VPI-E-73-11
April 1973



# Algebraically Growing Waves in Ducts with Sheared Mean Flow

**Ali Hasan Nayfeh and Demetri P. Telionis**
**Engineering Science and Mechanics Department**

**This research was supported by NASA Langley Research Center under Grant NGR47-004-109.**

**Approved for Public Release; distribution unlimited.**

**College of Engineering,**
**Virginia Polytechnic Institute and State University**

# ALGEBRAICALLY GROWING WAVES IN DUCTS WITH SHEARED MEAN FLOW

Ali Hasan Nayfeh*

and

Demetri P. Telionis**

Department of Engineering Science and Mechanics
College of Engineering
Virginia Polytechnic Institute
Blacksburg, Virginia 24061

---

* Professor

** Assistant Professor

## ABSTRACT

Standing or traveling waves which vary algebraically with the axial distance in uniform ducts with sheared mean velocity profiles are investigated. The results show that such waves are not possible for ducts with uniform cross sections and fully developed mean flows.

CONTENTS

## I. INTRODUCTION

Recently, the problem of sound propagation through ducts with mean flow has received considerable attention, as evidenced by the large number of contributions (see Refs. 1, 2, and 3). To predict the reduction of sound as it travels through a duct with a given length, most of the authors quoted in the above references assume that all acoustic flow quantities can be expressed in terms of the normal modes in the duct as

$$Q = \sum_{n=0}^{\infty} A_n \{q_n(y) \exp[i(k_n x - \omega t)] + \bar{q}_n(y) \exp[-i(k_n x - \omega t)]\} \tag{1}$$

Here, Q stands for any of the acoustic flow quantities, u(axial velocity), v (normal velocity), p (pressure); x is the axial coordinate; y is the normal coordinate; and t is time. The real and imaginary parts of $k_n$ represent, respectively, the wave number and attenuation rate of the nth mode. In writing Eq. 1, the above authors assumed that the eigenfunctions between the brackets in Eq. 1 form a complete set.

Recently, Möhring[4] suggested that these eigenfunctions are not complete and that there exist standing wave solutions of the acoustic equations of the form,

$$Q = q(x,y)e^{i\omega t} + \bar{q}(x,y)e^{-i\omega t} \tag{2}$$

where q(x,y) is an algebraic function of x, for the case of sheared mean flow profiles that do not vanish at the wall. Moreover, he suggested that q(x,y) behaves like $x^{3/2}$ in the far field. Whereas Nayfeh and Telionis[5] showed that algebraically varying modes are possible in ducts with varying cross sections, it is shown in this paper that such modes do not exist in uniform ducts, contrary to Möhring's suggestion.

## II. STANDING WAVES

The differential equations that govern the propagation of sound in ducts with parallel generators were derived by Pridmore-Brown[6] from the Euler equations. For a two-dimensional flow with the x-axis directed along the axis of the duct, these equations may be expressed in terms of the acoustic pressure p and the normal component v of the acoustic velocity as

$$(1-M^2)\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{2M}{c}\frac{\partial^2 p}{\partial x \partial t} + 2\rho_o c \frac{dM}{dy}\frac{\partial v}{\partial x} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad (3)$$

$$\frac{\partial v}{\partial t} + Mc\frac{\partial v}{\partial x} + \frac{1}{\rho_o}\frac{\partial p}{\partial y} = 0 \qquad (4)$$

where $\rho_o$ is the density of the mean flow, c is the speed of sound and M is the Mach number which is assumed to be a function of y only. For lined walls, the boundary conditions are

$$v/p = \beta/\rho_o c \quad \text{at } y = \pm d \qquad (5)$$

where β is the acoustic admittance of the walls and 2d is the width of the duct.

In the far field, we seek asymptotic solutions to Eqs. 3 and 4 in the form,

$$p = e^{i\omega t} x^r [F_o(y) + x^{-1} F_1(y) + x^{-2} F_2(y) + \ldots] \qquad (6)$$

$$v = e^{i\omega t} x^r [G_o(y) + x^{-1} G_1(y) + x^{-2} G_2(y) + \ldots] \qquad (7)$$

where r is any real number. We substitute expansions 6 and 7 into Eqs. 3 and 4, equate coefficients of like powers of x to zero, and obtain

$$\begin{cases} F_o'' + \left(\omega^2/c^2\right) F_o = 0 & (8) \\ F_o' + i\rho_o\omega\, G_o = 0 & (9) \end{cases}$$

$$\begin{cases} F_1'' + \left(\omega^2/c^2\right) F_1 = 2i\omega r \frac{M}{c} F_o - 2\rho_o c r \frac{dM}{dy} G_o & (10) \\ F_1' + i\rho_o\omega\, G_1 = -\rho_o r\, M_c G_o & (11) \end{cases}$$

$$\begin{cases} F_m'' + \left(\omega^2/c^2\right) F_m = \xi_m & (12) \\ \qquad\qquad\qquad\qquad \text{for } m \geq 2 \\ F_n' + i\rho_o\omega\, G_m = \zeta_m & (13) \end{cases}$$

where $\xi_m$ and $\zeta_m$ are known functions of $F_o$, $F_1, \ldots, F_{m-1}$ and $G_o$, $G_1, \ldots, G_m$.

