14p

NASA TECHNICAL NOTE

NASA TN D-7592

NASA TN D-7592

(NASA-TN-D-7592) INCLUSION OF KNOWN INTEGRALS IN THE OPTIMAL TRAJECTORY PROBLEM (NASA) ~~21~~ 20 p HC $2.75 CSCL 22C

N74-16537

Unclas

H1/30 29215

INCLUSION OF KNOWN INTEGRALS IN THE OPTIMAL TRAJECTORY PROBLEM

by Rowland E. Burns

George C. Marshall Space Flight Center
Marshall Space Flight Center, Ala. 35812

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • FEBRUARY 1974

1. REPORT NO. TN D-7592	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE Inclusion of Known Integrals in the Optimal Trajectory Problem		5. REPORT DATE February 1974
		6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) Rowland E. Burns		8. PERFORMING ORGANIZATION REPORT # M125
9. PERFORMING ORGANIZATION NAME AND ADDRESS George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812		10. WORK UNIT NO.
		11. CONTRACT OR GRANT NO.
		13. TYPE OF REPORT & PERIOD COVERED Technical Note
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D.C. 20546		14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Prepared by Aero-Astrodynamics Laboratory, Science and Engineering

16. ABSTRACT

The classical problem of determination of the rocket trajectory which minimizes mass expenditure during motion between two points in the field of a single gravitating body is analyzed.

The known integrals of the system are incorporated into the adjoint equations, resulting in a reduction from a seventh-order adjoint system to a third-order adjoint system. The first case which is treated is that of planar motion under specific end conditions. In this case a regularization of the recently derived equations is achieved.

The general three-dimensional case is also reduced from a seventh-order adjoint system to a third-order adjoint system. In this case a regularization has not been found.

17. KEY WORDS	18. DISTRIBUTION STATEMENT Cat.30		
19. SECURITY CLASSIF. (of this report) Unclassified	20. SECURITY CLASSIF. (of this page) Unclassified	21. NO. OF PAGES ~~19~~ 20	22. PRICE $ 2.75

MSFC - Form 3292 (Rev December 1972)

For sale by National Technical Information Service, Springfield, Virginia 22151

TABLE OF CONTENTS

PRECEDING PAGE BLANK NOT FILMED

INCLUSION OF KNOWN INTEGRALS IN THE OPTIMAL TRAJECTORY PROBLEM

INTRODUCTION

The optimal trajectory problem concerning a single attracting center – which is usually taken to be that trajectory which minimizes mass expenditure – has been treated in numerous publications.

Two important integrals of the problem were quickly isolated in the early literature. One of these integrals, vectorial in nature, corresponds to a sort of "angular momentum" conservation. The other, a scalar integral, is a sort of conservation of "energy." It has long been realized that, if full use could be made of the information contained in these integrals, a great reduction in the number of numerical integrations of the adjoint equations would be achieved. Thus, the iteration required would be faster.

The transformation which yields such a reduction has been elusive. In this paper the transformation is explicitly demonstrated. The transformation exacts a price, however, in that the resultant equations are ill-behaved from the numerical point of view.

In the case of a planar trajectory which has the magnitude of the radius and velocity vector, as well as the flight path angle, constrained at the terminal point, a regularization is achieved. In the general three-dimensional case, such a regularization has not been found.

The first section below deals with known results. Subsequently, the planar case is discussed in depth and the formulation of the three-dimensional case follows. A few concluding remarks on other unsuccessful lines of approach to achieve a general regularization are given.

