$

A similar expression is obtainable for the second of the adaptive equations.

From the appearance of (IV.25), it seems doubtful that the resulting filter offers much noise suppression in the adaptive law for the case where $\alpha_{n-1} \neq 0$, $\beta_{n-1} \neq 0$.

In this section the adaptive algorithm is altered so that it depends *directly* upon filters having transfer functions of the form of (IV.24), thus enhancing the noise suppression capability. In addition, the filter poles in the alternate algorithm may be chosen *distinct* from the observer eigenvalues if desired, and the *Gilbart and Monopoli adaptive law* ([46]; also refer to the section Degree of Stability in Chapter III) *may be implemented in order to improve the speed of response of the adaptive algorithm.*

Development of the Algorithm

The new algorithm is based upon the same form of the observer as the form of the observer in Section 4.2 with the exception that, in eqs. (IV.5), the matrix H is defined as

$$H = \text{diag}\,[h_1, h_2, \ldots, h_n] \text{ and } n\text{x}n;$$

that is to say, h_1 is here non-zero. Consequently the discussion of the transformation and the error equation is the same as in Section 4.2. The new algorithm is therefore developed from the starting point of the error equation (IV.4)

$$\dot{e} = K_0 e + (GC - \Delta A_0)x + \Delta B + Hu \qquad \text{(IV.4)}$$

in which

$$K_0 = \begin{bmatrix} -k_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -k_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$GC-\Delta A_0 = \begin{bmatrix} \phi_1 & 0 & 0 & \cdots & 0 \\ \phi_2 & 0 & 0 & \cdots & 0 \\ \phi_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_n & 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \qquad \text{(IV.26)}$$

$$\Delta B = [\psi_1, \psi_2, \ldots, \psi_n]^T$$

The vector error equation (IV.4) is collapsed (in the same manner as (IV.6)) to yield

$$\sum_{i=0}^{n} k_i e_1^{(i)} = \sum_{k=1}^{n} \left[\phi_k x_1^{(n-k)} + \sum_{j=0}^{n-k-1} p^j \left(\dot{\phi}_k x^{(n-k-j-1)}\right)\right]$$

$$+ \sum_{k=1}^{n} \left[\psi_k r^{(n-k)} + \sum_{j=0}^{n-k-1} p^j \left(\dot{\psi}_k r^{(n-k-j-1)}\right)\right]$$

$$+ \sum_{j=1}^{n} h_j u_j^{(n-j)} \qquad \text{(IV.27)}$$

in which $p^j \equiv \frac{d^j}{dt^j}$ and in which it is to be understood that the second summation is to be ignored when $n-k-1 < 0$.

The intent at this point is to define the vector v in such a manner so as to make the right side of (IV.27) equal

$$\left(\sum_{i=0}^{n-1} c_i p^i\right)\left[\sum_{i=1}^{n} (\phi_i v_i + \psi_i s_i)\right] + \sum_{j=2}^{n} h_j w_j^{(n-j)} \qquad \text{(IV.28)}$$

where v_i and s_i are filtered-output and filtered-input variables with transfer functions similar to (IV.24). w is a vector to be defined later.

The function v_i and s_i are accordingly defined as

$$\sum_{j=0}^{n-1} c_j v_i^{(j)} = x_1^{(n-i)}$$

$$\sum_{j=0}^{n-1} c_j s_i^{(j)} = r^{(n-i)} \qquad \text{(IV.29)}$$

$$i = 1, 2, 3, \ldots, n \qquad \text{and } c_{n-1} \equiv 1.$$

It is pointed out tha the 2n equations (IV.28) actually represent two low-pass filters of order n-1 each. This follows from the transfer function.

$$v_i = \frac{s^{n-i} x_1}{s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0}$$

$$s_i = \frac{s^{n-i} r}{s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0} \qquad \text{(IV.29A)}$$

$$i = 1, 2, 3, \ldots, n$$

from which it is seen that

$$\dot{v}_i = v_{i-1}$$

$$\dot{s}_i = s_{i-1} \qquad \text{(IV.30)}$$

From the identity that

$$p^i\,(\phi v) = \sum_{j=0}^{i-1} p^j\,(\dot{\phi} v^{(i-j-1)}) + \phi v^{(i)},$$

it follows that (IV.28) equals

$$\sum_{k-1}^{n} \sum_{i=1}^{n-1} c_i \left[\sum_{j=0}^{i-1} p^j\,(\dot{\phi}_k v_k^{(i-j-1)}) + \phi_k v_k^{(i)}\right] + \sum_{k-1}^{n} c_0 \phi_k v_k$$

$$+ \sum_{k=1}^{n} \sum_{i=1}^{n-1} c_i \left[\sum_{j=0}^{i-1} p^j\,(\dot{\psi}_k s_k^{(i-j-1)}) + \psi_k s_k^{(i)}\right] + \sum_{k=1}^{n} c_0 \psi_k s_k + \sum_{j=2}^{n} h_j w_j^{(n-j)}$$

This expression is set equal to the right side of (IV.27) and the vector u is subsequently determined. After employing (IV.29), the resulting expression is

$$\sum_{j=1}^{n} h_j u_j^{(n-j)} = \sum_{j=2}^{n} h_j w_j^{(n-j)} + \sum_{k=1}^{n} \sum_{j=n-k}^{n-2} p^j \Big(\sum_{i=j+1}^{n-1} c_i\, \dot{\phi}_k v_k^{(i-j-1)} \Big)$$

$$- \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} p^j [\dot{\phi}_k (x_1^{(n-k-j-1)} - \sum_{i=j+1}^{n-1} c_i v_k^{(i-j-1)})]$$

$$+ \sum_{k=1}^{n} \sum_{j=n-k}^{n-2} p^j \Big(\sum_{i=j+1}^{n-1} c_i\, \dot{\psi}_k s_k^{(i-j-1)} \Big)$$

$$- \sum_{k-1}^{n-1} \sum_{j=0}^{n-k-1} p^j\, [\dot{\psi}_k (r^{(n-k-j-1)} - \sum_{i=j+1}^{n-1} c_i s_k^{(i-j-1)})]$$

for which it is understood that a summation term is to be ignored if the upper index is less than the lower index, and that $n \geq 2$. This expression is simplified by use of (IV.29A) and (IV.30) so that

$$\sum_{j=0}^{n-1} h_{n-j} u_{n-j}^{(j)} = \sum_{j=0}^{n-2} h_{n-j} w_{n-j}^{(j)}$$

$$+ \sum_{k=2}^{n} \sum_{j=n-k}^{n-2} p^j \left(\sum_{i=j+1}^{n-1} c_i \dot{\phi}_k v_{k-i+j+1} \right)$$

$$- \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} p^j \left(\dot{\phi}_k \sum_{i=0}^{j} c_{j-i} v_{k+i+1} \right)$$

$$+ \sum_{k=2}^{n} \sum_{j=n-k}^{n-2} p^j \left(\sum_{i=j+1}^{n-1} c_i \dot{\psi}_k s_{k-i+j+1} \right)$$

$$- \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} p^j \left(\dot{\psi}_k \sum_{i=0}^{j} c_{j-i} s_{k+i+1} \right)$$

From the above, the vector u can easily be determined by equating terms containing equal values of j on both sides. The result is

$$h_n u_n = h_n w_n - c_0 \left[\sum_{k=1}^{n-1} (\dot{\phi}_k v_{k+1} + \dot{\psi}_k s_{k+1}) \right]$$

$$h_j u_j = \sum_{k=j}^{n} \sum_{i=n-j+1}^{n-1} c_i (\dot{\phi}_k v_{k-i+n-j+1} + \dot{\psi}_k s_{k-i+n-j+1})$$

$$- \sum_{k=1}^{j-1} \left[\dot{\phi}_k \sum_{i=0}^{n-j} c_{n-j-i} v_{k+i+1} + \dot{\psi}_k \sum_{i=0}^{n-j} c_{n-j-i} s_{k+i+1} \right] + h_j w_j \tag{IV.31}$$

for j = 1,2, ..., n-1. It is understood in (IV.31) that summation terms are to be ignored if the upper index is less than the lower index. The function w_j will be defined later.

By employing (IV.31), the collapsed error equation (IV.27) becomes

$$\sum_{i=0}^{n} k_i e_1^{(i)} = \left(\sum_{i=0}^{n-1} c_i p^i\right) \left[\sum_{k=1}^{n} (\phi_k v_k + \psi_k s_k)\right] + \sum_{j=0}^{n-1} h_{n-j} w_{n-j}^{(j)}$$

$$k_n \equiv 1 , \quad C_n \equiv 1 \qquad \text{(IV.32)}$$

The Laplace transform of (IV.32) is taken and both sides of the equation is divided by $\sum_{i=0}^{n-1} c_i s^i$ yielding

$$\frac{\sum_{i=0}^{n} k_i s^i}{\sum_{i=0}^{n-1} c_i s^i} e_1 = \mathcal{L}\left[\sum_{k=1}^{n} (\phi_k v_k + \psi_k s_k)\right] + \frac{\sum_{j=0}^{n-2} h_{n-j} s^j w_{n-j}}{\sum_{i=0}^{n-1} c_i s^i} + \frac{f(s)}{\sum_{i=0}^{n-1} c_i s^i} \qquad \text{(IV.33)}$$

for which f(s) depends upon the initial conditions of the variables in (IV.32). On the left side of (IV.33) is the ratio of polynomials

$$\frac{\sum_{i=0}^{n} k_i s^i}{\sum_{i=0}^{n-1} c_i s^i} = s + k_{n-1} - c_{n-2} + \frac{\sum_{i=0}^{n-2} \rho_i s^i}{\sum_{i=0}^{n-1} c_i s^i} \qquad \text{(IV.34)}$$

where

$$\rho_0 = k_0 - c_0(k_{n-1} - c_{n-2})$$

$$\rho_i = k_i - c_{i-1} - c_i(k_{n-1} - c_{n-2}) \qquad i = 1, 2, \ldots, n-2$$

Defining

$$h_{n-j}\, w_{n-j} = \rho_j\, e_1 \qquad j = 0,\ 1, 2,\ \ldots,\ n-2$$

then (IV.33) becomes

$$(s + k_{n-1} - c_{n-2})e_1 = [\sum_{k=1}^{n} \mathcal{L}(\phi_k v_k + \psi_k s_k)] + \frac{f(s)}{\sum_{i=0}^{n-1} c_i s^i} \tag{IV.35}$$

In the time domain (IV.35) is

$$\dot{e}_1 + (k_{n-1} - c_{n-2})e_1 = \sum_{k=1}^{n} (\phi_k v_k + \psi_k s_k) + \sum_{i=1}^{n-1} \chi_i \exp[-\lambda_i t] \tag{IV-36}$$

Here χ_i is a bounded function dependent upon the unknown initial condition of the equation (IV.27). Eq. (IV.36) is equivalent to (IV.27) and to (IV.4) with the exception that (IV.36) is a scalar equation in terms of e_1. A Liapunov function candidate can be constructed so that it depends upon e_1 but not the other elements of the vector error e. This function will be chosen according to [46] so as to allow a faster rate of convergence of the resulting adaptive law.

$$V = \frac{1}{2} e_1^2 + \sum_{i=1}^{n} \frac{1}{2m_i} (\phi_i + q_i v_i e_1)^2 + \sum_{i=1}^{n} \frac{1}{2\tilde{m}_i} (\psi_i + \tilde{q}_i s_i e_1)^2$$

The time derivative of V along the trajectory described by (IV.36) is

$$\dot{V} = -(k_{n-1} - c_{n-2})e_1^2 + \sum_{k=1}^{n} (\phi_k v_k + \psi_k s_k)e_1$$

$$+ \sum_{i=1}^{n} \frac{1}{m_i} (\phi_i + q_i v_i e_1)(\dot{\phi}_i + q_i \dot{\overline{v_i e_1}})$$

$$+ \sum_{i=1}^{n} \frac{1}{\tilde{m}_i} (\psi_i + \tilde{q}_i s_i e_1)(\dot{\psi}_i + \tilde{q}_i \dot{\overline{s_i e_1}})$$

$$+ e_1 \sum_{i=1}^{n-1} \chi_i \exp[-\lambda_i t]$$

Define

$$\dot{\phi}_i = - m_i v_i e_1 - q_i \dot{\overline{v_i e_1}}$$

$$\dot{\psi}_i = - \tilde{m}_i s_i e_1 - \tilde{q}_i \dot{\overline{s_i e_1}} \qquad \text{(IV.37)}$$

$$m_i > 0, \ \tilde{m}_i > 0, \ q_i \geq 0, \ \tilde{q}_i \geq 0 \quad i = 1, 2, \ldots, n$$

Then

$$\dot{V} = -(k_{n-1} - c_{n-2})e_1^2 - \sum_{i=1}^{n} (q_i v_i^2 e_1^2 + \tilde{q}_i s_i^2 e_1^2) + \sum_{i=1}^{n-1} \chi_i \exp[-\lambda_i t]$$

If the set of constants $\{c_i\}$ are chosen so that the filter poles all have negative real parts and if $k_{n-1} - c_{n-2} > 0$, then $\chi_i \exp[-\lambda_i t]$ vanishes exponentially and $\dot{V}$ is then negative semidefinite on the compact manifold (e, Φ, Ψ) but negative definite on the non-compact manifold over which V is defined. Consequently, $e_1 \to 0$.

The expansion to the noncompact manifold - i.e. require $e, \Phi, \Psi \to 0$ - has been discussed in Section 4.2. The results obtained there, in terms of a steady-state frequency requirement for the input signal r, applies equally here. The reader is referred to Section 4.2 for a discussion of this.

The adaptive equations (IV.37) appear to require the generation of a derivative for implementation whenever either q_i or $\tilde{q}_i$ is positive. Of course, in the actual adaptation of ϕ and ψ, no derivative is required. Since $\phi_i = g_i - \Delta a_i$ and $\psi_i = d_i - b_i$ where Δa_i and b_i are constant, (IV.37) defines

$$\dot{g}_i = -m_i v_i e_1 - q_i \dot{\overline{v_i e_1}} \qquad \text{(IV.37A)}$$

$$\dot{d}_i = -\tilde{m}_i s_i e_1 - \tilde{q}_i \dot{\overline{s_i e_1}}$$

or, what is the same

$$g_i = -q_i v_i e_1 - m_i \int_0^t v_i e_1 \, dt \qquad \text{(IV.37B)}$$

$$d_i = -q_i \dot{\overline{s_i e_1}} - \tilde{m}_i \int_0^t s_i e_1 \, dt$$

In (IV.37B) it is seen that the parameter estimates g_i and d_i for each i are available without recourse to differentiation.

However, implementation of (IV.31) seems to require the availability of $\dot{\phi}$ and $\dot{\psi}$ which are dependent upon a differentiation network if q_i or $\tilde{q}_i$ is positive. Nevertheless, the differentiation can be avoided by careful manipulation of the u_i terms in (IV.31). The basic notion here is, whenever $\dot{\phi}_i$ and $\dot{\psi}_i$ appear in a u_j term, to place the derivative part of $\dot{\phi}_i$ and $\dot{\psi}_i$ in u_{j-1} without implementing a derivative network. The relocation of the derivative part of $\dot{\psi}_i$ or $\dot{\phi}_i$ in a different u implicitely generates the differentiation.

