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Robot Arm Dynamics and Control

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PREFACE

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.

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ABSTRACT

This report treats two central topics related to the dynamical aspects of the control problem of the six degrees of freedom JPL Robot Research Project (RRP) manipulator: (a) variations in total inertia and gravity loads at the joint outputs, and (b) relative importance of gravity and acceleration-generated reaction torques or forces versus inertia torques or forces. The relation between the dynamical state equations in explicit terms and servoing the manipulator is briefly discussed in the framework of state variable feedback control which also forms the basis of adaptive manipulator control.

Exact state equations have been determined for total inertia and gravity loads at the joint outputs as a function of joint variables, using the constant inertial and geometric parameters of the individual links defined in the respective link coordinate frames. The range of maximum variations in total inertia and gravity loads at the joint outputs has been calculated for both no load and load in the hand.

The main result of this report is the construction of a set of greatly simplified state equations which describe total inertia and gravity load variations at the output of the six joints with an average error of less than 5%. The simplified state equations also show that most of the time the gravity terms are more important than the inertia terms in the torque or force equations for joint numbers 2, 3, 4, and 5. Further, the acceleration-generated reaction torques or forces, except from extreme arm motion patterns, are shown to have very low quantitative significance as compared to the straight inertial torques or forces in the dynamic equations restricted to simultaneous motions at the first three joints.

The results are summarized in four tables and nine figures. The report also contains all analytic tools and byproducts needed to arrive at the outlined conclusions. An important analytical byproduct is the simplification of the general matrix algorithm for manipulator dynamics.

JPL Robot Research Project Manipulator

I. INTRODUCTION

The purpose of control is to keep fixed or alter the dynamical behavior of a physical system in accordance with man's wishes formulated in terms of performance requirements and goals. The nature of the control problem comprises two distinct parts: (a) quantitative description of the dynamical behavior of the physical system (in our case, the manipulator) to be controlled and (b) specification of a "scheme" or control law for carrying out the desired controlled behavior (in our case, to accomplish a variety of manipulative tasks with specified performance). This report is mainly about the former part of the manipulator control problem:

> Modeling and evaluating the dynamical properties and behavior of the JPL Robot Research Project (RRP) manipulator.

The fundamental idea of control is that the inputs should be computed from the state. Of course, this idea is known as feedback. Thus, the natural framework for formulating and solving control problems is the state description of the physical system. The state incorporates all information necessary to determine the control action to be taken since, by definition of a dynamical system, the future evolution of the system is completely determined by its present state and the future inputs. The relation between explicit state equations for manipulator dynamics and servoing the manipulator is briefly treated in Section II.

The actual dynamical model for the six degrees of freedom JPL RRP manipulator can be obtained from known physical laws (from the laws of the Newtonian mechanics) and from physical measurements. This task amounts to the development of the equations of motion for the six manipulator joints in terms of specified (measured) geometric and inertial parameters of the links. Conventional procedures could then be applied to develop the actual motion equations. Instead of using conventional procedures, the equations of motion in this report are developed through the application of a general algorithmic description of manipulator dynamics. The algorithm is based on a specific representation of link coordinate frames in jointed mechanisms and the formalism of the Lagrangian mechanics. The features of the general algorithm together with the definitions of the involved functional symbols and mathematical operations are described in Section III. Section III also provides a general specification of the six equations of motion for the JPL RRP manipulator as well as a condensed physical explanation of the different terms appearing in the equations. Section III concludes with a compact vector/matrix description of the six motion equations.

The complete dynamical model of the JPL RRP manipulator is described by a set of six coupled nonlinear differential equations. Each equation contains a large number of torque or force terms classified into four groups: (a) inertial torque or force, (b) reaction torques or forces generated by acceleration at other joints, (c) velocity-generated (centripetal and Coriolis) reaction torques or forces, and (d) gravity torque or force. With few exceptions, each torque or force term depends on the instantaneous configuration (position) of several links. To gain analytic insight into the dynamical behavior of the manipulator in terms of explicit state equations while keeping the analysis manageable, well-defined and useful dynamical model restrictions are identified in Section IV. It is emphasized, however, that the model restrictions are introduced only for analytic purposes.

In Section V explicit state equations are presented for inertial, gravity, and acceleration-generated reaction torque/force terms for manipulator motions restricted to the first three joints. The last three (wrist) joints are thought to be temporally at rest in a known configuration. While in Section VI complete (unrestricted) explicit state equations are presented for inertial and gravity torques or forces acting at all six joint axes. The exact state equations developed in Sections V and VI form one part of the important results of this report.

Partial derivatives of the different link coordinate transformation matrices as well as the pseudo inertia matrices (together with numerical values of inertial components) utilized in the development of the explicit state equations are compiled in Appendices A and B. Modifications of the explicit and exact state equations for inertial and gravity terms when a load is emplaced in the hand are treated in Appendix C.

The concluding part of the report is Section VII organized in five subsections. From the exact state equations and numerical values of inertial components of the JPL RRP manipulator the following concluding computations are made:

- Maximum and minimum values of total inertia seen at all six joint axes are determined in subsection VII. A; the constant and varying components of total inertias are separated out in the computations.

- Maximum gravity load variations seen at the different joint axes are determined in subsection VII. B.

