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(NASA-CR-132902) ATTITUDE DYNAMIC OF SPIN-STABILIZED SATELLITES WITH FLEXIBLE APPENDAGES Final Report (Carnegie-Mellon Univ.) ~~266~~ 264 p HC $15.50 CSCL 22B

N74-15517

Unclas

G3/31 15646

CARNEGIE-MELLON UNIVERSITY

Applied Space Sciences Program

FINAL REPORT

NASA Grant NGR-39-087-026

"ATTITUDE DYNAMICS OF SPIN-STABILIZED SATELLITES WITH FLEXIBLE APPENDAGES"

Marc L. Renard

(Principal Investigator)

Pittsburgh, September 1973

Marc L. Renard

Dr. Marc L. Renard
Associate Professor of
Applied Space Sciences and
Electrical Engineering

ACKNOWLEDGEMENTS

The author of this report wishes to gratefully acknowledge the support of this work by the NATIONAL AERONAUTICS and SPACE ADMINISTRATION.

Of particular value to him were discussions held with Drs. J.V. Fedor, T. Flatley, R. Montgomery and S.J. Paddack of GSFC, on the subject of satellite IMP-I, and with Dr. D.L. Blanchard on UK-4.

Contributing to this project as Research Engineers were, in chronological order and for various lengths of time: J.E. Rakowski, P. Stakem, W. Keksz, R. Sridharan, J. Russial and N. Winowich. We are also indebted to Miss Cheryll Conaway, formerly of the Applied Space Sciences Program, for her help in typing and editing the present report.

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CHAPTER 1

Object of the Study

In recent years, the study of the attitude dynamics of a spacecraft considered as a partly rigid, partly elastic or articulated body has become of increasing importance[1-1]. At first, such work did not present such a degree of urgency, as many investigations concentrated on rotational and librational dynamics of essentially rigid spacecraft, as is apparent from the reviews of D.B. De Bra[1-2] and R.E. Roberson[1-3,1-4]. Any elastic body effects are conspicuously absent of V.V. Beletskii's classic book on the "Motion of an artificial satellite about its center of mass" who writes at the outset that "the discussion is confined to problems which fall within the scope of the dynamics of rigid bodies".

Satellites became increasingly "elastic", as booms were extended tens and hundreds of meters from the central body[1-5, 1-6, 1-7] or as large solar panels or manned toroidal space stations are considered[1-8]. Three methods are most commonly used in the study of the dynamics of the elastic spacecraft: discretization by modeling the continuous system by finite elements; modal representation; and the Likin's[1-9] method of hybrid coordinates.

The present work uses the modal approach. It is a study of the relevant equations and parameters in the dynamical analysis of the attitudes motion of a spin-stabilized spacecraft having flexible appendages. It is principally aimed at developing working tools, such as stability diagrams, tables or simulation analyses by means of computer programs. These programs are of low time-consumption, and their use is quite easy to learn. As such, it is hoped that they will prove valuable to the engineer engaged in the design of spin-stabilized elastic spacecraft.

CHAPTER 2

A Study of Modal Shapes and Eigenfrequencies of Flexible Appendages on a Spin-Stabilized Satellite

2.1 Introduction

In order to study the dynamics of the spin-stabilized satellite with flexible appendages, by the methods of generalized dynamics, the continuum of the elastic parts should be represented by generalized coordinates q_i $(i = 1,2,\dots)$. The q_i are functions of time describing the amplitude of the non-dimensional displacements, $\frac{w_k}{\ell}$, of boom k at abscissa $\xi \equiv \frac{x}{\ell}$, in terms of modes $\Phi_i(\xi)$

$$w_k(\xi)\,\frac{1}{\ell} = \sum_i q_i(t)\;\Phi_i(\xi) \qquad (2.1\text{-}1)$$

w_k will be (in the assumption of small displacements) along y for equatorial displacements (E) and along z for meridional displacements (M) (See Fig. 2.1).

ξ, η, ζ are the geometric coordinates x,y,z non-dimensionalized by ℓ, undeflected length of the boom. $\xi_o = \frac{x_o}{\ell}$ is the non-dimensional radius of the central hub.

The system of mode shapes, Φ_i, adopted here are the modes of the rotating structure corresponding to the boom's Etkin number[2-1] $\bar{\lambda} = \frac{\rho\ell^4}{EI}\,\omega_s^2$ and non-dimensional radius $\xi_o = \frac{x_o}{\ell}$: ρ is the (uniform) lineal density of the boom, in units of mass/length. E is the boom's Young modulus, in units of force/unit area, I is the geometric moment of inertia of the boom's cross section, in units of length^4, and ω_s is the spin rate, in rad./sec. Thus $\bar{\lambda}$ is non-dimensional. Finally, x_o is the radius of the central hub, at which distance the elastic boom is assumed to be cantilevered. As will be seen, these significantly depart in shape and frequency from those of the non-rotating structure corresponding to $\bar{\lambda} = 0$ and $\xi_o = 0$.

In the following, it is assumed that only antisymmetric motions are considered, or that the motion of the CM away from the origin is negligible. The latter amounts, as has been shown by F. Vigneron[2-2], to assuming that the central mass M_c is sufficiently large for terms of order

$$\rho \ell \left(\frac{\rho \ell}{M_c}\right)^2 \left[\int_{\text{boom}} w \, dx\right]^2$$

to be neglected in comparison with terms like

$$\rho \int_{\text{boom}} w^2 \, dx$$

Typically, for the ALOUETTE and ISIS satellites, Ref. [2-2] gives the values: $\frac{\rho \ell}{M_c}$ = 0.005 to 0.01 (copper-beryllium booms)

2.2 Equations of Motion: equatorial vibrations

2.2.1 Basic equation

We shall first consider motions in the "equatorial" plane of the satellite, i.e. (x,y) or (ξ,η). These were the first type of vibrations considered by this author and J.E. Rakowski[2-3].

Any section of boom located at ρ, of abscissa x, is in rotational equilibrium under the action of (Fig. 2.1).

- bending moment from the left, which for pure flexure in the equatorial plane, is

$$dM_{el} = -EI\frac{\partial^2 w(x)}{\partial x^2}\,\vec{1}_z \qquad (2.2\text{-}1)$$

in which w(x) is the assumed small displacement of the boom element in the y-direction, and $\vec{1}_z$ is the unit vector along the z-direction.

- the moment about ρ of inertia forces

$$d\vec{F}_{in} = -\rho\ dx_1\ \ddot{\vec{r}}_Q \qquad (2.2\text{-}2)$$

imparted by the particles of the boom to the right of ρ, i.e. having abscissa between x and ℓ.

Therefore,

$$EI\,\frac{\partial^2 w(x)}{\partial^2 x^2} = -\int_x^{\ell} \rho\, dx_1 \left\{\left[\vec{r}_Q - \vec{r}_P\right]\wedge \ddot{\vec{r}}_Q\right\}_z$$

In terms of their components, we have

$$\vec{r}_{PQ} = \vec{r}_Q - \vec{r}_P = (x_1 - x)\,\vec{1}_x + \left(w(x_1) - w(x)\right)\vec{1}_y = \begin{bmatrix} x_1 - x \\ w(x_1) - w(x) \\ 0 \end{bmatrix}$$

$$\vec{r}_Q = (x_0 + x_1)\,\vec{1}_x + w(x_1)\;\vec{1}_y = \begin{bmatrix} x_0 + x_1 \\ w(x_1) \\ 0 \end{bmatrix}$$

Thus

$$\mathring{\vec{r}}_Q = \mathring{w}(x_1) + \vec{\omega}\wedge\vec{r}_Q$$

$$= \begin{bmatrix} -w(x_1)\,\omega_z \\ \mathring{w}(x_1) + \omega_z\,(x_0 + x_1) \\ \omega_x\, w(x_1) - \omega_y\,(x_0 + x_1) \end{bmatrix}$$

Also

$$\ddot{\vec{r}}_Q = \frac{\partial}{\partial t}\left(\dot{\vec{r}}_Q\right) + \vec{\omega} \wedge \dot{\vec{r}}_Q = \begin{bmatrix} -\dot{\omega}_z\, W(x_1) - \omega_z\, \dot{W}(x_1) \\ \dot{\omega}_z\,(x_0+x_1) + \ddot{W}(x_1) \\ \dot{\omega}_x\, W(x_1) + \omega_x\, \dot{W}(x_1) - \dot{\omega}_y\,(x_0+x_1) \end{bmatrix}$$

$$+ \begin{Vmatrix} \omega_x & \omega_y & \omega_z \\ -W(x_1)\,\omega_z & \dot{W}(x_1) + \omega_z(x_0+x_1) & \omega_x\, W(x_1) - \omega_y\,(x_0+x_1) \end{Vmatrix}$$

$$= \begin{bmatrix} (\omega_x\,\omega_y - \dot{\omega}_z)\, W(x_1) - (\omega_y^2 + \omega_z^2)(x_0+x_1) - 2\,\omega_z\, \dot{W}(x_1) \\ \left(\ddot{W}(x_1) - \omega_z^2\, W(x_1)\right) - \omega_x^2\, W(x_1) + (\dot{\omega}_z + \omega_x\,\omega_y)(x_0+x_1) \\ (\omega_x\,\omega_z - \dot{\omega}_y)(x_0+x_1) + (\dot{\omega}_x + \omega_y\,\omega_z)\, W(x_1) + 2\,\omega_x\, \dot{W}(x_1) \end{bmatrix}$$

Under the assumption of small displacements and transverse angular rates, terms of order w^2, $\omega_x\omega_y$, ω_x^2, ω_y^2... are neglected, and $\ddot{\vec{r}}_Q$ reduces to

$$\ddot{\vec{r}}_Q = \begin{bmatrix} -\dot{\omega}_z W(x_1) - \omega_z^2 (x_0 + x_1) - 2\omega_z \dot{W}(x_1) \\ (\ddot{W}(x_1) - \omega_z^2 W(x_1)) + \dot{\omega}_z (x_0 + x_1) \\ (\omega_x \omega_z - \dot{\omega}_y)(x_0 + x_1) \end{bmatrix} \tag{2.2-3}$$

Finally, along $\vec{i}_z$

$$\left[(\vec{r}_Q - \vec{r}_P) \wedge \ddot{\vec{r}}_Q\right]_z = \begin{vmatrix} x_1 - x & W(x_1) - W(x) \\ -\dot{\omega}_z W(x_1) - \omega_z^2 (x_0 + x_1) - 2\omega_z \dot{W}(x_1) & \ddot{W}(x_1) - \omega_z^2 W(x_1) + \dot{\omega}_z (x_0 + x_1) \end{vmatrix}$$

and neglecting quantities of smaller order

$$\left[(\vec{r}_Q - \vec{r}_P) \wedge \ddot{\vec{r}}_Q\right]_z = (x_1 - x)\left[\ddot{W}(x_1) - \omega_z^2 W(x_1) + \dot{\omega}_z (x_0 + x_1)\right] + \left(W(x_1) - W(x)\right)\omega_z^2 (x_0 + x_1)$$

With the same notation as above, let $\xi_1 = \frac{x_1}{\ell}$, $\eta_1 = \frac{w(x_1)}{\ell}$, $\alpha = \frac{\rho \ell^4}{EI}$; this becomes

$$\frac{EI}{\ell} \frac{\partial^2 \eta}{\partial \xi^2} = -\rho \ell^3 \int_\xi^\ell \left\{ (\xi_1 - \xi)\left[(\ddot{\eta}_1 - \omega_z^2 \eta_1) + \dot{\omega}_z (\xi_0 + \xi_1)\right] + \omega_z^2 (\eta_1 - \eta) (\xi_0 + \xi_1) \right] d\xi$$

With the abbreviated notation $\underbrace{\eta_{\xi\ldots\xi}}_{k \text{ times}} = \dfrac{\partial^k \eta}{\partial \xi^k}$, we obtain

$$\eta_{\xi\xi} = -\alpha \int_{\xi}^{1} \left[(\eta_1 - \eta)\, \omega_z^2 (\xi_0 + \xi_1) + (\xi_1 - \xi)\left(\dot{\omega}_z(\xi_0 + \xi_1) + \ddot{\eta}_1 - \omega_z^2 \eta_1\right) \right] d\xi_1 \qquad (2.2\text{-}4)$$

Taking the derivative of (2.2-4) with respect to ξ, and using Leibniz's formula, $f(\xi_1,\xi)$ being the integrand,

$$\eta_{\xi\xi\xi} = \alpha f(\xi_1 = \xi, \xi) - \alpha \int_{\xi}^{1} \left[-\eta_{\xi}\, \omega_z^2 (\xi_0 + \xi_1) - (\ddot{\eta}_1 - \omega_z^2 \eta_1) - \dot{\omega}_z (\xi_0 + \xi_1) \right] d\xi_1$$
$$-\alpha \int_{\xi}^{1} \left[-\eta_{\xi}\, \omega_z^2 (\xi_0 + \xi_1) - (\ddot{\eta}_1 - \omega_z^2 \eta_1) - \dot{\omega}_z (\xi_0 + \xi_1) \right] d\xi_1$$

Finally

$$\eta_{\xi\xi\xi\xi} = -\alpha\, \eta_{\xi}\, \omega_z^2 (\xi_0 + \xi) - \alpha\, \dot{\omega}_z (\xi_0 + \xi) - \alpha(\ddot{\eta} - \omega_z^2 \eta) + \alpha \int_{\xi}^{1} \eta_{\xi\xi}\, \omega_z^2 (\xi_0 + \xi_1)\, d\xi_1 \qquad (2.2\text{-}5)$$
$$= -\alpha \eta_{\xi}\, \omega_z^2 (\xi_0 + \xi) - \alpha \dot{\omega}_z (\xi_0 + \xi) - \alpha(\ddot{\eta} - \omega_z^2 \eta) + \alpha \omega_z^2\, \eta_{\xi\xi} \left[\tfrac{1}{2}(1 - \xi^2) + \xi_0 (1 - \xi) \right]$$

The non-dimensionalization is completed by introducing the non-dimensional Etkin's number[2-1]

$$\bar{\lambda} = \alpha\, \omega_z^2 = \frac{\rho \ell^4}{EI}\, \omega_z^2 \doteq \left(\frac{\omega_z}{\omega_{CANT,NR}} \right)^2$$

where ω_{cant} is the first cantilever frequency of the non-rotating boom. It is to be stressed that $\bar{\lambda}$ is a constant only if ω_z, the satellite spin-rate, may be considered such. Equation (2.2-5) is rewritten in the form

$$\eta_{\xi\xi\xi\xi} - \bar{\lambda}\, \eta_{\xi\xi} \left[\tfrac{1}{2}(1 - \xi^2) + \xi_0 (1 - \xi) \right] + \bar{\lambda}\, \eta_{\xi} (\xi + \xi_0)$$
$$+ \alpha\, \ddot{\eta} - \bar{\lambda}\, \eta = -\alpha\, \dot{\omega}_z (\xi + \xi_0) \qquad (2.2\text{-}6)$$

So far, quantities which have been neglected were of order ε^2 of smallness, or smaller. Now $\alpha\dot{\omega}_z$ itself is of order ε^2, i.e. with $d\phi = \omega_z\, dt$

$$\frac{\bar{\lambda}}{\omega_z^2}\ \dot{\omega}_z \doteq \bar{\lambda}\ \frac{d\omega_z/\omega_z}{d\phi}$$

if the product $\bar{\lambda}$ x the percentage change of ω_z per unit angle of rotation is very much smaller than quantities assumed to be of order ε. Assuming that such is the case, we are then left with the homogeneous Equation (2.2-6) with a r.h. side equal to zero.

$$\eta_{\xi\xi\xi\xi} - \bar{\lambda}\,\eta_{\xi\xi}\left[\frac{1}{2}(1-\xi^2) + \xi_0(1-\xi)\right] + \bar{\lambda}\,\eta_{\xi}\,(\xi+\xi_0) - \bar{\lambda}\eta + \alpha\ddot{\eta} = 0 \qquad (2.2\text{-}7)$$

2.2.2 Solution of the basic equation

Using separation of variables, with $' = \frac{d}{d\tau} = \frac{d}{d(\omega_z t)}$;

$$\tau = \omega_z t \ ,$$

$$\eta_j = \Phi_j(\xi)\ T_j(\tau) \qquad (2.2\text{-}8)$$

Hence

$$\Phi_j^{(IV)} - \bar{\lambda}\ \Phi_j^{(2)}\left[\frac{1}{2}(1-\xi^2)+\xi_0(1-\xi)\right] + \bar{\lambda}\ \Phi_j^{(1)}(\xi+\xi_0) - \bar{\lambda}\left(1+\bar{\omega}_j^{\,2}\right)\Phi_j = 0$$

yielding

$$T_j = \begin{matrix}\sin\\ \cos\end{matrix}\left(\bar{\omega}_j\tau\right) = \begin{matrix}\sin\\ \cos\end{matrix}\left(\omega_j t\right)$$

where ω_j is the jth eigenfrequency of the equatorial vibrations associated with $(\xi_0, \bar{\lambda})$. This equation is in agreement with that obtained by Etkins and Hughes[2-1], in the special case $\xi_0 = 0$.

Determination of ω_j (or $\bar{\omega}_j$) from Equation (2) proceeds as follows. Equation (2) is linear, with ξ varying coefficients. Thus any linear combination of solutions of (2) is a solution of (2).

Let $\mathcal{S}_{3,j}$ be the solution satisfying the b.c.

$$\begin{array}{lcccc} \xi = 0 : & 0 & 0 & 1 & 0 \\ & \phi_j & \phi_j^{(1)} & \phi_j^{(2)} & \phi_j^{(3)} \end{array} \qquad (2.2\text{-}9)$$

and $\mathcal{S}_{4,j}$ be the solution satisfying

$$\begin{array}{lcccc} \xi = 0 : & 0 & 0 & 0 & 1 \\ & \phi_j & \phi_j^{(1)} & \phi_j^{(2)} & \phi_j^{(3)} \end{array} \qquad (2.2\text{-}10)$$

Therefore, the desired solution, which satisfies the "built-in, free" boundary conditions

$$\begin{array}{lcccc} & \phi_j & \phi_j^{(1)} & \phi_j^{(2)} & \phi_j^{(3)} \\ \xi = 0 & 0 & 0 & & \end{array} \qquad (2.2\text{-}11)$$

$$\begin{array}{lcccc} \xi = 1 & & & 0 & 0 \end{array} \qquad (2.2\text{-}12)$$

is of the form

$$C_3 \mathcal{S}_{3,j} + C_4 \mathcal{S}_{4,j} \qquad (2.2\text{-}13)$$

with C_3, C_4 unknown. (2.2-11) is automatically satisfied by (2.2-13). Expressing (2.2-12)

$$C_3 \left.\mathcal{S}_{3,j}^{(2)}\right]_{\xi=1} + C_4 \left.\mathcal{S}_{4,j}^{(2)}\right]_{\xi=1} = 0 \qquad (2.2\text{-}14)$$

$$C_3 \left.\mathcal{S}_{3,j}^{(3)}\right]_{\xi=1} + C_4 \left.\mathcal{S}_{4,j}^{(3)}\right]_{\xi=1} = 0 \qquad (2.2\text{-}15)$$

In order to be satisfied for non-zero values of C_3, C_4, $\bar{\omega}_j$ should be such that the determinant

$$D(\bar{\omega}_j) = \left[s_{3,j}^{(2)} s_{4,j}^{(3)} - s_{3,j}^{(3)} s_{4,j}^{(2)} \right]_{\bar{\omega}_j, \xi = 1} = 0 \qquad (2.2\text{-}16)$$

The successive eigenfrequencies, $\bar{\omega}_j$, are determined to any prescribed accuracy by iteration . $s_{1,j}$, $s_{2,j}$ are determined by numerical integration of differential equation (2), subject to b.c. (9) and (10.) respectively.

The modal shapes, $\Phi_j(\xi)$, which as expected are defined only to an arbitrary multiplicative constant, are determined, once $\bar{\omega}_j$ is known, as

$$\bar{\Phi}_j(\xi) = C_3 \left[s_{3,j} - \frac{s_{3,j}^{(2)}}{s_{4,j}^{(2)}} s_{4,j} \right]_{\bar{\omega}_j} \qquad (2.2\text{-}17)$$

2.23 Orthogonality of the mode shapes

It is now proven, that given $\bar{\lambda}$, $\xi_\circ \geq 0$, the modes ϕ_j, ϕ_k are orthogonal, i.e.

$$\langle \bar{\Phi}_j, \bar{\Phi}_k \rangle = \int_0^1 \bar{\Phi}_j(\xi)\, \bar{\Phi}_k(\xi)\, d\xi = 0 \qquad j \neq k$$

Note that $\langle \bar{\Phi}_j, \bar{\Phi}_j \rangle = \int_0^1 \bar{\Phi}_j^2 \, d\xi \underset{def}{=} m_{1,j} > 0$

Let $\mathcal{L}$ be the operator

$$\frac{\partial^4}{\partial \xi^4} - \bar{\lambda} \left[\frac{1}{2}(1-\xi^2) + \xi_0 (1-\xi) \right] \frac{\partial^2}{\partial \xi^2} + \bar{\lambda}(\xi + \xi_0) \frac{\partial}{\partial \xi} - \bar{\lambda}$$

Now

$$\mathcal{L}(\bar{\Phi}_j) = \bar{\lambda}\, \bar{\omega}_j^2\, \bar{\Phi}_j$$

and $\mathcal{L}(\bar{\Phi}_K) = \bar{\lambda}\, \bar{\omega}_K^2\, \bar{\Phi}_K$ (2.2-18)

Then, from multiplying by ϕ_k and ϕ_j respectively, and substracting

$$\bar{\Phi}_j^{(4)} \bar{\Phi}_k - \bar{\Phi}_k^{(4)} \bar{\Phi}_j - \bar{\lambda} \left[\frac{1}{2} \left(1-\xi^2\right) + \xi_0 \left(1-\xi\right) \right] \left[\bar{\Phi}_j^{(2)} \bar{\Phi}_k - \bar{\Phi}_k^{(2)} \bar{\Phi}_j \right]$$
$$+ \bar{\lambda} \left(\xi + \xi_0\right) \left(\bar{\Phi}_j^{(1)} \bar{\Phi}_k - \bar{\Phi}_k^{(1)} \bar{\Phi}_j \right) = \bar{\lambda} \left(\bar{\omega}_j^2 - \bar{\omega}_k^2 \right) \bar{\Phi}_j \bar{\Phi}_k \qquad (2.2\text{-}19)$$

Integrate with respect to ξ, from $\xi = 0$ (root) to $\xi = 1$ (tip),

$$\int_{boom} \bar{\Phi}_j^{(4)} \bar{\Phi}_k \, d\xi = \bar{\Phi}_j^{(3)} \bar{\Phi}_k \Big]_{root}^{tip} - \int_{boom} \bar{\Phi}_j^{(3)} \bar{\Phi}_k^{(1)} d\xi$$
$$= - \bar{\Phi}_j^{(2)} \bar{\Phi}_k^{(1)} \Big]_{root}^{tip} + \int_{boom} \bar{\Phi}_j^{(2)} \bar{\Phi}_k^{(2)} d\xi$$
$$= \int_{boom} \bar{\Phi}_k^{(4)} \bar{\Phi}_j \, d\xi$$

Thus

$$\int_{boom} \left\{ \bar{\Phi}_j^{(4)} \bar{\Phi}_k - \bar{\Phi}_k^{(4)} \bar{\Phi}_j \right\} d\xi = 0$$

Next compute

$$\int_{boom} \frac{1}{2} \left(1-\xi^2\right) \bar{\Phi}_j^{(2)} \bar{\Phi}_k \, d\xi = \frac{1}{2} \left(1-\xi^2\right) \bar{\Phi}_j^{(1)} \bar{\Phi}_k \Big]_{root}^{tip} + \int_{boom} \bar{\Phi}_j^{(1)} \Big[\underbrace{-\frac{1}{2} \bar{\Phi}_k^{(1)} \left(1-\xi^2\right)}_{1} + \underbrace{\xi \bar{\Phi}_k}_{2} \Big] d\xi$$
$$\int_{boom} \frac{1}{2} \left(1-\xi^2\right) \bar{\Phi}_k^{(2)} \bar{\Phi}_j \, d\xi = \frac{1}{2} \left(1-\xi^2\right) \bar{\Phi}_k^{(1)} \bar{\Phi}_j \Big]_{root}^{tip} + \int_{boom} \bar{\Phi}_k^{(1)} \Big[\underbrace{-\frac{1}{2} \bar{\Phi}_j^{(1)} \left(1-\xi^2\right)}_{3} + \underbrace{\xi \bar{\Phi}_j}_{4} \Big] d\xi$$

Terms corresponding to (1) and (3) will cancel in the difference. Terms (2) and (4) will cancel the terms resulting from the last term in ξ, in (2.2-19).

in the l.h. side of (2.2-19). Finally

$$\int_{boom} \xi_o (1-\xi)\, \Phi_j^{(2)} \Phi_K \, d\xi = \xi_o \Phi_j^{(1)} (1-\xi) \Phi_K \Big]_{root}^{tip} - \xi_o \int_{boom} \Phi_j^{(1)} \Big[\overset{(5)}{-\Phi_k} + \overset{(7)}{(1-\xi)\Phi_k^{(1)}} \Big] d\xi$$

$$\int_{boom} \xi_o (1-\xi)\, \Phi_k^{(2)} \Phi_j \, d\xi = \xi_o \Phi_K^{(1)} (1-\xi) \Phi_K \Big]_{root}^{tip} - \xi_o \int_{boom} \Phi_K^{(1)} \Big[\underset{(6)}{-\Phi_j} + \underset{(8)}{(1-\xi)\Phi_j^{(1)}} \Big] d\xi$$

Again, terms (7) and (8) will cancel in the difference. Terms (5) and (6) will cancel the terms resulting from the last term in ξ_o, in the l.h. side of (2.2-19). We are left with

$$\bar{\lambda} \left(\bar{\omega}_j^2 - \bar{\omega}_K^2 \right) \int_{boom} \Phi_j(\xi)\, \Phi_K(\xi)\, d\xi = 0$$

$$\int_{boom} \Phi_j \Phi_K \, d\xi = 0 \qquad (j \neq k) \tag{2.2-20}$$

The modal mass, m_j $(\bar{\lambda}, \xi_o)$, is defined, for j=k, as

$$m_{1,j} \underset{def}{\equiv} \int_{boom} \Phi_j^2 \, d\xi \tag{2.2-21}$$

in which $\Phi_j(\xi)$ is normalized to correspond to a unit deflection at the boom's tip, $\xi = 1$. The following quantity, to appear later, is also of interest

$$m_{2,j} \underset{def}{\equiv} \int_{boom} \xi_1 \; \Phi_j(\xi)\, d\xi \tag{2.2-22}$$

with $\xi_1 = \xi_o + \xi$, varying between ξ_o (root) and $\xi_o + 1$ (tip). It is readily determined when the modal shape, $\Phi_j(\xi)$, is known.

Also, for later use, two identities are given here, which are obtained by multiplying Eq. (2.2-19) written for ϕ_j, by ϕ_k, and integrating over the boom,

$$\frac{1}{\bar{\lambda}}\int_{boom}\bar{\Phi}_j^{(4)}\,\bar{\Phi}_k\,d\xi = \frac{1}{\bar{\lambda}}\int_{boom}\bar{\Phi}_j^{(2)}\,\bar{\Phi}_k^{(2)}\,d\xi = \int_{boom}\frac{1}{2}\left[\left(1-\xi^2\right)+2\,\xi_o\left(1-\xi\right)\right]$$

$$\bar{\Phi}_j^{(2)}\,\bar{\Phi}_k\,d\xi - \int_{boom}\left(\xi+\xi_o\right)\,\bar{\Phi}_j^{(1)}\,\bar{\Phi}_k\,d\xi$$

$$+\left(\bar{\omega}_s^{\,2}+\bar{\omega}_j^{\,2}\right)\int_{boom}\bar{\Phi}_j\,\bar{\Phi}_k\,d\xi$$

$$= -\int_{boom}\bar{\Phi}_j^{(1)}\,\bar{\Phi}_k^{(1)}\,\frac{1}{2}\left[\left(1-\xi^2\right)+2\,\xi_o\left(1-\xi\right)\right]d\xi$$

$$+\int_{boom}\bar{\Phi}_j^{(1)}\,\bar{\Phi}_k\,\left(\xi+\xi_o\right)d\xi - \int_{boom}\bar{\Phi}_j^{(1)}\,\bar{\Phi}_k$$

$$\left(\xi+\xi_o\right)d\xi + \left(\bar{\omega}_s^{\,2}+\bar{\omega}_j^{\,2}\right)\int_{boom}\bar{\Phi}_j\,\bar{\Phi}_k\,d\xi\,.$$

Thus, <u>for j ≠ k</u>

$$\frac{1}{\bar{\lambda}}\int_{boom}\bar{\Phi}_j^{(2)}\,\bar{\Phi}_k^{(2)}\,d\xi + \int_{boom}\bar{\Phi}_j^{(1)}\,\bar{\Phi}_k^{(1)}\,\frac{1}{2}\left[\left(1-\xi^2\right)+2\,\xi_o\left(1-\xi\right)\right]d\xi = 0 \qquad (2.2\text{-}23)$$

and <u>for j = k</u>

$$\frac{1}{\bar{\lambda}}\int_{boom}\bar{\Phi}_j^{(2)}\bar{\Phi}_j^{(2)}d\xi + \int_{boom}\bar{\Phi}_j^{(1)}\bar{\Phi}_j^{(1)}\frac{1}{2}\left[\left(1-\xi^2\right)+2\xi_o\left(1-\xi\right)\right]d\xi = \left(\bar{\omega}_s^{\,2}+\bar{\omega}_j^{\,2}\right)m_{jj} \qquad (2.2\text{-}24)$$

2.3 Equations of motion: meridional vibrations

The developments in the case of motions in the (x,z) plane, of a boom located along axis +x in its undeflected position, or "meridional" vibrations, closely parallels those for equatorial vibrations, given in Section 2.2. In the following, only those terms which depart from the ones in Section 2.2 will be given in detail.

2.3.1 Basic equation

The equation expressing the equilibrium, at any section "x" of the boom, at point P, between the flexure moment from the left and the moment , about P, of inertia forces imparted by the particles Q of the boom to the right of P (i.e. those having an abscissa x, between x and ℓ , reads

$$EI\,\frac{\partial^2 w(x)}{\partial x^2} = \int_x^{\ell} \rho\, dx_1 \left[\left(\vec{\pi}_Q - \vec{\pi}_P\right) \wedge \ddot{\vec{\pi}}_Q \right]_y \tag{2.3-1}$$

Now w(x) is an elastic displacement parallel to z. Computing the relevant quantities,

$$\vec{\pi}_{PQ} = (x_1 - x)\,\vec{i}_x + \left(w(x_1) - w(x)\right)\vec{i}_z$$

$$\vec{\pi}_Q = (x_0 + x_1)\,\vec{i}_x + w(x_1)\,\vec{i}_z$$

Thus

$$\dot{\vec{\pi}}_Q = \begin{bmatrix} \omega_y\, w(x_1) \\ \omega_z (x_0 + x_1) - \omega_x\, w(x_1) \\ \dot{w}(x_1) - \omega_y (x_0 + x_1) \end{bmatrix}$$

Also

$$\ddot{\vec{\pi}}_Q = \frac{\partial}{\partial t}\left(\dot{\vec{\pi}}_Q\right) + \vec{\omega} \wedge \dot{\vec{\pi}}_Q$$

$$\ddot{\vec{\pi}}_Q = \begin{bmatrix} \dot{\omega}_y\, w(x_1) + 2\omega_y \dot{w}(x_1) - (x_0 + x_1)(\omega_y^2 + \omega_z^2) + \omega_x \omega_z\, w(x_1) \\ -\dot{\omega}_x\, w(x_1) - 2\omega_x \dot{w}(x_1) + (x_0 + x_1)(\omega_x \omega_y + \dot{\omega}_z) + \omega_y \omega_z\, w(x_1) \\ \ddot{w}(x_1) - w(x_1)(\omega_x^2 + \omega_y^2) + (x_0 + x_1)(\omega_x \omega_z - \dot{\omega}_y) \end{bmatrix}$$

Again, under the assumption of small displacements and transverse angular rates, terms of order w^2, $\omega_x\omega_y$, $\omega_x^2, \omega_y^2 \ldots$ are neglected. $\ddot{\vec{r}}_Q$ reduces to

$$\ddot{\vec{r}}_Q = \begin{bmatrix} -(x_0+x_1)\,\omega_z^2 \\ (x_0+x_1)\,\dot{\omega}_z \\ \ddot{w}(x_1) + (x_0+x_1)(\omega_x\omega_z - \dot{\omega}_y) \end{bmatrix}$$

Along $\vec{i}_y$,

$$\left[(\vec{r}_Q - \vec{r}_P)\wedge \ddot{\vec{r}}_Q\right]_y = \begin{vmatrix} w(x_1) - w(x) & x_1 - x \\ \ddot{w}(x_1) + (x_0+x_1)(\omega_x\omega_z - \dot{\omega}_y) & -(x_0+x_1)\,\omega_z^2 \end{vmatrix}$$

Substituting into (2.3-1), and non-dimensionalizing

$$\frac{EI}{\ell}\frac{\partial^2\eta}{\partial\xi^2} = -\rho\,\ell^3\int_\xi^\ell \Big\{(\xi_1-\xi)\left(\ddot{\eta}_1 + (\xi_0+\xi_1)(\omega_x\omega_z - \dot{\omega}_y)\right) + \omega_z^2(\eta_1-\eta)(\xi_0+\xi_1)\Big\}\, d\xi$$

or

$$\eta_{\xi\xi} = -\alpha\int_\xi^\ell \Big\{(\eta_1-\eta)\,\omega_z^2(\xi_0+\xi_1) + (\xi_1-\xi)(\xi_0+\xi_1)(\omega_x\omega_z - \dot{\omega}_y) + (\xi_1-\xi)\,\ddot{\eta}_1\Big\}\, d\xi_1 \tag{2.3-2}$$

Comparing (2.3-2) to (2.2-4), it is seen that terms (b) and (c) in (2.3-2) differ in the following way from the corresponding ones in (2.2-4)

(b) here has a factor $(\omega_x\omega_z - \dot{\omega}_y)$ instead of $\dot{\omega}_z$

(c) here has a factor $\ddot{\eta}_1$ instead of $\ddot{\eta}_1 - \omega_z^2\eta_1$

Therefore, with these changes, the equation analogous to (2.2-6) which describes the meridional vibrations should be

$$\eta_{\xi\xi\xi\xi} - \bar{\lambda}\,\eta_{\xi\xi}\left[\tfrac{1}{2}(1-\xi^2) + \xi_0(1-\xi)\right] + \bar{\lambda}\,\eta_\xi(\xi+\xi_0) + \alpha\ddot{\eta} = -\alpha(\omega_x\omega_z - \dot{\omega}_y)(\xi+\xi_0)$$

(2.3-3)

So far, quantities neglected have been of order ε^2 of smallness, or smaller. Now, in order for the r.h. side of (2.3-3) to be of order ε^2, we should have

$$\alpha \omega_z^2 \frac{\omega_x}{\omega_z} = \bar{\lambda} \frac{\omega_x}{\omega_z}$$

$$\alpha \frac{d\omega_y}{dt} = \bar{\lambda} \frac{d\omega_y/\omega_z}{d\phi}$$

very small compared to quantities assumed to be of order E. If such is the case, we are left with homogeneous equation

$$\eta_{\xi\xi\xi\xi} - \bar{\lambda}\,\eta_{\xi\xi}\left[\frac{1}{2}(1-\xi^2) + \xi_o(1-\xi)\right] + \bar{\lambda}\,\eta_\xi(\xi+\xi_o) + \alpha\ddot{\eta} = 0 \qquad (2.3\text{-}4)$$

(2.3-4) differs from (2.2-7) only in that term $-\bar{\lambda}\eta$ of (2.2-7) is not present.

2.3.2 Solution of the basic equation

After separation of variables and non-dimensionalizing time by $\tau = \omega_z t$, the solution to (2.3-4) will be

$$\eta_j = \Phi_j(\xi)T_j(t)$$

in which $T_j = \begin{matrix}\sin\\ \cos\end{matrix}\ \bar{\omega}_j\tau = \begin{matrix}\sin\\ \cos\end{matrix}\ \bar{\omega}_j t$, and Φ_j satisfies the differential equation

$$\Phi_j^{(iv)} - \bar{\lambda}\,\Phi_j^{(2)}\left[\frac{1}{2}(1-\xi^2) + \xi_o(1-\xi)\right] + \bar{\lambda}\,\Phi_j^{(1)}(\xi+\xi_o) - \bar{\lambda}\,\bar{\omega}_j^2\,\Phi_j = 0 \qquad (2.3\text{-}5)$$

As expected, this equation is the same as that obtained in (2.2-8) for equatorial vibrations provided the substitution of

$\bar{\omega}_j^2$ in (2.3-5) is made for $[(1+\bar{\omega}_j^2)]$ in [2.2-8] (2.3-6)

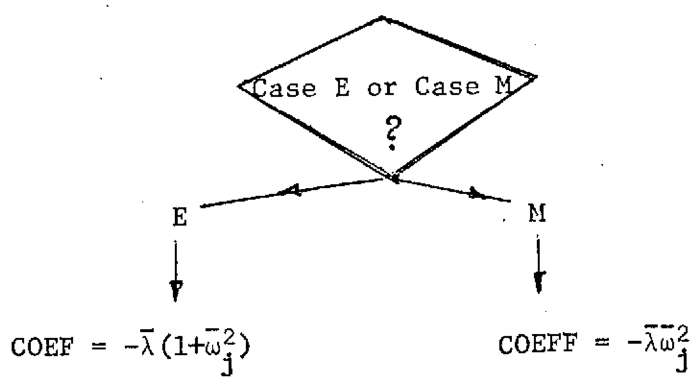
Therefore, the method outlined in Section (2.2.2) to solve for $\bar{\omega}_j$ can be adopted and followed without any other modification than that specified by (2.3-6). In fact, program SEARCH DP, which obtains the first three eigenvalues

$$\omega_1, \omega_2, \omega_3$$

given a pair $(\bar{\lambda}, \xi_\circ)$, iteratively solves an equation such as (2.3-5),

$$\Phi_j^{(IV)} - \bar{\lambda}\, \Phi_j^{(2)} \left[\frac{1}{2}\left(1-\xi^2\right) + \xi_\circ \left(1-\xi\right)\right] + \bar{\lambda}\, \Phi_j^{(1)} \left(\xi + \xi_\circ\right) + COEF * \Phi_j = 0 \qquad (2.3\text{-}7)$$

in which the coefficient "COEF" is determined as follows:



2.3.3 Orthogonality of the mode shapes

Modes $\phi_j(\xi)$ $(j = 1,2...)$ for meridional vibrations can be proven to be orthogonal, as in Section (2.2-3), since Equation (2.2-19) holds equally well in the present case. Thus

$$\int_{boom} \phi_j \phi_k d\xi = 0 \qquad j \neq k \qquad (2.3\text{-}8)$$

and we define, for case M,

$$m_{1,j} \stackrel{\equiv}{\text{def}} \int_{\text{boom}} \phi_j^2 \, d\xi > 0 \qquad (2.3\text{-}9)$$

$$m_{2,j} \stackrel{\equiv}{\text{def}} \int_{\text{boom}} \phi_j \xi_1 \, d\xi \qquad (2.3\text{-}10)$$

with $\xi_1 = \xi_\circ + \xi$.

With the substitution $\bar{\omega}_s^2 + \bar{\omega}_j^2$ in (2.2-8) $\rightarrow \bar{\omega}_j^2$ in (2.3-5), the following relations, valid for meridional vibrations, are deduced straightforwardly from Equations (2.2-23) and (2.2-24)

for $j \neq k$

$$\frac{1}{\bar{\lambda}} \int_{\text{boom}} \Phi_j^{(2)} \Phi_k^{(2)} \, d\xi \quad \int_{\text{boom}} \Phi_j^{(1)} \Phi_k^{(1)} \frac{1}{2} \left[(1-\xi^2) + 2\xi_0 (1-\xi) \right] d\xi = 0. \qquad (2.3\text{-}11)$$

and for $j=k$

$$\frac{1}{\bar{\lambda}} \int_{\text{boom}} \Phi_j^{(2)} \Phi_j^{(2)} d\xi + \int_{\text{boom}} \Phi_j^{(1)} \Phi_j^{(1)} \frac{1}{2} \left[(1-\xi^2) + 2\xi_0 (1-\xi) \right] d\xi = m_{1,j} \, \bar{\omega}_j^2. \qquad (2.3\text{-}12)$$

It should be noted here that for the <u>same</u> pair of values $(\bar{\lambda}, \xi_\circ)$, if $(\text{COEF})_j$ is the value to be given to COEF in (2.3-7), in order for the determinant (2.2-16) to vanish, then

$$(\text{COEF})_{j,E} = (\text{COEF})_{j,M} = \text{COEF}$$

or

$$\bar{\omega}_{j,E}^2(\bar{\lambda}, \xi_\circ) + 1 = \bar{\omega}_{j,M}^2(\bar{\lambda}, \xi_\circ) \qquad (2.3\text{-}13)$$

whereas the modal shapes determined from (2.3-7) with the value $(\text{COEF})_j$ of COEF have to be the same in cases E and M

$$\Phi_{j,E}(\bar{\lambda}, \xi_0, \bar{\omega}_{j,E}) = \Phi_{j,M}(\bar{\lambda}, \xi_0, \bar{\omega}_{j,M})$$

In (2.3-13), if it is found more convenient to non-dimensionalize by a quantity proportional to the 1st eigenfrequency of the non-rotating cantilever boom, namely

$$\omega^*_{NR} = (EI/\rho\ell^4)^{1/2}$$

then (2.3-13) becomes

$$\left(\frac{\omega_{j,E}}{\omega^*_{NR}}\right)^2 + \left(\frac{\omega_z}{\omega^*_{NR}}\right)^2 = \left(\frac{\omega_{j,E}}{\omega^*_{NR}}\right)^2 + \bar{\lambda} = \left(\frac{\omega_{j,M}}{\omega^*_{NR}}\right)^2 \qquad (2.3\text{-}14)$$

as illustrated in some examples of Section (2.8)

2.4 Program determining the modal frequencies for equatorial or meridional vibrations: SEARCH DP.

Program SEARCH DP, listed at the end of the present chapter, is written in FORTRAN V and implements the developments of Section 2.2 and 2.3.

The calculations are carried out in double precision, which suffices for values of $\bar{\lambda}$ up to about 5,000. For higher values of λ, an arbitrary N-precision, scheme had to be used: this is described in Section 2.7.

2.4.1 Description of the program

Number of statements (including comment cards): about 270

Input: - 1 card giving Q = E or M?; $\bar{\lambda}$; ξ_o in format (A1, F6.5, G5.4)

Output: 1) - A heading, specifying "Equatorial case" or "Meridional case"

2) - The values of a "frequency" number" defined as $\sqrt{\bar{\lambda}}$

- Lines giving the value of determinant of Equation (2.2-16), called here FE34 ; the value of $\sqrt{COEF}$, the value of index U, number of trials in μ before converging to the root of $\mathcal{D}(\bar{\omega}_j) = 0$

- Lines labeled KKK number of iterations, giving the successive values of the determinant as μ is changed to obtain convergence of the determinant to zero. The iteration stops when μ_{k+1} differs from μ_k by less than 10^{-4}.

- A statement that "MU converged" giving the value of FE34 and μ.

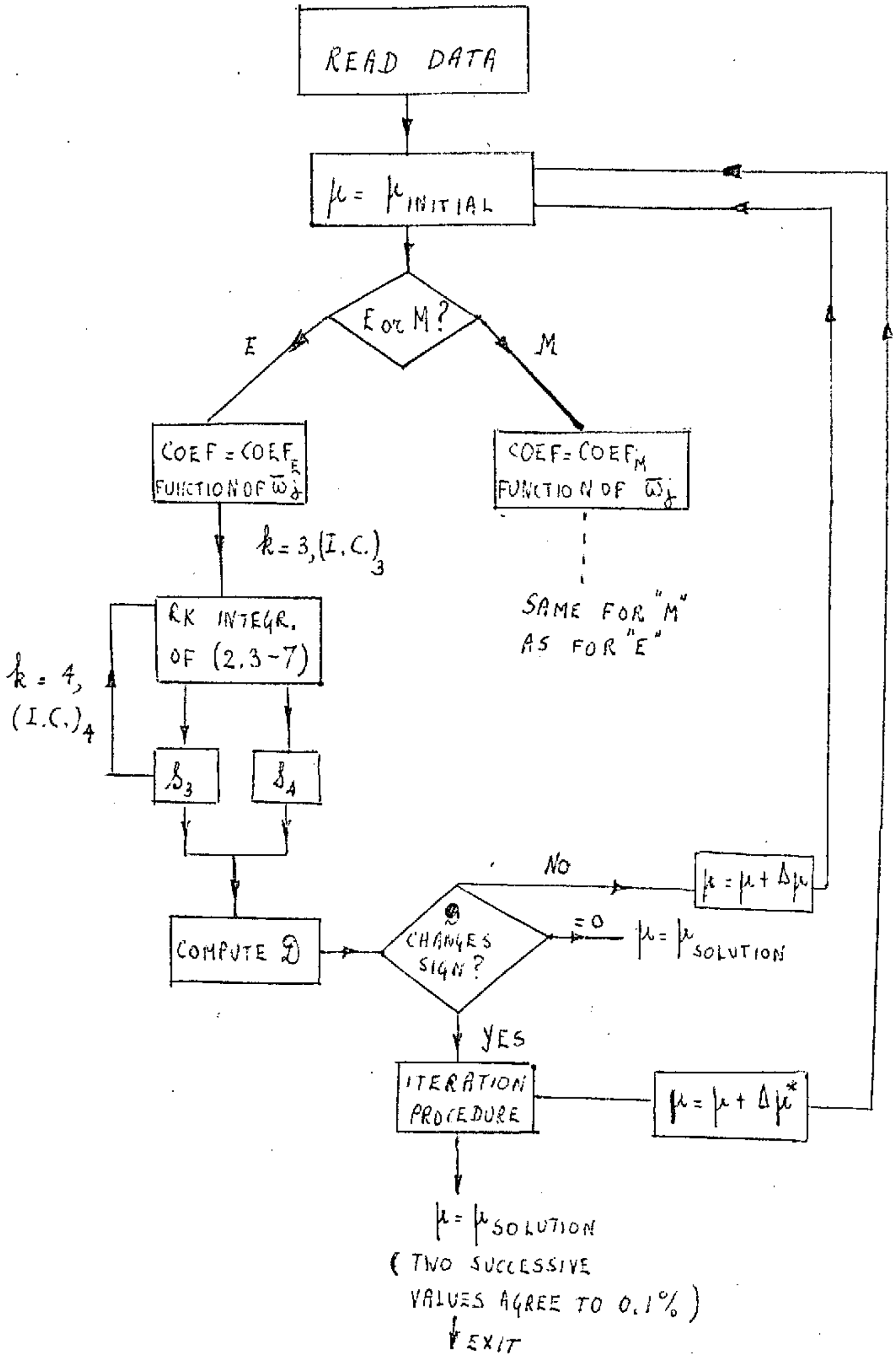
- A print-out of FE34, μ, $\bar{\lambda}$, and NATFRQ, defined as $\frac{\omega_j}{\omega^*_{NR}}$

- The value of the step in μ, DLT, and the value of the order of the eigenvalue, j or NOR

3) same for j = 2, 3, in that order.

2.4.2 Schematic flow chart:

The following flow chart schematically describes the main control flow in SEARCH DP.


READ DATA
μ = μ INITIAL
E or M?
E
M
COEF = COEF_E FUNCTION OF ϖj
COEF = COEF_M FUNCTION OF ϖj
k = 3, (I.C.)3
SAME FOR "M" AS FOR "E"
RK INTEGR. OF (2.3-7)
k = 4, (I.C.)4
δ3
δ4
COMPUTE D
D CHANGES SIGN?
NO
μ = μ + Δμ
= 0
μ = μ SOLUTION
YES
ITERATION PROCEDURE
μ = μ + Δμ*
μ = μ SOLUTION
(TWO SUCCESSIVE VALUES AGREE TO 0.1%)
EXIT

SEARCH - DP

2.4.3 Comments

a) It has been numerically determined[2-4] that 100 steps across the boom's length would suffice, over the range of $\bar{\lambda}$ and ξ_o investigated, to obtain eigenvalues agreeing up to the 5^{th} digit with those obtained with 200 steps across the boom's length. The "100-steps" are therefore incorporated as a "fixed" feature in program SEARCH DP.

b) A method of linear interpolation is used for finding the roots of $(\bar{\omega}_j) = 0$. The iteration on μ(or equivalently the eigenvalue to be) stops when two successive values of μ, in the iteration process, agree to at least 0.1%.

c) The integration method is a simple Runge-Kutta with fixed step, having a per step error of the order of Δx^5.

d) Using double-precision arithmetic, the number of significant digits retained in the two terms in $\mathcal{D}$, in Equation 2.2-16, does not suffice for values of $\bar{\lambda}$ larger than about 5,000, and an arbitrary precision package ("NP" - package, N > 0 integer) had to be developed and is described in Section 2.7.

2.4.4 Listing and sample output

A listing and a sample output of program SEARCH DP are given at the end of this chapter.

2.5 Program Determining the Modal Shaptes ϕ_j and "Masses" $m_{1,j}$, $m_{2,j}$: MODE

MODE is a Fortran-V, double precision program determining the modal shapes, normalized to unit deflection at the boom's tip,

$$\phi_j(\xi) \quad j = 1,2,3$$

which are solutions of Equation 2.3-7, in which

$\bar{\omega}_j$ is the jth eigenvalue determined by SEARCH DP

$$\text{COEF} = (\text{COEF})_{j,E} = (\text{COEF})_{j,M}$$

$$(\text{COEF})_{j,E} = 1 + \bar{\omega}^2_{j,E}$$

$$(\text{COEF})_{j,M} = \bar{\omega}^2_{j,M}$$

2.5.1 Description of program MODE

Number of statements (including comment cards): 158

Input: - 1 card giving IE - E or M?; j; $\bar{\lambda}$; $\xi_\circ$:

μ = COEF (to be used in Equation 2.3-7)

in (A1, I1, 3G12.6 format)

Output: 1) - A heading, specifying "Equatorial Case" or "Meridional Case"

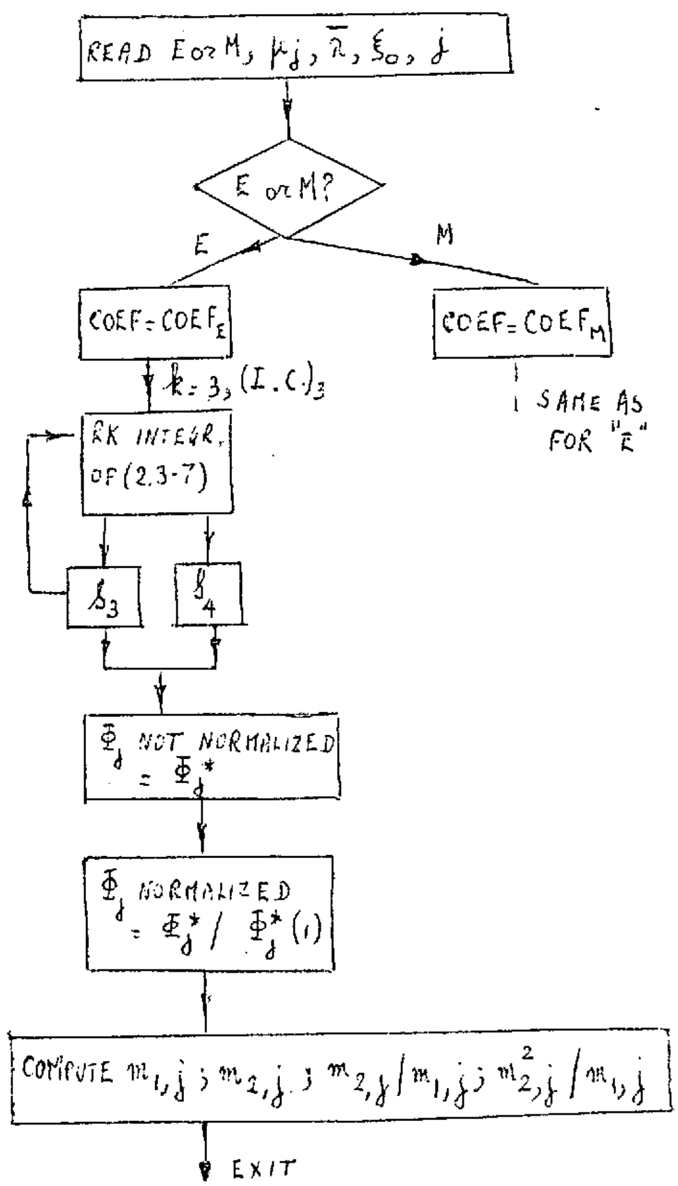
2) - The values of $\mu_j = \text{COEF}_j$ (as obtained from SEARCH DP), $\bar{\lambda}$, $\xi_\circ$, j (1, 2 or 3)

- The values of $m_{1,j} = \int_{boom} \Phi_j^2 \, d\xi$; $m_{2,j} = \int_{boom} (\xi_\circ + \xi)\Phi_j d\xi$;

$\frac{m_{2,j}}{m_{1,j}}$; $\frac{m^2_{2,j}}{m_{1,j}}$ which are of interest in the dynamical simulation of the evolution in time of the spacecraft angular rates (ω_x, ω_y, ω_z) and modal coordinates (q_j)

- The deflection $\Phi_j(\xi)$ as a function of ξ; I, the station index, varying from I = 1 (at the root) I = 101 (at the tip), in steps of 2.

2.5.2 Schematic flow diagram. The main control flow in MODE is as follows:



2.5.3 Comments

a) The number of steps of integration, experimentally determined to give values of μ agreeing up to the 5th digit when solving the step size, was found to be 100. As in 2.4.3 and SEARCH DP, the the 100 steps are a fixed feature incorporated in the program.

b) The method of integration is Runge-Kutta with fixed step.

c) The calculations are carried out in double-precision, which should suffice for values of $\bar{\lambda}$ of up to 10,000. The data μ_j, however, might have had to be determined with the use of "NP" arbitrary precision package.

2.5.4 Listing and sample output.

A listing and a sample output of program MODE are given at the end of this chapter.

2.6 Parametric Study of Eigenfrequencies and Modal Shapes as a Function of $\bar{\lambda}$ (Etkin's Number) and $\xi_\circ$ (Non-Dimensional Radius of the Hub)

Given the design parameters $\bar{\lambda}$ and $\xi_\circ$, the study of the eigenfrequencies ω_j,(which normalized to ω_s, are noted $\bar{\omega}_j$, and to $\omega^*_{NR} = (\frac{EI}{\rho \ell^4})^{1/2}$, are noted $\frac{\omega_j}{\omega^*_{NR}}$) will be made easier by using several programs described hereunder.

2.6.1 Preliminary Comment

First of all, it should be emphasized here that there is no point in comparing mode shapes ϕ_j for "E" and "M", since they are the <u>same</u> solutions to Equation (2.3-7), for $COEF_j = COEF_{j,E} = COEF_{j,M}$, once j has been chosen and $\bar{\lambda}$ and $\xi_\circ$ have been given. Any slight numerical departure, such as described in Ref. [2-4], 2-5] could only result from the inaccuracy is determining the eigenfrequency (0.1% relative accuracy on μ, in program SEARCH DP). Only the eigenfrequencies $\bar{\omega}_{j,E}$, $\bar{\omega}_{j,M}$ corresponding to these modal shapes will be different.

2.6.2 Program computing dynamical parameters, given $\bar{\lambda},\xi_\circ$: PARAM.

Program PARAM, written in FORTRAN-V, will permit to get a quick look at various relevant dynamical parameters, given Q = E or M, $\bar{\lambda}$ and $\xi_\circ$, namely

$m_{1,j}$

$m_{2,j}$

$(m_2/m_1)_j$ and $j = 1,2,3$

$(m_2^2/m_1)_j$

and also the sum over one, two, three modes

$$\sum_j \frac{m_{2,j}^2}{m_{1,j}}$$

a quantity to be used later in this work. It will also plot the mode shapes (up to j = 3) in the computer printout.

The data entered are

$\bar{\omega}_{j,M}$ (j = 1,2,3) obtained from SEARCH DP, case M (NDS = 0)

$\bar{\lambda}$

$\xi_\circ$

The program basically computes $\Phi_j(\xi)$ and the relevant integrals, $m_{1,j}$, $m_{2,j}$ etc... as defined before.

A listing and a sample output of program PARAM is given at the end of this chapter.

2.7 Arbitrary Precision Package: MP (for use on OS) and P (N-Precision Package), in Fortran.

2.7.1 Motivation

An earlier version of SEARCH DP had been written[2-4] to alleviate a problem of numerical stability at large values of $\bar{\lambda}$ (higher than about 5,000). This version used on IBM-library multiple precision (MP package). It was found, however, that this package was unavailable in a TSS environment. Therefore, an arbitrary precision package (NP) was written in Fortran V, and used for finding the eigenvalues $\bar{\omega}_j$ at values of $\bar{\lambda}$, and the accuracy of determinant $\mathcal{D}$ in Equation (2.2-16) will be critically affected when taking differences of very large numbers.

MPAP (Multiple Precision Arithmetic Package) is present in the Internal Library of the IBM-360. The routine calls on specialized subroutines to perform floating point calculations with precision to be specified by the programmer (typically, here, quadruple precision was required).

MP-SEARCH, as used in Ref. [2-4], and MP-MODE, are thus basically MPAP versions of SEARCH and MODE. Their one disadvantage, as expected, is an increased running time, of the order of 1.5 minutes for eigenvalue (IBM/360). For this reason, it is important that the eigenvalues or modal quantities of MP-MODE obtained for high $\bar{\lambda}$ be stored for later use in the simulation (Option MG1V = 1 in program FLEXAT, see Chapter 5), and that interpolation be used whenever possible.

2.7.3 Multiple precision in TSS: MP-package

Written in FORTRAN for case of conversion to any machine, N-PRES is a multiple-precision arithmetic system for scientific calculation It may be used on any machine which stores one integer per work, where a word is $\geq$ 31 bits long.

2.7.3.1 Short description of the program

2.7.3.1.1 Representable numbers.

Let N, M be integers

$$2 \leq N \leq 16$$

All numbers in the program are considered floating point constants of + N precision, expressed in scientific notation. Thus, for N = 3, or precision 4N = 12, we could have

.371246875003*10**8371

The exponent must always be an integer, positive, negative or zero and less than or equal to 4 decimal digits long. Thus a number such as $\pm d_1 d_2 \ldots d_{60} * 10^{\pm D_1 D_2 D_3 D_4}$

2.7.3.1.2 Internal Storage (Multiple-point, floating)

The mantissa is stored 4 digits to a word, in "N" digits. The exponent takes up the N+1 tion (Any $1 \leq N \leq 16$)

<u>Example:</u> for M = 3, N = 2

$8.4326 \times 10^4 = .84326000 \times 10^5$

$6.0 \times 10^4 = .6 \times 10^2$

represented as

	NUMBER 1	NUMBER 2	
	8	6	
	4	0	Word 1
	3	0	
	2	0	
	6	0	
	0	0	Word 2
	0	0	
	0	0	
EXPO→	0	0	
	0	0	
	0	0	
	5	2	

All operations are designed to handle such units, called N-CONS (for N constant).

2.7.3.1.3 Quick guide to operations and subroutines

Name	Subroutine Function (all operating with N cons)
INIT	Initialize the N-precision system
INPUT	Input
OUTPUT	Output
CIN	Convert integer to N-CON
CNI	Convert N-CON to integer
CFN	Convert floating point to N-CON
CNF	Convert N-CON to floating point
NABS	Mem(Add) ← ABS[Mem(Add)]
NPWR	Mem(Add) ← [Mem(Add)]**P with P a parameter to NPWR
NSCL	Mem(Add) ← [Mem(Add)]*10**S with S a parameter to NSCL

Name	Subroutine function (all operating with N cons)
NCMPR	if Mem(Add 1) = Mem(Add 2), A = B if Mem(Add 1) > Mem(Add 2), A > B if Mem(Add 1) < Mem(add 2), A < B with A, B, parameters to NCMPR
COPY	Mem(Add 2) ← Mem(Add 1)
RENORM } SHIFT	Internal use only
PUNCH	Output to punch
IMUL	Mem(Add) ← Mem(Add)*I I = |integer| ≤ limit
FDIV	Mem(Add) ← Mem(Add)/F F = floating point
MADD	Mem(Add 3) ← Mem(Add 1) + Mem(Add 2)
MSUB	Mem(Add 3) ← Mem(Add 1) - Mem(Add 2)
MMUL	Mem(Add 3) ← Mem(Add 1) * Mem(Add 2)
MDIV	Mem(Add 3) ← Mem(Add 1) / Mem(Add 2)

2.7.3.2 Some examples of N-precision programming

2.7.3.2 Square root

A. Algorithm: Newton-Raphson

Let $B = \sqrt{A}$, with old B = 1

the $B = \frac{1}{2} \left(\frac{A}{\text{Old } B} + \text{Old } B\right)$

If Abs(Old B - B) > B*10** limit

Old B = B

Else done

B. Fortran Progarm:

```
      Limit = - 12
      Read (5,1)A
1     Format (F10.2)
      Old B = 1.
2     B = (A/Old B + Old B)/2
      X = ABS(Old B-B)
      Y = B*10**Limit
      If (X, LE. Y) GØ TØ 3
      Old B = B
      GO TO 2
3     WRITE (6,4)B,A
4     FORMAT ('␣', F10.2, '␣ IS SQUARE ROOT OF, F10.2)
      STOP
      END
```

C. N-Precision Program — Comments

```
      IMPLICIT INTEGER (A-Z)        (all N-cons.)
      CALL INIT(1,4)               (16 digits of precision, N=4)
      CON V = -12                  (limit)
      A = 1
      B = 2                        (Allocation of variable
                                    names to N-con addresses)
      Old = 3
      TWO = 4
      X = 5
      Y = 6
      HALF = 7
      CALL Input(A)
      Call NSCL(A,1)
      CALL CIN(TWO,2)              (N-con at address TWO contains
                                    the value 2)
```

Call CIN(Old B,1)	(N-con. at address 'Old B' contains the value 1)
Call CFN(HALF,.5)	(Half contains 0.5)
Call Output(HALF)	(Conversion OK; print and check)
Call Output(A)	(Print input number)
Q = 1	(Iteration Counter)
Call MDIV(A, OldB,B)	(B = A/Old B)
Call MADD(B,Old B,B)	(B = B+Old B)
Call MMUL(B,HALF,B)	(B = B*.5)
Call Output(B)	(write partial answer)
Call HM	(How many subroutines called so far. Print it out.)
Call MSUB(B,Old B,X)	(X = B-Old B)
Call NABS(X)	(X = Abs X)
Call COPY (B,Y)	(Y - (B))
Call NSCL (Y,CONV)	(Y - (Y)*10** CONV)
Call NCMPR(X,Y,I,J)	(Result: If X > Y, I > J; X < Y, I < J; X = Y, I = J)
If(I. LE. J) GO TO 2x	(IF(ABS(B-Old B).LE. B*10**CONV) GO TO 2
Call COPY(B, Old B)	(Old B = (B))
GO TO 1	
CONTINUE	(Done!)
Call HM	(How many calls)
(Write out results)	

2.7.3.2.2 Conversion of a statement from SEARCH DP

Consider the FORTRAN statement of SEARCH DP:

```
     IF(FE34*DECID) 52, 51, 50
```

The N-PREC. translation would be

```
     CALL MMUL(DECID, FE34, TEMP)
     CALL NCMPR(TEMP,ZERO,I,J)
     If(I.LT. J) GO TO 52
     If(J.EQ. J) GO TO 51
50   CONTINUE
```

2.8 Results from programs SEARCH DP, MP and NP

The frequencies $\bar{\omega}_j$ (normalized to $\omega_s = 1$) for j = 2, are given for case M. Those for case E are immediately obtained from

$$\bar{\omega}^2_{E,j} = \bar{\omega}^2_{M,j} - 1$$

Also given below is the quantity $\sum_{j=1}^{3} \frac{m^2_{2,j}}{m_{1,j}}$, which will be of special importance in Chapters 4 and 5. The first non-dimensional frequency x $\omega_1 \sqrt{\rho \ell^4/EI}$ is also represented, for cases E and M, and various values of ξ_o, on Fig. 2.2.