This operation is accomplished by use of the identity

$$\overline{e\dot{v}_1}\ v_2 = \frac{d}{dt}(ev_1v_2) - ev_1\dot{v}_2$$

The results are found to be

$$h_n u_n = \sum_{k=1}^{n} c_0(m_k\ v_{k+1} - q_k\ v_k)v_k\ e_1 + h_n w_n$$

$$+ \sum_{k=1}^{n} c_0(\tilde{m}_k\ s_{k+1} - \tilde{q}_k\ s_k)\ s_k\ e_1$$

$$h_j u_j = \sum_{k=j}^{n} \sum_{i=n-j+1}^{n-1} c_i\ e_1\ v_k\ (q_k\ v_{k-i+n-j} - m_k\ v_{k-i+n-j+1})$$

$$+ \sum_{k=1}^{j-1} e_1\ v_k \sum_{i=0}^{n-j} (m_k\ c_{n-j-i}\ v_{k+i+1} - q_k\ c_{n-j-i}\ v_{k+i})$$

$$+ \sum_{k=1}^{j} q_k\ e_1\ v_k \sum_{i=0}^{n-j+1} c_{n-j-i-1}\ v_{k+i+1}$$

$$- \sum_{k=1+j}^{n} \sum_{i=n-j}^{n-1} c_i\ q_k\ e_1\ v_k\ v_{k-i+n-j-1}$$

$$+ \sum_{k=j}^{n} \sum_{i=n-j+1}^{n-1} c_i\ e_1\ s_k\ (\tilde{q}_k\ s_{k-i+n-j} - \tilde{m}_k\ s_{k-i+n-j+1})$$

$$+ \sum_{k=1}^{j-1} e_1\ s_k \sum_{i=0}^{n-j} (\tilde{m}_k\ c_{n-j-i}\ s_{k+i+1} - \tilde{q}_k\ c_{n-j-i}\ s_{k+i})$$

$$+\sum_{k=1}^{j} \tilde{q}_k\, e_1\, s_k \sum_{i=0}^{n-j+1} c_{n-j-i-1}\, s_{k+i+1}$$

(IV.38)

$$-\sum_{k=j+1}^{n} \sum_{i=n-j}^{n-1} c_i\, \tilde{q}_k\, e_1\, s_k\, s_{k-i+n-j-1} + h_j w_j$$

$$h_1 u_1 = -\sum_{k=2}^{n} q_k e_1 v_k v_{k-1} - q_1 e_1 v_1 \sum_{i=0}^{n} c_{n-i-2}\, v_{2+i}$$

$$-\sum_{k=2}^{n} \tilde{q}_k\, e_1\, s_k\, s_{k-1} - \tilde{q}_1\, e_1\, s_1 \sum_{i=0}^{n} c_{n-i-2}\, s_{2+i}$$

For reference with the use of the above, the definitions of w_j are repeated

$$h_{n-i}\, w_{n-i} = \rho_i\, e_1$$
$$\rho_0 = k_0 - c_0\, (k_{n-1} - c_{n-2}) \qquad \text{(IV.34)}$$
$$\rho_i = k_i - c_{i-1} - c_i\, (k_{n-1} - c_{n-2})$$
$$i = 0,\ 1,\ 2,\ \ldots,\ n-2$$

It is noted that u has the desirable property that $\lim_{t\to\infty} u = 0$.

To summarize, the theorem 4.4 is given.

Theorem 4.4 [*CARROLL*]

A full-order adaptive observer (IV.3A) can be constructed to observe the state of and to identify the parameters of system (IV.2) having unknown parameters in matrices $\tilde{A}$ and B iff ($\tilde{A}$, C) is a completely observable pair and, if the command input r is periodic, r possesses more steady state frequencies than half the number of parameters being adapted.

The adaptation is accomplished by the algorithm (IV.37B) dependent *upon low-pass filtration of the system input and output. The filters are given by (IV.29) or (IV.29A) and the poles of these filters may be freely chosen independently of the observer eigenvalues whenever*

a) $k_{n-1} - c_{n-2} > 0$

b) the filter poles are chosen so that all lie in left half plane, and

c) the control input u is constructed according to (IV.38) and (IV.34)

Example

The system of Section 4.2 is used here as on example of the altered adaptive law. Let the system be as (IV.1*) and the error equation is

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = \begin{bmatrix}-k_2 & 1 & 0\\ -k_1 & 0 & 1\\ -k_0 & 0 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix} + \begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} x_1 + \begin{bmatrix}0\\ \psi_2\\ \psi_3\end{bmatrix} r + \begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}$$

where

$$\phi_1 = g_1 + \alpha_2 \qquad \psi_2 = d_2 + \beta_1$$

$$\phi_2 = g_2 + \alpha_1 \qquad \psi_3 = d_3 + \beta_3$$

$$\phi_3 = g_3 + \alpha_0.$$

The collapsed error equation is

$$\dddot{e}_1 + k_2\ddot{e}_1 + k_1\dot{e}_1 + k_0 e_1 = p^2(\phi_1 x_1) + p(\phi_2 x) + \phi_3 x$$

$$+ p(\psi_2 r) + (\psi_3 r) + \ddot{u}_1 + \dot{u}_2 + u_3$$

The filters are given by

$$(p^2 + c_1p + c_0)v_1 = p^2x_1 \qquad (p^2 + c_1p + c_0)s_1 = p^2r$$

$$(p^2 + c_1p + c_0)v_2 = px_1 \qquad (p^2 + c_1p + c_0)s_2 = pr$$

$$(p^2 + c_1p + c_0)v_3 = x_1 \qquad (p^2 + c_1p + c_0)s_3 = r$$

which may be generated by

$$\begin{bmatrix} \dot{v}_3 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix} \begin{bmatrix} v_3 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1$$

$$\begin{bmatrix} \dot{s}_3 \\ \dot{s}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix} \begin{bmatrix} s_3 \\ s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1$$

$$v_1 = -c_1v_2 - c_0v_3 + x_1$$

$$s_1 = -c_1s_2 - c_0s_3 + x_1$$

The input vector u is given by

$$u_2 = -c_0(\dot{\phi}_1v_2 + \dot{\phi}_2v_3 + \dot{\psi}_2s_3) + [k_1 - c_0 - c_1 (k_2 - c_1)]e_1$$

$$u_3 = -\dot{\phi}_1(c_1v_2 + c_0v_3) + \dot{\phi}_2c_2v_2 + c_2\dot{\phi}_3v_3$$

$$+ c_2\dot{\psi}_2s_2 + c_2\dot{\psi}_3s_3 + [k_0 - c_0(k_2 - c_1)]e_1$$

According to (IV.37B) the adaptive laws are

$$g_1 = -q_1v_1e_1 - m_1 \int_0^t v_1e_1 \, d\tau$$

$$g_2 = -q_2v_2e_1 - m_2 \int_0^t v_2e_1 \, d\tau$$

$$g_3 = -q_3v_3e_1 - m_3 \int_0^t v_3e_1 \, d\tau$$

$$d_2 = -\tilde{q}_2 s_2 e_1 - \tilde{m}_2 \int_0^t s_2 e_1 \, d\tau$$

$$d_3 = -\tilde{q}_3 s_3 e_1 - \tilde{m}_3 \int_0^t s_3 e_1 \, d\tau$$

With this definition, v becomes

$$\begin{aligned} u_2 = {} & (-m_1 v_2 c_1 + q_1 v_1 c_1 - c_0 m_1 v_3 + q_1 c_0 v_2) e_1 v_1 \\ & + (-\tilde{m}_2 s_2 + \tilde{q}_2 s_1) c_2 e_1 s_2 + (-m_2 v_2 + q_2 v_1) c_2 e_1 v_2 \\ & + (\tilde{m}_3 s_3 + \tilde{q}_3 s_2) c_2 e_1 s_3 + (-m_3 v_3 + q_3 v_2) c_2 e_1 v_3 \\ & + c_0 \tilde{q}_2 e_1 s_2 s_3 + c_0 q_1 e_1 v_1 v_2 + c_0 q_2 e_1 v_2 v_3 + [k_1 - c_0 - c_1(k_2 - c_1)] e_1 \end{aligned}$$

$$\begin{aligned} u_3 = {} & -c_0 e_1 v_1 (q_1 v_1 - m_1 v_2) - c_0 e_1 v_2 (q_2 v_2 - m_2 v_3) \\ & - c_0 e_1 s_2 (\tilde{q}_2 s_2 - \tilde{m}_2 s_3) \;\; + [k_0 - c_0(k_2 - c_1)] e_1 \end{aligned}$$

$$\begin{aligned} u_1 = {} & -q_1 e_1 v_1 \, (c_1 v_2 + c_0 v_3) - c_2 e_1 (q_2 v_2^2 + q_3 v_3^2) \\ & - c_2 e_1 (\tilde{q}_2 s_2^2 + \tilde{q}_3 s_3^2) \end{aligned}$$

The poles of the filters may be arbitrarily placed so long as

$$k_2 - c_1 > 0.$$

The observer configuration and transformation reconstruction is the same as in Section 4.2.

CHAPTER V

SUBSEQUENT EXTENSIONS OF THE SINGLE-INPUT SINGLE-OUTPUT ADAPTIVE OBSERVER - PART II

5.1 New Canonical Form

Subsequent to the published appearance of the single-input single-output adaptive observer in [71], and earlier in a different context in [44], interest was generated among other researchers in the adaptive observer concept. This chapter examines the published effort of those researchers particularly in comparison with the adaptive observer of Chapter IV.

The first modification of the adaptive observer was by Lüders and Narendra [62]. They proposed a transformation of the system that is to be observed into a *different canonical form* than the output form of Chapter IV in order to *lower the order of the state variable filters.* Whereas, with the output-form observer, the filters are of the form

$$- a_i + \frac{s^i}{s^{n-1} + a_{n-1}s^{n-2} + \dots + a_1 s + a_0}, \quad 0 \leq i \leq n-2,$$

in which n is the order of the system, the new canonical form allows the state variable filter to be of the form

$$\frac{1}{s + \lambda_i}, \quad 0 \leq i \leq n - 1$$

In order to compare the two schemes it is supposed that n parameters in each of the input and system matrices of the system are to be adapted. Then by the method of Chapter IV there are required 2n-4 integrators to implement the filters whereas the system in the new canonical form requires 2n-2, certainly not an improvement! However,, if the number of parameters being adapted is q, the method of Chapter IV requires 2n-4 integration to implement the filters as compared to q-2 integrators for the method with the other canonical form. Thus with the second form there is a significant savings when the number of parameters being adapted is small. But there is a penalty for this savings: the eigenvalues of the observer (which are the same as the roots of the denominator polynomial of the state variable filters) must be both real and distinct with the canonical form of Lüders and Narendra where there is no such restirction in the output form representation.

The canonical form proposed by [62] is

$$\begin{bmatrix} \alpha_{n-1} & 1 & 1 & 1 & \cdots & 1 \\ \alpha_{n-2} & & & & & \\ \alpha_{n-3} & & & & & \\ \cdot & & & \Lambda & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \alpha_0 & & & & & \end{bmatrix}_{n\times n} \qquad (V.1)$$

where Λ is a square diagonal matrix of order n-1 having arbitrary but real and distinct eigenvalues. A system (IV.2) may be transformed so that the system matrix A is in the form (V.1) iff the system is completely observable through its single output [62]. The elements α_i of (V.1) represent the unknown parameters of the system.

It is noted that the unknown parameters of the system appear in the left-most column of the transformed matrix in both the output form and the form (V.1); *this is a necessary requirement for any canonical form used in an adaptive observer when the system output matrix is* $[C\ 0\ 0 \ldots 0]$, $C \neq 0$. The reason for the necessity is that the unknown parameters must appear in the range space of the available measurements, so that the parameters may be adapted by an algorithm dependent only upon available measurements. That is to say, the unknown parameters must appear in $\mathcal{R}[C^T]$ where C is the output matrix of a system (IV.2) and the symbol $\mathcal{R}$ denotes "the range of".

The observer of [62] is, analogous to (IV.3A), of the form

$$\dot{z} = Kz + GCx + Dr + Hu \qquad (V.2)$$

in which x is the state of the transformed system (that is, transformed so that the system matrix A of (IV.2A) is in the canonical form (V.1)) and K has the form

$$\begin{bmatrix} -\lambda_1 & 1 \quad 1 \quad 1 \quad \cdots \quad 1 \\ 0 & \\ 0 & \\ 0 & \Lambda \\ \vdots & \\ 0 & \end{bmatrix}_{n \times n} \qquad (V.3)$$

Then D and G are adjusted in (V.2) to make the error equation (IV.4) asymptotically stable in the sense of Liapunov. The control u is synthesized to enable this to be done.

It is apparent that the approach in [62] is very like that of the last chapter but with the exception that a new canonical form is used. The new canonical form allows, however, one difference in the development of Chapter IV. In the technique of Chapter IV, the vector error equation (IV.3) is first converted to a scalar differential equation (IV.6) and then integrated to a first-order differential equation (IV.14) by use of the filtered state variables. The canonical form of [62] allows the sequence of these steps to be *reversed*. By doing this the awkward expressions in (IV.10) and(IV.11) are avoided. For a detailed account of the procedure of [62], the reader is referred to the "adaptive law" section of Chapter VI.

The same researchers have proposed yet another canonical form [63], albeit a generalization of their form (V.1). This form is

$$\begin{bmatrix} \alpha_{n-1} & q^T \\ \hline \alpha_{n-2} & \\ \alpha_{n-3} & \\ \vdots & F \\ \alpha_0 & \end{bmatrix}_{n \times n} \qquad \text{(V.4)}$$

where q^T is a row vector of dimension 1 x (n-1) and F is a(known) square matrix of dimension (n-1)x(n-1). A requirement is that the pair (q^T, F) be completely observable. (V.1) is seen to be a special case of (V.4).

The advantage of (V.4) over (V.1) is that n-1 of the observer eigenvalues, which are the eigenvalues of F, need not be both real and distinct as required by the form (V.1). The corresponding disadvantage is that, when F is not diagonal, the resulting state variable filter inevitably requires more integrators for implementation, which puts the adaptive observer based upon (V.4) at a greater disadvantage over the output form observer when the parameters that are to be adapted are many, and lessens its advantage for few parameters.

Nevertheless, this author believes that the form (V.4) is a significant contribution because of the requirement that (q^T, F) need be completely observable, as opposed to the need, in (V.1), that q projects onto each axis of $\mathcal{E}^n$. As yet there has been no verification of the possibility that significant simplification of the observer structure can be accomplished by judiciously choosing F and q^T.

One final point should be made in comparing the form (V.4) or (V.1) with the output form of Chapter IV for developing adaptive observers. That is the fact that, despite the existence theorems for transformations to the diverse canonical forms, one must have available a *literal* transformation matrix in order to implement the retransformation of the observer output. In the output form, since the literal transformation from companion form to output form is easily produced for any order system, and since the van der Monde matrix couples the Jordan form to the companion form, the problem of obtaining a literal transformation matrix is simplified to the problem of finding a literal expression for the eigenvalues of the system. Although this is a difficult chore, it is perhaps a less difficult task than seeking for a literal transformation *directly* from the equation $\tilde{A} = T^{-1}AT$ as must be done with the other forms that have been suggested.

5.2 *Greater Filter Freedom*

In the adaptive observer of Chapter IV and the diverse forms of section 5.1, the poles of the state variable filters must be the same as the eigenvalues of the observer. Since the system in which the observer is to be used may contain noise, the requirement that the filter poles be equal to the corresponding observer eigenvalues might not allow so effective an adaptation as could be obtained if the eigenvalues of the observer and the poles of the filters could be *independently* chosen.

Kudva and Narendra have proposed [64] an observer of the same form as that in Chapter IV, i. e. having the observer matrix in output form, and have shown that the poles of the filters may be independently chosen from that of the observer eigenvalues.

This observation is proved by use of the Kalman-Yakubovich lemma [48]. This lemma says that if

1) the square matrix K is a stable matrix,

and if

2) $h^T (sI-K)^{-1}d$ is positive real,

where h and d are vectors,

then there exists a positive definite square matrix P and a vector q such that

$$K^TP + PK = - qq^T$$

and

$$Pd = h.$$

The development of the proof, that the poles of the filters can be freely chosen, picks up essentially at equation (IV.12). This scalar equation is then re-converted to a vector equation of the form

$$\dot{\varepsilon} = K\varepsilon + d\,\phi^T v \tag{V.5}$$

where $\varepsilon_1 = e_1$ and ϕ is the vector composed of the parameter differences α_i and β_i, $\phi = [\alpha_0\ \alpha_1\ \ldots\ \alpha_n \beta_0 \beta_1\ \ldots \beta_m]^T$. d is a vector with first element unity. The Liapunov candidate is now chosen as

$$V = \frac{1}{2}\{\varepsilon^T P\varepsilon + \Phi^T\Gamma\Phi\} \tag{V.6}$$

where P and Γ are both positive definite and Γ is diagonal. The differential of (V.6) along the trajectory described by (V.5) is

$$\dot{V} = \frac{1}{2}\varepsilon^T(K^TP + PK)\varepsilon + \varepsilon^TPd\,\Phi^Tv + \Phi^T\Gamma\dot{\Phi} \tag{V.7}$$

Therefore choosing

$$\dot{\Phi} = -\Gamma^{-1}\varepsilon^TPdv, \tag{V.8}$$

(V.7) becomes

$$\dot{V} = \frac{1}{2}\varepsilon^T(K^TP + PK)\varepsilon \tag{V.9}$$

By applying the Kalman-Yakubovich lemma the adaptive law is reduced to

$$\dot{\Phi} = -\Gamma^{-1}e_1v \tag{V.10}$$

since $\varepsilon_1 = e_1$. Equation (V.10) is similar to (IV.17).

