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NEW PLATE AND SHELL ELEMENTS FOR NASTRAN

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SUMMARY

A new higher order triangular plate-bending finite element is presented in this paper which possesses high accuracy for practical mesh subdivisions and which uses only translations and rotations as grid point degrees of freedom. The element has 18 degrees of freedom (d.o.f.), viz., the transverse displacement and two rotations at the vertices and mid-side grid points of the triangle. The transverse displacement within the element is approximated by a quintic polynomial; the bending strains thus vary cubically within the element. Transverse shear flexibility is taken into account in the stiffness formulation. Two examples of static and dynamic analysis are included to show the behavior of the element. Excellent accuracy is achieved in all cases.

This element, designated as TR-18, is demonstrated to be an ideal candidate for generation of a family of plate and shell elements for inclusion into NASTRAN. The following elements are specifically mentioned in this context, viz., (i) triangular plate element, (ii) quadrilateral plate element, (iii) curved triangular shell element, (iv) curved quadrilateral shell element and (v) plates with membrane-bending coupling and multilayered plates. The present paper describes the detailed theoretical derivations for the aforementioned elements. In addition, the behavior of the TR-18 element and associated quadrilateral plate element is illustrated by two sample problems. Comparisons with existing elements in the literature and the present NASTRAN quadrilateral elements are shown.

INTRODUCTION

NASTRAN presently (Level 15.5) has, in all, a total of nine different forms of plate elements in two different shapes (triangular and quadrilateral). The present NASTRAN basic bending element, TRBSC, the basic unit from which the bending properties of the other plate elements are formed, uses a cubic displacement field (with the x^2y term omitted). This constrains the normal slope (on the exterior edges of the TRPLT bending element) to vary linearly, which in turn makes the element overly stiff. A need thus exists for a more accurate plate bending element for NASTRAN.

A brief review of some of the more important plate bending elements is now made. Formulations of triangular plate bending finite elements were given as long ago as 1966 by Clough and Tocher (ref. 1) and by Bazeley et al. (ref. 2). The conforming elements presented therein allow only a linear variation of slope normal to an edge and have since been found to be overly

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stiff, whereas the nonconforming element given in ref. 2 uses a cubic polynomial for transverse displacement and is not of very high accuracy. Improvements to these elements have been made by using higher degree polynomials for transverse displacements; indeed elements of very high accuracy have been reported by Argyris (ref. 3), Bell (ref. 4) and Cowper et al. (ref. 5) using quintic polynomials for the displacements field. But these elements have strains, curvatures and/or higher order derivatives of displacements as grid point degrees of freedom (d.o.f.) which lead to an inconsistency when abrupt thickness or material property variation occurs. That is to say that the continuity of strains and curvatures implied by their use as degrees of freedom at grid points is violated wherever concentrated loads, changes in slope, changes in thickness, or connections to other structures occur. In short, the proper use of elements that assume continuity of strains and curvatures is restricted to regions where discontinuities do not occur. Further, the existence of higher order derivatives makes it difficult to impose boundary conditions on these and indeed the simple interpretation of energy derivatives as "nodal forces" disappears (ref. 6). Bell has also developed another element in ref. 4, designated T-15 by him, which has only displacements and rotations as degrees of freedom. But it has a major drawback in that not all grid points of the element have the same d.o.f.; consequently, it becomes difficult, if not impossible, to consider connections of this element with other finite elements. Thus the practical use of the T-15 element in general purpose programs is severely limited.

A need still exists to develop a new accurate plate bending finite element that has the advantages of the accuracy associated with a high order displacement polynomial but does not have the disadvantages discussed above and is therefore suitable for inclusion in general purpose computer programs like NASTRAN.

In this paper, a triangular element and an associated quadrilateral element are developed that use only displacements and rotations as grid point degrees of freedom and use a quintic polynomial for lateral displacement. The quadrilateral element is formed by four triangular elements. The stiffness, consistent mass and load matrices of the separate triangles are evaluated and added by the direct stiffness technique to form the respective matrices for the quadrilateral. The terms associated with the internal grid points are then eliminated by static condensation. None of the elements discussed in references 1 to 5 possess the property of transverse shear flexibility. This has been taken into account in the present paper by a procedure based on that used in NASTRAN (ref. 7).* The components of transvers shear strain are quadratic functions of position. Convergence to the limiting case of zero transverse shear strain is uniform.

In addition, three elements, viz., (i) a curved triangular shell element, (ii) a curved quadrilateral shell element, and (iii) a multilayered plate element can be derived from the TR-18 element. Together with the quadrilateral plate element, these elements constitute the TR-18 family of elements.

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*A similar procedure for incorporation of transverse shear flexibility into a quartic element was communicated to the author by Dr. R. H. MacNeal of MacNeal-Schwendler Corporation.

LIST OF SYMBOLS

{a}	Column vector of coefficients
a,b,c	Dimensions of triangular element in local co-ordinates (fig. 1)
a_1, a_2, a_3	Coefficients of quintic polynomial
$[B_1]$,$[B_2]$,$[B_3]$	Matrices relating strains and generalized displacements
[C]	Row vector relating transverse displacement to generalized displacement
[D]	Matrix relating bending stresses and bending strains
D	Plate flexural rigidity, $Et^3/12(1 - \nu^2)$
E	Elastic modulus
$[G_o]$	Matrix relating interior grid point displacement to exterior grid point displacements of quad-element
[J]	Matrix relating transverse shear forces and strains
[K]	Stiffness matrix
L	Length of side of plate
[M]	Consistent mass matrix
{M}	Vector of bending and twisting moments per unit length
N	Number of elements per side of plate
[Q]	Matrix relating grid-point displacement vector and vector of polynomial coefficients
[R]	Augmented matrix of Q and constraint relations
[S]	Matrix relating vector of polynomial coefficients and grid point displacement vector
T	Kinetic energy
$[T_1]$,$[T_2]$	Transformation matrices
t	Thickness of plate
[U]	Matrix of transformation of strain components
U	Strain energy
{V}	Vector of transverse shears per unit length
w	Lateral displacement
w_c	Central deflection
x,y,z	Co-ordinate axes in the local system
X,Y,Z	Co-ordinate axes in the global system
α	Rotation of xz plane at each grid point
β	Rotation of yz plane at each grid point
γ_{xz}, γ_{yz}	Transverse shear strains
$\{\gamma\}$	Vector of transverse shear strains
$\{\delta\}$,$\{\Delta\}$	Column vectors of grid point displacement in local or global system
ϕ	Inclination of material orientation axis to x-axis
$\{\phi\}$	Displacement vector of quadrilateral element
ν	Poisson's ratio
ρ	Mass density of plate material
λ	Non-dimensional parameter of eigenvalues, $\rho t\omega^2 L^4/D$
$[\lambda]$	Direction cosine matrix of quadrilateral median plane
ω	Circular frequency of plate vibration
$\{\chi\}$	Bending strains

$\chi_x, \chi_y, \chi_{xy}$ Bending strains

TRIANGULAR PLATE ELEMENT TR-18

In this section of the paper, the derivation of the stiffness matrix, consistent load vector and consistent mass matrix of the triangular plate element is given. The procedure for the derivation is described in detail in reference 8, and hence only essential details are presented here.

The element has 18 d.o.f., the transverse displacement and 2 rotations at each vertex and at the mid-point of each side. Three additional conditions are introduced, viz., the slope normal to each edge (hereinafter called normal slope) varies cubically along each edge. This establishes 3 constraint equation between the coefficients of the polynomial for displacements, which, together wi the 18 d.o.f., uniquely determine the 21 coefficients in the quintic polynomial. The variation of deflection along any edge is a quintic polynomial in the edgewise co-ordinate; the six coefficients of this polynomial are uniquely determined by deflection and edgewise slope at the 3 grid points of the edge. Displacements are thus continuous between two elements that have a common edge. The normal slope along each edge is constrained to vary cubically; however, since the normal slopes are defined only at 3 points along an edge, there is no normal slope continuity between 2 elements that have a common edge. The element thus belongs to the class of non-conforming elements. The development of this element follows closely that of Cowper et al. (ref. 5).