CASE M - FIRST NONDIMENSIONAL NATURAL FREQUENCY

	0.00	0.10	0.25	0.50
0	3.681	3.703	3.734	3.787
5	1.913	1.953	2.013	2.107
10	1.555	1.605	1.675	1.788
20	1.339	1.395	1.476	1.601
30	1.256	1.316	1.401	1.531
50	1.183	1.246	1.335	1.469
100	1.120	1.186	1.278	1.417
200	1.081	1.148	1.242	1.385
500	1.050	1.118	1.214	1.358
1000	1.034	1.104	1.201	1.346
3000	1.021	1.091	1.188	1.339
7000	1.016	1.087	1.184	1.329
10000	1.013	1.083	1.181	1.327

NOTE: $\tilde{\bar{\omega}}_E = \left(\bar{\omega}_M^{\,2} - 1\right)^{1/2}$

$$\left(\omega_1\right)_{\bar{\lambda}=0} = 3.518 \left(\frac{EI}{\rho \ell^4}\right)^{1/2}$$

CASE M - SECOND NONDIMENSIONAL NATURAL FREQUENCY $\bar{\omega}_2 = \omega_2 / \omega_S$

	0.00	0.10	0.25	0.50
1	22.18	22.20	22.23	22.78
5	10.236	10.276	10.339	10.447
10	7.419	7.476	7.561	7.703
20	5.546	5.624	5.736	5.921
30	4.760	4.849	4.981	5.191
50	4.023	4.128	4.281	4.523
100	3.364	3.488	3.665	3.941
200	2.976	3.113	3.308	3.606
500	2.707	2.855	3.060	3.373
1000	2.603	2.754	2.964	3.282
3000	2.520	2.673	2.886	3.195
7000	2.490	2.644	2.857	3.178
10000	2.482	2.635	2.849	3.171

NOTE: $\bar{\omega}_E = \left(\bar{\omega}_M^{\,2} - 1\right)^{1/2}$

$\left(\omega_2\right)_{\bar{\lambda}=0} = 21.91 \left(\frac{EI}{\rho l^4}\right)^{1/2}$

$m^2_{2,1}/m_{1,1}$

(ONE NODE)

$\bar{\lambda}$ \ ξ_o	0.00	0.10	0.25	0.50
0	0.3233	0.4190	0.5810	0.9250
10	0.3249	0.4212	0.5893	0.9325
20	0.3260	0.4231	0.5929	0.9400
30	0.3268	0.4246	0.5957	0.9457
50	0.3280	0.4268	0.5997	0.9538
100	0.3297	0.4301	0.6056	0.9652
200	0.3311	0.4330	0.6111	0.9757
500	0.3323	0.4357	0.6165	0.9861
1000	0.3328	0.4371	0.6194	0.9916
3000	0.3331	0.4385	0.6224	0.9950
10000	0.3332	0.4392	0.6241	1.001
	Δ = 0.3333	Δ = 0.4433	Δ = 0.6458	Δ = 1.0833

NONDIMENSIONAL DYNAMICAL PARAMETERS

SUM OVER 3 MODES

$\bar{\lambda}$ \ ξ_o	0.0	0.10	0.25
10	0.3328	0.4401	0.6336
100	0.3329	0.4404	0.6343
1000	0.3332	0.4413	0.6366
	Δ = 0.3333	Δ = 0.4433	Δ = 0.6458

$m_{2,1}{}^2/m_{1,1}$ (ONE MODE)

$\bar{\lambda}$ \ ξ_o	0.00	0.10	0.25	0.50
0	0.3233	0.4190	0.5810	0.9250
10	0.3249	0.4212	0.5893	0.9325
20	0.3260	0.4231	0.5929	0.9400
30	0.3268	0.4246	0.5957	0.9457
50	0.3280	0.4268	0.5997	0.9538
100	0.3297	0.4301	0.6056	0.9652
200	0.3311	0.4330	0.6111	0.9757
500	0.3323	0.4357	0.6165	0.9861
1000	0.3328	0.4371	0.6194	0.9916
3000	0.3331	0.4385	0.6224	0.9950
10000	0.3332	0.4392	0.6241	1.001
	Δ = 0.3333	Δ = 0.4433	Δ = 0.6458	Δ = 1.0833

REFERENCES - Chapter 2

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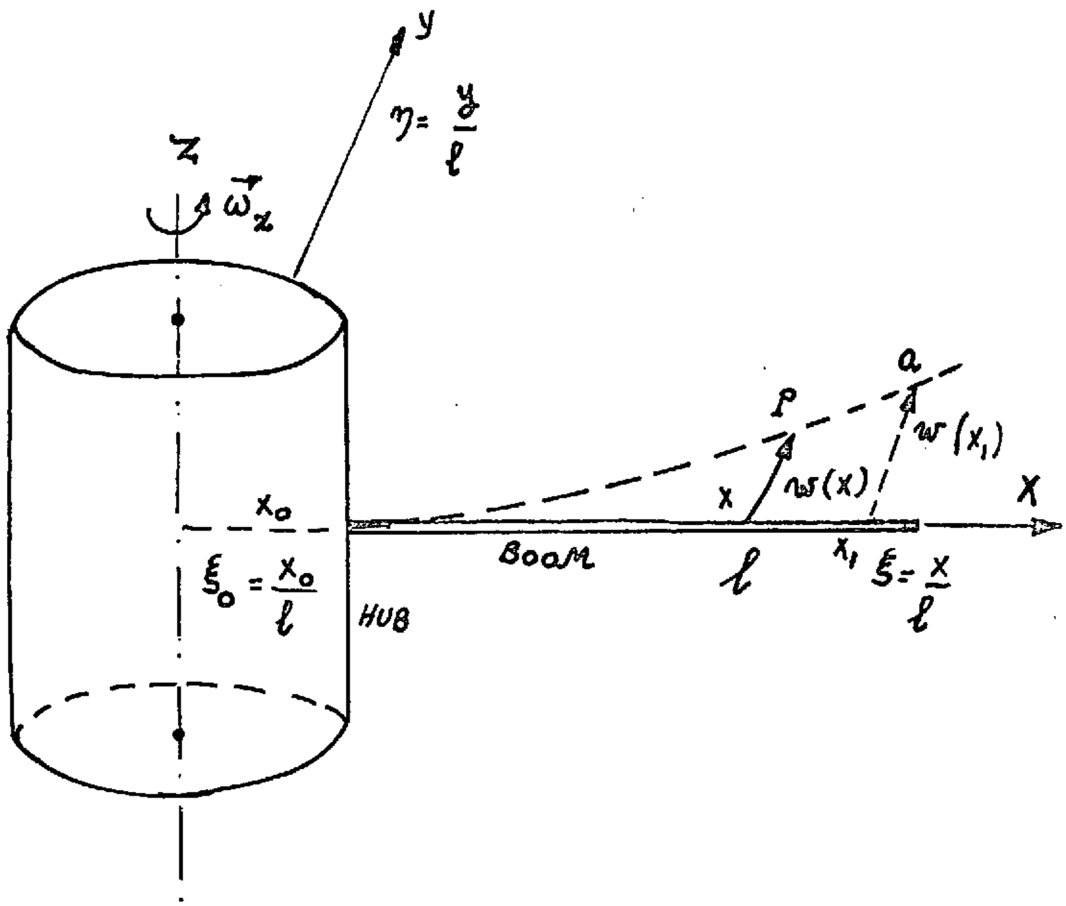
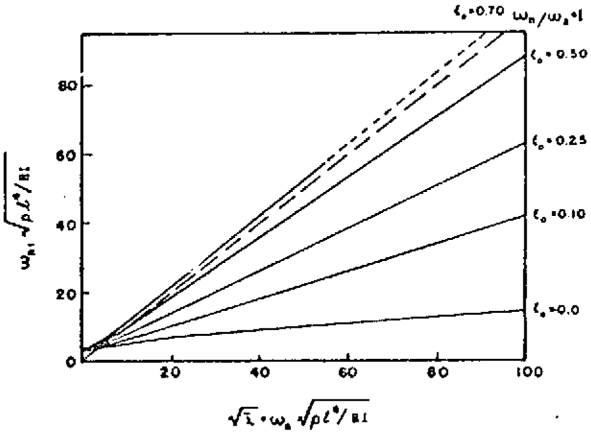
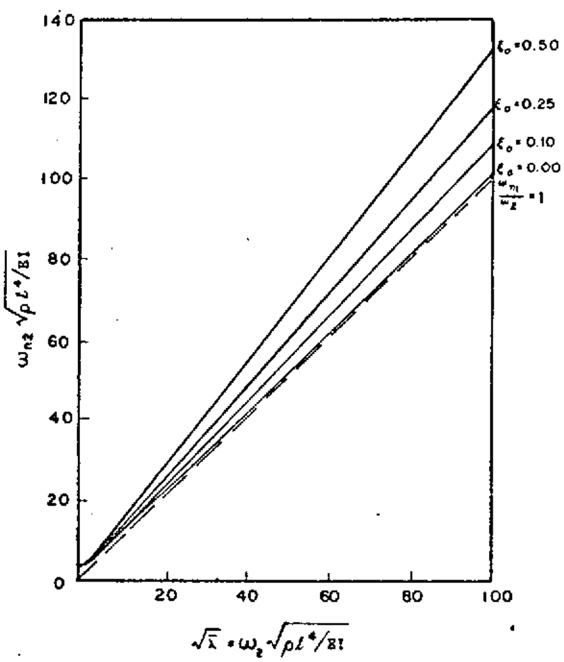


FIG. 2-1. GEOMETRY OF SPINNING SATELLITE WITH FLEXIBLE APPENDAGE.



Equatorial vibrations.



Meridional vibrations.

FIG. 2-2. FREQUENCY OF FUNDAMENTAL MODE VS $\bar{\lambda}$.

PROGRAM LISTING

AND

SAMPLE OUTPUT

SEARCH DP

```
C
C
C    THIS PROGRAM FINDS THE FIRST THREE EIGENVALUES (MU) FOR THE ROTATING
C    BOOM IN EITHER EQUATORIAL OR MERIDION.FLEXURE
C
C    IT WILL COMPUTE THESE ACCURATELY FOR VALUES OF LAMDA UP TO
C    APPROXIMATELY 5000
C
      DOUBLE PRECISION P(4),K(4),M(4),L(4),E3(101),E31P(101),E32P(101),
     1FE34,DECID,TVAL,HY,EMOM3,EMOM4,ESHR3,ESHR4,FEPRV,
     1E33P(101),E34P(101),E4(101),E41P(101),E42P(101),E43P(101),
     3E44P(101),A,B,C,E,A0,B0,C0,E0,MU1,MU,EPSB,EPSC,LAS,LASS,
     4 DLT,UP,DWN,SIO,SI
      REAL H,NN,LAM,NATFRQ
      INTEGER I,D,N,Z,  NINT,INTER,W,R,U,NOR,KKK,Q
       DATA/LM/1HM
C
C    INPUT DATA IS E OR M,LAMBDA,PSI-ZERO
C
 18    READ(5,20,END=22)Q,LAM,SIO
 20    FORMAT(A1,F6.5,G5.4)
       NDS=1
       IF(Q.EQ.LM)NDS=0
       IF(NDS.EQ.1)WRITE(6,16)
       IF(NDS.EQ.0)WRITE(6,17)
 16    FORMAT(' EQUATORIAL CASE'//)
 17    FORMAT(' MERIDIONAL CASE'//)
      WRITE(6,21) LAM,SIO
  21  FORMAT(1H ,'LAM=',F12.6,3X,'SIO=',D9.3/)
      KKK=1
      R=0
      U=1
      FE34=0.
       MU=1.0D-6
       EPSB=1.0D-14
      W=0
      FEPRV=0.
      EPSB=10.**-14
C
C   SET NOPT = 1 FOR REVERSED INTEGRATION (TIP TO ROOT)
C
      NOPT=1
C
C  NDS = DIRECTION SWITCH
C    WHEN NDS = 1, SEARCH FOR EQUATORIAL ROOTS,
C    WHEN NDS = 0, SEARCH FOR MERIDIAN ROOTS
C
      NOR= 1
      N=1
      DLT=1.
      IA=0
      NINT=100
      INTER=NINT+1
      ANFRQ=SQRT(LAM)
      WRITE(6,60) ANFRQ
   60 FORMAT(1H ,'NFRQ=SQRT LAM=',F10.5)
   99 SI=0.
      TVAL=FE34
      JJ=0
C  CLEAR ARRAYS
      DO 31 I=1,4
```

```
      K(I)=0.
      L(I)=0.
      M(I)=0.
   31 P(I)=0.
      DO 1 I=1,101
      E34P(I)=0.
      E44P(I)=0.
      E33P(I)=0.
      E43P(I)=0.
      E32P(I)=0.
      E42P(I)=0.
      E31P(I)=0.
      E41P(I)=0.
      E4(I)=0.
    1 E3(I)=0.
      H=1./FLOAT(NINT)
C
C   SET INITIAL CONDITIONS  ON THE S3 AND S4 SOLUTIONS
C
      D=3
    8 IF(D.EQ.4) GO TO 2
      E0=0.
      B0=0.
      IF(NOPT.GT.0) GO TO 12
      E32P(1)=1.
      B0=E32P(1)
      GO TO 13
   12 E3(1)=1.
      E0=E3(1)
   13 C0=0.
      A0=0.
      GO TO 3
    2 A0=0.
      C0=0.
      IF(NOPT.GT.0)GO TO 14
      E43P(1)=1.
      C0=E43P(1)
      GO TO 15
   14 E41P(1)=1.
      A0=E41P(1)
   15 B0=0.
      E0=0.
    3 A=A0
      B=B0
      C=C0
      E=E0
C
C  BEGIN RUNGA KUTTA INTEGRATION
C
      N=1
    4 I=1
      NN=N
      SI=(NN-1.)*H
    5 K(I)=H*A
      L(I)=H*B
      M(I)=H*C
      MU1=1.+MU*MU
      IF(NDS.EQ.1) GO TO 40
```

```
      MU1=MU1-1.
   40 P(I)=((1.-SI*SI+2.*SIO*(1.-SI))/2.*B-(SI+SIO)*A+MU1*E)*LAM*H
      IF(NOPT.GT.0) P(I)=((-SI*SI+2.*SI*(1.+SIO))/2.*B
     1+(1.-SI+SIO)*A+MU1*E)*LAM*H
      SI=(NN-1.)*H
      I=I+1
      IF(I.GT.3) GO TO 6
      Z=I-1
      E=E0+K(Z)/2.
      A=A0+L(Z)/2.
      B=B0+M(Z)/2.
      C=C0+P(Z)/2.
      SI=SI+H/2.
      GO TO 5
    6 IF(I.GT.4) GO TO 7
      E=E0+K(3)
      A=A0+L(3)
      B=B0+M(3)
      C=C0+P(3)
      SI=SI+H
      GO TO 5
    7 IF(D.EQ.4) GOTO 9
      SI=NN*H
      Z=N+1
      E3(Z)=E3(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E31P(Z)=E31P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E32P(Z)=E32P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E33P(Z)=E33P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E34P(Z)=LAM*(((SIO+1.)**2-(SI+SIO)**2)*E32P(Z)/2.
     1-(SI+SIO)*E31P(Z)+MU1*E3(Z))
      IF(NOPT.GT.0) E34P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E32P(Z)/2.+(1.-SI+SIO)*E31P(Z)+MU1*E3(Z))
      E=E3(N+1)
      A=E31P(N+1)
      B=E32P(N+1)
      C=E33P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM3=E32P(INTER)
      ESHR3=E33P(INTER)
      IF(NOPT.GT.0) EMOM3=E3(INTER)
      IF(NOPT.GT.0) ESHR3=E31P(INTER)
      D=4
      GO TO 8
    9 SI=NN*H
      Z=N+1
      E4(Z)=E4(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E41P(Z)=E41P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E42P(Z)=E42P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E43P(Z)=E43P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E44P(Z)=LAM*(((SIO+1.)**2-(SI+SIO)**2)*E42P(Z)/2.
     1-(SI+SIO)*E41P(Z)+MU1*E4(Z))
      IF(NOPT.GT.0) E44P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E42P(Z)/2.+(1.-SI+SIO)*E41P(Z)+MU1*E4(Z))
```

```
      E=E4(N+1)
      A=E41P(N+1)
      B=E42P(N+1)
      C=E43P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM4=E42P(INTER)
      ESHR4=E43P(INTER)
      IF(NOPT.GT.0) EMOM4=E4(INTER)
      IF(NOPT.GT.0) ESHR4=E41P(INTER)
C
C     RUNGA KUTTA FINISHED
C     NOW BEGIN LINEAR INTERPOLATION
C     FE34 IS THE VALUE OF THE DETERMINANT (S3 AND S4)
C
      FE34=EMOM3*ESHR4-ESHR3*EMOM4
      IF(R.EQ.1) GO TO 51
      IF(U.EQ.1) GO TO 50
      IF(FE34*DECID)52,51,50
   50 DECID=FE34
      LAS=MU
      LASS=LAS
      WRITE(6,85)FE34,MU,U
   85 FORMAT(1H ,'FE34',D12.6,5X,'MU=',D12.6,5X,'U=',I3)
      MU=MU+DLT
      U=U+1
      GO TO 99
   52 UP=MU
      DWN=LAS
      HY=FE34
      TVAL=FE34
   51 IF(ABS(FE34).LE.EPSB) GO TO 53
      IF(ABS(DECID).LE.EPSB) GO TO 42
      WRITE(6,88)KKK,FE34,MU
   88 FORMAT(1H ,'KKK',I3,3X,'FE34',D12.6,3X,'MU',D12.6)
      R=1
      IF(FE34*DECID)55,51,56
   55 UP=MU
      MU=DWN-(DWN-UP)*DECID/(DECID-FE34)
      KKK=KKK+1
      EPSC=ABS(ABS(MU)-ABS(LASS))
      IF(EPSC.LT.MU*10.**-4)GO TO 10
      LASS=MU
      GO TO 58
   56 DWN=MU
      DECID=FE34
      MU=DWN-(DWN-UP)*DECID/(DECID-HY)
      KKK=KKK+1
      EPSC=ABS(ABS(MU)-ABS(LASS))
      IF(EPSC.LT.MU*10.**-4)GO TO 10
      LASS=MU
   58 FE1=ABS(FEPRV)-ABS(FE34)
       IF(ABS(FE1).GT.1.0D-14)GOTO 82
      IA=IA+1
```

```
      IF(IA.LT.5) GO TO 82
      WRITE(6,83)FE34
   83 FORMAT(1H ,'STUCK ON THIS FE34',D12.6)
      IA=0
      FEPRV=0.
      GO TO 53
   82 FEPRV=FE34
      GO TO 99
   42 WRITE(6,43)DECID
   43 FORMAT(1H ,'NO GOOD DECID=',D12.6)
      GO TO 53
   10 WRITE(6,11) FE34,MU
   11 FORMAT(1H0,'MU CONVERGED FE34=',D24.18,3X,'MU=',D12.6)
   53 NATFRQ=MU*SQRT(LAM)
      WRITE(6,54) FE34,MU,LAM,NATFRQ,W
   54 FORMAT(1H0,'FE34',D12.6,5X,'MU=',D12.6,5X,'LAM=',F12.6,
     15X,'NATFRQ=',E12.6,5X,'W=',I3)
      WRITE(6,86) DLT,NOR
   86 FORMAT(1H0,'DLT=',D9.3,5X,'NOR=',I3)
      IF(NOR.EQ.3) GO TO 57
      NOR=NOR+1
      R=0
      U=1
      KKK=1
      MU=MU+DLT
      GO TO 99
   57  MU=1.0D-6
       WRITE(6,100)
  100  FORMAT('1')
      NOR=1
      KKK=1
      DLT=1.
      R=0
      U=1
       GOTO 18
   22  CALL EXIT
      END

END OF UNIVAC 1108 FORTRAN V COMPILATION.     0 *DIAGNOSTIC* MESSAGE(S)
```

```
EQUATORIAL CASE

LAM=  10.000000   SIO= .100+000

P0=SQRT LAM=   3.16228
E34 .170180+001      MU= .100000-005     U=  1
E34 .607250+000      MU= .100000+001     U=  2
KK  1   FE34-.239548+001   MU .200000+001
KK  2   FE34 .134973+000   MU .120223+001
KK  3   FE34 .258975-001   MU .124479+001
KK  4   FE34 .482325-002   MU .125286+001
KK  5   FE34 .893275-003   MU .125436+001
KK  6   FE34 .165264-003   MU .125464+001

U CONVERGED FE34= .165264106829071666-003    MU= .125469+001

E34 .165264-003      MU= .125469+001      LAM=   10.000000     NATFRQ=  .396769+01

L= .100+001        NOR=  1
E34-.337822+001      MU= .225469+001     U=  1
E34-.756432+001      MU= .325469+001     U=  2
E34-.111908+002      MU= .425469+001     U=  3
E34-.126038+002      MU= .525469+001     U=  4
E34-.100439+002      MU= .625469+001     U=  5
E34-.165669+001      MU= .725469+001     U=  6
KK  1   FE34 .132765+002   MU .825469+001
KK  2   FE34-.395210+000   MU .737738+001
KK  3   FE34-.798063-001   MU .740274+001
KK  4   FE34-.159426-001   MU .740783+001
KK  5   FE34-.317792-002   MU .740885+001

U CONVERGED FE34=-.317791832175018385-002    MU= .740905+001

E34-.317792-002      MU= .740905+001      LAM=   10.000000     NATFRQ=  .234295+02

L= .100+001        NOR=  2
E34 .162858+002      MU= .840905+001     U=  1
E34 .403199+002      MU= .940905+001     U=  2
E34 .718955+002      MU= .104091+002     U=  3
E34 .109783+003      MU= .114091+002     U=  4
E34 .151555+003      MU= .124091+002     U=  5
E34 .193483+003      MU= .134091+002     U=  6
E34 .230505+003      MU= .144091+002     U=  7
E34 .256265+003      MU= .154091+002     U=  8
E34 .263256+003      MU= .164091+002     U=  9
E34 .243054+003      MU= .174091+002     U= 10
E34 .186657+003      MU= .184091+002     U= 11
E34 .849254+002      MU= .194091+002     U= 12
KK  1   FE34-.708675+002   MU .204091+002
KK  2   FE34 .722558+001   MU .199542+002
KK  3   FE34 .519382+000   MU .199963+002
KK  4   FE34 .368619-001   MU .199993+002

U CONVERGED FE34= .368618862357834587-001    MU= .199995+002

E34 .368619-001      MU= .199995+002      LAM=   10.000000     NATFRQ=  .632439+02
```

```
ERIDIONAL CASE

LAM=  10.000000   SIO= .100+000

N  0=SQRT LAM=  3.16228
FE34 .284392+001      MU= .100000-005      U=  1
FE34 .170179+001      MU= .100000+001      U=  2
KK   1    FE34-.144094+001    MU .200000+001
KK   2    FE34 .207673+000    MU .154150+001
KK   3    FE34 .173932-001    MU .159926+001
KK   4    FE34 .140389-002    MU .160404+001
KK   5    FE34 .112971-003    MU .160442+001

U CONVERGED FE34= .11297085702363651 7-003    MU= .160446+001

E34 .112971-003       MU= .160446+001      LAM=   10.000000      NATFRQ=  .507373+01

L = .100+001       NOR=  1
E 4-.398435+001       MU= .260446+001      U=  1
E34-.841747+001       MU= .360446+001      U=  2
E 4-.117975+002       MU= .460446+001      U=  3
E 4-.123912+002       MU= .560446+001      U=  4
E34-.842668+001       MU= .660446+001      U=  5
K    1    FE34 .168603+001    MU .760446+001
K    2    FE34-.483714+000    MU .743773+001
KK   3    FE34-.175355-001    MU .747490+001
K    4    FE34-.623739-003    MU .747623+001

U CONVERGED FE34=-.62373942554749817 2-003    MU= .747628+001

E 4-.623739-003       MU= .747628+001      LAM=   10.000000      NATFRQ=  .236421+02

LT= .100+001       NOR=  2
E 4 .164506+002       MU= .847628+001      U=  1
E 4 .407204+002       MU= .947628+001      U=  2
E34 .725760+002       MU= .104763+002      U=  3
E34 .110735+003       MU= .114763+002      U=  4
E 4 .152709+003       MU= .124763+002      U=  5
E34 .194697+003       MU= .134763+002      U=  6
E34 .231561+003       MU= .144763+002      U=  7
E 4 .256871+003       MU= .154763+002      U=  8
E 4 .263048+003       MU= .164763+002      U=  9
E34 .241608+003       MU= .174763+002      U= 10
E 4 .183506+003       MU= .184763+002      U= 11
E 4 .795876+002       MU= .194763+002      U= 12
KK   1    FE34-.788630+002    MU .204763+002
K    2    FE34 .732752+001    MU .199786+002
K    3    FE34 .578262+000    MU .200209+002
KK   4    FE34 .450579-001    MU .200242+002

U CONVERGED FE34= .45057870441269187 7-001    MU= .200245+002

E34 .450579-001       MU= .200245+002      LAM=   10.000000      NATFRQ=  .633229+02

L  .100+001       NOR=  3
```

PROGRAM LISTING

AND

SAMPLE OUTPUT

MØDE

```
C      MODE
C        THIS PROGRAM CALCULATES THE MODE SHAPES
C        GIVEN THE EIGENVALUES,LAMBDA, AND PSI-ZERO
C        REVERSED INTEGRATION METHOD ONLY, CASE E OR CASE M -SPECIFY ON INPUT
C
       IMPLICIT DOUBLE PRECISION(A-H,O-Z)
       INTEGER D,ZI
       DOUBLE PRECISION MU,LAM,MU1
       DOUBLE PRECISION K(4),L(4),M(4),P(4)
       DIMENSION EG3(101),EG4(101),DBPT(101),BPT(101)
      DATA/LE/1HE
C     NPRI=1 FOR PRINTED OUTPUT/NPRI=0 FOR PUNCHED OUTPUT
      READ(5,180) NPRI
  180 FORMAT(I1)
C        MAKE SURE MU IS THE CORRECT ONE FOR EITHER THE E OR M CASE.
      READ(5,93) IE,JZ,LAM,SIO,MU
```

```
 93 FORMAT(A1,I1,3G12.6)
    WRITE(6,101) IE
101 FORMAT(1H1,'CASE',2X,A1,/)
    WRITE(6,95) MU,LAM,SIO,JZ
 95 FORMAT(' MU=',D12.6,' LAM=',D12.6,' SIO=',D12.6,' J=',I2)
    NDS=0
    IF(LE.EQ.IE)NDS=1
     NINT=100
     HH=1./FLOAT(NINT)
     INTER=NINT+1
     H=HH
     ANFRQ=DSQRT(LAM)
     WRITE(6,1)ANFRQ
     FORMAT(' ANFREQ='D10.5)
C
C       D=3  I.C.
C
     E0=1.
     A0=0.
     C0=0.
     B0=0.
     E=1.
     A=0.
     B=0.
     C=0.
     EV=1.
     EG3(1)=1.
C
C       D=3  INTEGRATION
C
     D=3
     N=1
     I=1
     NN=FLOAT(N)
     SI=(NN-1.)*HH
     K(I)=H*A
     L(I)=H*B
     M(I)=H*C
     MU1=MU*MU+1.
     IF(NDS.EQ.1)GOTO 40
     MU1=MU1-1.
40   P1=-(SI*SI)/2.
     P1=B *(P1+SI*(SI0+1.))
     P3=A*((SI0-SI)+1.)
     P4=E*MU1
     P(I)=H*LAM*(P4+P1+P3)
     SI=(NN-1.)*HH
     I=I+1
     IF(I.GT.3)GOTO 61
     ZI=I-1
     E=E0+K(ZI)/2.
     A=A0+L(ZI)/2.
     B=B0+M(ZI)/2.
     C=C0+P(ZI)/2.
     SI=SI+H/2.
     GOTO 5
     IF(I.GT.4)GOTO 7
     E=K(3)+E0
```

```
      A=A0+L(3)
      B=B0+M(3)
      C=C0+P(3)
      SI=SI+H
      GOTO 5
      SI=NN*HH
      EV=EV+(K(1)+K(4)+2.0*(K(2)+K(3)))/6.
      EV1=EV1+(L(1)+L(4)+2.0*(L(2)+L(3)))/6.
      EV2=EV2+(M(1)+M(4)+2.*(M(2)+M(3)))/6.
      EV3=EV3+(P(1)+P(4)+2.*(P(2)+P(3)))/6.
      P1=-SI*SI/2.
      P1=((SI0+1.)*SI+P1)*EV2
      P3=(SI0-SI+1.)*EV1
      P4=MU1*EV
      EV4=(P1+P3+P4)*LAM
      E=EV
      A=EV1
      B=EV2
      C=EV3
      E0=E
      A0=A
      B0=B
      C0=C
C        RUNGA KUTTA FINISHED
      N=N+1
      IF(D.EQ.4)GOTO 70
      EG3(N)=EV
      GOTO 71
70    EG4(N)=EV
71    IF(N.LT.INTER)GOTO 4
      IF(D.EQ.4)GOTO 9
      EMOM3=EV
C        RESET FOR D=4 INTEGRATION
      D=4
         CASE D=4 I.C.
      A0=1.
      E0=0.
      B0=0.
      C0=0.
      A=1.
      E=0.
      B=0.
      C=0.
      EV=0.
      EV2=0.
      EV3=0.
      EV4=0.
      EV1=A0
      N=1
      GOTO 4
9     EMOM4=EV
      ALF=EMOM3/EMOM4
      DO 72 LB=1,101
      LL=102-LB
```

```
      BBPT(LL)=EG3(LB)-ALF*EG4(LB)
72    CONTINUE
      BET=BBPT(101)
      DO 73 LC=1,101
      BPT(LC)=BBPT(LC)/BET
73    CONTINUE
      SM=0.0D0
      DO 216 I=2,101
      SM=SM+(BPT(I)+BPT(I-1))/2.*((FLOAT(I)-1.5)*HH+S10)*HH
216   CONTINUE
      AM2=SM
      SM=0.0D0
      DO 218 I=2,101
      SM=SM+(BPT(I)*BPT(I)+BPT(I-1)*BPT(I-1))/2.D0*HH
218   CONTINUE
      AM1=SM
      COR=AM2*AM2/AM1
     SQOT=AM2/AM1
     WRITE(6,74) AM1,AM2,SQOT,COR
  74 FORMAT(' M1= ',D15.6,3X,' M2= ',D15.6,3X,' M2/M1= ',D15.6,
    13X,' M2**2/M1= ',D15.6)
     IF(NPRI) 181,181,182
 182 CONTINUE
     WRITE(6,76) (BPT(I),I,I=1,101,2)
     GO TO 75
 181 PUNCH 77,(BPT(I),I=1,101,2)
75    CONTINUE
  76 FORMAT(' BPT= ',D12.6,3X,'I= ',I3)
  77 FORMAT(G11.5)
     STOP
      END
```

iPILATION: NO DIAGNOSTICS.

CASE E

MU= .200245+002 LAM= .100000+002 SIO= .100000+000 J= 3
ANFREQ=.31623+001
M1= .248946+000 M2= .282874-001 M2/M1= .113628+000

M2**2/M1= .321424-002

BPT=	.888178-015	I=	1
BPT=	.110301-001	I=	3
BPT=	.430495-001	I=	5
BPT=	.920902-001	I=	7
BPT=	.154254+000	I=	9
BPT=	.225733+000	I=	11
BPT=	.302842+000	I=	13
BPT=	.382053+000	I=	15
BPT=	.460035+000	I=	17
BPT=	.533701+000	I=	19
BPT=	.600246+000	I=	21
BPT=	.657190+000	I=	23
BPT=	.702417+000	I=	25
BPT=	.734203+000	I=	27
BPT=	.751246+000	I=	29
BPT=	.752680+000	I=	31
BPT=	.738089+000	I=	33
BPT=	.707505+000	I=	35
BPT=	.661400+000	I=	37
BPT=	.600672+000	I=	39
BPT=	.526613+000	I=	41
BPT=	.440877+000	I=	43
BPT=	.345438+000	I=	45
BPT=	.242535+000	I=	47
BPT=	.134621+000	I=	49
BPT=	.243010-001	I=	51
BPT=	-.857352-001	I=	53
BPT=	-.192775+000	I=	55
BPT=	-.294150+000	I=	57
BPT=	-.387303+000	I=	59
BPT=	-.469844+000	I=	61
BPT=	-.539613+000	I=	63
BPT=	-.594726+000	I=	65
BPT=	-.633625+000	I=	67
BPT=	-.655103+000	I=	69
BPT=	-.658338+000	I=	71
BPT=	-.642905+000	I=	73
BPT=	-.608779+000	I=	75
BPT=	-.554327+000	I=	77
BPT=	-.486292+000	I=	79
BPT=	-.399758+000	I=	81
BPT=	-.298114+000	I=	83
BPT=	-.182998+000	I=	85
BPT=	-.562353-001	I=	87
BPT=	.802290-001	I=	89
BPT=	.224411+000	I=	91
BPT=	.374373+000	I=	93
BPT=	.528312+000	I=	95
BPT=	.684655+000	I=	97
BPT=	.842156+000	I=	99
BPT=	.100000+001	I=	101

PROGRAM LISTING

AND

SAMPLE OUTPUT

PARAM

```
      TEST OF SUBROUTINE PARAM
      INTEGER PLOTS
      REAL LAM
      DOUBLE PRECISION MU(3),SIO
      COMMON/ONE/LAM,SIO
      COMMON/TWO/MU
      COMMON/FIVE/NAUX2,LH,AUX1
      COMMON/NINE/MODE
      COMMON/GRAF/PLOTS
      READ(5,201) MODE,LAM,SIO,PLOTS
  201 FORMAT(I5,F12.5,G12.6,A3)
      READ(5,202) (MU(K),K=1,MODE)
  202 FORMAT(3D12.6)
      CALL PARAM
      STOP
      END
```

```
      SUBROUTINE PARAM

      SUBROUTINE PARAM IS DESIGNED TO COMPUTE AND OUTPUT PARAMETERS M1,
           M2, M2/M1, AND M2SQUARED/M1 AND THEIR SUMS THROUGHOUT THE
           THE FIRST THREE MODES.    ALSO WILL PLOT OUT THE FIRST THREE
           MODES IF DESIRED.    CALLED THRU MAIN JUST AS CASEM
```

```
C     MU(MODE) IS ENTERED IN COMMON

      COMMON/ONE/LAM,SIO
      COMMON/TWO/MU
       COMMON/FIVE/NAUX2,IB,AUX1
      COMMON/NINE/MODE
      COMMON/GRAF/PLOTS
      REAL NN,LAM
      INTEGER I,D,N,Z,NINT,INTER,PLOTS
      DIMENSION OUTPUT(23),/MU(3)
      DOUBLE PRECISION P(4),K(4),M(4),L(4),E3(101),E31P(101),E32P(101),
     1EMOM3,EMOM4,ALFA,BETA,          MMX3(101),MMX4(101),BPT(101,4),
     1E33P(101),E34P(101),E4(101),E41P(101),E42P(101),E43P(101),
     1E44P(101),A,B,C,E,AO,BO,CO,EO,MU1,MU(3),SIO,SI
      DOUBLE PRECISION AM1(4),AM2(4),SM

      NAUX2IS A PLOT CONTROL PARAMETER
C     NDS=0 FOR MERIDIONAL CASE/NDS=1 FOR EQUATORIAL CASE

      NAUX2=101
       WRITE(6,347) PLOTS
  347 FORMAT(1H1,'PARAM                PLOT=',A6//)
      NOPT=1
       NDS=0
      MODES=3
      NINT=100
      INTER=NINT+1
      DO 34 MODE=1,MODES
   99 SI=0.
      N=1
      H=1./FLOAT(NINT)

C CLEAR ARRAYS

      DO 31 I=1,4
       K(I)=0.
      L(I)=0.
      M(I)=0.
   31 P(I)=0.
      DO 1 I=1,101
      E34P(I)=0.
      E44P(I)=0.
      E33P(I)=0.
      E43P(I)=0.
      E32P(I)=0.
      E42P(I)=0.
      E31P(I)=0.
      E41P(I)=0.
      E4(I)=0.
    1 E3(I)=0.
C
C  THIS SECTION COMPUTES THE FIRST MODE SHAPE AND THEN THE MODE SHAPE
C  PARAMETERS M1AND M2 FOR CASE A
C
      D=3
    8 IF(D.EQ.4) GO TO 2
      EO=0.
      BO=0.
```

```
      IF(NOPT.GT.0) GO TO 12
      E32P(1)=1.
      B0=E32P(1)
      GO TO 13
12    E3(1)=1.
      E0=E3(1)
13    C0=0.
      A0=0.
      GO TO 3
2     A0=0.
      C0=0.
      IF(NOPT.GT.0)GO TO 14
      E43P(1)=1.
      C0=E43P(1)
      GO TO 15
14    E41P(1)=1.
      A0=E41P(1)
15    B0=0.
      E0=0.
3     A=A0
      B=B0
      C=C0
      E=E0
      N=1
9     I=1
      NN=N
      SI=(NN-1.)*H
5     K(I)=H*A
      L(I)=H*B
      M(I)=H*C
      MUI=1.+MU(MODE)*MU(MODE)
      IF(NOS.EQ.1) GO TO 40
      MUI=MUI-1.
40    P(I)=((1.-SI*SI+2.*SI0*(1.-SI))/2.*B-(SI+SI0)*A+MUI*E)*LAM*H
      IF(NOPT.GT.0) P(I)=((-SI*SI+2.*SI*(1.+SI0))/2.*B
     1+(1.-SI+SI0)*A+MUI*E)*LAM*H
      SI=(NN-1.)*H
      I=I+1
      IF(I.GT.3) GO TO 6
      Z=I-1
      E=E0+K(Z)/2.
      A=A0+L(Z)/2.
      B=B0+M(Z)/2.
      C=C0+P(Z)/2.
      SI=SI+H/2.
      GO TO 5
6     IF(I.GT.4) GO TO 7
      E=E0+K(3)
      A=A0+L(3)
      B=B0+M(3)
      C=C0+P(3)
      SI=SI+H
      GO TO 5
7     IF(D.EQ.4) GOTO 9
      SI=NN*H
      Z=N+1
      E3(Z)=E3(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E31P(Z)=E31P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
```

```
      E32P(Z)=E32P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E33P(Z)=E33P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E34P(Z)=LAM*(((SIO+1.)**2-(SI+SIO)**2)*E32P(Z)/2.
     1-(SI+SIO)*E31P(Z)+MU1*E3(Z))
      IF(NOPT.GT.0) E34P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E32P(Z)/2.+(1.-SI+SIO)*E31P(Z)+MU1*E3(Z))
      E=E3(N+1)
      A=E31P(N+1)
      B=E32P(N+1)
      C=E33P(N+1)
      EO=E
      AO=A
      BO=B
      CO=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM3=E32P(INTER)
      IF(NOPT.GT.0) EMOM3=E3(INTER)
      DO 30 I=1,INTER
   30 MMX3(I)=E3(I)
      D=4
      GO TO 8
    9 SI=MN*H
      Z=N+1
      E4(Z)=E4(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E41P(Z)=E41P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E42P(Z)=E42P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E43P(Z)=E43P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E44P(Z)=LAM*(((SIO+1.)**2-(SI+SIO)**2)*E42P(Z)/2.
     1-(SI+SIO)*E41P(Z)+MU1*E4(Z))
      IF(NOPT.GT.0) E44P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E42P(Z)/2.+(1.-SI+SIO)*E41P(Z)+MU1*E4(Z))
      E=E4(N+1)
      A=E41P(N+1)
      B=E42P(N+1)
      C=E43P(N+1)
      EO=E
      AO=A
      BO=B
      CO=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM4=E42P(INTER)
      IF(NOPT.GT.0) EMOM4=E4(INTER)
      DO 32 I=1,INTER
   32 MMX4(I)=E4(I)
      ALFA=EMOM3/EMOM4
      BETA=MMX3(101)-ALFA*MMX4(101)
      IF(NOPT.GT.0) BETA=MMX3(1)-ALFA*MMX4(1)
      DO 102 LB=1,101
      LL=LB
      IF(NOPT.GT.0) LL=102-LB
  102 BPT(LL,MODE)=(MMX3(LB)-ALFA*MMX4(LB))/BETA
      DO 512 LB=1,101
      AUX1=BPT(LB,MODE)
      IF(PLOTS.EQ.'YES') CALL PLOT
    2 CONTINUE
      SM=0.
```

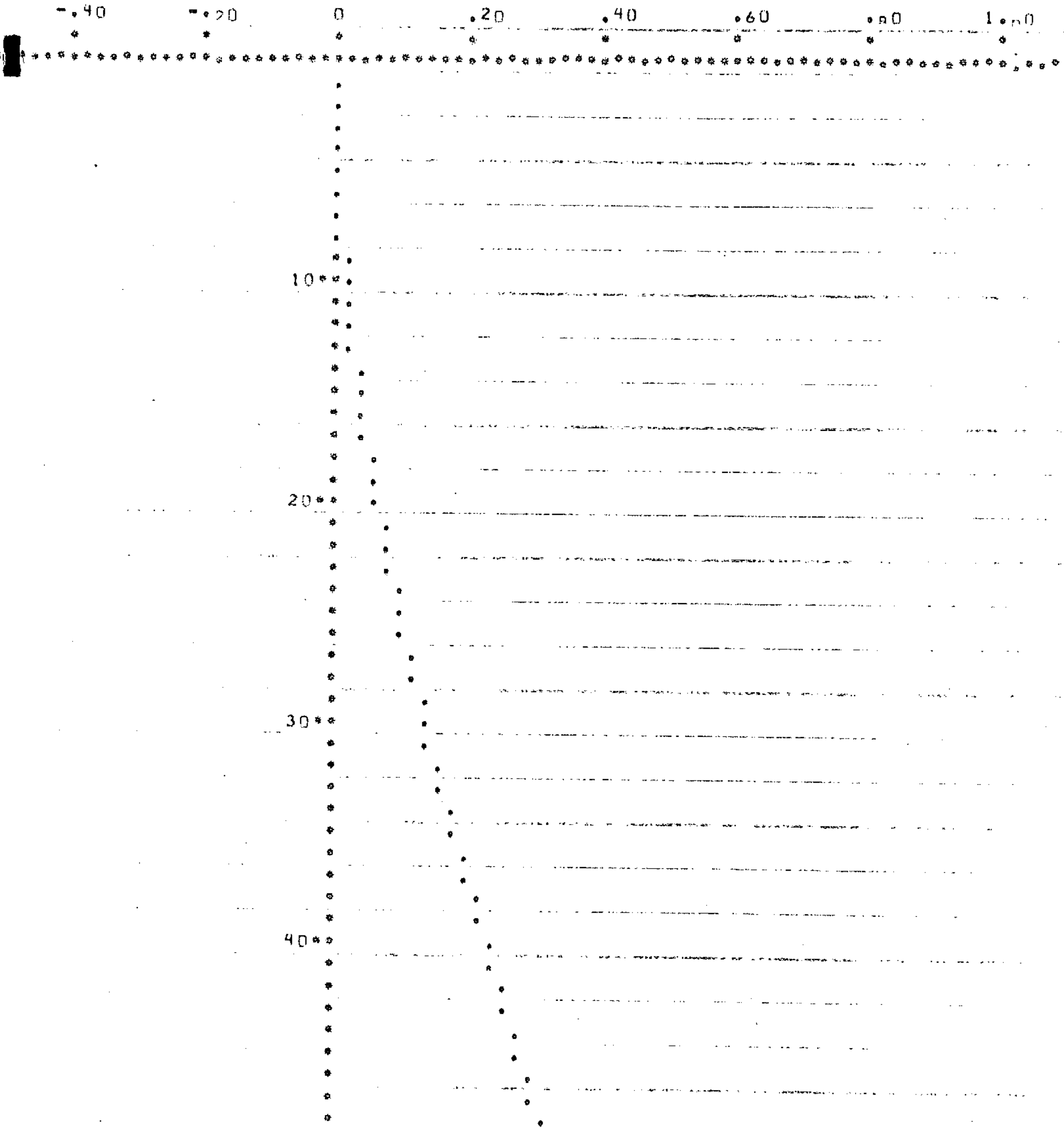
```
      DO 216 I=2,101
  216 SM=SM+(BPT(I,MODE)+BPT(I-1,MODE))/2.*((FLOAT(I)-1.5)*H+SIO)*H
      AM2(MODE)=SM
      SM=0.
      DO 218 I=2,101
  218 SM=SM+(BPT(I,MODE)*BPT(I,MODE)+BPT(I-1,MODE)*BPT(I-1,MODE))/2.*H
      AM1(MODE)=SM
   34 CONTINUE
C
C     END OF MODE SHAPE AND MODE PARAMETER CALCULATION
C
      OUTPUT( 1   )=SIO
      OUTPUT( 2   )=LAM
      OUTPUT( 3   )=MU(1)
      OUTPUT( 4   )=MU(2)
      OUTPUT( 5   )=MU(3)
      ZMU(1)=SQRT(OUTPUT(3)*OUTPUT(3)-1.)
      ZMU(2)=SQRT(OUTPUT(4)*OUTPUT(4)-1.)
      ZMU(3)=SQRT(OUTPUT(5)*OUTPUT(5)-1.)
      OUTPUT( 6)=AM1(1)
      OUTPUT( 7)=AM1(2)
      OUTPUT( 8)=AM1(3)
      OUTPUT( 9)=AM2(1)
      OUTPUT(10)=AM2(2)
      OUTPUT(11)=AM2(3)
      OUTPUT(12)=AM2(1)/AM1(1)
      OUTPUT(13)=AM2(2)/AM1(2)
      OUTPUT(14)=AM2(3)/AM1(3)
      OUTPUT(15)=OUTPUT(12)*AM2(1)
      OUTPUT(16)=OUTPUT(13)*AM2(2)
      OUTPUT(17)=OUTPUT(14)*AM2(3)
      OUTPUT(18)=OUTPUT(12)
      OUTPUT(19)=OUTPUT(18)+OUTPUT(13)
      OUTPUT(20)=OUTPUT(19)+OUTPUT(14)
      OUTPUT(21)=OUTPUT(15)
      OUTPUT(22)=OUTPUT(21)+OUTPUT(16)
      OUTPUT(23)=OUTPUT(22)+OUTPUT(17)
      WRITE(6,501) OUTPUT
  501 FORMAT(1H1,'KSI-ZERO.......................       ',F4.3/1H ,'LAMBDA...
     B.......................',F6.0////,1H ,'MU',1H','S ARE FOR CASE M',
     C/1H ,'MUM1...............................',F8.4/1H ,'MUM2..............
     D............',F8.4/1H ,'MUM3..............................',F8.4/1H ,
     E////1H ,'M 1  MODE 1.......................',F11.8/1H ,'M 1  MODE 2...
     F.................',F11.8/1H ,'M 1  MODE 3.......................',F11.8
     G/1H ,                                    ////1H ,'M 2  MODE 1...
     H.................',F11.8/1H ,'M 2  MODE 2.......................',F11.8
     I/1H ,'M 2  MODE 3.......................',F11.8/1H ,
     J                    ////1H ,'M2 OVER M1  MODE 1...............',F11.8
     K/1H ,'M2 OVER M1  MODE 2..............',F11.8/1H ,'M2 OVER M1  MODE
     L3..............',F11.8/1H ,                                       ///
     M/1H ,'M2 SQUARED OVER M1 MODE 1.....',F11.8/1H ,'M2 SQUARED OVER M
     N1 MODE 2......',F11.8/1H ,'M2 SQUARED OVER M1 MODE 3.....',F11.8/1H
     O ,                                     ////1H ,'SUM OVER 1 MODE O
     PF M2/M1.......',F11.8/1H ,'SUM OVER 2 MODES OF M2/M1.....',F11.8/1H
     Q ,'SUM OVER 3 MODES OF M2/M1......',F11.8/1H ,
     R                 ////1H ,'SUM OVER 1 MODE OF M2*M2/M1....',F11.8/1H
     S ,'SUM OVER 2 MODES OF M2*M2/M1...',F11.8/1H ,'SUM OVER 3 MODES OF
```

```
    TM2.M2/M1..',F11.8/1H ,////)
     WRITE(6,502) (ZMU(I),I=1,3)
 502 FORMAT(1H ,'MUE1..............................',F8.4/1H ,
    X'MUE2............................',F8.4/1H ,
    X'MUE3............................',F8.4)
     RETURN
     END

LATION:          NO  DIAGNOSTICS.
```

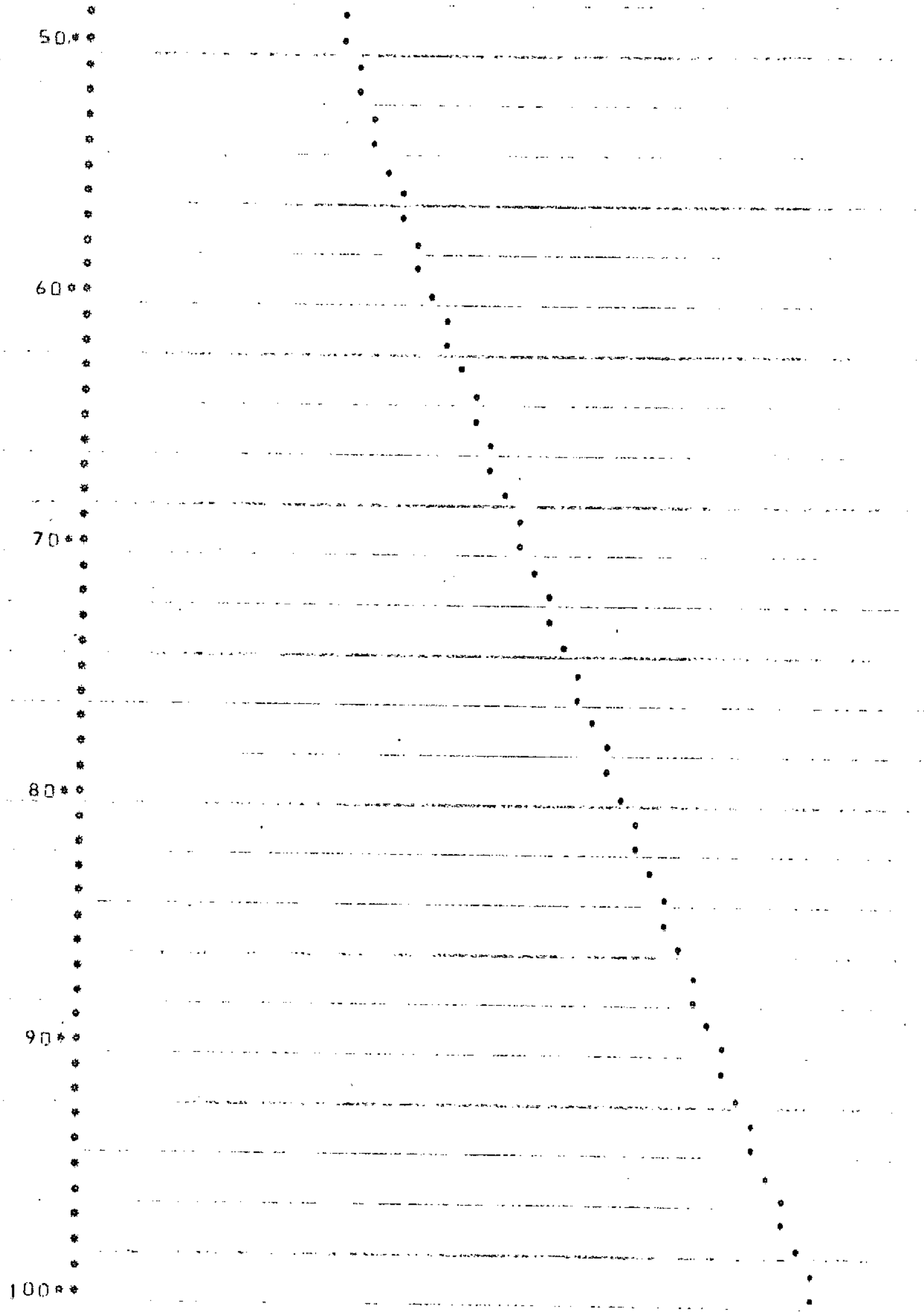
```
                                                                        19:35:04.4
AN V LEVEL 2206 0026       (EXEC8 LEVEL E1201-0011)
S DONE ON 06 SEP 73 AT 19:35:03

T        ENTRY POINT 000341

CODE(1) 000347; DATA(0) 000425; BLANK COMMON(2) 000000

:

00006
00003
000001

ES (BLOCK, NAME)



ENT   (BLOCK, TYPE, RELATIVE LOCATION, NAME)

1 L        0001     000013 121G        0000     000402 14F         0001     000320 15L       0
164G       0001     000131 171G        0001     000137 177G        0001     000022 2L        0
2 5G       0001     000205 223G        0001     000213 230G        0001     000220 235G      0
3          0000     000347 4F          0000     000370 7F          0000     000377 9F        0
AN         0000 R   000323 AP          0000 R   000344 A1          0000 R   000336 BLANK     0
D T        0003     000003 GAMA        0000     000414 INJP$       0000 I   000342 I1        0
K          0003 R   000000 LAM         0000 R   000001 LINE        0000 R   000000 MAX       0
MODF       0000 I   000346 M1          0004 I   000001 N           0000 I   000340 N1        0
PKY        0000 R   000157 SAVE        0003 D   000001 SI0         0000 R   000343 SI0R      0


      SUBROUTINE PLOT
      THIS SUBROUTINE PLOTS NUTATION ANGLE VS N
      COMMON/ONE/LAM,SI0,GAMA,PKX,PKY
      COMMON/FIVE/MK,N,CA
      COMMON/NINE/MODE
      DOUBLE PRECISION SI0
      REAL MAX,LAM,LINE
      DIMENSION SAVE(100),LINE(110),AP(5),AN(5)
      DATA BLANK,STAR,DOT/1H ,1H*,1H./
      IF(N.NE.1) GO TO 2
      N1=(MK+50)/100
      DO 1 J1=1,100
      SAVE(J1)=0.
      J1=0
      MAX=0.
      I1=0
      SI0R=SI0
      IF(N/N1*N1.NE.N) GO TO 3
```

```
      JI=JI+1
      SAVE(JI)=CA
      IF(ABS(CA).GT.MAX) MAX=ABS(CA)
 3    IF(N.NE.NK) RETURN
      WRITE (6,4)
 4    FORMAT(1H1,'PLOT OF MODESHAPE FOR')
      WRITE(6,95)LAM,SIOR,MODE
 95   FORMAT(1H ,'LAMBDA=',F6.0,/1H ,'SI-ZERO=',F4.2,/1H ,'MODE=',I2///)
      AI=MAX/50.
      DO 6 II=1,5
      AN(II)=-AI*(60.-10.*II)
 6    AP(6-II)=-AN(II)
      II=0
      WRITE(6,7) AN,II,AP
 7    FORMAT(1H ,14X,5(F6.2,4X),3X,I1,2X,5(5X,F5.2))
      DO 8 JI=1,110
      LINE(JI)=BLANK
 8    IF((JI+4)/10*10.EQ.(JI+4)) LINE(JI)=STAR
      WRITE(6,9) LINE
 9    FORMAT(1H ,12X,110A1)
      DO 10 JI=1,110
 10   LINE(JI)=STAR
      WRITE(6,9)LINE
      DO 11 JI=1,110
 11   LINE(JI)=BLANK
      DO 13 KI=1,100
      JI=SAVE(KI)/AI+56.5
      LINE(56)=STAR
      IF(KI/10*10.NE.KI) GO TO 12
      LINE(55)=STAR
 12   LINE(JI)=DOT
      WRITE(6,9) LINE
      IF(KI/10*10.NE.KI) GO TO 15
      IF(JI.GE.50.AND.JI.LE.54) GO TO 15
      MI=NI*KI
      WRITE(6,14) MI
 14   FORMAT(1H+,61X,I5)
 15   LINE(JI) =BLANK
 13   LINE(55)=BLANK
      RETURN
      END

  LATION:         NO  DIAGNOSTICS.
```

PLOT OF MODESHAPE FOR
LAMBDA= 10.
SI-ZERO= .10
MODE= 1

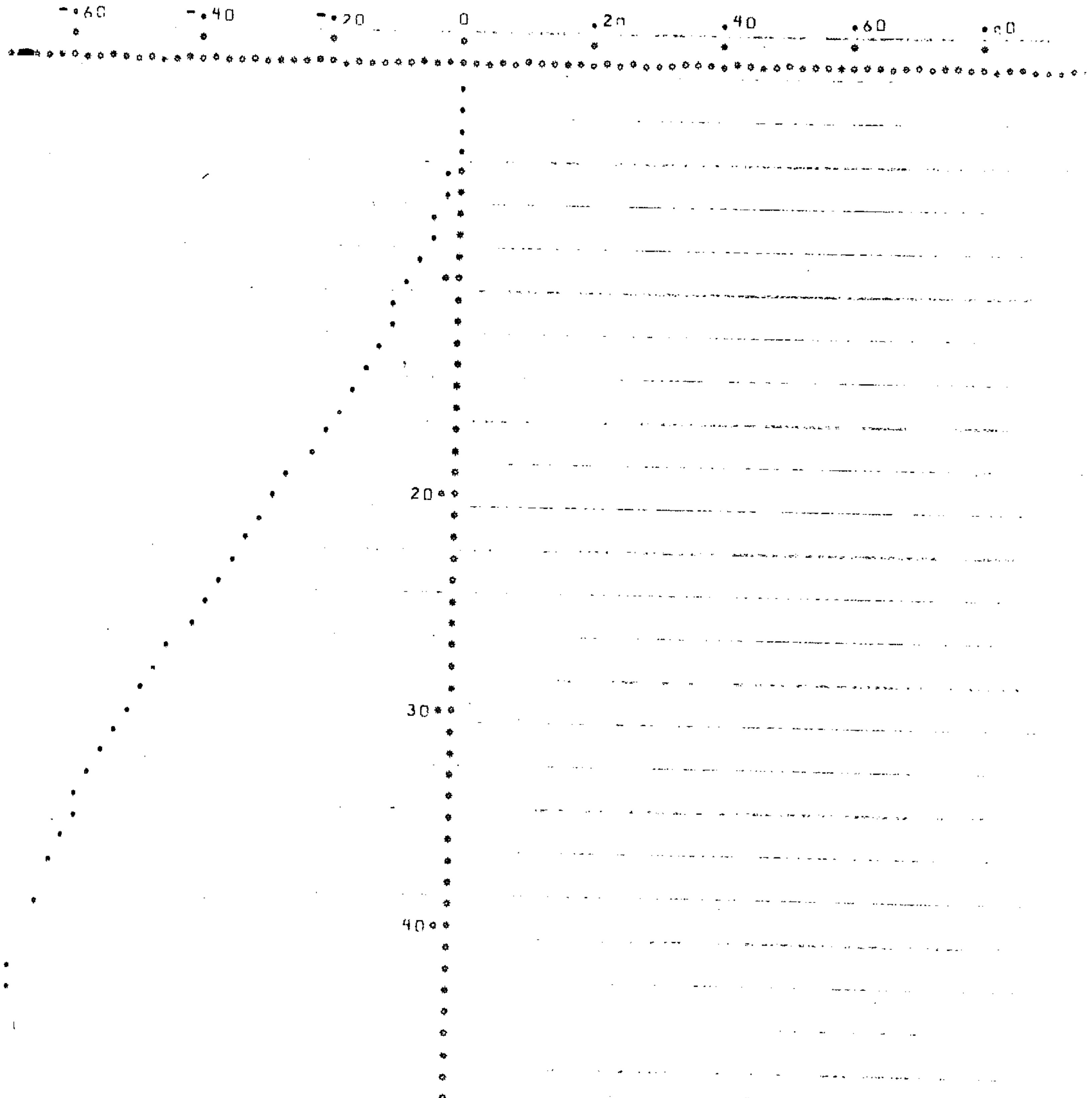


FIRST MODE

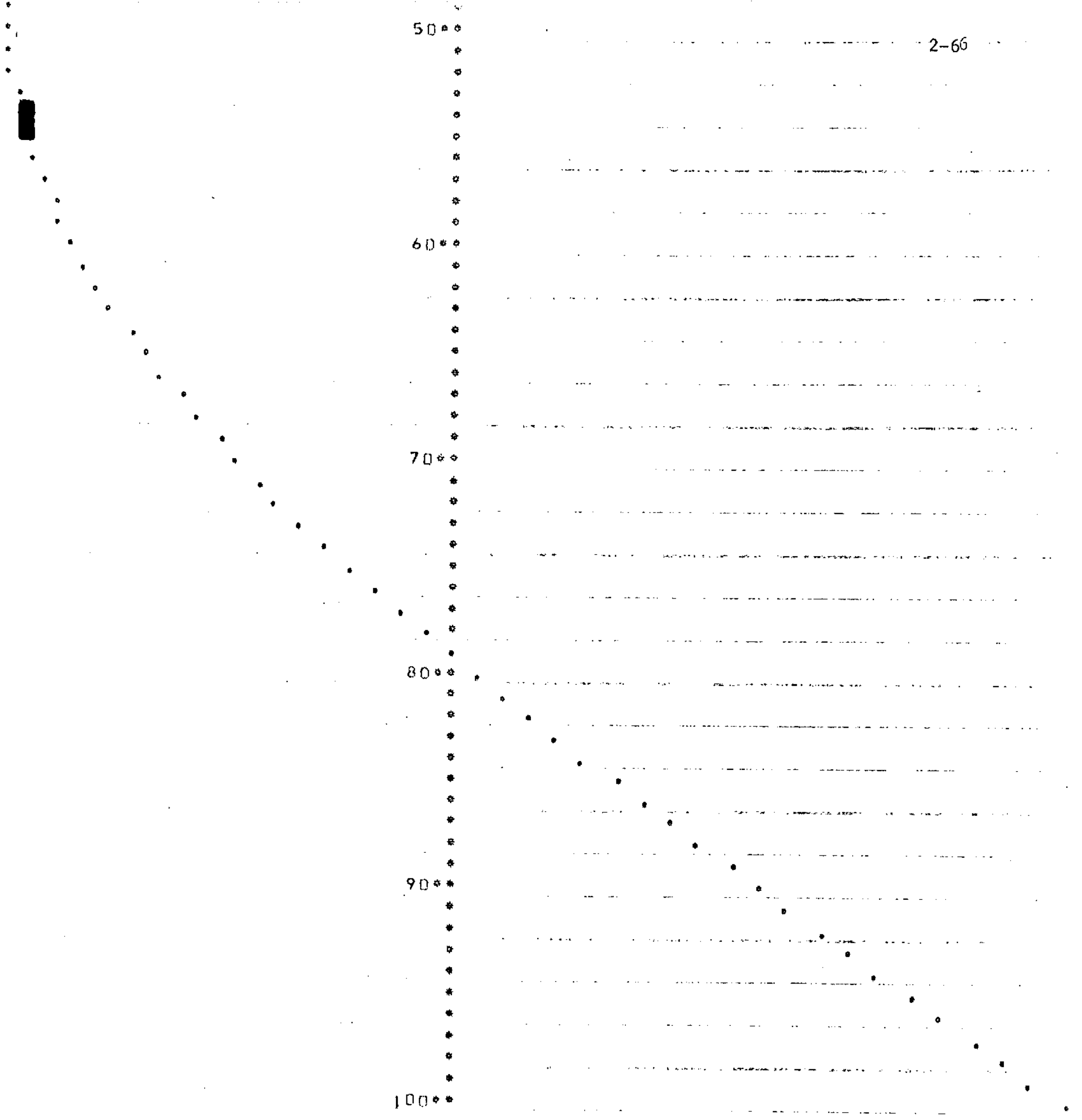


FIRST MODE(CONTINUED)