The poles of the state variable filters are determined by d in the sense that each N_i has the form

$$N_i = \frac{s^{n-i}}{s^{n-1} + d_2s^{n-2} + d_3s^{n-3} + \ldots + d_n}$$

On the other hand, the eigenvalues of the observer are the eigenvalues of the matrix K in (V.5), which is in output form. Thus the poles of the filters and the eigenvalues of the observer are independently chosen.

5.3 *Cyclic Multivariable Adaptive Observer*

One may recognize the distinction between adapting (and identifying) parameters in the system matrix A of a system (IV.2) and adapting (and identifying) parameters in the input matrix B by means of the adaptive observer. That is to say, the adapting of parameters in the system matrix

A is accomplished by entirely depending upon a transformation T to place A in a canonical form, while adapting parameters of the input matrix B is accomplished by adapting parameters of $T^{-1}B$ without regard to which T, dependent only upon A and the chosen canonical form, has been chosen.

This being so, the single-input single-output adaptive observer can readily be extended to a multi-input system (because no change is required in selecting a T which delivers A into canonical form).

Lüders and Narendra reported [65] just such an extension based upon their previously reported single-input single-output adaptive observer [62] (see section 5.1). In addition they noted that if the system has multiple outputs and if it is possible to combine these outputs in a manner so that the system is completely observable by this single combined measurement, then the single-output multiple-input observer can serve as a multivariable observer.

Not all multivariable systems can be observed through a single output even though it is completely observable through multiple outputs. In this, the system matrix A plays a crucial role [66]. Let $0 \neq b \in \mathcal{E}^n$ and let range be denoted by the symbol $\mathcal{R}$. Then the space $\mathcal{E}_b$ is called the *cyclic subspace generated by b* when

$$\mathcal{E}_b = [A:\mathcal{R}(b)]$$

where $[A:\mathcal{R}(b)] \equiv \mathcal{R}(b) + A\mathcal{R}(b) + A^2\mathcal{R}(b) + \ldots + A^{n-1}\mathcal{R}(b)$.

The space $\mathcal{E}^n$ is *cyclic* if there exists a $b \in \mathcal{E}^n$ such that $\mathcal{E}_b = \mathcal{E}^n$. It is common usuage to say that A is cyclic in order to imply that $\mathcal{E}^n$ is cyclic.

A matrix A need not be cyclic. A trivial example is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; b = [b_1 \;\; b_2]^T \neq 0$$

Then $\mathcal{E}_b = \mathcal{R}(b) \neq \mathcal{E}^2$.

From the preceeding discussion, it is apparent that the multivariable system of [65] is limited to cyclic systems.

One interesting observation made in [65] is the following. Suppose that the system (IV.2) can be obtained, by transformation if necessary, so that the uppermost row of the output matrix C is of the form [1 0 0 ... 0], suppose that there are unknown elements in the remainder of C, and finally suppose that the system is observable through the first output y_1 (i.e. the pair [1 0 0 ... 0] and A are completely observable). Since the state of the system can be determined through the single output y_1, the unknown elements of C may be determined, and this may be done adaptively.

Choosing

$$V = \mathrm{tr}\,[(\Delta C)^T(\Delta C)]$$

then

$$\begin{aligned} \dot{V} &= -\mathrm{tr}\,[(\Delta C)^T e_y z^T] \\ &= -e_y^T e_y + e_y h_1 e_1 \end{aligned} \qquad (V.11)$$

where ΔC is the difference between the estimate of the output matrix and the true value and e_y equals the error between the observer estimate of the output and the output itself. Since $e_1 \to 0$ in (V.11), $\dot{V}$ is negative semi-definite. Hopefully* this implies that $\Delta C \to 0$, thereby estimating the unknown values of C.

It is apparent that the cyclic multivariable observer [65] is of n^{th} order and the output measurements are therefore generated even though they are available for measurement. The next chapter describes a reduced adaptive observer which generates the state of the system by an observer of order dependent upon the number of output measurements.

* Note that $\dot{V}$ in (V.11) is only semidefinite on the space of V.

CHAPTER VI

THE REDUCED ADAPTIVE OBSERVER

6.1 Comparison with Cyclic Adaptive Observer

The adaptive observer concept is a scheme for determining the state of a system possessing unknown parameters when only the system input, output, and form are known. The first reported adaptive observer, for single-input single-output time invariant linear systems, appeared in [44] and [71]. A modification of this observer to simplify the adaptive dynamics was subsequently reported [62]. Both these schemes exhibit the desirable properties that the eigenvalues of the observer matrix may be freely chosen (an important capability for systems with measurements corrupted by noise), that the simple Lyapunov adaptive algorithm is implemented entirely on line during system operation, that no derivative networks are required in the adaptive algorithm, and that both the state of the system under observation and the unknown parameters of that system are progressively determined regardless of the magnitude of parameter ignorance.

In [65] the single-input single-output adaptive observer was extended to cyclic multivariable systems by introducing a suitable transformation that converts the system to a single-output system. Consequently the multivariable adaptive observer in this scheme is of the same order as the system regardless of the number of system outputs available, and the number of adaptive gains needed to implement the observer algorithm equals at least the sum of the system order and the number of input parameters being adapted.

In this chapter an adaptive observer for multivariable systems is reported for which the dynamic order of the observer is reduced, subject to mild restrictions given in Theorem 6.1, to n-p+1 where n is the order of the system being observed and p is the number of independent output measurements. The observer structure which is developed here depends directly upon the multivariable structure of the system rather than a transformation to a single-output system. The number of adaptive gains is at most the sum of the order of the system and the number of input parameters being adapted. Moreover, for the relatively frequent specific cases for which the number of required adaptive gains is less than the sum of system order and input parameters, the number of these gains is easily determined by inspection of the system structure. This adaptive observer possesses all the properties ascribed earlier to the single-input single-output adaptive observer. Like the other adaptive observers mentioned, some restriction is required of the allowable system command input to guarantee convergence of the adaptive algorithm, but the restriction is more lenient than that required by the full-order multivariable observer in [65]. Finally, this reduced observer is not restricted to cyclic systems as is [65].

6.2 *Development of The Reduced Observer*

The Problem to be Solved

An observable and controllable linear time-invarient dynamical system described by

$$\dot{\bar{x}} = A\bar{x} + Br \qquad \bar{x}(t_0) = \bar{x}^o$$
$$y = C\bar{x} \tag{VI.1}$$

is considered, where $\bar{x}(t)\varepsilon\ \mathcal{E}^n$ is the state of the system, $r(t)\varepsilon\ \mathcal{E}^m$ is the command input and $y(t)\varepsilon\ \mathcal{E}^p$ is the output. For purposes of this chapter, (VI.1) is multivariable with $n>p\geq 1$ and $m\geq 1$, the pair (C,A) is completely observable, and the pair (A,B) is completely controllable. A and B are appropriately - dimensioned matrices having parameters of unknown value. C is a known matrix of dimension pxn.

The problem is to determine the state $\bar{x}$ of (VI.1) using only the input r, the output y, and the structure, but not the values, of matrices A and B. This is to be accomplished by a generating process which duplicates as little as possible the state information available in the output; thus the generating process is said to be a reduced adaptive observer.

The reduced adaptive observer is of the form

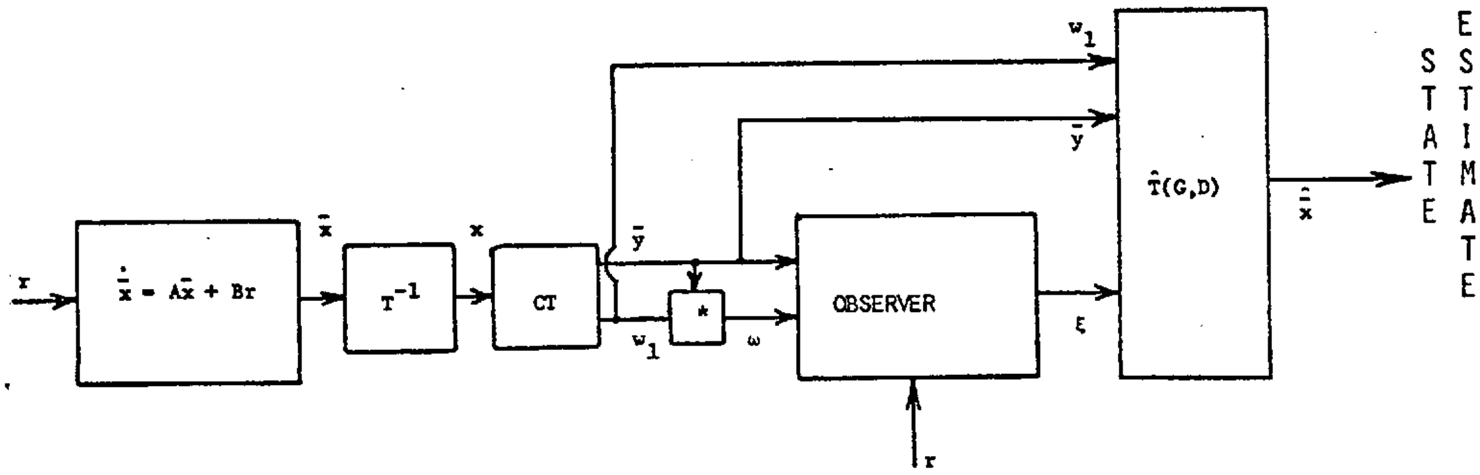
$$\dot{\xi} = F\xi + Gy + Dr + Hu$$
$$\xi(t_0) = \xi^o \qquad \text{(VI.2)}$$

where $\xi\varepsilon\ \mathcal{E}^{n-p+1}$ is the estimate of the missing state information in the output of (VI.1). The matrices G and D and the vector u are to be adaptively manipulated so as to guarantee that ξ asymptotically equals a transformation of the unknown state variables in (VI.1). F may have arbitrary distinct eigenvalues.

The state $\bar{x}$ can be ultimately constructed once the transformation has been identified. Figure 6.1 illustrates the situation.

The Strategy of the Solution

The transformation T, indicated by T^{-1} in Figure 6.1, allows the system (VI.1) to be assumed to be in a form suitable for constructing an adaptive law based upon Lyapunov synthesis techniques.



* Output Transformation (c.f. Section 6)

FIGURE 6.1 ADAPTIVE OBSERVER SCHEME

The strategy for solving the problem posed in the section labeled "The Problem to be Solved" is to first determine the effects of parameter uncertainty in the system upon the accuracy of the observer estimate of the system state. In the section labeled "The Error Equation" an error vector is defined as a comparison between the transformed system state and the observer estimate; subsequently an error equation is derived reflecting the influences of parameter uncertainty in the system. Theorem 6.1 of the next section defines sufficient conditions under which (VI.1) may be transformed into a form suitable for a Lyapunov synthesis technique. It is seen in this section that with this form the error equation may be considerably simplified. In the section labeled "The Adaptive Law" a Lyapunov adaptive synthesis technique is used to derive an adaptive law. The essence of this method is to define the adaptable parameters in such a way as to insure, by means of a Lyapunov function, that the error equation is asymptotically stable. Due to the fact that the resulting Lyapunov function chosen here (as may be seen by equation (VI.18) and (VI.20)) is defined on a noncompact manifold, Theorems 6.2 and 6.3 give sufficient restriction upon the system input to insure that the error equation is asymptotically stable on the compact manifold. Thus an estimate of the system state, which asymptotically converges to the true system state, may be obtained by inverting the original transformation of the system as indicated in the appropriate section.

As an illustration of the technique of this chapter, an example is given and a computer simulation of this example appears subsequently.

It is propitious to collect here certain definitions which allow brevity in the remaining sections of this chapter. The motivations for these definitions will be discussed in the appropriate locations.

Definition 6.1

$\mathcal{J}_{n,p}$ hereafter refers to the collection of all non-singular square matrices T of dimension nxn having the following properties

T may be partitioned as

$$T = \begin{bmatrix} T_{11} & \vdots & T_{12} \\ \cdots & \cdot & \cdots \\ T_{21} & \vdots & T_{22} \end{bmatrix}$$

with T_{11}: (p-1)x(p-1), T_{12}: (p-1)x(n-p+1), T_{21}: (n-p+1)x(p-1), T_{22}: (n-p+1)x(n-p+1)

wherein a) $T_{12} = 0$;

b) each element in the uppermost row of T_{21} is independent of any system parameter.

and c) the uppermost row of T_{22} is $[C\ \ 0\ \ 0\ \ldots\ \ 0]$ with $C \neq 0$.

When there is no possibility of confusion, $\mathcal{J}_{n,p}$ will be referred to as $\mathcal{J}$.

Definition 6.2

The "adaptive canonical form" refers to all matrices $\tilde{A}$ of dimension nxn having the following properties:

in partition

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

a) $\tilde{A}_{22}$ has the form

$$\begin{bmatrix} \alpha_1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ \cdot & & & \Lambda_{n-p} & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & & & & & \end{bmatrix}_{(n-p+1)x(n-p+1)}$$

where Λ_{n-p} is a diagonal matrix with distinct eigenvalues of dimension n-p and α_1 any real number,

and b) $\tilde{A}_{21}$ has no more than n-1 non-zero elements.

Definition 6.3

A square matrix A is said to be cyclic if there exists a vector c such that the pair (c,A) is completely observable. Otherwise A is said to be non-cyclic.

The Error Equation

The development of the error equation is somewhat similar to that in [7] for systems with known parameters.

Without restriction if C is known, it may be assumed in (VI.1)

that $C = [I_p \vdots 0]$ where I_p is a pxp identity matrix. For example, the transformation $\bar{x} = [\cdot\overset{C}{\underset{H}{\vdots}}\cdot]^{-1}x$ transforms (VI.1) into this form for any C, where H is selected to make the transformation matrix non-singular. Then in partitioned form (VI.1) is written as

$$\dot{\bar{y}} = A_{11}\bar{y} + A_{12}w + B_1 r$$

$$\dot{w} = A_{21}\bar{y} + A_{22}w + B_2 r$$

$$y = \begin{bmatrix} \bar{y} \\ \\ w_1 \end{bmatrix} \quad x = \begin{bmatrix} \bar{y} \\ \\ w \end{bmatrix} \quad w_1 \equiv \text{uppermost element in } w \qquad \text{(VI.3)}$$

where $\bar{y} \varepsilon\, \mathcal{E}^{p-1}$ and $w \varepsilon\, \mathcal{E}^{n-p+1}$. The dimensions of $\bar{y}$ and w indicate the dimensions of the partitions in (VI.3). Since only w is to be estimated by the adaptive observer, the dimension of the vector w is chosen as small as possible while still retaining an element of the output, which is essential for implementation of the adaptive law.

The adaptive observer is initially described by

$$\dot{\xi} = F\xi + (FK + G\text{-}KM)\bar{y} - \dot{K}\bar{y} + (D\text{-}KB_1)r + Hu \qquad \text{(VI.4)}$$

in which $\xi \varepsilon\, \mathcal{E}^{n-p+1}$. If at this point (VI.4) is taken as a hypothesis for a generator of w, it will be shown that the error between w and a function of ξ can be made to vanish by adaptively adjusting G, D, and K. It will be subsequently shown that a suitable transformation of (VI.3) allows (VI.4) to be rendered unto (VI.2).