Element Geometry

Rectangular cartesian co-ordinates are used in the formulation. An arbitrary triangular element is shown in figure 1, where X, Y, and Z are a system of global co-ordinates and x, y, z are the system of local co-ordinates for the triangular element. The grid points of the element are numbered in counterclockwise direction as shown. The following relationship between the dimensions of the triangular element a, b, c, the inclination θ between the X and x axes and the co-ordinates of the vertices of the element can be easily derived (see fig. 1):

$$\cos\theta = (X_3 - X_1)/r \qquad \sin\theta = (Y_3 - Y_1)/r \tag{1}$$

where

$$r = [(X_3 - X_1)^2 + (Y_3 - Y_1)^2]^{1/2} \tag{2}$$

$$a = (X_3 - X_5)\cos\theta - (Y_5 - Y_3)\sin\theta$$

$$= \{(X_3 - X_5)(X_3 - X_1) + (Y_3 - Y_5)(Y_3 - Y_1)\}/r \tag{3}$$

and similarly,

$$b = \{(X_5 - X_1)(X_3 - X_1) + (Y_5 - Y_1)(Y_3 - Y_1)\}/r \quad (4)$$

$$c = \{(X_3 - X_1)(Y_5 - Y_1) - (Y_3 - Y_1)(X_5 - X_1)\}/r \quad (5)$$

Displacement Function

The deflection $w(x, y)$ within the triangular element is assumed to vary as a quintic polynomial in the local co-ordinates, i.e.,

$$\begin{aligned} w(x,y) = {} & a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + \\ & a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^4 + a_{12}x^3y + a_{13}x^2y^2 + \\ & a_{14}xy^3 + a_{15}y^4 + a_{16}x^5 + a_{17}x^4y + a_{18}x^3y^2 + a_{19}x^2y^3 + \\ & a_{20}xy^4 + a_{21}y^5 \end{aligned} \quad (6)$$

There are 21 constants, a_1 to a_{21}. These are evaluated as follows:

The element has 18 d.o.f. At each grid point there are 3 displacement components as d.o.f., viz., w, displacement in z-direction, α, rotation about the x-axis and β, rotation about y-axis. The rotations α and β are obtained from the definitions of transverse shear strains γ_{xz} and γ_{yz}, i.e.,

$$\left.\begin{aligned} \gamma_{xz} &= \frac{\partial w}{\partial x} + \beta \\ \gamma_{yz} &= \frac{\partial w}{\partial y} - \alpha \end{aligned}\right\} \quad (7)$$

It can be shown (ref. 8) that γ_{xz} and γ_{yz}, and hence α and β, at any grid point can be expresred in terms of the constants a_1 to a_{21}. Thus 18 relations between grid point displacement values and the constants are obtained. Three constraints among the coefficients in the above polynomial (eq. (6)) are now introduced so that the normal slope varies cubically along each edge. It is clear that the three constraint equations will involve only the coefficients of the fifth degree terms in equation (6), since the lower degree terms satisfy the condition of cubic normal slope automatically. Moreover the conditions depend only on the direction of an edge and not on its position. Along the edge defined by grid points 1 and 3, where $y = 0$, the condition of cubic normal slope requires that

$$a_{17} = 0 \tag{8}$$

It can be shown (ref. 8) that the condition for cubic variation of normal slope along edge 1-5 is

$$5b^4c\ a_{16} + (4b^3c^2 - b^5)a_{17} + (3b^2c^3 - 2b^4c)a_{18} + (2bc^4 - 3b^3c^2)a_{19} +$$

$$(c^5 - 4b^2c^3)a_{20} - 5bc^4\ a_{21} = 0 \tag{9}$$

and the condition for cubic variation of the normal slope along the edge 3-5 (see fig. 1) is

$$5a^4c\ a_{16} + (-4a^3c^2 + a^5)a_{17} + (3a^2c^3 - 2a^4c)a_{18} + (-2\ ac^4 + 3a^3c^2)a_{19} +$$

$$(c^5 - 4a^2c^3)\ a_{20} + 5ac^4a_{21} = 0 \tag{10}$$

The 18 relations between grid point displacements (w, α and β at each of the six grid points) and the coefficients of the polynomial, together with the three constraint equations (8), (9), and (10), uniquely determine the coefficients a_1 to a_{21}. The following equations can therefore be written:

$$\{\delta\} = [Q]\ \{a\} \tag{11}$$

and

$$\{a\} = [S]\ \{\delta\} \tag{12}$$

where [Q] is the 18 x 21 matrix involving the co-ordinates of grid points substituted into the function w (eq. (6)) and the appropriate expressions of α and β; $\{a\}$ is the column vector of coefficients a_1 to a_{21}, and [S] is a 21 x 18 matrix and consists of the first 18 columns of the inverse of an augmented matrix of [Q] and the three constraint equations (8), (9), and (10).

Stiffness Matrix

The following relationships are obtained from the theory of deformation for plates (ref. 9). In the present notation, the curvatures are defined by

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial\beta}{\partial x} \\ \dfrac{\partial\alpha}{\partial y} \\ \dfrac{\partial\alpha}{\partial x} - \dfrac{\partial\beta}{\partial y} \end{Bmatrix} \tag{13}$$

Bending and twisting moments are related to curvatures by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \tag{14}$$

where [D] is, in general, a full symmetric matrix of elastic coefficients.

Shear forces (and hence shear strains) are proportional to the third derivatives of the displacements. Since the displacement within the element is assumed to vary as a quintic polynomial, shear strains are expressed by a quadratic polynomial as follows:

$$\left.\begin{aligned} \gamma_x &= b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 \\ \gamma_y &= c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 \end{aligned}\right\} \tag{15}$$

The shear forces V_x, V_y are related to γ_x, γ_y by

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = t^* \, [G] \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \tag{16}$$

where G is in general a full 2 x 2 symmetric matrix and t^* is an effective thickness of the element.

It can be shown that b_1 to b_6 and c_1 to c_6 can be expressed in terms of the coefficients a_1 to a_{21} (ref. 8) and hence $\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$ can be expressed as

$$\{\gamma\} = [B_1]\ \{a\} \tag{17}$$

where $[B_1]$ is as given in reference 8. The curvature $\{\chi\}$ is now split into 2 parts, i.e.,

$$\{\chi\} = \{\chi_1\} + \{\chi_2\} \tag{18}$$

where

$$\{\chi_1\} = \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\,\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \qquad \{\chi_2\} = \begin{Bmatrix} -\dfrac{\partial \gamma_{xz}}{\partial x} \\ -\dfrac{\partial \gamma_{yz}}{\partial y} \\ -\dfrac{\partial \gamma_{xz}}{\partial y} - \dfrac{\partial \gamma_{yz}}{\partial x} \end{Bmatrix} \tag{19}$$

It follows that $\{\chi_1\}$ is the vector of curvatures in the absence of transverse shear and $\{\chi_2\}$ is the contribution of transverse shear to the vector of curvatures. Now $\{\chi_1\}$ and $\{\chi_2\}$ are expressed in terms of generalized co-ordinates $\{a\}$ as

$$\{\chi_1\} = [B_2]\ \{a\} \tag{20}$$

and

$$\{\chi_2\} = [B_3]\ \{a\} \tag{21}$$

where $[B_2]$ and $[B_3]$ are given in reference 8. Thus,

$$\{\chi\} = \{\chi_1\} + \{\chi_2\} = \left([B_2] + [B_3]\right)\{a\} \tag{22}$$

The generalized stiffness matrix can be obtained as (ref. 8):

$$[K]_{gen} = \iint \left\{\left([B_2] + [B_3]\right)^T [D] \left([B_2] + [B_3]\right) + [B_1]^T\ [C][B_1]\right\} dx\, dy \tag{23}$$

The element stiffness matrix in the local co-ordinate system, $[K]_e$, is, by virtue of equation (12),

$$[K]_e = [S]^T [K]_{gen} [S] \tag{24}$$

The element stiffness matrix in the global co-ordinate system, $[K]_g$, is

$$[K]_g = [T_2]^T [K]_e [T_2] \tag{25}$$

where $[T_2]$ is the transformation matrix of displacement vectors from global to local co-ordinates of element.