Substituting expansions 6 and 7 into Eq. 5 and again equating coefficients of like powers of x, we obtain

$$\frac{G_m(d)}{F_m(d)} = \frac{\beta}{\rho_o c} \quad , \quad m = 0,1,2\ldots \tag{14}$$

The symmetric solution of Eqs. 8 and 9 subject to the Boundary Condition 14 with m = 0 is

$$\begin{cases} F_o = A_n \cos \frac{\omega y}{c} \\ G_o = -\frac{iA_n}{\rho_o c} \sin \frac{\omega y}{c} \end{cases} \tag{15}$$

provided that

$$\tan \frac{\omega d}{c} = i\beta \tag{16}$$

The antisymmetric counterpart of Eqs. 15 and 16 is

$$\begin{cases} F_o = A_n \sin \frac{\omega y}{c} \\ G_o = \frac{iA_n}{\rho_o c} \cos \frac{\omega y}{c} \end{cases} \tag{17}$$

$$\cot \frac{\omega d}{c} = -i\beta \tag{18}$$

Equations 16 and 18 can be satisfied only when $\beta$ is purely imaginary; that is, only when the resistance of the lining material is zero. Lining materials with such a property, though, are considered unrealistic. Therefore, we conclude that standing waves which vary algebraically with the axial distance may be possible only for hard-walled ducts, provided that the frequency is an element of the infinite discrete set

$$\omega_n = n\pi c/d \qquad \text{for symmetric modes}$$
$$\omega_n = (n+\tfrac{1}{2})\,\pi c/d \qquad \text{for antisymmetric modes} \qquad (19)$$

With $\beta = 0$ an inspection of Eqs. 4-7 reveals that the Boundary Conditions 14 reduce to

$$F_m'(d) = 0 \quad , \quad m = 0,1,2,\ldots \qquad (20)$$

Taking $\omega = \omega_n = n\pi c/d$, we can write Eq. 10 as

$$F_1'' + \frac{n^2\pi^2}{d^2} F_1 = 2irA_n \left[\frac{n\pi}{d} M \cos\frac{n\pi y}{d} + \frac{dM}{dy}\sin\frac{n\pi y}{d}\right] \qquad (21)$$

Equation 21 has a solution if, and only if, its nonhomogeneous part is orthogonal to the solution of the adjoint homogeneous problem $\cos(n\pi y/d)$; that is, if, and only if,

$$\frac{n\pi}{d}\int_0^d M(y)\cos^2\frac{n\pi y}{d}\,dy + \frac{1}{2}\int_0^d \frac{dM}{dy}\sin\frac{2n\pi y}{d}\,dy = 0 \qquad (22)$$

Integrating by parts, we can rewrite Eq. 22 as

$$S \equiv \int_0^d M(y)\,dy - \int_0^d M(y)\cos\frac{2n\pi y}{d}\,dy = 0 \qquad (23)$$

The case $n = 0$ corresponds to a time independent solution which is not of interest. For $n \geq 1$, we assume that the mean Mach number profile is uniform and equal to $M_0$ in the core of the duct, and possesses a boundary layer profile $M(y)$ in a thin layer of thickness $\delta$ next to the walls. Then,

$$S \equiv M_0(d-\delta) + \int_{d-\delta}^{\delta} M(y)dy - M_0 \frac{d}{2n\pi} \sin \frac{2n\pi(d-\delta)}{d}$$

$$- \int_{d-\delta}^{\delta} M \cos \frac{2n\pi y}{d} dy \tag{24}$$

A lower bound on S is given by

$$\frac{S}{M_0 d} > 1 - \frac{\delta}{d} - \frac{1}{2n\pi} \tag{25}$$

As long as $\delta/d$ is less than 0.85, which is a value very much larger than those encountered in practical applications, $S > 0$. Therefore, there cannot exist standing waves that vary algebraically with the axial distance.