Let an estimate of w be $\xi + K\bar{y}$. Then defining, as in [7], the error

$$e = \xi + K\bar{y} - w \qquad \text{(VI.5)}$$

on the reduced space $\mathcal{E}^{n-p+1}$, it follows that

$$\dot{e} = \dot{\xi} + K\dot{\bar{y}} + \dot{K}\bar{y} - \dot{w}$$

$$= F(\xi + K\bar{y} - w) + (F - A_{22})w + (G - A_{21})\bar{y}$$

$$+ K(\dot{\bar{y}} - M\bar{y}) + (D\text{-}B_2 - KB_1)r + Hu$$

Defining $M = \tilde{M}\text{-}A_{11}$ so that $\dot{\bar{y}} - M\bar{y} = \dot{\bar{y}} - A_{11}\bar{y} - \tilde{M}\bar{y} = A_{12}w + B_1 r - \tilde{M}\bar{y}$,

then $$\dot{e} = Fe + (F - A_{22} + KA_{12})w + (G - A_{21} - K\tilde{M})\bar{y} \qquad \text{(VI.6)}$$

$$+ (D - B_2)r + Hu$$

in which it is seen that the reduced error depends upon both the measurable vector $\bar{y}$ and the unmeasurable (save the first element) vector w. It is impossible in a manner similar to the Luenberger observer [3, 4] to define $F = A_{22} - KA_{12}$, $M = A_{11}$, $D = B_2$, $G = A_{21}$, and $H = 0$ to eliminate these dependences from (VI.6), since A and B are here unknown.

Rather, it is desired to adaptively adjust the triple (G, D, K) so that the coefficients of w, $\bar{y}$, and r in (VI.6) eventually vanish. Then if F is chosen with eigenvalues all with negative real parts and if $u \to 0$, the reduced error e vanishes.

The Transformation

If it is possible to show with respect to (VI.3) that a suitable transformation matrix T exists so that $T_{11} = CT$ and that $\tilde{A} = T^{-1}AT$ is in adaptive canonical form with the partition element $\tilde{A}_{22} = T_{22}^{-1} (A_{22} - T_{21} A_{12}) T_{22}$ having arbitrary specified eigenvalues, then setting $F = \tilde{A}_{22}$ in the equation (analogous to (VI.6))

$$\dot{e} = Fe + (F - \tilde{A}_{22} + K\tilde{A}_{12})w + (G - \tilde{A}_{21} - K\tilde{M})\bar{y} + (D - \tilde{B}_2)r + Hu \tag{VI.7}$$

permits defining $K = 0$. Doing this is advantageous since the influence of w in (VI.7) is eliminated, the necessity of adapting K is removed, and (since $\tilde{M}$ is related to $\tilde{A}_{11}$) the influence of unknown elements of $\tilde{A}_{11} = T_{11}^{-1} (A_{11} T_{11} + A_{12} T_{21})$ is diminished.

As will be seen, under some restrictions on A a transformation T can be found that satisfies the preceeding requirement and the additional requirement that $T_{11} = CT$ be independent of parameters of A. By virtue of this latter requirement the outputs can be treated as transformed state variables without specifically identifying T.

For a suitable definition of u, the transformation which satisfies these requirements is a member of the collection $\mathcal{J}$ and the transformed matrix $\tilde{A} = T^{-1}AT$ is in adaptive canonical form. Theorem 6.1 gives sufficient conditions on A for the existence of such a $T \varepsilon \mathcal{J}$.

In the following theorem, let the symbol $\mathcal{R}[\chi]$ denote the range of χ, let $Q \equiv [C \ 0 \ 0 \ \ldots \ 0]$, $C \neq 0$ be a row vector of dimension 1x(n-p+1) and let $\tilde{A}_{22}$ denote the (n-p+1)x(n-p+1) partition of the adaptive canonical form

THEOREM 6.1 [*CARROLL*]

Let the pair (A_{12}, A_{22}) *of the matrix A be completely observable. Then there exists a* $T \varepsilon \mathcal{J}$ *that transforms A into the adaptive canonical form in which n-p eigenvalues of* $\tilde{A}_{22}$ *may be almost arbitrarily chosen.*

If in addition, $\mathcal{R}[Q^T] \subset \mathcal{R}[A_{12}^T]$, *then the n-p eigenvalues of* $\tilde{A}_{22}$ *may be arbitrarily chosen.*

PROOF: The proof is in two parts: to show that a $T \varepsilon \mathcal{J}$ exists that puts $\tilde{A}_{22}$ into adaptive canonical form with the desired properties, and that $\tilde{A}_{21}$ also satisfies the requirements of the adaptive canonical form.

Suppose that (A_{22}, A_{12}) of the matrix A is completely observable. According to the definition of $\mathcal{J}$, the (n-p+1) x (n-p+1) partition T_{22} of $T \varepsilon \mathcal{J}$ is arbitrary except for the uppermost row which is $Q \equiv [C \ 0 \ 0 \ \ldots \ 0]$, $C \neq 0$. Since $\tilde{A}_{22} = T_{22}^{-1} (A_{22} - T_{21}A_{12})T_{22}$, where T_{21} is the (n-p+1) x (p-1) partition of $T \varepsilon \mathcal{J}$, it must be shown that $\tilde{A}_{22}$ is of the form required by definition of the adaptive canonical form, and that by choosing T_{21} the n-p eigenvalues can be freely chosen. It has been shown [62] that there exists a matrix T_{22} of the required form which transforms a cyclic matrix P into $\tilde{A}_{22} + L$, where L is a matrix having only the leftmost column non-zero, if and only if (Q,P) is completely observable. In the present context, $P \equiv A_{22}-T_{21}A_{12}$. Thus if by choice of T_{21}, P can have n-p eigenvalues equal to the desired eigenvalues of A_{22} and if (Q,P) is completely observable for this choice of T_{21}, then L = 0 (except perhaps for the element in the upper left corner, which is irrelevant by definition of the adaptive canonical form).

Suppose first that $\mathcal{R}[Q^T] \subset \mathcal{R}[A_{12}^T]$. Then for any choice of T_{21} the pair $(Q, A_{22} - T_{21}A_{12})$ is completely observable and at least n-p eigenvalues of $A_{22} - T_{21}A_{12}$ can be arbitrarily chosen [67]. Therefore $\tilde{A}_{22} = T_{22}^{-1} (A_{22} - T_{21}A_{12})T_{22}$ is in adaptive canonical form with arbitrary eigenvalues for some choice of T_{21} and T_{22} of $T \in \mathcal{J}$.

Suppose now that $\mathcal{R}[Q^T] \not\subset \mathcal{R}[A_{12}^T]$. Since the pair (A_{12},A_{22}) is completely observable, at least n-p eigenvalues of $P = A_{22} - T_{21}A_{12}$ can be arbitrarily chosen but (Q,P) may not be observable. A trivial extension of theorem 4 of [68] says that the set $\mathcal{K} \equiv \{T_{21} | (A_{22} - T_{21}A_{12}, Q)$ not observable$\}$ is either an empty set or a hypersurface in the parameter space of T_{21} when the pair (A_{22}, A_{12}) is completely observable. Consequently A_{22} is in adaptive canonical form with almost arbitrary eigenvalues for some choice of T_{22} and T_{21} of $T \in \mathcal{J}$, since the choices of T_{21} is limited to those $T_{21} \notin \mathcal{K}$. Thus the first part of the theorem is proved.

Now it is shown that $\tilde{A}_{21}$ has not more than n-1 non-zero elements with the appropriate choice of $T \in \mathcal{J}$. In the $(n-p+1) \times (p-1)$ partition T_{21} of $T \in \mathcal{J}$ there are $(n-p+1)(p-1) - (p-1) = (n-p)(p-1)$ parameter-dependent elements. At most n-p of these elements are needed to specify the n-p eigenvalues of $A_{22} - T_{21}A_{12}$. Therefore, at least $(n-p)(p-1) - (n-p) = (n-p)(p-2)$ parameter-dependent elements of T_{21}

are unspecified. Each unspecified element may be specified so as to make an element of $\tilde{A}_{21} = T_{22}^{-1} (A_{21}T_{11} + A_{22}T_{21} - T_{21}T_{11}^{-1}A_{11}T_{11} - T_{21}A_{12}T_{21})$ zero. Since there are at most (n-p+1)(p-1) non-zero elements in $\tilde{A}_{12}$, eliminating (n-p)(p-2) of them leaves at most (n-p+1)(p-1) - (n-p)(p-2) = n-1 non-zero elements in A_{21}. Thus the theorem is proved.

Remark

The stipulation that the eigenvalues are "almost" arbitrary when $\mathcal{R}(Q^T) \not\subset \mathcal{R}(A_{12}^T)$ may be illustrated as follows. Let

$$A_{22} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \qquad A_{12} = [0 \quad 0 \quad 1]$$

$$a_{32} \neq 0, \; a_{21} \neq 0$$

Then the observability matrix of the pair $(Q, A_{22} - T_{21}A_{12})$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{11} & 0 & -t_{11} \\ a_{11}^2 & -a_{32}t_{11} & t_{11}(t_{31}-a_{11}) \end{bmatrix}$$

From the observability matrix it is seen that the pair $(Q, A_{22}-T_{21}A_{12})$ is completely observable for any choice of T_{21} except $t_{11} = 0$. This implies there exists a hypersurface of dimension unity for which the observer eigenvalues may not be located.

COROLLARY

If in addition to the requirements of the theorem the pair (Q, A_{22}) *is completely observable where* $Q = [C \ 0 \ 0 \ \dots \ 0]$, $C \neq 0$, *then the uppermost row of the partition* T_{21} *of* $T \in \mathcal{J}$ *may be chosen as zero.*

PROOF:

The proof of the theorem requires that $(Q, A_{22} - T_{21}A_{12})$ be completely observable for some choice of T_{21}. If (Q, A_{22}) is observable, then $(Q, A_{22} - T_{21}A_{12})$ is completely observable by the trivial choice $T_{21} = 0$. However, since n-r eigenvalues of $A_{22} - T_{21}A_{12}$ are to be arbitrarily chosen by choice of T_{21} and at least (n-p)(p-2) elements of $\tilde{A}_{21}$ are to be chosen zero by choice of T_{21}, it generally requires all but p-1 non-zero elements of T_{21}. Generally these elements must be parameter-dependent; thus only the p-1 parameter-independent elements appearing in the uppermost row of T_{21} may be zero.

The Adaptive Law

It is assumed that (VI.3) satisfies the conditions of Theorem 6.1 and consequently may be written as

$$\dot{\bar{y}} = \tilde{A}_{11}\bar{y} + \tilde{A}_{12}w + \tilde{B}_1 r$$
$$\dot{w} = \tilde{A}_{21}\bar{y} + \tilde{A}_{22}w + \tilde{B}_2 r$$
$$y = \begin{bmatrix} \bar{y} \\ \omega \end{bmatrix} \qquad \text{(VI.8)}$$

where $\tilde{A}$ is of adaptive canonical form and the scalar ω is a linear combination of w_1 and elements of $\bar{y}$. The scalar ω is constructed externally to the system in accordance with the upper row of T_{21} so that the transformed system output matrix is in the form assumed in (VI.3). According to the corollary, $\omega = w_1$ if (Q, A_{22}) is completely observable.

F in (VI.2) is taken at

$$F = \begin{bmatrix} -\lambda_1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ \vdots & & & \Lambda_{n-p} & & \\ 0 & & & & & \end{bmatrix}_{(n-p+1) \times (n-p+1)} \qquad \text{(VI.9)}$$

for λ_1 any real number distinct from the distinct eigenvalues of the diagonal matrix Λ_{n-p}. Let $h_1 u_1 = (\alpha + \lambda_1 - g_1)w_1 \equiv \phi_1 w_1$ and (VI.7) may be written

$$\dot{e} = Fe + \Phi\tilde{y} + \Psi r + Hu \qquad \text{(VI.10)}$$

$$e(t_0) = e^o$$

where $\Phi = G - \tilde{A}_{21}$ and $\Psi = D - \tilde{B}_2$. The other elements of u will be defined later.

The adaptive law for Φ and Ψ in (VI.10) must be defined in terms of only those variables which are available for measurement. Consequently (VI.10) will be manipulated in a way to obtain a scalar equation, equivalent to (VI.10), for which such an adaptive law can be formulated.

Let the (n-p+1) x (p-1) matrix V be defined as

$$V = \begin{bmatrix} y^{-T} \\ \hline (pI-\Lambda)^{-1} \; h_1 y^{-T} \end{bmatrix} \qquad \text{(VI.11)}$$

in which p means " $\frac{d}{dt}$" and $h_1^T = [1 \;\; 1 \;\; 1 \;\; \ldots \;\; 1]$ of appropriate dimension. Clearly the (n-p) x (p-1) submatrix $(pI-\Lambda_{n-p})^{-1}h_1\bar{y}^T$ is composed of filtered output variables.

In a similar manner, let the (n-p+1) x m matrix S be defined as

$$S = \begin{bmatrix} r^T \\ \hline (pI-\Lambda)^{-1} h_2 r^T \end{bmatrix} \qquad \text{(VI.12)}$$

in which $h_2^T = [1 \ 1 \ 1 \ \ldots \ 1]$ of appropriate dimension. It will later be shown that the adaptive law requires at most n-1 elements of V.

Consider now the lowermost n-p scalar equations of (VI.10). The i^{th} equation, $2 \leq i \leq n-p+1$, is

$$\dot{e}_i = -\lambda_i e_i + \sum_{j=1}^{p-1} \phi_{ij} y_j + \sum_{j=1}^{m} \psi_{ij} r_j + h_i u_i \qquad \text{(VI.13)}$$

If in (VI.13) $h_i u_i$, $2 \leq i \leq n-p+1$, is defined as

$$h_1 u_1 = \phi_1 w_1$$

$$h_i u_i = \sum_{j=1}^{p-1} \dot{\phi}_{ij} v_{ij} + \sum_{j=1}^{m} \dot{\psi}_{ij} s_{ij} \qquad \text{(VI.14)}$$

$$2 \leq i \leq n-p+1$$

then (VI.13) is a separable differential equation for each i. To show this, the identities for each i

$$\sum_{j=1}^{p-1} \dot{\phi}_{ij} v_{ij} = \frac{d}{dt} [\sum_{j=1}^{p-1} \phi_{ij} v_{ij}] - \sum_{j=1}^{p-1} \phi_{ij} \dot{v}_{ij}$$

$$\sum_{j=1}^{m} \dot{\psi}_{ij} s_{ij} = \frac{d}{dt} [\sum_{j=1}^{m} \psi_{ij} s_{ij}] - \sum_{j=1}^{m} \psi_{ij} \dot{s}_{ij}$$

are needed. Using them, (VI.13) becomes

$$\dot{e}_i = -\lambda_i e_i + \sum_{j=1}^{p-1} \phi_{ij} y_j + \sum_{j=1}^{m} \psi_{ij} r_j$$

$$- \sum_{j=1}^{p-1} \phi_{ij} \dot{v}_{ij} - \sum_{j=1}^{m} \psi_{ij} \dot{s}_{ij} -$$

$$+ \frac{d}{dt} [\sum_{j=1}^{p-1} \phi_{ij} v_{ij} + \sum_{j=1}^{m} \psi_{ij} s_{ij}]$$

Substituting (VI.11) and (VI.12) into the above yields

$$\frac{d}{dt} [e_i - \sum_{j=1}^{p-1} \phi_{ij} v_{ij} - \sum_{j=1}^{m} \psi_{ij} s_{ij}] = -\lambda_i [e_i - \sum_{j=1}^{p-1} \phi_{ij} v_{ij} - \sum_{j=1}^{m} \psi_{ij} s_{ij}] \tag{VI.15}$$

(VI.15) is integrated to yield

$$e_i = \sum_{j=1}^{p-1} \phi_{ij} v_{ij} + \sum_{j=1}^{m} \psi_{ij} s_{ij} + \theta_i \exp[-\lambda_i t] \tag{VI.16}$$

where

$$\theta_i = e_i^0 - \sum_{j=1}^{p-1} \phi_{ij}(t) v_{ij}(t) - \sum_{j=1}^{m} \psi_{ij}(t) s_{ij}(t)$$

at $t = t_0$.