The evaluation of the elements of the generalized stiffness matrix, $[K]_{gen}$ of equation (23), in closed form is, though straightforward, very tedious. This is due to the lengthy expressions involved in the triple matrix products. The integration involved in equation (23) is now split into 5 integrals as follows:

$$\begin{aligned}[K]_{gen} = & \iint [B_2]^T [D] [B_2] \, dx \, dy \\ & + \iint [B_2]^T [D] [B_3] \, dx \, dy + \iint [B_3]^T [D] [B_2] \, dx \, dy \\ & + \iint [B_3]^T [D] [B_3] \, dx \, dy + \iint [B_1]^T [G] [B_1] \, dx \, dy \end{aligned} \tag{26}$$

The first term $\iint [B_2]^T [D][B_2] \, dx \, dy$ is evaluated in closed form; the other four terms are evaluated using numerical integration. The numerical integration formulas used are listed in ref. 8. If the plate is assumed to be rigid in transverse shear, the matrices $[B_1]$ and $[B_3]$ are null and the last four terms of equation (26) vanish.

Consistent Mass Matrix

It can be shown that the generalized consistent mass matrix is (ref. 8)

$$[M]_{gen} = \rho t \iint [C]^T [C] \, dx \, dy \tag{27}$$

where $[C] = [1 \; x \; y \; x^2 \; xy \; y^2 \; ..... \; y^5]$.

The mass matrix can be transformed to element co-ordinates and global co-ordinates by the same transformations as those used for stiffness matrix. Thus,

$$[M]_e = [S]^T \, [M]_{gen} \, [S] \tag{28}$$

and

$$[M]_g = [T_2]^T \, [M]_e \, [T_2] \tag{29}$$

where the subscripts e and g on $[M]$ stand for element and global system, respectively.

Consistent Load Vector

It can be shown that the generalized consistent load vector is, (ref. 8)

$$[P]_{gen} = \iint \, [C]^T \, q \, dx \, dy \tag{30}$$

where q is the distributed loading.

The consistent load vecto[illegible]n now be transformed to element and global co-ordinates by

$$[P]_e = [S]^T \, [P]_{gen} \tag{31}$$

$$[P]_g = [T_2]^T \, [P]_e \tag{32}$$

THE QUADRILATERAL PLATE ELEMENT

The quadrilateral element is formed from four of the triangular elements just described. Two arrangements of the quadrilateral element are shown in Figures 3(a) and 3(b).

The quadrilateral element has eight grid points on its edges. In the arrangement of the quadrilateral element shown in Figure 3(a), which will be designated as QUAD1, the quadrilateral is divided, first into 2 triangles by one diagonal and then again into 2 more triangles by the other diagonal. In each case one additional grid point, at the mid-point of the diagonal, is introduced; the stiffness, mass and load matrices of the triangles are evaluated and added and the terms associated with the internal grid point are eliminated by static condensation. The stiffness, mass and load matrices of the quadrilateral element are obtained by adding one-half the contribution of each case. In the arrangement of the quadrilateral element shown in Fig. 3(b), designated as QUAD5, five additional grid points are introduced internally so that the quadrilateral is divided into four triangular elements. The eight grid points on the edges are numbered 1 to 8. Grid point 9 is located at the intersection of lines joining mid-points of opposite edges. Grid points 10 to 13 are located at the middle of the lines joining grid point 9 to each of the corners of the quadrilateral. The stiffness, mass and load matrices of the triangular elements are evaluated, as described previously, and added by the direct stiffness technique to form the respective matrices for the quadrilateral. The internal grid points are then eliminated by static condensation.

In a preliminary operation, the grid points of the quadrilateral are adjusted to lie in a median plane. The median plane is selected to be parallel to, and midway between, the diagonals of the quadrilateral. The adjusted quadrilateral is the normal projection of the given quadrilateral on the median plane. The short line segments joining the corners of the original and projected quadrilateral elements are assumed to be rigid in bending and extension. The quadrilateral element and its projection onto the median plane is shown in Fig. 3(c).

FORMULATION AND SOLUTION OF EQUATIONS

The global stiffness matrices, load vectors, and mass matrices for the complete structure modeled by these elements are assembled from the corresponding matrices of the individual elements by standard methods (ref. 6) to form the matrix equation

$$[K]\ \{U\} = \{P\} \tag{33}$$

Because the d.o.f. at grid points consist of displacements and rotations, it presents no difficulty to specify the appropriate geometric boundary conditions at any irregular and/or complex boundary. After the boundary conditions are applied, the matrix equation (33) is solved by Gaussian elimination to obtain the global displacement vector $\{U\}$.

DISCUSSION OF RESULTS

The triangular and quadrilateral elements are used to solve two problems in statics and dynamics of thin isotropic plates. Only the results for the simply supported plate are presented here; the interested reader may consult ref. 8 for details. The problem analyzed is that of the statics and dynamics of a square plate with edges simply supported. All calculations were carried out on the CDC 6400/6600 series of computers with SCOPE operating system of the Langley Research Center. Single precision arithmetic was used throughout. A value of Poisson's ratio of 0.0 is used in all problems. It is mentioned in this context that other finite-element analyses in the literature use 0.3 as the value of Poisson's ratio.

Static Analysis of a Square Plate

The arrangement of the finite elements in a quarter of the square plate is shown in Fig. 4. The number of subdivisions of the edge of the square is denoted by N. Due to symmetry, only one-quarter of the plate is analyzed. The calculated values of the deflection at the center of the simply supported plate are given in Table 1 and compared with the exact solution given by Timoshenko (ref. 9). These values together with other known finite element analyses available in the literature (refs. 3, 4, 5 and 10) are also compared in Figures 5 and 6 in plots of deflection versus mesh size using a linear scale for N^{-1}.

As seen from table 1, the "Q" arrangement is found to give better results than the "P" arrangement for the uniformly distributed loading; however, the "P" arrangement is found to be better, in general, for concentrated loads. For the clamped plate, the "P" arrangements are found to be slightly better than the "Q" arrangements, as noted from ref. 8. For the quadrilateral element, QUAD1 is found to be superior to QUAD5.

The high accuracy achieved with the present elements (triangular and quadrilateral), even for the coarsest mesh, is evident from Table 1 and Figures 5 and 6 for the simply supported plate. In the case of the clamped plate, the results for the coarsest grid are not as accurate as in the case of the simply supported plate (ref. 8); however, as the element size is decreased the values of deflection obtained with the present elements approach very rapidly the exact results.

Free Vibration of a Square Plate

The natural frequencies of a simply supported square plate were determined using the triangular and quadrilateral elements. The non-dimensional eigenvalues are

$$\lambda = \rho t \omega^2 L^4 / D \qquad (34)$$

ρ = mass density

t = thickness of plate

ω = circular frequency

L = length of side of square plate

$D = Et^3/12(1 - \nu^2)$, the flexural rigidity of the plate.

The exact eigenvalues for the simply supported plate are given by

$$\lambda = (r^2 + s^2)^2 \pi^4 \quad (35)$$

where r and s refer to the number of half-waves parallel to the edge directions.

The lowest 6 values obtained using the present elements and the exact results are shown in Table 2. The eigenvalue problems were solved using a Jacobi routine that produced the complete set of eigenvalues and eigenvectors. Consistent mass matrix was used for treatment of inertia. It is seen that the lowest eigenvalue is calculated to within 1% of exact result. Good agreement is noticed for higher eigenvalues as well.

THE TR-18 FAMILY OF ELEMENTS

A number of finite element formulations for doubly curved shells are presently available, the notable among them being the works of Ahmad, Irons and Zienkiewicz (Ref. 11), Bonnes, Dhatt, Giroux, and Robichand (Ref. 12), Strickland and Loden (Ref. 13), Key and Beisinger (Ref. 14), Dhatt (Ref. 15), and Olson and Lindberg (Ref. 16). Some of these have neglected transverse shear deformations whereas some others use sub-triangles and/or second and higher order derivatives of the displacements of the element as degrees of freedom, thus complicating the formulation. A need still exists for an accurate shell element that has only translations and rotations as d.o.f.