PLOT OF MODESHAPE FOR
LAMBDA= 10.
SI-ZERO= .10
MODE= 2

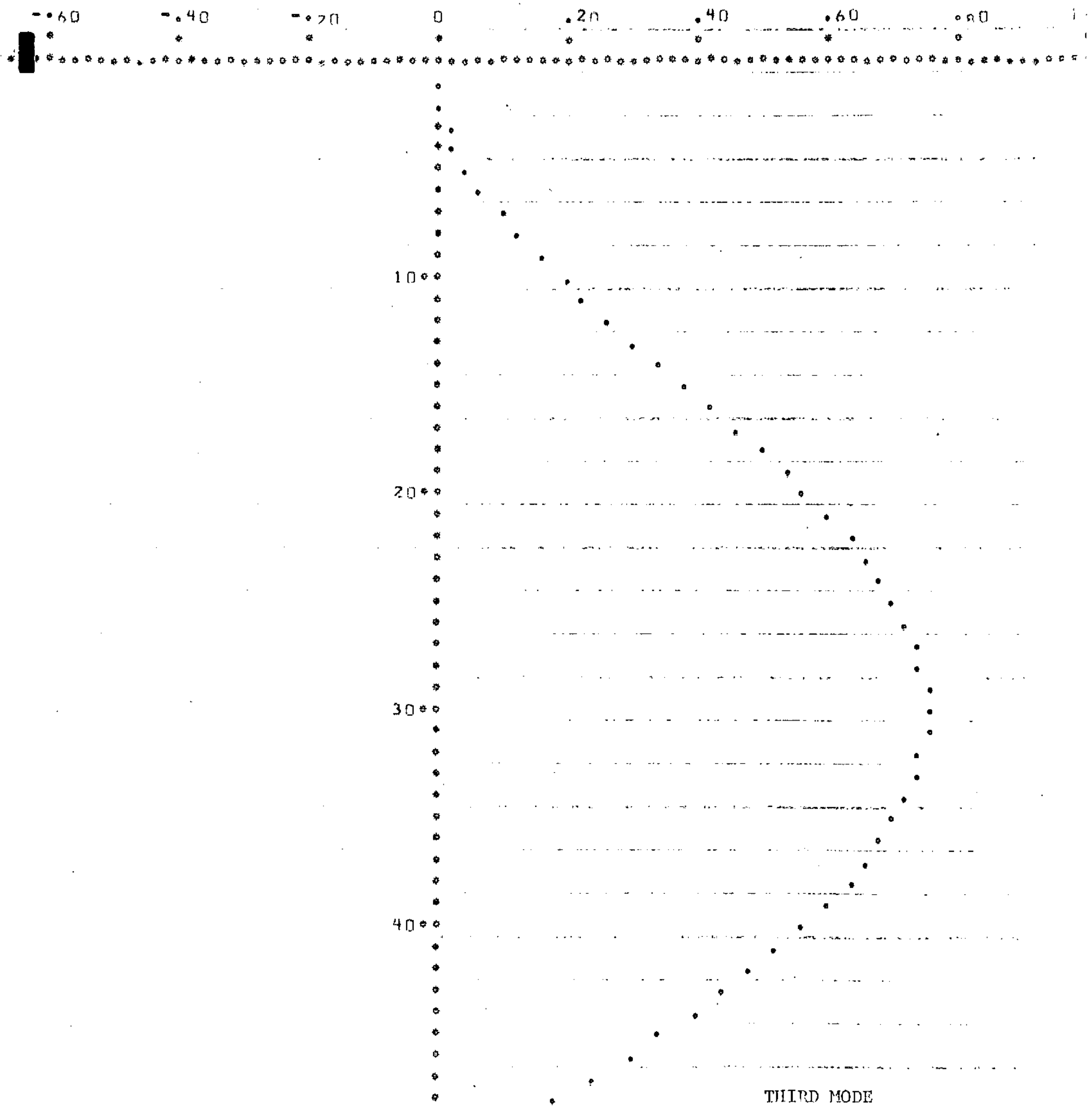


SECOND MODE

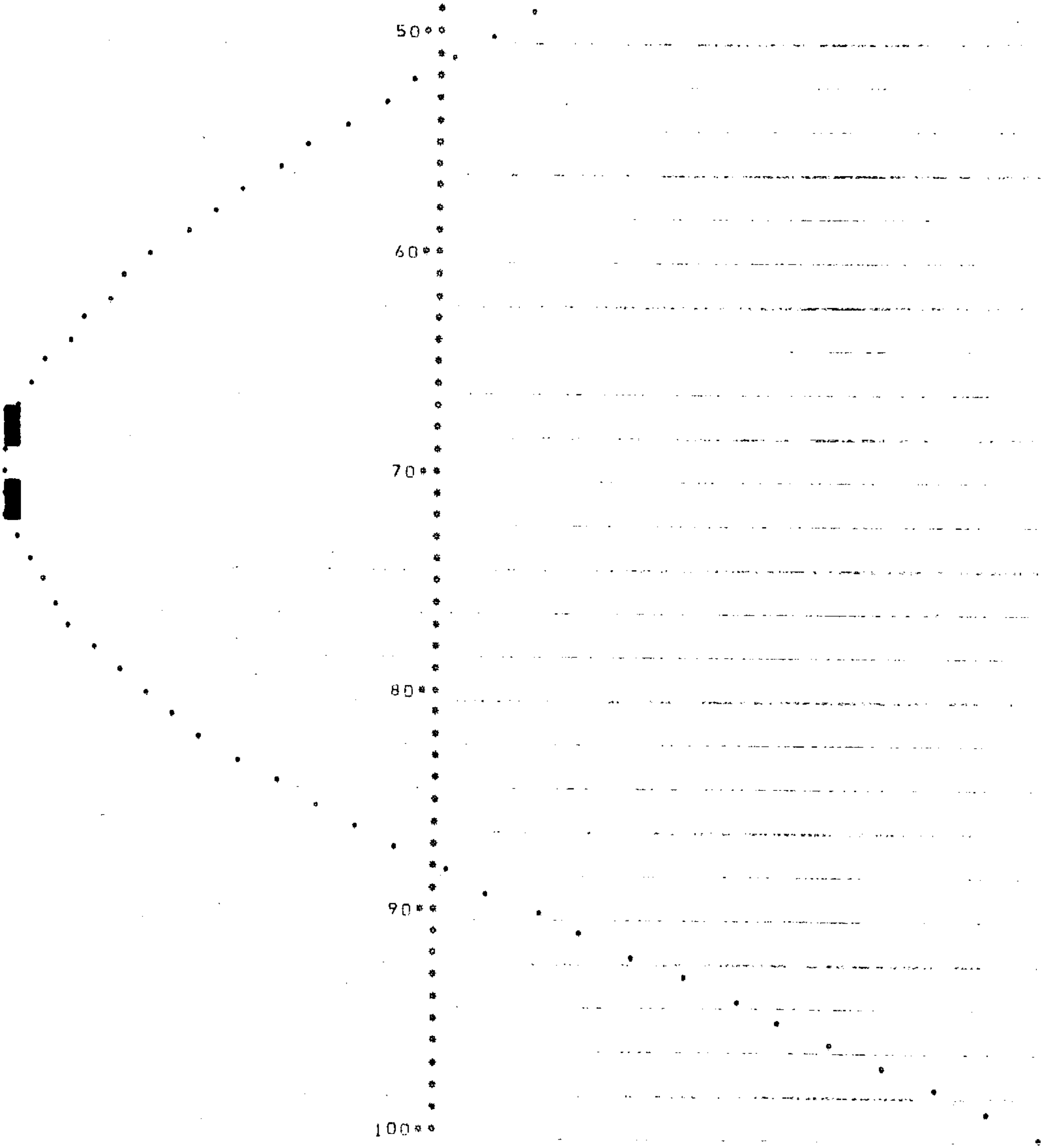


SECOND MODE(CONTINUED)

PLOT OF MODESHAPE FOR
LAMBDA= 10.
SI-ZERO= .10
MODE= 3



THIRD MODE



THIRD MODE(CONTINUED)

```
KSI-ZERO.......................... .100
LAMBDA............................ 10.

MU'S ARE FOR CASE M
MUM1.............................. 1.6045
MUM2.............................. 7.4763
MUM3.............................. 20.0245

M 1  MODE 1....................... .25606434
M 1  MODE 2....................... .24199705
M 1  MODE 3....................... .24910518

M 2  MODE 1....................... .32841758
M 2  MODE 2....................... -.06161919
M 2  MODE 3....................... .02823317

M2 OVER M1  MODE 1................ 1.28255880
M2 OVER M1  MODE 2................ -.25462787
M2 OVER M1  MODE 3................ .11333833

M2 SQUARED OVER M1 MODE 1......... .42121486
M2 SQUARED OVER M1 MODE 2......... .01568996
M2 SQUARED OVER M1 MODE 3......... .00319990

SUM OVER 1 MODE OF M2/M1.......... 1.28255880
SUM OVER 2 MODES OF M2/M1......... 1.02793093
SUM OVER 3 MODES OF M2/M1......... 1.14126925

SUM OVER 1 MODE OF M2*M2/M1....... .42121486
SUM OVER 2 MODES OF M2*M2/M1...... .43690482
SUM OVER 3 MODES OF M2*M2/M1...... .44010472

MUE1.............................. 1.2547
MUE2.............................. 7.4091
MUE3.............................. 19.9995
```

CHAPTER 3

Application to Some Problems of Satellite Dynamics

The present chapter considers the use that can be made of the results of the previous chapter in some problems of interest in satellite dynamics[3-1]. A first field of application is in studying the nutational divergence of a satellite equipped with flexible appenaages, but this is the topic of Chapter 4 and 5. We shall be considering here some other problems, such as the simulation of free oscillations, thermal flutter and variation of the spin rate due to the booms motion.

3.1 Simulation of free oscillations

3.1.1 Generalities

In a motion of type E (equatorial) or M (meridional), the free oscillations can be simulated in the following manner. Given N modes, $\Phi_j(\xi)$, $\xi = 1,\ldots N$, with associated frequencies $\bar{\omega}_j$, and given an initial distribution of displacements and velocities ($t = \frac{t_{dim}}{1/\omega_s}$)

$$\eta(\xi,0) = \eta_0(\xi) \; ; \; \eta_t(\xi,0) = \left(\frac{\partial \eta}{\partial t}\right)_{t=0} = \eta_{t,0}(\xi)$$

the displacement is written as a sum of modes

$$\eta = \sum_{j=1}^{N} \phi_j(\xi)\left(c_j \cos \bar{\omega}_j t + s_j \sin \bar{\omega}_j t\right)$$

$$\eta_t = \sum_{j=1}^{N} \phi_j(\xi)\, \bar{\omega}_j \left(-c_j \sin \bar{\omega}_j t + s_j \cos \bar{\omega}_j t\right)$$

Then

$$c_j = \frac{1}{m_{1,j}} \int_{boom} \eta_0 \, \phi_j(\xi)\, d\xi$$

$$s_j = \frac{1}{m_{1,j}\, \bar{\omega}_j} \int_{boom} \eta_{t,0}\, \phi_j(\xi)\, d\xi$$

As an example, Figures 3.1 and 3.2 are meant to illustrate that starting with an initial shape identical to the first mode at $\bar{\lambda} = 10$, $\xi_0 = 0.1$, with no initial velocities, the stationary wave which exists in this case cannot be maintained if $\bar{\lambda}$ is changed to 100. Not only has the frequency changed appreciably (τ_i is the period of the first mode oscillation for $\bar{\lambda} = 10$, $\xi_0 = 0.1$), but the second mode is present to an appreciable extent.

3.1.2 Application to Satellite UK-4

From data received through NASA GSFC on satellite UK-4, we computed the eigenfrequencies and modal shapes for satellite UK-4. This satellite has the following physical characteristics:

UK-4 Computations

ω_s = 30 rpm; 15 rpm; 6 rpm.

ρ = 0.00058 lb mass/in

$$= \frac{5.8 \times 10^{-4}}{2.54} \times 10^2 \times .45359 \text{ kg/m}$$

$$= 1.036 \times 10^{-2} \text{ kg/m}$$

$EI = 10^3$ lbf x in^2

$= 10^3 \times 4.448$ newton in^2

$= 10^3 \times 4.448 \times 2.54^2 \times 10^{-4}$ newton m^2

$= 2.869$ newton -m^2

$x_o = 11.6$ inches

$\ell = 276$ inches $= 7.01$ m

$\xi_o = \frac{x_o}{\ell} = .042$

$I_{zh} = 18.348$ slug ft^2

1 slug = 14.5938 kg mass

$I_{zh} = 18.348 \times 32.1741 \times .4539$ kg ft^2

$= 24.876$ kgm^2

$I_{xh} = 17.41$ slug ft^2

$I_{yh} = 16.54$ slug ft^2

ETKIN'S NUMBER: $\bar{\lambda} = \omega_s{}^2 \; \frac{\rho \ell^4}{EI}$

$$\bar{\lambda}_{30\ \text{rpm}} = \left(\frac{30 \times 2\pi}{60}\right)^2 \times 1.036 \times 10^{-2} \times \frac{(7.01)^4}{2.869}$$

$$= (3.141592)^2 \times 1.036 \times 10^{-2} \times 7.01^4/2.869$$

$$= 86.06$$

$$\bar{\lambda}_{15rpm} = \frac{1}{4}(\bar{\lambda})_{30rpm} = \frac{86.06}{4} = 21.515$$

$$\bar{\lambda}_{6rpm} = \frac{86.06}{25} = 3.44$$

Data for programs:

$\bar{\lambda}$ = 1; 3.44; 10; 16.8; 21.515; 50; 86.06; 100

ξ_e = .042

$\bar{I}_p$ determined from

$$\left[\left(\frac{I_{zh}}{I_{xh}} - 1\right)\left(\frac{I_{zh}}{I_{yh}} - 1\right)\right]^{1/2} = \frac{I_z}{\bar{I}_p} - 1$$

giving

$$\frac{I_{zh}}{\bar{I}_p} = 1.0767$$

Results (see graphs)

Graph 1: Resonance on thermal flutter at $\sqrt{\bar{\lambda}} = .4$ or $\bar{\lambda} = 16.$, i.e. at spin rate

$\omega_s = \omega_{1,rot} = 1.35$ rad/sec = 12.9 rpm

Graph 2 : Mode shapes

ω_s=6;15;30 r.p.m.

SATELLITE UK4: ATTITUDE STABILITY

Table I: Case M

ω(rpm)	6	10.2	13.25	15	22.8	30	32.3
$\bar{\lambda}$	3.44	10.0	16.8	21.515	50	86.06	100.0
$\sqrt{\bar{\lambda}}$	1.84	3.162	4.1	4.64	7.07	9.28	10.0
$\omega_{n_1}\sqrt{\frac{\rho\ell^4}{EI}}$	4.08	4.98	5.76	6.24	8.56	10.75	11.48
ω_{n_1}	1.38	1.685	1.945	2.11	2.895	3.63	3.88

Table II: Case E

ω(rpm)	3.23	6.0	10.2	13.25	15	22.8	30	32.3
$\bar{\lambda}$	1.0	3.44	10.0	16.8	21.515	50	86.06	100.0
$\sqrt{\bar{\lambda}}$	1.0	1.84	3.162	4.1	4.64	7.07	9.28	10.0
$\omega_{n1}\sqrt{\rho\ell^4/EI}$	3.55	3.64	3.85	4.05	4.18	4.82	5.44	5.64
ω_{n_1}(Hz)	1.2	1.23	1.30	1.37	1.415	1.63	1.84	1.905

$\sqrt{\rho\ell^4/EI} = \sqrt{8.715} = 2.96$

Resonance on Thermal Flutter at $\sqrt{\bar{\lambda}} = 4.00$, $\bar{\lambda} = 16.0$

$$\omega_s = \omega_n = 4.0/2.96 = 1.35\text{(Hz)}$$
$$= 12.90 \text{ rpm.}$$

$K_p > 1$ No posigrade resonance

No nutational instability

Var. of spin rate for 10% defl.

.57%	30 RPM
.785%	15 RPM

Fig. 3.3 represents the first mode of vibration for the three values of the spin rate being contemplated. Centrifugal effects are noted as Etkin's number $\bar{\lambda}$ is increased.

3.2 Resonant thermal flutter

3.2.1 Determination of resonant frequency

It has been shown by Etkins and Hughes[3-2] that assuming a relatively simple model for the boom's thermal curvature $\bar{\kappa}_{TO}$ (independent of ξ) due to the sun's heat input during the spinning motion, the steady-state oscillation of the booms would be described by

$$\eta = \bar{\kappa}_{T_0} \cos \omega_s (t - t_0)$$

In order to find for which spin rate ω_s the motion will diverge (have an amplitude tending to infinity), these authors solved equation for boundary conditions,

$$E(0) = 0 \quad E'(0) = 0 \quad E''(0) = \bar{\kappa}_{TO} \quad E'''(0) = 0$$

and vary $\bar{\lambda}$ until very large values of $\Phi(1)$ are observed. The analysis was limited to satellites of zero radius.

An alternative approach was proposed[3-1], which is recalled here. If in Equation (2.2- 8), we let ω_1 tend continuously to ω_s along the eigenfrequencies curves $\bar{\omega}_1(\bar{\lambda}, \xi_0)$; the spatial part of a solution to Equation (2.2-7), normalized to unity at the tip, satisfies b.c.

$$\phi(0) = 0 \quad \phi'(0) = 0 \quad \phi''(\ ) = 0 \quad \phi'''(1) = 0$$

In order to also admit boundary conditions 0,0 for the zeroth and first derivatives at $\xi = 0, \bar{\mathcal{K}}_{TO}$, 0 for the second the third derivatives at $\xi = 1$, $\bar{\phi}(\bar{\lambda}, \xi_o)$ should be scaled up by an infinite factor, i.e., the amplitude at the tip tends to infinity. Thus, resonance on thermal flutter will correspond to the intersection of the curve, for given ξ_o

$$\omega_1 \left(\frac{\rho \ell^4}{EI} \right)^{1/2} = f\left(\sqrt{\bar{\lambda}}\right)$$

with the bisectrix of the first quadrant (Fig. 3.4)

$$\frac{\omega_1}{\omega_s} = 1$$

No thermal flutter resonance can occur for

a) $\xi_o \gtrsim 0.7$

b) second or higher modes

as is shown on Fig. 2 . 2.

3.2.2 Application to UK-4

Using the above data for UK-4, the thermal flutter resonance point was found at (Fig. 3.5)

$$\bar{\lambda} = 16.0 \qquad (\xi_o = 0.042)$$

and for the physical characteristics of the satellite, this translates to

$$\omega_{s,\,critical} = 1.35\ Hz = 12.90\ r.p.m.$$

a spin rate to be avoided for steady-state operation.

3.3 Variation of the spin rate due to the free oscillations

3.3.1 Method of calculation

It is often of interest to satellite users to know what amount of spin rate variation can be expected, due to the vibrations of the boom. The equatorial vibrations will cause a very slight variation of the spin rate described by

$$I_{hub} \frac{d\omega_s}{dt} = T$$

where T is a torque due to the moment at the oot of t boom and to the she r force acting through the central hub radius. This is described in non-dimensional form by ($\bar{\tau} \equiv \omega_* t$)

$$\frac{d\bar{\omega}}{d\bar{\tau}} = \frac{1}{I_{hub}\,\omega_{NR}^{4}} \frac{EI}{\ell} \eta_{max} \left[\eta_{\xi\xi} - \xi_0 \eta_{\xi\xi\xi}\right]_{\xi=0}$$

$$= \frac{\rho \ell^3}{I_{hub}} \frac{1}{\sqrt{\bar{\lambda}}} \eta_{max} \bar{T}$$

$$\bar{T} \underset{def}{\equiv} \left[\eta_{\xi\xi} - \xi_0 \eta_{\xi\xi\xi}\right]_{\xi=0}$$

or, after integration, and with $\Gamma \underset{def}{\equiv} \dfrac{\rho \ell^3}{I_{hub}}$,

$$\left[\frac{\Delta\omega_s}{\omega_s}\right]_{max} \frac{1}{\Gamma\,\eta_{max}} = \frac{1}{\sqrt{\bar{\lambda}}} \int_0^{\bar{\tau}_*} \bar{T}\, d\bar{\tau} \qquad (3.3\text{-}1)$$

in which $\bar{\tau}^*$ is the value of $\bar{\tau}$ maximizing the integral. This value can be obtained using a program such as SIM, which is listed at the end of this chapter.

3.3.2 Application to UK-4

Using the above data for satellite UK-4, the maximum variation of the spin rate for an assumed 10% deflection of the boom was determined to be

0.57% at ω_s = 30 r.p.m.

0.755% at ω_s = 15 r.p.m.

REFERENCES - Chapter 3

[3-1] RENARD, M.L. and RAKOWSKI, J.E.: "Equatorial Vibrations of a Long Flexible Boom on a Spin-Stabilized Satellite of Non-Zero Radius," Proc. of the Astronautical Congress, October 1969. Vol. 1, pp. 35-53, E. Lunc (Editor), Pergamon Press, 1971.

[3-2] ETKIN, B. and HUGHES, P.C.: "Explanation of the anomolous spin behavior of satellites with long flexible antennae," Jour. of Spacecraft and Rockets, 4, 9, 1139-1145.

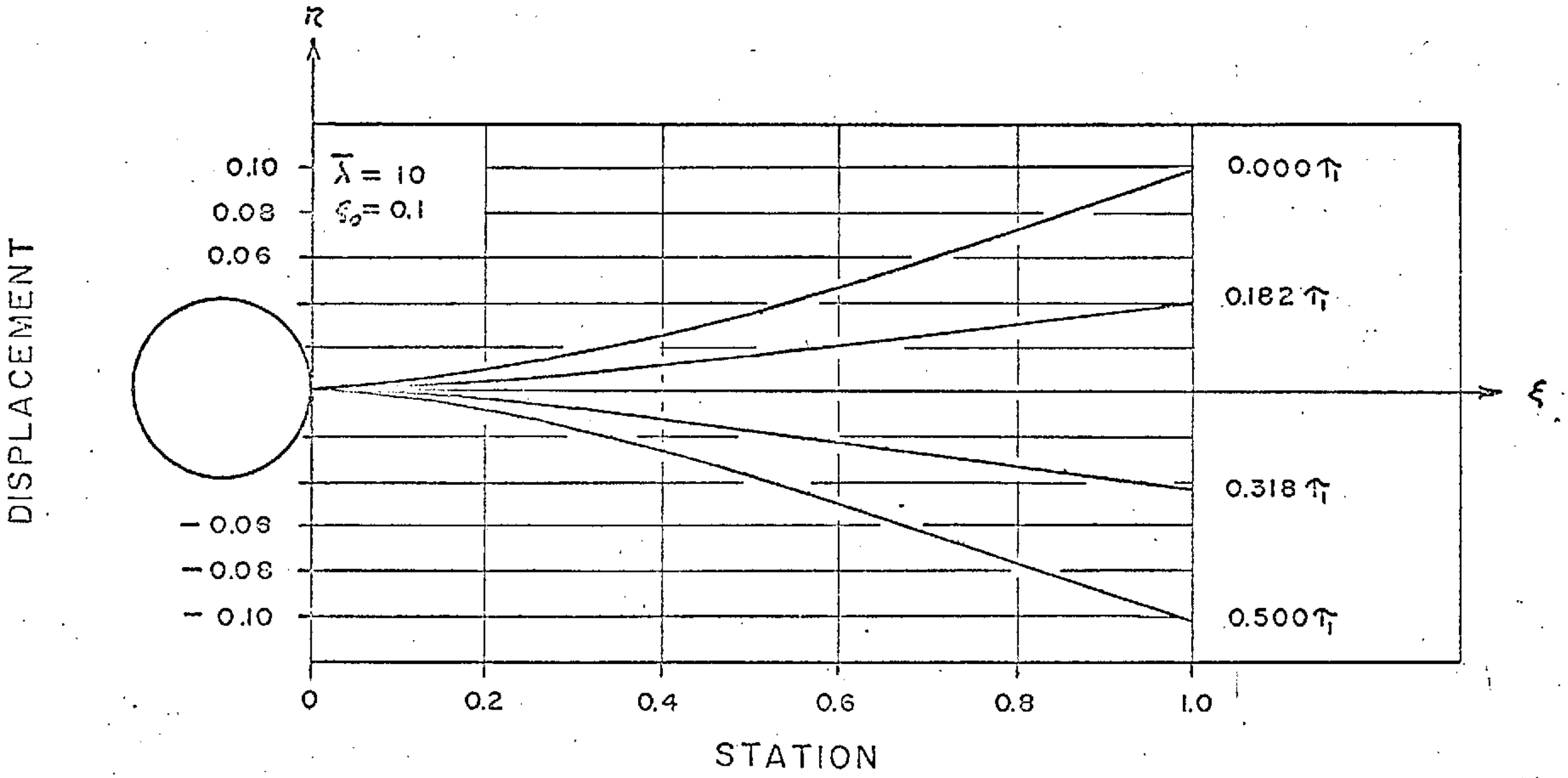


FIG. 3-1. FIRST MODE VIBRATION IN ROTATING AXES FOR THE REFERENCE CASE.

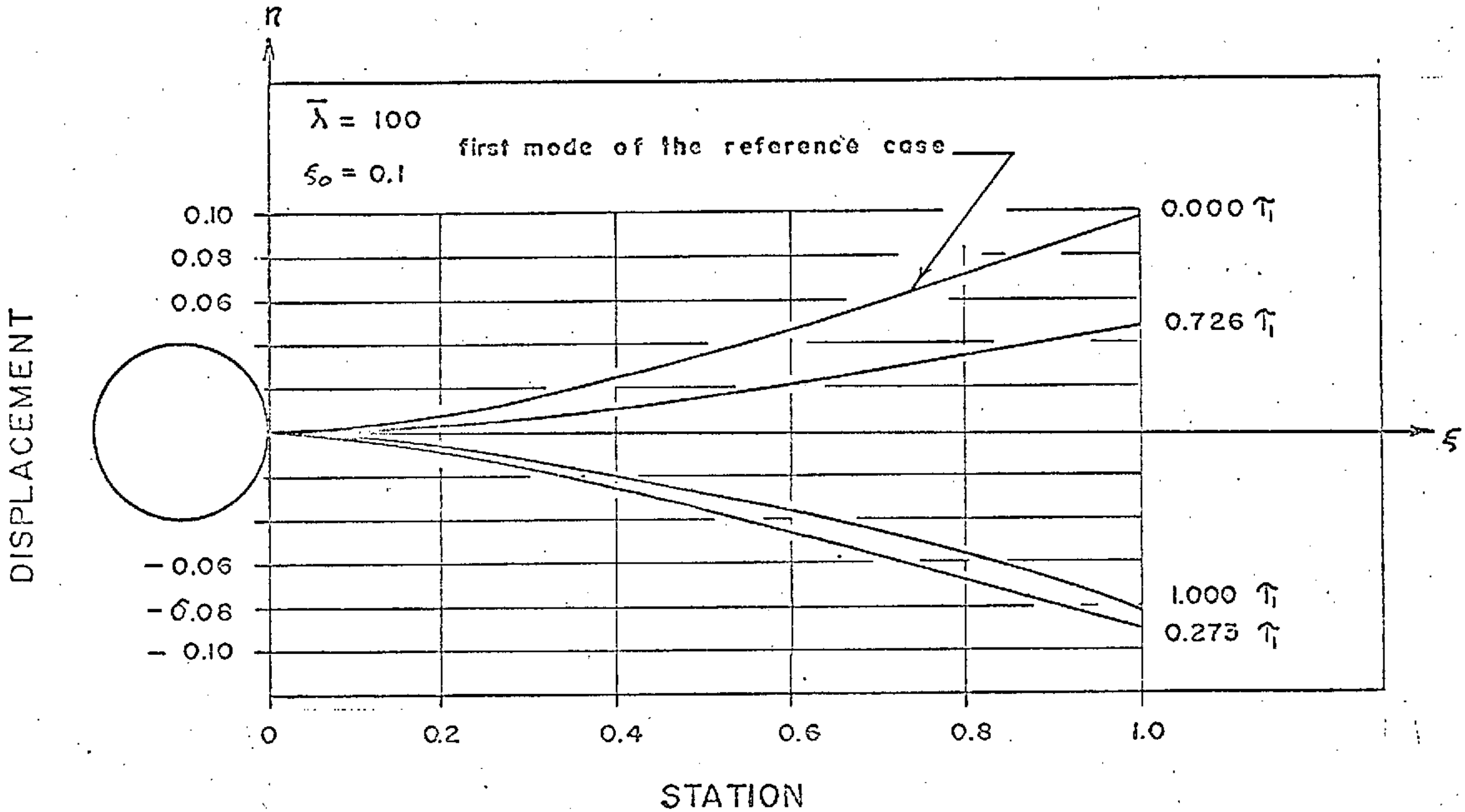


FIG. 3-2. VIBRATION IN ROTATING AXES WITH THE REFERENCE CASE FIRST MODE AS INITIAL SHAPE

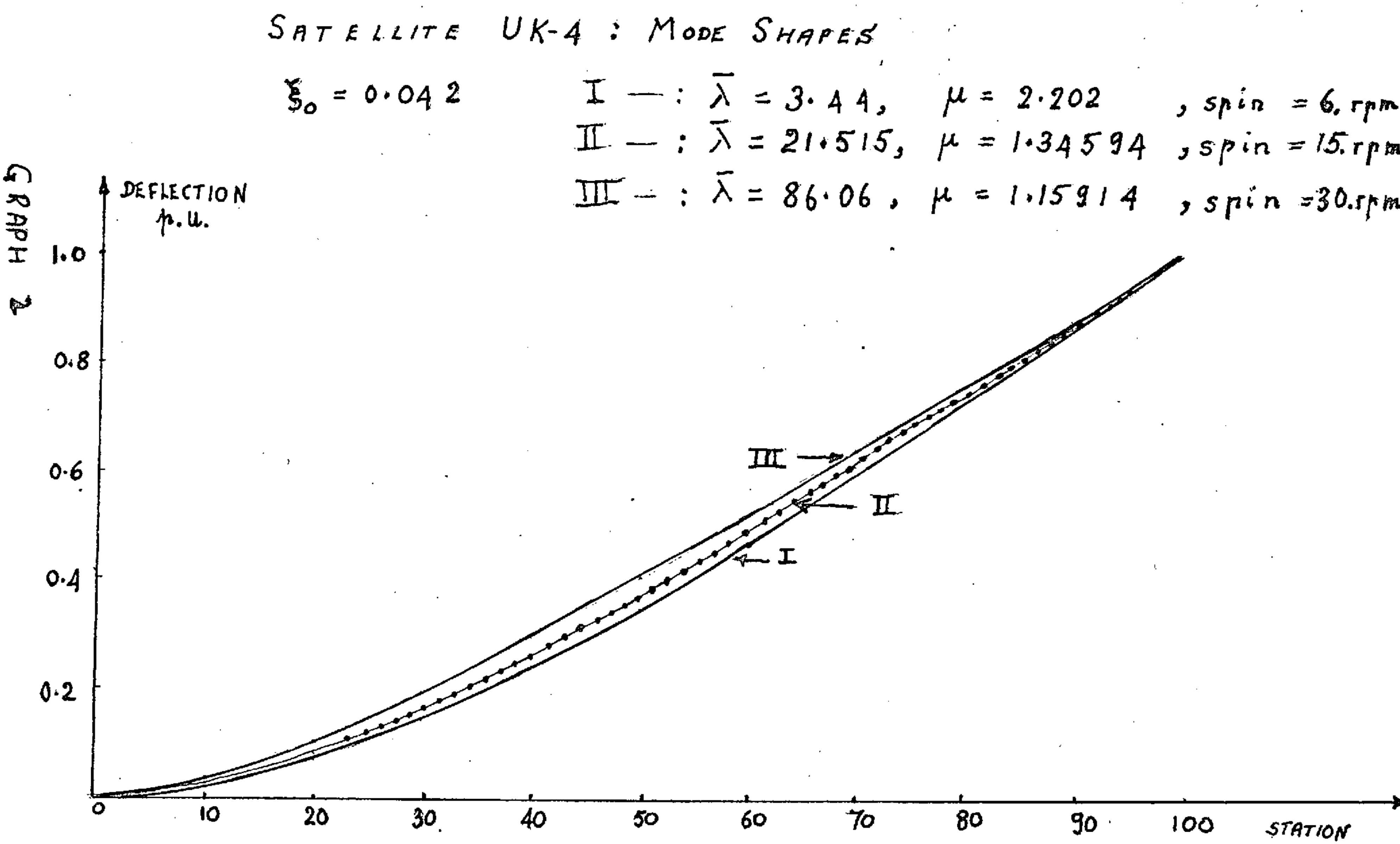


FIG. 3-3. FIRST MODE SHAPE FOR UK-4.

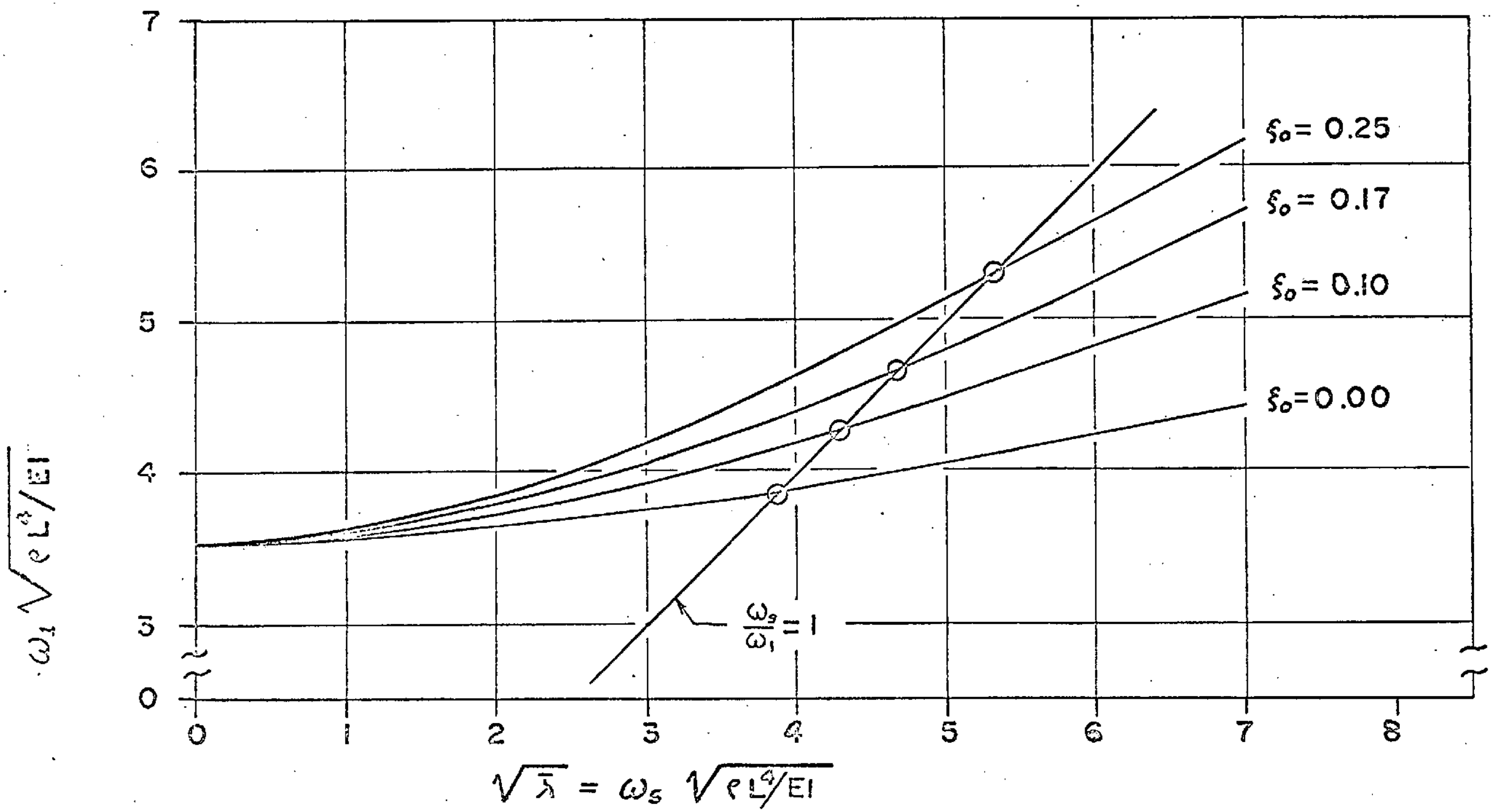


FIG. 3-4 RESONANT VALUES OF THE FUNDAMENTAL FREQUENCY

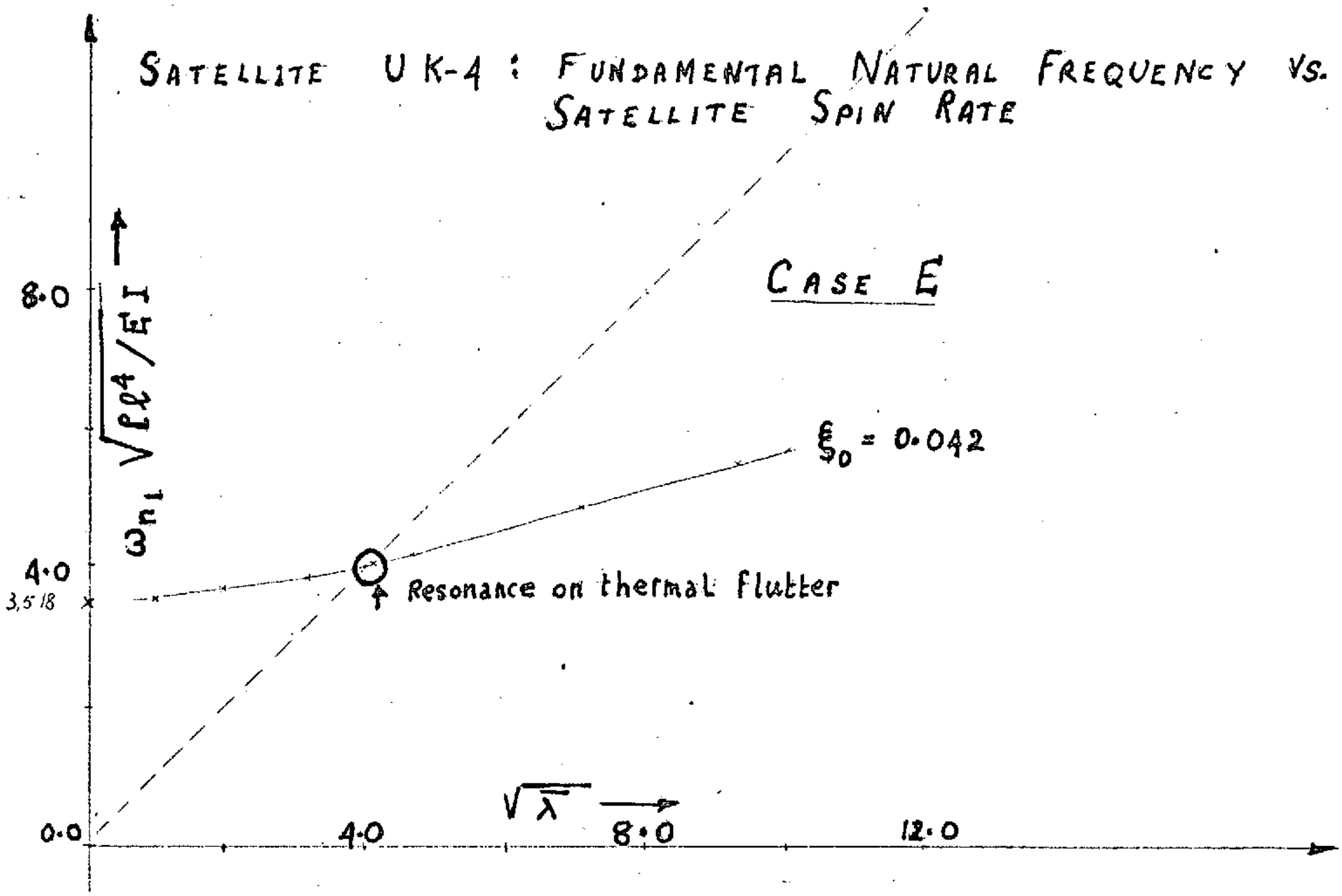


FIG. 3-5. RESONANCE ON THERMAL FLUTTER: UK-4.

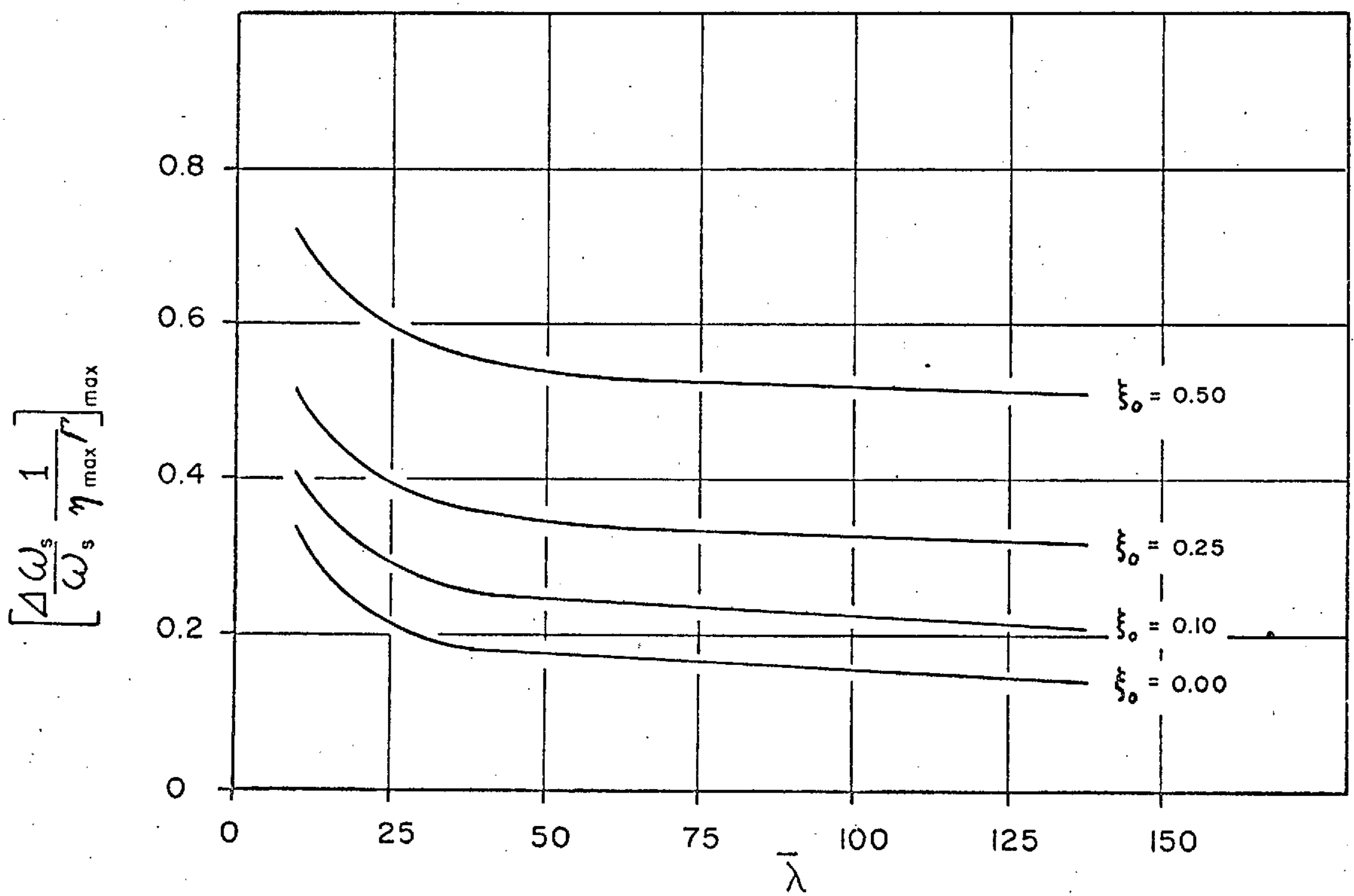


FIG. 3-6

SPIN VARIATION versus $\bar{\lambda}$

PROGRAM SIM

LISTING

```
$DUP E730MR18, ,PRINT,BCD
@ FOR SIM
      SUBROUTINE SIM(NDS)
      DOUBLE PRECISION MU,XNO,YNO
      REAL NU
      INTEGER SAM
C                                    SIM
C
C     THIS PROGRAM COMPUTES THE DISPLACEMENT OF THE FREELY VIBRATING
C     ROTATING BOOM AT SPECIFIED STATIONS, STARTING FROM A GIVEN
C     DISPLACEMENT INITIAL CONDITION
C
C     IT COMPUTES THE NONDIMENSIONAL SPIN VARIATION AND INDUCED NUTATION
C     ANGLE FOR EITHER THE EQUATORIAL OR MERIDIAN BOOM (SEE NDS BELOW)
C
      COMMON/ONE/MU,LAM,SIO,RGAMA,PK,NU,XNO,YNO
      DOUBLE PRECISION COR,ME1,ME2
      DOUBLE PRECISION P(4),K(4),M(4),L(4),E3(101),E31P(101),E32P(101),
     1E33P(101),E34P(101),E4(101),E41P(101),E42P(101),E43P(101),
     3E44P(101),A,B,C,E,AO,BO,CO,EO,
     4SIO,SI
      DOUBLE PRECISION EDA(2),EIG(2)
      REAL H,NN,LAM
      INTEGER I,D,N,Z,  NINT,INTER
      DOUBLE PRECISION ELSM3(2),ELSM4(2)
      DOUBLE PRECISION EMOM3(2),EMOM4(2),ESHR3(2),ESHR4(2),
     1RMOM3(2),RMOM4(2),RSHR3(2),RSHR4(2),BPT(101,2),TFAC(2),
     2COFDIS(2,2),ALFA(2),       MMX3(101,2),MMX4(101,2),BETA(2),MU1
      DIMENSION AA(2,2),BBB(2),CCC(2),GAMA(2),CC2(2)
      INTEGER PPP(2),QQQ(2)
      DOUBLE PRECISION RTORK,DTORK,EDA2P1,EDA3P1
      REAL DLTT,MODE
      GAMA(1)=RGAMA
      GAMA(2)=0.0
      EIG(1)=MU
      EIG(2)=0.
      EDA(1)=XNO
      EDA(2)=YNO
      IEND=1
      MAXI=27
      STP=25.
      NR=2
C*** THIS PROGRAM USES REVERSED INTEGRATION ONLY
      WRITE(6,300)
 300  FORMAT('OSIMULATION ENTERED')
C  NDS = DIRECTION SWITCH
C  NDS = 1 IN PLANE, = 0 OUT OF PLANE
C*** SET IBIG=1 TO READ BIGMODE 1 ONLY
      IVAR=1
      IBIG=1
      IFIRST=1
      KK=1
      N=1
      NINT=100
```

```
      INTER=NINT+1
   74 ANFRQ=SQRT(LAM)
      WRITE(6,60) ANFRQ
   60 FORMAT(1H ,'NFRQ=SQRT LAM=',F10.5)
   99 SI=0.
C
C  CLEAR ARRAYS
C
      DO 31 I=1,4
      K(I)=0.
      L(I)=0.
      M(I)=0.
   31 P(I)=0.
      DO 1 I=1,101
      E34P(I)=0.
      E44P(I)=0.
      E33P(I)=0.
      E43P(I)=0.
      E32P(I)=0.
      E42P(I)=0.
      E31P(I)=0.
      E41P(I)=0.
      E4(I)=0.
    1 E3(I)=0.
      H=1./FLOAT(NINT)
      D=3
    8 IF(D.EQ.4) GO TO 2
      B0=0.
      E3(1)=1.
      E0=E3(1)
      C0=0.
      A0=0.
      GO TO 3
    2 C0=0.
      E41P(1)=1.
      A0=E41P(1)
      B0=0.
      E0=0.
    3 A=A0
      B=B0
      C=C0
      E=E0
      N=1
    4 I=1
      NN=N
      SI=(NN-1.)*H
    5 K(I)=H*A
      L(I)=H*B
      M(I)=H*C
      MU1=1.+EIG(KK)*EIG(KK)
      IF(NDS.EQ.1) GO TO 40
      MU1=MU1-1.
   40                  P(I)=((-SI*SI+2.*SI*(1.+SI0))/2.*B
     1+(1.-SI+SI0)*A+MU1*E)*LAM*H
```

```
      SI=(NN-1.)*H
      I=I+1
      IF(I.GT.3) GO TO 6
      Z=I-1
      E=EO+K(Z)/2.
      A=AO+L(Z)/2.
      B=BO+M(Z)/2.
      C=CO+P(Z)/2.
      SI=SI+H/2.
      GO TO 5
      IF(I.GT.4) GO TO 7
      E=EO+K(3)
      A=AO+L(3)
      B=BO+M(3)
      C=CO+P(3)
      SI=SI+H
      GO TO 5
      IF(D.EQ.4) GOTO 9
      SI=NN*H
      Z=N+1
      E3(Z)=E3(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E31P(Z)=E31P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E32P(Z)=E32P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E33P(Z)=E33P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
                              E34P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E32P(Z)/2.+(1.-SI+SIO)*E31P(Z)+MU1*E3(Z))
      E=E3(N+1)
      A=E31P(N+1)
      B=E32P(N+1)
      C=E33P(N+1)
      AO=A
      BO=B
      EO=E
      CO=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM3(KK)=E3(INTER)
      ELSM3(KK)=E32P(1)
      RMOM3(KK)=E32P(INTER)
      ESHR3(KK)=E33P(1)
      RSHR3(KK)=E33P(INTER)
      DO 28 I=1,101
   28 MMX3(I,KK)=E3(I)
      D=4
       GOTU 2
    9 SI=NN*H
      Z=N+1
      E4(Z)=E4(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E41P(Z)=E41P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E42P(Z)=E42P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E43P(Z)=E43P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
                              E44P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E42P(Z)/2.+(1.-SI+SIO)*E41P(Z)+MU1*E4(Z))
      E=E4(N+1)
```

```
      A=E41P(N+1)
      B=E42P(N+1)
      C=E43P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM4(KK)=E4(INTER)
      ELSM4(KK)=E42P(1)
      RMOM4(KK)=E42P(INTER)
      ESHR4(KK)=E43P(1)
      RSHR4(KK)=E43P(INTER)
      DO 29 I=1,101
   29 MMX4(I,KK)=E4(I)
   94 CONTINUE
      ALFA(KK)=EMOM3(KK)/EMOM4(KK)
                          BETA(KK)=MMX3(1,KK)-ALFA(KK)*MMX4(1,KK)
      GAMA(KK)=EIG(KK)*SQRT(LAM)
      DO 102 LB=1,101
      LL=102-LB
  102 BPT(LL,KK)=(MMX3(LB,KK)-ALFA(KK)*MMX4(LB,KK))/BETA(KK)
      SUM=0.
      DO 216 I=2,101
  216 SUM=SUM+(BPT(I,1)+BPT(I-1,1))/2.*((FLOAT(I)-1.5)*H+SIO)*H
      ME2=SUM
      SUM=0.
      DO 218 I=2,101
  218 SUM=SUM+(BPT(I,1)**2+BPT(I-1,1)**2)/2.*H
      ME1=SUM
      COR=ME2*ME2/ME1
      IF(IFRST.NE.1) GO TO 92
      EDA(1)=BPT(51,1)
      EDA(2)=BPT(101,1)
   92 WRITE(6,98) EDA(1),EDA(2)
   98 FORMAT(1H ,'INITIAL DISPLACEMENTS',6X,F12.6,3X,F12.6/)
      WRITE(6,105) ME1,ME2,COR,NDS
  105 FORMAT(1H ,'M1=',D12.6,3X,'M2= ',D12.6,3X,'COR= ',D12.6,
     13X,'NDS=',I3/)
   72 IF(IBIG.NE.1) GO TO 73
      GAMA(1)=EIG(1)*ANFRQ
      GAMA(2)=0.
      CC2(1)=1.
      CC2(2)=0.
      ALFA(2)=0.
      RMOM3(2)=0.
      RMOM4(2)=0.
      RSHR3(2)=0.
      RSHR4(2)=0.
      ELSM3(2)=0.
      ELSM4(2)=0.
      ESHR3(2)=0.
      EIG(2)=0.
```

```
   73  DO 213  I=1,2
  213 WRITE(6,214)ALFA(I),I
  214 FORMAT(1H ,'ALFA=',D12.6,3X,'KK=',I3)
      IF(IBIG.EQ.1) GO TO 109
      DO 104 KK=1,2
      MM=51
      DO 104 LL=1,2
C
C    THESE COEFFICIENTS FIT THE BOOM TO INITIAL DISPLACEMENTS
C
      COFDIS(LL,KK)=BPT(MM,KK)
      AA(LL,KK)=COFDIS(LL,KK)
  104 MM=MM+50
C
C
C    GJR IS USED TO INVERT THE COEFFICIENT MATRIX
C
C
      CALL GJR(AA,2,BBB,CCC,PPP,QQQ,$113)
      DO 161  I=1,NR
      SUMM=0.
      DO 160 J=1,NR
  160 SUMM=SUMM+AA(I,J)*EDA(J)
      CC2(I)=SUMM
  161 WRITE(6,162)CC2(I),I
  162 FORMAT(1H ,'CC2=',E12.6,3X,'I=',I3)
C
C MODAL SOLUTION FORM HERE
C
  109 FF=0.
      FEE=0.
      RTORK=0.
      DTORK=0.
      TINT=0.
      DLTT=1.0/(STP*GAMA(1))*2.*3.1416
      WRITE(6,50)DLTT
   50 FORMAT(1H ,'DLTT=',F10.5/)
      SFAC=1./EIG(1)
      WRITE(6,131) SFAC
  131 FORMAT(1H ,'THE NO. OF ROT CYC PER FUND VIB CYC IS',F10.4/)
      DO 122  I=1,MAXT
      T=DLTT*FLOAT(I-1)
      DO 129 KK=1,2
  129 TFAC(KK)=CC2(KK)*COS(GAMA(KK)*T)
      EDA2P1=0.
      EDA3P1=0.
      EDAE2=0.
      EDAE3=0.
      DO 119  KK=1,2
      SUM2=(RMOM3(KK)-ALFA(KK)*RMOM4(KK))*TFAC(KK)
      SUM3=(RSHR3(KK)-ALFA(KK)*RSHR4(KK))*TFAC(KK)
      SUM6=(ELSM3(KK)-ALFA(KK)*ELSM4(KK))*TFAC(KK)
      SUM7=(ESHR3(KK)-ALFA(KK)*ESHR4(KK))*TFAC(KK)
      EDAE2=EDAE2+SUM6
```

```
      EDAE3=EDAE3-SUM7
      EDA2P1=EDA2P1+SUM2
  119 EDA3P1=EDA3P1-SUM3
      RTORK=EDA2P1-SIO*EDA3P1
      AVTK=(RTORK+DTORK)/2.
      DTORK=RTORK
      IF(I.NE.1) GO TO 134
      AVTK=0.
  134 TINT=TINT+AVTK*DLTT/ANFRQ
      FEE=360.*SFAC*FLOAT(I-1)/STP
      IF(FEE-360.)140,140,117
  117 FF=FF+1.
      FEE=FEE-360.
  140 IF(I.NE.1) GO TO 44
      WRITE(6,215)T,FEE,FF
  215 FORMAT(1H ,'TIME IN SEC',F9.3,3X,' FEE=',F9.4,3X,'NROT=',F9.4/)
      WRITE(6,130)EDA2P1,EDAE2,EDA3P1,EDAE3
  130 FORMAT(1H ,'ELAS RMOM',D12.6,3X,'TIP MOM',E12.6,3X,
     1'RSHEAR=',D12.6,3X,'TIP SHEAR',E12.6/)
      WRITE(6,41)TINT
  41  FORMAT(1H ,'1/SQRT(LAM)*INTEGRAL(M*DLTT) =',E12.6//)
  44  DO 111 LL=1,101,50
      SUM = 0.
      DO 112 KK=1,2
      MODE=BPT(LL,KK)*TFAC(KK)
  112 SUM=SUM+MODE
      IF(I.EQ.1) GO TO 114
      IF(IVAR.EQ.1) GO TO 111
  114 WRITE(6,115) SUM,LL
  111 CONTINUE
  115 FORMAT(1H ,'ADA',E12.6,3X,'STATION',I3)
      IF(IVAR.NE.1) GO TO 122
      WRITE(6,121) SUM,TINT,T,I
  122 CONTINUE
  121 FORMAT(1H ,'ENDISP',F12.6,3X,'TINT',E12.6,3X,'T',F9.3,3X,'I',I3)
  120 WRITE(6,568)
  568 FORMAT(1H /)
      GO TO 100
  113 SAM=2
      WRITE(6,116) SAM
  116 FORMAT(1H ,'GJR DUMPED SAM=',I3)
  100 CONTINUE
      GO TO 42
 42    RETURN
C
       END
CARDS IN =    316, CARDS OUT =       0, PAGES OUT =       6
```

```
      SUBROUTINE GJR(A,N,B,C,P,Q,*)
      INTEGER N,P(N),Q(N)
      DIMENSION A(N,N),B(N),C(N)
      EPS=1.*(10.)**-7
 101  EPS=EPS/10.
      IF(EPS.LT.10.**-15) GO TO 102
      DO 35 K=1,N
      PIVOT=0.
      DO 5 I=K,N
      DO 5 J=K,N
      IF(ABS(A(I,J)).LE.ABS(PIVOT)) GO TO 5
      PIVOT=A(I,J)
      P(K)=I
      Q(K)=J
 5    CONTINUE
      IF(ABS(PIVOT).LE.EPS)GO TO 101
      IF(P(K)-K)6,10,6
 6    DO 7 J=1,N
      L=P(K)
      Z=A(L,J)
      A(L,J)=A(K,J)
 7    A(K,J)=Z
 10   IF(Q(K)-K)11,15,11
 11   DO 12 I=1,N
      L=Q(K)
      Z=A(I,L)
      A(I,L)=A(I,K)
 12   A(I,K)=Z
 15   DO 25 J=1,N
      IF(J-K)20,16,20
 16   B(J)=1./PIVOT
      C(J)=1.
      GO TO 22
 20   B(J)=-A(K,J)/PIVOT
      C(J)=A(J,K)
 22   A(K,J)=0.
      A(J,K)=0.
 25   CONTINUE
      DO 30 I=1,N
      DO 30 J=1,N
 30   A(I,J)=A(I,J)+C(I)*B(J)
 35   CONTINUE
      DO 50 M=1,N
      K=N-M+1
      IF(P(K)-K)40,45,40
 40   DO 43 I=1,N
      L=P(K)
      Z=A(I,L)
      A(I,L)=A(I,K)
      A(I,K)=Z
 43   CONTINUE
 45   IF(Q(K)-K)46,50,46
 46   DO 48 J=1,N
      L=Q(K)
      Z=A(L,J)
      A(L,J)=A(K,J)
      A(K,J)=Z
 48   CONTINUE
 50   CONTINUE
      RETURN
 102  WRITE(6,103)
 103  FORMAT(21H0 GJR COULD NOT DO IT)
      RETURN 7
      END
```

CHAPTER 4

Simulation of the Motion of The Central Rigid Body and its Elastic Appendages

4.1 Introduction

In the previous chapters, the problem of determining the modal shapes and frequencies of the rotating structure was examined, and applications were studied in which these modes are utilized.

In the present chapter, equations of motion are written for the generalized coordinates representing the flexible structure and for the angular rates of the central rigid body. A simulation of the spacecraft motion is then possible. Various cases of simulation are examined, and the effect of modal truncation and of nonlinear terms is discussed.

4.2 Modal Equations of Motion: equatorial vibrations (Case "E", for equatorial)

4.2.1 Constancy of $\vec{H}$.

In what follows, it is assumed that the motion of the center of mass of the spacecraft is negligible (or that only antisymmetric motions of the booms are considered) and that the "limited approach" is taken[4-1], i.e. the motion of the spacecraft's center of mass in inertial space can be determined independently of the attitude.

If over the time of interest, i.e. a few tens of spin periods or so, the torque-impulse due to all environmental attitude perturbing torques (gravity-gradient, solar pressure, magnetic, etc.) can be considered as negligible, then very sensibly the moment of momentum $\vec{H}$ about the center of mass remains constant:

$$\vec{H} = \vec{H}(o) = \text{constant vector} \qquad (4.2\text{-}1)$$

in which $\vec{H}$ is the value of $\vec{H}$ at t = 0.

4.2.2 Representation of the elastic appendages

Consider a particle of a boom, having non-dimensional abscissa ξ, located along axis +x in its undeflected position. Its elastic displacement, $\eta = \frac{w(x)}{\ell}$, is represented in terms of the modes $\Phi_j(\xi = \frac{x}{\ell})$

$$\eta_{+x} = \sum_{j=1}^{N} q_j(\bar{t})\ \Phi_j(\xi) \qquad (4.2\text{-}2)$$

in which the q_j are non-dimensional amplitudes, dependent on the non-dimensional time $\bar{t} = \omega_s t$, with $\omega_s = \frac{2\pi}{\tau_s}$ the angular spin rate of the satellite in its nominal motion. N is some positive integer, which specifies the number of terms after which the series is truncated.

We recall that the $\Phi_j(\xi)$ are orthogonal modes, normalized to unit deflection at the boom's tip, so that

$$\int_{\text{boom}} \Phi_j(\xi)\ \Phi_k(\xi) = 0 \qquad j \neq k \qquad (4.2\text{-}3)$$

$$m_{1,j} \ \overset{\equiv}{\text{def}} \int_{\text{boom}} \Phi_j^2(\xi)d\xi > 0 \qquad j = k \qquad (4.2\text{-}4)$$

$$\xi_1 = \xi + \xi_o;$$

both $m_{1,j}$, $m_{2,j}$ are assumed to be known quantities, determined as in Chapter 2.