Equation (VI.16) is applied to the first equation of (VI.10) giving

$$\dot{e}_1 = -\lambda_1 e_1 + \text{tr } \Phi^T V + \text{tr } \Psi^T S + \phi_1 y_p + \sum_{i=2}^{n-p+1} \theta_i \exp[-\lambda_i t] \tag{VI.17}$$

It is thus seen that (VI.10) and (VI.17) are equivalent, with definition of u in (VI.14), but with the difference that (VI.17) is a scalar equation. The adaptive law, dependent upon measurable variables only, may now be formulated.

A Lyapunov function candidate is selected as

$$V = e_1^2 + \delta_1^2 \phi_1^2 + \text{tr } (\Lambda \circledast \Phi^T \Phi) + \text{tr } (\Gamma \circledast \Phi^T \Phi) \tag{VI.18}$$

in which Λ and Γ are matrices having no non-positive element and the symbol $\circledast$ represents element-by-element multiplication of matrices.

The time derivative of (VI.18) along the trajectory described by VI.17) is

$$2\dot{V} = -\lambda_1 e_1^2 + (\delta_1^2 \;\dot{\phi}_1 + y_p e_1)\phi_1$$

$$+ \operatorname{tr} \Phi^T (\Delta \circledast \dot{\Phi} + Ve_1) + \operatorname{tr} \Psi^T(\Gamma \circledast \dot{\Psi} + Se_1) \tag{VI.19}$$

$$+ \sum_{i=2}^{n-p+1} \theta_i \exp[-\lambda_i t] e_1$$

Then (VI.19) can be made

$$2\dot{V} = -\lambda_1 e_1^2 + \sum_{i=2}^{n-p+1} \theta_i \; \exp[-\lambda_i t] e_1 \tag{VI.20}$$

whenever $\dot{\phi}_1$, $\dot{\Phi}$, and $\dot{\Psi}$ are defined as

$$\delta_1^2 \;\dot{\phi}_1 = -y_p e_1$$

$$\Delta \circledast \dot{\Phi} = -\; Ve_1 \tag{VI.21}$$

$$\Gamma \circledast \dot{\Psi} = -\; Se_1$$

Equations (VI.21) may be also written in scalar form as

$$\dot{\phi}_1 = -\; \frac{1}{\delta_1^2} y_p e_1 = -\dot{g}_1$$

$$\dot{\phi}_{ij} = -\; \frac{1}{\delta_{ij}^2} v_{ij}\; e_1 = -\dot{g}_{ij} \tag{VI.21a}$$

$$\dot{\psi}_{ij} = -\; \frac{1}{\gamma_{ij}^2} s_{ij}\; e_1 = -\; \dot{d}_{ij}$$

for each i, j in their proper domains. Equations (VI.21) or (VI.21a) are the adaptive laws sought.

$\dot{V}$ is eventually negative definite whenever all the eigenvalues $-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-p+1}$ have negative real parts since then the initial condition disturbances θ_i decay exponentially. Consequently e_1 is asymptotically stable in the sense of Lyapunov.

It is desired that $\lim_{t\to\infty} e(t) = 0$ in order for the adaptive observer to generate the system state. If some restriction on the input vector r is imposed, it can be shown that $e_1 \to 0$ implies $e \to 0$.

To see this, consider the limiting value of (VI.17) which is

$$o = \text{tr}\ \Phi^T V + \text{tr}\ \Psi^T S + \phi_1 y_p \tag{VI.22}$$

If by suitably restricting r, or equivalently V, S, and y_p, so that (VI.22) implies in the limit that $\Phi^T = 0$, $\Psi^T = 0$, and $\phi_1 = 0$, then (VI.10) is

$$\dot{e} = Fe$$

implying $e \to 0$ since F is an asymptotically stable matrix. The above equation follows from (VI.10) since u_i, $2 \leq i \leq n-p+1$, is zero in the limit as evident from (VI.14) and (VI.21).

The following theorems define the restriction on r guaranteeing $\Phi = 0$, $\Psi = 0$, and $\phi_1 = 0$ for $e_1 = 0$ when the steady state r is periodic.

THEOREM 6.2 [*CARROLL*]

Let q be the number of adaptive parameters in the observer (VI.2), let the observer matrix F have eigenvalues all with negative real parts, and let the system (VI.3) be completely controllable through each column vector in the input matrix B. If the collection of inputs $\{r_1, r_2, \ldots, r_m\}$ *possesses no fewer than* $[(q)/2]$ *distinct steady-state frequencies then (VI.2) generates the system state.*

PROOF:

The proof is by induction. It is shown [71,62] that the theorem holds for m=1. Assuming that the theorem holds for $m=m_1$, it will be shown that it holds for $m=m_1+1$.

Let each y_j, $1 \leq j \leq p$, be related to the inputs $r_1, r_2, \ldots, r_{m_1+1}$

by

$$y_j = \sum_{k=1}^{m_1+1} h_{jk}(p)\, r_k$$

where $p \equiv d/dt$.

Then (VI.22) is

$$0= \sum_{k=1}^{m_1+1} \{\psi_{1k} + \sum_{j=1}^{p-1} \phi_{ij}\, h_{jk}(p) + \sum_{i=2}^{n-p+1} \sum_{j=1}^{p-1} (\phi_{ij} \frac{h_{jk}(p)}{p+\lambda_i} + \frac{\psi_{ik}}{p+\lambda_i})$$

$$+ \phi_1\, h_{pk}(p)\}\, r_k \qquad \text{(VI.23)}$$

Since, by (VI.20), $e_1 \to 0$ and, by (VI.21), ϕ_{ij}, ψ_{ij}, and ϕ_1 are constants, (VI.23) may be written

$$H_1(p)r_1 + H_2(p)r_2 + \ldots + H_{m_1}(p)r_{m_1} = - H_{m_1+1}(p)r_{m_1+1} \qquad \text{(VI.24)}$$

where $H_k(p)$ are the terms in brackets in (VI.23) for each k, $1 \leq k \leq m_1+1$. Let the number of distinct adaptive coefficients in the left side of (VI.24) be q_1 and the number of distinct adaptive coefficients in $H_{m_1+1}(p)$ be q_2. By definition $q=q_1 + q_2$. By assumption $\{r_1, r_2, \ldots, r_{m_1}\}$ contains $[(\ q_1\ )/2]$ distinct frequencies and the left side of (VI.24) is zero since $H_1(p) = H_2(p) = \ldots = H_{m_1}(p) = 0$ and

$$0 = H_{m_1+1}(p)r_{m_1+1}$$

Therefore only the distinct coefficients of $H_{m_1+1}(p)$ are non-zero. By inspection of (VI.23), these are the ψ_{ik} terms which are q_2 in number. Thus by [71,62] if r_{m_1+1} contains at least $[(\ q_2\ )/2]$ distinct frequencies (i.e. distinct from the frequencies of $\{r_1, r_2, \ldots, r_{m_1}\}$) then $H_{m_1+1}(p) = 0$. Consequently $\{r_1, r_2, \ldots, r_{m_1+1}\}$

containing $[(q)/2]$ distinct frequencies implies that $H_1(p) = H_2(p) = \ldots = H_{m_1+1}(p) = 0$ which was to be proved.

THEOREM 6.3 [CARROLL]

Let the conditions on the observer (VI.2) be as stated in Theorem 6.2, but let there be no requirement upon the column vector of the input matrix B of the system (VI.3). Then it is sufficient that each input $r_i \in r$ each possess $[(q)/2]$ distinct steady-state frequencies in order for (VI.2) to generate the system state.

PROOF:

The proof follows from equation (VI.23).

When any $h_{jk}(p)$ is zero or linearily dependent, then the parameters ϕ_{ij} and ψ_{ij} are not fully "coupled" with each of the inputs v_k of equation (VI.23). This in general requires that frequencies must be assigned to each r_k depending upon the degree of freedom in the coefficient of r_k in section (VI.23). Assuming complete"decoupling" of each ϕ and ψ with respect to each v_k, it is clearly sufficient that each v_k must possess $[(q)/2]$ frequencies from equation (VI.24).

Remark:

The sufficient conditions stated in Theorem 6.3 are noted to be very conservative as a cursory glance at the proof of this theorem reveals. It is suspected by the author that under the conditions of Theorem 6.3 the requirement for state generation may be liberalized to allow only the collection of inputs $\{r_1, r_2, \ldots, r_m\}$ possess $[(q)/2]$ steady-state frequencies, as in Theorem 6.2, but with the additional restriction that the frequencies must be assigned in some way depending upon the controllability structure of system (VI.3).

At the time of this writing, however, the above speculation has not been proved.

Reconstruction of the System State

The observer (VI.2) generates the state of the transformed system (VI.8). To obtain the state of the system (VI.1) the observer estimate ξ must be transformed by

$$\hat{x} = T\begin{bmatrix} \bar{y} \\ \xi \end{bmatrix}$$

where $\hat{x}$ is the estimate of the system state x. T cannot be immediately written since it contains unknown elements of A; however, sufficient identification of the system matrix A occurs as a result of the adaptive laws (VI.21) to allow T to be determined. Consequently, the time-varying matrix $\hat{T}(G,D)$ may be constructed so that

$$\hat{x} = \hat{T}(G,D)\begin{bmatrix} \bar{y} \\ \xi \end{bmatrix}$$

is the observer estimate of x. Since $\lim_{t\to\infty} \hat{T}(G,D) = T$, the state x is obtained.

Theorem 6.4 summarizes the results of this Chapter

THEOREM 6.4

The state of system (VI.1) may be adaptively constructed by the observer (VI.2) by employing the adaptive algorithm (VI.21) and the control vector u of (VI.14), both subject to definitions (VI.11) and (VI.12) if

a) in (VI.1) the partition (A_{12}, A_{22}) is completely observable, and *b) the number of distinct frequencies in the system command input r is no fewer than [(q)/2] where q is the number of parameters to be adapted. Moreover, the number of parameters to be adapted is not greater than n plus the number of input parameters.*

Example

A specific example is given here to illustrate the design of a reduced-order adaptive observer.

Suppose the system is represented by

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 0 & b_2 \end{bmatrix} r$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} \qquad \text{(VI.1*)}$$

with a_0, a_1, a_2, a_3, b_1, and b_2 unknown constants. (This is, of course, not the most general input matrix.)

It is seen that $(A_{12}, A_{22}) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$ is completely observable. Therefore there exists a non-singular square transformation $T \in \mathcal{J}$ that puts (VI.1*) into adaptive canonical form. Such a matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & -\lambda_2^2 & \lambda_2 & 1 \end{bmatrix}$$

Note that the uppermost row of T_{21} is zero since ([1 0], A_{22}) is a completely observable pair.

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ 0 & -\lambda_2^2 & \lambda_2 & 1 \\ \tau & \lambda_2^3-a_1 & 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ b_1\lambda_2^2 & b_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{VI.8*}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

where $\tau = a_1 a_3 - a_2 \lambda_2^2 - a_0$

From the form of (VI.3*) it is seen that

$$\begin{aligned} \phi_{21} &= g_{21} - \tau \\ \phi_{22} &= g_{22} + (a_1 - \lambda_2^3) \\ \phi_{21} &= d_{21} - b_1\lambda_2^2 \\ \phi_{22} &= d_{22} - b_2 \end{aligned} \tag{VI.10*}$$

is to be adapted. Note that only 4 parameters need to be adapted for which in (VI.1*) there are 6 unknowns.

The adaptive laws are

$$\dot{g}_{21} = -\frac{1}{\delta_{21}} v_{21} e_1$$

$$\dot{g}_{22} = -\frac{1}{\delta_{22}} v_{22} e_1$$

$$\dot{d}_{21} = -\frac{1}{\gamma_{21}} s_{21} e_1 \qquad \text{(VI.21*)}$$

$$\dot{d}_{22} = -\frac{1}{\gamma_{22}} s_{22} e_1$$

in which $e_1 = \xi_1 - y_3$, and the reduced observer is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 & -\lambda_2^2 \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \text{(VI.2*)}$$

with

$$u_1 = +(\lambda_1 + \lambda_2) y_3$$

$$u_2 = \dot{\phi}_{21} v_{21} + \dot{\phi}_{22} v_{22} + \dot{\psi}_{21} s_{21} + \dot{\psi}_{22} s_{22} \qquad \text{(VI.14*)}$$

$$= -\left(\frac{1}{\delta_{21}} v_{21}^2 + \frac{1}{\delta_{22}} v_{22}^2 + \frac{1}{\gamma_{21}} s_{21}^2 + \frac{1}{\gamma_{22}} s_{22}^2 \right) e_1$$

and

$$\dot{v}_{21} + \lambda_2 v_{21} = y_1$$

$$\dot{v}_{22} + \lambda_2 v_{22} = y_2 \qquad \text{(VI.11*)}$$

$$\dot{s}_{21} + \lambda_2 s_{21} = r_1$$
$$\dot{s}_{22} + \lambda_2 s_{22} = r_2 \qquad \text{(VI.12*)}$$

The observer eigenvalues, $-\lambda_1$ and $-\lambda_2$, are arbitrary but distinct negative numbers.

The state $\bar{x}$ of system (VI.1*) may be constructed by the equation

$$\bar{x} = \hat{T}(G,D) \begin{bmatrix} \bar{y} \\ \xi \end{bmatrix}$$

where

$$T = \lim_{t\to\infty} \hat{T}(G,D) = \lim_{t\to\infty} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ g_{22}(t)+\lambda_2^3 & -\lambda_2^2, & \lambda_2 & 1 \end{bmatrix}$$

Computer Simulation

The system of section "Example" was simulated on a digital computer. The system parameters assumed unknown were b_2, a_3, a_{21}, and a_0. The following values were chosen for simulation:

$a_0 = 15$	$b_1 = 1$	$\gamma_{21} = 1/10$
$a_1 = 33.5$	$b_2 = 2$	$\delta_{22} = 1/250$
$a_2 = 26.0$	$\lambda_1 = 10$	$\phi_{21}(0) = 180.25$
$a_3 = 8.5$	$\lambda_2 = 5$	$\psi_{22}(0) = -100$

The inputs r_1 and r_2 were chosen as sine waves with frequencies of 3.5 and 5 rad/sec. respectively. The behavior of the two adaptive parameters ϕ_{21} and ψ_{22} are shown in Figure 6.2 and the (transformed) observer error e_2 is shown in Figure 6.3.