Such shell elements can be derived using the TR-18 plate element; the formulation presented here is simple and includes transverse shear deformations; it is based on the linear shear deformation theory of thin shells as given by Washizu (Ref. 17).

Using shallow shell theory, flat plate elements can be easily converted to curved shell elements. The linear strain triangular membrane element, known as TRIM6 in the literature, can be combined with the TR-18 plate element to develop a doubly curved shallow shell triangular element. The surface of the shell will be approximated by a quadratic polynomial of the position coordinates of the base triangle. By a procedure analogous to that discussed for the quadrilateral plate element, a quadrilateral shallow shell element can be developed. Multilayered plates, and plates with coupled membrane and bending deformations, can be designed using TR-18 plate elements.

Fig. 7 shows a differential element dA on the middle surface of the doubly curved shell with orthogonal curvilinear surface co-ordinates ξ_1, ξ_2, ξ_3. A right handed cartesian co-or te system X, Y, Z is also shown. In Fig. 8 and Fig. 9 the curved triangular shell element is shown in basic and local coordinate systems. The differential surface element is expressed as

$$dA = \alpha_1 \ \alpha_2 \ d\xi_1 \ d\xi_2 \tag{36}$$

where α_1 and α_2 are the Lamé parameters.

If the surface $z(x,y)$ of an element is shallow, the following relations are valid

$$\left(z_{,x}\right)^2 \ll 1 \qquad \left(z_{,y}\right)^2 \ll 1 \qquad \left|z_{,x}\, z_{,y}\right| \ll 1 \tag{37}$$

where

$$z_{,x} = \frac{\partial z}{\partial x} \qquad z_{,y} = \frac{\partial z}{\partial y}$$

The set of orthogonal curvilinear co-ordinates (ξ_1, ξ_2, ξ_3) over the surface of the shallow element dA can be replaced by a set of shallow cartesian co-ordinates (x,y,z) where

$$\xi_1 = x \qquad \xi_2 = y \tag{38}$$

and Lamé parameters $\qquad \alpha_1 = \alpha_2 \approx 1$ (39)

From eq. (36), (38) and (39), $\qquad dA = dx\ dy$ (40)

The curvatures of the shallow element can then be approximated by

$$\frac{1}{R_{11}} = -z_{,xx} \tag{41}$$

$$\frac{1}{R_{22}} = -z_{,yy} \tag{42}$$

$$[illegible] = -[illegible] \quad [illegible]$$

The curved triangular element, shown in Fig. [illegible], is defined by grid points 1 to 6, the points being numbered in a counter-clockwise direction. The points 2, 4, and 6 are at the middle of the sides of the element. The basic co-ordinates of the nodes are X_1, Y_1, Z_1 through X_6, Y_6, Z_6. The x-y plane of the local co-ordinate system is parallel to the plane of the three corners of the triangle (points 1, 3 and 5 of the element). The x-axis is parallel to the side 1-3 of the triangular element. Each point on the surface of the triangular element determines, and is determined by, its projection in the x-y plane.

The transformation matrix between local and basic co-ordinates is given by

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} X_0 \\ Y_0 \\ Z_0 \end{Bmatrix} \qquad (44)$$

where $\begin{Bmatrix} X_0 \\ Y_0 \\ Z_0 \end{Bmatrix}$ is the vector from the origin of the basic co-ordinate system to the origin of the local co-ordinate system. The distances X_0, Y_0, Z_0 are not involved in the calculation of the stiffness matrix of the element since only the differences of co-ordinates are used; hence they are discarded. The inversion of equation (44) yields

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \qquad (45)$$

It may be seen that λ_{11}, λ_{21} and λ_{31} are components in X, Y and Z directions of a unit vector along the x direction; and so on for λ_{12}, λ_{13}, etc.

An analytical description of the surface of the element suitable for application of shallow shell theory is obtained by assuming that the elevation of the shell middle surface may be expressed as a quadratic polynomial in the local co-ordinates of the element, i.e.,

$$z(x,y) = f_1 + f_2x + f_3y + f_4x^2 + f_5xy + f_6y^2 \qquad (46)$$

This implies that the shell element has constant curvatures and is consistent with the approximations of shallow shell theory. Knowing the co-ordinates x, y, z of the six points of the triangular element, the constants f_1 to f_6 can be evaluated.

Symbolically

$$\{z\} = [Q_1] \{f\} \tag{47}$$

or

$$\{f\} = [Q_1]^{-1} \{z\} \tag{48}$$

where $[Q_1]$ is a 6 x 6 matrix of the co-ordinates of the six points of the element substituted into equation (46).

<u>Degrees of freedom and assumed displacement function</u>.- The element has 30 degrees of freedom (d.o.f.), with 5 d.o.f. per grid point. These are the three translations u. v, w in the x, y, and z directions and the rotations of the xz and yz planes, α and β. The displacements u, v, w are positive in the positive co-ordinate directions; the slopes are positive when they cause compression at the top of the surface. The u and v d.o.f. are assumed to vary over the element by a full quadratic polynomial of local co-ordinates, as follows:

$$u = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \tag{49}$$

$$v = a_7 + a_8x + a_9y + a_{10}x^2 + a_{11}xy + a_{12}y^2 \tag{50}$$

The deflection w will be defined by a quintic polynomial as in equation (6). The coefficients a_1 to a_{21} of equation (6) will be renumbered a_{13} to a_{33} respectively. The 33 coefficients a_1 to a_{33} can be uniquely determined from the 30 d.o.f. of the shell element (5 d.o.f. each at six grid points) together with the 3 constraint equations (8), (9), and (10).

<u>Strain-displacement relations</u>.- The expressions for transverse shear strains and bending strains for the curved shell element are the same as those for the TR-18 element (eqs. (7) and (13)). The membrane strains are

$$\left.\begin{aligned} e_x &= \frac{\partial u}{\partial x} - w\,\frac{\partial^2 z}{\partial x^2} \\ e_y &= \frac{\partial v}{\partial y} - w\,\frac{\partial^2 z}{\partial y^2} \\ e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2w\,\frac{\partial^2 z}{\partial x \partial y} \end{aligned}\right\} \tag{51}$$

Stiffness matrix.- The stiffness matrix can be evaluated by the standard procedures (ref. 6). The element can then be tested against other elements (refs. 11 to 16) for suitability as well as accuracy. At the time of writing of this paper, the calculations for the element have not been completed.

Curved Quadrilateral Shell Element

A curved quadrilateral shell element can be constructed from the curved triangular shell elements by a procedure analogous to that of the construction of the quadrilateral plate element from the TR-18 element.

Plates With Membrane-Bending Coupling

Plates with coupled membrane and bending deformations and multilayered plates (fig. 10) can be analyzed by means of the elements presented earlier herein. Multilayered plates will produce coupling between membrane and bending deformations when the plate is not symmetrical with respect to its middle surface. A general form of the coupled stress-strain relationship can be expressed as

$$\begin{Bmatrix} \{F\} \\ \{M\} \\ \{V\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & 0 \\ [B]^T & [D] & 0 \\ 0 & 0 & t^*[G] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_m\} \\ \{\chi\} \\ \{\gamma\} \end{Bmatrix} \tag{52}$$

where

{F} is a vector of membrane force components F_x, F_y, F_{xy}

{M} is a vector of bending and twisting moments M_x, M_y, M_{xy}

{V} is a vector of transverse shear components V_x, V_y

$\{\varepsilon_m\}$ is a vector of membrane strain components ε_x, ε_y, ε_{xy}

$\{\chi\}$ is a vector of curvatures χ_x, χ_y, χ_{xy}

$\{\gamma\}$ is a vector of average transverse shear strain γ_x, γ_y

[A] is a 3 x 3 matrix, $\sum_{k=1}^{N} [G_e] (t_k - t_{k-1})$

[B] is a 3 x 3 matrix, $\sum_{k=1}^{N} [G_e] \frac{t_k^2 - t_{k-1}^2}{2}$

[D] is a 3 x 3 matrix, $\sum_{k=1}^{N} [G_e] \frac{t_k^3 - t_{k-1}^3}{3}$

[G] is a 2 x 2 transverse shear matrix

$[G_e]$ is a 3 x 3 matrix of elastic coefficients

t_k is the distance to the outer edge of plate (or layer in a multilayer plate) from reference surface

t_{k-1} is the distance to the inner edge of plate (or layer in a multilayered plate) from reference surface

t^* is an effective thickness for the element

The inplane strain vector at any point is

$$\{\varepsilon\} = \{\varepsilon_m\} - z\,\{\chi\} \tag{53}$$

where z is the distance from the reference surface. The strain energy of the plate element is

$$U = \frac{1}{2} \int [\{F\}^T\{\varepsilon_m\} + \{M\}^T\{\chi\} + \{V\}^T\{\gamma\}]dA \tag{54}$$

where the integration is carried out over the surface of the element. The stiffness matrix for the triangular and quadrilateral elements can be evaluated by the usual procedures (refs. 6 and 8).