KNOWN RESULTS

The general equation for rocket motion about a single attracting center is

$$\ddot{\vec{r}} = \frac{\vec{T}}{m} - \frac{\mu \vec{r}}{r^3} \tag{1}$$

where $\vec{r}$ is the position vector of the rocket from the attracting center, $\vec{T}$ is the vectorial thrust, m is the mass of the rocket, and μ is the gravitational parameter of the attracting center. Derivatives with respect to time are shown by the dot notation. The rate of change of the mass is related to the magnitude of the thrust, T, and the exhaust velocity, c, via

$$\dot{m} = - T/c \qquad . \tag{2}$$

If one defines $\vec{\lambda}$ as the solution of the vectorial differential equation

$$\ddot{\vec{\lambda}} = -\mu\left(\frac{\vec{\lambda}}{r^3} - 3\frac{\vec{\lambda}\cdot\vec{r}}{r^5}\,\vec{r}\right) \tag{3}$$

and further defines

$$S = \frac{\lambda}{m} - \frac{\sigma}{c} \tag{4}$$

where

$$\dot{\sigma} = \frac{T\lambda}{m^2} \qquad , \tag{5}$$

then it is well known [1] that an optimal trajectory (in the sense of minimizing mass expenditure) will result if the direction and magnitude of thrust are chosen according to

$$\vec{T} = T\frac{\vec{\lambda}}{\lambda} \tag{6}$$

$$T = \begin{cases} T_{max} & \text{if} \quad S > 0 \\ T_{min} & \text{if} \quad S < 0 \qquad . \end{cases} \tag{7}$$

The case where $S \equiv 0$ for a nonzero interval of time will not be considered here (see Reference 2).

Two important integrals of the system of equations represented by equations (1) through (6) are given by

$$\dot{\vec{\lambda}} \times \vec{r} - \vec{\lambda} \times \dot{\vec{r}} = \vec{C} \tag{8}$$

and

$$TS - \frac{\mu}{r^3} \vec{\lambda} \cdot \vec{r} - \dot{\vec{\lambda}} \cdot \dot{\vec{r}} = C_4 \tag{9}$$

where $\vec{C} = (C_1, C_2, C_3)$ and C_4 are integration constants.

If $|\vec{r}|$, $|\dot{\vec{r}}|$, and θ $\left(\theta = \cos^{-1}\frac{\vec{r} \cdot \dot{\vec{r}}}{r\,\dot{r}}\right)$ are fixed at the final point and only these quantities are fixed, then it may be shown that [1]

$$\vec{C} = (C_1, C_2, C_3) = \vec{O} = (0, 0, 0) \quad . \tag{10}$$

Furthermore, if the final time is unspecified, it is known that [1]

$$C_4 = 0 \quad . \tag{11}$$

THE PLANAR CASE

The case represented by equations (10) and (11) will be discussed first. Assuming that equation (10) is valid, take the dot product of $\vec{r}$ with equation (8) and interchange the order to yield

$$\vec{\lambda} \cdot (\vec{r} \times \dot{\vec{r}}) = 0 \quad . \tag{12}$$

Geometrically, this result can be interpreted to indicate that the λ vector will always remain orthogonal to a vector which is orthogonal to the plane formed by $\vec{r}$ and $\dot{\vec{r}}$. Thus, one can write

$$\vec{\lambda} = \xi \vec{r} + \eta \vec{v} \tag{13}$$

where ξ and η are scalar functions of time. The degrees of freedom represented by ξ and η are utilized by requiring that equations (8) and (9) be satisfied identically. In Reference 1 it is shown that

$$\dot{\xi} - \xi \left(\frac{v^2 - \frac{\mu}{r}}{\vec{r} \cdot \vec{v}} \right) = T \left(\frac{S}{\vec{r} \cdot \vec{v}} - \frac{\xi \eta}{m \lambda} \right) \tag{14}$$

and

$$\dot{\eta} + \frac{T}{m \lambda} \eta^2 = -2 \xi \tag{15}$$

where

$$\vec{v} = \dot{\vec{r}} \tag{16}$$

and

$$\lambda = \sqrt{\xi^2 r^2 + \eta^2 v^2 + 2 \xi \eta \vec{r} \cdot \vec{v}} \quad . \tag{17}$$

The numerical application of the theory developed to this point is now clear. From appropriate starting values of the required quantities, equations (5), (14), and (15) could be integrated forward in time. Equation (13) then yields the λ vector as a function of $\vec{r}$ and $\vec{v}$, so that equations (6) and (7) give the direction and magnitude of the thrust vector. Equation (1) may then be integrated, numerically, to yield updated values of $\vec{r}$ and $\vec{v}$.