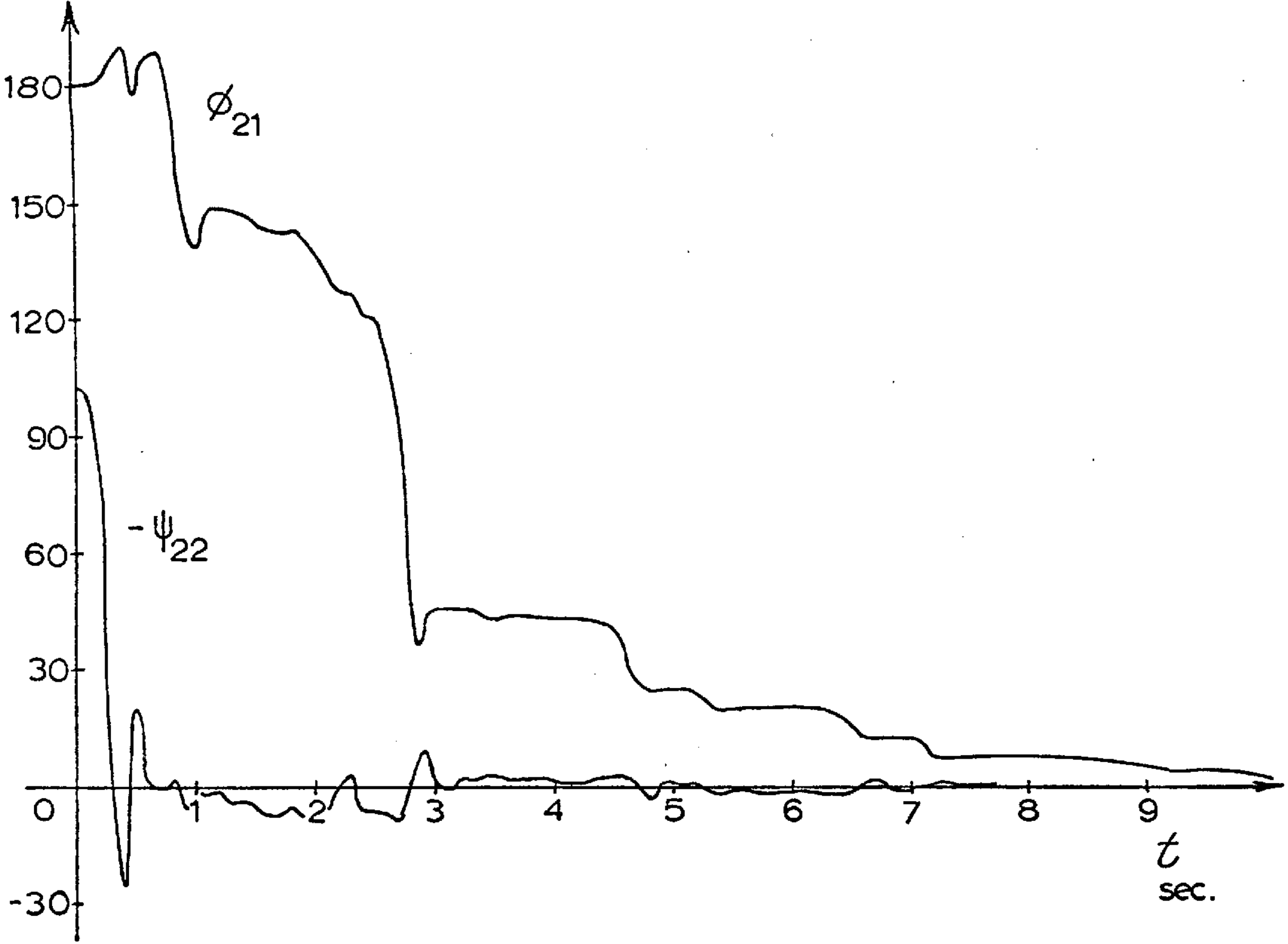


FIGURE 6.2 PARAMETER FUNCTIONS VS. TIME

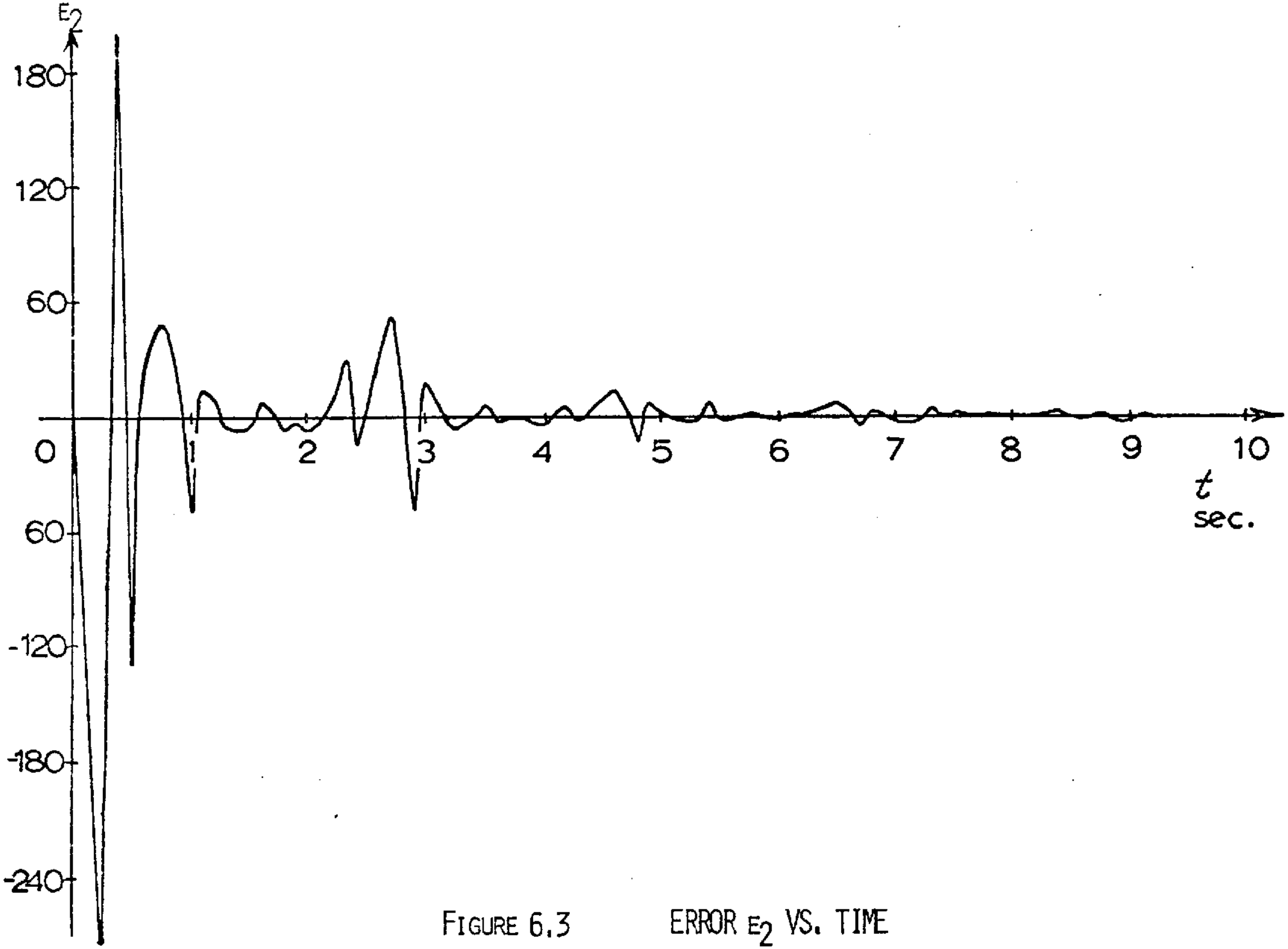


FIGURE 6.3 ERROR E_2 VS. TIME

CHAPTER VII

UNRESOLVED QUESTIONS

Due to the newness of the adaptive observer concept and the apparent richness of this concept, there remains much to be done with the adaptive observer. With reference to Chapters IV-VI, the thrust of investigation has been to attain greater simplicity in design (Chapters V and VI) and to allow greater freedom in eigenvalue selection of the adaptive network (Section 5.2). Certainly even better results may be attained in this endeavor; however there are a number of different questions that should be resolved. This chapter discusses some of them and, in the section entitled "Design Considerations", the complexity of choosing gain parameters is illustrated by simulation examples.

7.1 A Separation Theorem

Throughout the literature concerning the adaptive observer, the technique has been to require the eventual generation of state of an unknown linear system given the input and output measurements of the system and its structure. This is essentially an *open-loop* process. Since in the full order observer the unknown parameters are identified as a by-product of adaptation, the adaptive observer may be placed in a closed loop in order to control the system. In this case both the state of the system and the observer estimate of the parameters of the system are used in the feedback control law. Notwithstanding the certainty that the adaptive observer eventually becomes a form of the Luenberger observer and therefore comes to possess the closed loop properties of the

Luenberger observer (see section 2.4), it has not been shown that such a feedback system is stable throughout the adaptive process.

The difficulty in showing this lies in the fact that, before adaptation is essentially complete, the observer is both time-varying and non-linear; consequently a simple pole location analysis in order to determine over-all system stability cannot be made as was done with the Luenberger observer. Rather, investigations into this subject must employ a more sophisticated stability analysis.

7.2 A Functional Adaptive Observer

Should the observer be used in a closed loop to generate a control of the form $r = a^T x$, it was seen in Section 2.5 that a Luenberger functional observer could be constructed of order $\nu-1$ to generate r rather than the complete state vector x. Since ν may be much smaller than n-p, this represents a great savings in complexity.

An adaptive functional observer might therefore considerably reduce not only the order of the observer but also the complexity of the associated adaptive law. It is the complexity of the adaptive law - in terms of the number of integrators and number of multipliers needed to construct the law - that is a serious factor in applications.

However, the theory of the functional observer must account not only for the generation of a linear function $a^T x$, as the Luenberger linear function observer does but also for the *identifiability* of the parameters associated with a^T. The reason for this is that, in general, the vector a^T is dependent upon the unknown parameters of the system when $r = a^T x$ is used for many controls. Consequently the functional adaptive observer must be defined on the adjoined space of the system state and the system parameter space. No attempt, to the author's knowledge, has been made to develop a functional adaptive observer.

7.3 Canonical Forms

As previously remarked in Section 5.1, the more general canonical form offered in [63] offers exciting investigation in the adaptive laws resulting from the many choices of canonical forms.

In addition to the various improvements, if any, in the quality of different adaptive laws synthesized, it is hoped that a general theory will be developed whereby one may determine *a priori* at least some of the qualities of adaptive laws in relation to the canonical form chosen.

7.4 The Effects of Noise

This dissertation has emphasized the importance of the observer, whether of the Luenberger kind or the adaptive kind, to generate the state of a linear system in the presence of noise. However, there has been no reported work with the adaptive observer in terms of its ability to do this. Investigations are needed to determine the degree of susceptibility of the adaptive law to noise, to determine the relative merit of the different canonical forms proposed in regard to this degree of susceptability to noise, and the effects of observer, and filter, pole placement in minimizing that susceptability. By susceptability, one means the bound on the region for which the error between observer estimate of state and the state of the unknown system cannot be guaranteed to diminish, and the effect of noise upon convergence rate outside this bound. (That, in fact, a bound exists has been demonstrated in relation to the model-reference Liapunov adaptive problem).

7.5 Modeling

One may wonder what happens to the adaptive observer when an output of a non-linear system is substituted for that of a linear one, or if an output of a system of higher order than the (full order) adaptive observer is substituted. That this is an important question is indicated by the observation that few, if any, *real* systems are linear, and that many real systems are of unknown order.

It has been maintained by some that the resulting configuration of the adaptive observer in each instance discussed above will be the "best" linear model of the process being observed. It is unknown to the author whether in fact this occurs. The difficulty in determining this is the vagueness of the term "best linear model". Nevertheless, modeling is an important factor which should be the subject of an interesting investigation.

7.6 Design Consideration - Speed of Response

As has been previously noted, one of the salient features of the Liapunov synthesis technique is that the resulting adaptive algorithm is guaranteed to converge regardless of the magnitude of parameter ignorance. In addition, this algorithm is guaranteed to converge *regardless of the choice in magnitude of the positive adaptive gain constants,* appearing in Section 4.2 as m_s/m_i and in Section 6.2 as Γ and Δ.

However, as the third chapter makes evident, this choice of the adaptive gain constants affects the rate of convergence of the parameter estimates and the error between state estimate and system state. In the context of the adaptive observer, the accelerated adaptive laws due to Phillipson and Monopoli (See chapter III) cannot be implemented because of the requirement that $\lim_{t\to\infty} u(t) \neq 0$. Consequently the choice of the adaptive gain constants Γ, Δ, or m_s/m_i play a significant role in determining the rate of convergence of the observer estimates to their proper values.

However, the criteria for determining a "good" choice of gain constants is not defined. It is difficult to do so because of the non-linear nature of the formulation of the adaptive observer, and the consequent interaction between the various other constants to be chosen in the observer.

Other constants which may affect the rate of convergence are the observer eigenvalues, the filter poles, the magnitude and frequency of the system input, the initial conditions upon the parameter ignorance, and the amount of noise inherent in the measurements.

To gain some understanding of the relationship among these various factors and the rate of convergence, the example of Chapter VI has been investigated in the absence of noise by computer simulation. The remainder of this chapter reports some of these results. It is not intended that these results establish a relationship for all adaptive observer structures or even to establish "rule of thumb" design techniques; rather these results are given to indicate the complexity of the problem and to provide a starting point for future investigations into the relationship between rate of convergence and parameter gains.

In Figure 7.1, the parameter estimate ϕ_{22} for the plant parameter $a_1 - \lambda_2^3$ of the example in Chapter VI is plotted against time and as a function of the gain constant δ_{21} when ϕ_{21} *is the only parameter being adapted* (the other being preset to their proper values). The observer eigenvalues were -4 and -5 and the input was a square wave of frequency 3.5 and 5 rad/sec and magnitude 50. It is noted that for $1/\delta_{21}=$ 100 the curve decays almost exponentially, but for larger gains there is some oscillation. The convergence time for all choices of gains is about 3 seconds.

In Figure 7.2, only the input parameter ψ_{22} is adapted. The data for eigenvalues and input is as before. It is noted that even for low gains the behavior is oscillatory, yet faster in convergence than the plant parameter. This behavior characterizes the input parameter behavior. One additional point: for high gains of $1/\gamma_{22}$ the convergence time increases. This implies an existence of an "optimal" gain.

The major observation drawn from Figures 7-1 and 7-2 in that, with but one parameter being adapted in the example of Chapter VI, the reduced order observer, the convergence rate is very acceptable for a wide range of adaptive gains.

When two parameters are adapted simultanteously the situation is more complicated. This stems from the fact that interaction between the two parameters affects the rate of convergence. In Figure 7-3 both ϕ_{21} and ψ_{22} are simultaneously adaptive; the curves shown here are the effect on the plant parameter ϕ_{21} as the plant gain constant is changed but the input gain constant, for ψ_{22}, is held constant at $1/\gamma_{22} = 10$. The input is a sine wave of frequency 3.5 and 5.0 rad/sec. and the observer eigenvalues are -5 and -10. Figure 6-2 may also be compared with these. In this case, there is a radical change in the behavior of ϕ_{21} as a function of the corresponding gain. The better gain seems to be for $1/\gamma_{21} = 500$. The fact that the curve for

$1/\delta_{21} = 1000$ is slower to converge indicates the existence here of an "optimal" gain.

To explicitly see the effects of the interaction between gains, Figure 7.4 is offered. This is the plot of ψ_{22} corresponding to Figure 7.3. It is emphasized that in Figure 7.4 the input gain constant $1/\gamma_{22}$ is the same for each of the curves; the difference in the curves illustrated here is due entirely to interaction with the change in plant gain $1/\delta_{21}$. It is seen that the effects of interaction are marked.

The insight to be gained from Figures 7.3 and 7.4 seems to be that convergence time is much longer for two adaptive parameters, that there is marked interaction between the two parameters, and that an optimal choice of gain exists in order to minimize convergence time.

In Figure 7.5, the plant parameter ϕ_{21} is shown when the plant adaptive gain $1/\delta_{21} = 50$ but the input adaptive gain $1/\gamma_{22}$ changes. The input is as in Figures 7.3 and 7.4, but it is warned that the two curves shown in Figure 7.5 are not exactly compatible since the initial condition on ψ_{22} was also increased when $1/\gamma_{22} = 50$. Nevertheless, the figure tends to indicate that a faster convergence rate is obtained if the ratio γ_{22}/δ_{21} is high.

The effects of changes in the quality of system input frequency is shown in Figures 7.6, 7.7, and 7.8. In Figure 7.6, the plant parameter ϕ_{21} is shown for an input adaptive gain $1/\gamma_{22} = 10$ and plant adaptive gain $1/\delta_{21} = 50$ when the system input is either a sine wave or a square wave of frequency 3.5 and 5 rad/sec. and magnitude of 50. The plant eigenvalues were -5 and -10. It is seen that the square wave input forces ϕ_{21} to converge faster than the sine wave input for these choices of plant and input gains. The same effect is seen in

ψ_{22}, not shown here.

Contrarily, for gains in which $1/\delta_{21} = 1000$ and $1/\gamma_{22} = 10$, Figure 7.7 shows that the sine wave input converges faster than the square wave input. The curve for the square wave input is much more oscillatory than the sine wave input; no such effect is seen in Figure 7.6. The same effect is seen in Figure 7.8 for the input parameter ψ_{22}: for gains $1/\delta_{21} = 1000$ and $1/\gamma_{22} = 10$, the sine wave input forces ψ_{22} to converge faster without as much oscillation as the square wave input, unlike the case for which $1/\delta_{21} = 50$ and $1/\gamma_{22} = 10$.