CONCLUDING REMARKS

New triangular elements and associated quadrileteral elements for plate and shell analysis having only displacement and rotations as grid point degrees of freedom are described in this paper. The examples presented for plate elements demonstrate that high accuracy is achievable using these elements for practical subdivisions.

The effect of transverse shear deformations is included in the element formulation. Transverse shear strains vary quadratically within the element; convergence to the limiting case of zero transverse shear strain is uniform. The present elements are expected to give better approximations than most displacement model plate bending elements for solving problems where transverse shear effects are significant.

Finally, it is remarked that these elements are ideally suited for inclusion into general purpose computer programs due to (i) simplicity of formulation, (ii) use of only displacements and rotations as grid point degrees of freedom, (iii) high accuracy for practical mesh subdivisions and (iv) inclusion of transverse shear flexibility in the element properties.

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TABLE 1

Coefficients for Central Deflection of Simply Supported Square Plate of Side L

$\left[E = 1.0 \times 10^8; \quad \nu = 0.0\right]$

Number of elements per side, N	Coefficient $1000\, w_c D/PL^2$ due to central concentrated load P				Coefficient $1000\, w_c D/q_o L^4$ due to uniformly distributed load q_o			
	Triangular element		Quadrilateral element		Triangular element		Quadrilateral element	
	Q arrangement	P arrangement	QUAD 1	QUAD 5	Q arrangement	P arrangement	QUAD 1	QUAD 5
2	11.827079	12.064944	11.886797	12.294189	4.080739	4.173045	4.115169	4.177057
4	11.766154	11.705526	11.704203	11.760712	4.066626	4.074954	4.068629	4.078833
6	11.690134	11.647158	11.653533	11.673154	4.064244	4.066794	4.064797	4.068828
8	11.656354	11.627134	11.632881	11.642431	4.063427	4.064620	4.063666	4.065832
12	11.628511	11.612820	11.616516	11.620020	4.062843	4.063288	4.062927	4.063869
Exact (ref. 9)	11.600				4.062			

TABLE 2

Nondimensional Eigenvalue Numbers of Simply Supported Square Plate

Mode, (r,s)	Number of elements per side, N								Exact solution
	2				4				
	Triangular element		Quadrilateral element		Triangular element		Quadrilateral element		
	Q arrangement	P arrangement	QUAD 1	QUAD 5	Q arrangement	P arrangement	QUAD 1	QUAD 5	
(1,1)	386.14	384.97	388.10	387.20	388.93	388.74	389.08	388.53	389.64
(1,2)	2477.59	2477.59	2558.41	2628.19	2415.26	2415.26	2420.91	2427.66	2435.23
(2,2)	6541.93	6541.94	7997.35	8910.19	6179.56	6179.56	6209.53	6195.16	6234.18
(1,3)	7149.37	10008.09	10116.30	10413.34	9482.08	9334.84	9620.47	9769.04	9740.91
	9922.22	10458.52	10201.76	11160.88	9704.72	9548.69	9620.47	9769.04	
(2,3)					16144.46	16144.65	16358.12	16436.21	16462.14
(3,3)					30843.18	30676.12	31361.90	31498.11	31560.55

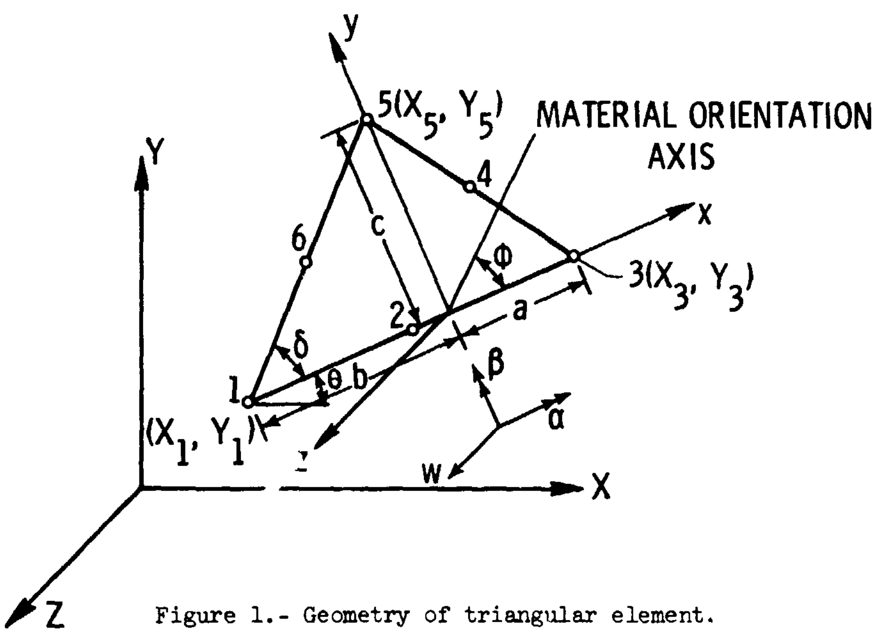


Figure 1.- Geometry of triangular element.