4.2.3 Kinetic energy contained in the elastic structure

The total kinetic energy, T, is made out of two parts: one is independent of the generalized coordinates q_j and the other one, T_1, depends on the q_j and appears as the integral of a density $\mathcal{D}_{T_1}$. More specifically (Fig. 4.1)

$$T = \sum_{\substack{\text{all particles}\\ m}} m \frac{\vec{v}_m^{\,2}}{2} \tag{4.2-6}$$

with
$$\vec{v}_m = \vec{\omega} \wedge (\vec{r}_{m,0} + \vec{\delta}) + \dot{\vec{\delta}} \tag{4.2-7}$$

in which $\vec{\omega}$ is the instantaneous rotation, $\vec{r}_{m,0}$ is the vector coordinate to m in its reference position and $\vec{\delta}$ is the elastic displacement from $\vec{r}_{m,0}$.

Computing T,

$$T = \sum_{\substack{\text{RIGID + FLEXIBLE}\\ \text{PARTS}}} \frac{m}{2} \left(\vec{\omega} \wedge \vec{r}_{m,0}\right)^2 + \sum_{\substack{\text{FLEXIBLE}\\ \text{PARTS}}} \left\{ \frac{1}{2} m \left[(\vec{\omega} \wedge \vec{\delta})^2 + 2 (\vec{\omega} \wedge \vec{r}_{m,0}) \cdot (\vec{\omega} \wedge \vec{\delta}) + 2 \left(\vec{\omega} \wedge \vec{r}_{m,0}\right) \cdot \dot{\vec{\delta}} + 2 \left(\vec{\omega} \wedge \vec{\delta}\right) \cdot \dot{\vec{\delta}} + \dot{\vec{\delta}}^{\,2} \right\}$$

Now, for small linear displacements of the elastic parts

$$\vec{\delta} = w \, \vec{1}_y$$

$$\vec{\omega} \wedge \vec{\delta} = \begin{pmatrix} -w\omega_z \\ 0 \\ w\omega_x \end{pmatrix}$$

Thus

$$\sum_{\substack{\text{ALL ELASTIC}\\ \text{PARTS}}} \frac{1}{2} m \left(\vec{\omega} \wedge \vec{\delta}\right)^2 = \sum \frac{1}{2} m \left(\omega_z^2 + \omega_x^2\right) w^2 \tag{4.2-9}$$

If ω_x, ω_y, w are assumed to be of first order of smallness, (4.2-9) is rewritten

$$\sum \frac{1}{2} w^2(\omega_z^2) + O(\epsilon^4) = \sum \frac{1}{2} w^2 \omega_s^2 + O(\epsilon^3) \tag{4.2-10}$$

in which ω_s is the (constant) nominal value of the spin rate.

$$\vec{\omega} \wedge \vec{r}_{m,0} = \begin{pmatrix} 0 \\ \omega_z x_1 \\ -\omega_y x_1 \end{pmatrix}$$

Then

$$\frac{1}{2} \sum_{\substack{\text{ALL} \\ \text{ELASTIC} \\ \text{PARTS}}} 2m \left(\vec{\omega} \wedge \vec{r}_{m,0}\right) \cdot \left(\vec{\omega} \wedge \vec{\delta}\right) = \sum m \begin{bmatrix} 0 & \omega_z x & -\omega_y x \end{bmatrix} \begin{bmatrix} -W\omega_z \\ 0 \\ W\omega_x \end{bmatrix} = O(\epsilon^3) \tag{4.2-11}$$

Furthermore,

$$\sum_{\substack{\text{ALL} \\ \text{EL. PARTS}}} \frac{1}{2} m \dot{\vec{\delta}}^{\,2} = \sum \frac{1}{2} m \dot{W}^2 \tag{4.2-12}$$

and

$$\frac{1}{2} \sum_{\substack{\text{ALL} \\ \text{EL. PARTS}}} 2m \left(\vec{\omega} \wedge \vec{r}_{m,0}\right) \cdot \dot{\vec{\delta}} = \sum m \omega_z x_1 \dot{W} \tag{4.2-13}$$

Finally,

$$\frac{1}{2} \sum_{\substack{\text{ALL} \\ \text{EL. PARTS}}} 2m \left(\vec{\omega} \wedge \vec{\delta}\right) \cdot \dot{\vec{\delta}} = 0 \tag{4.2-14}$$

Introducing expressions (4.8-10) through (4.2-14) in Equations (4.2-8), we obtain

$$T = T_\circ + \frac{1}{2}\int_0^{2} \rho \, ds(w^2\omega_s^2 + \dot{\omega}^2 + 2\ \omega_z x_1 \dot{w}) + O(\varepsilon^3) \qquad (4.2\text{-}15)$$

Since the element of curvilinear abscissa, ds, is related to dx by

$$ds^2 = dx^2 + dy^2 = dx^2(1 + (\frac{\partial w}{\partial x})^2)$$

or

$$ds = dx(1 + (\frac{\partial w}{\partial x})^2)^{1/2}$$

$$= dx(1 + \frac{1}{2}(\frac{\partial w}{\partial x})^2 + \ldots)$$

and

$$dx = ds(1 - \frac{1}{2}(\frac{\partial w}{\partial x})^2 \ldots)$$

Therefore, consistent with the order of magnitudes retained explicitly, (4.2-15) can be rewritten with x instead of s as the integration variable,

$$T = T_\circ + \frac{1}{2}\int_0^{\ell} \rho \ dx(w^2\omega_s + \dot{\omega}^2 + 2\ \omega_z x_1 \dot{w}) + O(\varepsilon^3) \qquad (4.2\text{-}16)$$

The "flexible body" part of $T_\circ$, however, has to include a correction term, since for terms involving

$$\int_0^{\ell} x^2 \ ds = \int_0^{\ell\,\text{upper}} x^2 \ dx + \frac{1}{2}\int_0^{\ell\,\text{upper}} x^2 (\frac{\partial w}{\partial x})^2 \ dx$$

i.e. with an integrand of zero-th order of magnitude, we can write

$$\int_0^{\ell} x^2\, ds = \int_0^{\ell_{upper}} x^2\, dx + \frac{1}{2}\int_0^{\ell_{upper}} x^2 \left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2\text{-}17)$$

Neglecting terms of 3rd order of smallness,

$$\int_0^{\ell} x^2\, ds = \int_0^{\ell} x^2\, dx - \ell^2(\ell - \ell_{upper}) + \frac{1}{2}\int_0^{\ell} x^2 \left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2\text{-}18)$$

and using (4.2-17),

$$\int_0^{\ell} x^2\, ds = \int_0^{\ell} x^2\, dx - \frac{1}{2}\int_0^{\ell} (\ell^2 - x^2)\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2\text{-}19)$$

Now, if the integrand is

$$x_1^2 = (x_o + x)^2 = x^2 + 2x_o x + x_o^2$$

we obtain

$$\int_0^{\ell} x_o^2\, ds = x_o^2\, \ell = x_o^2 \int_0^{\ell} dx$$

$$\int_0^{\ell} x\, ds = \int_0^{\ell_{upper}} x\, dx + \frac{1}{2}\int_0^{\ell_{upper}} x \left(\frac{\partial w}{\partial x}\right)^2 dx$$

or neglecting terms of third order of smallness

$$\begin{aligned}\int_0^{\ell} x\, ds &= \int_0^{\ell} x\, dx - \ell(\ell - \ell_{upper}) + \frac{1}{2}\int_0^{\ell} x \left(\frac{\partial w}{\partial x}\right)^2 dx \\ &= \int_0^{\ell} x\, dx - \frac{1}{2}\int_0^{\ell} (\ell - x)\left(\frac{\partial w}{\partial x}\right)^2 dx\end{aligned}$$

Finally,

$$\int_0^{\ell} x_1^2\, ds = \int_0^{\ell} x_1^2\, dx - \frac{1}{2}\int_0^{\ell} \left[(\ell^2 - x^2) + 2x_o(\ell - x)\right]\left(\frac{\partial w}{\partial x}\right)^2 dx$$

Therefore, T_0 is rewritten, with $\Phi(\vec{\omega})$ the inertia dyadic of the rigidified, undeflected total reference body as

$$T_0 = \frac{1}{2}\,\vec{\omega}\cdot\Phi(\vec{\omega}) - \frac{1}{2}\int_0^\ell \rho\,dx\,\frac{1}{2}\left[(\ell^2-x^2)+2x_0(\ell-x)\right]\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (4.2\text{-}20)$$

Collecting (4.2-16) and (4.2-20)

$$T = \frac{1}{2}\,\vec{\omega}\cdot\Phi(\vec{\omega}) + \frac{1}{2}\int_0^\ell \rho\,dx\,\Big(w^2\omega_3 + \dot{w}^2 + 2\omega_z x_i\,\dot{w} - \frac{1}{2}\left[(\ell^2-x^2)\right.$$
$$\left. + 2x_0(\ell-x)\right]\left(\frac{\partial w}{\partial x}\right)^2\Big)\,dx + O(\epsilon^3) \qquad (4.2\text{-}21).$$

4.2.4 Potential energy of the elastic structure

For pure flexure in the (x,y) plane, the potential energy is given by

$$V = \frac{EI}{2}\int_0^\ell \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + O(\varepsilon^3)$$

where it is legitimate to use x, instead of s, as the integration variable, to the order of the terms explicitly retained.

4.2.5 Equations of motion for the elastic modes, equatorial vibrations.

4.2.5.1 Equation for the jth coordinate, q_j

At this point we introduce the modal representation (4.1-2). For the sake of simplicity, let the bars on $\bar{t} = [\frac{t}{1/\omega_s}]$, be dropped. Furthermore, let the energies be non-dimensionalized by

$\rho\ell^3\omega_s^2$ and the lengths by ℓ

Although $\bar{\omega}_s$, non-dimensional value of the nominal satellite spin-rate, is 1, we shall for clarity retain it in the equations.

In non-dimensional form, with $\dot{}$ designating derivatives with respect to $\bar{t}$, and $\xi_1 = \xi + \xi_o$,

$$\bar{T}_1 = \frac{1}{\rho \ell^3 \omega_s^2}\left(T - \frac{1}{2}\vec{\omega}\cdot\bar{\Phi}(\vec{\omega})\right) = \frac{1}{2}\int_{boom}\Big(\dot{\eta}^2 + \eta^2\bar{\omega}_s^2 + 2\bar{\omega}_z\dot{\eta}\xi_1 - \frac{1}{2}\left[(1-\xi^2) + 2\xi_o(1-\xi)\right]\left(\frac{\partial\eta}{\partial\xi}\right)^2 d\xi \tag{4.2-22}$$

$$\bar{V} = \frac{1}{2\bar{\lambda}}\int_{boom}\left[\frac{\partial^2\eta}{\partial\xi^2}\right]^2 d\xi \tag{4.2-23}$$

with $\bar{\lambda} = \frac{\rho\ell^4}{EI}\omega_s^2$, as in Chapter 2.

Now let, with t the non-dimensional time,

$$\eta = \sum_{j=1}^{N} q_j(t)\,\bar{\Phi}_j(\xi)$$

Then

$$\dot{\eta} = \sum_{j=1}^{N}\dot{q}_j\,\bar{\Phi}_j$$

$$\eta^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} q_j\,q_k\,\bar{\Phi}_j\,\bar{\Phi}_k$$

$$\dot{\eta}^2 = \sum_{j=1}^{N}\sum_{k=1}^{N}\dot{q}_j\,\dot{q}_k\,\bar{\Phi}_j\,\bar{\Phi}_k$$

$$\frac{\partial\eta}{\partial\xi} = \sum_{j=1}^{N} q_j\,\frac{d\bar{\Phi}_j}{d\xi}$$

$$\left(\frac{\partial\eta}{\partial\xi}\right)^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} q_j\,q_k\,\frac{d\bar{\Phi}_j}{d\xi}\,\frac{d\bar{\Phi}_k}{d\xi}$$

Similarly,

$$\left(\frac{\partial^2\eta}{\partial\xi^2}\right)^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} q_j\,q_k\,\frac{d^2\bar{\Phi}_j}{d\xi^2}\,\frac{d^2\bar{\Phi}_k}{d\xi^2}$$

Thus, the Lagrangian function is

$$\bar{\mathcal{L}} \equiv \frac{1}{2}\vec{\omega}\cdot\bar{\Phi}(\vec{\omega}) + \bar{T}_1 - \bar{V} = \frac{1}{2}\int_{boom}\sum_{j=1}^{N}\sum_{k=1}^{N}\Big[\mathring{q}_j\mathring{q}_k\bar{\Phi}_j\bar{\Phi}_k + \bar{\omega}_s^2 q_j q_k \bar{\Phi}_j\bar{\Phi}_k + 2\omega_z\mathring{q}_j\xi_1\bar{\Phi}_j$$
$$-\frac{1}{2}\left[(1-\xi^2)+2\xi_o(1-\xi)\right]q_j q_k\frac{d\phi_j}{d\xi}\frac{d\phi_k}{d\xi} - \frac{1}{\bar{\lambda}}q_j q_k\frac{d^2\bar{\Phi}_j}{d\xi^2}\frac{d^2\bar{\Phi}_k}{d\xi^2}\Big]d\xi \quad (4.2\text{-}24)$$

Now, for any i = 1, 2...,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial\dot{q}_i} = 0 \qquad (4.2\text{-}25)$$

We recall that, as in (4.2-3), (4.2-4), (4.2-5)

$$\int_{boom}\phi_j\phi_k\,d\xi = 0 \qquad (j \neq k)$$

$$\int_{boom}\phi_j^2\,d\xi \equiv m_{1,j}$$

$$\int_{boom}\phi_j\xi_1\,d\xi \equiv m_{2,j}$$

the ϕ_j having been previously normalized to unit deflection at the boom's tip. Using these relations, and (4.2-25) for i = j, after defining

$$a_{jk} = a_{kj} \overset{\equiv}{\text{def}} \frac{1}{2}\int_{boom}\left[(1-\xi^2) + 2\xi_o(1-\xi)\right]\frac{d\Phi_j}{\partial\xi}\frac{d\Phi_k}{d\xi}\,d\xi \qquad (4.2\text{-}26)$$

$$b_{jk} = b_{kj} \overset{\equiv}{\text{def}} \int_{boom}\frac{d^2\phi_j}{d\xi^2}\frac{d^2\phi_k}{d\xi^2}\,d\xi \qquad (4.2\text{-}27)$$

we obtain, for the jth modal coordinate

$$m_{1,j}\ddot{q}_j - m_{1,j}\bar{\omega}_s^2 q_j + \sum_{k=1}^{N}a_{jk}q_k + \frac{1}{\bar{\lambda}}\sum_{k=1}^{N}b_{jk}q_k = -m_{2,j}\dot{\bar{\omega}}_z \qquad (4.2\text{-}28)$$

4.2.5.2 Evaluation of the coefficient of q_j

We now evaluate the coefficient of q_j in (4.2-28), say c_j,

$$c_j \underset{\text{def}}{\equiv} \frac{1}{\lambda} \sum_{k=1}^{N} b_{jk} + \sum_{k=1}^{N} a_{jk} - m_{1,j} \, \bar{\omega}_s^2 \tag{4.2-29}$$

From Equation (2.2-23), written in terms of a_{jk}, b_{jk},

$$\frac{1}{\bar{\lambda}} b_{jk} + a_{jk} = 0 \qquad j \neq k$$

$$\frac{1}{\bar{\lambda}} b_{jj} + a_{jj} = m_{1,j} \left(\bar{\omega}_j^2 + \bar{\omega}_s^2\right) \qquad j = k$$

Thus (4.2-28) takes the simple form

$$m_{1,j}\ddot{q}_j + m_{1,j}\bar{\omega}_j^2 q_j = -\frac{m_{2,j}}{m_{1,j}} \dot{\bar{\omega}}_z$$

or

$$\ddot{q}_j + \bar{\omega}_j^2 \, q_j = -\frac{m_{2,j}}{m_{1,j}} \dot{\bar{\omega}}_z \tag{4.2-30}$$

A few remarks should be made regarding (4.2-30) First of all, the modal equations for the j^{th} coordinate reduce to a harmonic motion, in the case where $\bar{\omega}_z$ is constant. Second, as has been seen in Chapter 3, the "driving amplitude", measured by the non-dimensional ratio

$$\frac{m_{2,j}}{m_{1,j}}$$

is strongly a function of ξ_o, and to a lesser extent of $\bar{\lambda}$, Etkin's number. Thirdly, it should be noted that it is only because, for the sake of consistency, the difference of an integral in $\underline{\underline{s}}$ (curvilinear abscissa) and $\underline{\underline{x}}$ was carefully considered when the integrand was a quantity of zeroth order (as detailed above), that term

$$- m_{1,j} \; \bar{\omega}_s^2 \; q_j$$

could finally be cancelled in Equation (4.2-30). Failure to make this distinction leads to having this extra term still present in the final equation and in order to "fall back" on (4.2-30), one has to introduce, rather belatedly, an additional term due to a "rotational potential"[4-2]. Finally, if linear distributed damping is introduced, Equation (4.2-30) takes the form

$$\ddot{q}_j + 2\nu\bar{\omega}_j \dot{q}_j + \bar{\omega}_j^2 q_j = - \frac{m_{2,j}}{m_{1,j}} \dot{\bar{\omega}}_z \tag{4.2-31}$$

whose derivatives are taken with respect to non-dimensional time.

4.3 Modal Equations of Motion: meridional vibrations (Case "M", for meridional)

Without repeating in the same detail the explanations of Section 4.2, we now derive the modal equations in the case of motions parallel to axis-z (meridional vibrations). Only the relevant differences are underlined.

4.3.1 Constancy of $\vec{H}$

In the absence of attitude perturbing torques, the torque-free motion has the integral

$$\vec{H} = \vec{H}_o \tag{4.3-1}$$

where $\vec{H}_o$ is the value of the moment of momentum at t = 0.

4.3.2 Representation of the elastic appendages

The displacement $\eta(x) = \frac{w(x)}{\ell}$ of an element of boom located at $\xi = \frac{x}{\ell}$ of axis +x in terms of the modes $\Phi_j(\xi)$ for <u>meridional</u> motions is

$$\eta_{+x} = \sum_{j=1}^{N} q_j(\bar{t}) \; \Phi_j(\xi) \tag{4.3-2}$$

Again q_j are non-dimensional amplitudes, functions of the mon-dimensional time $\bar{t} = \frac{t}{1/\omega_s}$. N is positive integer specifying the number of terms after which the series is truncated.

The "meridional" modes are orthogonal

$$\int_{boom} \phi_j(\xi)\phi_k(\xi)d\xi = 0 \quad j \neq k \tag{4.3-3}$$

and we have defined

$$m_{1,j} \equiv \int_{boom} \phi_j^2(\xi)d\xi > 0 \tag{4.3-4}$$

$$m_{2,j} \equiv \int_{boom} \xi_1\phi_j(\xi)d\xi \quad \text{with } \xi_1 = \xi + \xi_o \tag{4.3-5}$$

These quantities are known as functions of $\bar{\lambda}$, Etkin's number, and $\xi_o = \frac{x_o}{\ell}$, hub non-dimensional radius.

4.3.3 Kinetic energy contained in the elastic structure

With the same notations as in Section 4.2.3,

$$T = T_o \text{ (rigid part + flexible part) } + T_1$$

Now

$$\vec{\delta} = w\,\vec{1}_z$$

$$\vec{\omega} \wedge \vec{\delta} = \begin{pmatrix} \omega_y w \\ -\omega_z w \\ 0 \end{pmatrix}$$

and $$\sum_{\substack{\text{quel.} \\ \text{parts}}} \frac{1}{2} m \left(\vec{\omega} \wedge \vec{\delta}\right)^2 = \Sigma \frac{1}{2} m W^2 \left(\omega_x^2 + \omega_y^2\right) = O(\epsilon^4) \tag{4.3-6}$$

Let $x_1 = x + x_\circ$. Then

$$\vec{\omega} \wedge \vec{r}_{m,0} = \begin{pmatrix} 0 \\ \omega_z x_1 \\ -\omega_y x_1 \end{pmatrix}$$

The next terms are

$$\frac{1}{2} \sum_{\substack{\text{ALL} \\ \text{EL. PARTS}}} 2m(\vec{\omega} \wedge \vec{r}_{m,0}) \cdot (\vec{\omega} \wedge \vec{\delta}) = \frac{1}{2} \sum \left(-2\omega_s \omega_x x_1 w\right) + O(\epsilon^3) \qquad (4.3\text{-}7)$$

$$\sum_{\substack{\text{ALL EL.} \\ \text{PARTS}}} \frac{1}{2} m \overset{\circ}{\vec{\delta}}{}^2 = \sum \frac{1}{2} m \dot{w}^2 \qquad (4.3\text{-}8)$$

$$\frac{1}{2} \sum_{\substack{\text{ALL} \\ \text{EL. PARTS}}} 2m(\vec{\omega} \wedge \vec{r}_{m,0}) \cdot \overset{\circ}{\vec{\delta}} = -\frac{1}{2} \sum 2m\, \omega_y x_1 \overset{\circ}{w} \qquad (4.3\text{-}9)$$

$$\frac{1}{2} \sum_{\substack{\text{ALL} \\ \text{EL. PARTS}}} 2m\, (\vec{\omega} \wedge \vec{\delta}) \cdot \overset{\circ}{\vec{\delta}} = 0 \,. \qquad (4.3\text{-}10)$$

Introducing expressions (4.3-6) through (4.3-10) into (4.2-8), and since we can substitute dx for ds when the integrand is of first order of smallness, or smaller,

$$T = \frac{1}{2}\vec{\omega} \cdot \Phi(\vec{\omega}) + \frac{1}{2}\int_0^\ell \rho \, dx(\dot{w}^2 - 2\omega_s \omega_x x_1 w - 2\omega_y x_1 \dot{w} - \frac{1}{2}[(\ell^2 - x^2) + 2x_\circ(\ell - x)](\frac{\partial w}{\partial x})^2) + O(\varepsilon^3) \qquad (4.3\text{-}12)$$

4.3.4 Potential energy of the elastic structure

For pure flexure in the (x,z) plane, the potential energy is

$$V = \frac{EI}{2}\int_0^{\ell}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + 0(\varepsilon^3)$$

Again, it is legitimate to use x, instead of s, as the integration, to the order of the terms explicitly retained.

4.3.5 Equations of motion: elastic modes, meridional vibrations

5.3.5.1 Equation for the jth coordinate, q_j

The kinetic energy, T, and potential energy, V, are non-dimensionalized by the quantity $\rho\ell^3\omega_s^2$. Note that, although $\bar{\omega}_s$, non-dimensional value of the nominal satellite spin rate, is 1, it is retained as "$\bar{\omega}_s$" in the equations. Let $T_{rb} = \frac{1}{2}\vec{\omega}\cdot\Phi(\vec{\omega})$.

$$\bar{T} \underset{def}{\equiv} \frac{T}{\rho\ell^3\omega_s^2} = \frac{T_{rb}}{\rho\ell^3\omega_s^2} + \frac{1}{2}\int_{boom}\left(\dot{\eta}^2 - 2\bar{\omega}_s\bar{\omega}_x\xi_1\eta - 2\omega_y\xi_1\dot{\eta} - \frac{1}{2}\left[(1-\xi^2) + 2\xi_o(1-\xi)\right]\left(\frac{\partial\eta}{\partial\xi}\right)^2\right)d\xi$$

$$\bar{V} \underset{def}{\equiv} \frac{V}{\rho\ell^3\omega_s^2} = \frac{1}{2\bar{\lambda}}\int_{boom}\left(\frac{\partial^2\eta}{\partial\xi^2}\right)^2 d\xi$$

With the same substitutions as in (4.2), we obtain the Lagrangian

$$\bar{\mathcal{L}} = \frac{T_{rb}}{\rho\ell^3\omega_s^2} + \frac{1}{2}\int_0^1\sum_{j=1}^{N}\sum_{k=1}^{N}\Big[\dot{q}_j\dot{q}_k\Phi_j\Phi_k - 2\bar{\omega}_s\bar{\omega}_x\xi_1 q_j\Phi_j - 2\bar{\omega}_y\xi_1\dot{q}_j\Phi_j - \frac{1}{2}\left[(1-\xi^2) + 2\xi_o(1-\xi)\right]q_j q_k\frac{d\phi_j}{d\xi}\frac{d\phi_k}{d\xi} - \frac{1}{\bar{\lambda}}q_j q_k\frac{d^2\Phi_j}{d\xi^2}\frac{d^2\Phi_k}{d\xi^2}\Big]d\xi \qquad (4.3\text{-}13)$$

Now define

$$a_{jk} = a_{kj} \underset{def}{\equiv} \frac{1}{2}\int_{boom}\left[(1-\xi^2) + 2\xi_o(1-\xi)\right]\frac{d\phi_j}{d\xi}\frac{d\phi_k}{d\xi}\,d\xi$$

$$b_{jk} = b_{kj} \stackrel{\text{def}}{\equiv} \int_{\text{boom}} \frac{d^2\phi_j}{d\xi^2} \frac{d^2\phi_k}{d\xi^2} \, d\xi$$

For modal coordinate q_j, the Lagrangian equation is, with Equations (4.3-3) through (4.3-5),

$$m_{1,j} \ddot{q}_j + \frac{1}{\bar{\bar{\lambda}}} \sum_{k=1}^{N} b_{jk} + \sum_{k=1}^{N} a_{jk} = + m_{2,j} (\dot{\bar{\omega}}_y - \bar{\omega}_s \bar{\omega}_x) \qquad (4.3\text{-}14)$$

4.3.5.2 Evaluation of the coefficient of q_j

From Equations (2.3-11,12) , Section 2.23, we obtain

$$\frac{1}{\bar{\bar{\lambda}}} b_{jk} + a_{jk} = 0 \qquad j \neq k$$

$$\frac{1}{\lambda} b_{jj} + a_{jj} = \bar{\omega}_j^2 m_{1,j} \qquad j=k$$

Thus Equation (4.3-14) can be rewritten in the form

$$\ddot{q}_j + \bar{\omega}_j^2 q_j = \frac{m_{2,j}}{m_{1,j}} (\dot{\bar{\omega}}_y - \bar{\omega}_s \bar{\omega}_x) \qquad (4.3\text{-}15)$$

in which $\bar{\omega}_j$ is the j^{th} eigenfrequency of meridional vibrations, a function of $\bar{\lambda}$ and ξ_o.

Again, if linear distributed damping is introduced, the equation of motion becomes

$$\ddot{q}_j + 2\nu\bar{\omega}_j \dot{q}_j + \bar{\omega}_j^2 q_j = \frac{m_{2,j}}{m_{1,j}} (\bar{\omega}_y - \dot{\bar{\omega}}_s \bar{\omega}_x) \qquad (4.3\text{-}16)$$

$\frac{m_{2,j}}{m_{1,j}}$ thus appears as a "driving amplitude". As seen in Chapter 2, it is also strongly dependent on ξ_o, and to a lesser extent on Etkin's number $\bar{\lambda}$.

4.4 Equations for the rates: equatorial vibrations (Case "E")

The equation for the time derivatives of the rates are now derived from the constancy of the moment of momentum for the torque-free motion, as given in (4.2-1).

Since, about the center of mass,

$$\vec{H} = \vec{H}(o) = \int \vec{r} \wedge \dot{\vec{r}}\, dm$$

$$\dot{\vec{H}} = \int \vec{r} \wedge \ddot{\vec{r}}\, dm$$

Computing, with the same notations as in 4.2 and 4.3,

$$\dot{\vec{r}} = \dot{\vec{\delta}} + \vec{\omega} \wedge (\vec{r}_{m,0} + \vec{\delta})$$

$$\ddot{\vec{r}} = \ddot{\vec{\delta}} + 2\vec{\omega} \wedge \dot{\vec{\delta}} + \dot{\vec{\omega}} \wedge \vec{r}_{m,0} + \dot{\vec{\omega}} \wedge \vec{\delta} + \vec{\omega} \wedge (\vec{\omega}_s \wedge \vec{r}_{m,0}) + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{\delta})$$

Let $\dot{\vec{H}}$ be divided between a part "relating to $\vec{r}_{m,0}$", $\dot{\vec{H}}_I$, and a part relating to $\vec{\delta}$, $\dot{\vec{H}}_{II}$

$$\dot{\vec{H}} = \dot{\vec{H}}_I + \dot{\vec{H}}_{II}$$

and

$$\dot{\vec{H}}_I = \int_{boom} \vec{r}_{m,0} \wedge \{\dot{\vec{\omega}} \wedge \vec{r}_{m,0} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}_{m,0})\}\, ds \qquad (4.4\text{-}1)$$

$$\dot{\vec{H}}_{II} = \int_{boom} [\vec{r} \wedge \{\ddot{\vec{\delta}} + 2\vec{\omega} \wedge \dot{\vec{\delta}} + \dot{\vec{\omega}} \wedge \vec{\delta} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{\delta})\} \qquad (4.4\text{-}2)$$

$$+ \vec{\delta} \wedge (\dot{\vec{\omega}} \wedge \vec{r}_{m,0}) + \vec{\delta}\,(\vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}_{m,0}))]\, ds$$

Note that as has been seen in Section (4.3) and (4.3)

$$ds = dx\left(1 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) + \text{higher order terms} \qquad (4.4\text{-}3)$$

in which the elastic displacement (along +y) is

$$\vec{\delta} = w(x)\vec{1}_y \qquad (4.4\text{-}4)$$

Therefore, if $\bar{\omega}_x$, $\bar{\omega}_y$ (normalized to ω_s, nominal value of the satellite spin rate) and $\eta(x) = \frac{w(x)}{\ell}$ are considered to be of first order of smallness ($O(\varepsilon)$), the equations for the rates deduced from (4.4-1) should be written with

$$-\int_{boom}[\cdots]ds = \int_{boom}[\cdots]dx \ ,$$

for integrands of zeroth order of smallness, or smaller, <u>if only quantities of first-order of smallness, or larger, are retained</u>.

Thus, neglecting terms of order 3 of smallness, or smaller (with $\int_{boom}[\cdots]ds = \int_{boom}[\cdots]dx$ for an integrand of first order of smallness, or smaller),

$$(\dot{\vec{H}}_{II})_x = \int_{boom} -x_1 \dot{\omega}_y \, w \, dx$$

$$(\dot{\vec{H}}_{II})_y = \int_{boom} x_1 \left(-2\omega_x \dot{w} - \dot{\omega}_x w - \omega_y \omega_x w\right) dx$$

$$(\dot{\vec{H}}_{II})_z = -\int_{boom} x_1 \ddot{w} \, dx$$

Now, neglecting terms of order 2 of smallness, or smaller (with $\int_{boom}[\cdots]ds = \int_{boom}[\cdots]dx$ for an integrand of *first* order of smallness or smaller)

$$(\dot{\vec{H}}_{II})_x = (\vec{H}_{II})_y = 0$$

in the analysis.

$$\int_{boom}[\cdots]ds = \int_{boom}[\cdots][1 + \tfrac{1}{2}(\tfrac{\partial w}{\partial x})^2]dx,$$ in a manner similar to the one used in Section 4.2 and 4.3, if the integrand is of zeroth order of smallness, and <u>if quantities of second order of smallness, or larger, are to be retained in the analysis</u>.

With this qualification in mind, the various terms in the integrand of (4.4-2) are computed without eliminating smaller terms at this point.

$$\vec{\delta} = W \vec{I}_y$$

$$\vec{\pi} \wedge \ddot{\vec{\delta}} = \begin{bmatrix} 0 \\ 0 \\ x_1 \ddot{W} \end{bmatrix}$$

$$2 \vec{\omega} \wedge \dot{\vec{\delta}} = 2 \begin{bmatrix} -\omega_x \dot{W} \\ 0 \\ \omega_x \dot{W} \end{bmatrix}$$

$$2 \vec{\pi} \wedge (\vec{\omega} \wedge \dot{\vec{\delta}}) = 2 \begin{bmatrix} \omega_x W \dot{W} \\ -x_1 \omega_x \dot{W} \\ \omega_z W \dot{W} \end{bmatrix}$$

$$\vec{\pi} \wedge (\dot{\vec{\omega}} \wedge \vec{\delta}) = \begin{bmatrix} \dot{\omega}_x W^2 \\ -x_1 \dot{\omega}_x W \\ \dot{\omega}_z W^2 \end{bmatrix}$$

$$\vec{\delta} \wedge (\dot{\vec{\omega}} \wedge \vec{\pi}_{m,0}) = \begin{bmatrix} -x_1 \dot{\omega}_y W \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\delta} \wedge (\vec{\omega} \wedge (\vec{\omega} \wedge \vec{\pi}_{m,0})) = \begin{bmatrix} x_1 \omega_x \omega_z W \\ 0 \\ x_1 (\omega_y^2 + \omega_z) \omega_z \end{bmatrix}$$

$$\vec{\pi} \wedge (\vec{\omega} \wedge (\vec{\omega} \wedge \vec{\delta})) = \begin{bmatrix} \omega_z \omega_y W^2 \\ -x_1 \omega_y \omega_z W \\ -x_1 (\omega_x^2 + \omega_z^2) W \end{bmatrix}$$

$$(\dot{\vec{H}}_{II})_z = \int_{boom} x_1 \ddot{W} \, dx + O(\epsilon^2)$$

$$(\dot{\vec{H}}_I) = \left[I_x \dot{\omega}_x + (I_z - I_y) \omega_y \omega_z \right] \vec{I}_x + \left[I_y \dot{\omega}_y + (I_x - I_z) \omega_z \omega_x \right] \vec{I}_y + \left[I_z \dot{\omega}_z + (I_y - I_x) \omega_x \omega_y \right] \vec{I}_z \qquad (4.4\text{-}5)$$

if it can be assumed that x, y, z are principal axes of inertia of the total, rigidified spacecraft, of total moments of inertia I_x, I_y, I_z about the corresponding axes.

To summarize, we have, to order ε, the following equations for the rates, in case E,

$$\dot{\omega}_x = -\frac{I_z - I_y}{I_x}\,\omega_y\omega_s$$

$$\dot{\omega}_y = -\frac{I_x - I_z}{I_y}\,\omega_s\omega_x$$

$$\dot{\omega}_z + \frac{1}{I_z}\int_{\text{boom}} \ddot{w}x_1\,dx = -\frac{I_y - I_x}{I_z}\,\omega_x\omega_y \approx 0, \text{ to } O(\varepsilon) \qquad (4.4\text{-}6)$$

Let the time, t, be non-dimensionalized as $\bar{t} = \frac{t}{1/\omega_s}$ (from now on, $\cdot$ will designate derivatives with respect to $\bar{t}$); the lengths are non-dimensionalized by ℓ , length of the boom, and $\xi = \frac{x}{\ell}$, $\xi_o = \frac{x_o}{\ell}$, $\eta = \frac{w}{\ell}$; $k_x \equiv \frac{I_x}{I_z}$, $k_y \equiv \frac{I_y}{I_z}$. We obtain

$$\dot{\bar{\omega}}_x = -\frac{1-k_y}{k_x}\,\bar{\omega}_y$$

$$\dot{\bar{\omega}}_y = -\frac{k_x - 1}{k_y}\,\bar{\omega}_x$$

$$\dot{\bar{\omega}}_z = -\frac{\rho\ell^3}{I_z}\int_{\text{boom}} \ddot{\eta}\,\xi_1\,d\xi \qquad (4.4\text{-}7)$$

Using the modal expansion in terms of $\phi_j(\xi)$, having eigenfrequencies $\bar{\omega}_{j,E}$, with $\bar{\Phi}_j$ and $\bar{\omega}_{j,E}$ functions of $\bar{\lambda}$ and ξ_o,

$$\dot{\bar{\omega}}_z = -\frac{\rho \ell^3}{I_z} \int_{boom} \sum_{j=1}^{N} \ddot{q}_j \, \phi_j(\xi) \, \xi_1 \, d\xi \qquad (4.4\text{-}7)$$

Now, from Equation (4.2-31), with $\bar{\omega}_j = \bar{\omega}_{j,E}$

$$\ddot{q}_j = -\frac{m_{2,j}}{m_{1,j}} \dot{\bar{\omega}}_z - 2\nu \bar{\omega}_j \dot{q}_j - \bar{\omega}_j^2 q_j$$

Substituting

$$\dot{\bar{\omega}}_z \left(1 - \frac{\rho \ell^3}{I_z} \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}} \right) = \frac{\rho \ell^3}{I_z} \sum_{j=1}^{N} \left[2\nu \bar{\omega}_j \dot{q}_j + \bar{\omega}_j^2 q_j \right] m_{2,j} \qquad (4.4\text{-}8)$$

Thus the normalized moment of inertia is apparently reduced from the value 1, due to the flexibility of the boom, by an amount equal to

$$\frac{\rho \ell^3}{I_z} \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}} \qquad (4.4\text{-}9)$$

or writing, with I_{zh} = moment of inertia, about z, of the central hub, and

$$\Delta \underset{def}{\equiv} \xi_o^2 + \xi_o + \frac{1}{3} \qquad (4.4\text{-}10)$$

$$I_z = I_{zh} + \rho \ell^3 \Delta \qquad (4.4\text{-}11)$$

the non-dimensional inertia correction becomes, in Equation (4.4-8), with $\Gamma \underset{def}{\equiv} \frac{\rho \ell^3}{I_{zh}}$,

$$-\frac{\Gamma}{1 + \Gamma \Delta} \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}} \qquad (4.4\text{-}12)$$

and (4.4-8) is rewritten (Rate equations for case E)

$$\mathring{\bar{\omega}}_x = - \frac{1-k_y}{k_x} \bar{\omega}_y$$

$$\mathring{\bar{\omega}}_y = - \frac{k_x - 1}{k_y} \bar{\omega}_x$$

$$\mathring{\bar{\omega}}_z \left(1 - \frac{\Gamma}{1+\Gamma\Delta} \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}} \right) = \frac{\Gamma}{1+\Gamma\Delta} \sum_{j=1}^{N} \left[2\nu \bar{\omega}_j \mathring{q}_j + \bar{\omega}_j^2 q_j \right] m_{2,j} \qquad (4.4\text{-}13)$$

The stability of the motion, in the presence of equatorial vibrations, as studied in Chapter 5, can be done on the basis of equations

(4.2-25) for the modal coordinates

(4.4-13) for the angular rates

with N = 1, 2 or 3, depending on the number of modes retained in the analysis.

4.5 Equations for the rates: meridional vibrations (Case "M")

4.5.1 Equations for the rates, boom along the direction

Without repeating the development of Section 4.1.1, the components of $\dot{\vec{H}}_{II}$, in expression (4.4-2) are rederived for an elastic displacement $\vec{\delta}$ parallel to axis-z.

$$\vec{\delta} = w(x) \vec{i}_z$$

$$\vec{\pi} \wedge \ddot{\vec{\delta}} = \begin{bmatrix} 0 \\ -x_i \ddot{w} \\ 0 \end{bmatrix}$$

$$2\vec{\pi} \wedge (\vec{\omega} \wedge \dot{\vec{\delta}}) = 2 \begin{bmatrix} \omega_x w \dot{w} \\ \omega_y w \dot{w} \\ -x_1 \omega_x \dot{w} \end{bmatrix}$$

$$\vec{\pi} \wedge (\dot{\vec{\omega}} \wedge \vec{\delta}) = \begin{bmatrix} \dot{\omega}_x W^2 \\ \dot{\omega}_y W^2 \\ -x_1 \dot{\omega}_x W \end{bmatrix}$$

$$\vec{\pi} \wedge (\vec{\omega} \wedge (\vec{\omega} \wedge \vec{\delta})) = \begin{bmatrix} -\omega_z \omega_y W^2 \\ -\omega_x \omega_z W^2 + x_1(\omega_x^2 + \omega_y^2) W \\ x_1 \omega_z \omega_y W \end{bmatrix}$$

$$\vec{\delta} \wedge (\dot{\vec{\omega}} \wedge \vec{\pi}_{m,o}) = \begin{bmatrix} x_1 W \dot{\omega}_z \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\delta} \wedge (\vec{\omega} \wedge (\vec{\omega} \wedge \vec{\pi}_{m,o})) = \begin{bmatrix} -x_1 \omega_x \omega_y W \\ -x_1 (\omega_y^2 + \omega_z^2) W \\ 0 \end{bmatrix}$$

Neglecting terms of order 3 of smallness,

$$(\dot{\vec{H}}_{II})_z = \int_{boom} x_1 W \dot{\omega}_z \, \rho \, dx$$

$$(\dot{\vec{H}}_{II})_y = \int_{boom} x_1 (\ddot{W} - W)(\omega_z^2 + \omega_y^2) \, \rho \, dx$$

$$(\dot{\vec{H}}_{II})_z = \int_{boom} x_1 (\omega_z \omega_y W - \omega_x \dot{W} - \dot{\omega}_x W) \, \rho \, dx$$

Neglecting terms of order 2 of smallness,

$$(\dot{\vec{H}}_{II})_x = (\dot{\vec{H}}_{II})_z = 0$$

$$(\dot{\vec{H}}_{II})_y = -\int_{boom} x_1 (\ddot{W} + W \omega_s^2) \, \rho \, dx$$

$$(\dot{\vec{H}}_I) = [I_x \dot{\omega}_x + (I_z - I_y)\omega_y \omega_z] \vec{i}_x + [I_y \dot{\omega}_y + (I_x - I_z)\omega_z \omega_x] \vec{i}_y + [I_z \dot{\omega}_z + (I_y - I_x)\omega_x \omega_y] \vec{i}_z$$

with the same assumption as in 4.4

To summarize, we have to order ε, after non-dimensionalization of time by $1/\omega_s$, and with $k_x = \frac{I_x}{I_z}$, $k_y = \frac{I_y}{I_z}$,

$$\begin{aligned} \dot{\bar{\omega}}_x &= -\frac{1-k_y}{k_x} \bar{\omega}_y \\ \dot{\bar{\omega}}_y &= \quad - \frac{k_x - 1}{k_y} + \frac{\rho \ell^3}{I_y} \int_{boom} (\ddot{\eta} + \eta) \xi_1 \, d\xi \qquad (4.5\text{-}1) \\ \dot{\bar{\omega}}_z &= 0 \end{aligned}$$

In the second of equations (4.5-1),

$$\frac{\rho \ell^3}{I_y} = \frac{\rho \ell^3}{I_z k_y} = \frac{1}{k_y} \frac{\Gamma}{1+\Gamma\Delta}$$

Using the modal expression for $\eta_M = \sum_{j=1}^{N} q_j(t)\Phi_{jm}(\xi)$, $\phi_j(\xi)$ being the jth modal shape having associated frequency $\bar{\omega}_{j,M}$,

$$\int_{boom} \sum_{j=1}^{N} \left(\ddot{q}_j + q_j\right) \Phi_j \xi_1 \, d\xi = \sum_{j=1}^{N} \ddot{q}_j m_{2,j} + \sum_{j=1}^{N} q_j m_{2,j}$$

Since, from Equation (4.3-16), with $\bar{\omega}_s = 1$,

$$\ddot{q}_j = -\bar{\omega}_j^2 q_j + \frac{m_{2,j}}{m_{1,j}} \left(\dot{\bar{\omega}}_y - \bar{\omega}_x\right) - 2\nu\bar{\omega}_j \dot{q}_j$$

we obtain in (4.5-1)

$$\dot{\bar{\omega}}_x = -\frac{1-k_y}{k_x} \bar{\omega}_y$$

$$\dot{\bar{\omega}}_y \left(1 - \frac{\Gamma}{1+\Gamma\Delta} \frac{1}{k_y} \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right) = -\bar{\omega}_x \left(\frac{k_x - 1}{k_y} + \frac{1}{k_y} \frac{\Gamma}{1+\Gamma\Delta} \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right)$$
$$+ \frac{\Gamma}{1+\Gamma\Delta} \frac{1}{k_y} \sum_{j=1}^{N} m_{2,j} \left\{ (1-\bar{\omega}_j^2) q_j - 2\nu\bar{\omega}_j \dot{q}_j \right\} \qquad (4.5\text{-}2)$$

$$\dot{\bar{\omega}}_z = 0$$

Investigation of the stability of the motion in the presence of meridional vibrations, as studied in Chapter 5, will be carried out on the basis of equations

(4.3-16) for the modal coordinates

(4.5-2) for the angular rates

with N = 1, 2 or 3, depending on the number of modes retained in the analysis. Since so far we have been considering a single boom located along the +x axis, it is of importance to generalize the analysis to multi-booms configurations. This is done in the following section.

4.6 Generalization to Multiple-Boom Geometry

The equations for therates and modal coordinates were given, for equatorial vibrations, by Equations (4.4-13) and (4.2-25), respectively, and for meridional vibrations by Equations (4.5-2) and (4.3-16) respectively, in the case of a single boom located along the +x axis. In the present section, we proceed to generalize the developments to the case of multiple-boom arrangements located in plane (x,y) (A plane containing axes x_p, y_p, two principal axes of inertia of the ellipsoid in inertia of the rigidified, total spacecraft) (Fig. 4.1).

In order to allow for various possibilities, the following definitions and notations are used

- $k_x \overset{\equiv}{\text{def}} \dfrac{I_x}{I_z}$, $k_y \overset{\equiv}{\text{def}} \dfrac{I_y}{I_z}$ are ratios, smaller than one for quasi-rigid body stability, which relate to principal moments of inertia I_x, I_y, I_z of the total, rigidified structure.

- given the Etkin's number, $\bar{\lambda}_k$, and non-dimensional radius $\xi_{0,k}$, for boom "k", the notation :

- $\Phi_{j,k}(\bar{\lambda}_k, \xi_{0,k})$ is used for the j^{th} modal shape corresponding to these values of $\bar{\lambda}$ and ξ_0 (there is no necessity to distinguish between Φ_j for the equatorial vibrations as opposed to Φ_j for meridional vibration, since they are the same)

- $m_{2,j,k} \equiv \int_{\text{boom } k} \xi_{1,k} \, \Phi_{j,k}(\xi_K) \, d\xi_K$

  $m_{1,j,k} \equiv \int_{\text{boom } k} \bar{\Phi}^2_{j,k}(\xi_K) \, d\xi_K$ $\qquad \xi_{1,k} \underset{\text{def}}{\equiv} \xi_{0,k} + \xi_K$

  and all $\Phi_{j,k}$ are normalized to a unit deflection at the tip.

- $\bar{\omega}_{j,k}$, a function of $\bar{\lambda}_k$, $\xi_{0,k}$, for given j, is the jth eigenfrequency for equatorial vibrations.whereas $\bar{\bar{\omega}}_{j,k}$ is the jth eigenfrequency for meridional vibrations. For the same pair $(\lambda_k, \xi_{0,k})$, we have from Equation

$$\bar{\omega}^2_{j,k} + 1 = \bar{\bar{\omega}}^2_{j,k} \quad \text{(all j, k)}$$

- $q_{j,k}$ is the jth modal coordinate (of type E, or M depending on which equations contain it) for boom k.

- ζ_k is the angle between the boom's undeflected position (an axis normal to $Z_p \equiv z$, thus contained in plane x_p, y_p) and axis x_p of the ellipsoid of inertia.

- $\Gamma_k \equiv \dfrac{(\rho \ell^3)_k}{(I_{z,k})_k}$; $\Delta_k \equiv \frac{1}{3} + \xi_{0,k} + \xi^2_{0,k}$

4.6.1 2 pairs of booms at right angle, along two principal axes of inertia

In this case, we assume that booms (+x, -x) are aligned on x_p, principal axis of inertia, and that booms (+y, -y) are normal to

(+x, -x), thus aligned on principal axis of inertia y_p (Fig. 4-1).

In order to generalize the previously obtained equations for the modal coordinates and angular rates, we observe that in these equations,

(x,y,z) are a r.h.s. system, with

$$\text{BOOM ALONG } +x\text{, deflection } q \text{ along } \begin{cases} +y \text{ in case E} \\ +z \text{ in case M} \end{cases}$$

Now consider the boom along -x. Equations analogous to these derived for the +x boom will apply, substituting

for	the expression
axis x	axis -x
axis y	axis -y
axis z	axis -z
q_x along z	q_{-x} along z

since (-x, -y, -z) is a direct system. The deflection q_{-x}, along -y (i.e. in case E), will be measured, for the sake of convenience, along axis +y, in the same manner as q_{+x} is measured. Therefore, in the analogous equations, written for case E, substitute

for	the expression	
q_{+x} along +y	$-q_{-x}$ along +y	(4.6-1)

Similarly, the substitutions needed are, in the following cases:

<u>boom along +y axis</u> Substitute

for	the expression
axis x	axis y
axis y	axis -x
axis z	axis z
q_{+x} along z	q_{-y} along z
q_{+x} along +y	$-q_{y}$ along +x

(4.6-2)

<u>boom along -y axis</u> Substitute

for	the expression
axis x	axis -y
axis y	axis x
axis z	axis z
q_{+x} along z	q_{-y} along z
q_{+x} along y	q_{-y} along +z

(4.6-3)

Effecting these substitutions in Equations (4.4-13) and (4.2-31), we obtain

<u>Equatorial vibrations</u> (case E)

It should be recalled that $\bar{\omega}_{j,k}$ refers to "E" type, jth eigenfrequency of boom "k". Although this is not done explicitly, the 'ν' could be subscripted to account for different damping ratios in the various booms.

<u>Rates</u>

Booms -x, +y, -y: the equations for $\dot{\bar{\omega}}_x$, $\dot{\bar{\omega}}_y$ in (4.4-13) remain unchanged.

The equations for $\dot{\bar{\omega}}_z$, in (4.4-13), read

-x boom:

$$\dot{\bar{\omega}}_z \left(1 - \frac{\Gamma_{-x}}{1+\Gamma_{-x}\Delta_{-x}} \sum_{j=1}^{N} \frac{m_{2,j,-x}^2}{m_{1,j,-x}}\right) = -\frac{\Gamma_{-x}}{1+\Gamma_{-x}\Delta_{-x}} \sum_{j=1}^{N} \left(2\nu\bar{\omega}_{j,-x}\, \mathring{q}_{j,-x} + \bar{\omega}_{j,-x}^2\, q_{j,-x}\right)$$

+y boom:

$$\dot{\bar{\omega}}_z \left(1 - \frac{\Gamma_y}{1+\Gamma_y\Delta_y} \sum_{j=1}^{N} \frac{m_{2,j,y}^2}{m_{1,j,y}}\right) = -\frac{\Gamma_y}{1+\Gamma_y\Delta_y} \sum_{j=1}^{N} \left(2\nu\bar{\omega}_{j,y}\, \mathring{q}_{j,y} + \bar{\omega}_{j,y}^2\, q_{j,y}\right)$$

-y boom: (4.6-4)

$$\dot{\bar{\omega}}_z \left(1 - \frac{\Gamma_{-y}}{1+\Gamma_{-y}\Delta_{-y}} \sum \frac{m_{2,j,-y}^2}{m_{1,j,-y}}\right) = \frac{\Gamma_{-y}}{1+\Gamma_y\Delta_y} \sum_{j=1}^{N} \left(2\nu\bar{\omega}_{j,-y}\, \mathring{q}_{j,-y} + \bar{\omega}_{j,-y}^2\, q_{j,-y}\right)$$

Modal Coordinates:

-x boom:

$$\ddot{q}_{j,-x} + 2\nu\bar{\omega}_{j,-x}\, \mathring{q}_{j,-x} + \bar{\omega}_j^2 q_{j,-x} = \frac{m_{2,j,-x}}{m_{1,j,-x}} \dot{\bar{\omega}}_z \qquad (4.6\text{-}5)$$

+y boom:

$$\ddot{q}_{j,y} + 2\nu\bar{\omega}_{j,y}\, \mathring{q}_{j,y} + \bar{\omega}_j^2 q_{j,y} = \frac{m_{2,j,y}}{m_{1,j,y}} \dot{\bar{\omega}}_z$$

-y boom:

$$\ddot{q}_{j,-y} + 2\nu\bar{\omega}_{j,-y}\, \mathring{q}_{j,-y} + \bar{\omega}_j^2 q_{j,-y} = -\frac{m_{2,j,-y}}{m_{1,j,-y}} \dot{\bar{\omega}}_z$$

Meridional vibrations (case M)

Again, $\bar{\omega}_{j,k}$ refers to "M" type, jth eigenfrequency of boom "k", and although this is not explicitly done, the ν's could be subscripted to account for different damping ration in the various booms.

Rates

Booms -x, +y, -y: the equation for $\dot{\bar{\omega}}_z$ remains unchanged, in (4.5-2)

The equations for $\dot{\bar{\omega}}_x$, $\dot{\bar{\omega}}_y$ (4.5-2) read, with $b_k \equiv \dfrac{\Gamma_k}{1+\Gamma_k\Delta_k}$; $k = +x, -x, +y, -y$

-x boom:

$$\dot{\bar{\omega}}_x = -\frac{1-k_y}{k_x}\bar{\omega}_y$$

$$\dot{\bar{\omega}}_y \left(1 - \frac{b_{-x}}{k_y} \sum_{j=1}^{N} \frac{m_{2,j,-x}^2}{m_{1,j,-x}}\right) = -\bar{\omega}_x \left(\frac{k_x - 1}{k_y} + b_{-x}\frac{1}{k_y} \sum_{j=1}^{N} \frac{m_{2,j,-x}^2}{m_{1,j,-x}}\right)$$
$$- \frac{b_{-x}}{k_y} \sum_{j=1}^{N} m_{2,j,-x} \left\{ (1-\bar{\omega}_{j,-x}^2) q_{j,-x} - 2\nu\bar{\omega}_{j,-x}\, \mathring{q}_{j,-x} \right\}$$

+y boom.

$$\mathring{\bar{\omega}}_x \left(1 - b_y \frac{1}{k_x} \sum_{j=1}^{N} \frac{m_{2,j,y}^2}{m_{1,j,y}}\right) = -\bar{\omega}_y \left(\frac{1-k_y}{k_x} - \frac{1}{k_x} b_y \sum_{j=1}^{N} \frac{m_{2,j,y}^2}{m_{1,j,y}} \right)$$

$$- b_y \frac{1}{k_x} \sum_{j=1}^{N} m_{2,j,y} \left\{ \left(1-\bar{\omega}_{j,y}^2\right) q_{j,y} - 2\nu \bar{\omega}_{j,y} \mathring{q}_{j,y} \right\}$$

$$\mathring{\bar{\omega}}_y = - \frac{k_x - 1}{k_y} \bar{\omega}_x$$

-y boom.

$$\mathring{\bar{\omega}}_x \left(1 - \frac{b_{-y}}{k_x} \sum_{j=1}^{N} \frac{m_{2,j,-y}^2}{m_{1,j,-y}}\right) = -\bar{\omega}_y \left(\frac{1-k_y}{k_x} - \frac{b_{-y}}{k_x} \sum_{j=1}^{N} \frac{m_{2,j,-y}^2}{m_{1,j,-y}} \right)$$

$$+ \frac{b_{-y}}{k_x} \sum_{j=1}^{N} m_{2,j,-y} \left\{ \left(1-\bar{\omega}_{j,-y}^2\right) q_{j,-y} - 2\nu \bar{\omega}_j \mathring{q}_{j,-y} \right\}$$

$$\mathring{\bar{\omega}}_y = - \frac{k_x - 1}{k_y} \bar{\omega}_x \qquad (4.6\text{-}6)$$

Modal Coordinates:

-x boom:

$$\ddot{q}_{j,-x} + 2\nu\bar{\omega}_{j,x}\dot{q}_{j,-x} + \bar{\omega}^2_{j,x} q_{j,-x} = -\frac{m_{2,j,-x}}{m_{1,j,-x}}\left(\dot{\bar{\omega}}_y - \bar{\omega}_x\right)$$

+y boom

$$\ddot{q}_{j,y} + 2\nu\bar{\omega}_{j,y}\dot{q}_{j,y} + \bar{\omega}^2_{j,y} q_{j,y} = -\frac{m_{2,j,y}}{m_{1,j,y}}\left(\dot{\bar{\omega}}_x + \bar{\omega}_y\right)$$

-y boom:

$$\ddot{q}_{j,-y} + 2\nu\bar{\omega}_{j,-y}\dot{q}_{j,-y} + \bar{\omega}^2_{j,-y} q_{j,-y} = \frac{m_{2,j,-y}}{m_{1,j,-y}}\left(\dot{\bar{\omega}}_x + \bar{\omega}_y\right) \qquad (4.6\text{-}7)$$

4.6.1.1 The Four Different booms

Let

$$b_k \overset{\equiv}{\text{def}} \frac{(\rho\ell^3)_k}{I_z} \qquad \Gamma_k = \frac{(\rho\ell^3)_k}{(I_z h)_k} \qquad \Delta_k = \frac{1}{3} + \xi_{0,k} + \xi^2_{0,k}$$

The equations of motion become

Equatorial vibrations (case E):

Rates $\quad k = +x, -x, +y, -y \;;\; j = 1, 2, \ldots N;$

$$\dot{\bar{\omega}}_x = -\frac{1-k_y}{k_x}\bar{\omega}_y$$

$$\dot{\bar{\omega}}_y = -\frac{k_x - 1}{k_y}\bar{\omega}_x$$

$$\dot{\bar{\omega}}_z \left(1 - \sum_k b_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}\right) = \sum_{j=1}^{N}\Big[b_x\big[2\nu\bar{\omega}_{j,x}\,\dot{q}_{j,x} + \bar{\omega}_{j,x}^2\, q_{j,x}\big] - b_{-x}\big[2\nu\bar{\omega}_{j,-x}\,\dot{q}_{j,-x}$$

$$+\bar{\omega}_{j,-x}^2\, q_{j,-x}\big] - \sum_{j=1}^{N}\Big[b_y\big[2\nu\bar{\omega}_{j,y}\,\dot{q}_{j,y} + \bar{\omega}_j^2 q_{j,y}\big] - b_{-y}\big[2\nu\bar{\omega}_{j,-y}\,\dot{q}_{j,-y} + \bar{\omega}_{j,-y}^2\, q_{j,-y}\big]\Big]$$

(4.6-8)

Modal coordinates:

For j = 1,2,....N; k = +x, -y

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_{j,k}\,\dot{q}_{j,k} + \bar{\omega}_j^2\, q_{j,k} = -\frac{m_{2,j,k}}{m_{1,j,k}}\,\dot{\bar{\omega}}_z \qquad (4.6\text{-}9)$$

For j = 1,2,...N; k = -x, +y

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_{j,k}\,\dot{q}_{j,k} + \bar{\omega}_j^2\, q_{j,k} = \frac{m_{2,j,k}}{m_{1,j,k}}\,\dot{\bar{\omega}}_z$$

Meridional vibrations (case M)

Rates: With k taking the values indicated; j = 1,2,...N;

$$\dot{\bar{\omega}}_x\left(1 - \frac{1}{k_x}\sum_{k=+y,-y} b_k \sum_{j=1}^{N}\frac{m_{2,j,k}^2}{m_{1,j,k}}\right) = -\bar{\omega}_y\left(\frac{1-k_y}{k_x} - \frac{1}{k_x}\sum_{k=+y,-y} b_k\sum_{j=1}^{N}\frac{m_{2,j,k}}{m_{1,j,k}}\right)$$

$$-\frac{1}{k_x}\Big[b_y\sum_{j=1}^{N} m_{2,j,y}\big\{(1-\bar{\omega}_{j,y}^2)q_{j,y} - 2\nu\bar{\omega}_{j,y}\,\dot{q}_{j,y}\big\} \qquad (4.6\text{-}10)$$

$$-b_{-y}\sum_{j=1}^{N} m_{2,j,-y}\big\{(1-\bar{\omega}_{j,-y}^2)q_{j,-y} - 2\nu\bar{\omega}_{j,-y}\,\dot{q}_{j,-y}\big\}\Big]$$

$$\dot{\bar{\omega}}_y\left(1 - \frac{1}{k_y}\sum_{k=+x,-x} b_k\sum_{j=1}^{N}\frac{m_{2,j,k}^2}{m_{1,j,k}}\right) = -\bar{\omega}_x\left(\frac{k_x-1}{k_y} + \frac{1}{k_y}\sum_{k=+x,-x} b_k\sum_{j=1}^{N}\frac{m_{2,j,k}^2}{m_{1,j,k}}\right)$$

$$+\frac{1}{k_y}\Big[b_x\sum_{j=1}^{N} m_{2,j,x}\big\{(1-\bar{\omega}_{j,x}^2)q_{j,x} - 2\nu\bar{\omega}_{j,x}\,\dot{q}_{j,x}\big\}$$

$$-b_{-x}\sum_{j=1}^{N} m_{2,j,-y}\big\{(1-\bar{\omega}_{j,-x}^2)q_{j,-x} - 2\nu\bar{\omega}_{j,-x}\,\dot{q}_{j,-x}\big\}\Big]$$

$$\dot{\bar{\omega}}_z = 0$$

(4.6-10)

Modal coordinates

For j = 1, 2,...N; k = +x, -x, +y, -y;

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_{j,k}\,\dot{q}_{j,k} + \bar{\omega}_j^2\, q_{j,k} = \frac{m_{2,j,k}}{m_{1,j,k}}\left(\dot{\bar{\omega}}_y - \bar{\omega}_x\right) \qquad k = x$$

or

$$= -\frac{m_{2,j,k}}{m_{1,j,k}}\left(\dot{\bar{\omega}}_y - \bar{\omega}_x\right) \qquad k = -x$$

$$= -\frac{m_{2,j,k}}{m_{1,j,k}}\left(\dot{\bar{\omega}}_x + \bar{\omega}_y\right) \qquad k = y \qquad (4.6\text{-}11)$$

$$= \frac{m_{2,j,k}}{m_{1,j,k}}\left(\dot{\bar{\omega}}_x + \bar{\omega}_y\right) \qquad k = -y$$

4.6.1.8 Aligned booms identical; different booms along (+x, +y).

Equations (4.6-8) and (4.6-9), or (4.6-10) and (4.6-11) are simplified, in view of the relations

$$\Gamma_x = \Gamma_{-x}\,;\ \xi_{o,x} = \xi_{o,-x}\,;\ \Delta_x = \Delta_{-x}\,;\ \ell_x = \ell_{-x}\,;\ \bar{\omega}_{j,x} = \bar{\omega}_{j,-x}\,;\ m_{2,j,x} = m_{2,j,-x}\,;\ m_{1,j,x} = m_{1,j,-x}$$

and similar ones for subscripts y, -y. (4.6-12)

Equatorial case:

Rates: With j = 1, 2,...N;

$$\dot{\bar{\omega}}_x = -\frac{1-k_y}{k_x}\,\bar{\omega}_y$$

$$\dot{\bar{\omega}}_y = -\frac{k_x-1}{k_y}\,\bar{\omega}_x$$

$$\dot{\bar{\omega}}_z\left(1 - 2\ell_x\sum_{j=1}^{N}\frac{m_{2,j,x}^2}{m_{1,j,x}} - 2b_y\sum_{j=1}^{N}\frac{m_{2,j,y}^2}{m_{1,j,y}}\right) = \ell_x\sum_{j=1}^{N} m_{2,j,x}\Big[\left(2\nu\bar{\omega}_{j,x}\,\dot{q}_{j,x} + \bar{\omega}_{j,x}^2\, q_{j,x}\right) - \left(2\nu\bar{\omega}_{j,x}\,\dot{q}_{j,-x} + \bar{\omega}_{j,x}^2\, q_{j,-x}\right)\Big] - \ell_y\sum_{j=1}^{N} m_{2,j,y}\Big[\left(2\nu\bar{\omega}_{j,y}\,\dot{q}_{j,y} + \bar{\omega}_j^2\, q_{j,y}\right) - \left(2\nu\bar{\omega}_{j,y}\,\dot{q}_{j,-y} + \bar{\omega}_{j,y}^2\, q_{j,-y}\right)\Big] \qquad (4.6\text{-}13)$$

Modal coordinates:

Same as (4.6-9), with Equations (4.6-12). (4.6-14)

Meridional case:

Rates: With j = 1, 2,....N;

$$\dot{\bar\omega}_x \left(1 - \frac{2}{k_x} \ell_y \sum_{j=1}^{N} \frac{m_{2,j,y}^2}{m_{1,j,y}}\right) = -\bar\omega_y \left(\frac{1-k_y}{k_x} - \frac{2}{k_x} \sum_{j=1}^{N} \frac{m_{2,j,y}^2}{m_{1,j,y}}\right)$$
$$- \frac{1}{k_x} \ell_y \sum_{j=1}^{N} m_{2,j,y} \left\{ \left[\left(1-\bar\omega_{j,y}^2\right) q_{j,y} - 2\nu\bar\omega_{j,y} \dot{q}_{j,y} \right] - \left[\left(1-\bar\omega_{j,y}^2\right) q_{j,-y} - 2\nu\bar\omega_{j,y} \dot{q}_{j,-y} \right] \right\}$$

$$\dot{\bar\omega}_y \left(1 - \frac{2}{k_y} \ell_x \sum_{j=1}^{N} \frac{m_{2,j,x}^2}{m_{1,j,x}}\right) = -\bar\omega_x \left(\frac{k_x-1}{k_y} + \frac{2}{k_y} \ell_x \sum_{j=1}^{N} \frac{m_{2,j,x}^2}{m_{1,j,x}}\right) \tag{4.6-15}$$
$$+ \frac{1}{k_y} \ell_x \sum_{j=1}^{N} m_{2,j,x} \left\{ \left[\left(1-\bar\omega_{j,x}^2\right) q_{j,x} - 2\nu\bar\omega_{j,x} \dot{q}_{j,x} \right] - \left[\left(1-\bar\omega_{j,x}^2\right) q_{j,-x} - 2\nu\bar\omega_{j,x} \dot{q}_{j,-x} \right] \right\}$$

$$\dot{\bar\omega}_z = 0$$

Modal coordinates:

Same as (4.6-11), with Equations (4.6-12) (4.6-16)

4.6.1.3 Identical booms along x, -x, y, -y

In this case, we can use in common for all booms, the notations

$$\bar\omega_j,\ m_{1,j},\ m_{2,j},\ \Gamma,\ \Delta,\ \xi_0,\ \ell$$

Thus Equations (4.6-15) and (4.6-16) are simplified as follows:

Equatorial case:

Rates: With j = 1, 2,...N;

$$\dot{\bar\omega}_x = -\frac{1-k_y}{k_x}\bar\omega_y$$

$$\dot{\bar\omega}_y = -\frac{k_x-1}{k_y}\bar\omega_x$$

$$\dot{\bar\omega}_z \left(1 - 4\ell \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right) = 2\ell \sum_{j=1}^{N} m_{2,j} \left[\bar\omega_j^2 \left(q_{j,x} + q_{j,-y} - q_{j,-x} - q_{j,-y}\right) \right. \tag{4.6-17}$$
$$\left. + 2\nu\bar\omega_j \left(\dot{q}_{j,x} + \dot{q}_{j,-y} - q_{j,-x} - q_{j,y}\right) \right]$$

Modal coordinates: With j = 1, 2,....N;

$$\ddot{q}_{j,x} + 2\nu\bar{\omega}_j\,\dot{q}_{j,x} + \bar{\omega}_j^2\,q_{j,x} = -\frac{m_{2,j}}{m_{1,j}}\,\dot{\bar{\omega}}_z$$

$$\ddot{q}_{j,-x} + 2\nu\bar{\omega}_j\,\dot{q}_{j,-x} + \bar{\omega}_j^2\,q_{j,-x} = \frac{m_{2,j}}{m_{1,j}}\,\dot{\bar{\omega}}_z$$

$$\ddot{q}_{j,y} + 2\nu\bar{\omega}_j\,\dot{q}_{j,y} + \bar{\omega}_j^2\,q_{j,y} = \frac{m_{2,j}}{m_{1,j}}\,\dot{\bar{\omega}}_z \qquad (4.6\text{-}18)$$

$$\ddot{q}_{j,-y} + 2\nu\bar{\omega}_j\,\dot{q}_{j,-y} + \bar{\omega}_j^2\,q_{j,-y} = -\frac{m_{2,j}}{m_{1,j}}\,\dot{\bar{\omega}}_z$$

Meridional case:

Rates: With j = 1, 2,....N;

$$\dot{\bar{\omega}}_x\left(1-\frac{2}{k_x}\,\ell\sum_{j=1}^{N}\frac{m_{2,j}^2}{m_{1,j}}\right) = -\bar{\omega}_y\left(\frac{1-k_y}{k_x} - \frac{2}{k_x}\,\ell\sum_{j=1}^{N}\frac{m_{2,j}^2}{m_{1,j}}\right) - \frac{1}{k_x}\,\ell\sum_{j=1}^{N}m_{2,j}\left\{(1-\bar{\omega}_j^2)(q_{j,y}-q_{j,-y}) - 2\nu\bar{\omega}_j(\dot{q}_{j,y}-\dot{q}_{j,-y})\right\} \qquad (4.6\text{-}19)$$

$$\dot{\bar{\omega}}_x\left(1-\frac{2}{k_y}\,\ell\sum_{j=1}^{N}\frac{m_{2,j}^2}{m_{1,j}}\right) = -\bar{\omega}_x\left(\frac{k_x-1}{k_y} + \frac{2}{k_y}\,\ell\sum_{j=1}^{N}\frac{m_{2,j}^2}{m_{1,j}}\right) + \frac{1}{k_y}\,\ell\sum_{j=1}^{N}m_{2,j}\left\{(1-\bar{\omega}_j^2)(q_{j,x}-q_{j,-x}) - 2\nu\bar{\omega}_j(\dot{q}_{j,x}-\dot{q}_{j,-x})\right\}$$

$$\dot{\bar{\omega}}_z = 0$$

Modes: With j = 1, 2.....N; k = +x, -x, +y, -y

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_j\,\dot{q}_{j,k} + \bar{\omega}_j^2\,q_{j,k} = \frac{m_{2,j}}{m_{1,j}}\left(\dot{\bar{\omega}}_y - \bar{\omega}_x\right) \quad k = x \qquad (4.6\text{-}20)$$

or

$$= \frac{m_{2,j}}{m_{1,j}}\left(\dot{\bar{\omega}}_y - \bar{\omega}_x\right) \quad k = -x$$

or

$$= -\frac{m_{2,j}}{m_{1,j}}\left(\dot{\bar{\omega}}_x + \bar{\omega}_y\right) \quad k = y$$

or

$$= \frac{m_{2,j}}{m_{1,j}}\left(\dot{\bar{\omega}}_x + \bar{\omega}_y\right) \quad k = -y$$

An alternate form of the equations for the rates has been used in the computer programs described in Chapter 5.