The advantage of using equations (14) and (15) in lieu of integrating equation (3) directly is that of dimensionality. Specifically, equation (3) represents three second-order equations which are not necessarily simple to integrate. Equations (14) and (15) are two first-order equations – i.e., a reduction from 7 to 3 in the number of integrations necessary to determine the steering program has been achieved. Furthermore, the transversality conditions [equation (8)] are identically satisfied; likewise, the fact that the Hamiltonian [equation (9)] will remain identically zero is assured.

The advantage of reducing the number of required integrations becomes even more important in light of the two-point boundary value problem associated with the trajectory optimization procedure. The satisfaction of the four equations represented by equations (8) and (9) is guaranteed no matter what initial values are chosen for the variables ξ and η. Thus, a vast reduction in iteration accompanies the reduction in integration.

The difficulty in using equations (14) and (15) is that the occurrence of $\vec{r} \cdot \vec{v}$ in the denominator of equation (14) is a near-singularity. $\vec{r} \cdot \vec{v}$ will become sufficiently small that overflow is almost certain to be encountered during the optimal trajectory calculations. Therefore, the next process is the regularization of equation (14).

Set

$$\xi = \phi \eta \quad . \tag{18}$$

Then

$$\lambda = \eta \sqrt{\phi^2 r^2 + v^2 + 2\phi \vec{r} \cdot \vec{v}} \quad . \tag{19}$$

Define, for convenience,

$$\lambda^* = \sqrt{\phi^2 r^2 + v^2 + 2\phi \vec{r} \cdot \vec{v}} \quad . \tag{20}$$

Using equations (18) and (20) in equation (15) gives

$$\dot{\eta} = -\left(2\phi + \frac{T}{m \lambda^*}\right) \eta \quad . \tag{21}$$

Next, differentiate equation (18) to yield

$$\dot{\xi} = \dot{\phi} \eta + \phi \dot{\eta} = \left(\dot{\phi} - 2\phi^2 - \frac{T\phi}{m \lambda^*}\right) \eta \tag{22}$$

where equation (21) was used for $\dot{\eta}$.

Substituting equation (22) into equation (14) and employing equation (18) to eliminate ξ gives

$$\eta \left[(\vec{r} \cdot \vec{v}) (\dot{\phi} - 2 \phi^2) - \phi \left(v^2 - \frac{\mu}{r} \right) - \frac{T}{m} \lambda^* \right] = - \frac{T}{c} \sigma = \dot{m} \sigma \quad . \tag{23}$$

The expression on the right side of the first equality sign comes from separating S into its elements via equation (4). Subsequently, equation (2) was employed.

To simplify the algebraic calculations, abbreviate

$$\Phi \equiv (\vec{r} \cdot \vec{v}) (\dot{\phi} - 2 \phi^2) - \phi \left(v^2 - \frac{\mu}{r} \right) - \frac{T}{m} \lambda^* \tag{24}$$

so that equation (23) can be written as

$$\eta \Phi = \dot{m} \sigma \quad . \tag{25}$$

Since the singular arc case has been excluded via choice, the thrust magnitude must be either T_{max} or T_{min}. In either case $\dot{m}$ is a constant. Thus, equation (25) can be differentiated to give

$$\dot{\eta} \Phi + \eta \dot{\Phi} = \dot{m} \dot{\sigma} = \dot{m} \frac{T \lambda^* \eta}{m^2} \tag{26}$$

from equation (5).