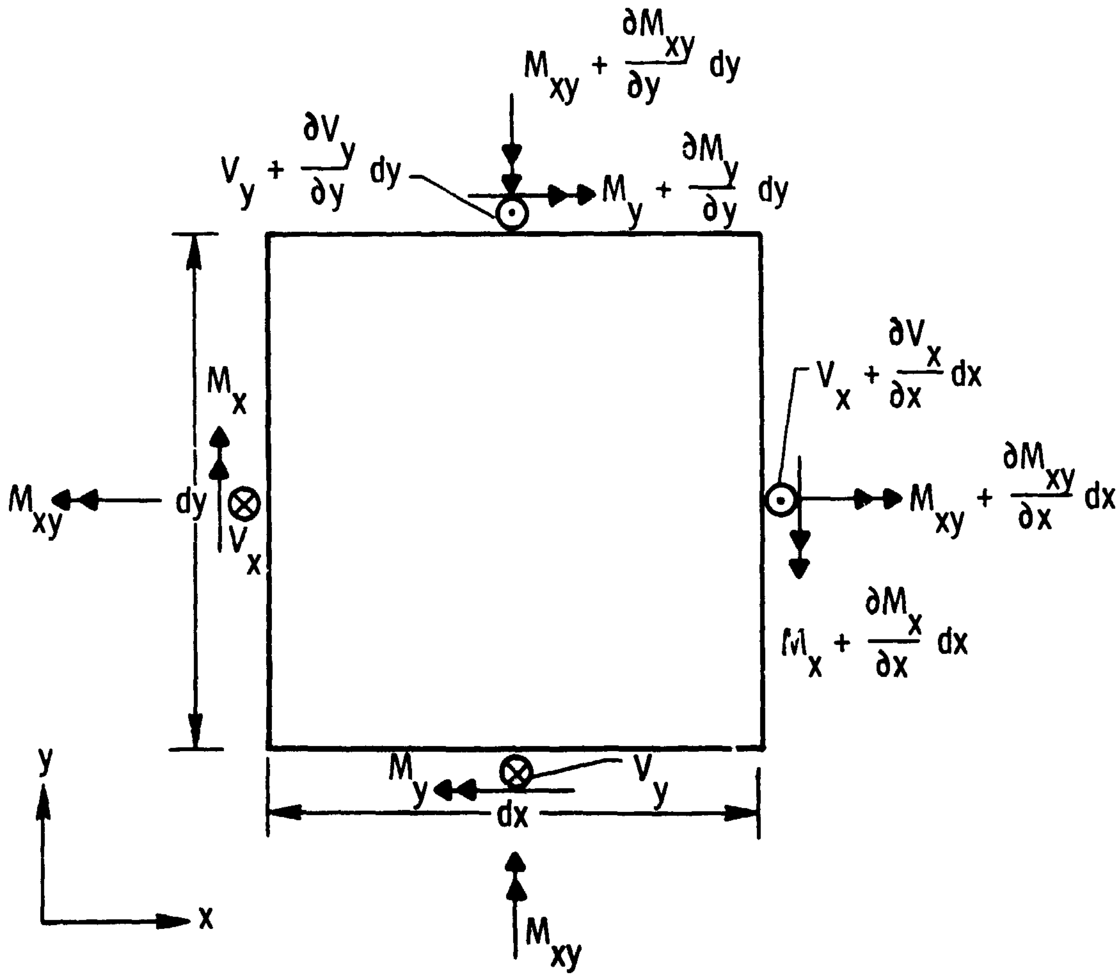


Figure 2.- Sign convention for moments and shears.

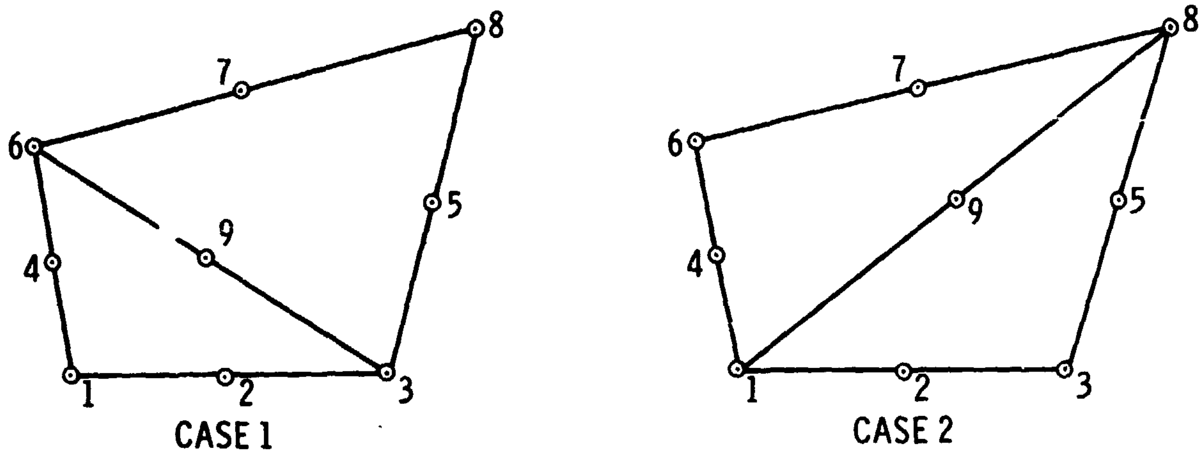


Figure 3(a).- Quadrilateral element (QUAD1) geometry.

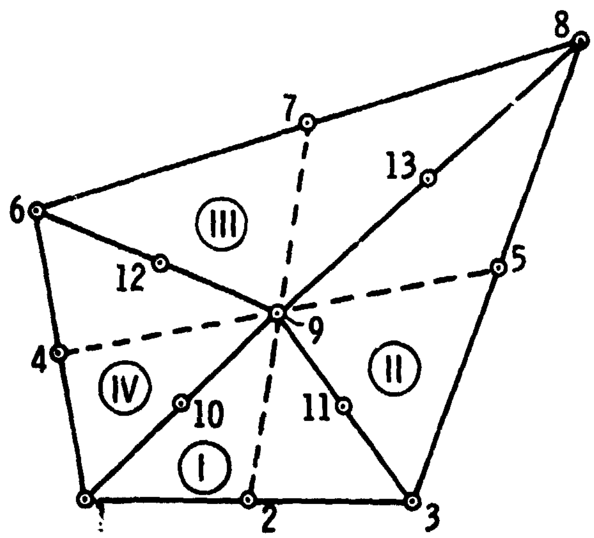


Figure 3(b).- Quadrilateral element (QUAD5) geometry.

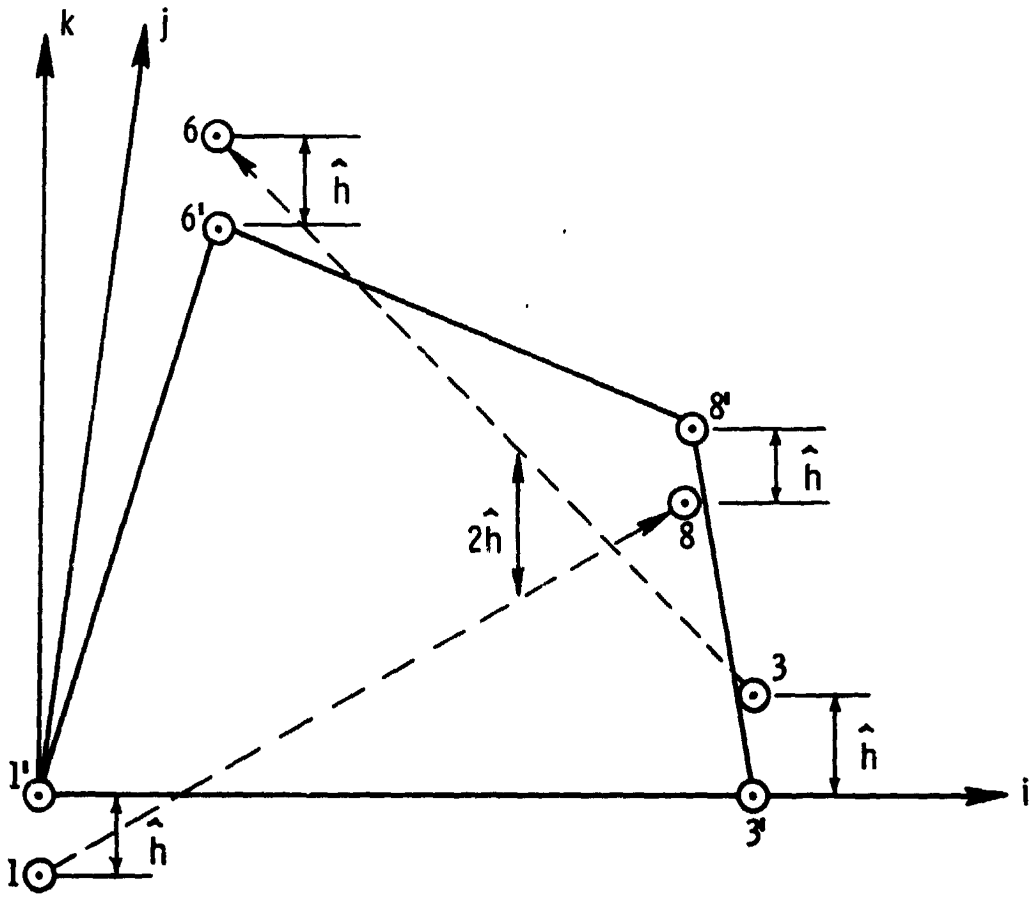
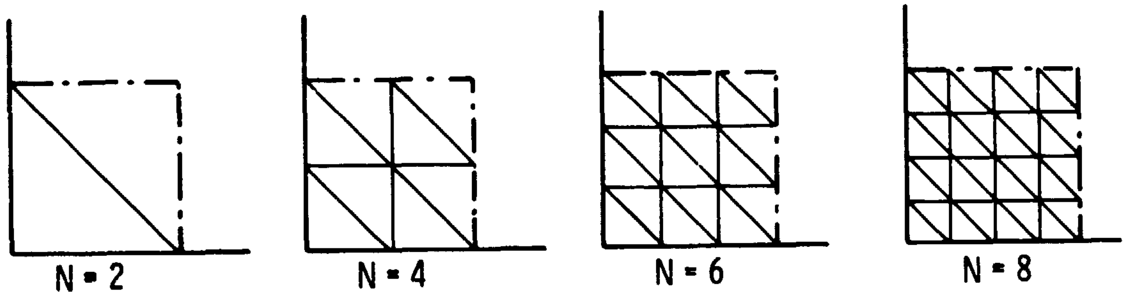
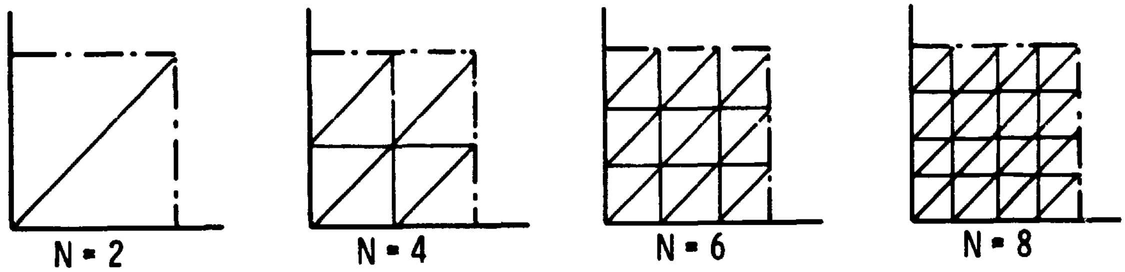