## III. TRAVELING WAVES

Next, let us determine if traveling waves that vary algebraically along the duct can exist. We seek expansions in the far field of the form,

$$p = e^{i\theta} x^r[F_0(y) + x^{-1} F_1(y) + \ldots] \tag{26}$$

$$v = e^{i\theta} x^r[G_0(y) + x^{-1} G_1(y) + \ldots] \tag{27}$$

where $\theta = kx - \omega t$. Substituting expansions 26 and 27 into Eqs. 3 and 4, and equating coefficients of like powers of x, we obtain

$$F_0'' + [\frac{\omega^2}{c^2} - k^2(1-M^2) + 2M \frac{\omega}{c} k] F_0 + 2i\rho_0 ck \frac{dM}{dy} G_0 = 0 \tag{28}$$

$$F_0' + i\rho_0(\omega + kMc) G_0 = 0 \tag{29}$$

$$F_1'' + [\frac{\omega^2}{c^2} - k^2(1 - M^2) + 2M \frac{\omega}{c} k] F_1 + 2\rho_0 c \frac{dM}{dy} ikG =$$

$$= \frac{2Mi\omega}{c} r F_0 - 2\rho_0 cr \frac{dM}{dy} G_0 - 2(1-M^2) ikr F_0 \qquad (30)$$

$$F_1' + i\rho_0 (\omega + kMc) G_1 = -Mcr G_0 \qquad (31)$$

$$F_m' + [\frac{\omega^2}{c^2} - k^2(1-M^2) + 2M \frac{\omega}{c} k] F_m + 2\rho_0 c \frac{dM}{dy} ik G = \xi_m \qquad (32)$$

for $m = 2,3,\ldots$

$$F_m' + i\rho_0(\omega + kMc) G_m = \zeta_m \qquad (33)$$

For a mean velocity profile that vanishes at the walls, the Boundary Conditions 14 still hold.

Eliminating $G_0$ from Eqs. 28 and 29,we have

$$L(F_0) \equiv F_0'' - \frac{2kc}{\omega+Mkc} \frac{dM}{dy} F_0' + [\frac{\omega^2}{c^2} - k^2(1-M^2) + 2M \frac{\omega}{c} k] F_0 = 0 \qquad (34)$$

In this case for each frequency $\omega$ there exists an infinite number of eigenvalues k and eigenfunctions $F_0$ that satisfy Eq. 33 , the Boundary Condition 14 and $F_0(0) = 0$ or $F_0'(0) = 0$. These eigenfunctions can be obtained either analytically or numerically.

Eliminating $G_1$ from Eqs. 30 and 31 and using Eq. 34,we obtain

$$L(F_1) = \psi_1[M(y), F_0(y)] \qquad (35)$$

where $\psi_1$ is a known function of $M(y)$ and $F_0(y)$. For a given frequency, the nonhomogeneous first-order problem has a solution if, and only if, its nonhomogeneous part is orthogonal to the solution of the adjoint homogeneous problem. This solvability condition imposes a restriction on the mean Mach number profile. Carrying the expansion to higher order, we find more restrictions on $M(y)$. If $r$ is a positive integer, there will be $r$ restrictions on $M(y)$; otherwise, there will be an infinite number of restrictions on $M(y)$. Since $M(y)$ must satisfy the Navier-Stokes equations, it will not satisfy, in general, any other restriction.

## IV. RESULTS AND DISCUSSION

The results show that there cannot exist standing waves which vary algebraically along a uniform duct if the walls have a finite resistance. Since Nayfeh[7] showed that the acoustic boundary-layer produces an effective finite admittance at the wall even if it is rigid, we conclude that there are no standing waves which vary algebraically along a duct with uniform cross section.

Also, the results show that traveling waves which vary algebraically along the duct do not exist unless the mean velocity profile satisfies one or more restrictions besides satisfying the Navier-Stokes equations. Clearly, such velocity profiles are not realistic.

REFERENCES

1. J. F. Unruh and W. Eversman, "The Transmission of Sound in an Acoustically treated Rectangular Duct with Boundary-Layer". Wichita State University, Aeronautical Report, 72-2 (May 1972).

2. H. H. Hubbard, D. L. Lansing and H. L. Runyan, "A Review of Rotating Blade Technology", J. Sound Vib., 19, 227-249 (1971).

3. J. Lighthill, "The Fourth Annual Fairey Lecture: The Propagation of Sound Through Moving Fluids", J. Sound Vib., 24, 482-492 (1972).

4. W. Möhring, "On the Resolution into Modes of Sound Fields in Ducts with Shear Flow", presented at the Symposium on "Acoustics of Flow Ducts", Institute of Sound and Vibration Research, University of Southampton (January 1972).

5. A. H. Nayfeh and D. P. Telionis, "Acoustic Propagation in Ducts with Varying Cross Sections", VPI Engineering Report VPI-E-73-7, March 1973.

6. D. C. Pridmore-Brown, "Sound Propagation in a Fluid Flowing through an Attenuating Duct", J. Fluid Mech., 4, 393-406 (1958).

7. A. H. Nayfeh, "Effect of the Acoustic Boundary Layer on the Wave Propagation in Ducts", VPI Engineering Report, VPI-E-73-10, April 1973.