Eliminating $\dot{\eta}$ by use of equation (21) and assuming that

$$\eta \neq 0 \tag{27}$$

allows equation (26) to be written as

$$\dot{\Phi} - \left(2 \phi + \frac{T}{m \lambda^*} \right) \Phi = \dot{m} \frac{T \lambda^*}{m^2} \quad . \tag{28}$$

The next task is to calculate $\dot{\Phi}$ from equation (24). This calculation is somewhat complicated but the result is

$$\dot{\Phi} = \left[\left(v^2 - \frac{\mu}{r}\right) + \frac{T}{m\,\lambda^*}(\phi\, r^2 + \vec{r}\cdot\vec{v})\right](\dot{\phi} - 2\,\phi^2) + (\vec{r}\cdot\vec{v})(\ddot{\phi} - 4\,\phi\,\dot{\phi})$$
$$- \dot{\phi}\left(v^2 - \frac{\mu}{r}\right) - \phi\left[2\frac{T}{m\,\lambda^*}(\phi\,\vec{r}\cdot\vec{v} + v^2) - \mu\frac{\vec{r}\cdot\vec{v}}{r^3}\right] + \frac{T\,\dot{m}\,\lambda^*}{m^2}$$
$$- \frac{T}{m\,\lambda^*}\left[\phi\,\dot{\phi}\,r^2 + \dot{\phi}\,(\vec{r}\cdot\vec{v}) + \phi^2\,(\vec{r}\cdot\vec{v}) + \phi\left(v^2 - \frac{\mu}{r}\right) + \frac{T}{m\,\lambda^*}(\phi\,\vec{r} + \vec{v})^2 - \frac{\mu\,\vec{r}\cdot\vec{v}}{r^3}\right] \tag{29}$$

Inserting equations (29) and (24) into equation (28), gathering, and simplifying yields

$$(\vec{r}\cdot\vec{v})\,\ddot{\phi} - \dot{\phi}\,(\vec{r}\cdot\vec{v})\left(6\,\phi + \frac{T}{m\,\lambda^*}\right) + 4\,\phi^3\,(\vec{r}\cdot\vec{v}) - \phi^2\left(3\,\vec{r}\cdot\vec{v} + 2\,\phi\,r^2\right)\frac{T}{m\,\lambda^*}$$
$$+ \phi\left(\frac{2\,T\,\lambda^*}{m} - \frac{2\,T\,v^2}{m\,\lambda^*} + \mu\frac{\vec{r}\cdot\vec{v}}{r^3}\right) + \frac{T}{m\,\lambda^*}\left(\frac{\mu\,\vec{r}\cdot\vec{v}}{r^3}\right) = 0 \quad . \tag{30}$$

Further simplifications of equation (30) are possible. Notice that

$$\frac{2\,T}{m}\lambda^* - \frac{2\,T}{m\,\lambda^*}v^2 = \frac{2\,T}{m\,\lambda^*}\,\phi\,(\phi\,r^2 + 2\,\vec{r}\cdot\vec{v}) \quad . \tag{31}$$

Then equation (30) may be written as

$$\left[\ddot{\phi} - \dot{\phi}\left(6\,\phi + \frac{T}{m\,\lambda^*}\right) + 4\,\phi^3 + \frac{T}{m\,\lambda^*}\,\phi^2 + \frac{\mu\,\phi}{r^3} + \frac{T}{m\,\lambda^*}\,\frac{\mu}{r^3}\right](\vec{r}\cdot\vec{v}) = 0 \quad . \tag{32}$$

In general $\vec{r}\cdot\vec{v}\neq 0$ so the first factor of equation (32) must vanish. Then equation (32) becomes

$$\ddot{\phi} - 5\,\phi\,\dot{\phi} + 3\,\phi^3 + \left(\phi + \frac{T}{m\,\lambda^*}\right)\left(-\,\dot{\phi} + \phi^2 + \frac{\mu}{r^3}\right) = 0 \quad . \tag{33}$$

This is the equation which was originally sought. There is no longer a singularity involving $\vec{r}\cdot\vec{v}$.