Figure 3(c).- Median plane for quadrilateral element.



P arrangements of triangular element mesh idealization



Q arrangements of triangular element mesh idealization

Figure 4.- Finite element idealization of square plate.

Notation	Element Shape	Reference
N	T (Q-mesh)	Present paper
P	T (P-mesh)	Present paper
Q1	Q	Present paper
Q5	Q	Present paper
ACM	R	1
HCT	T	1
Z	T	2
TUBA-6	T	3
B-2	T (T-18)	4,10
B-3	T (T-21)	4,10
C-N	T (Q-mesh)	5
C-P	T (P-mesh)	5
NQ	Q	7
CFQ	Q	9

T - Triangular; Q - Quadrilateral; R - Rectangular

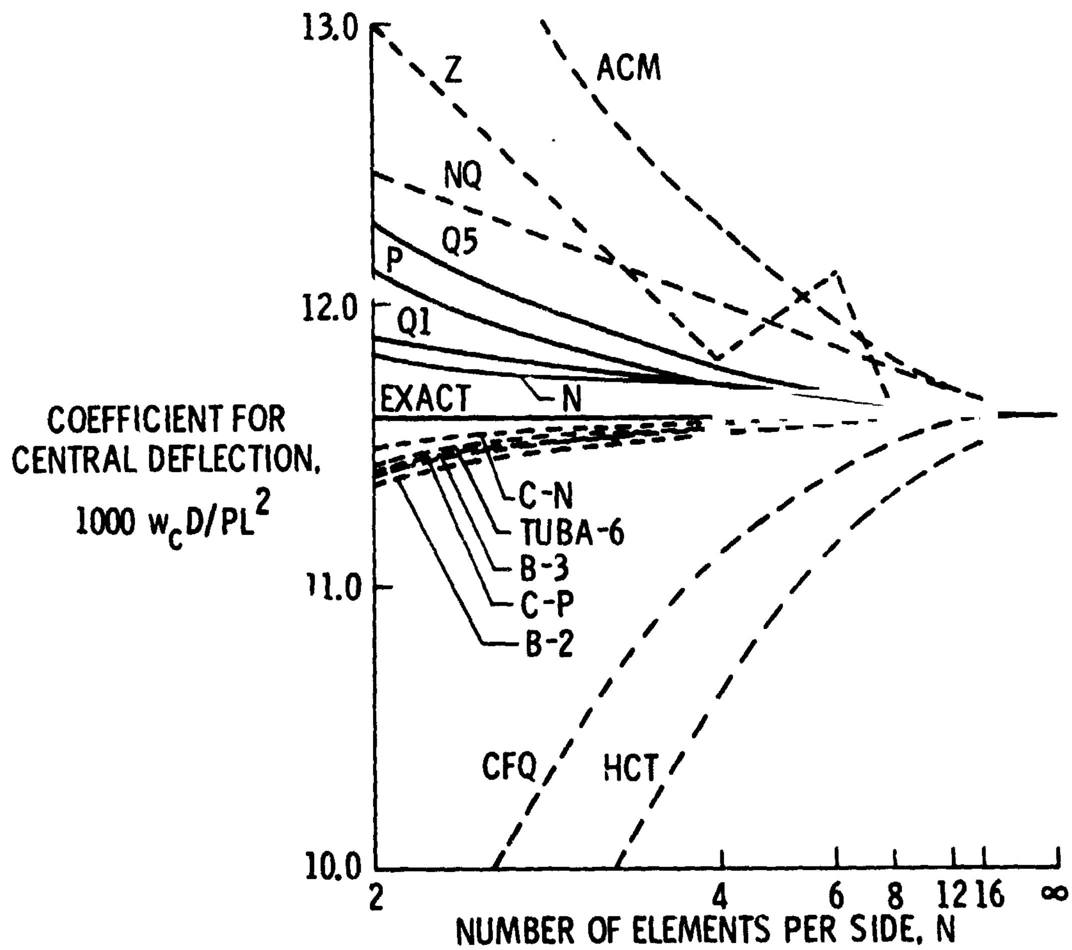


Figure 5.- Simply supported square plate: central deflection w_c under central point load P.

Notation	Element Shape	Reference
N	T (Q-mesh)	Present paper
P	T (P-mesh)	Present paper
Q1	Q	Present paper
Q5	Q	Present paper
ACM	R	1
HCT	T	1
Z	T	2
TUBA-6	T	3
B-1	T (T-15)	4
B-2	T (T-18)	4,10
B-3	T (T-21)	4,10
C-N	T (Q-mesh)	5
C-P	T (P-mesh)	5
NQ	Q	7
CFQ	Q	9

T - Triangular; Q - Quadrilateral; R - Rectangular

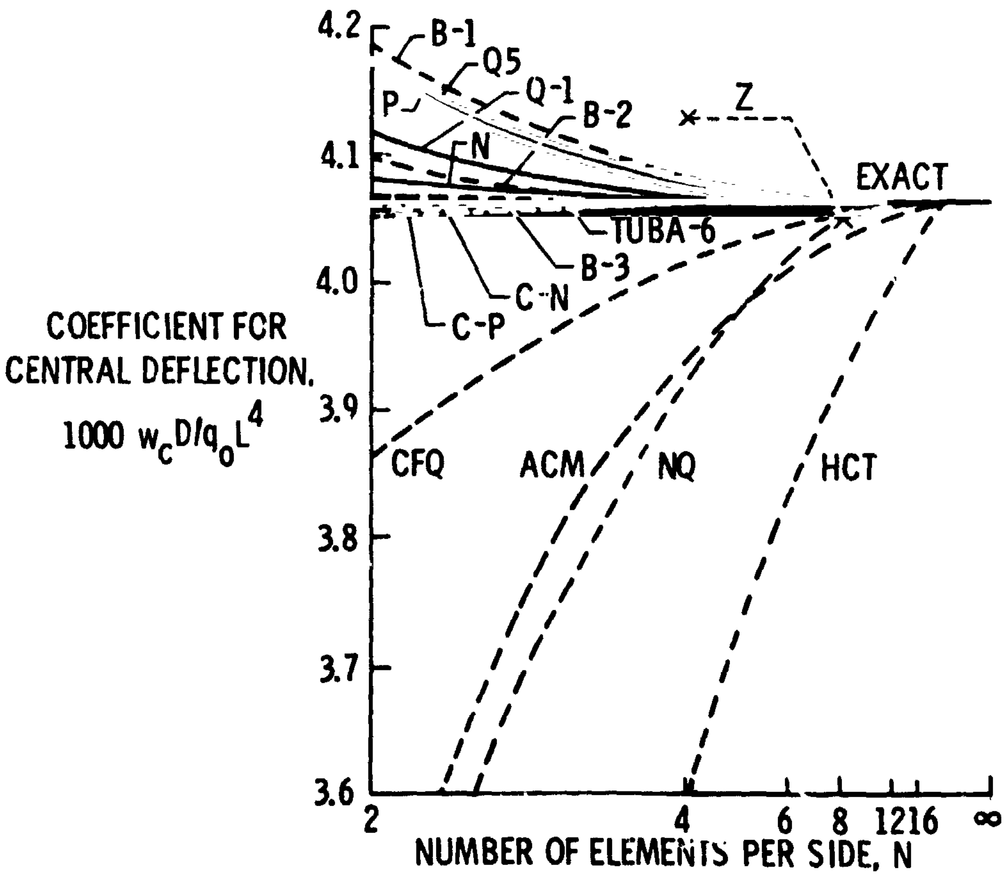


Figure 6.- Simply supported square plate: central deflection w_c under uniformly distributed load q_o.