Let

$$K_{px} \underset{def}{\equiv} \frac{I_{z,\,hub}}{I_{x,\,hub}}$$

$$K_{py} \underset{def}{\equiv} \frac{I_{z,\,hub}}{I_{y,\,hub}}$$

Assume furthermore that the motion is antisymmetric, i.e. $q = -q_x$, $q_{-y}, = -q_y$: Then

$$k_x = \frac{I_x}{I_z} = \frac{I_{x,\,hub} + 2\Delta\rho\ell^3}{I_{z,\,hub} + 4\Delta\rho\ell^3} = \frac{\frac{1}{K_{px}} + 2\Gamma\Delta}{1 + 4\Gamma\Delta}$$

$$k_y = \frac{I_y}{I_z} = \frac{\frac{1}{K_{py}} + 2\Gamma\Delta}{1 + 4\Gamma\Delta}$$

and the rates, as given in (4.6-17) and (4.6-19), respectively, can be rewritten:

Case E:

$$\dot{\bar{\omega}}_x = - \frac{1 - \frac{1}{K_{py}} + 2\Gamma'\Delta}{\frac{1}{K_{px}} + 2\Gamma'\Delta}\,\bar{\omega}_y \qquad (4.6\text{-}21)$$

$$\dot{\bar{\omega}}_y = - \frac{\frac{1}{K_{px}} - 2\Gamma\Delta - 1}{\frac{1}{K_{py}} + 2\Gamma'\Delta}\,\bar{\omega}_x$$

$$\dot{\bar{\omega}}_z = - \frac{2\Gamma'\left[\sum_{j=1}^{N} m_{2,j}\left[\bar{\omega}_j^{\,2}(q_y - q_x) + 2\nu\bar{\omega}_j(\dot{q}_y - \dot{q}_x)\right.\right.}{1 + 4\Gamma'\left(\Delta - \sum_{j=1}^{N}\frac{m_{2,j}^2}{m_{1,j}}\right)}$$

Case M:

$$\dot{\bar{\omega}}_x = -\frac{\left[1 - \frac{1}{K_{py}} + 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right) + 2\Gamma \sum_{j=1}^{N} m_{2,j}\left[(1-\bar{\omega}_j^2)q_{j,y} - 2\nu\bar{\omega}_j \dot{q}_{j,y}\right]\right.}{\;}$$

$$\dot{\bar{\omega}}_y = -\frac{\frac{1}{K_{px}} - 1 - 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right) - 2\Gamma \sum_{j} m_{2,j}\left[(1-\bar{\omega}_j^2)q_{j,x} - 2\nu\bar{\omega}_j \dot{q}_{j,x}\right]}{\frac{1}{K_{py}} + 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right)}$$

$$\dot{\bar{\omega}}_z = 0 \qquad (4.6\text{-}22)$$

If furthermore, the transverse moments of inertia of the hub are equal, i.e.

$$I_{x,hub} = I_{y,hub}$$

$$K_p = K_{px} = K_{py}$$

the above equations for the rates simplify to

Case E:

$$\dot{\bar{\omega}}_x = -\frac{1 - \frac{1}{K_p} + 2\Gamma\Delta}{\frac{1}{K_p} + 2\Gamma\Delta}\,\bar{\omega}_y$$

$$\dot{\bar{\omega}}_y = -\frac{\frac{1}{K_p} - 2\Gamma\Delta - 1}{\frac{1}{K_p} + 2\Gamma\Delta}\,\bar{\omega}_x$$

$$\dot{\bar{\omega}}_z = -\frac{2\Gamma\left[\sum_{j=1}^{N} m_{2,j}\left[\bar{\omega}_j^2(q_y - q_x) + 2\nu\bar{\omega}_j(\dot{q}_y - \dot{q}_x)\right]\right]}{1 + 4\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right)} \qquad (4.6\text{-}23)$$

Case M:

$$\dot{\bar{\omega}}_x = -\frac{1 - \frac{1}{K_p} + 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right) + 2\Gamma \sum_{j=1}^{N} m_{2,j}\left[(1-\bar{\omega}_j^2)q_{j,y} - 2\nu\bar{\omega}_j \dot{q}_{j,y}\right]}{\frac{1}{K_p} + 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right)}$$

$$(4.6\text{-}24)$$

$$\dot{\bar{\omega}}_y = -\frac{\frac{1}{K_p} - 1 - 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right) - 2\Gamma \sum_{j=1}^{N} m_{2,j}\left[(1-\bar{\omega}_j^2)q_{j,x} - 2\nu\bar{\omega}_j \dot{q}_{j,x}\right]}{\frac{1}{K_p} + 2\Gamma\left(\Delta - \sum_{j=1}^{N} \frac{m_{2,j}^2}{m_{1,j}}\right)}$$

4.6.2 "B" booms in x, y plane, necessarily along principal axes of inertia.

The booms are all contained in plane x_p, y_p, with x_p, y_p as two transverse axes of inertia of the rigidified structure, and are normal to z_p, satellite spin axis. With the notations introduced in the beginning of Section 4.6, ζ_k is the angle between the axis of the boom and axis x.

Let $q_{j,k}$ be the jth modal coordinate of boom k. The equations for the modal coordinates and the rates, written in Sections 4.2 to 4.5 for the "+x boom", along a principal axis of inertia, will be modified as follows:

4.6.2.1 +x boom "k"; angle ζ_k with x_p

Equatorial case.

Modal coordinate:

$$\ddot{q}_{j,k} + 2\nu\,\bar{\omega}_{j,k}\,\dot{q}_{j,k} + \bar{\omega}_j^2\, q_{j,k} = -\frac{m_{2,j,k}}{m_{1,j,k}}\,\mathring{\bar{\omega}}_z \qquad (4.6\text{-}25)$$

Rates: $X \equiv x_p$, $Y \equiv y_p$, $Z \equiv z_p$

$$\mathring{\bar{\omega}}_X = -\frac{1-k_Y}{k_X}\,\bar{\omega}_Y$$

$$\dot{\bar{\omega}}_Y = -\frac{k_X-1}{k_Y}\,\bar{\omega}_X$$

$$\dot{\bar{\omega}}_Z\left(1-\frac{(\rho\ell^3)_k}{I_Z}\sum_{j=1}^{N}\frac{m_{2,j,k}^2}{m_{1,j,k}}\right)=\frac{(\rho\ell^3)_k}{I_Z}\sum_{j=1}^{N} m_{2,j,k}\left[2\nu\bar{\omega}_{j,k}\dot{q}_{j,k}+\bar{\omega}_{j,k}^2\,q_{j,k}\right] \qquad (4.6\text{-}26)$$

Meridional case

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_{j,k}\,\dot{q}_{j,k} + \bar{\omega}_j^2\, q_{j,k} = -\frac{m_{2,j,k}}{m_{1,j,k}}\left(-\mathring{\bar{\omega}}_X \sin\zeta_k + \mathring{\bar{\omega}}_Y\cos\zeta_k - \bar{\omega}_X\cos\zeta_k - \bar{\omega}_Y\sin\zeta_k\right) \qquad (4.6\text{-}27)$$

since $\dot{\bar{\omega}}_y$ in (4.3-16) becomes $-\dot{\bar{\omega}}_x \sin\zeta_k + \dot{\bar{\omega}}_y \cos\zeta_k$

$\bar{\omega}_x$ in (4.3-16) becomes $\bar{\omega}_x \cos\zeta_k + \bar{\omega}_y \sin\zeta_k$

Rates:

In the case of a "+x" boom, Equation (4.5-1) shows that the vibrations parallel to the z-axis generate a torque along the direction normal to "+x", i.e. "+y" having projections:

$$\left(\rho \ell^3\right)_k \int_{\text{boom } k} \left(\ddot{\eta}+\eta\right) \xi_1 \, d\xi \times \left(-\sin\zeta_k\right) \qquad \text{ALONG } x_p$$

$$\left(\rho \ell^3\right)_k \int_{\text{boom } k} \left(\ddot{\eta}+\eta\right) \xi_1 \, d\xi \times \left(\cos\zeta_k\right) \qquad \text{ALONG } y_p$$

The equations for the rates will read, before dividing by I_x, I_y respectively,

$$I_x \dot{\bar{\omega}}_x + \left(I_z - I_y\right)\bar{\omega}_y = \left(\rho \ell^3\right)_k \left(-\sin\zeta_k\right) \sum_{j=1}^{N} m_{2,j,k} \left[\left(1-\bar{\omega}_{j,k}^2\right) q_{j,k} - 2\nu\bar{\omega}_{j,k} \dot{q}_{j,k}\right]$$

$$+\left(\rho \ell^3\right)_k \left(-\sin\zeta_k\right) \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}} \left(-\dot{\bar{\omega}}_x \sin\zeta_k + \dot{\bar{\omega}}_y \cos\zeta_k - \bar{\omega}_x \cos\zeta_k - \bar{\omega}_y \sin\zeta_k\right)$$

$$I_y \dot{\bar{\omega}}_y + \left(I_x - I_z\right)\bar{\omega}_x = \left(\rho \ell^3\right)_k \left(\cos\zeta_k\right) \sum_{j=1}^{N} m_{2,j,k} \left[\left(1-\bar{\omega}_{j,k}^2\right) q_{j,k} \right. \tag{4.6-28}$$

$$\left. - 2\nu\bar{\omega}_{j,k} \dot{q}_{j,k}\right] + \left(\rho \ell^3\right)_k \left(\cos\zeta_k\right) \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}} \left(-\dot{\bar{\omega}}_x \sin\zeta_k + \dot{\bar{\omega}}_y \cos\zeta_k - \bar{\omega}_x \cos\zeta_k\right.$$

$$\left. - \bar{\omega}_y \sin\zeta_k\right)$$

$$\dot{\bar{\omega}}_z = 0$$

Now define the following coefficients

$$A_{11} \underset{def}{\equiv} I_x - (\rho \ell^3)_k \sin^2 \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$A_{12} \underset{def}{\equiv} (\rho \ell^3)_k \sin \zeta_k \cos \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$A_{21} \underset{def}{\equiv} (\rho \ell^3)_k \sin \zeta_k \cos \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$A_{22} \underset{def}{\equiv} I_y - (\rho \ell^3)_k \cos^2 \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$a_{11} \underset{def}{\equiv} \frac{A_{11}}{I_x}$$

$$a_{12} \underset{def}{\equiv} \frac{A_{12}}{I_x}$$

$$a_{21} \underset{def}{\equiv} \frac{A_{21}}{I_y}$$

$$a_{22} \underset{def}{\equiv} \frac{A_{22}}{I_y}$$

Let

$$f_{1,k} \underset{\text{def}}{\equiv} \frac{(\rho\ell^3)_k}{I_X}\left(-\sin\zeta_k\right)\sum_{j=1}^{N}\Big\{m_{2,j,k}\left[\left(1-\bar{\omega}_{j,k}^{2}\right)q_{j,k}-2\nu\bar{\omega}_{j,k}\dot{q}_{j,k}\right] + \frac{m_{2,j,k}^{2}}{m_{1,j,k}}\left(-\bar{\omega}_X\cos\zeta_k-\bar{\omega}_y\sin\zeta_k\right)$$

$$f_{2,k} \underset{\text{def}}{\equiv} \frac{(\rho\ell^3)_k}{I_y}\left(\cos\zeta_k\right)\sum_{j=1}^{N}\Big\{m_{2,j,k}\left[\left(1-\bar{\omega}_{j,k}^{2}\right)q_{j,k}-2\nu\bar{\omega}_{j,k}\dot{q}_{j,k}\right] + \frac{m_{2,j,k}^{2}}{m_{1,j,k}}\left(-\bar{\omega}_X\cos\zeta_k-\bar{\omega}_y\sin\zeta_k\right)\Big\}$$

System (4.6-28) is rewritten

$$a_{11}\overset{\circ}{\bar{\omega}}_X + a_{12}\overset{\circ}{\bar{\omega}}_y = f_{1,k}$$

$$a_{21}\overset{\circ}{\bar{\omega}}_X + a_{22}\overset{\circ}{\bar{\omega}}_y = f_{2,k}$$

Let $D_k = a_{11}a_{22}-a_{21}a_{12} = 1-(\rho\ell^3)_k\left(\frac{\sin^2\zeta_k}{I_X}+\frac{\cos^2\zeta_k}{I_Y}\right)\sum_{j=1}^{N}\frac{m_{2,j,k}^{2}}{m_{1,j,k}}$

Then the equations for the rates are

$$\overset{\circ}{\bar{\omega}}_X = \frac{a_{22}f_{1,k}-a_{12}f_{2,k}}{D_k}$$

$$\overset{\circ}{\bar{\omega}}_y = \frac{a_{11}f_{2,k}-a_{21}f_{1,k}}{D_k} \qquad (4.6\text{-}29)$$

$$\overset{\circ}{\bar{\omega}}_Z = 0$$

4.6.2.2 General case: "B" booms, making angles $\zeta_k (k = 1,\ldots,B)$ with x_p.

Equations (4.6-28) will, in the general case of B booms, at angles ζ_k $(k = 1,\ldots,B)$, have r.h. sides with sums over k, in addition to the summation over j. Important note: all modal displacements are referred to the +z axis (case M) or to the normal "y_k" to the boom "x_k" (in case E) such that (x_k, y_k, z) is a direct system.

Equatorial case:

Modal coordinates:

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_{j,k}\,\mathring{q}_{j,k} + \bar{\omega}^2_{j,k}\,q_{j,k} = -\frac{m_{2,j,k}}{m_{1,j,k}}\,\mathring{\bar{\omega}}_z \qquad (4.6\text{-}30)$$

in which expression (4.6-31) is substituted.

Rates:

$$\mathring{\bar{\omega}}_x = -\frac{1-k_y}{k_x}\,\bar{\omega}_y$$

$$\mathring{\bar{\omega}}_y = -\frac{k_x-1}{k_y}\,\bar{\omega}_x \qquad (4.6\text{-}31)$$

$$\mathring{\bar{\omega}}_z\left(1-\frac{1}{I_z}\sum_{k=1}^{B}(\rho\ell^3)_k\sum_{j=1}^{N}\frac{m^2_{2,j,k}}{m_{1,j,k}}\right) = \frac{1}{I_z}\sum_{k=1}^{B}\left\{(\rho\ell^3)_k\sum_{j=1}^{N}m_{2,j,k}\left[2\nu\bar{\omega}_{j,k}\,\mathring{q}_{j,k} + \bar{\omega}^2_{j,k}\,q_{j,k}\right]\right\}$$

Meridional case:

Modal coordinates:

$$\ddot{q}_{j,k} + 2\nu\bar{\omega}_{j,k}\,\mathring{q}_{j,k} + \bar{\omega}^2_j\,q_{j,k} = \frac{m_{2,j,k}}{m_{1,j,k}}\left(-\mathring{\bar{\omega}}_x\sin\zeta_k + \mathring{\bar{\omega}}_y\cos\zeta_k - \bar{\omega}_x\cos\zeta_k - \bar{\omega}_y\sin\zeta_k\right) \qquad (4.6\text{-}32)$$

in which expression (4.6-33) is substituted

Defining

$$c_{11} \underset{\text{def}}{\equiv} 1 - \sum_{k=1}^{B} \frac{(\rho \ell^3)_k}{I_X} \sin^2 \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$c_{12} \underset{\text{def}}{\equiv} \sum_{k=1}^{B} \frac{(\rho \ell^3)_k}{I_X} \sin \zeta_k \cos \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$c_{21} \underset{\text{def}}{\equiv} \sum_{k=1}^{B} \frac{(\rho \ell^3)_k}{I_Y} \sin \zeta_k \cos \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$c_{22} \underset{\text{def}}{\equiv} 1 - \sum_{k=1}^{B} \frac{(\rho \ell^3)_k}{I_Y} \cos^2 \zeta_k \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$D \underset{\text{def}}{\equiv} c_{11} c_{22} - c_{21} c_{12}$$

$$= 1 - \sum_{k=1}^{B} (\rho \ell^3)_k \left(\frac{\sin^2 \zeta_k}{I_X} + \frac{\cos^2 \zeta_k}{I_Y} \right) \sum_{j=1}^{N} \frac{m_{2,j,k}^2}{m_{1,j,k}}$$

$$f_1 \underset{\text{def}}{\equiv} \sum_{k=1}^{B} \frac{(\rho \ell^3)_k}{I_X} (-\sin \zeta_k) \sum_{j=1}^{N} \Big\{ m_{2,j,k} \left[(1 - \bar{\omega}_j^2) q_{j,k} - 2\nu \bar{\omega}_{j,k} \dot{q}_{j,k} \right] + \frac{m_{2,j,k}^2}{m_{1,j,k}} \left(-\bar{\omega}_X \cos \zeta_k - \bar{\omega}_Y \sin \zeta_k \right) \Big\}$$

$$f_2 \underset{\text{def}}{\equiv} \sum_{k=1}^{B} \frac{(\rho \ell^3)_k}{I_X} (\cos \zeta_k) \sum_{j=1}^{N} \Big\{ m_{2,j,k} \left[(1 - \bar{\omega}_j^2) q_{j,k} - 2\nu \bar{\omega}_{j,k} \dot{q}_{j,k} \right] + \frac{m_{2,j,k}^2}{m_{1,j,k}} \left(-\bar{\omega}_X \cos \zeta_k - \bar{\omega}_Y \sin \zeta_k \right) \Big\}$$

we obtain the equations

$$\begin{aligned} \dot{\bar{\omega}}_X &= \frac{c_{22} f_1 - c_{12} f_2}{D} \\ \dot{\bar{\omega}}_Y &= \frac{c_{11} f_2 - c_{21} f_1}{D} \\ \dot{\bar{\omega}}_Z &= 0 \end{aligned} \qquad (4.6\text{-}33)$$

The above formulation is the most general that will be considered in this work.

4.6.3 System considering meridional and equatorial vibrations simultaneously. (B booms in X, Y plane, not necessarily along a principal axis of inertia).

In the Lagrangian formulation, for an elastic displacement

$$\vec{W}_{total} = W_E(x)\,\vec{I}_y + W_M(x)\,\vec{I}_z$$

$$V = \sum_{\substack{ALL\\BOOMS}} \frac{EI}{2} \int_{boom} \left[\left(\frac{\partial^2 W_E}{\partial x^2} \right)^2 + \left(\frac{\partial^2 W_M}{\partial x^2} \right)^2 \right] ds + O(\epsilon^3)$$

and the kinetic energy is, if $\eta_E = \dfrac{w_E}{\ell_k}$, $\eta_M = \dfrac{w_M}{\ell_k}$, etc.,

$$T = T_{\substack{RIGID\\BODY}} + \frac{1}{2} \sum_{\substack{ALL\\BOOMS}} (\rho \ell^3)_k\, \omega_s^2 \int_{boom} \Big(\dot{\eta}_{E,k}^2 + \dot{\eta}_{M,k}^2 + \eta_{E,k}^2$$

$$+ 2\bar{\omega}_z \xi_{1,k} \dot{\eta}_{E,k} - 2\bar{\omega}_x \xi_{1,k} \eta_{M,k} - 2\bar{\omega}_y \xi_{1,k} \dot{\eta}_{M,k} - \frac{1}{2}\Big[(1-\xi^2)$$

$$+ 2\xi_0(1-\xi) \Big] \left[\left(\frac{\partial \eta_{M,k}}{\partial \xi} \right)^2 + \left(\frac{\partial \eta_{E,k}}{\partial \xi} \right)^2 \right] \Big) d\xi$$

It can readily be seen that when η_E, η_M are expanded in their modes $\Phi(\bar{\lambda}, \xi_o)$, with associated frequencies $\bar{\omega}_{\underset{E}{j,h}}$, $\bar{\omega}_{\underset{M}{j,k}}$, the corresponding modal equations for

$$q_{\underset{E}{j,k}} \quad q_{\underset{M}{j,k}}$$

are uncoupled. For "E", only $\dot{\bar{\omega}}_z$ will appear in the r.h. side, and this quantity is a function of the $q_{\underset{E}{j,k}}$ only. For "M", only $\bar{\omega}_x$, $\bar{\omega}_y$, $\dot{\omega}_x$, $\dot{\omega}_y$ will appear in the r.h. sides, and these quantities are functions of the $q_{\underset{M}{j,k}}$. Hence, in the total system,

- the two first equations of (4.6-33) are those for $\dot{\omega}_X$, $\dot{\omega}_Y$

- the last equation of (4.6-31) is the one for $\dot{\omega}_z$
- the modal coordinate equations for q_{j,k_M} are given by (4.6-32)
- the modal coordinate equations for q_{j,k_E} are given by (4.6-30)

4.7 Conclusion

The equations of motion of the spinning spacecraft having flexible appendages have been derived in a rather general case, using the modes of the rotating structure at the nominal spin rate, and for a central hub of non-zero radius. They were found to be in agreement with some other published results[4-3] in the limit case of a central body of zero radius, and can be used with profit in the numerical simulation of flexible spacecraft motions.

REFERENCES - Chapter 4

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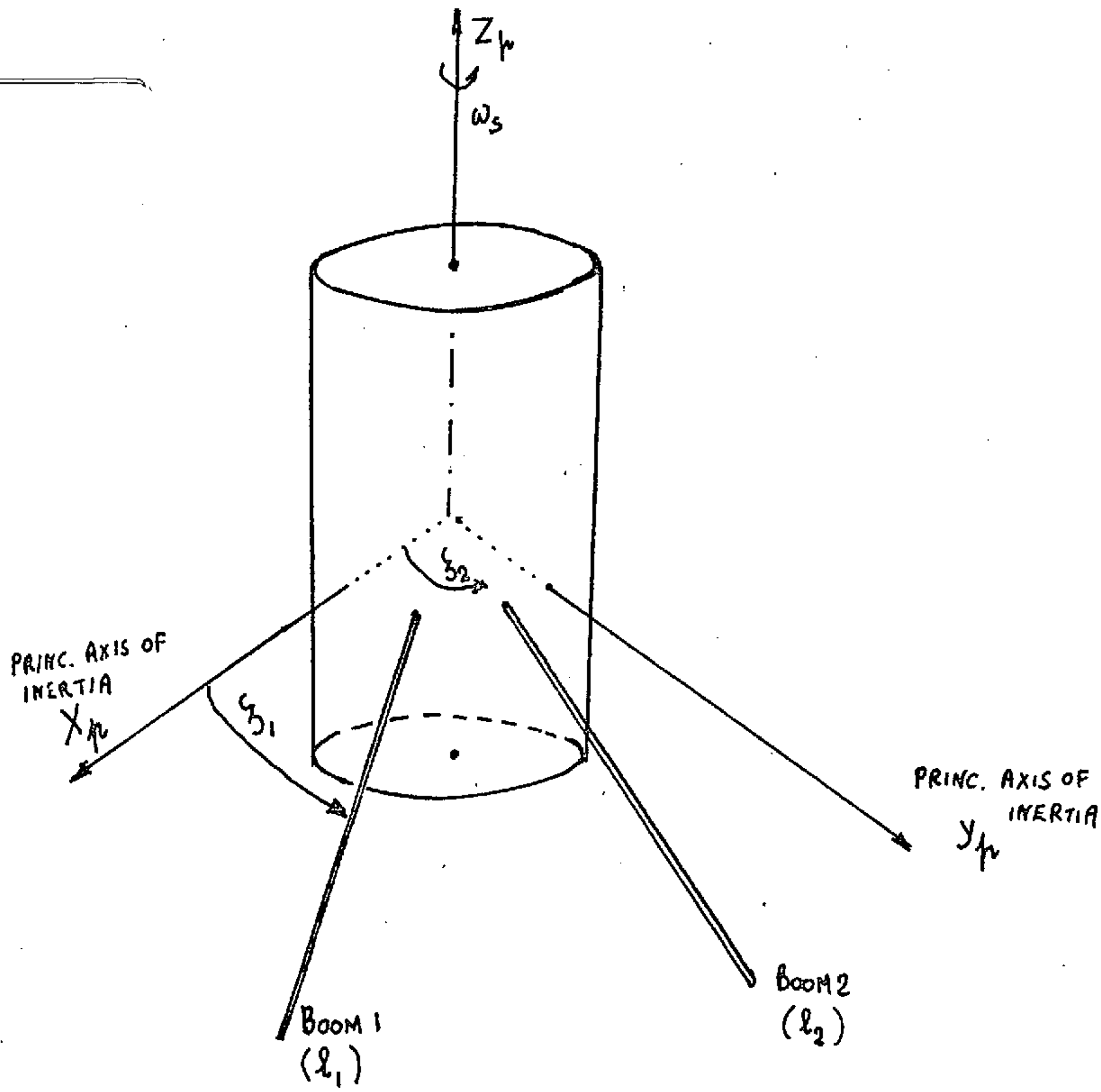


FIG. 4- 1 MULTI-BOOM GEOMETRY

CHAPTER 5

Simulation of the Satellite Attitude Motion and Stability Studies

5.1 Motivation

In the present chapter, we present a simulation study of the evolution with time of the satellite attitude, from which stability charts can be obtained for use by the satellite designer. Of particular interest is the "nutational divergence" phenomenon, in which the satellite, although stable if it were "quasi-rigid", exhibits a steadily increasing nutation angle. Its spin axis thus drifts away from the invariant angular momentum vector, on which it is assumed to be aligned initially. This instability is due only to the dissipative motion of the elastic appendage.

To this effect, a set of computer programs, "FLEXAT", has been developed which numerically integrates the equations of motion and prints or graphically outputs the variables of interest. This program quite markedly differs from earlier versions we have used in the work, as will be explained later. The version given here accommodates three modes of the rotating structure and a dissymetric central body, and since it permits an easy visualization of the qualitative features of the attitude motion, it should appeal to the satellite project engineer.

5.2 A Package for the Simulation of the Spacecraft with Flexible Appendages.

5.2.1 Generalities

FLEXAT is a set of programs, written in FORTRAN V, which were mostly run on the UNIVAC 1108 at Carnegie-Mellon University. It is composed of the following parts:

a) A short "MAIN" program calling on the relevant SUBROUTINES.

b) A subroutine CASEM 2 called upon to study the stability of the meridional vibrations. This subroutine internally calls on its own subroutine RATES, which computes the angular rates $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$.

c) A subroutine CASEE 2 called upon to simulate the equatorial vibrations. Again, this subroutine internally calls on its own subroutine RATES, which computes the angular rates $\dot{\bar{\omega}}_x$, $\dot{\bar{\omega}}_y$. In particular, this subroutine can be used to simulate the nutational divergence occurring when the GMI (greater moment of inertia) rule is violated, for the rigidified body.

d) A subroutine SEARCH (NDS) called by the MAIN program and yielding the eigenfrequencies $\bar{\omega}_j$ (up to j=3, if required) of the rotating structure, corresponding to the specified values of $\bar{\lambda}$, ξ_0. This subroutine, for the essential part, is the same as that described in Section 2.4.

e) A subroutine PLOT, called internally in either CASEM 2 or CASEE 2, giving a graphical output of the evolution with time of the satellite nutation angle, over a number of satellite spin periods (generally taken to be 10 to 20).

Each of these parts is now discussed in more detail.

5.2.2 Program MAIN

In this program, COMMON, DIMENSION etc. are given. Then the "unchanging parameters" are specified by cards. The listing given at the end of this chapter, for example, specifies

NSKP : skip the printing of 60% of the results is desired (NSKP = 1) else NSKP = 1; all results plotted in both cases

NØRU=NSUP=3: include 3 modal coordinates for each boom.

XNØ(1)=0.05: the "x-boom" and the "-x-boom" have modal deflections (1st mode) equal to $\pm$ 0.05 times the length of the boom

XNØ(2) ...YNØ(3): the$\pm$"x-boom" and the$\pm$"y-boom" have zero modal deflections, for the 2nd and 3rd modes.

NU(1) NU(3)=0.05: same damping ratio on the 3 modes

CASE = 'M' : meridional vibrations

SIO = 0 : value of ξ_o

LAM = 10 : value of $\bar{\lambda}$

MGIV : a switch. If equal to 1, the eigenfrequencies $\bar{\omega}_j$ and $m_{1,j}$, $m_{2,j}$ are given as data (they are assumed to be known from a previous study, or from a table). If equal to 0, the $\bar{\omega}_j$ and the other quantities will be obtained "on line" by calling SEARCH(1) (in case E) or SEARCH (0) (in case M)

GAM : Γ in the developments of Chapter 4.

PKX,PKY : ratios $K_{px} = I_{z,hub}/I_{x,hub}$; $K_{py} = I_{z,hub}/I_{y,hub}$ These measure dissymmetry of the ellipsoid of inertia of the central body.

PREC: : the integration interval in time is equal to $\frac{\tau_{spin}}{75}$ or $\frac{2\pi/\omega_j}{75}$ (with j = PREC) whichever the smaller. It has been found sufficient to take PREC = 1.

MAXP : maximum number of such periods (defined under PREC) to be considered.

MODES : 3 (should be the same as NØRU, NSUP). Three modes are retained.

5.2.3 Subroutine SEARCH (NDS)

This subroutine has already been described in Chapter 2. It obtains $\bar{\omega}_j$ in the relevant case (E or M) for j = 1,2,...NSUP. Note that

a) NDS is an argument given in MAIN (0 for case M;1 for case E)

b) SEARCH is bypassed if MGIV = 1, i.e. if the eigenfrequencies in the case of interest are externally given, other than completed on line.

5.2.4 Subroutines CASEM2, CASEE2

This subroutine, fed with the $\bar{\omega}_j$, $m_{1,j}$, $m_{2,j}$ values obtained from data or computed in SEARCH, proceeds to integrate equations (2.2-8) or (2.3-5), as the case may be, if <u>MGIV = 0</u>, and bypasses the procedure if <u>MGIV = 1</u>.

It then proceeds to compute the quantity

$$\sum_{j=1}^{NSUP} \frac{m_{2,j}^2}{m_{1,j}}$$

The equations which are integrated are those for

$q_{x,j}$, $q_{y,j}$, $\bar{\omega}_x$, $\bar{\omega}_y$, $\bar{\omega}_z$

The system is thus of order 4 NSUP + 3. The rates are computed in an internal subroutine "RATES". Antisymmetric vibrations are assumed,so that $q_{x,j} = -q_{-x,j}$; $q_{y,j} = -q_{-y,j}$. The four booms are assumed to have the same geometric and structural properties (thus same ξ_o, $\bar{\lambda}$, Γ, $\rho\ell^3$), to be along the principal axes of inertia of the rigidified structure ($\zeta_k = 0, \pi$ for the x-booms, $\zeta_k = \frac{\pi}{2}, \frac{3\pi}{2}$ for the y-boom, in Chapter 4). The ellipsoid of inertia need not be of revolution ($K_{px} \neq K_{py}$). Thus the relevant equations have been written as equations

(4.6-20) and (4.6-24) for program CASEM2

(4.6-18) and (4.6-23) for program CASEE2

Different assumptions (booms of different length, structural properties) could easily be considered by the user, for any special application, after a rather simple rewrite of the equations, as given in Chapter 4, or a suitable distinction between "Γ_x", "Γ_y",... etc. rather than the common "Γ"... adopted here.

The method of integration is RUNGE-KUTTA with fixed step ,
the latter being computed in the program as some function of the spin period or of the vibration period of the jth mode, as precised in 5.2.2. under "PREC".

The output consists of a print of the case data, of the quantities $\sum_{j=1}^{NSUP} \frac{m_{2,j}^2}{m_{1,j}}$; Δ, ν_1, ν_2, ν_3; $\bar{\omega}_j$, $m_{1,j}$, $m_{2,j}$ (j=1,...NSUP); H initial= $\frac{|\vec{H}_o|}{I_{zh}}$ (assuming $\vec{H}_o$ and z are initially aligned); then tables giving

$q_{x,1}$	$q_{y,1}$	$q_{x,2}$	$q_{y,2}$	$q_{x,3}$	$q_{y,3}$	$\bar{\omega}_x$	θ (angle of nutation, degrees)	STEP

There exists an option to skip the printing of the first 60% of the results over the time interval considered, which makes sense if one is only interested at looking at the long-term behavior.

5.2.5 Subroutine PLØT

The PLOT routine graphically presents the results of the above computation. PLOT is internal to CASEM2 or CASEE2, as the case may be.

5.3 Results from simulation study, using FLEXAT

5.3.1 Comparison between the present and some previous results

As compared to the approach previously taken by J. Rakowski and the present author[5-1,5-2], the equations used in the present simulation do not include "extra" non-linear terms such as q_x^2, q_y^2, $\omega_x\omega_y$... Including these terms, although they appear in the derivations of Chapter 4, did not seem fully consistent with writing the contributions to the kinetic and elastic energy with some terms of order 3 of smallness neglected (such would be the case, for instance, if $\int\{\cdots\}dx = \int\{\cdots\}ds$, with the integrand of first order of smallness).

However, strictly for the sake of comparison, the stability boundaries, derived as explained in 5.3.2, were compared in a large number of cases using, on one hand, the equation with the extra non-linear terms, and on the other hand the equations obtained in Chapter 4.

In no cases were the differences of much significance. All were well within the sampling interval ($K_p \pm .016$).

5.3.2 Parametrization of the stability chart

Following the notation adopted earlier [5-1,5-2], it is proposed to define a stability chart as follows, in the symmetric case ($K_{px} = K_{py} = K_p$)(See Fig. 5.1)

- abscissa: $K_p = \frac{I_{p,hub}}{I_{z,hub}}$, a measure of the asymmetry of the ellipsoid of inertia of the central body.

- ordinate: $\Gamma = \frac{\rho \ell^3}{I_{z,h}}$, a measure of the relative importance of the inertia of a boom ($\frac{1}{3}\rho\ell^3$, if $\xi_o = 0$), and the inertia of the hub. All things being equal, small booms of small mass will give small values of Γ.

- parameter of the plane:

  ξ_o = fixed non-dimensional radius of the hub (referred to the booms length)

- parameters of the curves:

  $\bar{\lambda} = \frac{\pi\ell^4}{EI}\omega_s^2 \div \left(\frac{\omega_s}{\omega_{cant}}\right)^2$, a ratio of certrifugal to elastic forces, large for high spin rates or very flexible booms (E, I small; $\rho\ell^4$ large)

Thus $\bar{\lambda}$ = constant curves will be drawn on the (K_p,Γ) plane, for ξ_o = constant, corresponding to the observed limit of stability, i.e. a point, at given Γ, ξ_o, $\bar{\lambda}$, such that any slight increase in K_p causes stability of the observed motion,the nutation angle tending asymptotically to zero; whereas to the left of it (decreasing K_p), the motion is observed to be unstable, the nutation angle steadily increasing with time.

In the asymmetric case, one more degree of freedom exists, and the chart will draw $\bar{\lambda}$ = constant curves, corresponding to the observed limit of stability, for given ξ_o, K_{py}, in a (Γ, K_{px}) plane of representation.

5.3.3 The GMI rule

As described in [5-3], a rigid body undergoing a torque-free motion about its center of mass, but having internal energy dissipation, has a stable spinning motion only about its maximum axis of inertia, i.e. if

$$\frac{I_z}{I_x} \text{ and } \frac{I_z}{I_y} > 1 \tag{5.3-1}$$

If one of these ratios was one, there would be no preferred axis of rotation about which the satellite would spin after an initial nutation has been removed by energy dissipation. Condition (5.3-1) is commonly referred to as the GMI rule (or greatest moment of inertia rule).

In the stability chart, planes described above, condition (5-1) will be represented, in the symmetrical case

$$I_x = I_y = I_p$$

by a locus of equation

$$2\Gamma\Delta > \frac{1}{K_p} - 1 \tag{5.3-2}$$

or

$$2\Gamma\left(\frac{1}{3} + \xi_o + \xi_o^2\right) > \frac{1}{K_p} - 1 \tag{5.3-3}$$

These curves will, whatever the value of ξ_o, tend to the common point

$$\Gamma \to 0 \qquad K_p \to 1$$

which they should not include. This corresponds to the case where the satellite has no flexible appendages ($\rho \ell^3 \to 0$) and a spherical ellipsoid of inertia. The curves are shifted to the left as ξ_o increases (Fig. 5.1). Their $\bar{\lambda}$ parameter is $\bar{\lambda} = 0$.

Conclusion

For the stability of the satellite with perfectly rigid appendages, and of the satellite with flexible appendages in the presence of equatorial vibrations (as explained in 5.3.4), the greatest moment of inertia rule

$$I_z > I_x , I_y$$

should be satisfied for the total, rigidified satellite. On the (Γ, K_p) stability charts, the design point

$$(\Gamma, K_p)$$

for given

$$\xi_o , \bar{\lambda}$$

should be to the right (i.e. in the region not including the origin) of the Quasi-Rigid (QR) locus given by Equation (5-3).

5.3.4 Stability with equatorial vibrations

Stability in the presence of equatorial vibrations, was found to be equivalent to quasi-rigid body stability. The stability condition for case E is thus the same as the Q.R. body condition given in Equation (5.3-3). This result is in good agreement with Hughes and Fung[5-4] analysis in the case where $\xi_o = 0$. Two examples are given in Fig. 5-2 and 5-3.

5.3.5.1 Stability charts (case M), using three-mode analysis

Using the FLEXAT program with subroutine CASEM2, and retaining the three modes in the simulation, figures such as 5-4 to 5.7 can be produced. Each of them corresponds to the same value of $\Gamma = 10$ and $\xi_o = 0.1$. For $\bar{\lambda} = 100$, two values of K_p are considered corresponding to a slightly unstable or a slightly stable condition (Fig. 5.4, 5.6). The same applies to a higher $\bar{\lambda}$ case ($\bar{\lambda} = 1{,}000$) (Fig. 5.6, 5.7)

The final results of the three-mode stability analysis in the presence of meridional vibrations are summarized on charts 5-8, 5-9, 5-10 for values of $\bar{\lambda} = 0$ (Quasi-rigid body case) to $\bar{\lambda} = 10{,}000$, and for $\xi_o = 0, 0.1, 0.25$.

IMPORTANT NOTE: When using program FLEXAT, with subroutines SEARCH and CASEM2, for $\bar{\lambda} \gtrsim 5{,}000$, the values of the relevant frequency and modal quantities:

$$\bar{\omega}_{j,M}, \quad m_{1,j}, \quad m_{2,j} \tag{5.3-4}$$

should be given as input data, using option MGIV = 1, or described in Section 5.2.2. Quantities (5.3-4) cannot be obtained on line using program SEARCH DP, for such high values of $\bar{\lambda}$. They have been obtained using a multiple precision version (OS-MP or NP-package) of SEARCH, which is rather time-consuming and should be run only to set up tables such as in Section 2.8, for interpolation purposes.

5.3.5.2 Effect of higher modes, and of modal truncation

As the tables in Section 2.8 show, the effect of higher order modes ($j = 2,3$) on the motion parameters is as follows:

a) $\sum_{j=1}^{NSUP} m_{2,j}^2 / m_{1,j}$

For small values of $\bar{\lambda}$, the changes of this sum by increasing NSUP from 1 to 2,3 is at most 2.5% for $\xi_o = 0$, and 9% for $\xi_o = 0.25$.

For large values of $\bar{\lambda}$ ($\bar{\lambda}$ = 5,000), the corresponding changes are 0.03% for ξ_o = 0, and 0.5% for ξ_o = 0.25.

b) $\frac{m_{2,j}}{m_{1,j}}$ (amplitude in r.h. side of jth modal equation)

It can be seen that this ratio is at most 25% (for j = 2) of the value corresponding to j = 1, when j is increased to 2,3 .

c) $m_{2,j}$ (amplitude of some terms in the r.h. side of the rate equations).

The same comments apply to $m_{,j}$.

To assess the effect of higher modes qualitatively, it should be remembered that, when non-dimensionalized by ω_z,

$$\bar{\omega}_{j,M} > 1 \qquad\qquad j = 1, \ldots NSUP$$

and the forcing frequency (precision frequency in body-fixed axes) on the terms would be, for $|q_x|$, $|q_y|$ << 1,

$$\bar{\omega}_{F,el.} \approx \frac{1 - \frac{1}{K_p} + 2\Gamma\left(\Delta - \frac{m_2^2}{m_1}\right)}{\frac{1}{K_p} + 2\Gamma\left(\Delta - \frac{m_2^2}{m_1}\right)} < 1$$

as opposed to

$$\bar{\omega}_{F,rigid} = \frac{1 - \frac{1}{K_p} + 2\Gamma\Delta}{\frac{1}{K_p} + 2\Gamma\Delta} < 1$$

for a quasi-rigid body.

Note that $\frac{m_2^2}{m_1}$ is always smaller than Δ . Typically, for

$\xi_o = 0.1$, $\Delta = 0.443$

$\frac{m_2^2}{m_1}$ = 0.419 for $\bar{\lambda} = 0$; 0.430 for $\bar{\lambda} = 100$; 0.437 for $\bar{\lambda} = 1{,}000$.

Therefore, in an approximate sense, it can be said that angular rates $\bar{\omega}_F$ will not appreciably excite modes 2,3,... which are larger than $\bar{\omega}_1$ by a factor of several units at least.

With these observations in mind, we now discuss the conclusions of a detailed study of the effect of modal truncation on the stability charts (Γ, K_p; constant $\bar{\lambda}$, ξ_o).

It was indeed observed in the simulation that higher modes never developed to amplitudes of more than a few % of the amplitudes of the first mode, assuming i.e. which can be considered as "normal" for the initial deflection, namely close to the shape of the first mode Φ (ξ).

Within the accuracy retained in establishing the stability charts ($K_p \pm 0.015$), no noticeable difference could be reported between the stability chart determined here on the basis of three modal coordinates for each boom, and that we obtained on the basis of a single modal coordinate . Se ing times, however, were larger.

The results of the 3-mode analysis, using program FLEXAT, are summarized in Figures 5.8, 5.9, 5.10.

5.3.5.3 Effect of some higher order terms

As was mentioned in 5.1, there was a lack of consistency in retaining some non-linear terms of order 2 in the equations and neglecting some others. Equations (4.6-20) and (4.6-24) were used in the present stability simulation. It should be noted that little difference resulted in the stability charts. The angles of nutation, however, are computed here by

$$\theta = \text{nutation angle} = \arcsin\left(\frac{\sqrt{H_x^2 + H_y^2}}{H_{tot}}\right) = O(\varepsilon)$$

and since they involve quantities of first order of smallness, should be accurate, whereas the use of formula

$$\cos\theta = \frac{H_z}{H_{tot}} = 1 - O(\epsilon^2)$$

will see θ critically effected by terms of $O(\varepsilon^2)$, none of which should then have been neglected.

5.3.5.4 Parametric studies for $I_x \neq I_y$ (Ellipsoid of inertia not of revolution)

With the particular geometry considered here,

$I_y < I_x < I_z$ implies that

$$I_{y,hub} < I_{x,hub}$$

or

$$K_{py} > K_{px}$$

A set of parameters is chosen, namely

$$\xi_o,\ \bar{\lambda},\ \Gamma,\ \text{number of modes.}$$

In the (K_{py}, K_{px}) plane, the bisectrix of the first quadrant, $K_{px} = K_{py}$, will correspond to the symmetric case,

$$K_{px} = K_{py} = K_p$$

and the limit of stability K_{p*}, such that $K_p > K_{p*}$ will ensure stability of the motion, was found previously. Furthermore, in order to satisfy the GMI rule, we must have

$$K_{px} \text{ and } K_{py} > \frac{1}{1+2\Gamma\Delta}$$

In order to determine the parameter region to be studied with program FLEXAT, it is useful to note that

$$-1 < \frac{I_{zh} - I_{yh}}{I_{xh}} < 1$$

or
$$\frac{1}{K_{py}} - \frac{1}{K_{px}} < 1$$

and
$$\frac{1}{K_{py}} + \frac{1}{K_{px}} > 1$$

Similarly, from

$$-1 < \frac{I_{zh} - I_{xh}}{I_{yh}} < 1$$

$$\frac{1}{K_{px}} - \frac{1}{K_{py}} < 1$$

This is most conveniently represented on a $(1/K_{px}, 1/K_{py})$ plane. (Fig. 5.11). Thus, if

$$X_* \equiv \frac{1}{K_{px}} \qquad y_* \equiv \frac{1}{K_{py}}$$

the admissible domain of study is bounded by

$$y_* - X_* < 1 \qquad X_* < 1 + 2\Gamma\Delta$$

$$y_* + X_* > 1 \qquad y_* < 1 + 2\Gamma\Delta$$

$$X_* - y_* < 1$$

$$X_* > 0 \quad y_* > 0$$

In particular, for a constant ratio of $\frac{K_{px}}{K_{py}}$ (or $\frac{I_{yh}}{I_{xh}}$), the limits are shown by circles on Fig. 5.11.

5.4 Conclusions

A program has been developed for stability studies and simulation of the nutational motion of a spinning satellite with flexible appendages. The results of this program can be used with profit in the preliminary attitude design, to ascertain stability, determine the importance of structural damping and study the rate at which nutation is generated or removed from the system.

REFERENCES - Chapter 5

[5-1] Rakowski, J.E. and Renard, M.L.: "A Study of the Nutational Behavior of a Flexible Spinning Satellite Using Natural Frequencies and Modes of the Rotating Structure," Paper 70-1046, AAS/AIAA Astrodynamics Conference. Santa Barbara, August 1970.

[5-2] Rakowski, J.E.: "A Study of the Attitude Dynamics of a Spin-Stabilized Satellite having Flexible Appendages," Ph.D. Thesis, Mechanical Engineering, Carnegie-Mellon University, December 1970.

[5-3] Thomson, W.T.: Space Dynamics, John Wiley Ed., 1963.

[5-4] Hughes, P.C. and Fung, J.C.: "Liapunov Stability of Spinning Satellites with Long Flexible Appendages." Celestial Mechanics, 4, 295-308, 1971.

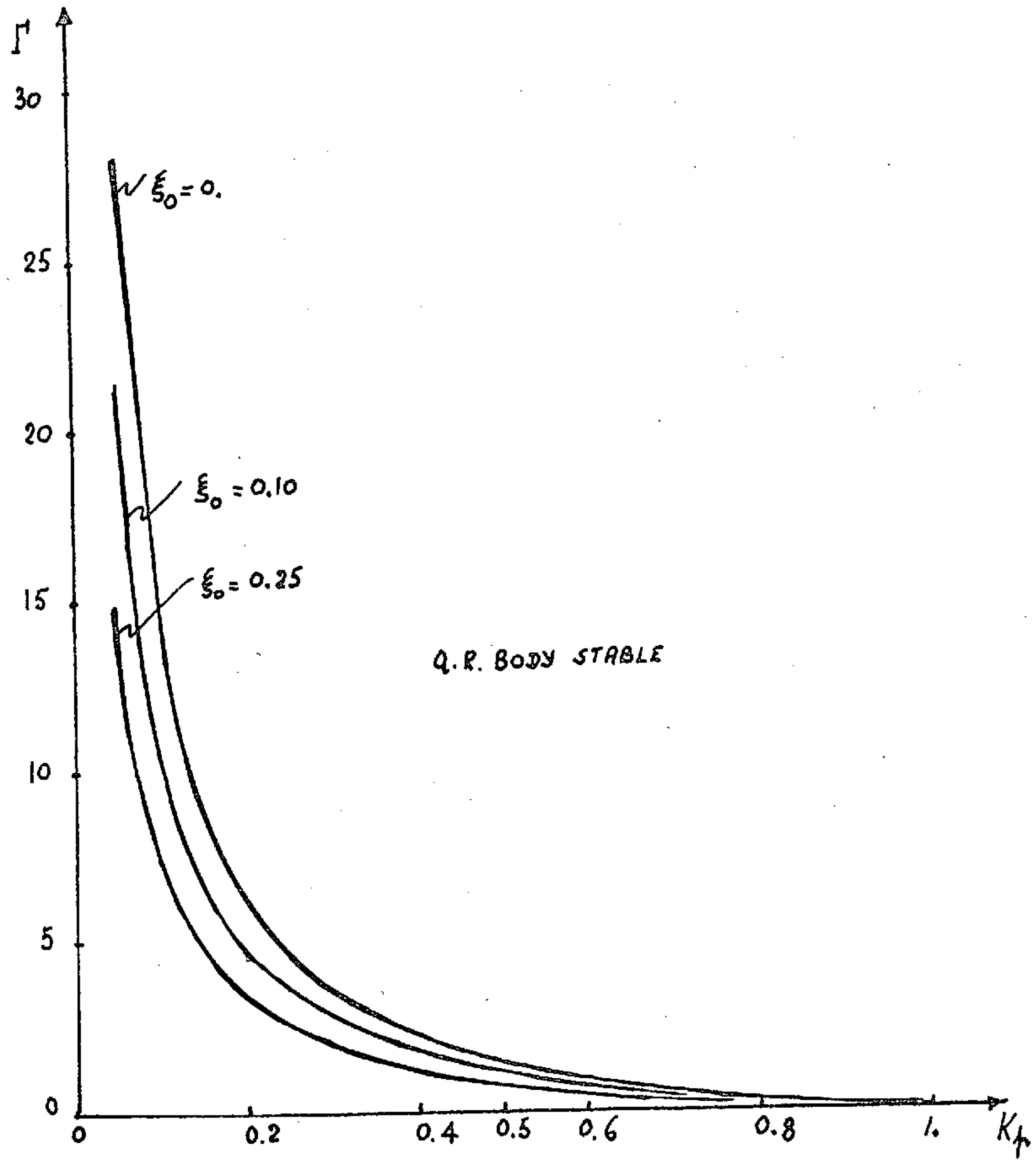


FIG.5-1. QUASI-RIGID BODY STABILITY

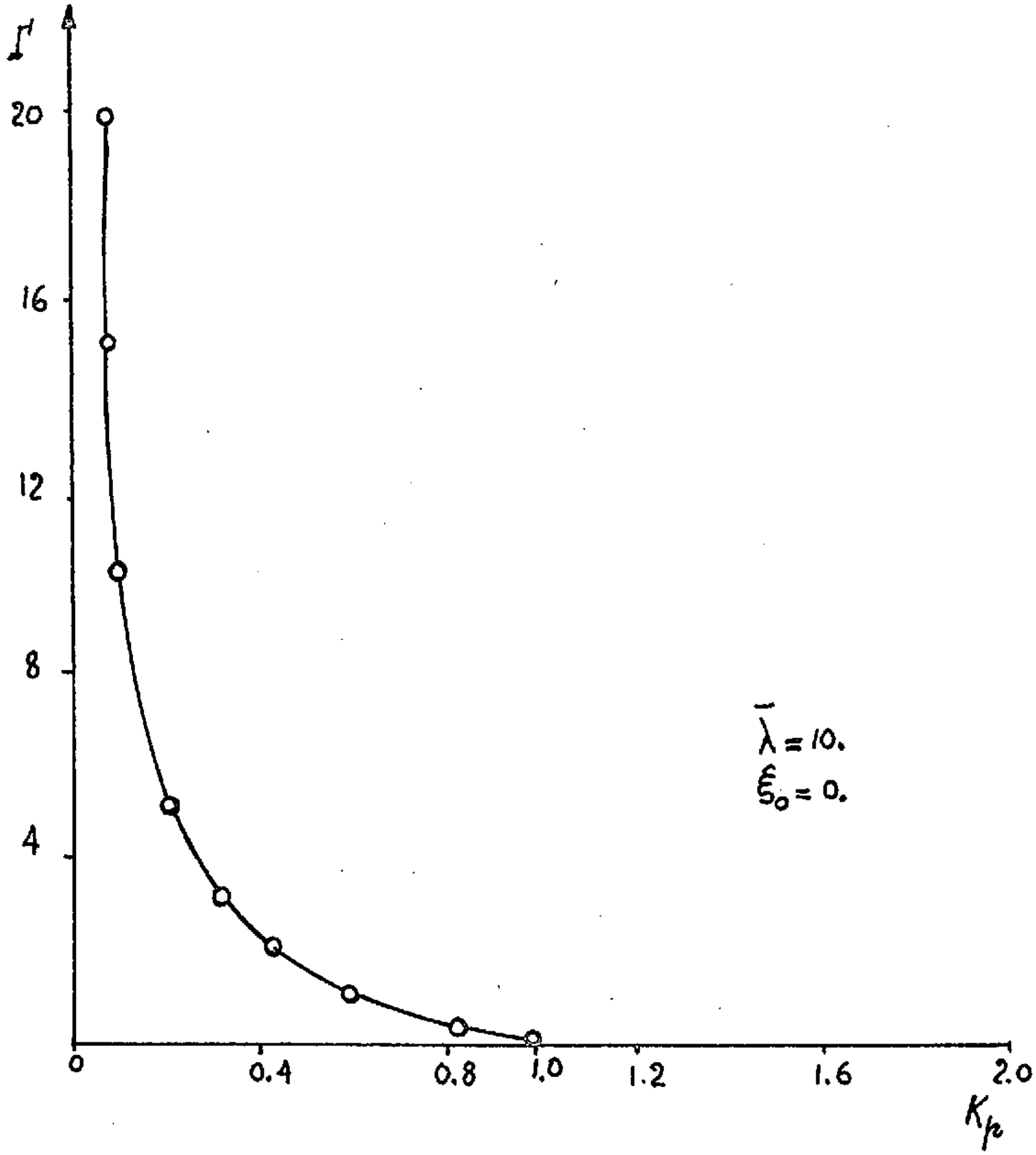


FIG. 5-2.Stability chart.Case E.

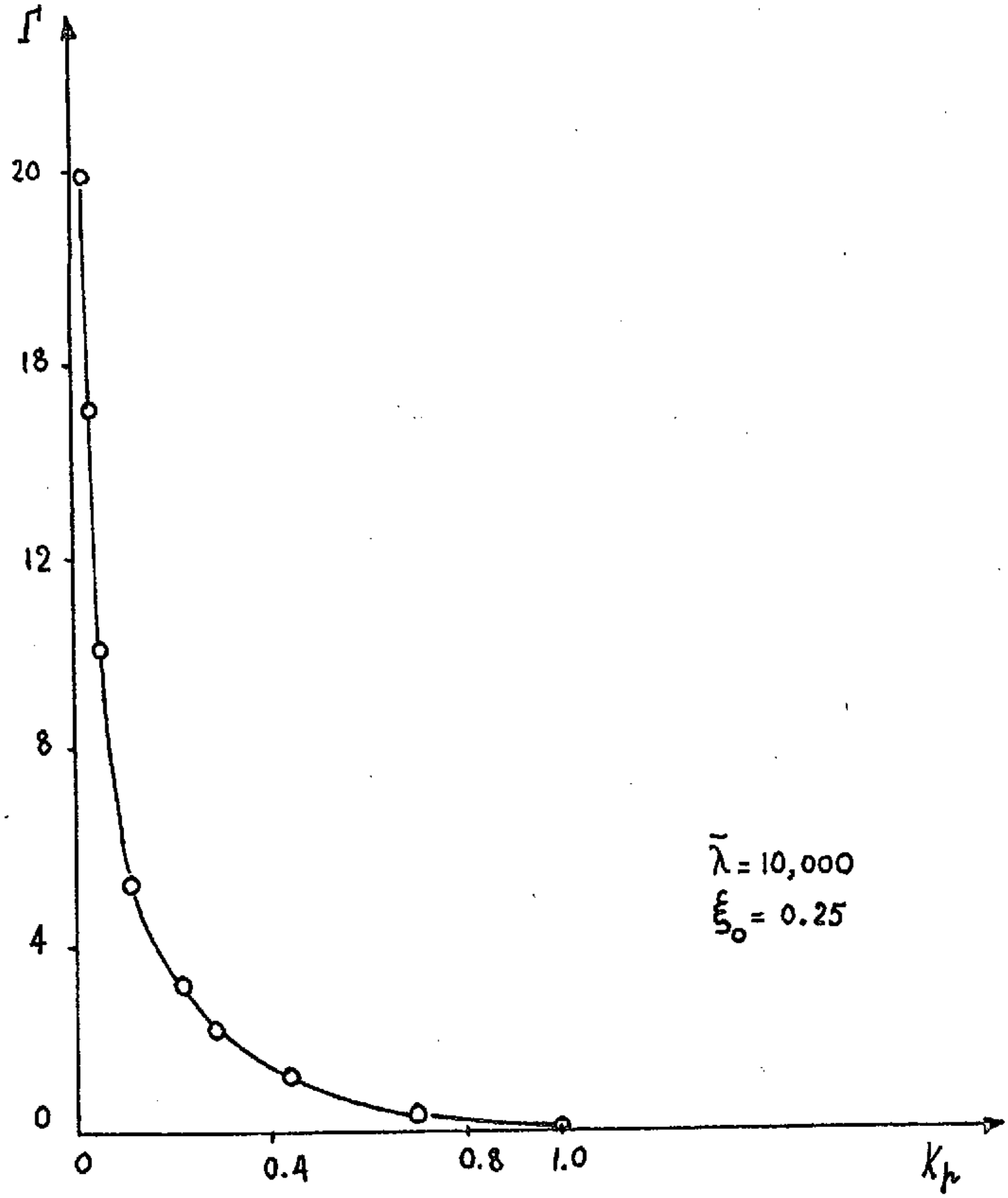


FIG. 5 -3. Stability chart. Case E.