As a result of Figures 7.6, 7.7, and 7.8, *it may be concluded that the convergence rate is frequency dependent and that the choice of "good" adaptive gains depends not only upon the interaction between all the gains but also upon the frequency of the input signal.*

Other investigations not illustrated here tend to show that the convergence rate is very dependent upon the choice of observer eigenvalues λ_i.

These studies serve to emphasize both the need and the difficulty of finding a rule whereby a choice of adaptive gains may be chosen in order to yield an acceptable convergence time. Lacking such a rule, it is a very difficult task, when there are many adaptive gains to be chosen, to choose the gains because of the interaction between all the gain constants, between the gain constants and the input frequency, and between the gain constants and the observer eigenvalues.

The effects of initial conditions (dependent upon parameter ignorance) of the adaptive integration have not been investigated.

ϕ_{21} only parameters being adapted

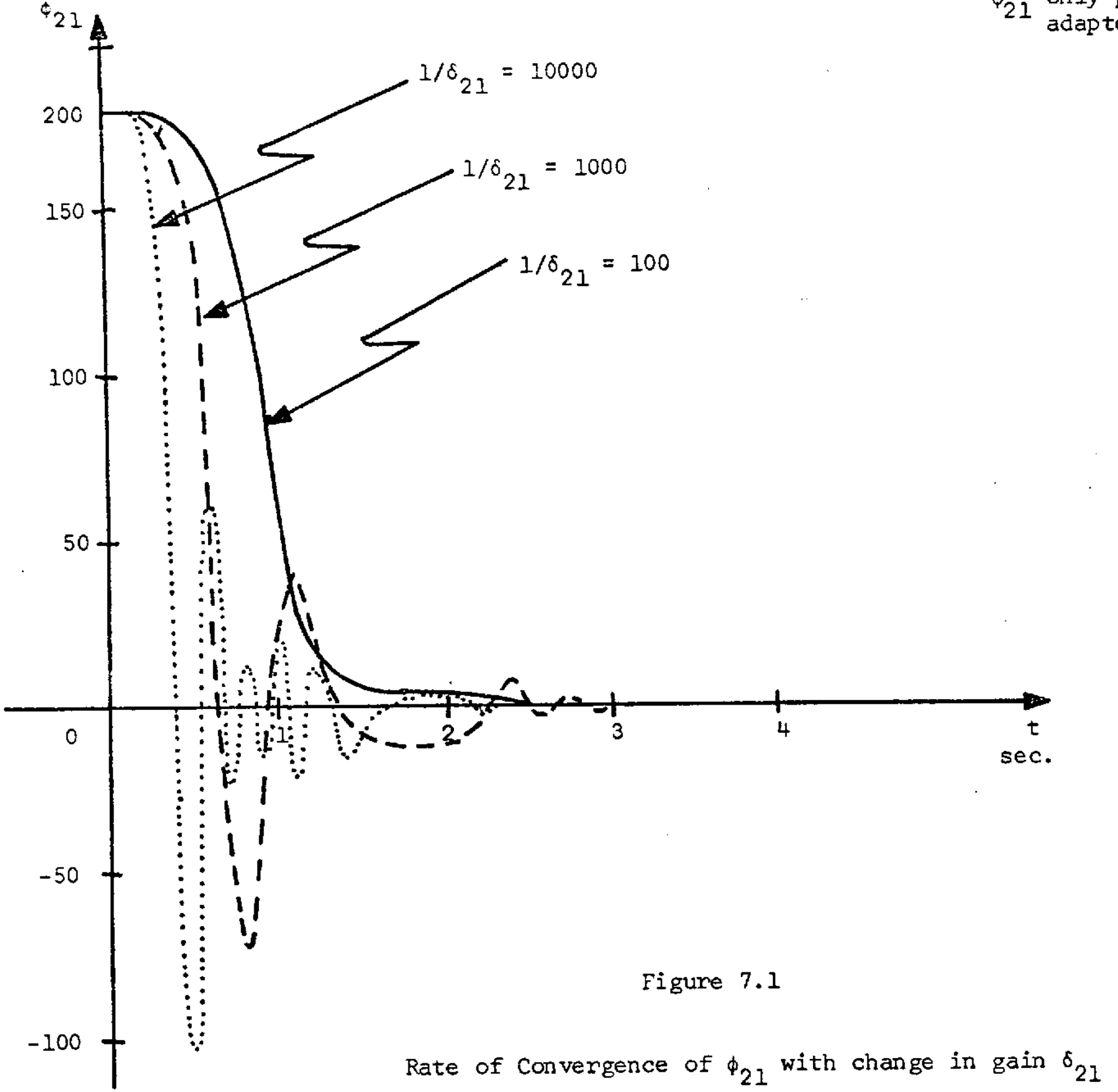


Figure 7.1

Rate of Convergence of ϕ_{21} with change in gain δ_{21}

ψ_{22} only adaptive parameter

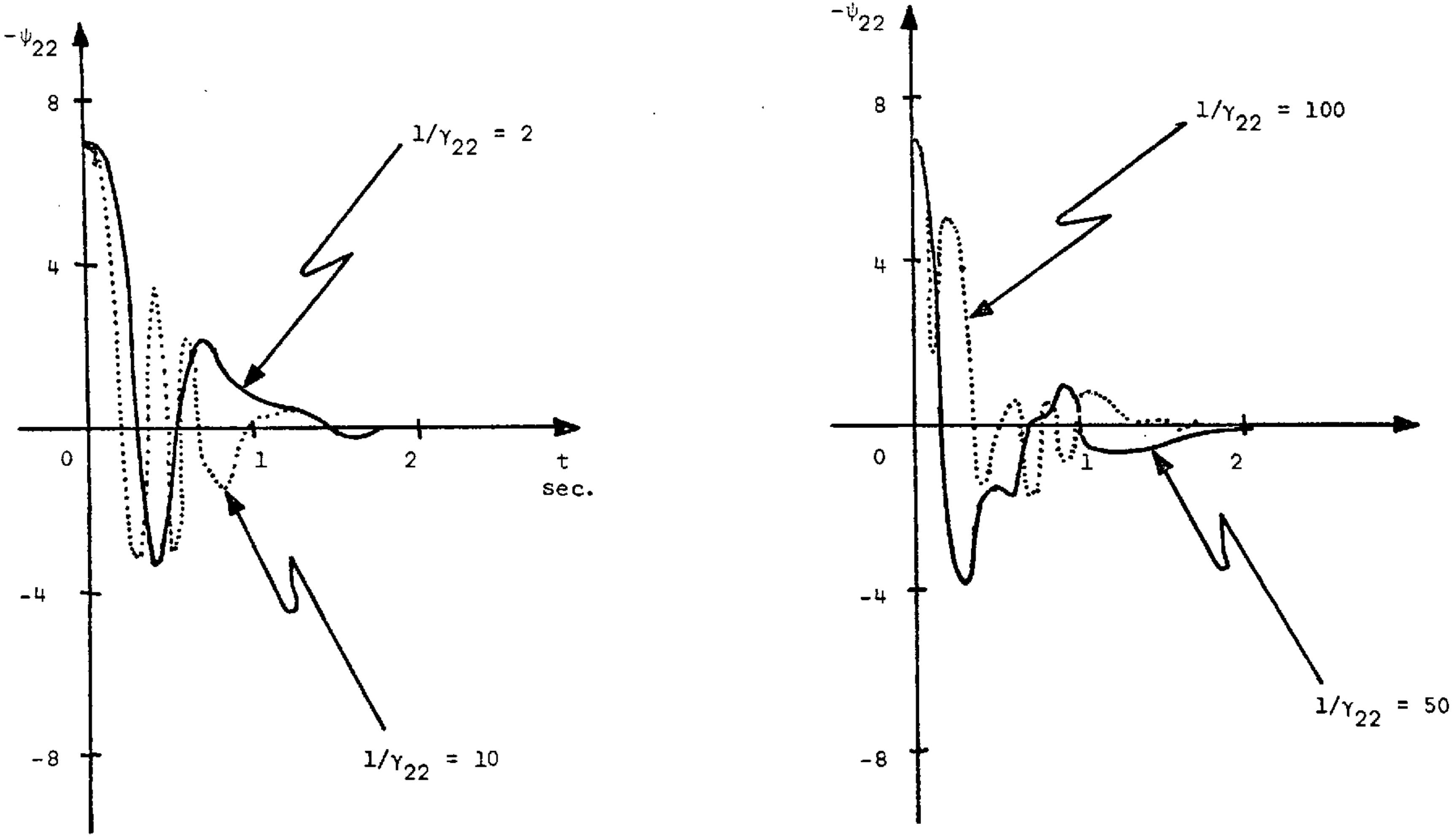


Figure 7.2

Rate of Convergence of ψ_{22} With Change in Gain γ_{22}

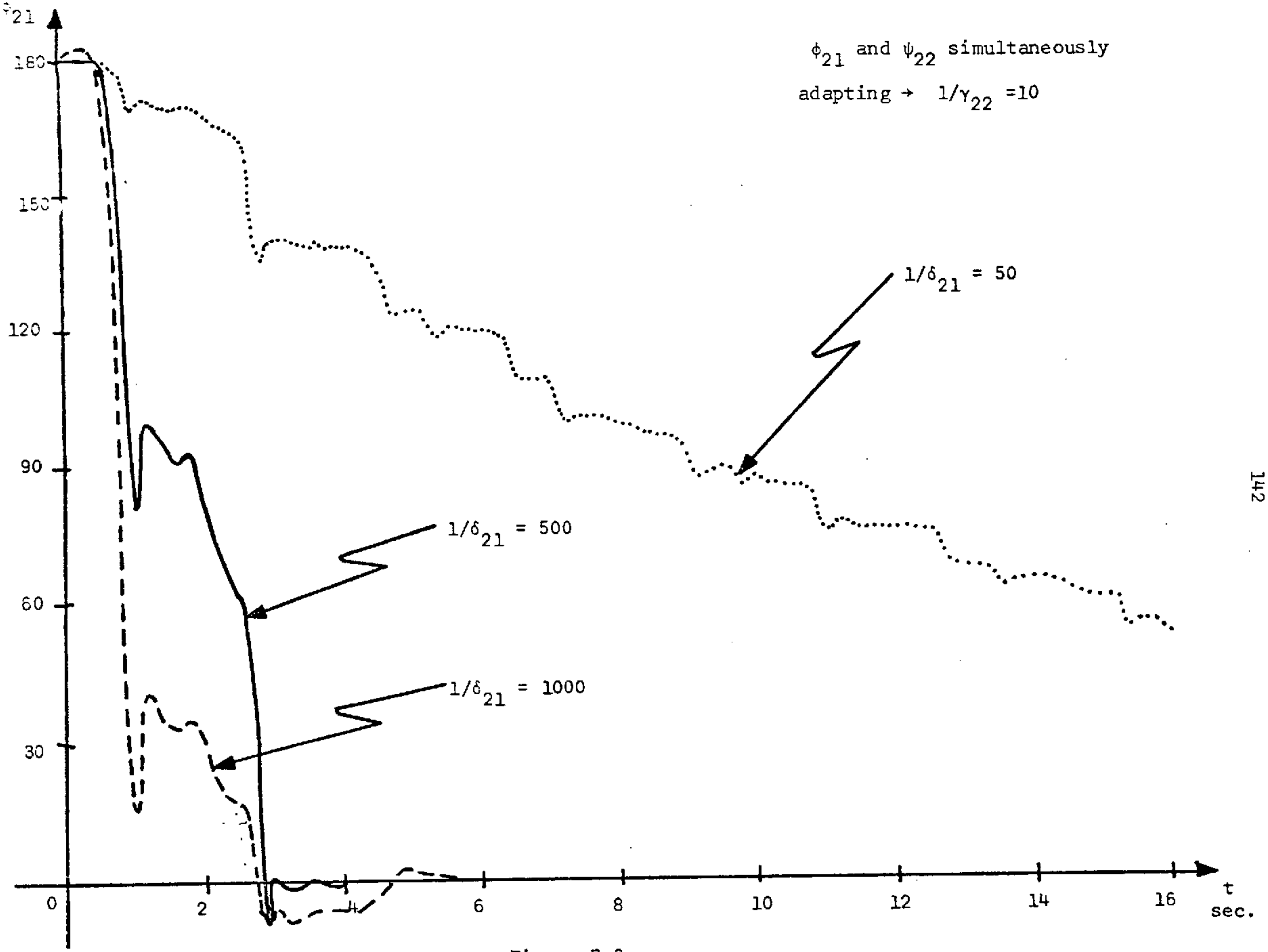


Figure 7.3

Rate of Convergence of ϕ_{21} With Change in Gain δ_{21}

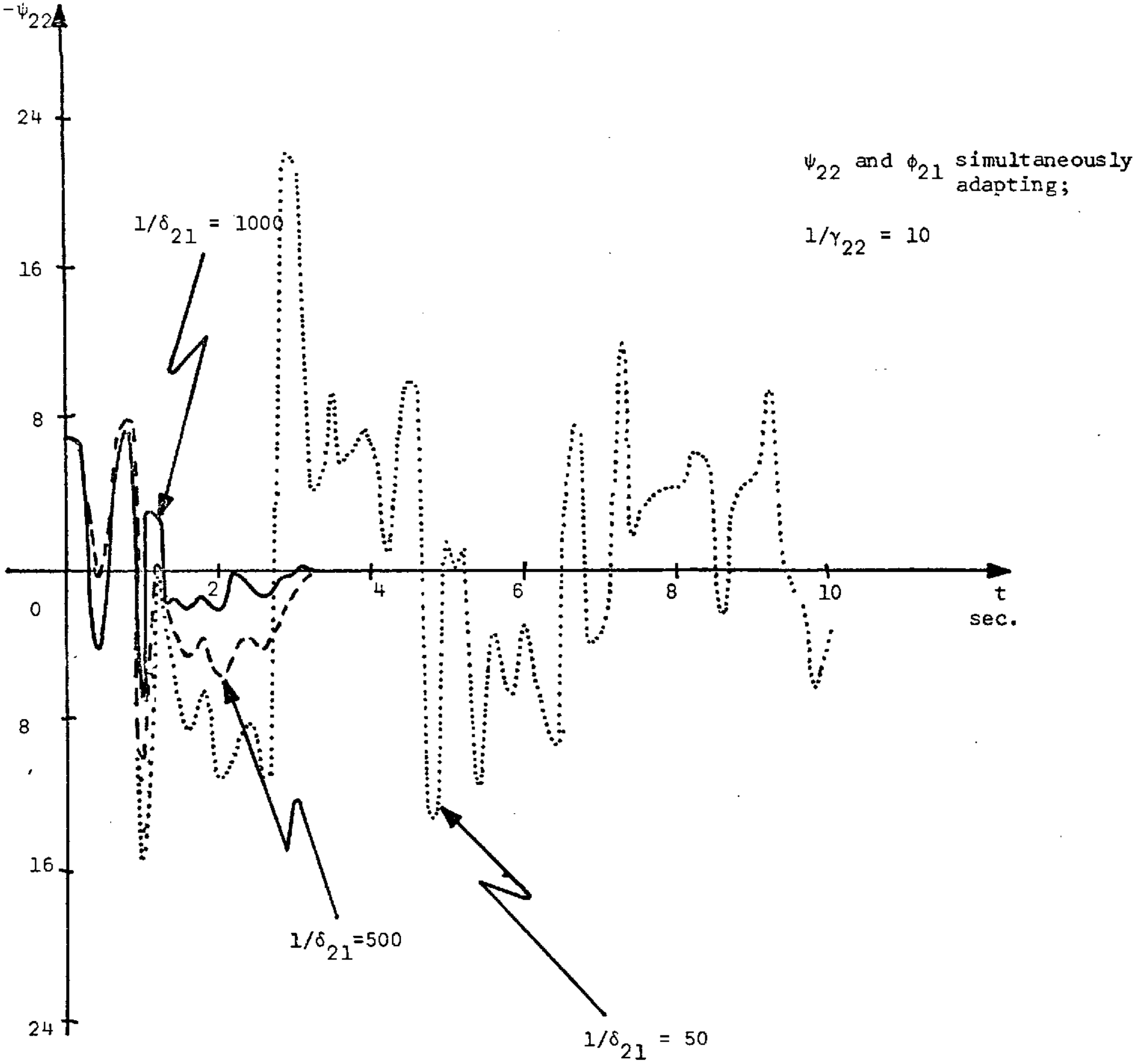


Figure 7.4
Rate of Convergence of ψ_{22} Demonstrating Interaction

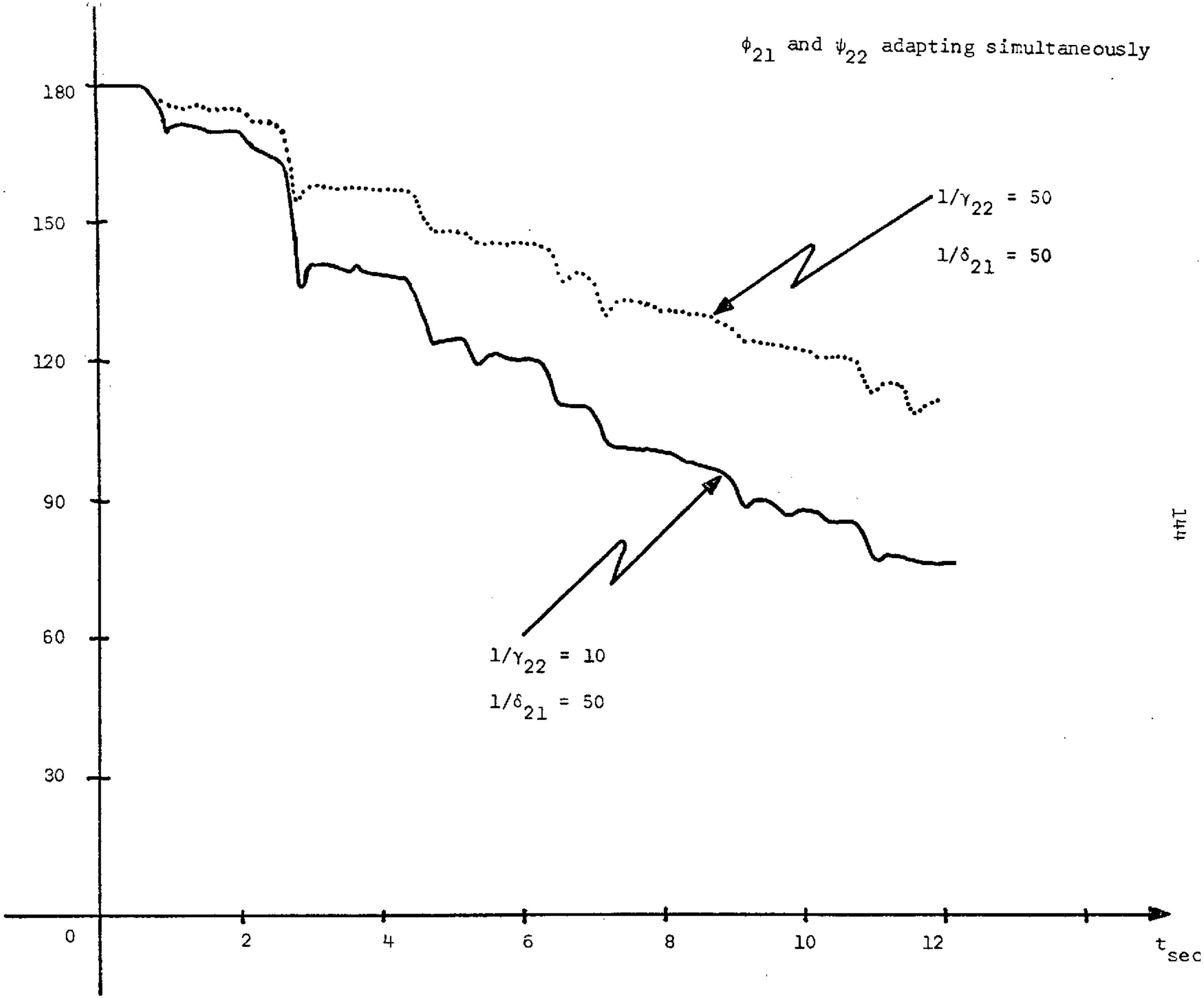


Figure 7.5

Dependence of Rate of Response of ϕ_{21} upon Gain Ratios

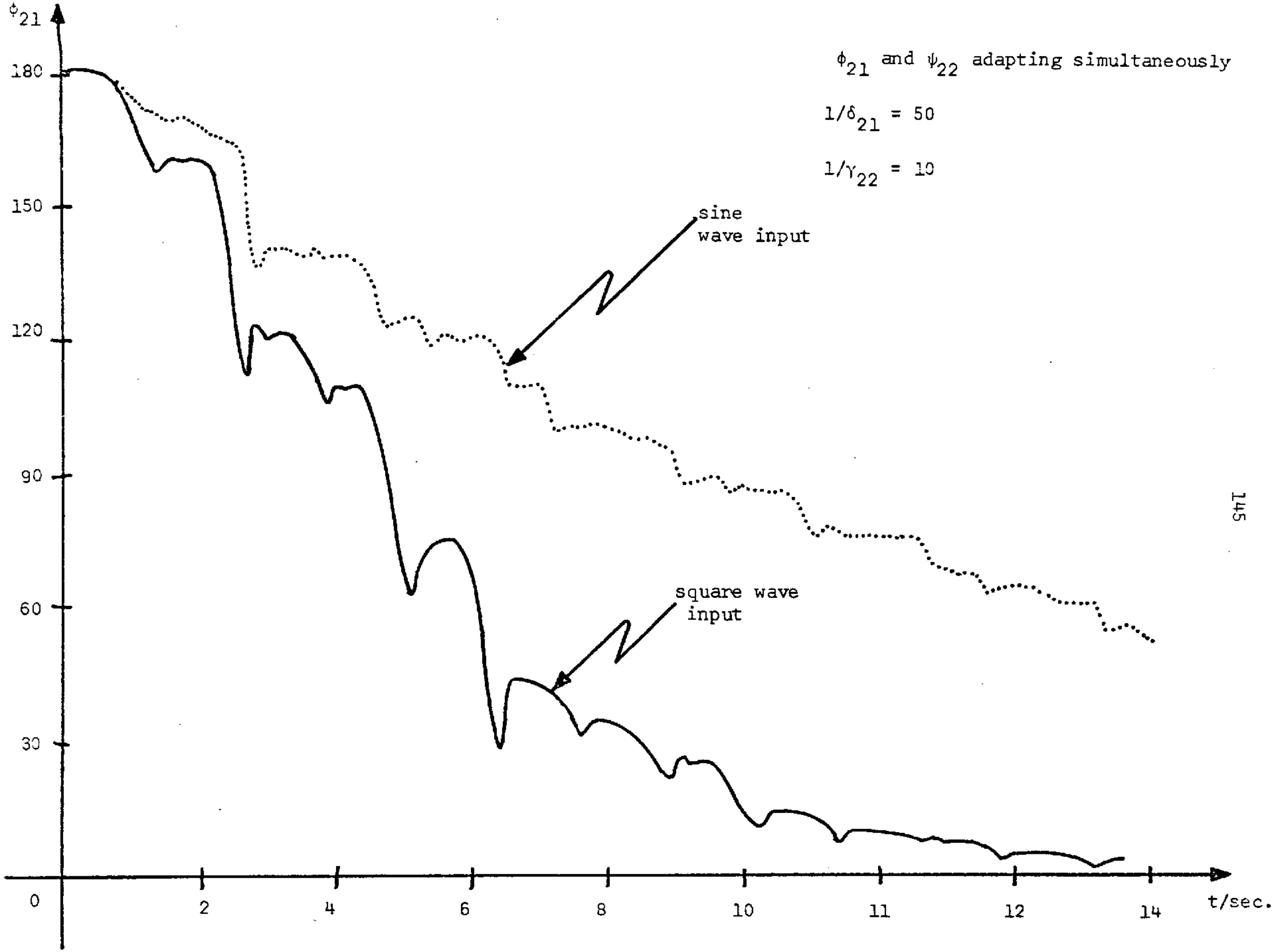


Figure 7.6
Dependence of rate of convergence of ϕ_{21} with input frequency

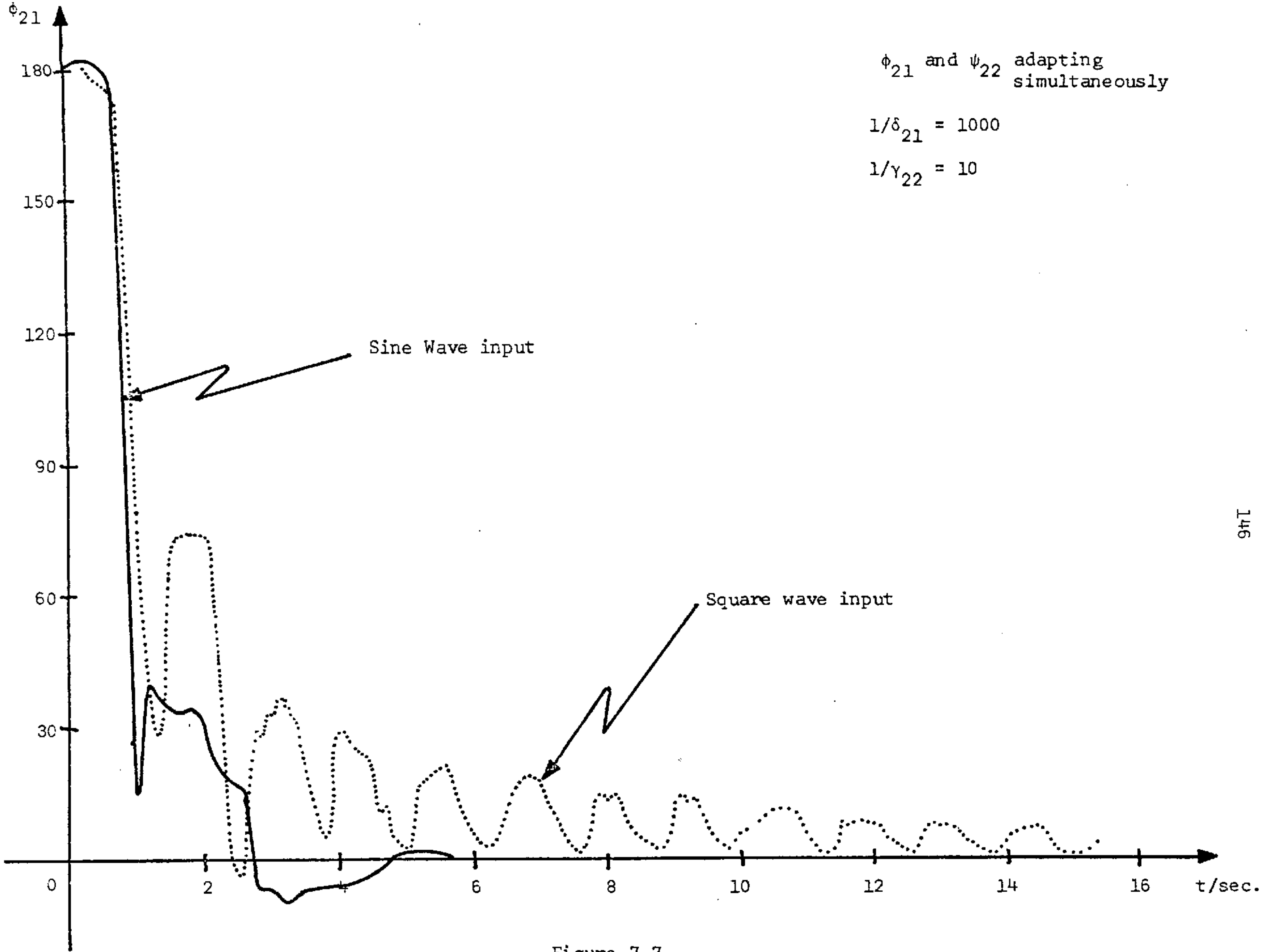


Figure 7.7

Dependence of Rate of Convergence of ϕ_{21} with input frequency

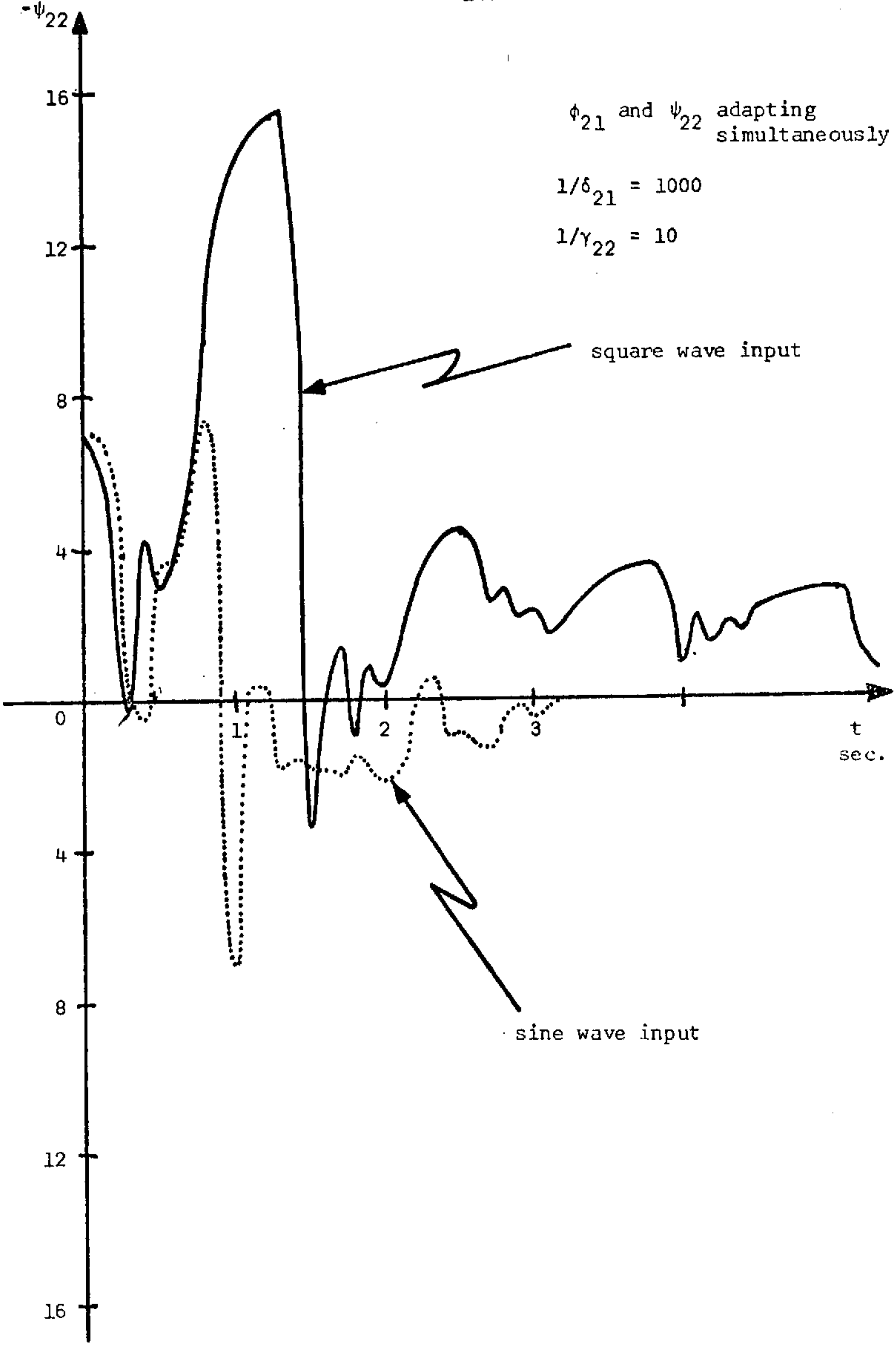


Figure 7.8

Dependence of rate of convergence of ψ_{22} upon input frequency

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