The substitution represented by equation (18) also converts the state equation (1) into a very convenient form. Using equations (18), (19), and (20) in equation (1) yields

$$\ddot{\vec{r}} = \frac{T}{m\,\lambda^*}\,(\phi\,\vec{r} + \vec{v}) - \frac{\mu\,\vec{r}}{r^3} \quad . \tag{34}$$

Equations (33), (34), and (5) now constitute a complete set of differential equations for the optimal trajectory problem.

Equation (34) may be put into an interesting form as follows:

$$\ddot{\vec{r}} \times \vec{r} = \frac{T}{m\,\lambda^*}\,\vec{v} \times \vec{r} \quad , \tag{35}$$

but

$$\ddot{\vec{r}} \times \vec{r} = \frac{d}{dt}\,(\vec{v} \times \vec{r}) \quad . \tag{36}$$

Writing

$$\vec{L} = \vec{r} \times \vec{v} \quad , \tag{37}$$

equation (35) becomes

$$\dot{\vec{L}} = \frac{T}{m\lambda^*}\vec{L} \qquad . \tag{38}$$

This equation easily integrates to yield

$$\vec{L} = \vec{L}_0 \, e^{\int_0^t \frac{T}{m\lambda^*} dt} \tag{39}$$

where $\vec{L}_0$ is the initial value of the angular momentum. Thus, $\vec{L}$ is always parallel to $\vec{L}_0$ (which was known anyway) and equation (39) can be written in scalar form as

$$L = L_0 \, e^{\int_0^t \frac{T}{m\lambda^*} dt} \qquad . \tag{40}$$

A modification of equation (33) which makes the problem more numerically tractable follows [3]:

Set

$$\dot{\phi} = \rho \qquad . \tag{41}$$

Then, from equation (33),

$$\dot{\rho} = \ddot{\phi} = 5\phi\rho - 3\phi^3 + \left(\phi + \frac{T}{m\lambda^*}\right)\left(\rho - \phi^2 - \frac{\mu}{r^3}\right) \qquad . \tag{42}$$

Thus

$$\frac{\left(\frac{d\rho}{dt}\right)}{\left(\frac{d\phi}{dt}\right)} = \frac{d\rho}{d\phi} = \frac{5\phi\rho - 3\phi^3 + \left(\phi + \frac{T}{m\lambda^*}\right)\left(\rho - \phi^2 - \frac{\mu}{r^3}\right)}{\rho} \qquad , \tag{43}$$

or

$$\frac{d\rho}{d\phi} = 5\phi - \frac{3\phi^3}{\rho} + \left(\phi + \frac{T}{m\lambda^*}\right)\left(1 - \frac{\phi^2}{\rho} - \frac{\mu}{\rho r^3}\right) \quad . \tag{44}$$

Next, set

$$\phi = e^{\nu} \tag{45}$$

and

$$\rho = w e^{2\nu} \quad . \tag{46}$$

Then

$$\frac{d\rho}{d\phi} = e^{\nu}\frac{dw}{d\nu} + 2we^{\nu} = 5e^{\nu} - 3\frac{e^{3\nu}}{we^{2\nu}} + \left(e^{\nu} + \frac{T}{m\lambda^*}\right)\left(1 - \frac{e^{2\nu}}{we^{2\nu}} - \frac{\mu}{we^{2\nu}r^3}\right) \quad . \tag{47}$$

Clearing and collecting yields

$$w\frac{dw}{d\nu} + 2\left(w - \frac{3}{2}\right)(w - 1) = \left(1 + \frac{Te^{-\nu}}{m\lambda^*}\right)\left(w - 1 - \frac{\mu e^{-2\nu}}{r^3}\right) \quad . \tag{48}$$

For the trivial case of field-free coast, equation (48) solves to yield

$$\nu = \ln C_5 \left[\frac{(w - 1)^{1/2}}{(w - 2)^{1/4}}\right] \tag{49}$$

where C_5 is an integration constant.