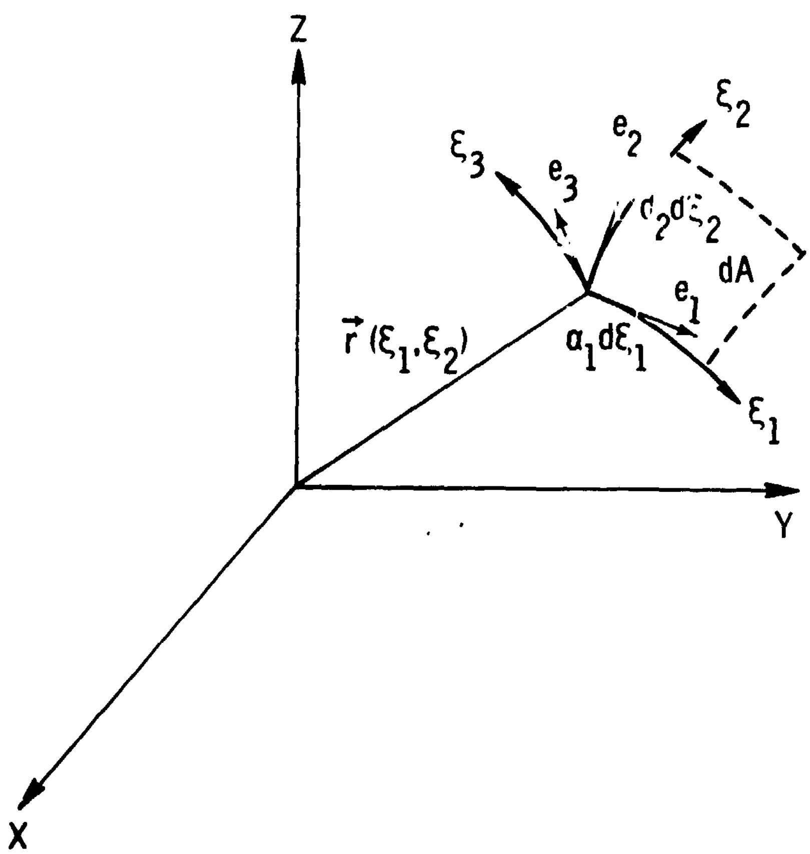


Figure 7.- Differential element.

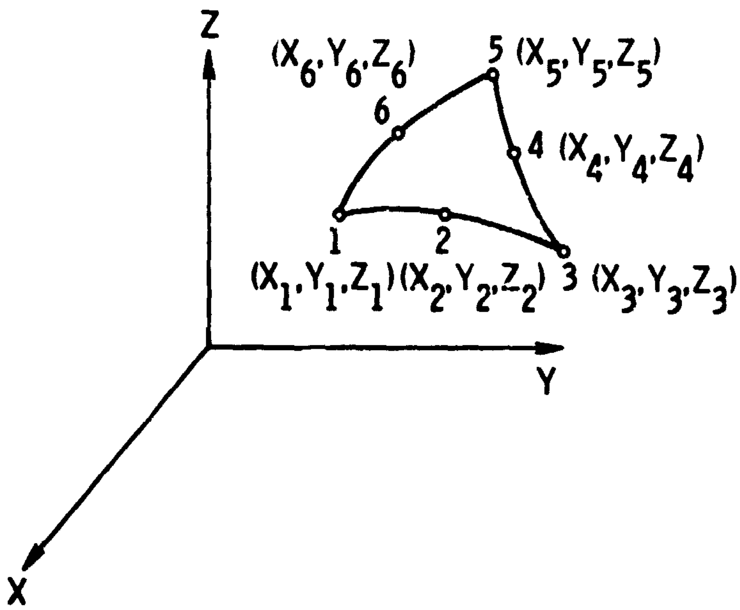


Figure 8.- Curved triangular shell element in basic co-ordinate system.

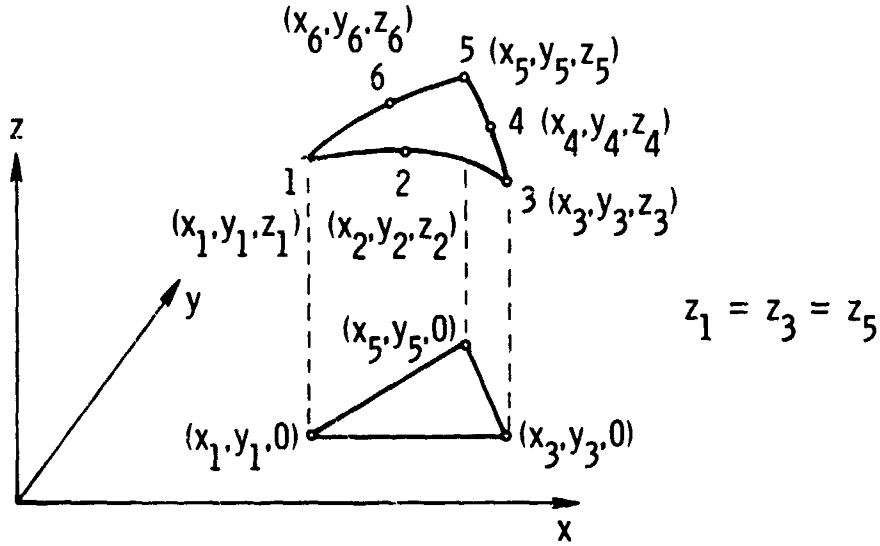


Figure 9.- Curved triangular shell element in local co-ordinate system.

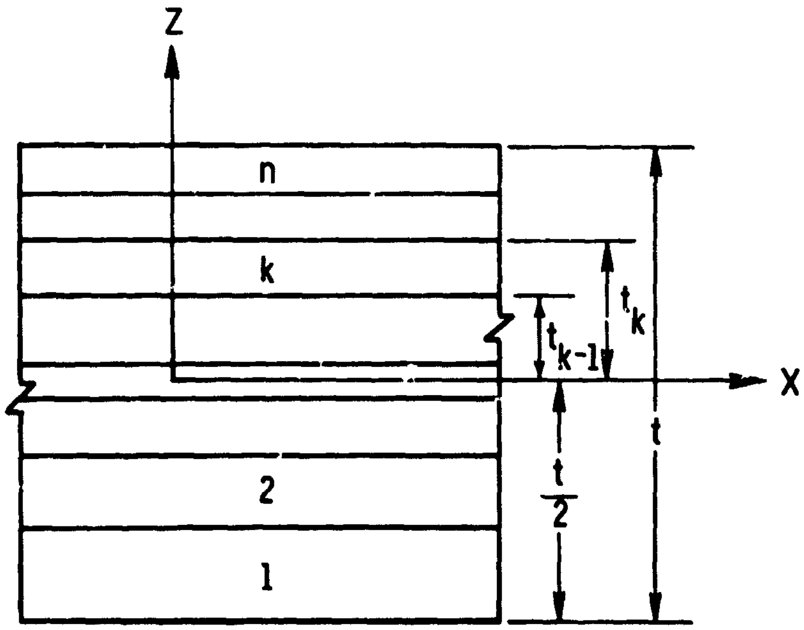


Figure 10.- Multilayered plate geometry.