PLOT OF NUTATION ANGLE IN DEGREES VS N FOR
LAMBDA= 100.
SI-ZERO= .10
GAMA= 10.000
PKX= .2200
PKY= .2200
PREC= 1
MAXP= 15
MODES= 3

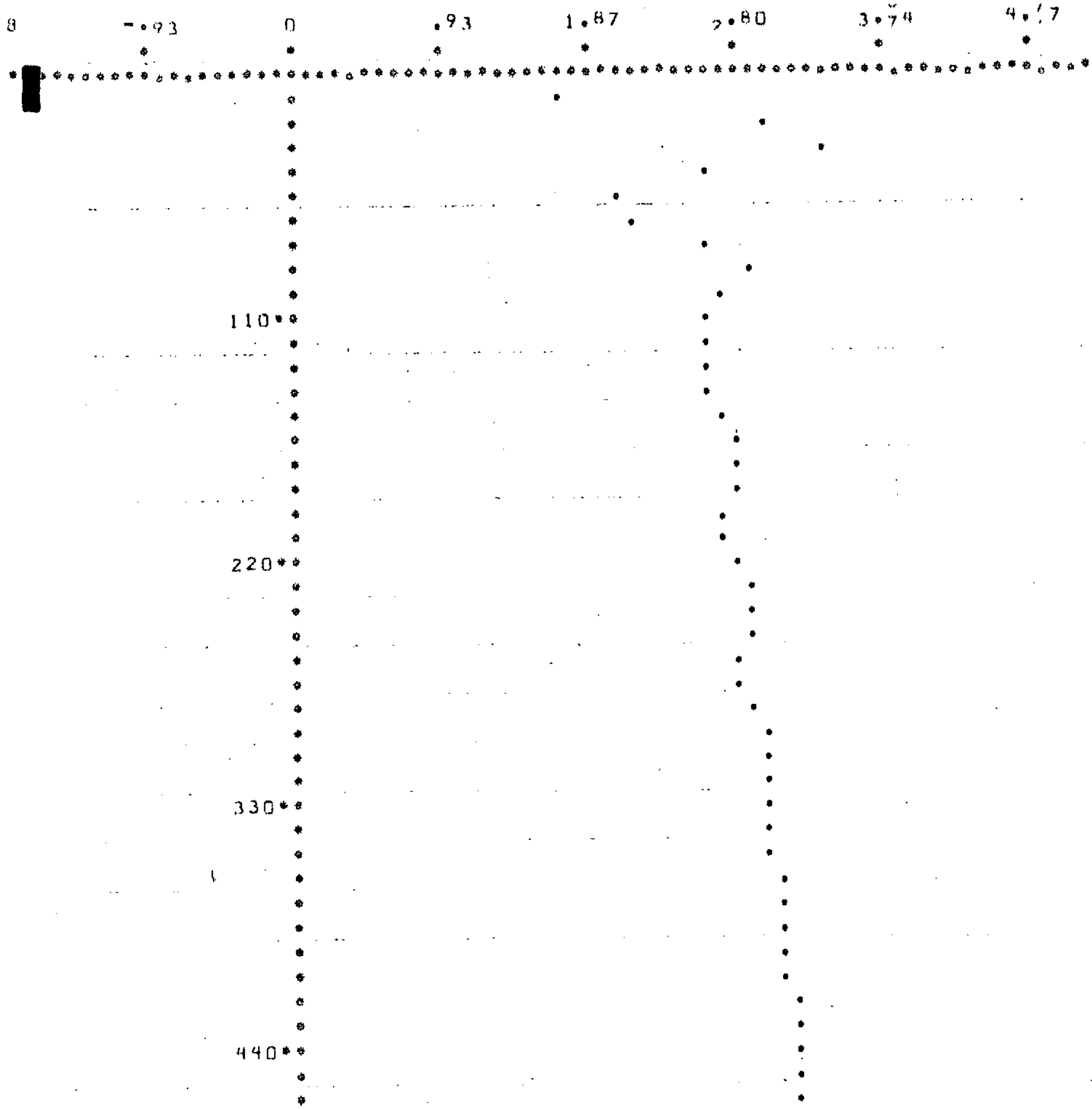


FIG. 5-4

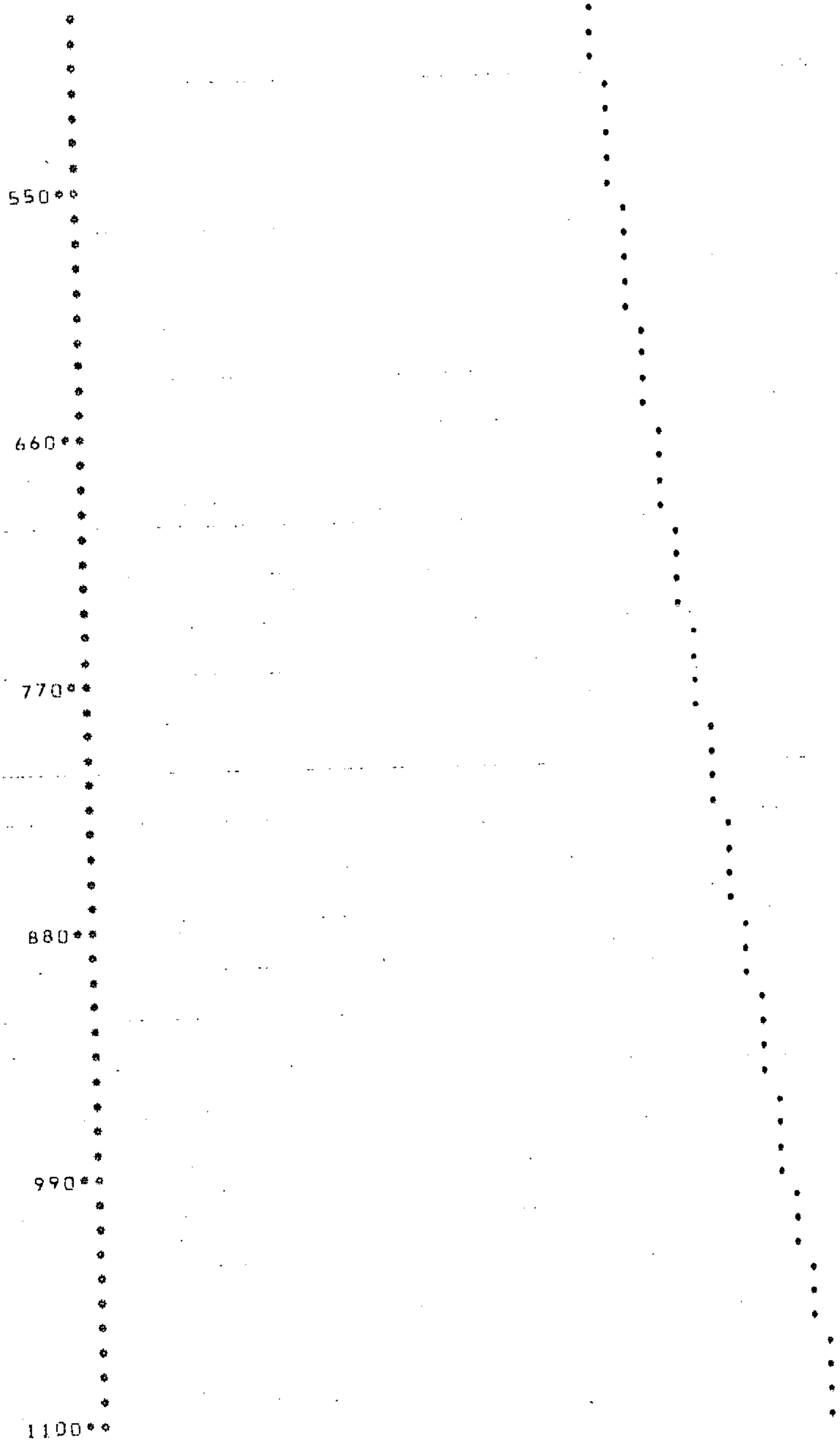


FIG.5-4
(Continued)

PLOT OF NUTATION ANGLE IN DEGREES VS N FOR
LAMBDA= 100.
SI-ZERO= .10
GAMA= 10.000
PKX= .2800
PKY= .2800
PREC= 1
MAXP= 15
MODES= 3

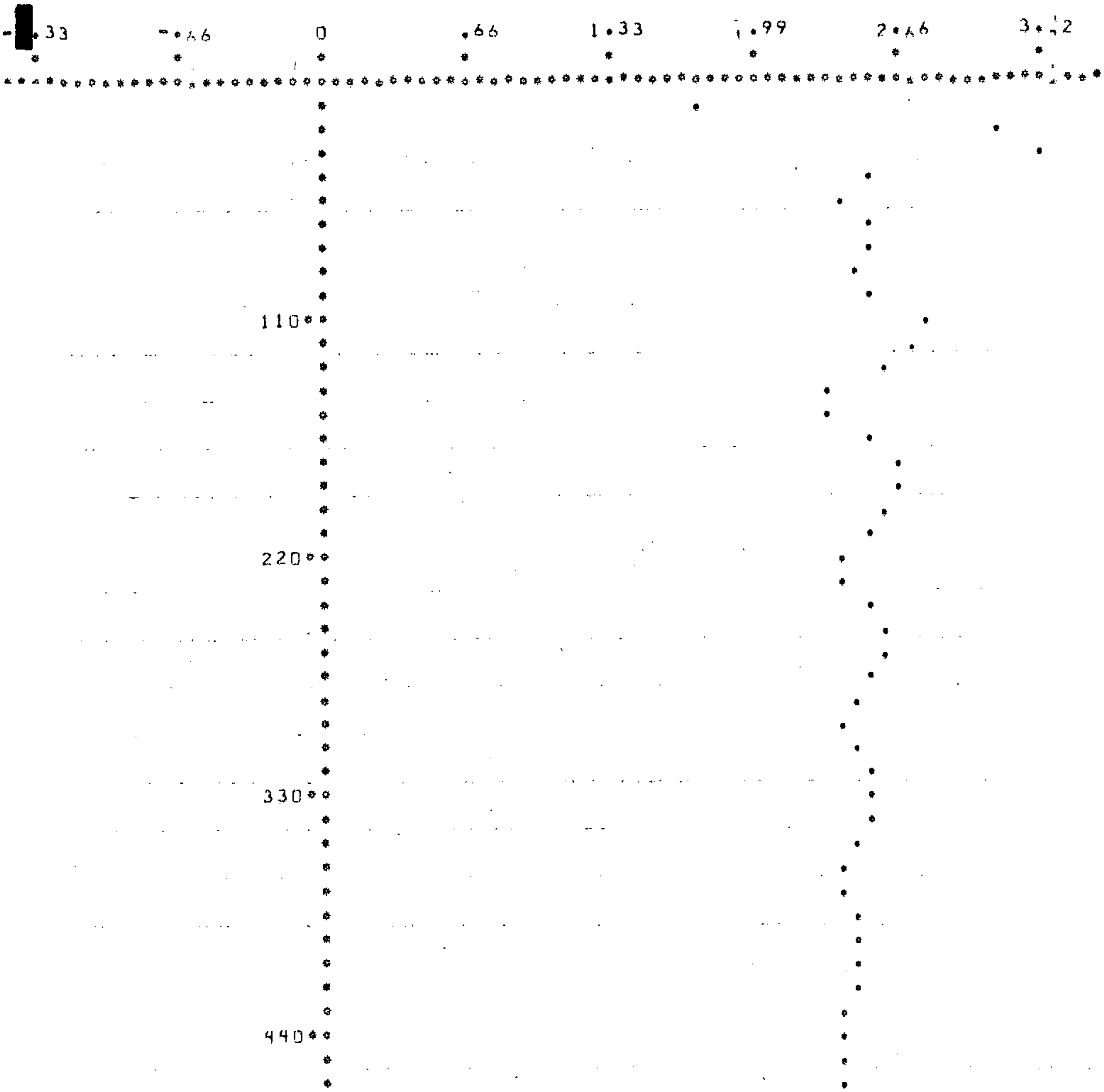


FIG.5-5

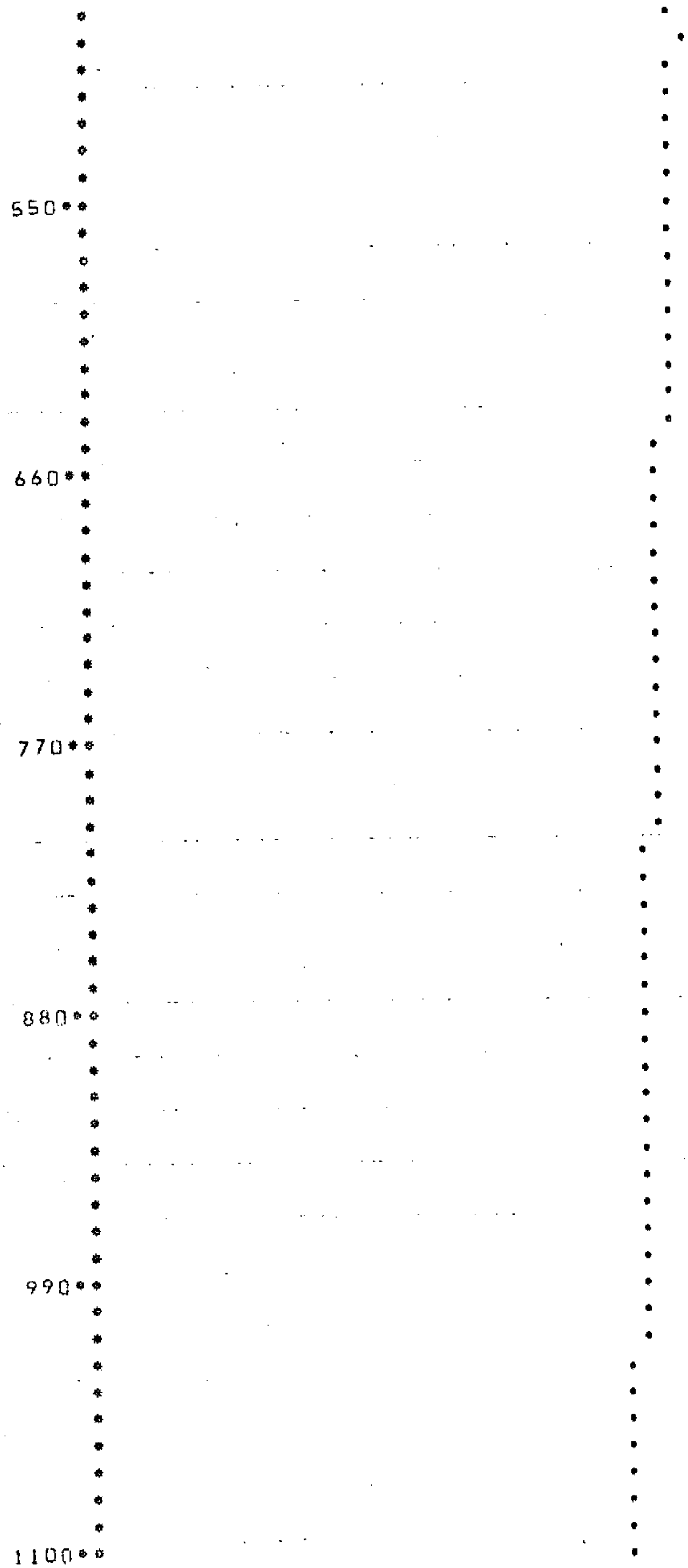


FIG. 5-5
(Continued)

PLOT OF NUTATION ANGLE IN DEGREES VS N FOR
LAMBDA= 1000.
SI-ZERO= .10
GAMA= 10.000
PKX= .3500
PKY= .3500
PREC= 1
MAXP= 15
MODES= 3

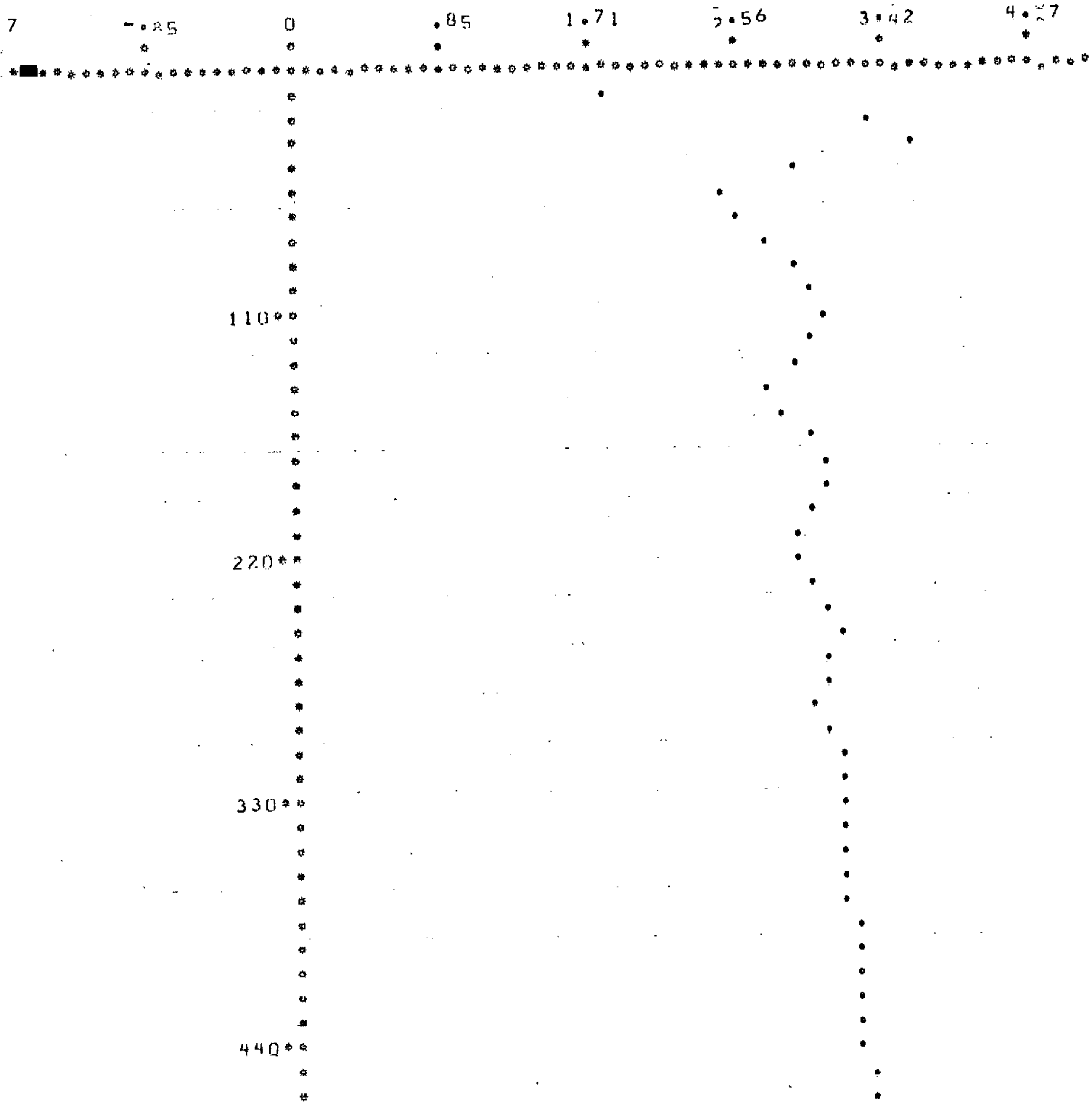


FIG.5-6

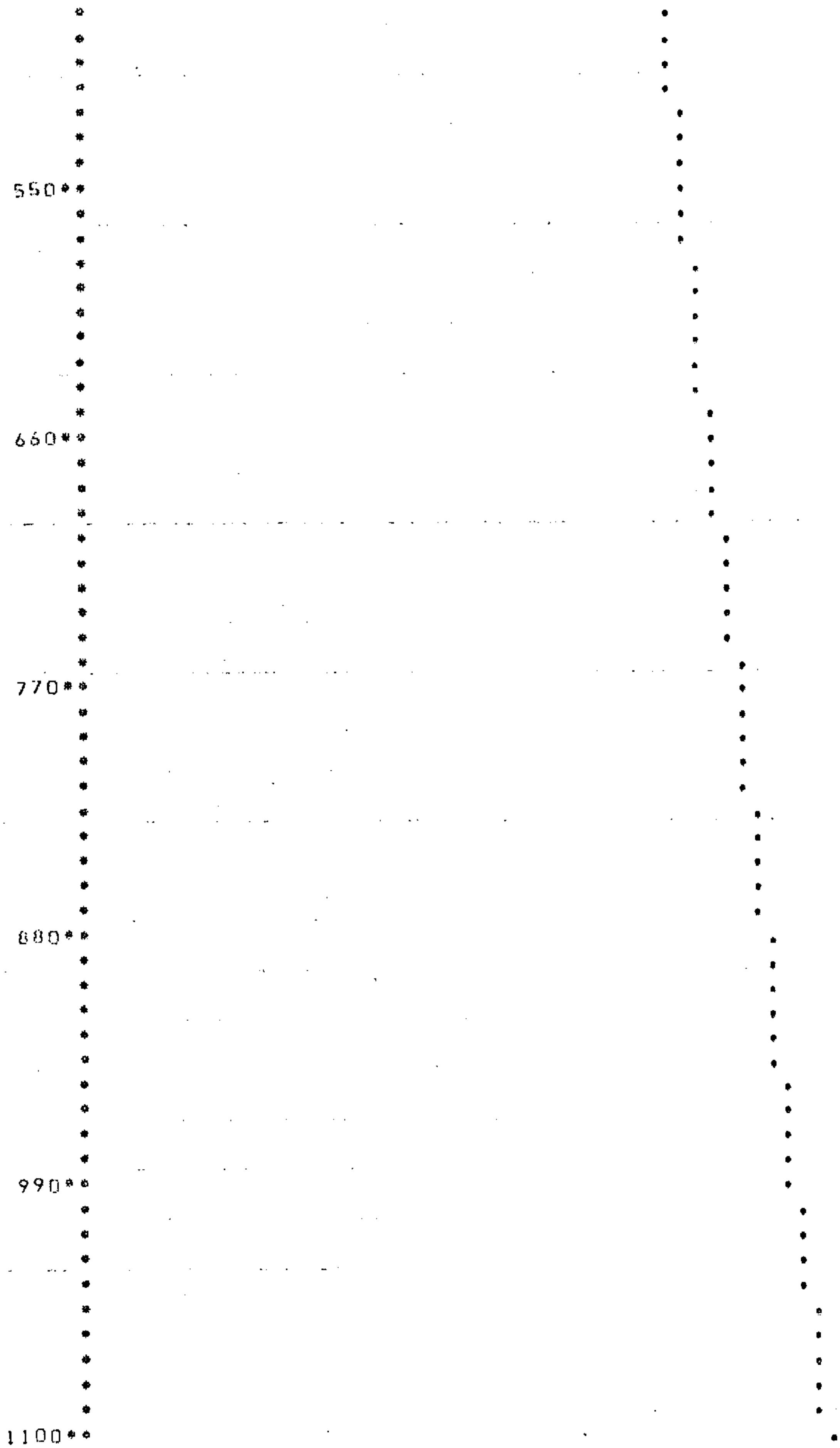


FIG.5-6
(Continued)

PLOT OF NUTATION ANGLE IN DEGREES VS N FOR
LAMBDA= 1000.
SI-ZERO= .10
GAMA= 10.000
PKX= .4000
PKY= .4000
PREC= 1
MAXP= 15
MODES= 3

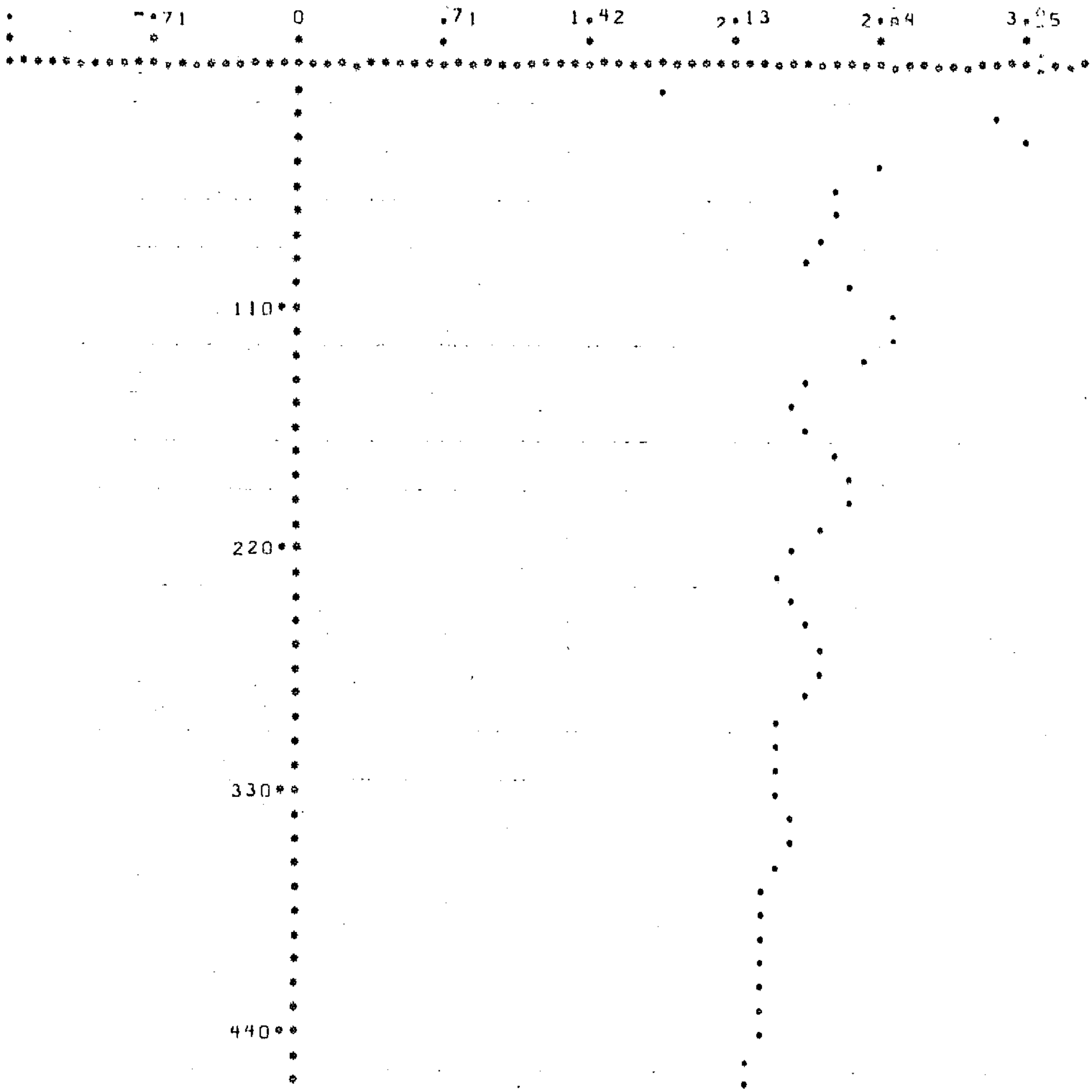


FIG.5-7

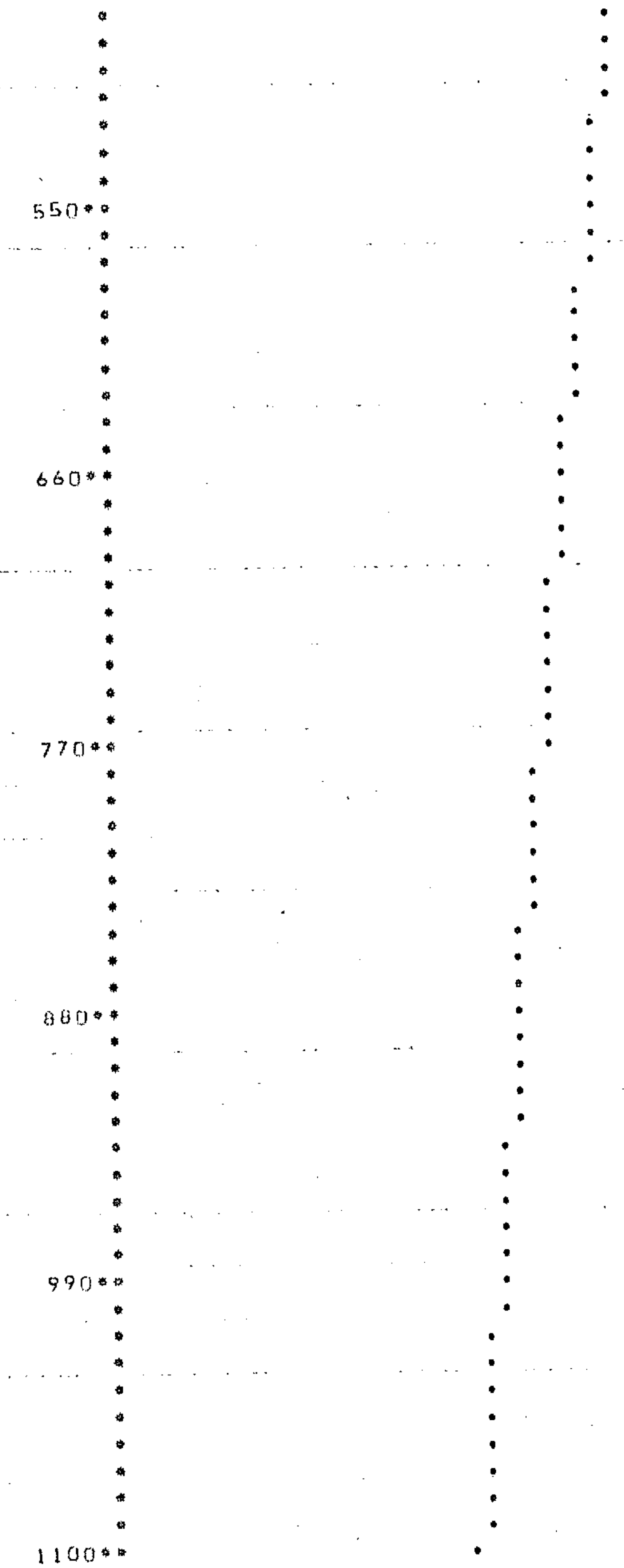


FIG.5-7
(Continued)

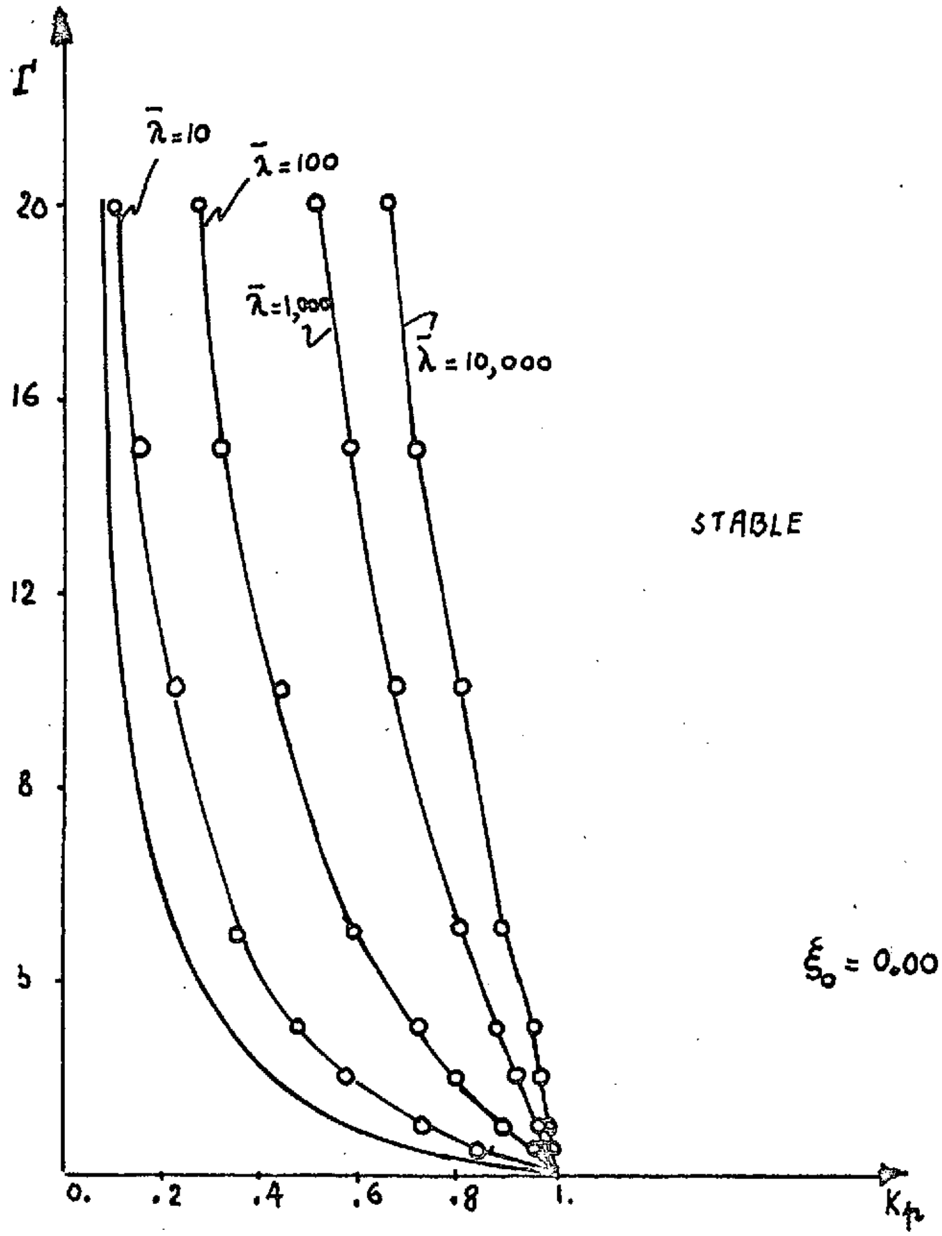


FIG. 5.8 Stability Diagram. Case M. 3 Modes.

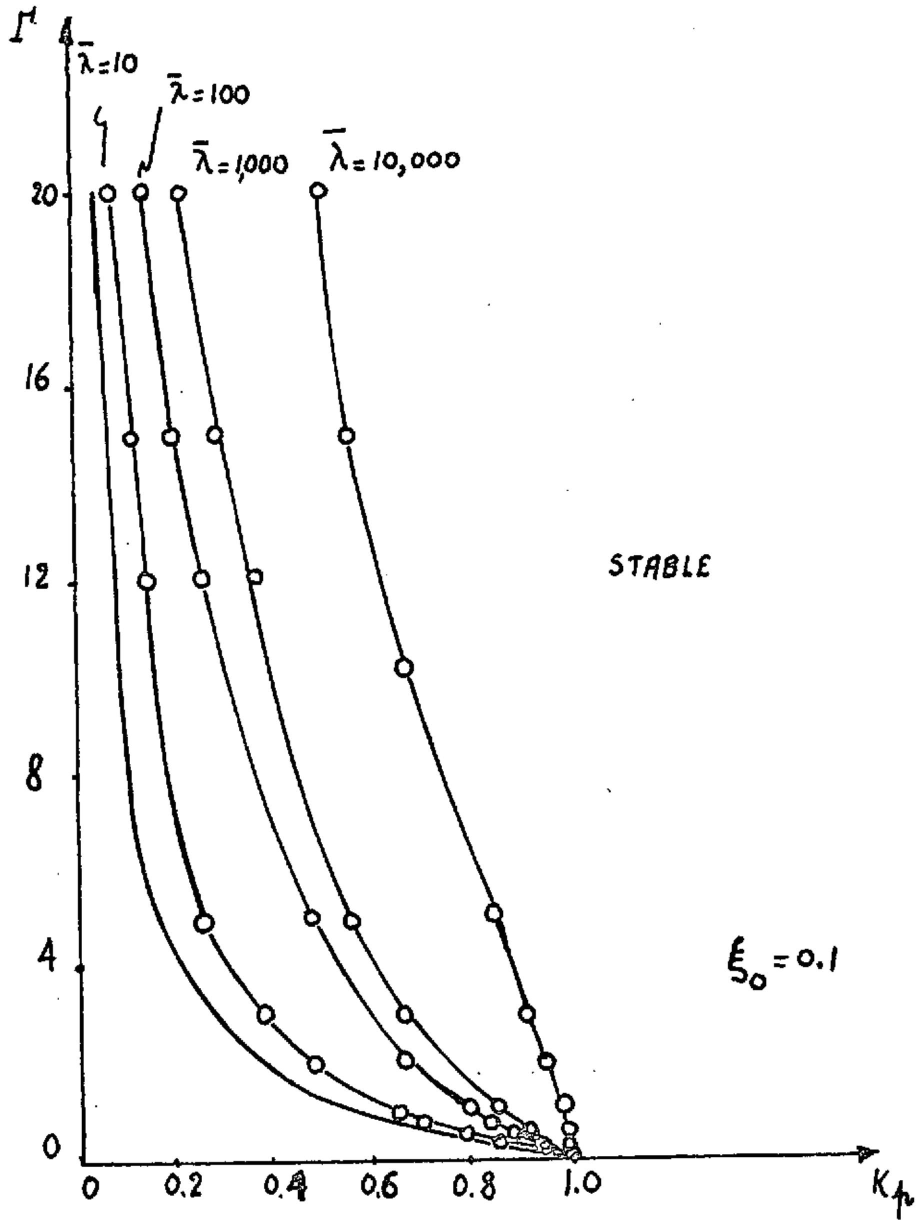


FIG. 5.9 Stability Diagram. Case M. 3 Modes.

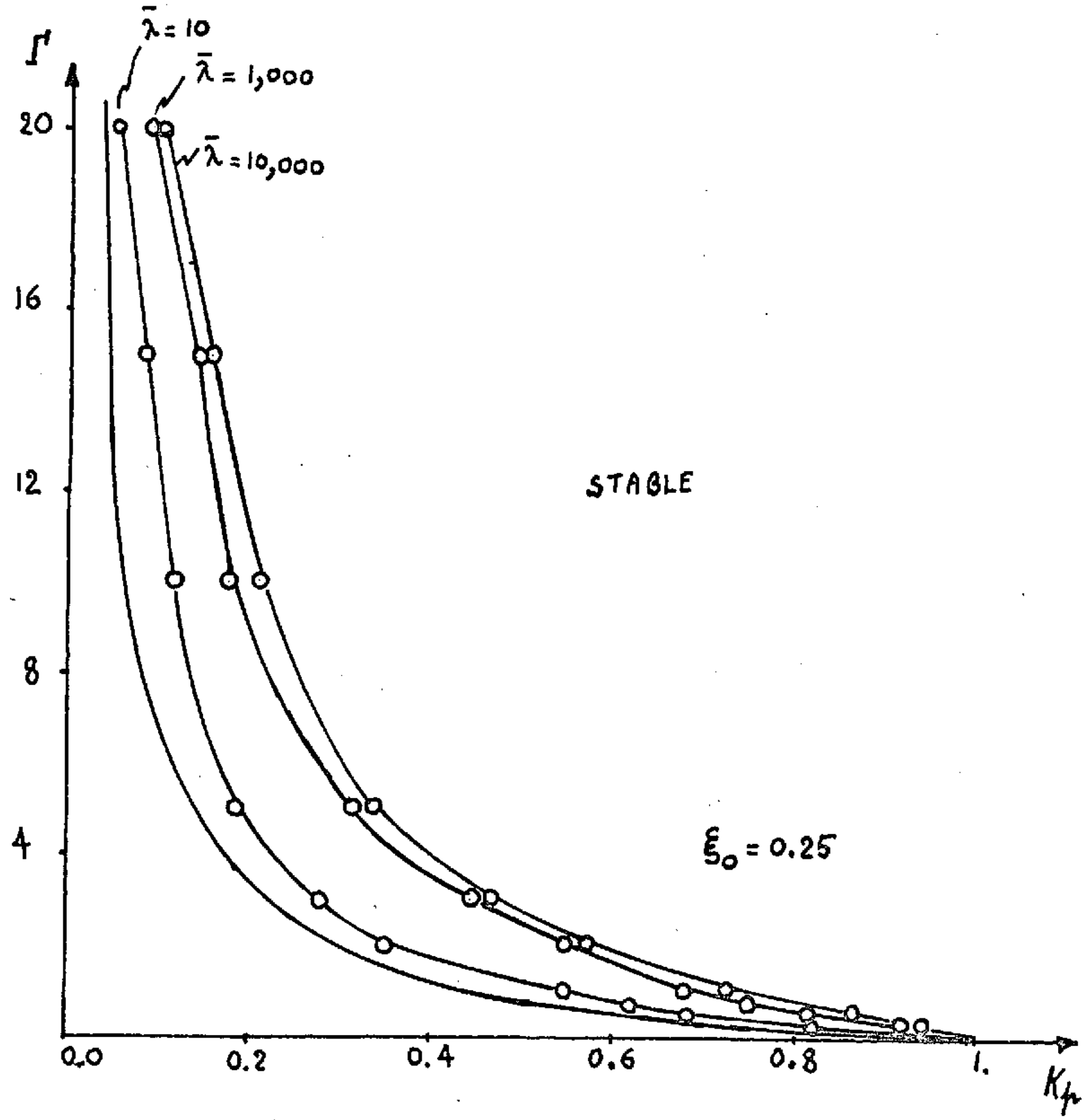


FIG. 5-10. Stability Diagram. Case M. 3 Modes.

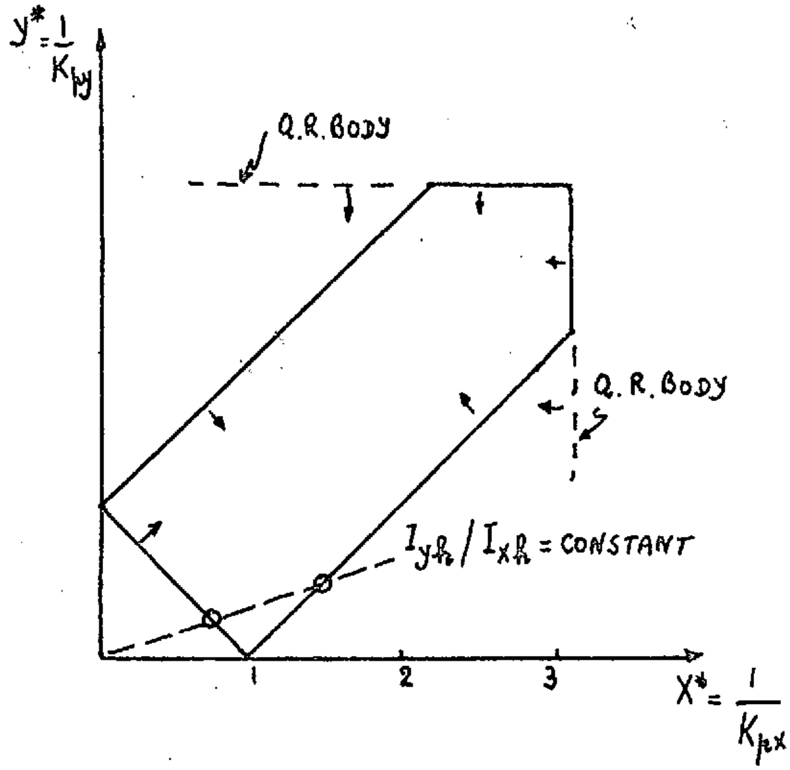


FIG. 5-11. X^*, Y^* DIAGRAM.

(Casem2)

```
THIS IS THE MAIN PROGRAM WHICH INPUTS DATA AND CALLS THE SUBPROGRAMS
DOUBLE PRECISION XNO(3),YNO(3),OMU(3),SIO
REAL LAM,NU,PKX,PKY,GAMA
INTEGER PREC,CASE
DIMENSION NU(3)
COMMON/ONE/GAMA,PKX,PKY,XNO,YNO,NU
COMMON/LEN/LAM,SIO
COMMON/ZEN/OMU
COMMON/THREE/MODES
COMMON/THIRD/NSUP
COMMON/FOUR/IQ,LM,NORU
COMMON/SIX/MAXP,PREC
COMMON/EDIT/NSKP,MGIV
EQUIVALENCE(CASE,IQ)
DATA/LM/1HM
NSKP=0  SKIP NO PRINTING,PLOT ALL
NSKP=1 SKIP 60 PERCENT OF PRINT,INITIALLY,PLOT ALL
NSKP=0

UNCHANGING PARAMETERS


NORU=3
NSUP=3
MODES=3
NU(1)=0.05
NU(2)=0.05
NU(3)=0.05
XNO(1)=.02
XNO(2)=0.
XNO(3)=0.
YNO(1)=-.02
YNO(2)=0.
YNO(3)=0.
PREC=1
MAXP=15
```

```
C
C     MGIV=1 MEANS MODAL QUANTITIES ARE INPUT DATA
      MGIV=0
      IF(MGIV.EQ.0) GO TO 10
C     INSERT VALUES OF OMEGA-J,AM1-J,AM2-J HERE IF MGIV=1
C     AS MANY CARDS NEEDED AS THREE TIMES NUMBER OF MODES
   10 CONTINUE
      CASE='M'
      LAM=1000.
      SIO=0.1
      IZ=1
      IF(IQ.EQ.LM) IZ=0
      CALL SEARCH(IZ)
      GAMA=10.
      PKX=0.4
      PKY=0.4
      CALL CASEM2
      PKY=0.35
      PKY=0.35
      CALL CASEM2
      LAM=100.
      SIO=0.1
      IZ=1
      IF(IQ.EQ.LM) IZ=0
      CALL SEARCH(IZ)
      PKX=0.22
      PKY=0.22
      CALL CASEM2
      PKX=0.28
      PKY=0.28
      CALL CASEM2
      STOP
      END
```

```
      SUBROUTINE SEARCH(NDS)
C
C     THIS PROGRAM FINDS THE FIRST THREE EIGENVALUES (MU) FOR THE ROTATING
C       BOOM IN EITHER EQUATORIAL OR MERIDION FLEXURE
C     IT WILL COMPUTE THESE ACCURATELY FOR VALUES OF LAMDA UP TO
C     APPROXIMATELY 5000
       DOUBLE PRECISION P(4),K(4),M(4),L(4),E3(101),E31P(101),E32P(101),
     1FE34,DECID,TVAL,HY,EMOM3,EMOM4,ESHR3,ESHR4,FEPRV,
     1E33P(101),E34P(101),E4(101),E41P(101),E42P(101),E43P(101),
     3E44P(101),A,B,C,E,AO,BO,CO,EO,MU1,MU,EPSB,EPSC,LAS,LASS,
     4 DLT,UP,DWN,SIO,SI
      DIMENSION OMU(3)
      REAL H,NN,LAM,NATFRQ
      INTEGER I,D,N,Z,  NINT,INTER,W,R,U,NOR,KKK
      COMMON/ZEN/OMU
      COMMON/THIRD/NSUP
      COMMON/LEN/LAM,SIO
C
C     SET NOPT = 1 FOR REVERSED INTEGRATION (TIP TO ROOT)
C     NDS = DIRECTION SWITCH
C      WHEN NDS = 1, SEARCH FOR EQUATORIAL ROOTS,
C       WHEN NDS = 0, SEARCH FOR MERIDION ROOTS
C
C
       IF(NDS.EQ.1)WRITE(6,16)
       IF(NDS.EQ.0)WRITE(6,17)
 16    FORMAT('1EQUATORIAL CASE'//)
 17    FORMAT('1MERIDIONAL CASE'//)
      WRITE(6,21) LAM,SIO,NSUP
 21   FORMAT(1H ,'LAM=',F12.6,3X,'SIO=',D9.3,3X,'NSUP=',I5,/)
      KKK=1
      R=0
      U=1
      FE34=0.
      MU=1.0D-6
       EPSB=1.0D-14
      W=0
      FEPRV=0.
      EPSB=10.**-14
      NOP1=1
      NOR=1
      N=1
      DLT=1.
      IA=0
      NINT=100
      INTER=NINT+1
      ANFRQ=SQRT(LAM)
      WRITE(6,60) ANFRQ
 60   FORMAT(1H ,'NFRQ=SQRT LAM=',F10.5)
      DO 786 JL=1,NSUP
 99   SI=0.
      TVAL=FE34
      JJ=0
      DO 31 I=1,4
       K(I)=0.
      L(I)=0.
      M(I)=0.
 31   P(I)=0.
      DO 1 I=1,101
      E34P(I)=0.
      E44P(I)=0.
      E33P(I)=0.
      E43P(I)=0.
      E32P(I)=0.
```

```
      E42P(I)=0.
      E31P(I)=0.
      E41P(I)=0.
      E4(I)=0.
      E3(I)=0.
      H=1./FLOAT(NINT)
C
C     SET INITIAL CONDITIONS  ON THE S3 AND S4 SOLUTIONS
C
      D=3
      IF(D.EQ.4) GO TO 2
      E0=0.
      B0=0.
      IF(NOPT.GT.0) GO TO 12
      E32P(1)=1.
      B0=E32P(1)
      GO TO 13
      E3(1)=1.
      E0=E3(1)
      C0=0.
      A0=0.
      GO TO 3
    2 A0=0.
      C0=0.
      IF(NOPT.GT.0)GO TO 14
      E43P(1)=1.
      C0=E43P(1)
      GO TO 15
   14 E41P(1)=1.
      A0=E41P(1)
      B0=0.
      E0=0.
      A=A0
      B=B0
      C=C0
      E=E0
C
C     BEGIN RUNGA KUTTA INTEGRATION
C
      N=1
      I=1
      NN=N
      SI=(NN-1.)*H
      K(I)=H*A
      L(I)=H*B
      M(I)=H*C
      MU1=1.+MU*MU
      IF(NDS.EQ.1) GO TO 40
      MU1=MU1-1.
      P(I)=((1.-SI*SI+2.*SI0*(1.-SI))/2.*B-(SI+SI0)*A+MU1*E)*LAM*H
      IF(NOPT.GT.0) P(I)=((-SI*SI+2.*SI*(1.+SI0))/2.*B
     1+(1.-SI+SI0)*A+MU1*E)*LAM*H
      SI=(NN-1.)*H
      I=I+1
      IF(I.GT.3) GO TO 6
      Z=I-1
      E=E0+K(Z)/2.
      A=A0+L(Z)/2.
```

```
      B=B0+M(2)/2.
      C=C0+P(2)/2.
      SI=SI+H/2.
      GO TO 5
      IF(I.GT.4) GO TO 7
      E=E0+K(3)
      A=A0+L(3)
      B=B0+M(3)
      C=C0+P(3)
      SI=SI+H
      GO TO 5
      IF(D.EQ.4) GOTO 9
      SI=NN*H
      Z=N+1
      E3(Z)=E3(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E31P(Z)=E31P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E32P(Z)=E32P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E33P(Z)=E33P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E34P(Z)=LAM*(((SI0+1.)**2-(SI+SI0)**2)*E32P(Z)/2.
     1-(SI+SI0)*E31P(Z)+MU1*E3(Z))
      IF(NOPT.GT.0) E34P(Z)=LAM*(((SI0+1.)**2-(1.-SI+SI0)**2)
     1*E32P(Z)/2.+(1.-SI+SI0)*E31P(Z)+MU1*E3(Z))
      E=E3(N+1)
      A=E31P(N+1)
      B=E32P(N+1)
      C=E33P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM3=E32P(INTER)
      ESHR3=E33P(INTER)
      IF(NOPT.GT.0) EMOM3=E3(INTER)
      IF(NOPT.GT.0) ESHR3=E31P(INTER)
      D=4
      GO TO 8
      SI=NN*H
      Z=N+1
      E4(Z)=E4(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E41P(Z)=E41P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E42P(Z)=E42P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E43P(Z)=E43P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E44P(Z)=LAM*(((SI0+1.)**2-(SI+SI0)**2)*E42P(Z)/2.
     1-(SI+SI0)*E41P(Z)+MU1*E4(Z))
      IF(NOPT.GT.0) E44P(Z)=LAM*(((SI0+1.)**2-(1.-SI+SI0)**2)
     1*E42P(Z)/2.+(1.-SI+SI0)*E41P(Z)+MU1*E4(Z))
      E=E4(N+1)
      A=E41P(N+1)
      B=E42P(N+1)
      C=E43P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
```

```
      EMOM4=E42P(INTER)
      ESHR4=E43P(INTER)
      IF(NOPT.GT.0) EMOM4=E4(INTER)
      IF(NOPT.GT.0) ESHR4=E41P(INTER)
C
C     RUNGA KUTTA FINISHED
C     NOW BEGIN LINEAR INTERPOLATION
C     FE34 IS THE VALUE OF THE DETERMINANT (S3 AND S4)
C
      FE34=EMOM3*ESHR4-ESHR3*EMOM4
      IF(R.EQ.1) GO TO 51
      IF(U.EQ.1) GO TO 50
      IF(FE34*DECID)52,51,50
 50   DECID=FE34
      LAS=MU
      LASS=LAS
      WRITE(6,85)FE34,MU,U
 85   FORMAT(1H ,'FE34',D12.6,5X,'MU=',D12.6,5X,'U=',I3)
      MU=MU+DLT
      U=U+1
      GO TO 99
 52   UP=MU
      DWN=LAS
      HY=FE34
      TVAL=FE34
 51   IF(ABS(FE34).LE.EPSA) GO TO 53
      IF(ABS(DECID).LE.EPSB) GO TO 42
      R=1
      IF(FE34*DECID)55,51,56
 55   UP=MU
      MU=DWN-(DWN-UP)*DECID/(DECID-FE34)
      KKK=KKK+1
      EPSC=ABS(ABS(MU)-ABS(LASS))
      IF(EPSC.LT.MU*10.**-4)GO TO 10
      LASS=MU
      GO TO 58
 56   DWN=MU
      DECID=FE34
      MU=DWN-(DWN-UP)*DECID/(DECID-HY)
      KKK=KKK+1
      EPSC=ABS(ABS(MU)-ABS(LASS))
      IF(EPSC.LT.MU*10.**-4)GO TO 10
      LASS=MU
 58   FE1=ABS(FEPRV)-ABS(FE34)
       IF(ABS(FE1).GT.1.0D-14)GOTO 82
      IA=IA+1
      IF(IA.LT.5) GO TO 82
      WRITE(6,83)FE34
 83   FORMAT(1H ,'STUCK ON THIS FE34',D12.6)
      IA=0
      FEPRV=0.
      GO TO 53
 82   FEPRV=FE34
      GO TO 99
 42   WRITE(6,43)DECID
 43   FORMAT(1H ,'NO GOOD DECID=',D12.6)
      GO TO 53
 10   WRITE(6,11) FE34,MU
```

```
11   FORMAT(1H0,'MU CONVERGED FE34=',D24.18,3X,'MU=',D12.6)
53   NATFRQ=MU*SQRT(LAM)
     WRITE(6,54) FE34,MU,LAM,NATFRQ,W
     OMU(JL)=MU
54   FORMAT(1H0,'FE34',D12.6,5X,'MU=',D12.6,5X,'LAM=',F12.6,
    15X,'NATFRQ=',E12.6,5X,'W=',I3)
     WRITE(6,86) DLT,NOR
86   FORMAT(1H0,'DLT=',D9.3,5X,'NOR=',I3)
     NOR=NOR+1
     K=0
     U=1
     KKK=1
     MU=MU+DLT
786  CONTINUE
2     RETURN
     END

ILATION:           NO  DIAGNOSTICS.
```

```
EG     0002 D 000204 DELTA     0002 D 000204 DELTA     0000 R 000014 DMU
V      0002 D 000074 DV        0000 D 006437 E         0000 D 001631 EMOM3
EO     0000 D 004473 E3        0000 D 001005 E31P      0000 D 001317 E32P
34P    0000 D 004447 E4        0000 D 004761 E41P      0000 D 005273 E42P
44P    0003 R 000000 GAMA      0003 R 000000 GAMA      0002 D 000206 GAMA2
H      0000 R 006520 HSQ       0002 I 000210 I         0002 I 000210 I
NJPS   0000   007345 INJPS     0000 I 000005 INTER     0000 I 006473 IR
       0000 I 006463 JI        0000 I 006501 JM        0000 I 007077 J1
KF     0000 I 000001 KMAX      0000 I 007102 KI        0000 D 000463 L
LAM    0000 I 006471 LB        0000 R 006537 LINE      0000 I 006472 LL
AX     0007 I 000000 MAXP      0007 I 000000 MAXP      0012 I 000001 MGIV
K      0000 D 001641 MMX3      0000 D 002153 MMX4      0000 I 006467 MODE
MODES  0005 I 000000 MODES     0000 D 006524 MU        0000 D 006453 MU
I      0006 I 000001 N         0006 I 000001 N         0000 I 006466 NDS
N      0000 I 006465 NOPT      0000 I 006513 NP        0000 I 006474 NPI
NSUP   0011 I 000000 NSUP      0003   000005 NU        0003 R 000017 NU
I      0000 R 006521 OMXY      0002 R 000217 OMEGA     0002 R 000217 OMEGA
MU     0000 R 006507 OMXDO     0000 R 006504 OMXO      0000 R 006510 OMYDO
OMZDO  0000 R 006506 OMZO      0000 D 000433 P         0000 R 006477 PER
PI     0003 R 000001 PKX       0003 R 000001 PKX       0003 R 000002 PKY
REC    0007 R 000001 PREC      0000 D 000407 PM        0000 R 006715 SAVE
SIO    0004 D 000001 SIO       0000 R 006464 SIOR      0000 D 000431 SM
UN2    0002 D 000211 UN2       0002 D 000000 V         0002 D 000000 V
NO     0003 D 000003 XNO       0002 D 000200 XI        0002 D 000200 XI
NO     0003 D 000011 YNO       0002 D 000202 YI        0002 D 000202 YI
ZKI
```

```
      SUBROUTINE CASEM2
C
C     CASEM2 SIMULATES THE NUTATIONAL MOTION OF THE FLEXIBLE SPIN-
C     STABILIZED SATELLITE WITH BOOMS IN MERIDION.VIBRATION, GIVEN A SET OF
C     2 I.C.'S ON THE BOOMS AND 3 INITIAL ANGULAR RATES
C     AND THEIR DERRIVITIVES
C     MODIFIED VERSION USES THE FIRST THREE MODES IN THE SIMULATION
C     AND USES THE EQUATIONS WITH FIRST ORDER TERMS RETAINED
C     THE OUTPUT CONSISTS OF TRANSVERSE RATES, BOOM TIP DISPLACEMENTS,
C     VARIATIONS ON ANGULAR MOMENTUM AND MECHANICAL ENERGY
C     AND THE NUTATION ANGLE IN DEGREES
C
      DIMENSION OMU(3)
      COMMON V,DV,COR,AM2,XI,YI,DELTA,GAMA2,I,UN2,OMEGA
      COMMON/ONE/GAMA,PKX,PKY,XNO,YNO,NU
      COMMON/LEN/LAM,SIO
      COMMON/THREE/MODES
      COMMON/FIVE/MK,M,CA
      COMMON/SIX/MAXP,PREC
      COMMON/ZEN/OMU
      COMMON/THIRD/NSUP
      COMMON/EDIT/NSKP,MGIV
      INTEGER PREC
      REAL MN,LAM,NU
      INTEGER MAXP,KMAX,MK,I,D,N,Z,NINT,I TER
      DIMENSION NU(3),DAM1(3),DAM2(3),DMU(3)
      DOUBLE PRECISION V(10,3),DV(10,3)
      DOUBLE PRECISION COR,GAMA2,DELTA,XI,YI,ZKI
```

```
      DOUBLE PRECISION XNDO(3),YNDO(3)
      DOUBLE PRECISION AM1(3),AM2(3),A1M(3),A2M(3)
      DOUBLE PRECISION UN2(3),AM21(3),                       XNO(3),YNO(3)
      DOUBLE PRECISION AK(7,4,3),CC(7,3),PW(3),ACX(3)
      DOUBLE PRECISION ACY(3),SM
      DOUBLE PRECISION P(4),K(4),M(4),L(4),E3(101),E31P(101),E32P(101),
     1EMOM3,EMOM4,ALFA,BETA,               MMX3(101),MMX4(101),OPT(101,3),
     1E33P(101),E34P(101),E4(101),E41P(101),E42P(101),E43P(101),
     1E44P(101),A,B,C,E,AO,BO,CO,EO,MU1,MU(3),SIO,S)
      DO 711 JI=1,NSUP
  711 MU(JI)=OMU(JI)
      SIOR=SIO
      WRITE(6,347)
  347 FORMAT(1H1,'CASE M        PROGRAM VERSION FIRST ORDER ',/)
C
C  SET PARAMETERS TO CONTROL SIMULATION
C  SET NOPT=1 FOR REVERSED INTEGRATION
C  NDS = DIRECTION SWITCH
C  NDS = 1 IN PLANE, = 0 OUT OF PLANE
C
      NOPT=1
      NDS=0
      NINT=100
      INTER=NINT+1
      WRITE(6,95)LAM,SIOR,GAMA,PKX,PKY,PREC,MAXP,MODES
   95 FORMAT(1H ,'LAMBDA=',F6.0,/1H ,'XI-ZERO=',F4.2,/1H ,'GAMA=',F7.3,/
     X1H ,'PKX=',F6.4,/1H ,'PKY=',F6.4/
     X         1H ,'PREC=',I2/1H ,'MAXP=',I3/1H ,'MODES=',I2////)
      IF(MGIV.EQ.1) GO TO 340
      DO 34 MODE=1,MODES
   99 SI=0.
      N=1
      H=1./FLOAT(NINT)
C
C  CLEAR ARRAYS
C
      DO 31 I=1,4
      K(I)=0.
      L(I)=0.
      M(I)=0.
   31 P(I)=0.
      DO 1 I=1,101
      E34P(I)=0.
      E44P(I)=0.
      E33P(I)=0.
      E43P(I)=0.
      E32P(I)=0.
      E42P(I)=0.
      E31P(I)=0.
      E41P(I)=0.
      E4(I)=0.
    1 E3(I)=0.
C
C     THIS SECTION COMPUTES THE FIRST MODE SHAPE AND THEN THE MODE SHAPE
C     PARAMETERS M1AND M2 FOR CASE M
C
      D=3
    8 IF(D.EQ.4) GO TO 2
```

```
      E0=0.
      B0=0.
      IF(NOPT.GT.0) GO TO 12
      E32P(1)=1.
      B0=E32P(1)
      GO TO 13
12    E3(1)=1.
      E0=E3(1)
13    C0=0.
      A0=0.
      GO TO 3
2     A0=0.
      C0=0.
      IF(NOPT.GT.0)GO TO 14
      E43P(1)=1.
      C0=E43P(1)
      GO TO 15
14    E41P(1)=1.
      A0=E41P(1)
15    B0=0.
      E0=0.
3     A=A0
      B=B0
      C=C0
      E=E0
      N=1
4     I=1
      NN=N
      SI=(NN-1.)*H
5     K(I)=H*A
      L(I)=H*B
      M(I)=H*C
      MUI=1.+MU(MODE)*MU(MODE)
      IF(NDS.EQ.1) GO TO 40
      MUI=MUI-1.
40    P(I)=((1.-SI*SI+2.*SI0*(1.-SI))/2.*B-(SI+SI0)*A+MUI*E)*LAM*H
      IF(NOPT.GT.0) P(I)=((-SI*SI+2.*SI*(1.+SI0))/2.*B
     1+(1.-SI+SI0)*A+MUI*E)*LAM*H
      SI=(NN-1.)*H
      I=I+1
      IF(I.GT.3) GO TO 6
      Z=I-1
      E=E0+K(Z)/2.
      A=A0+L(Z)/2.
      B=B0+M(Z)/2.
      C=C0+P(Z)/2.
      SI=SI+H/2.
      GO TO 5
6     IF(I.GT.4) GO TO 7
      E=E0+K(3)
      A=A0+L(3)
      B=B0+M(3)
      C=C0+P(3)
      SI=SI+H
      GO TO 5
7     IF(D.EQ.4) GOTO 9
      SI=NN*H
      Z=0+1
```

```
      E3(Z)=E3(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E31P(Z)=E31P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E32P(Z)=E32P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E33P(Z)=E33P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E34P(Z)=LAM*(((SIO+1.)**2-(SI+SIO)**2)*E32P(Z)/2.
     1-(SI+SIO)*E31P(Z)+MU1*E3(Z))
      IF(NOPT.GT.0) E34P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E32P(Z)/2.+(1.-SI+SIO)*E31P(Z)+MU1*E3(Z))
      E=E3(N+1)
      A=E31P(N+1)
      B=E32P(N+1)
      C=E33P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM3=E32P(INTER)
      IF(NOPT.GT.0) EMOM3=E3(INTER)
      DO 30 I=1,INTER
30    MMX3(I)=E3(I)
      D=4
      GO TO 8
9     SI=NO*H
      Z=N+1
      E4(Z)=E4(N)+(K(1)+2.*K(2)+2.*K(3)+K(4))/6.
      E41P(Z)=E41P(N)+(L(1)+2.*L(2)+2.*L(3)+L(4))/6.
      E42P(Z)=E42P(N)+(M(1)+2.*M(2)+2.*M(3)+M(4))/6.
      E43P(Z)=E43P(N)+(P(1)+2.*P(2)+2.*P(3)+P(4))/6.
      E44P(Z)=LAM*(((SIO+1.)**2-(SI+SIO)**2)*E42P(Z)/2.
     1-(SI+SIO)*E41P(Z)+MU1*E4(Z))
      IF(NOPT.GT.0) E44P(Z)=LAM*(((SIO+1.)**2-(1.-SI+SIO)**2)
     1*E42P(Z)/2.+(1.-SI+SIO)*E41P(Z)+MU1*E4(Z))
      E=E4(N+1)
      A=E41P(N+1)
      B=E42P(N+1)
      C=E43P(N+1)
      E0=E
      A0=A
      B0=B
      C0=C
      N=N+1
      IF(N.LT.INTER) GO TO 4
      EMOM4=E42P(INTER)
      IF(NOPT.GT.0) EMOM4=E4(INTER)
      DO 32 I=1,INTER
32    MMX4(I)=E4(I)
      ALFA=EMOM3/EMOM4
      BETA=MMX3(101)-ALFA*MMX4(101)
      IF(NOPT.GT.0) BETA=MMX3(1)-ALFA*MMX4(1)
      DO 102 LB=1,101
      LL=LB
      IF(NOPT.GT.0) LL=102-LB
102   BPT(LL,MODE)=(MMX3(LB)-ALFA*MMX4(LB))/BETA
      SM=0.
      DO 216 I=2,101
216   SM=SM+(BPT(I,MODE)+BPT(I-1,MODE))/2.*((FLOAT(I)-1.5)*H+SIO)*H
```

```
      AM2(MODE)=SM
      SM=0.
      DO 218 I=2,101
  218 SM=SM+(BPT(I,MODE)*BPT(I,MODE)+BPT(I-1,MODE)*BPT(I-1,MODE))/2.*H
      AM1(MODE)=SM
   34 CONTINUE
C
C     END OF MODE SHAPE AND MODE PARAMETER CALCULATION
C     ALL VARIABLES ARE NON-DIMENSIONAL
C     NP IS THE STEP DUMMY
C     NPI IS THE INTERNALLY CALCULATED PRINT INTERVAL
C     KMAX IS THE NUMBER OF SEGMENTS IN THE SMALLEST PERIOD
C     NX DETERMINES HOW MUCH OF THE SIMULATION IS PRINTED OUT
C     GAMA=RHO*L**3/I-HUB-2
C     COR=SUM OVER N MODES OF M2 SQUAERD OVER M1
C
  340 CONTINUE
      KMAX=75
      IR=MU(PREC)
      IF(PREC.NE.1) NPI=NPI*IR
      BFRQ=MU(PREC)
      PFAC=1./FLOAT(KMAX)
      PER=8.*(ATAN(1.))
      IF(BFRQ.GT.1.) PER=PER/MU(PREC)
      H=PER*PFAC
      MK=MAXP*KMAX
      IF(PREC.NE.1) MK=MAXP*KMAX*IR
      NX=.6*MK
      NPI=(MK-NX)/60
      COR=0.
      DO 202 JW=1,MODES
      AM21(JW)=AM2(JW)/AM1(JW)
      A1M(JW)=2.*AM1(JW)
      A2M(JW)=2.*AM2(JW)
  202 COR=COR+AM2(JW)*AM21(JW)
      DELTA=SIO*SIO+SIO+.3333333
      DDELT=DELTA
      DCOR=COR
      WRITE(6,97) DCOR,DDELT,NU
   97 FORMAT(1H ,'COR=',E11.5/1H ,'DELTA=',F 9.5/1H ,'NU1=',F10.4/1H ,
     X          'NU2=',F10.4,/1H ,'NU3=',F10.4////)
      DO 203 JW=1,MODES
      DAM1(JW)=AM1(JW)
      DAM2(JW)=AM2(JW)
      DMU(JW)=MU(JW)
      UN2(JW)=2.*NU(JW)*MU(JW)
      WRITE(6,400)JW,DMU(JW),JW,DAM1(JW),JW,DAM2(JW)
  400 FORMAT(1H ,'MU',I1,'=',E11.5,5X,'M1',I1,'=',E11.5,5X,
     X          'M2',I1,'=',E11.5/)
  203 CONTINUE
      DO 406 JW=1,MODES
  406 PW(JW)=MU(JW)*MU(JW)
      GAMA2=2.*GAMA
      ZK1=1.+2.*GAMA2*DELTA
   36 X1=1./PKX
      Y1=1./PKY
C
C     DEFINE INITIAL CONDITIONS
```

```
      OMXO=0.
      OMYO=0.
      OMZO=1.
      OMEGA=1.
      OMXDO=0.
      OMYDO=0.
      OMZDO=0.
C     THESE I.C. ALIGN Z-AXIS ON H-VECTOR
      DO 404 JW=1,MODES
      OMXO=OMXO+GAMA2*OMZO*XNO(JW)/(X1+2.*GAMA*DELTA)
  404 OMYO=OMYO+GAMA2*OMZO*YNO(JW)/(Y1+2.*GAMA*DELTA)
      DO 405 JW=1,3
      XNDO(JW)=0.
      YNDO(JW)=0.
      V(1,JW)=XNO(JW)
      V(2,JW)=YNO(JW)
      V(3,JW)=XNDO(JW)
  405 V(4,JW)=YNDO(JW)
      V(5,1)=OMXO
      V(6,1)=OMYO
      V(7,1)=OMZO
      V(8,1)=OMXDO
      V(9,1)=OMYDO
      V(10,1)=OMZDO
      DO 98 J=1,10
      DO 98 JW=1,MODES
   98 DV(J,JW)=V(J,JW)
      N=1
      NP=1
   88 I=1
   50 CONTINUE

C     RUNGE KUTTA INTEGRATION

      DO 407 JW=1,MODES
      ACX(JW)=-UN2(JW)*DV(3,JW)-DV(1,JW)*PW(JW)-AM21(JW)*
     X        (OMEGA*DV(5,1)-DV(9,1))
      ACY(JW)=-UN2(JW)*DV(4,JW)-DV(2,JW)*PW(JW)-AM21(JW)*
     X        (OMEGA*DV(6,1)+DV(8,1))
      AK(1,I,JW)=H*DV(3,JW)
      AK(2,I,JW)=H*DV(4,JW)
      AK(3,I,JW)=H*ACX(JW)
  407 AK(4,I,JW)=H*ACY(JW)
      AK(5,I,1 )=H*DV(8,1)
      AK(6,I,1 )=H*DV(9,1)
      AK(7,I,1 )=H*DV(10,1)
      I=I+1
      IF(I.GT.3) GO TO 60
      Z=I-1
      DO 91 J=1,7
      DO 91 JW=1,MODES
   91 DV(J,JW)=V(J,JW)+AK(J,Z,JW)/2.
      CALL RATES
      GO TO 50
   60 IF(I.GT.4) GO TO 90
      DO 10 J=1,7
      DO 10 JW=1,MODES
```

```
10    DV(J,JW)=V(J,JW)+AK (J,3,JW)
      CALL RATES
      GO TO 50
90    DO 11 J=1,7
      DO 11 JW=1,MODES
      CC(J,JW)=AK(J,1,JW)+2.*(AK(J,2,JW)+AK(J,3,JW))+AK(J,4,JW)
11    V(J,JW)=V(J,JW)+CC(J,JW)/6.
      DO 89 J=1,7
      DO 89 JW=1,MODES
89    DV(J,JW)=V(J,JW)
      CALL RATES

   RUNGE KUTTA FINISHED
   NOW CALCULATE OUTPUT VARIABLES

      IF(NP.NE.1) GO TO 41
      COMPUTE COMPONENTS OF H-VECTOR IN BODY-FIXED AXES
      AMX=V(5,1)*(X1+GAMA2*DELTA)
      AMY=V(6,1)*(Y1+GAMA2*DELTA)
      DO 451 KF=1,NSUP
      AMX=AMX+AM2(KF)*(V(4,KF)-V(7,KF)*V(1,KF))*GAMA2
451   AMY=AMY-AM2(KF)*(V(3,KF)+V(7,KF)*V(2,KF))*GAMA2
      AMZ=ZKI*V(7,1)
      HSQ=AMX*AMX+AMY*AMY+AMZ*AMZ
      HSQ=SQRT(HSQ)
      OHXY=AMX*AMX+AMY*AMY
      OHXY=SQRT(OHXY)
      PI=3.141592
      DEG=180./PI
      CA=DEG*ASIN(OHXY/HSQ)
      IF(NSKP.EQ.0) GO TO 427
      IF(N.GT.1.AND.N.LT.NX) GO TO 41
427   CONTINUE
      IF(N.NE.1) GO TO 29
      WRITE(6,33) HSQ
33    FORMAT(/1H ,'INITIAL H=',E12.6)
      WRITE(6,94)
94    FORMAT(////1H ,' QX MODE1     QY MODE1     QX MODE2     QY MODE2
     XQX MODE3     QY MODE3        OMEGA X        C ANG         N',/)
29    CONTINUE
      IF(NP.NE.1) GO TO 41

   WRITE OUTPUT VARIABLES

      WRITE(6,22)V(1,1),V(2,1),V(1,2),V(2,2),V(1,3),V(2,3),V(5,1),
     1CA,N
22    FORMAT(1H ,6(D11.5,1X),   D12.6,1X,F11.6,1X,I5)
41    NP=NP+1
      IF(NP.EQ.NPI) NP=1
      CALL PLOT
      N=N+1
      IF(N.GT.NK) RETURN
      GO TO 88


      SUBROUTINE RATES
      THIS ROUTINE CALCULATES THE DERIVATIVES  OF THE ANGULAR RATES
```

```
      COMMON V,DV,COR,AM2,X1,Y1,DELTA,GAMA2,I,UN2,OMEGA
      COMMON/ZEN/OMU
      COMMON/THREE/MODES
      COMMON/THIRD/NSUP
      DIMENSION OMU(3)
      DOUBLE PRECISION V(10,3),DV(10,3),COR,AM2(3),
     X    UN2(3),GAMA2,DELTA,X1,Y1,MU(3),AUX1,AUX2
      DO 711 JI=1,NSUP
  711 MU(JI)=OMU(JI)
      AUX1    =-1./(X1+GAMA2*(DELTA-COR))
      DV(8,1)=AUX1* (1.-Y1+GAMA2*(DELTA-COR))*OMEGA*DV(6,1)
      AUX2    =-1./(Y1+GAMA2*(DELTA-COR))
      DV(9,1)=AUX2* (X1-1.-GAMA2*(DELTA-COR))*OMEGA*DV(5,1)
      DV(10,1)=0.
      DO 12 J=1,MODES
      DV(8,1)=DV(8,1)+AUX1*GAMA2*AM2(J)*(DV(2,J)*(OMEGA*OMEGA-
     X            MU(J)*MU(J))-UN2(J)*DV(4,J))
      DV(9,1)=DV(9,1)-AUX2*GAMA2*AM2(J)*(DV(1,J)*(OMEGA*OMEGA-
     X            MU(J)*MU(J))-UN2(J)*DV(3,J))
   12 CONTINUE
      IF(I.LE.4) RETURN
      V(8,1)=DV(8,1)
      V(9,1)=DV(9,1)
       V(10,1)=DV(10,1)
      RETURN


      SUBROUTINE PLOT
      THIS SUBROUTINE PLOTS NUTATION ANGLE VS N
      COMMON/ONE/GAMA,PKX,PKY,XNO,YNO,NU
      COMMON/LEN/LAM,SIO
      COMMON/THREE/MODES
      COMMON/FIVE/MK,N,CA
      COMMON/SIX/MAXP,PREC
      DOUBLE PRECISION SIO
      REAL MAX,LAM,LINE
      DIMENSION SAVE(100),LINE(110),AP(5),AN(5)
      DATA BLANK,STAR,DOT/1H ,1H*,1H./
      IF(N.NE.1) GO TO 2
      N1=(MK+50)/100
      DO 1 J1=1,100
    1 SAVE(J1)=0.
      J1=0
      MAX=0.
      I1=0
      SIOR=SIO
    2 IF(N/N1*N1.NE.N) GO TO 3
      J1=J1+1
      SAVE(J1)=CA
      IF(ABS(CA).GT.MAX) MAX=ABS(CA)
    3 IF(N.NE.MK) RETURN
      WRITE (6,4)
    4 FORMAT(1H1,'PLOT OF NUTATION ANGLE IN DEGREES VS N FOR ')
      WRITE(6,95)LAM,SIOR,GAMA,PKX,PKY,PREC,MAXP,MODES
   95 FORMAT(1H ,'LAMBDA=',F6.0,/1H ,'SI-ZERO=',F9.2,/1H ,'GAMA=',F7.3,/
     X1H ,'PKX=',F6.4,/1H ,'PKY=',F6.4/
     X       1H ,'PREC=',I2/1H ,'MAXP=',I3/1H ,'MODES=',I2////)
```

```
      A1=MAX/50.
      DO 6 I1=1,5
      AN(I1)=-A1*(60.-10.*I1)
 6    AP(6-I1)=-AN(I1)
      I1=0
      WRITE(6,7) AN,I1,AP
 7    FORMAT(1H ,19X,5(F6.2,4X),3X,I1,2X,5(5X,F5.2))
      DO 8 J1=1,110
      LINE(J1)=BLANK
 8    IF((J1+4)/10*10.EQ.(J1+4)) LINE(J1)=STAR
      WRITE(6,9) LINE
 9    FORMAT(1H ,12X,110A1)
      DO 10 J1=1,110
 10   LINE(J1)=STAR
      WRITE(6,9)LINE
      DO 11 J1=1,110
 11   LINE(J1)=BLANK
      DO 13 K1=1,100
      J1=SAVE(K1)/A1+56.5
      LINE(56)=STAR
      IF(K1/10*10.NE.K1) GO TO 12
      LINE(55)=STAR
 12   LINE(J1)=DOT
      WRITE(6,9) LINE
      IF(K1/10*10.NE.K1) GO TO 15
      IF(J1.GE.50.AND.J1.LE.54) GO TO 15
      M1=N1*K1
      WRITE(6,14) M1
 14   FORMAT(1H+,61X,I5)
 15   LINE(J1) =BLANK
 13   LINE(55)=BLANK
      RETURN
      END
```