THE NONPLANAR CASE

Although the initial set of equations, equations (1) through (9), apply equally well to either two- or three-dimensional trajectories, equation (12) involves the assumption of planarity. The next goal is to relax this assumption by choosing a nonzero value for $\vec{C}$. Even so, the time-open problem ($C_4 = 0$) will be assumed.

In order to allow for nonplanar steering, write

$$\vec{\lambda} = \xi \vec{r} + \eta \vec{v} + \zeta (\vec{r} \times \vec{v}) \tag{50}$$

where ξ, η, and ζ are scalar functions of time. The variables ξ and η as used in this section agree with the variables ξ and η of the last section if $\zeta \equiv 0$. From equation (50),

$$\dot{\vec{\lambda}} = \dot{\xi} \vec{r} + \xi \vec{v} + \dot{\eta} \vec{v} + \eta \dot{\vec{v}} + \dot{\zeta} (\vec{r} \times \vec{v}) + \zeta (\vec{r} \times \dot{\vec{v}}) \tag{51}$$

and, from equations (1) and (50),

$$\dot{\vec{v}} = \frac{T}{m\lambda}\left[\xi \vec{r} + \eta \vec{v} + \zeta (\vec{r} \times \vec{v})\right] - \frac{\mu \vec{r}}{r^3} \quad . \tag{52}$$

Calculating $\vec{r} \times \dot{\vec{v}}$ and inserting the result along with equation (52) into equation (51) yields

$$\dot{\vec{\lambda}} = \left(\dot{\xi} + \frac{T}{m\lambda}\xi\eta - \frac{\mu\eta}{r^3} + \frac{T\zeta^2}{m\lambda}\vec{r} \cdot \vec{v}\right)\vec{r} + \left(\dot{\eta} + \xi + \frac{T}{m\lambda}\eta^2 - \frac{T\zeta^2}{m\lambda}r^2\right)\vec{v}$$

$$+\left(\dot{\zeta} + 2\frac{T}{m\lambda}\eta\zeta\right)(\vec{r} \times \vec{v}) \quad . \tag{53}$$

It is now possible to calculate $\dot{\vec{\lambda}} \times \vec{r}$ as

$$\dot{\vec{\lambda}} \times \vec{r} = \left(\dot{\eta} + \xi + \frac{T}{m\lambda}\eta^2 - \frac{T\zeta^2}{m\lambda}r^2\right)\vec{v} \times \vec{r} + \left(\dot{\zeta} + 2\frac{T}{m\lambda}\eta\zeta\right)[r^2\vec{v} - (\vec{r} \cdot \vec{v})\vec{r}] \quad , \tag{54}$$

while from equation (50)

$$\vec{\lambda} \times \vec{v} = \xi(\vec{r} \times \vec{v}) + \zeta[(\vec{r} \cdot \vec{v})\vec{v} - v^2 \vec{r}] \quad . \tag{55}$$

Using equations (54) and (55) in equation (8) now gives

$$\left(\dot{\eta} + 2\xi + \frac{T}{m\lambda}\eta^2 - \frac{T\zeta^2 r^2}{m\lambda}\right)(\vec{v} \times \vec{r}) + \left[\left(\dot{\zeta} + \frac{2T}{m\lambda}\eta\zeta\right)r^2 - \zeta(\vec{r} \cdot \vec{v})\right]\vec{v}$$

$$+ \left[-\left(\dot{\zeta} + \frac{2T}{m\lambda}\eta\zeta\right)(\vec{r} \cdot \vec{v}) + v^2\zeta\right]\vec{r} = \vec{C} \quad . \tag{56}$$