..ILATION: NO DIAGNOSTICS.

CHAPTER 6

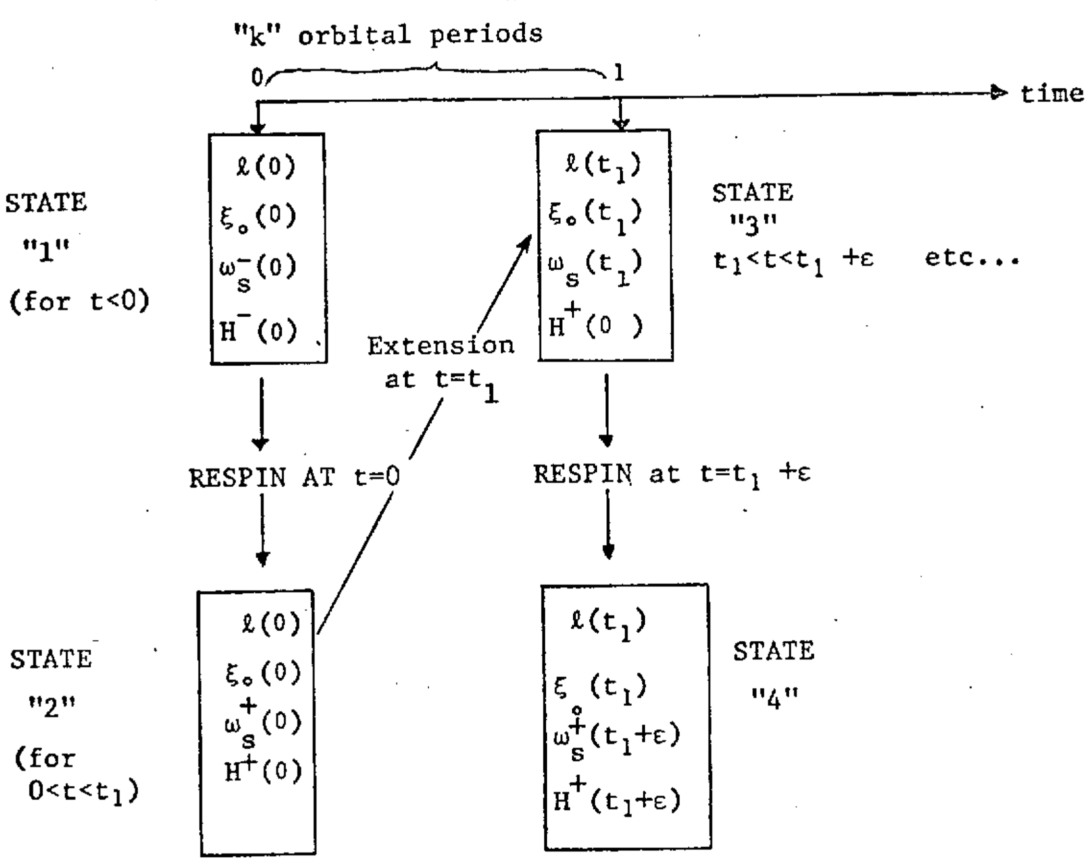
Other Topics

The present chapter contains a short note on the use of the stability charts in deployment dynamics, and two bibliographical reports on passive nutation damping devices.

6.1 Stability charts and deployment dynamics

6.11 Dynamic parameters during deployment

A deployment phase such as the one for IMP-I may be summarized as follows, if H, ℓ, $\xi_\circ$, ω_s designate the angular momentum, length of booms, non-dimensional radius of the hub $(\frac{x_\circ}{\ell})$, and spin rate ω_s $^{-\text{ or }+}$



We define a "state" as a set of values ℓ, $\xi_\circ$, ω_s, H. If non-dimensional variables are used, let

$$H_\circ \underset{\text{def}}{\equiv} H(\ell=0) = I_{zh}\,\omega_{s_\circ}$$

$$h(\text{any state}) \underset{\text{def}}{\equiv} \frac{H(\text{any state})}{H_0} \qquad h(\ell=0) = 1$$

$$\omega_s(0^-) \underset{\text{def}}{\equiv} \omega_{s0}$$

Thus

$$h = \frac{I_z\omega_s}{H_0} = \frac{(1 + 4\Gamma\Delta)I_{zh}}{I_{zh}\omega_{s0}} = (1 + 4\Gamma(\xi_\circ^2 + \xi_\circ + \tfrac{1}{3}))\frac{\omega_s}{\omega_{s(0)}} \tag{1}$$

with

$$\Gamma = \frac{\rho\ell^3}{I_{z,h}}$$

$$\Delta = \xi_\circ^2 + \xi_\circ + \frac{1}{3}$$

$I_{z,h}$ = moment of inertia of central hub about "z".

ρ = linear density of boom

In view of these definitions, an <u>extension</u> maneuver at $\underline{\underline{t}}$ corresponds to

$$h(t + 0) = h(t - 0)$$

$$\ell(t + 0) = \ell(t - 0) + \Delta\ell$$

$$\frac{\omega_s(t \pm 0)}{\omega_s(t - 0)} = \frac{(1 + 4\Gamma\Delta)t-0}{(1 + 4\Gamma\Delta)t+0}$$

in which $\Delta\ell$ is <u>specified</u>.

A <u>respin</u> maneuver at $\underline{\underline{t}}$ will give

$$h(t + 0) = h(t - 0) + (1 + 4\Gamma(\xi_\circ^2 + \xi_\circ + \tfrac{1}{3}))_{t+0}\frac{\omega_s(t+0)-\omega_s(t-0)}{\omega^-_{s(0)}}$$

$$(1 + 4\Gamma(\xi_\circ^2 + \xi_\circ + \tfrac{1}{3}))_{t+0} = (1 + 4\Gamma(\xi_\circ^2 + \xi_\circ + \tfrac{1}{3}))_{t-0}$$

in which $\delta\omega_s(t) = \omega_s(t + 0) - \omega_s(t - 0)$ is specified.

For a satellite of given hub (x_0, I_{zh} specified)

$$\xi_o = \frac{x_o}{\ell} = x_o \cdot \left(\frac{\rho}{\Gamma I_{zh}}\right)^{1/3} = \frac{x_o \rho^{1/3}}{I_{zh}^{1/3}} \frac{1}{\Gamma^{1/3}} = \bar{S} \frac{1}{\Gamma^{1/3}} \tag{2}$$

with $\bar{S}$ a fixed non-dimensional number $\underset{\text{def}}{\equiv} \dfrac{x_o \rho^{1/3}}{I_{zh}^{1/3}}$

Now, substituting (2) for ξ_o in Equation (1)

$$\omega_s = \omega_{s_0} [1 + 4\Gamma(\bar{S}^2\ \Gamma^{-2/3} + \bar{S}\ \Gamma^{-1/3} + \tfrac{1}{3})]^{-1} h$$

If ℓ is specified in any state,

$$\Gamma = \frac{\rho \ell^3}{I_{z,h}}$$

can be computed.

To that state there corresponds an Etkin's number

$$\bar{\lambda} = \frac{\rho \ell^4}{EI}\ \omega_s^2 = \frac{\rho \ell^4 \omega_{s_o}^2}{EI x_o^4}\ x_o^4 [1 + 4(\bar{S}^2\Gamma^{1/3} + \bar{S}\Gamma^{2/3} + \tfrac{1}{3}\Gamma)]^{-2} h^2$$

The quantity $\dfrac{\rho x_o^4}{EI}$ is specified for a given design. Let $\bar{R}$ be the non-dimensional quantity

$$\bar{R} \underset{\text{def}}{\equiv} \left(\frac{\omega_{s_o}}{\omega_*}\right)^2$$

$$\omega_*^2 \underset{\text{def}}{\equiv} \frac{EI}{\rho x_o^4}$$

Then

$$\bar{\lambda} = \bar{R}\ \xi_o^{-4} \left[1 + 4\left(\bar{S}^2\ \Gamma^{1/3} + \bar{S}\ \Gamma^{2/3} + \tfrac{1}{3}\Gamma\right)\right]^{-2} h^2$$

$$= \frac{\bar{R}}{\bar{S}^4}\ \Gamma^{4/3} \left[1 + 4\left(\bar{S}^2\ \Gamma^{1/3} + \bar{S}\ \Gamma^{2/3} + \tfrac{1}{3}\Gamma\right)\right]^{-2} h^2 \tag{3}$$

(2) and (3) thus give ξ_0, $\bar{\lambda}$ during the "states" of deployment as functions of boom's length and angular momentum. In these relations, $\bar{R}/\omega_{s0}$ and $\bar{S}$ are fixed for any given design.

6.1.2 Stability during deployment

The determination of the stability during deployment will thus proceed as follows:

a) $\bar{R}/\omega_{s_o}$ in computed (a fixed quantity), then $\bar{R}$ for ω_{s_o} given.

b) $\bar{S}$, a fixed quantity, is computed;

given the state ω_s, ℓ, ξ_o and H for some t:

c) compute h;

d) compute Γ ( );

e) compute $\Delta(\xi_o)$;

using the relevant formulae for either respin or extension maneuver

f) compute $\bar{\lambda}$ from (3);

g) determine the stability of the corresponding (K_p, Γ) point on the stability chart corresponding to the computed values of $\bar{\lambda}$ and ξ_o, using program FLEXAT of Chapter 5.

C3

6.2 A SURVEY OF PASSIVE NUTATION DAMPING TECHNIQUES

Prepared by

William O. Keksz

I. Introduction

In this paper, several methods of passive nutation damping are surveyed. In a review of rigid body dynamics, conditions of stability are presented. Ball, pendulum, and fluid dampers are surveyed, among others, along with effects of magnetic and gravitational torques and structural hysteresis energy dissipation. Finally, a few active and semipassive systems are mentioned in the way of comparison.

A, B, C	Moments of inertia about x, y, z axes
D′, E′, F′	Products of inertia for xy, xz, yz planes
x, y, z	Body-fixed axes, z along spin axis
X, Y, Z	Inertial axes, Z along $\vec{H}$
$\vec{\omega}$	Total angular velocity
p, q, r	Components of $\vec{\omega}$ along x, y, z axes
$(\dot{\ })$	d( )/dt
$\vec{H}$	Angular momentum
q_i	Generalized coordinates
L_{qi}	Moment in direction of q_i
ψ, θ, φ	Euler's angles
$\dot{\psi}$	Precession rate
$\dot{\varphi}$	Spin rate
θ	Nutation angle
γ	Magnetic or structural hysteresis factor
ω_T	$(p^2 + r^2)$ = component of $\vec{\omega}$ in xy plane
λ	(C − A)/A
Ω	λr = forcing frequency
i	$(-1)^{1/2}$
$\hat{i}, \hat{j}, \hat{k}$	Unit vectors along x, y, z
M	Mass of main body
m	Damper mass
s	Radius of gyration

Other symbols are defined throughout the text as needed.

II. Review of Rigid Body Dynamics [33]

A. Definitions

1. Euler's Angles

If X,Y,Z is fixed in space and x,y,z is the body fixed system, we define the Euler Angles ψ, θ, and φ in Fig. II-1. The spin axis is along z, and:

$\dot{\psi}$ = precession rate

θ = nutation angle

$\dot{\varphi}$ = spin rate

The unit vectors $\hat{i},\hat{j},\hat{k}$ lie along x,y,z.

We have:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (\cos\varphi\ \cos\psi - \sin\varphi\ \cos\theta\ \sin\psi) & (-\sin\varphi\ \cos\psi - \sin\psi\ \cos\theta\ \cos\varphi) & (\sin\theta\ \sin\psi) \\ (\cos\varphi\ \sin\psi + \sin\varphi\ \cos\theta\ \cos\psi) & (-\sin\varphi\ \sin\psi + \cos\varphi\ \cos\theta\ \cos\psi) & (-\sin\theta\ \cos\psi) \\ (\sin\theta\ \sin\varphi) & (\sin\theta\ \cos\varphi) & (\cos\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2. Angular Velocity

If $\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$ is the total angular velocity of the body, then:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} (\sin\theta\ \sin\varphi) & (0) & (\cos\varphi) \\ (\sin\theta\ \cos\varphi) & (0) & (-\sin\varphi) \\ (\cos\theta) & (1) & (0) \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\varphi} \\ \dot{\theta} \end{bmatrix}$$

Note that r is not the spin rate.

In most cases, the linear velocity of the center of mass is ignored for damper analysis.

3. Angular Momentum

$$\vec{H} = \begin{bmatrix} \hat{i}\hat{i}A & -\hat{i}\hat{j}D' & -\hat{i}\hat{k}E' \\ -\hat{j}\hat{i}D' & \hat{j}\hat{j}B & -\hat{j}\hat{k}F' \\ -\hat{k}\hat{i}E' & -\hat{k}\hat{j}F' & \hat{k}\hat{k}C \end{bmatrix} \cdot \vec{\omega}$$

The above is the angular momentum in x,y,z. For most cases, we can ignore the external torques produced by electromagnetic fields and gravitational gradients. Thus $\vec{H}$ is constant in inertial space (X,Y,Z), and thus we can align the Z axis along $\vec{H}$. If x,y,z are aligned along the principle axes of the body, $D',E',F' = 0$, and: $\vec{H} = Ap\hat{i} + Bq\hat{j} + Cr\hat{k}$
where p,q,r are the $\omega_x, \omega_y, \omega_z$ for alignment with the principle axes.

4. Kinetic Energy

The kinetic energy of the body is:

$$T = \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2) - D\omega_x\omega_y - E\omega_x\omega_z - F\omega_y\omega_z$$

and for the principle axes:

$$T = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2)$$

5. Euler's Equations

Here Euler's Equations are presented only for a principle axis x,y,z:

$$L_1 = A\dot{p} + qr(C-B)$$
$$L_2 = B\dot{q} + pr(A-C)$$
$$L_3 = C\dot{r} + pq(B-A)$$

where L_1, L_2, and L_3 are the external moments about the corresponding principle axes; here they will usually be zero.

6. Poinsot Ellipsoid

For a rigid body, T = constant, and thus:

$$\frac{\vec{\omega}\cdot\vec{H}}{H} = \frac{2T}{H} = Q$$

This must be the component of $\vec{\omega}$ along $\vec{H}$ and Z. If both sides of the energy relationship are divided by T, we get:

$$1 = \frac{p^2}{2T/A} + \frac{q^2}{2T/B} + \frac{r^2}{2T/C}$$

This is the Poinsot ellipsoid. If a plane is placed perpendicular to $\vec{H}$ a distance Q from the center of this ellipsoid, we see the Poinsot ellipsoid rolls on the plane (called the invariant plane), without slipping. The contact point is the tip of $\vec{\omega}$ (Fig.II-2). The curve traced out by the contact point on the plane is the herpolhode, and that on the ellipsoid is the polhode.

7. Body and Space Cones (Axisymmetric Body)

From the above we see that $\vec{\omega}$ sweeps out a surface in both the x,y,z and X,Y,Z frames. If $\theta = 0$ and we have an axisymmetric body (A=B) then these are both right circular cones. From the relations between p,q,r and $\dot{\psi}, \dot{\theta}, \dot{\varphi}$ substituted into the Euler moment equations, we have:

$$\dot{\psi} = \frac{C\,\dot{\varphi}}{(A-C)\cos\theta}$$

(a) C>A: $\dot{\psi}$ and $\dot{\varphi}$ are opposite in sign, and this is known as retrograde precession.

(b) C<A: $\dot{\psi}$ and $\dot{\varphi}$ have the same sign, and this is known as direct or posigrade precession.

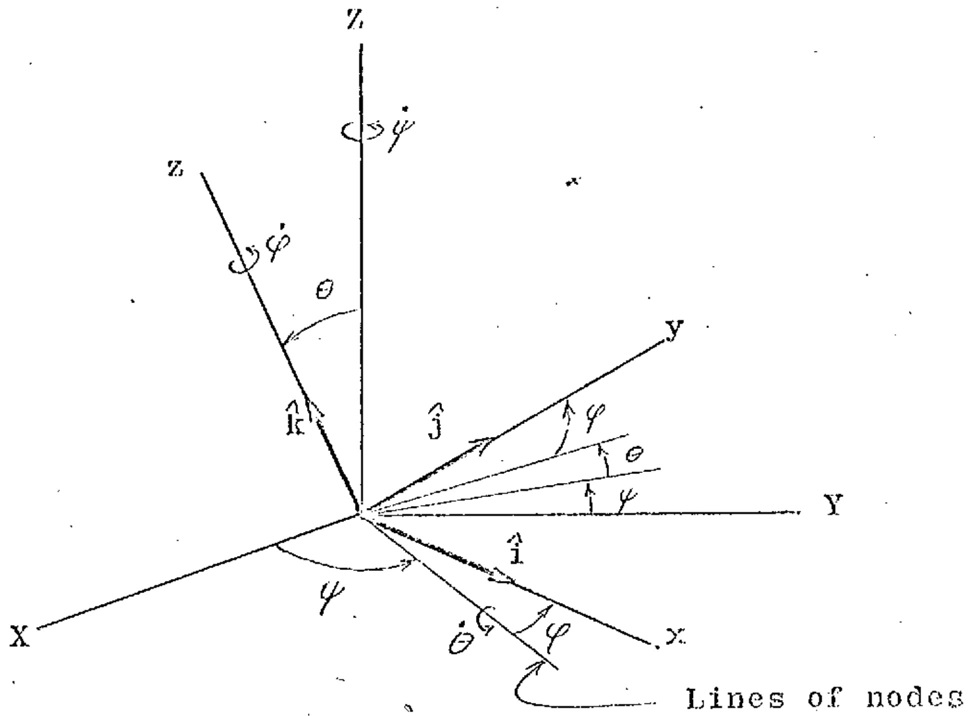


Fig. II-1: Euler's angles.

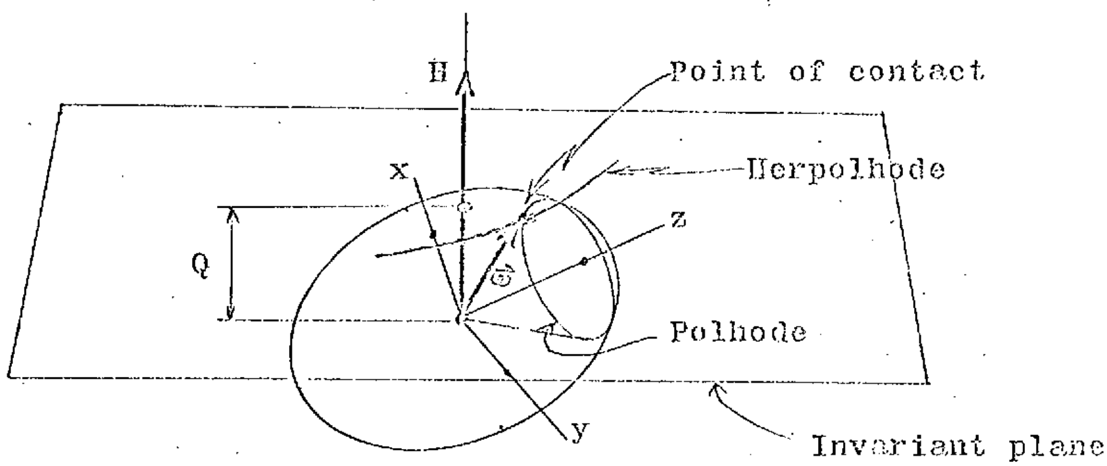
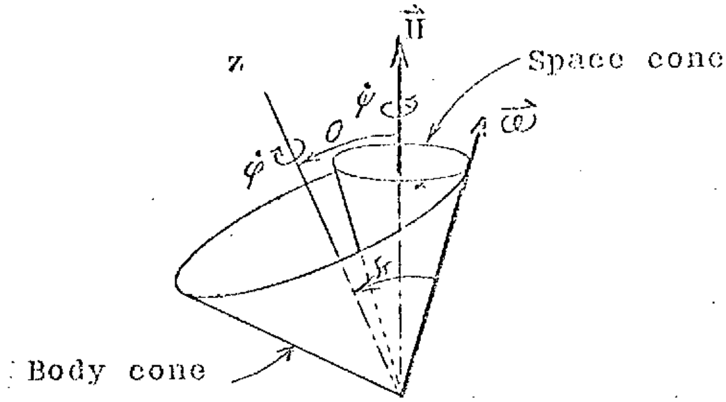
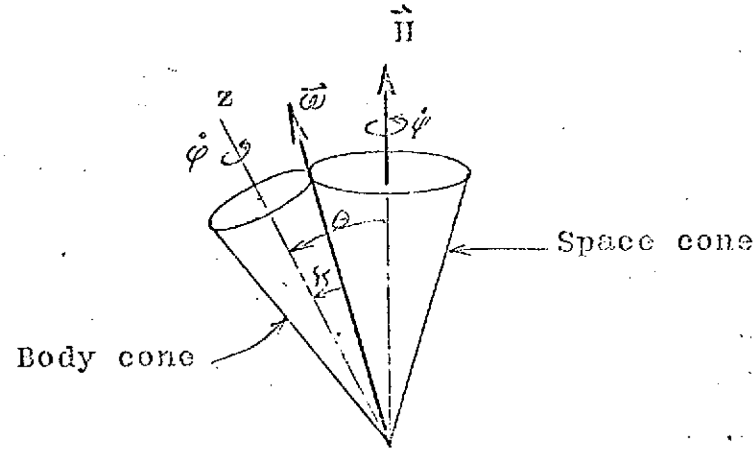


Fig. II-2: Poinsot ellipsoid.



(a) Retrograde precession; C > A.



(b) Direct precession; C < A.

Fig. II-3: Precession of body cone rolling on space cone. $\vec{\omega}$ is along the line of contact.

The body cone rolling on the space cone for each of these cases is illustrated in Fig. II-3. The angle between $\vec{\dot{\varphi}}$ and $\vec{\omega}$ is γ:

$$\tan\gamma = (p^2+q^2)^{1/2}/r = \omega_T/r$$

where ω_T is the component of $\vec{\omega}$ lying in the x,y plane. The angle between $\vec{\dot{\varphi}}$ and $\vec{H}$ is θ:

$$\tan\theta = (A/C)/(\omega_T/r)$$

By substituting $\Omega = [(C-A)/A]r = \lambda r$ into the Euler equations, we have:

$$\dot{p} + \Omega q = 0 \quad \longrightarrow \quad \ddot{p} = -\Omega\dot{q}$$

$$\dot{q} - \Omega p = 0$$

Thus $\ddot{p} + \Omega^2 p = 0$

and

$$p = p_0 \cos\Omega t + (\dot{p}_0/\Omega)\sin\Omega t$$

$$q = p_0 \sin\Omega t - (\dot{p}_0/\Omega)\cos\Omega t$$

These last imply that $\omega_T = (p^2+q^2)^{1/2}$ rotates about the z axis at the rate Ω.

By using a complex analysis, Ames and Murnaghan show that [1]:

$$\omega_T = \left[\frac{H^2 - 2CT}{A(A-C)}\right]^{1/2} e^{i\Omega t}$$

8. A Note on Unsymmetrical Bodies

The relations for $\vec{\omega}$ are given by Thomson for the case $A>B>C$ and $H^2<2TB$, a body spinning about its axis of least inertia [33a]:

$$p = \left[\frac{H^2 - 2TC}{A(A-C)}\right]^{1/2} \mathrm{cn}\, f(t - t_0)$$

$$q = \left[\frac{H^2 - 2TC}{B(B - C)}\right]^{1/2} \mathrm{sn}\, f(t - t_0)$$

$$r = -\left[\frac{2TA - H^2}{C(A - C)}\right]^{1/2} \mathrm{dn}\, f(t - t_0)$$

where $f = \left[\frac{(B - C)(2TA - H^2)}{ABC}\right]^{1/2}$

and the modulus of the elliptic functions is:

$$k = \left[\frac{(A - B)}{(B - C)} \frac{(H^2 - 2TC)}{(2TA - H^2)}\right]^{1/2}$$

This results in spin about the z axis with a superimposed wobble, with a θ_{max} and θ_{min}:

$$\cos^2\theta_{max} = C(2TB - H^2)/(B - C)H^2$$

$$\cos^2\theta_{min} = C(2TA - H^2)/(A - C)H^2$$

B. Miscellaneous Concepts

1. Stability of a Rigid Body

For a rigid body, T is a constant. If we let the initial condition be:

$p = p_1 + \epsilon$

q, r small

where ϵ is small, we can differentiate the Euler equations and substitute for p,q,r and $\dot{p},\dot{q},\dot{r}$. Then:

$$\ddot{q} + p_1^2 q(A - B)(A - C)/BC = 0$$

$$\ddot{r} + p_1^2 r(A - B)(A - C)/BC = 0$$

These are stable only if $(A - B)$ and $(A - C)$ are of the same sign. Thus they are unstable only if A is the intermediate rotational inertia.

2. Energy and Stability

In a real spacecraft, there is always an energy loss due to flexure of nonrigid parts, magnetic hysteresis, etc. Thus we have $\dot{T}<0$.

For an axisymmetric body, we have:

$$2T = A\omega_T^2 + Cr^2$$

$$H^2 = A^2\omega_T^2 + C^2r^2$$

Since $Cr = H\cos\theta$:

$$H^2 - 2TA = \cos^2\theta\, H^2(C - A)/C$$

or $$T = H^2\left[1 - \cos^2\theta\,(C - A)/C\right]/2A$$

Since there are no external torques, H is constant, and

$$\dot{T} = \left[H^2(C - A)/AC\right](\sin\theta\cos\theta)\dot{\theta}$$

$$= (H^2\lambda/C)(\sin\theta\cos\theta)\dot{\theta}$$

Thus, for decreasing T, θ decreases only if C>A, and the satellite is spinning about its axis of maximum inertia. This is the stable condition. For a prolate body, there must be an energy input for stability, which implies an active nutation control.

The change in energy required to stabilize a precessing body can easily be found. The desired energy state is:

$$T_f = \tfrac{1}{2}Cr_f^2$$

where the subscript f denotes final condition. Since H^2 is constant [36]:

$$H^2 = A^2\omega_T^2 + C^2r^2 = C^2r_f^2 = H_f^2$$

Then $r_f^2 = (A/C)\omega_T^2 + r^2$

Thus $\Delta T = \left|T - T_f\right| = \left|\tfrac{1}{2}A(1 - A/C)\omega_T^2\right|$

For an oblate body (A<C), this is the precessional energy, the amount to be removed; for a prolate body, it is the amount to be added.

Also $\omega_T = (r - \Omega) \tan\theta = (C/A)\, r \tan\theta$

III. Passive Dampers

Unless stated otherwise, the satellite will be assumed axisymmetric about the z (spin) axis, A = B, and oblate (A<C) for the below.

A. Ball-type [4, 24, 36]

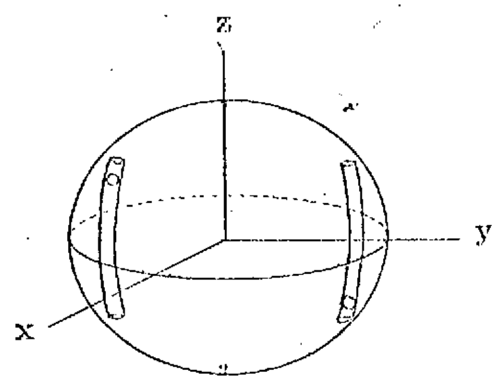
1. Mounted in the Meridian Plane

This type was first used in Telstar and later in ESRO II. These consist of a ball allowed to roll inside a circular cross section curved tube which is filled with a gas. Two are used, diametrically opposed, to maintain symmetry, and mounted in a plane through the spin axis. Energy dissapation comes about through viscous friction between the ball and gas, rolling friction between the ball and tube wall, and collision of the ball with the tube end, the latter only at large nutation angles.

Such a system is shown in Fig. III-1. According to Yu, the rotational motion of the ball (of radius a) is given:

$$(2/5)ma^2(\ddot{\omega}R/r) = ka - N$$

where k is the friction force at the contact point, and N the rolling friction torque. N is approximately an order of magnitude smaller than the viscous term. Neglecting N and assuming θ small, the motion of the ball



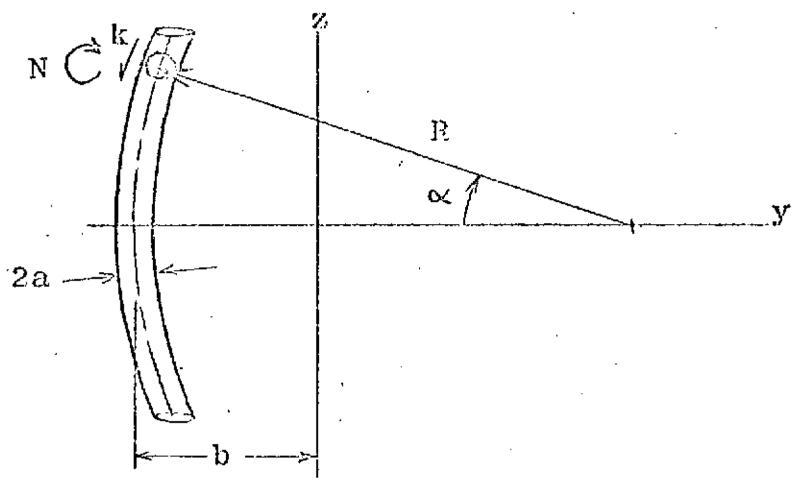


Fig. III-1: Ball-type damper mounted in meridian plane.

is described by:

$$\ddot{\alpha} + (5c/7m)\dot{\alpha} + (5br^2/7R)\alpha = (5b/7R)\cos\Omega t$$

where c is the coefficient of viscous friction. The time average rate of energy dissipation is, for viscous friction:

$$d\bar{T}_v/dt = T_v\Omega/2\pi = -cR^2\Omega^2\alpha_0{}^2/2$$

where $\alpha_0 = \theta(1 - \Omega^2/r^2)\left[(1 - \Omega^2/P^2)^2 + 4n^2\Omega^2/P^4\right]^{-\frac{1}{2}}$

$n = 5c/14m$

and $P = (5br^2/7R)^{\frac{1}{2}}$ is the natural frequency, the square root of the α coefficient.

We end up with an expotential damping:

$$\theta = \theta_0\ e^{-t/\tau}$$

and $\tau = \dfrac{5c\left[(1 - \Omega^2/P^2)^2 + 4n^2\Omega^2/P^4\right]}{7nmR^2\lambda\ (\lambda + 1)^2(1 - \lambda)^2}$

If rolling friction dissipation is included:

$$d\bar{T}_r/dt = 2FR\alpha_0|\Omega|/\pi$$

where F is the rolling friction, and:

$$\theta = (\theta_0 + \mu)e^{-t/\tau'} - \mu$$

$$\tau' = \tau\ln\left[(1 + \theta_0/\mu)/(1 + \theta_0/\mu e)\right]$$

Numerical computations show that μ is substantially less than one degree. Thus the viscous-only results can be used if θ_0 is somewhat greater than one degree.

The damping time can be greatly reduced by designing a resonant system, making $P = \Omega$. Then:

$$R_{res} = 5b/7\lambda^2$$

The time constant is then:

$$\tau_{res} = 28nC\lambda/5mb^2(\lambda + 1)^2r^2(1 - \lambda)^2$$

A resonant damper could not be used in Telstar because λ was close to zero and room had to be made for an electronics package, preventing a small value of b.

It is possible to conceive of dampers using straight tubes or tubes concave outward. It is easily seen, however, that the equilibrium position for the ball during nutation would be at the ends of the tubes, and the final spin axis would not coincide with that of the satellite without the balls.

The parameters for Telstar were A/C = .95, $\dot{\varphi}$ = 20-180 rpm, R = 15 ft, m = 0.0021 slug, a = 0.242 in (tungsten for its large density), c = 0.00193 lb-sec/ft (neon for its high viscosity). The theoretical damping time was calculated to be a maximum of about three minutes.

Note that a gas of low viscosity should be used for a tuned (resonant) damper, as n, proportional to c, appears in the numerator of the expression for τ_{res}.

The problems in this analysis are due to the assumed small θ and linearization of the equations. G.T. Kossyk devised a ground test of a model supported at its center of gravity which showed that the experimental τ was about four times that calculated using the mean value of the transverse inertia moments, and nine times that using the minimum value. Taking these factors into account, the τ for Telstar was calculated to be no more than thirty minutes.

2. Mounted in a Plane Parallel to the Equatorial

Two of this type were mounted in FR-I, and one in the HEOS spacecraft, which also used a liquid damper.

If h is the distance from the damper plane to the center of gravity, Routh criteria applied to the Euler equations indicate that $b/R < 1 - mh^2/A$ is necessary for stability. Also, optimum damping (minimum τ) is given by a viscous friction coefficient of:

$$c_{opt} = mR^2r\left[14mh^2(\lambda + 1)^3/5A\lambda\right]^{1/2}$$

This results in:

$$\tau_{opt} = (1/r)\left[56A\lambda/mh^2(\lambda + 1)^3\right]^{1/2}$$

Experimental results agree well with the theoretical. For two dampers and $\lambda = 0.61$, $h = 0.15$m, $R = 0.2$m, $r = 0.2$ rad/sec, and 250 gm give a maximum τ of 120 sec for reasonable θ. The experimental result was 130 sec. With all parameters equal, the efficiency ratio of the equatorial to meridian damper is $\left[(1 + \lambda)/(1 - \lambda)\right]^2$.

B. TEAM Damper [24,25]

The TEAM damper, used in Tiros, is essentially the same in concept as the meridian-mounted ball damper. A small mass fitted with rollers is allowed to run along a curved monorail (Fig. III-2). The difference lies in that there is no fluid involved, so only rolling friction exists. From the ball damper analysis, it can be

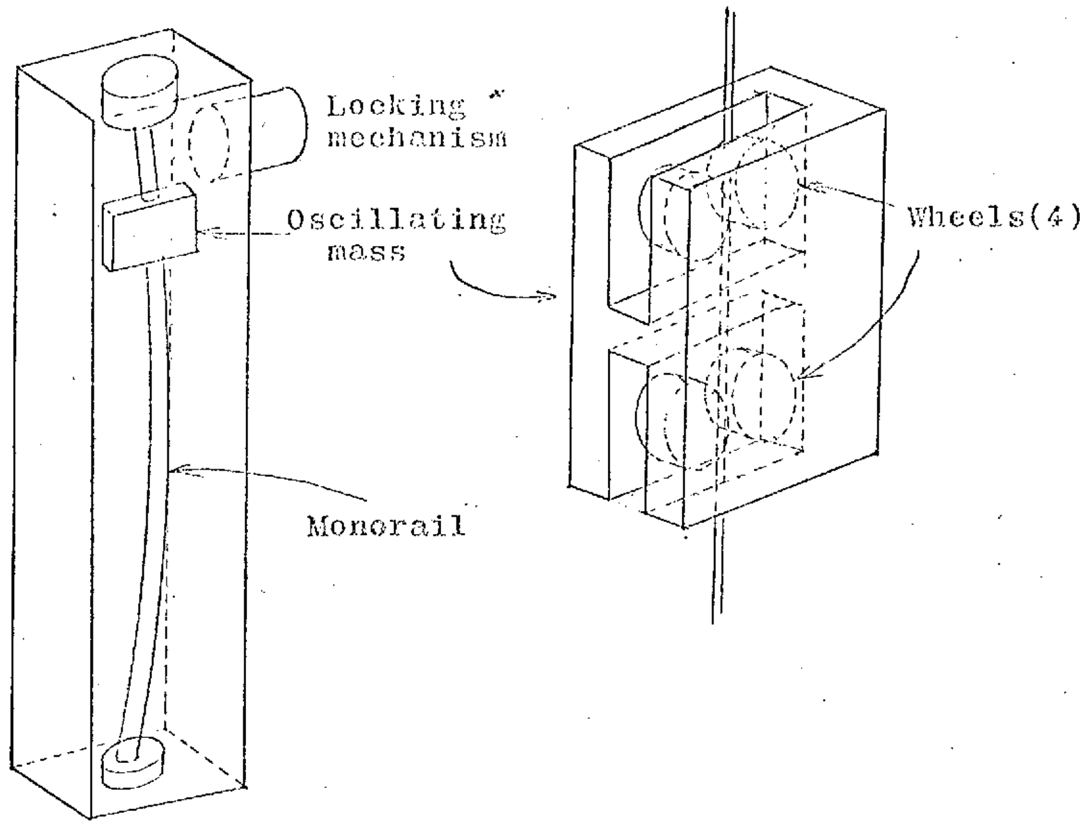


Fig. III-3: Tiros TEAM damper.

seen that this would behave well only at small θ.

For Tiros, the damper mass was about 0.001 of the total satellite mass, and assured a θ of less than 0.5 degrees. The time to damp from 2.5 to 0.5 degrees was about one minute. It was chosen because tests showed that the tube radius R of the ball damper would be greater than the track radius of TEAM. Also, it was found that the ball damper required an A/C not less than 1.6($\lambda \nless$ 0.375), where A/C for Tiros was 1.45 (λ = 0.31).

C. Pendulum Damper

1. Spin Axis Pivoted

The motion for a satellite with a pendulum pivoted on the spin axis, and moving in a plane perpendicular to the axis was described by Cartwright, Massingill, and Trueblood [6]. The driving frequency of the pendulum is the frequency of the acceleration due to nutation, $\Omega = \lambda r$. Without friction the pendulum would oscillate in synchronism opposite $\vec{\omega}_T$ at Ω, as in Fig.III-3a. However, if the pivot exerts a frictional torque, the pendulum lags behind this position by an angle δ (Fig.III-3b). The resulting torque on the axisymmetric main body causes the damping. As this lag angle increases, so does the damping, producing the convex portion of Fig.III-3b, and called the "nutation synchronous" mode.

When δ reaches 90 degrees, however, the pendulum is no longer in sync with $\vec{\omega}_T$, but is driven toward synchronism with r. This is a decreasing-rate decay with a superimposed convergent oscillation. It would be desirable to make the transition between the two modes at as small a θ as possible.

If the mass is assumed small so that the $\vec{\omega}_T$ rotates precisely at the nutation rate $(\lambda + 1)r = \Omega + r$ in inertial space, and Ω small, we have:

$$\dot{\theta} = (m\ell h/A)(\lambda + 1)r \sin\alpha$$

$$\ddot{\alpha} + (c_p/m)\dot{\alpha} + (h/\ell)(\lambda + 1)r^2\theta\sin\alpha = -(c_p/m)\lambda r$$

where α is the angle between the x axis and $\vec{\omega}_T$, assumed approximately equal to δ. Also, c_p is the friction coefficient of the relative velocity between the pendulum and main body.

Computer analysis has shown that the $\dot{\alpha}$ term can be neglected. To find the time and nutation angle at transition between modes, we set $\alpha = \pi/2$ and integrate the above equations. Thus:

$$\theta_* = -c_p(\ell/hmr)\lambda/(1 + \lambda)^2$$

$$t_* = \left[(\theta_0^2 - \theta_*^2)/c_p\ell^2\right](C/2\lambda)$$

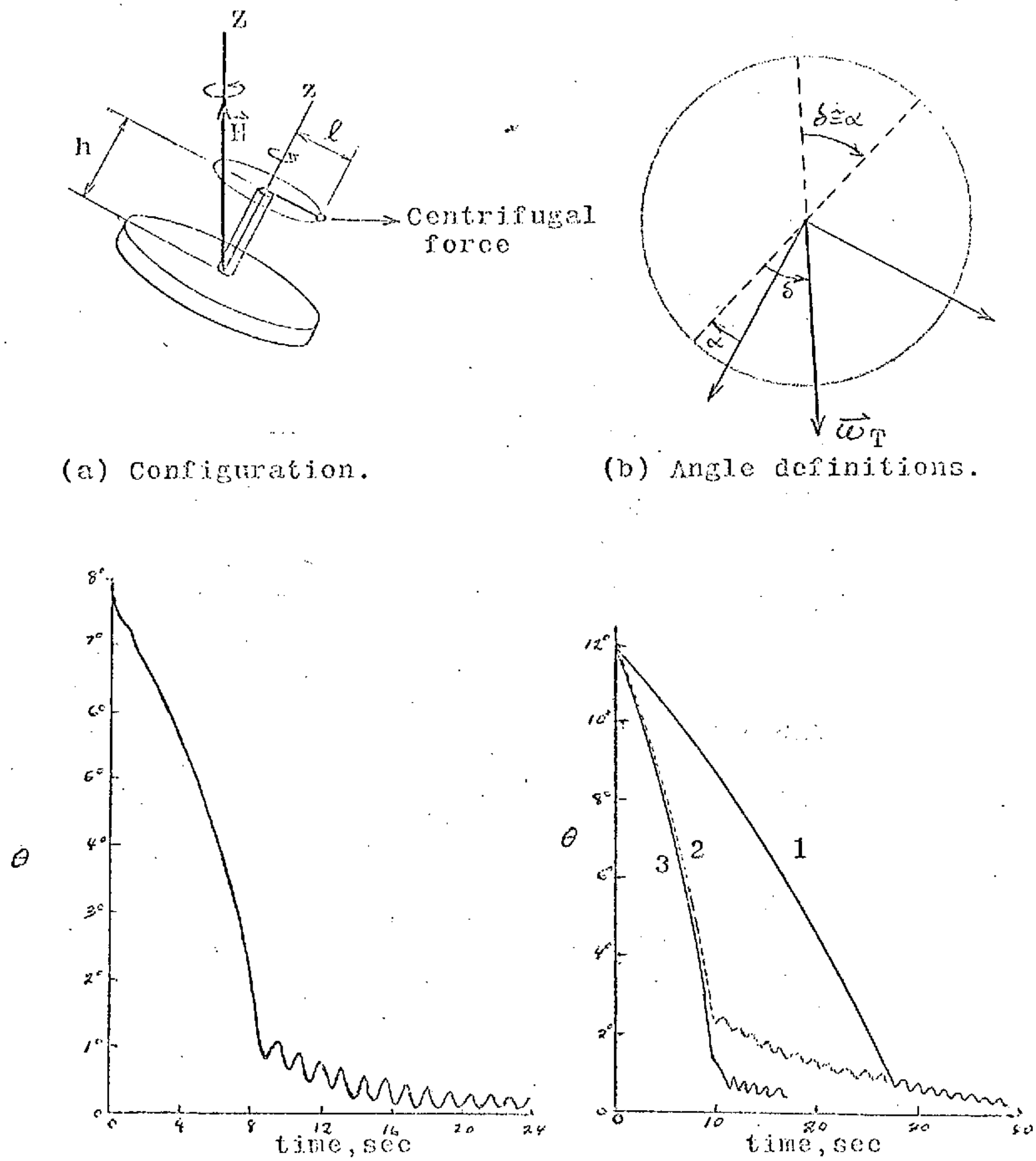
Numerical integration of exact equations show that the first equation overestimates θ_* by as much as a factor of 2, andthe second underestimates t_* by as much as a factor of 2. Also, for these equations to be valid, α must be near zero at time $t = 0$; thus:

$$m\ell^2 << C\theta_0^2(A/C)/(1 - A/C)$$

is a necessary condition for their validity.

Because of its nonsymmetry, there will be a small final nutation angle when only one damper is used:

$$\theta_f \doteq (m\ell h/C)(\lambda + 1)/\lambda$$



(a) Configuration. (b) Angle definitions.

(c) Experimental example. (d) Experimental comparsion:
1. Small ℓ
2. Large ℓ
3. Double

Fig. III-3: Axially-mounted pendulum.

If c_p is large, t_* decreases but θ_* increases. If c_p is an increasing function of velocity, there will be strong damping at the beginning. As the relative velocity decreases, so does c_p, and the damper-main body system is decoupled enough to delay transition.

Another improvement would be to use two pendulums of different radii. Experimental results show these to act independently, the long one damping quickly at large θ (Fig.III-3d), the short at small θ.

2. Pivoted Away From the Spin Axis

The problem of a pendulum moving in a plane perpendicular to the spin axis and pivoted at a point away from the axis have been studied by Haseltine [16, 17] and Newkirk, Haseltine, and Pratt [23].

If η is the rotation required to reach a point on the body, the kinetic energy of the system is [23]:

$$T = \tfrac{1}{2}\left[C\xi^2 + A\omega_T^2 + \bar{m}(\xi \times r_m)^2\right]$$

where $\xi = r + \dot{\eta}$

$$\bar{m} = Mm/(M + m)$$

and r_m is the distance from the center of gravity to the damper mass.

Using a set of modified Lagrangian equations:
$d(\partial T/\partial \dot{q}_i)/dt = L_{qi}$

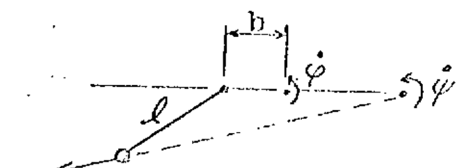
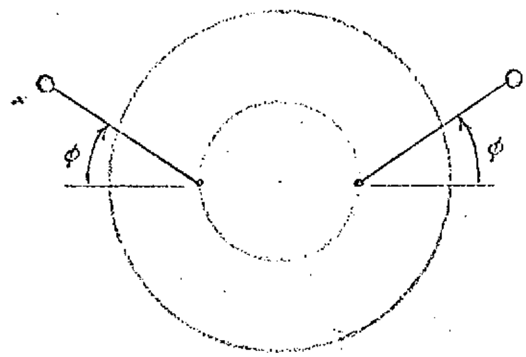
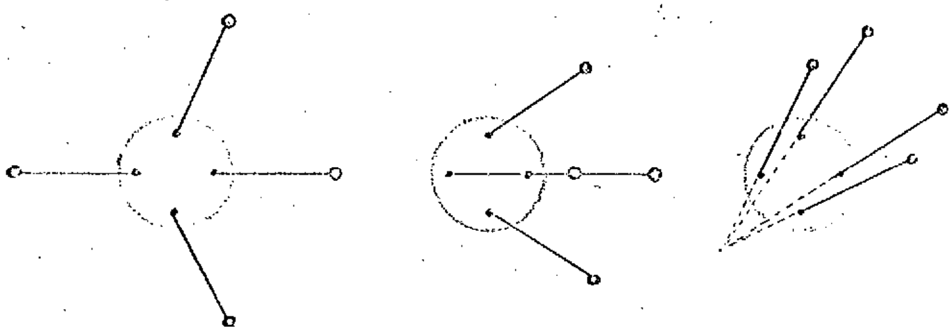


Fig.III-4: Offset.



(a) Off-design equilibrium
for two offset pendulums.



(b) Off-design equilibrium for four
pendulums.

Fig. III-5: Offset pendulum dampers.

in which all the L_{qi} are zero except:

$$L_\eta = -c_p \dot{\eta}$$

The following equations result:

$$\dot{\Phi} = -(c_p/A)(\dot{\xi} - r)$$

$$\dot{r} = \bar{A}c_p(\dot{\xi} - r)/(\bar{A}\bar{C} - \bar{D}^2) + C\bar{D}\,\Omega\, p/(\bar{A}\bar{C} - \bar{D}^2) + pq$$

$$\dot{q} = -pr + C\bar{C}\Phi p/(\bar{A}\bar{C} - \bar{D}^2) + \bar{D}c_p(\dot{\Phi} - r)/(\bar{A}\bar{C} - \bar{D}^2)$$

$$\dot{p} = (\bar{A} - \bar{C})qr/(\bar{A} + \bar{C}) + C\Phi q/(\bar{A} + \bar{C}) - \bar{D}(r^2 - q^2)/(\bar{A} + \bar{C})$$

where $\bar{A} = A + \bar{m}y^2$

$\bar{C} = \bar{m}h^2$

$\bar{D} = \bar{m}hy$

and y is the y coordinate of the mass. No small angle assumptions have been made. If however, θ and m are small, Φ constant, and other limiting assumptions are made:

$$\ddot{S} = -c_p(\dot{S} - \dot{\Phi})/\bar{C} + (C\bar{D}/2A\bar{C})\Phi(\dot{U}e^{-iS} + \dot{U}e^{iS})$$

$$\ddot{U} - i(C/A)\Phi U = -(D/A)\dot{S}^2 e^{iS}$$

where $S = \varphi + \psi$

$\dot{S} \cong r$

$U = \sin\theta\,(\cos\varphi + i\sin\varphi)$

Thus $|U| = \sin\theta \cong \theta$.