If the dot product equation (56) with $\vec{v} \times \vec{r}$ is calculated, one finds

$$\dot{\eta} + 2\xi + \frac{T}{m\lambda}(\eta^2 - \zeta^2 r^2) = \frac{\vec{C} \cdot (\vec{v} \times \vec{r})}{|\vec{r} \times \vec{v}|^2} \quad . \tag{57}$$

Dotting equation (56) with $\vec{v}$ yields

$$\dot{\zeta} + \frac{2T}{m\lambda}\zeta\eta = \frac{\vec{C} \cdot \vec{v}}{|\vec{r} \times \vec{v}|^2} \quad . \tag{58}$$

Finally, dotting equation (56) with $\vec{r}$ yields

$$\zeta = \frac{\vec{C} \cdot \vec{r}}{|\vec{r} \times \vec{v}|^2} \quad . \tag{59}$$

Since two expressions which govern the variable ζ have been found [equations (58) and (59)], it is necessary to show that they are compatible. This is most easily accomplished as follows. From equation (52)

$$\vec{r} \times \dot{\vec{v}} = \frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{T}{m\lambda}\,[\eta\,(\vec{r} \times \vec{v}) + \zeta\,\vec{r} \times (\vec{r} \times \vec{v})] \tag{60}$$

so that

$$(\vec{r} \times \vec{v}) \cdot \frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{T}{m\lambda}\,\eta\,(\vec{r} \times \vec{v})^2 \quad . \tag{61}$$

Then calculate

$$\frac{d}{dt}\left[\frac{1}{(\vec{r} \times \vec{v})^2}\right] = -\,\frac{2}{(\vec{r} \times \vec{v})^4}\left[(\vec{r} \times \vec{v}) \cdot \frac{d}{dt}(\vec{r} \times \vec{v})\right] = \frac{-\,2\,\eta\,T}{m\lambda\,(\vec{r} \times \vec{v})^2} \quad . \tag{62}$$

Assuming equation (59), one finds

$$\dot{\zeta} = \frac{\vec{C} \cdot \vec{v}}{(\vec{r} \times \vec{v})^2} + \vec{C} \cdot \vec{r}\,\frac{d}{dt}\left[\frac{1}{(\vec{r} \times \vec{v})^2}\right] = \frac{\vec{C} \cdot \vec{v}}{(\vec{r} \times \vec{v})^2} - \frac{2\,\eta\,T\vec{C} \cdot \vec{r}}{m\lambda\,(\vec{r} \times \vec{v})^2} \quad . \tag{63}$$

Substituting for the combination of variables represented by equation (59) gives

$$\dot{\zeta} + \frac{2T}{m\lambda}\,\eta\,\zeta = \frac{\vec{C} \cdot \vec{v}}{(\vec{r} \times \vec{v})^2} \quad .$$

Thus equations (58) and (59) are compatible. From this point onward the discussion deals only with equation (59) since equation (58) is redundant.

It is still necessary to derive the equation governing ξ. To do this, return to equation (9). Calculating $\vec{\lambda} \cdot \vec{v}$ from equation (53) yields

$$\vec{\dot{\lambda}} \cdot \vec{v} = \left(\dot{\xi} + \frac{T}{m\lambda}\xi\eta - \frac{\mu\eta}{r^3} + \frac{T\zeta^2}{m\lambda}\vec{r}\cdot\vec{v}\right)(\vec{r}\cdot\vec{v}) + \left(\dot{\eta} + \xi + \frac{T}{m\lambda}\eta^2 - \frac{T\zeta^2}{m\lambda}r^2\right)v^2 \quad . \tag{64}$$

From equation (50),

$$\vec{\lambda}\cdot\vec{r} = \xi r^2 + \eta\vec{r}\cdot\vec{v} \quad . \tag{65}$$