Three different solutions were tried for this set of equations:

(a) The stable solution in which the damper does not rotate relative to the main body. Then $|U| \cong \theta_0 =$ constant.

(b) "Slow damping" in which the damper has a small oscillation about a fixed point on the body, resulting in:

$$\theta = \theta_0 e^{-t/\tau}$$

$$\tau = \left\{ \frac{2(C - A)A\bar{C}^2 \left[(c_p/\bar{C})^2 + (C - A)^2 \bar{\Phi}^2/\bar{C}^2 \right]}{C\bar{D}^2 \bar{\Phi} c_p} \right\}$$

(c) "Fast damping" in which the damper rotates at the nutation frequency Cr/A. This solution is good only when θ is not small:

$$\theta^2 = \theta_0^2 - \left[2c_p(C - A)/CA - 2\pi\bar{D}^2 C\dot{\phi}/A^3 \right] t$$

The advantage of offsetting the pivot point from the axis is that it would appear that the pendulum will align itself radially outward from the pivot. Then a countermass could be mounted from the equilibrium position to preserve the symmetry of the satellite, with no residual wobble. An alternate is to employ two diametrically opposed pendulums (Fig.III-4).

For a pendulum offset a distance b and of arm length ℓ, the frequency is:

$$\frac{\omega}{2\pi}\sqrt{\frac{b}{\ell} + \frac{hC}{\ell A}\tan\theta}$$

For resonance:

$$\ell \cong [b + (\lambda + 1)h\theta]/\lambda^2 \rightarrow b/\lambda^2$$

assuming $\omega = r$, for small θ. If λ is only slightly greater than zero, ℓ can be large. A solution is to use a pendulum of radius of gyration s. Then:

$$\ell = [b + (\lambda + 1)h\theta]/\lambda^2(1 + s^2/\ell^2)$$

Haseltine [16] has shown that, when:

$$(\ell/b)[mh^2/(C - A) - (m/M)] > ½$$

The angle between the two dampers will not be 180 degrees in the steady state (Fig.III-5a). Then:

$\cos\phi$ = lesser of one or $b(C - A)/2mh^2\ell$ and the apparent wobble angle is approximately $(2mh\ell\sin\phi)/(C - A)$.

Haseltine also studied the motion with four identical pendulums mounted 90 degrees apart. Again, experimental results showed possible equilibrium positions resulting in a residual wobble (Fig.III-5b).

D. Liquid Dampers

1. Spin Axis Concentric

The use of an annulus partially filled with a dense, high viscosity liquid, usually mercury, has proven very popular; it was first used in Syncom and the Explorer series [24]. The basic theory was laid out by Carrier and Miles [5] for lamilar flow. The equations of motion for the body are similar to those for the pendulum, since both systems are circularly constrained. The dimensions of the system are given in Fig. III-6. For small θ, it was assumed that the liquid was in contact with the entire outer surface of the annulus. The rate at which energy is dissipated throughout the fluid is, if ρ is the density:

$$\dot{T} = -\rho\nu\iiint(\vec{\nabla} \times \vec{v})^2 dV$$

where $\vec{v}$ is the fluid velocity, ν is the kinematic viscosity, and dV is a differential element volume. Assuming the irrotational component of velocity cannot contribute to the integral:

$$\dot{T} = -8^{1/2}\pi d R^2 \rho (\dot{\varphi} h\theta)^2 |\dot{\varphi}| \, n_*^2 |G| / |\Delta|^2$$

where $n_* = (1 + \lambda)^2(1 - a_*/R)^2$

$$G = (i\dot{\varphi}R^2/\nu)^{1/2}$$

$$\Delta = (n_* - n)G + (n_* + n)$$

$$n = 1 + 2\lambda - \lambda^2$$

This results in a time constant of decay for θ of:

$$\tau = \Gamma A / dR^2 h^2 \rho |\dot{\varphi}|$$

and $$\Gamma = \frac{\lambda\left[(n_* - n)^2|G|^2 + 2^{1/2}(n_*^2 - n^2)|G| + (n_* + n)^2\right]}{32^{1/2}\pi(\lambda + 1)n_*^2|G|}$$

This is at a minimum in the neighborhood of $n_* = n$. Then:

$$\Gamma_{min} = \lambda/8^{1/2}\pi(\lambda + 1)|G|$$

and, if $a_*/R \ll 1$:

$$a_* \cong R[\lambda^2/(1+\lambda)^2 + 1/2^{1/2}|G|]$$

is the resonant condition. The variation in thickness of a_* has been assumed small.

For large θ, the fluid completely fills the cross section of the annulus over an angle Θ (Fig.III-7). The energy dissipation is then:

$$\dot{T} = -4|\dot{\varphi}|^{2.5}R^3\rho\nu^{1/2}(a + d)\Theta^{1/2}$$

The time constant for large is:

$$\tau = A|\dot{\psi}|\theta^2/8|\dot{\varphi}|(|\dot{\varphi}|\nu)^{1/2}R^3\rho(a + d)\Theta^{1/2}$$

For R = 10 cm, h = 10 cm, d = 0.25 cm, $a_* = 0.05$ cm, $A = 1.3$ kg - m^2, $\lambda = 1/3$, $\omega = 12$ rad/sec, $\rho = 13.6$ gm/cc, and $\nu = 10^{-3}$ cm^2/sec give a damping time of 14 sec for small θ. If a resonant damper were designed, a_* would be 0.637 cm and $\tau = 0.00044$ sec.

The large θ result for the above parameters and $\theta_0 = 1/6$, $\Theta = 5$, and $(a + d) = ½$ cm gives $\tau = 200$ sec. However, the Reynolds number is past critical for these, and the increased friction would reduce τ to about 70 sec.

The above would indicate that it would be desirable to design the damper for resonance. However, a study by Fitzgibbon and Smith [35] show that significant energy can be stored in the surface waves on the fluid

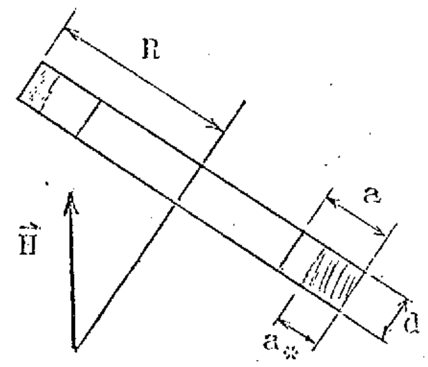


Fig.III-6: Damper parameters.

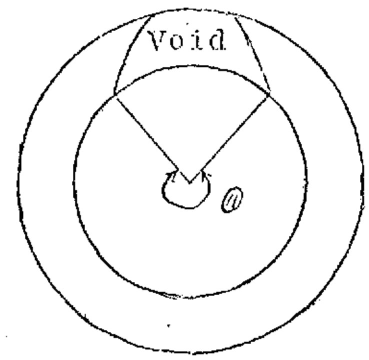


Fig.III-7: Large .

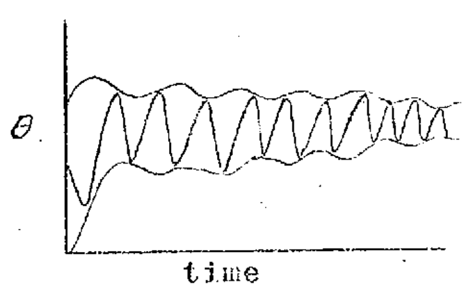


Fig.III-8: Wobble near resonance.

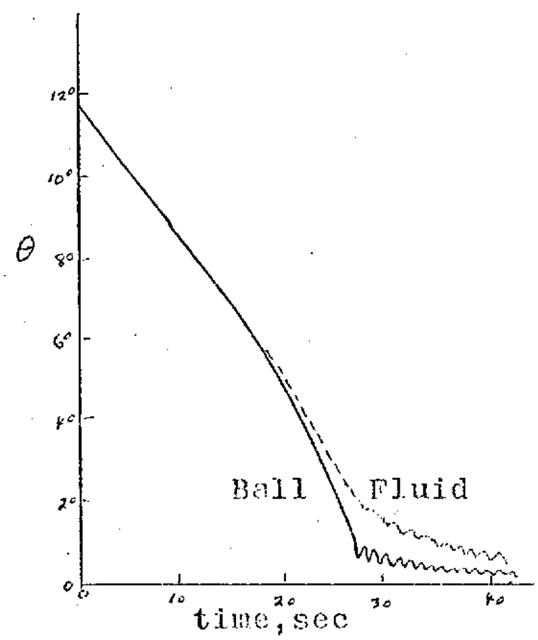


Fig.III-9: Experimental comparison of ball & fluid dampers of equal mass.

near resonance, with the result that energy is traded back and forth between liquid and rigid body. This can result if the damper mass is as little as 2% of the main body, resulting in a history of θ as shown in Fig.III-8 [21]. This can be overcome by damping the wave motion by the use of baffles, filling the void with a light liquid such as alcohol, or using enough damping fluid so that the void is small and the waves impact the inner surface of the damper. Also, damper masses are usually much smaller than 2% of the main body weight.

The advantage of this configuration is that it assures symmetry in the steady state, with no apparent residual wobble, as is the case with single, and some multiple, pendulums. A comparison of a fluid damper and single spin axis pivoted pendulum damper of equal mass from experimental results is shown in Fig.III-9[6].

The HEOS used a spin axis concentric mercury and alcohol damper for small θ, less than half a degree, and one equatorial ball damper for fast damping at larger θ.

2. Unsymmetrically Mounted

Ayache and Lynch analized toroidal and rectangular dampers of circular cross section and a U-shaped resonant damper mounted in planes parallel to the spin axis [2] in terms of a frictional coupling factor f_{Ds} inversely proportional to the time constant. Only the results are presented here. This is for small Ω only, the spacecraft nearly despun. Most of the stabilization is due to a flywheel on the spin axis.

For a toroidal damper as described in Fig.III-10a:

$$f_{Ds} = \tfrac{1}{2}\mathcal{A}_{imag} i / \left[(1 - K)^2 + K^2|\mathcal{A}|^2 + 2K(1 - K)\mathcal{A}_{real}\right]$$

where $\mathcal{A} = 2J_1(\beta_0)/\beta_0 J_0(\beta_0)$

$$\beta_0 = a_t(i\Omega/\nu)^{1/2} \qquad K = (r/\Omega)^2(\pi - \tfrac{1}{2}\theta)/\pi$$

J_0, J_1 = Bessel functions of order zero and one, respectively.

This is plotted vs. $a_t(\Omega/\nu)^{1/2}$ for various bubble sizes in Fig. III-10b, where a_t is the tube inside radius.

The rectangular damper has a frictional coupling factor $(1 - W^2)$ that of the toroidal, where (Fig.III-10c):

$$W = (a - b)/(a + b)$$

This means a greater time constant.

For the U-shaped damper (Fig.III-11):

$$f_{Ds} = \tfrac{1}{2}\ \mathrm{imag}\left[\frac{2\mathcal{A} - (K_* A_*/A_t)(1 - \mathcal{A})}{2 - 2K_*(1 - \mathcal{A})}\right]$$

where $K_* = 2(A_t/A_*)(r/\Omega)^2\ell_s/\ell$

$$A_* = (v_t + \Psi)/J_0(\beta_0)$$

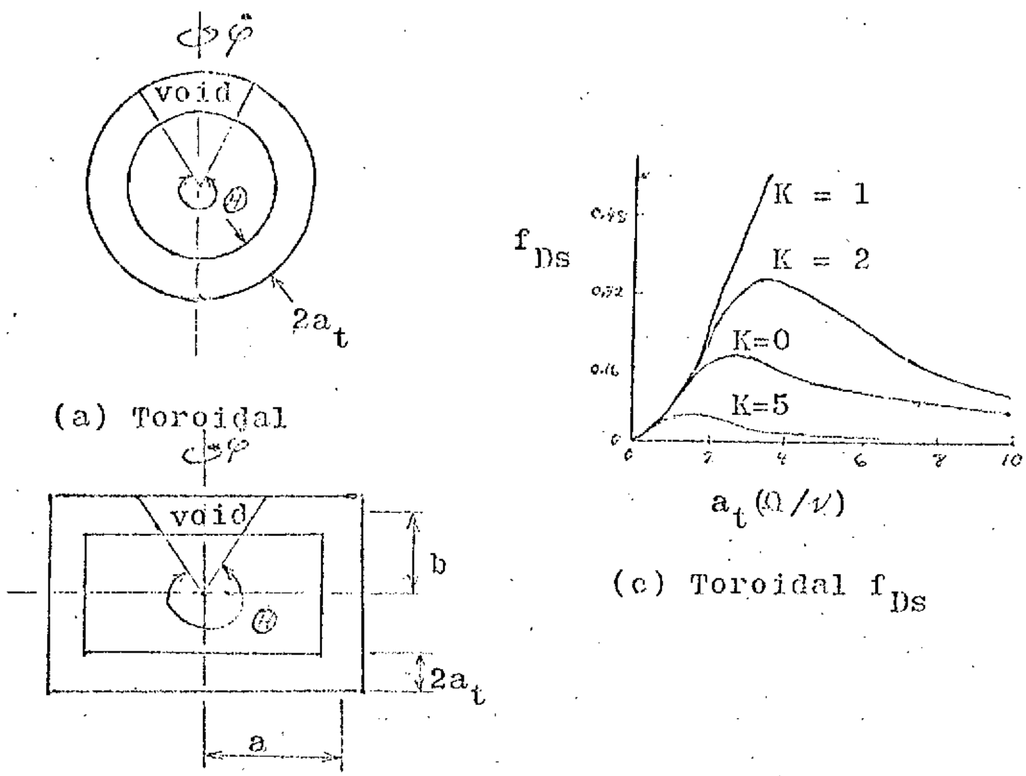


Fig. III-10: Toroidal and rectangular liquid dampers mounted along transverse axis.

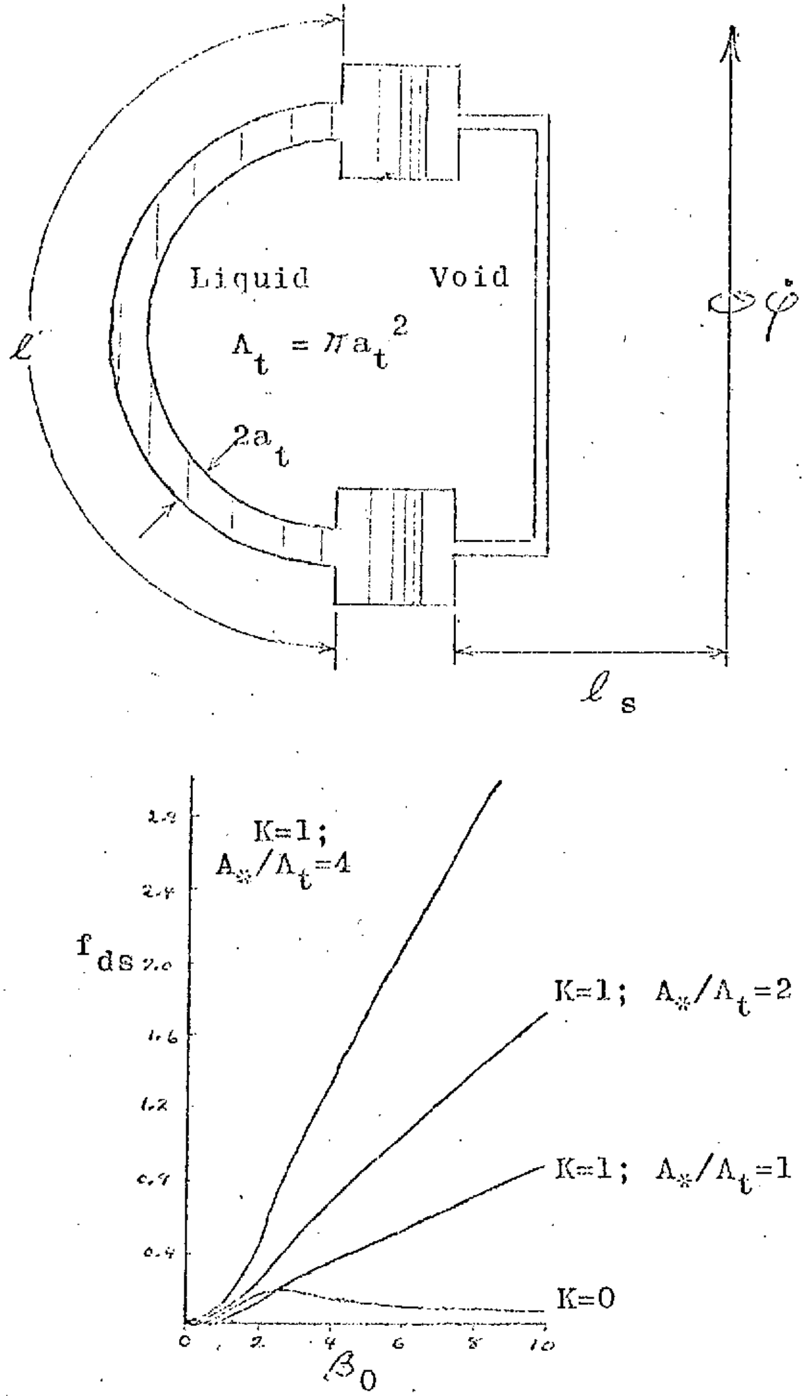


Fig. III-11: U-shaped liquid damper.

v_t = velocity of the tube wall

$$\Psi = (N - v_t c_*)/(-i\Omega^2)\rho$$

$$c_* = \rho r^2 \ell/\ell_s$$

$$N = (2c_*/a_t^2)\int_0^{a_t} v r_* \, dr_*$$

r_* = radial distance from tube center

E. Disk Type [26]

In this, a disk is mounted on a ball and socket at the center of gravity. For best results, the friction should be small. To my knowledge, this type has only been used in a test model by Perkel.

When the entire body is spinning smoothly and then disturbed, the disk damps down more quickly than the main body. For small friction, the damper plane is perpendicular to the precession cone axis. Up to a point, greater friction causes faster damping. The limit is when stiction occurs, freezing the damper.

The damping is exponential:

$$\theta = \theta_0 e^{-t/\tau}$$

$$\tau^{-1} = \frac{\theta_{1D}}{\theta_0} \frac{C_D}{A} \left[1 - \frac{CA_D}{C_D A}\right] r \left[1 - \left(\frac{\theta_{1D}}{\theta_0}\right)^2\right]^{1/2}$$

where C_D, A_D = polar and transverse moments of inertia of the damper

θ_{1D} = initial angle between $\vec{H}$ and disk axis

If the angle between the disk and body axes is small, r may be approximated by H/C.

The stiction problem can be overcome by using a lubricated bearing. The viscous friction constant for minimum τ is:

$$K_0 = C_D(1 - CA_D/C_DA)(\lambda + 1)r/\lambda.$$

F. Mass-Spring Systems

1. Perpendicular to Spin Axis

Wadleigh, Galloway, and Mathur have treated a spring-mass system mounted on and perpendicular to the spin axis [35]. If K is the spring constant, c the damping, and ω_n the natural frequency:

$$\dot{r} = 0$$

$$\dot{p} + \Omega q = 0$$

$$\dot{q} - \Omega p - 2(c/c_c)(\omega_n mh/A)\dot{x} - (Kh/A)x = 0$$

$$\ddot{x} + 2(c/c_c)\omega_n\dot{x} - (q^2 + r^2 - \omega_n^2)x + hrp + h\dot{q} = 0$$

$$\omega_n = (K/m)^{1/2} \text{ and } c_c = 2/(Km)^{1/2}$$

If it is assumed that the sinusoidal character of the spinning body is not affected:

$$p = p_0 \exp(-F_* t/2) \cos \Omega t$$

where F_* is the Rayleigh dissipation function:

$$F_*/2 = \frac{mh^2(c/c_c)\omega_n\lambda(\lambda + 1)(1 + \lambda^2)}{A\left\{\left[(\omega_n/r)^2 - 1 - \lambda^2\right]^2 + 4(c/c_c)^2(\omega_n/r)^2\lambda^2\right\}}$$

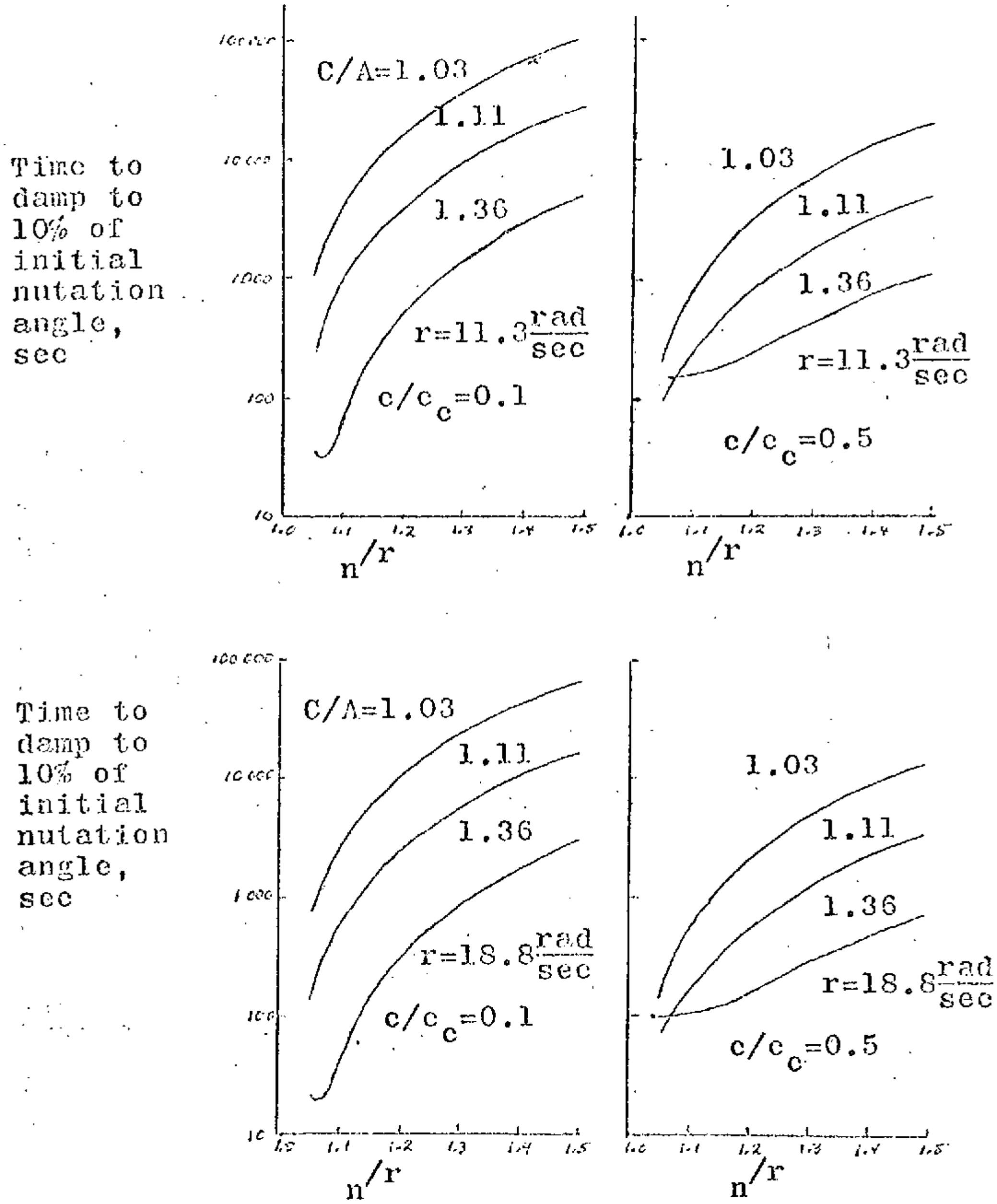


Fig. III-12: Performance of mass-spring dampers mounted on and perpendicular to the spin axis.

The maximum amplitude of the mass oscillation is:

$$x_{max} = q_0 h(r + \Omega)/2(c/c_c)\omega_n \Omega$$

Finally, the nutation angle is expressed:

$$\theta^2 = \left[q_0/(r + \Omega)\right]^2 \left| \exp\left[-F_* t/2 + i(r + \Omega)t\right] - 1 \right|^2$$

Because this system will be slightly asymmetric, this converges to an apparent wobble angle of $q_0/(r + \Omega)$.

In a laboratory test, with $\omega_n \cong 3$cps, $c/c_c \cong 0.5$, $\omega_n/r \cong 1.1$, $mh^2/A = 0.00135$, and an initial spin of 3 cps, all nutation damped out in 6 sec. See Fig.III-12.

2. Parallel to Spin Axis

Such a damper is inherently unbalanced. The nutation angle is a decreasing exponential with a superimposed convergent oscillation. Again, however, the apparent residual wobble is small [21]. The damper is not on the spin axis.

G. Spherical

A pendulum pivoted in a ball and socket and immersed in a fluid was mounted on the despun portion of OSO [9]. However, it will work for a single body satellite for $C/A > 1$.

If $p_* = (C/A)\dot{\varphi}$ and c is the damping constant of the fluid:

$$\tau = \tfrac{1}{2}Ac/h^2m^2p_*^2$$

For resonance:

$$(C/A)\dot{\varphi} = 1.745(TI_d)^{1/2}/(m\ell^3 + 2.467m\ell s^2)^{1/2}$$

where ℓ is the pendulum length, I_d is the diametral moment of inertia of the pendulum wire, and s is the transverse radius of gyration of the bob.

For OSO, there was no evidence of nutation for 8000 orbital passes.

H. Mass-Drum System

This is another system devised by Perkel [26], and consists of two masses strung on wires which are wrapped around a drum. The drum is connected to the main body concentric with the spin axis by a tortional spring-damper system. When nutation occurs, there is a restoring torque due to the relative deflection of the wires in addition to energy dissipation in the dampers. Fig.III-13.

Experimental work on a lab model indicated this system was capable of damping the nutation of a prolate body. Of course, if the cables were long enough, the actual polar inertia moment could be greater than that of the prolate main body alone, possibly greater than the transverse moment of inertia.

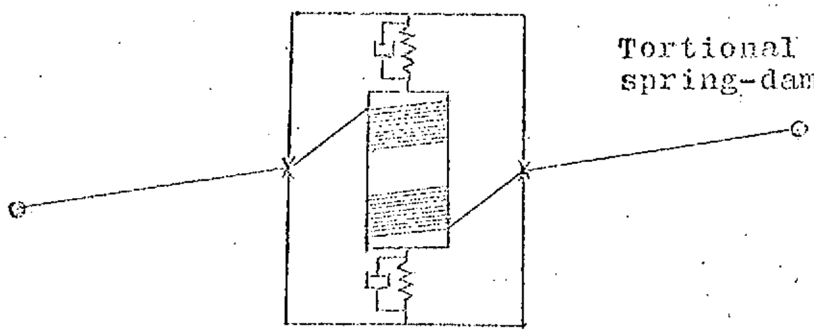


Fig. III-13: Mass-drum nutation damper and
spin rate control.

Another possibility along these lines would be to dispense with the drum, mounting masses on damped springs on the outside of the spacecraft, opposite each other. In this case there would be no direct coupling of the motion of the two dampers.

I. Magnetic Damping

One method is used to align the spin axis of a spacecraft along the local external magnetic field. A strong, permanent magnet is mounted in the spacecraft along the axis. This method was used in TRANSIT 1B and 2A. The spin had to be reduced to below 0.1 rps. Otherwise, the oblate spacecraft could have overcome the magnetic torque and assumed an attitude fixed in space [13].

Energy dissipation also comes about through eddy currents and magnetic hysteresis. If a rod is rotating about a transverse axis perpendicular to the external field, the component of the field along the rod is a function of time, and thus there must be an induced current. This eddy current causes heat to be radiated due to the resistance of the members. For a spacecraft of polar moment C, n number of permeable rods of volume V and diameter D, and spinning perpendicular to the local field initially at ω_0:

$$\omega = \omega_0 \exp\left[(-k_e/4\pi^2 C)t\right]$$

$$k_e = 6.25\pi^2 \sigma_e n \rho_*^{-1} (B_m{}^2)_e V D^2 \times 10^{-11} \text{ erg-sec}$$

where σ_e = separation effect due to distance between rods (σ_e = 1 for ∞)

ρ_* = resistivity of rad (ohm-cm)

$(B_m^2)_e$ = average of square of maximum flux density over entire length of rod for one orbit (gauss^2).

Hysteresis damping is due to the friction between the magnetic domains in the spacecraft. This results in a linear damping.

Note that in all of the above, there are external torques, and angular momentum is not conserved. Since there are energy losses, however, they can be applied to nutation damping. The latter two methods will generally cause energy loss no matter what the orientation of the satellite is intended to be, fixed in space.

In general, the magnetic torques are disturbances that must be overcome by other nutation dampers, and thus are beneficial only for spin removal and alignment with the local magnetic field.

J. Gravity Gradient

As in the above, this can be used for nutation damping only when the spin is very low, and the spin axis (always a prolate body in this case) oriented toward earth. For this type of spacecraft, no spin is usually desired along this axis. In satellites not meant to be gravity gradient stabilized, it is a disturbance to be overcome by the nutation damper. [20,32].

According to Thomson, the torque on a satellite with spin axis perpendicular to the orbital plane is:

$$L = 3\omega_*^2(A - C)\theta_e$$

where θ_e = deviation of spin axis from normal to orbital plane towards earth (small)

ω_* = orbital angular velocity

and L is about the axis tangent to the orbit. Conditions for stability are defined in terms of:

$$b/2\omega_*^2 = \tfrac{1}{2}\left[-5\lambda - (1 - \lambda)^2\right] + (\dot{\varphi}_1/\omega_*)(\lambda + 1)\lambda$$
$$-\tfrac{1}{2}(\dot{\varphi}_1/\omega_*)^2(\lambda + 1)^2$$

$$c/\omega_*^4 = 4\lambda^2 + 5(\dot{\varphi}_1/\omega_*)\lambda(\lambda + 1) + (\dot{\varphi}_1/\omega_*)^2(\lambda + 1)^2$$

$\dot{\varphi}_1$ = spin relative to the tangent to the orbit

For stability: $b^2/2\omega_*^2 < 0$

$$c^2/\omega_*^4 > 0$$

$$(b/2\omega_*^2)^2 > c/\omega_*^2$$

K. Structural Energy Dissipation

No structure is perfectly rigid, and the accelerations on a precessing spacecraft will cause energy loss through mechanical hysteresis. Usually, however, part or parts of the spacecraft can be considered rigid with energy dissipation only from the relatively flexible parts, such as antennae or solar panels. Two examples have been worked by Thomson [31, 33].

We have already shown that, for no external torques:

$$\dot{T} = (H^2\lambda/C)(\sin\theta\ \cos\theta)\dot{\theta}$$

for an axisymmetric body. The energy loss per cycle of stress per unit volume is:

$$\gamma\sigma^2/2E$$

where E is Young's modulus, σ the normal stress, and γ the hysteresis factor. Integrating this over the whole structure, for period of stress oscillation t_0:

$$\int (\gamma\sigma^2/2Et_0)dV = \dot{T}$$

Considering an arbitrary point on the spacecraft at coordinates (x,x), we can compute the acceleration at that point, which is the excitation. If $\dot{\theta}$ is comparatively small:

$$\vec{\omega} = (\mathring{\varphi}\sin\theta\ \sin\varphi)\hat{i} + (\mathring{\varphi}\sin\theta\ \cos\varphi)\hat{j} + (\dot{\varphi} + \mathring{\varphi}\cos\theta)\hat{k}$$

$$\dot{\vec{\omega}} = \dot{\varphi}\mathring{\varphi}\sin\theta(\cos\varphi\ \hat{i} - \sin\varphi\ \hat{j})$$

Note that (x, z) does define an arbitrary point in the spacecraft, not restricted to one plane, because of the axisymmetry. If the relative motion of the points on the spacecraft can be considered small, the acceleration at a point is:

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{\ell}) + \dot{\vec{\omega}} \times \vec{\ell}$$

where $\vec{\ell} = x\hat{i} + z\hat{k}$.

For an example, let us assume that the elastic part of a spacecraft serves only the function of energy dissipation, and the deflections cause no changes in the inertia. The satellite in Fig.III-14 consists of two disks, each of inertia C_1 and A_1, and mass M_1, connected by a flexible tube of radius x_1 and length 2ℓ. The gyroscopic moment required by each disk is:

$$L_g = C_1(\dot{\varphi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta - A_1\dot{\psi}^2 \sin\theta \cos\theta$$

and $\quad C = 2C_1$

$$A \cong 2(A_1 + M_1\ell^2)$$

Then $L_g = M_1\ell^2\dot{\psi}^2 \sin\theta \cos\theta$

The moment distribution is linear:

$$L_z = L_g z/\ell$$

and thus the maximum stress is:

$$\sigma = L_z x_1/I$$

where I is the cross-sectional moment of inertia of the tube. Substituting these into the energy dissipation equation:

$$\dot{\theta} \approx (\gamma/24\pi E)(M_1\ell^2 x_1/I)^2(V/C)(C/A)^4\omega_0^3 \sin\theta \cos^2\theta$$

$$= K \sin\theta \cos^2\theta$$

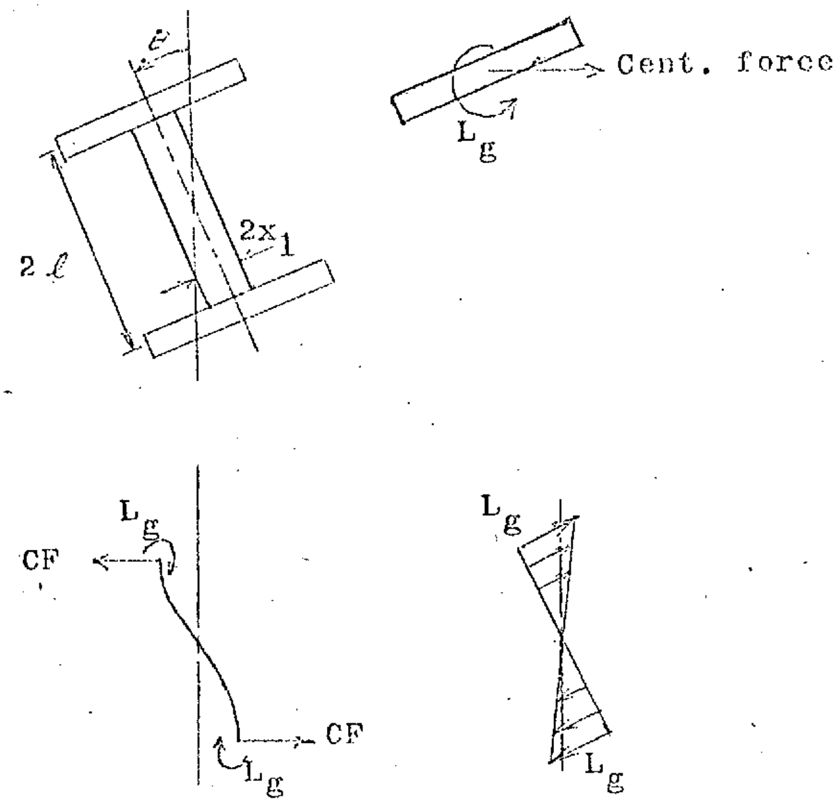


Fig. III-14: Structural energy dissipation example.

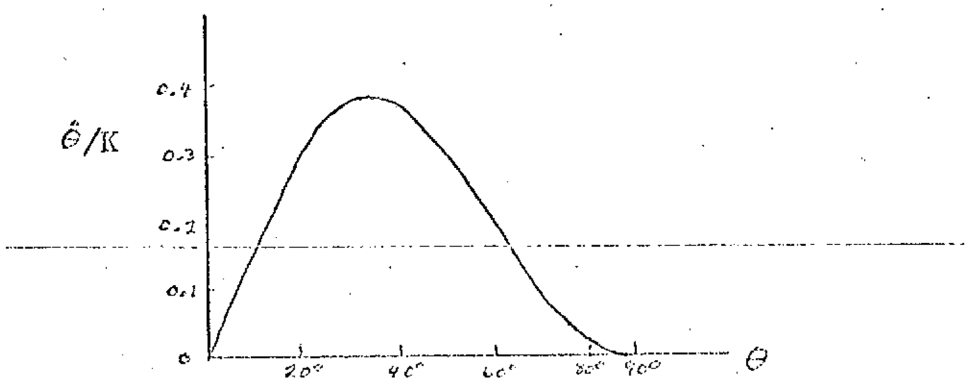


Fig. III-15: Variation in rate of tumbling.

This is shown in Fig.III-15. V is the volume of the stressed material and ω_0 the initial angular velocity.

If $A >> C$, such as for a missile:

$$\vec{a} = 2\omega_0^2 x(C/A)\ \sin\theta\ \cos\theta\ \sin\varphi\ \hat{k} = a_w\hat{k}$$

If we consider the inertia of the deflected members, resonance is observed. Fig.III-16 shows a cylindrical spacecraft of radius R, with four beams of length ℓ. If the elastic deformations $w(\xi,t)$ are assumed small and in the z direction only:

$$EI\ \frac{\partial^4 w}{\partial \xi^4} + \frac{m}{\ell}\ \frac{\partial^2 w}{\partial t^2} = a_w\ \frac{m}{\ell}$$

where m and I are the mass and cross-sectional moments of inertia for the beams. This gives:

$$\dot{\theta} = K_*\ \sin\theta\ \cos^2\theta\ / \left[(1 - \gamma^2\ \cos^2\theta)^2 + (\gamma/2\pi)^2\right]$$

$$K_* = 16\ Cm\gamma(\alpha_1\beta_1 R + 1)^2\omega_0^3/\pi A^2\beta_1^4\ell^2\Omega_1^2$$

$$\gamma = (1 - C/A)\omega_0/\Omega$$

$$\Omega_1 = \text{first natural frequency of beams}$$

Also, α_1 and β_1 are tabulated in [37]. This is shown graphically in Fig.III-17, which clearly shows resonance effects. The envelope of this curve is the same shape as the curve in Fig.III-15.

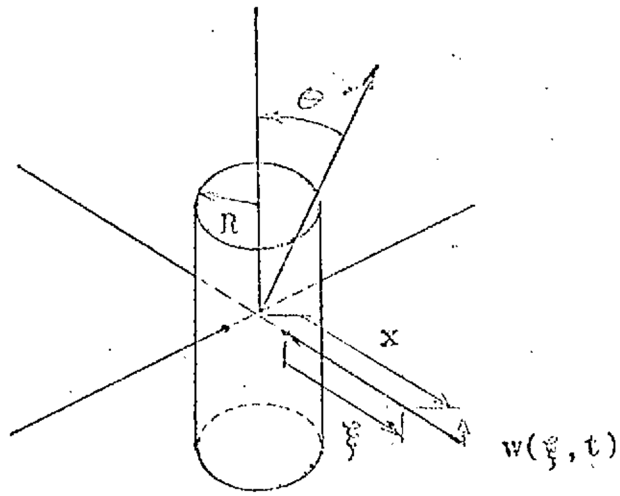


Fig. III-16: Satellite with four elastic beams.

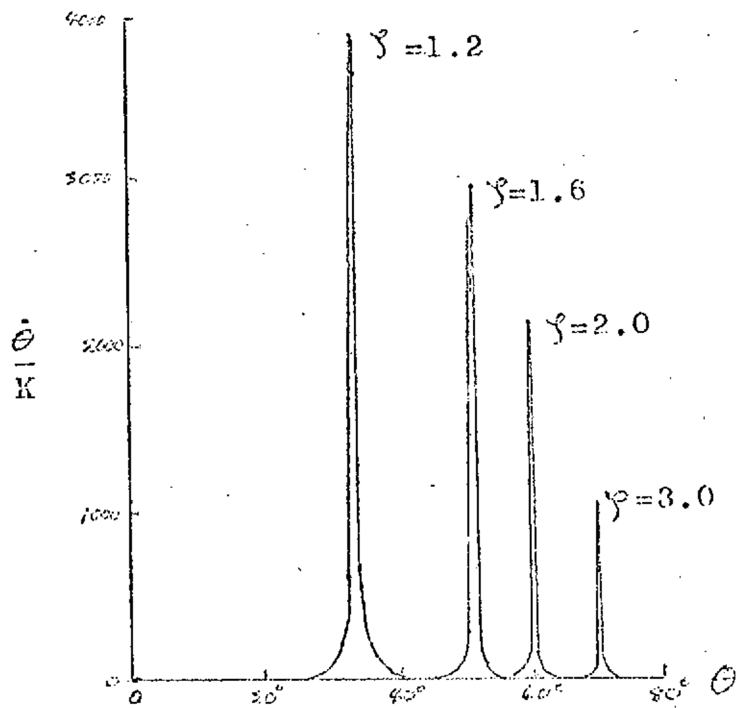


Fig. III-17: Resonance effects.

IV. Semipassive and Active Systems

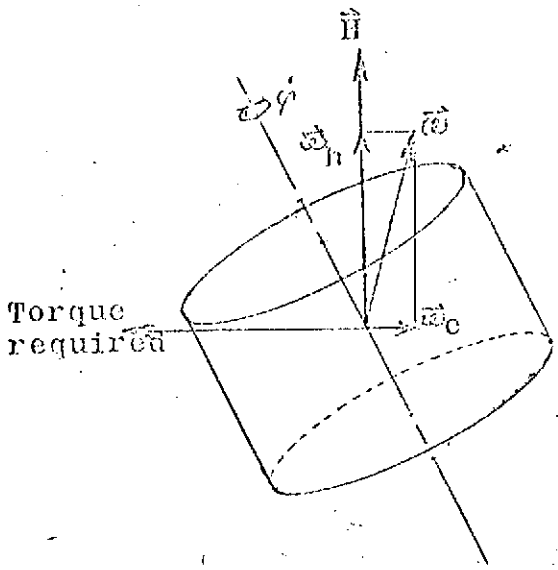
Such systems will be mentioned here only in passing. Active systems have energy sources activated either by on-board sensing equipment or ground command. Semipassive have energy sources that either remain constant or react naturally to attitude changes.

A. Oscillating Mass

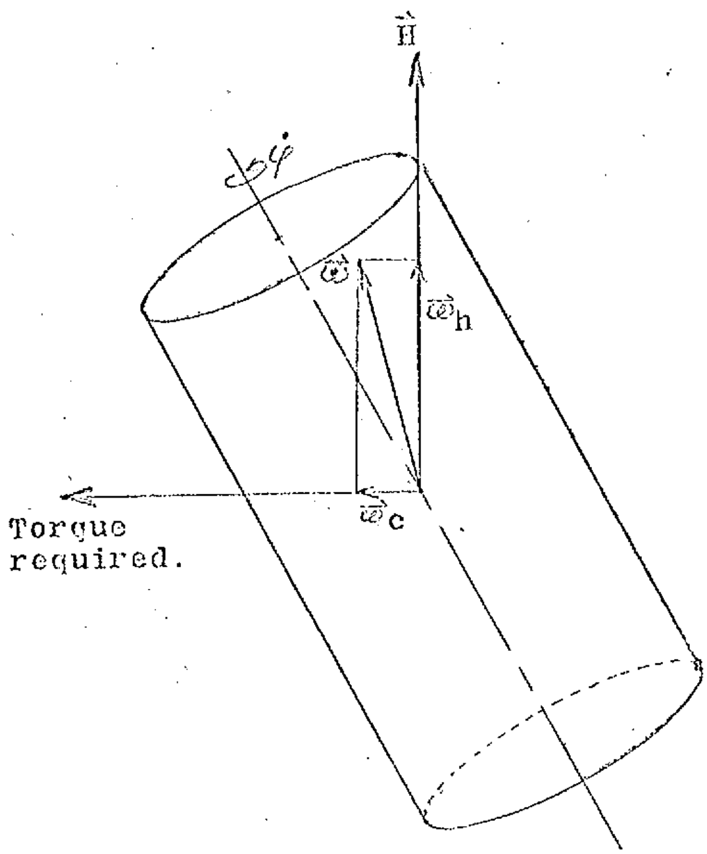
This was proposed by Kane and Sobala [19]. Two masses, diametrically opposed, are forced to oscillate back and forth along the spin axis at constant frequency. The spin axis (axis of symmetry) is normal to the orbital plane. This is capable of maintaining attitude at very low spin rates.

B. Dual Spin

The general reasoning behind dual spin spacecraft with despun dampers was most recently outlined by Tonkin [34]. Fig.IV-1 shows the position of $\vec{\omega}$ relative to $\vec{H}$ for oblate and prolate bodies. We have ω_h and ω_c as the components of $\vec{\omega}$ parallel and normal to $\vec{H}$, respectively. Note that the ω_c in each case is opposite in sign. An internal torque to reduce nutation must be of zero average value ($\vec{H}$ conserved) and remove energy for oblate bodies or inject it for prolate. The first requirement means that the required torque is



(a) Oblate spacecraft.



(b) Prolate spacecraft.

Fig. III-1: Torques required for damping.

normal to $\vec{H}$ and spinning with $\dot{\varphi}$ in inertial space. Since power is the scalar product of torque and angular velocity, the torque required must be opposite ω_c for oblate bodies and of the same sense as ω_c for prolate bodies.

The torques produced by a damper dissipate energy, thus the component normal to H is opposite ω_c. There is also a component along the spin axis, thus changing the spin of the body upon which the damper is mounted. If the damper is despun, the motor must compensate for this speed differential. This is the source of energy injection for the prolate body. Several references are presented in the bibliography.

C. Magnetic

As was shown before, eddy currents induced by the earth's magnetic field can cause torques on a spacecraft. This can be overcome by supplying a torquing coil whose axis is normal to the spin axis with a current 180 degrees out of phase with the externally induced EMF [14]. The spin axis can be oriented by another coil whose axis is parallel to the spin axis. The current in this is switched on and off, and the torque being a sinusoid while on, to give zero average torque on one transverse axis, and a resultant torque on the other.

D. Jet Pulse

Another method of supplying torque is to activate a single attitude motor aligned parallel with the spin axis. The pulsing is controlled by an on-board nutation sensor, firing when the motor is inside the body cone [15].

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6.3 EFFECTS OF A TOROIDAL LIQUID NUTATION DAMPER MOUNTED ON A TRANSVERSE AXIS OF AN AXISYMMETRIC SINGLE-SPIN SATELLITE

Prepared by
William O. Keksz

ABSTRACT

In this paper, an attempt is made to discover the parameters relevant to the performance of a toroidal liquid nutation damper mounted with its axis along a transverse axis of a single-spin satellite. Small initial values of the nutation angle ($< 12^{\circ}$) were assumed. By describing dissipation of energy by the fluid, a time constant for the nutation angle is found as a function of a Bessel function of complex argument.

NOMENCLATURE

a	Small radius of torus
a_*	Radial coordinate within torus
A	Function of a_* only
C	Constant
H	Magnitude of angular momentum of satellite
$\hat{1}_x, \hat{1}_y$	Unit vectors along transverse axes of satellite
$\hat{1}_z$	Unit vector along spin axis
I_t, I_t, I_z	Moments of inertia with respect to transverse and spin axes, respectively
j	$(-1)^{1/2}$
J_0	Bessel function of order zero
K	Kinetic energy of satellite
q	Argument of Bessel function $= a_*(j\dot{\varphi}/\nu)^{1/2}$
r_t	Large radius of torus
S	Surface area of control volume
t	Time
T	Function of time only
v	Fluid velocity
v_a	Fluid velocity at wall ($a_* = a$)
v_2	Complex fluid velocity for two dampers
V	Volume

W	Work done on fluid
ω	Angular velocity
ψ	Precession angle
φ	Spin angle
ρ	Fluid density
μ	Absolute viscosity
ν	Kinematic viscosity
τ	Shear stress in fluid
$(\dot{\ })$	Time derivative except for T
$(\ )'$	Derivative for A and T
$\lvert(\ )\rvert$	Magnitude of complex ( )
θ	Angle of nutation

The minimum energy condition for a spinning axisymmetric body is when the angular velocity is aligned with the axis of maximum inertia. When there are no external torques, the magnitude of the angular momentum,

$$H = (I_t^2(\omega_x^2 + \omega_y^2) + I_z^2\omega_z^2)^{1/2} \quad ,$$

is constant. Values for ω_x and ω_y exist when the nutation angle is not zero. Here we have assumed the momentum of the damper to be small. We also have

$$2K = I_t(\omega_x^2 + \omega_y^2) + I_z\omega_z^2$$

as twice the kinetic energy, again assuming the motion of the damper small. If the nutation angle decreases slowly, we have the angular velocity in the satellite frame given by

$$\vec{\omega} = \hat{i}_x(\dot{\psi}\sin\theta \ \sin\varphi) + \hat{i}_y(\dot{\psi}\sin\theta \ \cos\varphi) + \hat{i}_z(\dot{\varphi} + \dot{\psi}\cos\theta) \quad ,$$

and the precession speed,

$$\dot{\psi} = I_z\omega_z/I_t \ \cos\theta = H/I_t .$$

Combining these equations gives

$$H^2 - 2KI_t = (H^2/I_z)(I_z - I_t)\cos^2\theta .$$

If there is energy dissipation, as with a damper, H remains constant while K decreases. Thus (5),

$$K = (H^2/I_t I_z)(I_z - I_t)(\sin\theta \ \cos\theta)\dot{\theta} .$$

With a liquid damper, energy dissipation occurs because of viscous effects, being represented by the time rate of work done on the fluid by the wall of its container (2),

$$\dot{W} = \int_S \vec{\tau} \cdot \vec{v}\, dS$$

$$= \frac{d}{dt} \int_V \frac{v^2}{2} \rho\, dV$$

$$= -\dot{K} \quad .$$

The damper is illustrated in Fig.1. From the above, it is seen that the velocity distribution of the liquid must be found. Assuming that $a \ll r_t$, the velocity of the tube wall relative to the liquid is given by $v_a = r_t \omega_x$, or

$$v_a = r_t \dot{\psi} \sin\theta \quad \sin\varphi \; .$$

It is assumed that the velocity will be entirely tangential to the torus, and pressure variations due to centrifugal body forces are small. Thus the fluid momentum differential vector equation (for a particular θ),

$$\rho\, \partial v/\partial t = -\vec{\nabla} P + \mu \nabla^2 \vec{v},$$

becomes $\rho\, \partial v/\partial t = \mu \nabla^2 \vec{v}$,

or, introducing the kinematic viscosity $\nu = \mu/\rho$,

$$\partial v/\partial t = \nu \nabla^2 v.$$

In cylindrical coordinates, this is

$$\frac{\partial v}{\partial t} = \frac{\nu}{a_*}\frac{\partial}{\partial a_*}\left(a_*\frac{\partial v}{\partial a_*}\right).$$

Assuming that the solution is a product of a function A of a_* only, and T of t only, we have

$$AT' = T(a_*A')'/a_*,$$

or $\quad a_*AT' = T(A' + a_*A'')$.

If transients due to initial conditions are considered to damp out quickly, the time function will be in phase with v_a. Thus we let $T = \sin\varphi$. Then,

$$a_*A\dot{\varphi}\cos\varphi = \nu\sin\varphi\,(A' + a_*A'') \quad .$$

Rearranging terms result in

$$a_*A'' + A' - (\dot{\varphi}/\nu)\cot\varphi\; a_*A = 0,$$

which is rather difficult to solve. The coefficient in the third term is a function of time; it cannot be averaged over one revolution for φ to give a constant, for $\cot\varphi$ has values of infinity.

A simpler model may be had by assuming we have another damper, identical to the first, mounted on the y axis. Then we can represent the rotation of the transverse component of the angular velocity vector using complex variables. Since

$$\omega_x = \dot{\psi}\sin\theta\;\sin\varphi,$$
$$\omega_y = \dot{\psi}\sin\theta\;\cos\varphi,$$

and $\quad e^{-j\varphi} = \cos\varphi - j\sin\varphi$,

the velocity component in the xy plane can be described by a phasor,

$$j\omega_x - \omega_y = -\omega_{xy}e^{-j\varphi} ,$$

where $\omega_{xy} = (\omega_x^{\,2} + \omega_y^{\,2})^{1/2}$

$$= \dot{\varphi}\sin\theta$$

is the phasor magnitude.

Since we now have the term $e^{-j\varphi}$ as the excitation, we let this be equal to T_2 . Then,

$$T_2' = -j\dot{\varphi}e^{-j\varphi} = -j\dot{\varphi}T_2 ,$$

and the equation for the fluid velocity position depend- function A becomes

$$a_*A'' + A' + (j\dot{\varphi}/\nu)a_*A = 0 ,$$

or $\quad a_*^{\,2}A'' + a_*A' + (j\dot{\varphi}/\nu)a_*^{\,2}A = 0$.

This is a complex Bessel equation, the solution of which is (1)

$$A = CJ_0(q) ,$$

where $q = a_*(j\dot{\varphi}/\nu)^{1/2}$

and C is a constant, which may be found by examining the velocity at the wall, which is given by

$$v_{a2} = -r_t\omega_{xy}e^{-j\varphi} .$$

If we define A at $a_* = a$ as $J_0(q_a)$ where

$$q_a = a(j\dot{\varphi}/\nu)^{1/2}$$

then $C = r_t\omega_{xy}/J_0(q_a)$.

Thus we have

$$v_2 = AT_2 = r_t\omega_{xy}(J_0(q)/J_0(q_a))e^{-j\varphi} \quad .$$

Using this expression in the integral for work done on the fluid will result in a complex function. This would represent two energy flows for dissipation, ninety degrees out of phase with each other. Averaged over one revolution for φ , the work done could be represented by the magnitude of the complex work. Also at this point we can say that this is twice the energy dissipation rate for a single damper. Thus for the single damper,

$$\begin{aligned} \dot{W} &= \tfrac{1}{2}\left|\dot{W}_2\right| \\ &= \tfrac{1}{2}\frac{d}{dt}\int_V \tfrac{1}{2}\rho\left|v_2\right|^2 dV_2 \\ &= \frac{\rho}{4}\,\frac{d}{dt}\int_V\left|r_t\omega_{xy}(J_0(q)/J_0(q_a))e^{-j\varphi}\right|^2 dV_2 \\ &= \frac{\rho r_t^{\,2}\omega_{xy}^{\,2}\dot{\varphi}}{4\left|J_0(q_a)\right|^2}\int_0^a\left|J_0(q)e^{-j\varphi}\right|^2(4\pi r_t)(2\pi a_*)da_* \\ &= 2\rho r_t^{\,3}\omega_{xy}^{\,2}\pi^2\dot{\varphi}\left|e^{-j\varphi}/J_0(q_a)\right|^2\int_0^a\left|J_0(q)\right|^2 a_* da_* \quad . \end{aligned}$$

At any instant,

$$\left|e^{-j\varphi}\right|^2 = \cos^2\varphi + \sin^2\varphi = 1 \quad .$$

Making this substitution, we have

$$\dot{W} = 2\rho r_t{}^3 \omega_{xy}{}^2 \pi^2 \dot{\varphi} \left|J_0(q_a)\right|^{-2} \int_0^a \left|J_0(q)\right|^2 a_* da_* \ .$$

For a given value of θ, we have

$$\dot{\psi} = I_z\dot{\varphi}/(I_t - I_z)\cos\theta$$

and $\omega_z = \dot{\varphi} + \dot{\psi} \cos\theta$.

Thus $\omega_z = \dot{\varphi} + I_z\dot{\varphi}/(I_t - I_z)$

$$= \dot{\varphi}(1 + I_z/(I_t - I_z))$$

$$= \dot{\varphi} I_t/(I_t - I_z) \ .$$

Also, again for a particular θ with $\dot{\theta}$ small,

$$\tan\theta = I_t\omega_{xy}/I_z\omega_z \ .$$

Rearranging terms gives

$$\omega_{xy} = \omega_z(I_z/I_t)\tan\theta \ .$$

Substituting the relation between ω_z and $\dot{\varphi}$, we have

$$\omega_{xy} = \dot{\varphi}(I_z \tan\theta)/(I_t - I_z) \ .$$

If we assume no external torques and the momentum of the damper relative to the satellite main body small, the Euler equations for the satellite are

$$0 = I_t\dot{\omega}_x + (I_z - I_t)\omega_y\omega_z \ ,$$

$$0 = I_t\dot{\omega}_y - (I_z - I_t)\omega_x\omega_z \ ,$$

and $0 = I_z\dot{\omega}_z$.

Thus we can take ω_z constant during the damping action, and therefore $\dot{\varphi}$, φ, and ω_{xy} may also be held constant. Note that this requires that H_x and H_y be small compared to H_z, thus meaning θ is small $(<12^0)$; therefore

$$\sin\theta \cong \theta \quad ,$$

$$\cos\theta \cong 1 \quad ,$$

and $$\tan\theta \cong \theta \quad .$$

The equation for ω_{xy} is then

$$\omega_{xy} = \dot{\varphi}\theta I_z/(I_t - I_z),$$

and substituting this into the expression for $\dot{W}$,

$$\dot{W} \cong \frac{2\rho r_y{}^3\pi^2\dot{\varphi}^3 I_z{}^2\theta^2}{(I_t - I_z)^2 \left|J_0(q_a)\right|^2}\int_0^a \left|J_0(q)\right|^2 a_* da_* \quad .$$

Also, the equation for $\dot{K}$ becomes

$$\dot{K} \cong (H^2/I_t I_z)(I_z - I_t)\theta\dot{\theta} \quad .$$

But, for small θ ,

$$H \cong I_z\omega_z = \dot{\varphi} I_t I_z/(I_t - I_z) \quad .$$

Thus $$\dot{K} \cong \left[\frac{I_t I_z}{I_t - I_z}\right]^2 \left[\frac{I_z - I_t}{I_t I_z}\right]\theta\dot{\theta}$$

$$= \dot{\varphi}^2\theta\dot{\theta} I_t I_z/(I_t - I_z) \quad .$$

Setting $\dot{W} = -\dot{K}$, we have

$$-\dot{\theta} I_t = \frac{2\rho r_t{}^3\pi^2 I_z\,\theta\,\dot{\varphi}}{(I_t - I_z)\left|J_0(q_a)\right|^2}\int_0^a \left|J_0(q)\right|^2 a_* da_* \quad ,$$

$$\text{or } \dot{\theta} + \left[\frac{2\rho r_t{}^3\pi^2 I_z \dot{\varphi}}{I_t(I_t - I_z)\left|J_0(q_a)\right|^2} \int_0^a \left|J_0(q)\right|^2 a_* da_* \right] \theta = 0 \ .$$

The term in brackets is almost constant for small nutation angles, and is thus the inverse of the time constant for a decreasing exponential solution. Thus

$$\theta = \theta_0 \exp(-t/t_c) \quad ,$$

$$\text{where } t_c^{-1} = \frac{2\, r_t{}^3\pi^2 I_z \dot{\varphi}}{I_t(I_t - I_z)\left|J_0(q_a)\right|^2} \int_0^a \left|J_0(q)\right|^2 a_* da_* \ .$$

and θ_0 is the initial nutation angle. This may also be expressed using the approximation for angular momentum by

$$t_c^{-1} = (2\, r_t{}^3\pi^2 H / I_t{}^2 \left|J_0(q_a)\right|^2) \int_0^a \left|J_0(q)\right|^2 a_* da_* \ .$$

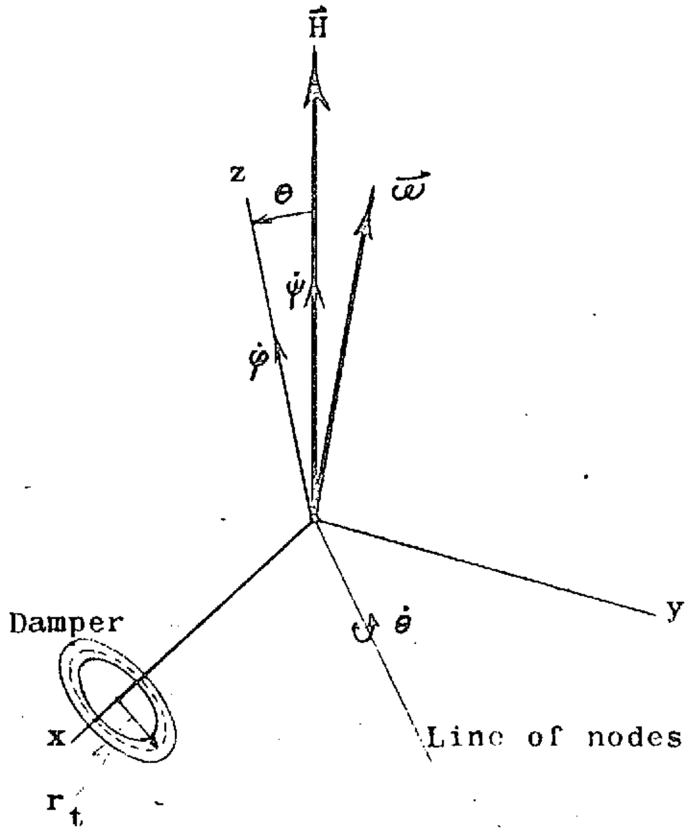

Fig. 1. Positioning of damper.

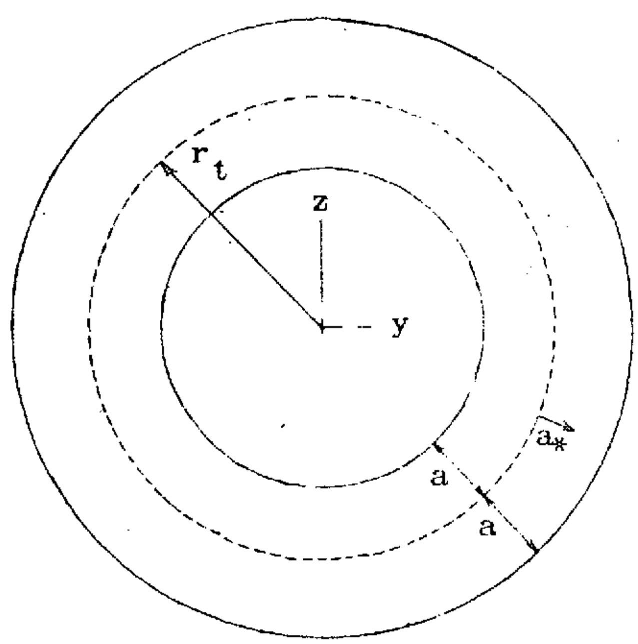

Fig. 2. Coordinates within damper. Not to scale, as actually $r_t >> a$.

CONCLUSION

The above is valid for small initial nutation angles for an axisymmetric single-spin satellite. Values for complex Bessel functions may be found in references 3 and 4. However, these are good only for Bessel functions in which the magnitude of the complex argument q is less than ten. However, the best liquid for use in the damper is mercury because of its high density; its kinematic viscosity is $(0.5)10^{-6}$ ft^2/sec at $75^{o}F$. Since q is inversely proportional to the square root of ν, its magnitude will be on the order of 10^2 or 10^3 for reasonable values of $\dot{\varphi}$. Bessel functions for complex arguments of these magnitudes have not been tabulated, and must be calculated.

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CHAPTER 7

General Conclusions

As a result of the present study, equations of motion and computer programs have been developed for analyzing the motion of a spin-stabilized spacecraft having long, flexible appendages. Stability charts were derived, or can be redrawn with the desired accuracy for any particular set of design parameters. Simulation graphs of variables of interest are readily obtainable on line using program FLEXAT. Finally, applications to actual satellites, such as UK-4 and IMP-I have been considered.