Then equation (9) may be written as

$$\left[\dot{\xi} + \frac{T}{m\lambda}\xi\eta + \frac{T}{m\lambda}\zeta^2(\vec{r}\cdot\vec{v})\right](\vec{r}\cdot\vec{v}) - \xi v^2 + \frac{\vec{C}\cdot(\vec{r}\times\vec{v})}{|\vec{r}\times\vec{v}|^2}v^2 + \xi\frac{\mu}{r} = TS \tag{66}$$

where equation (57) was used to eliminate $\dot{\eta}$. Assuming that $\vec{r}\cdot\vec{v} \neq 0$ one finds

$$\dot{\xi} - \xi\left(\frac{v^2 - \frac{\mu}{r}}{\vec{r}\cdot\vec{v}}\right) + \frac{\vec{C}\cdot(\vec{r}\times\vec{v})}{(\vec{r}\cdot\vec{v})\,|\vec{r}\times\vec{v}|^2}v^2 = T\left(\frac{S}{\vec{r}\cdot\vec{v}} - \frac{\xi\eta}{m\lambda} - \frac{\zeta^2\,\vec{r}\cdot\vec{v}}{m\lambda}\right) \quad . \tag{67}$$

Equations (57), (59), and (67), along with equations (1) and (5) now define an optimal three-dimensional trajectory. All of the advantages that were iterated in the planar case are still valid; i.e., there are but two adjoint equations which need be integrated. The nonplanar component of the steering vector, given by equation (59), is carried algebraically, so no further integration is involved in the three-dimensionalization of the problem.

The previously stated disadvantage of $\vec{r}\cdot\vec{v}$ appearing in the denominator of equation (67) has returned. The obvious need is for a regularization akin to equation (18). But in this case such a regularization has not been found. The primary reason for this is, simply, the algebraic complexities which result when one attempts to deal with the three-dimensional equations.

A fairly natural approach to avoid the singularity without the difficulties of excessive algebraic manipulations is to modify the fundamental representation of the λ vector, i.e., equation (50). $\vec{\lambda}$ could be assumed to be written with a vector basis formed by the sets $[\vec{v}, \vec{r} \times (\vec{r} \times \vec{v}), (\vec{r} \times \vec{v})]$, $[\vec{r}, \vec{v} \times (\vec{r} \times \vec{v}), (\vec{r} \times \vec{v})]$, $[\vec{r}, \vec{r} \times (\vec{r} \times \vec{v}), (\vec{r} \times \vec{v})]$, or $[\vec{v}, \vec{v} \times (\vec{r} \times \vec{v}), (\vec{r} \times \vec{v})]$ just as equation (50) assumed $(\vec{r}, \vec{v}, \vec{r} \times \vec{v})$ as a spanning set. In each case the reciprocal $\vec{r} \cdot \vec{v}$ occurs, however, so this approach was not successful.

It is hoped that the difficulty of the singularity will be removed in future research.

CONCLUSIONS

It has been known for many years that integrals of the minimum mass expenditure optimal trajectory about a single gravitating center exist. These integrals have existed as a separate part of the literature and no direct use has been made of them.

In this paper it has been demonstrated that the use of these integrals yields a reduced set of adjoint equations which is to be integrated. This reduced the time required for two-point boundary value problem iterations.

In the planar case, a regularization has been found which improves numerical properties of the equations after the inclusion of known integrals. Such a regularization has not been found in the three-dimensional case.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Marshall Space Flight Center, Alabama, November 1973

CAT-30

REFERENCES

1. Burns, Rowland E: A Study on the Optimal Trajectory Problem. Dissertation, University of Alabama, University, Alabama, 1971.

2. Burns, Rowland E: An Explicit Solution for the Lagrange Multipliers on the Singular Subarc of an Optimal Trajectory. NASA TN D-4119, National Aeronautics and Space Administration, Washington, D.C., Sept. 1967.

3. Siefert, H.: Deutche Mathematiker Vereinigung Jahresbericht. vol. 5, 1942, pp. 75-79.

NASA-Langley, 1974