The maximum total inertia and gravity load variations have been calculated for both no load and load in the hand. (The load is a 1.8 kg, 442 cm^3 cube placed with its mass center at the origin of the hand coordinate frame.) Utilizing the exact state equations restricted to simultaneous motions at the first three manipulator joints,

- The relative importance of acceleration-generated reaction torques/forces versus inertial torques forces is quantitatively evaluated in subsection VII. C.

The main result of this report is

- The development of simplified state equations for total inertial and gravity loads at all six joint axes, presented and evaluated regarding accuracy in subsection VII. D.

Parameters dependent on a load in the hand are separated out in the simplified state equations. Utilizing the simplified state equations,

- The relative importance of gravity load versus inertial load in the torque/force equations is quantitatively evaluated in subsection VII. E, normalized to unit acceleration.

It is shown that the gravity terms in most of the time of normal (not too fast) arm operation are more important than the inertial terms for joints Nos. 2, 3, 4, and 5 in the gravity field of Earth.

The development and evaluation of explicit state equations for total inertial and gravity loads acting at the six arm joint axes form the basic dynamical model for the JPL RRP manipulator under operating conditions when acceleration- and velocity-generated reaction torques or forces can be neglected. The relative significance of the different reaction terms in the complete torque/force equations for fast arm movements will be evaluated in a separate report after the determination of explicit state equations for all existing reaction torques and forces.

General simplification of the algorithmic definitions for all dynamic coefficients of any manipulator is introduced and mathematically justified in Appendix D at the end of the report.

II. DYNAMICAL MODEL AND CONTROL SYSTEM DESIGN

The RRP manipulator under consideration is a coupled electromechanical system. The inputs to the system (with which control is accomplished) are torque generated by motors driving the joints. The outputs are joint position and motor shaft velocity measurements. This input/output description forms the definition of the manipulator as a dynamical system. To make this dynamical system definition (or dynamical model) quantitative, mathematical relations are required which relate input to output. The mathematical relation between input (torque) and output (position and velocity) is obtained by the specification of state equations (differential equations) governing the manipulator motion.

The execution of purposeful manipulative tasks requires two types of performance from the viewpoint of servo control: (1) positioning the manipulator, and (2) exerting torques or forces on objects through the manipulator. Manipulator positioning is a task of controlling the relative displacement of several links connected by single degree of freedom joints. The positioning control problem can be subdivided into two classes: (a) point-to-point control, and (b) continuous path control. In point-to-point control mode only the final (terminal) joint variable values are specified as "desired output". While in continuous path control mode the "desired output" is a closed time history (time sequence) of joint variable values. The strict space-time coordination of several joint variable values defines a continuous path for manipulator motion in the work space.

The objective of closed loop (feedback) control is to reduce the effect of external disturbances and system parameter changes on the desired system output. In the case of position-servoing a manipulator, the notion "external disturbances" can be used in a broad sense: they can include known effects deliberately neglected in the mathematical form of the dynamical model. (For instance, neglected small reaction torques or forces, neglected small link mass center offsets, etc.) There is, however, a limit on the range of changes in system parameters and disturbances which can be tolerated without deterioration of desired servo performance. In general, the acceptable variation in system parameters can be extended by readjusting (varying) feedback gains.

In particular, for a critically damped position servo using velocity feedback, an essential system parameter is the total effective inertia J_t: if J_t is decreased by a factor "n" relative to a nominal value the damping and natural frequency are both increased by a factor $\sqrt{n}$; but reduction of the velocity feedback constant to $\sqrt{n}$ to its original (nominal) value will restore critical damping. (See, for instance, Refs. 1, 2, 3.) Manipulator motion in general and load handling in particular imply considerable variations in total effective inertia J_t as seen at the different joint drives. Therefore, to maintain a required servo performance despite variations in J_t, J_t must be known explicitly as a function of joint variables (or implicitly as a function of time for a given motion program).

A strict continuous path control requires a uniform servo performance. Thus, it is important to obtain an appropriate state description for total effective inertia variations as seen at the different joint drives. One outcome of the manipulator dynamic model analysis contained in this report is the specification of state functions for variations in total effective inertias, with or without load in the hand.

The gravity load acting at the different joint drives during arm motion is an important dynamic factor in commanding torques to obtain a desired manipulator position output in a continuous path control mode. Another outcome of the manipulator dynamic model analysis of this report is the specification of state functions for variations in gravity loads as seen at the different joint drives during arm motion, with or without load in the hand.

The speed of arm motion can be interpreted both kinematically and dynamically. The kinematic interpretation considers only the time required to move for instance the fingertip from one point to another in the workspace, while the dynamic interpretation of arm speed considers the torques or forces acting at both the different joints and the fingertip during arm motion. A useful dynamic definition for "fast" or "slow" arm motion can be formulated in terms of reaction torques or forces induced by the arm motion: the arm motion is "slow" if the effect of induced reaction torques or forces can be neglected in commanding torques to obtain desired position outputs; if not, then the motion is "fast" in a dynamic sense. It is noted that an arm motion can be "slow" in a dynamic sense but still be "fast" in a kinematic sense. Another outcome of manipulator dynamic model analysis is to contribute to the establishment of the boundary between dynamically "slow" and "fast' arm motion with respect to control system performance.

Figure 1 shows schematically the RRP manipulator position servo control under development, indicating also the relation of manipulator dynamical model and servo design.

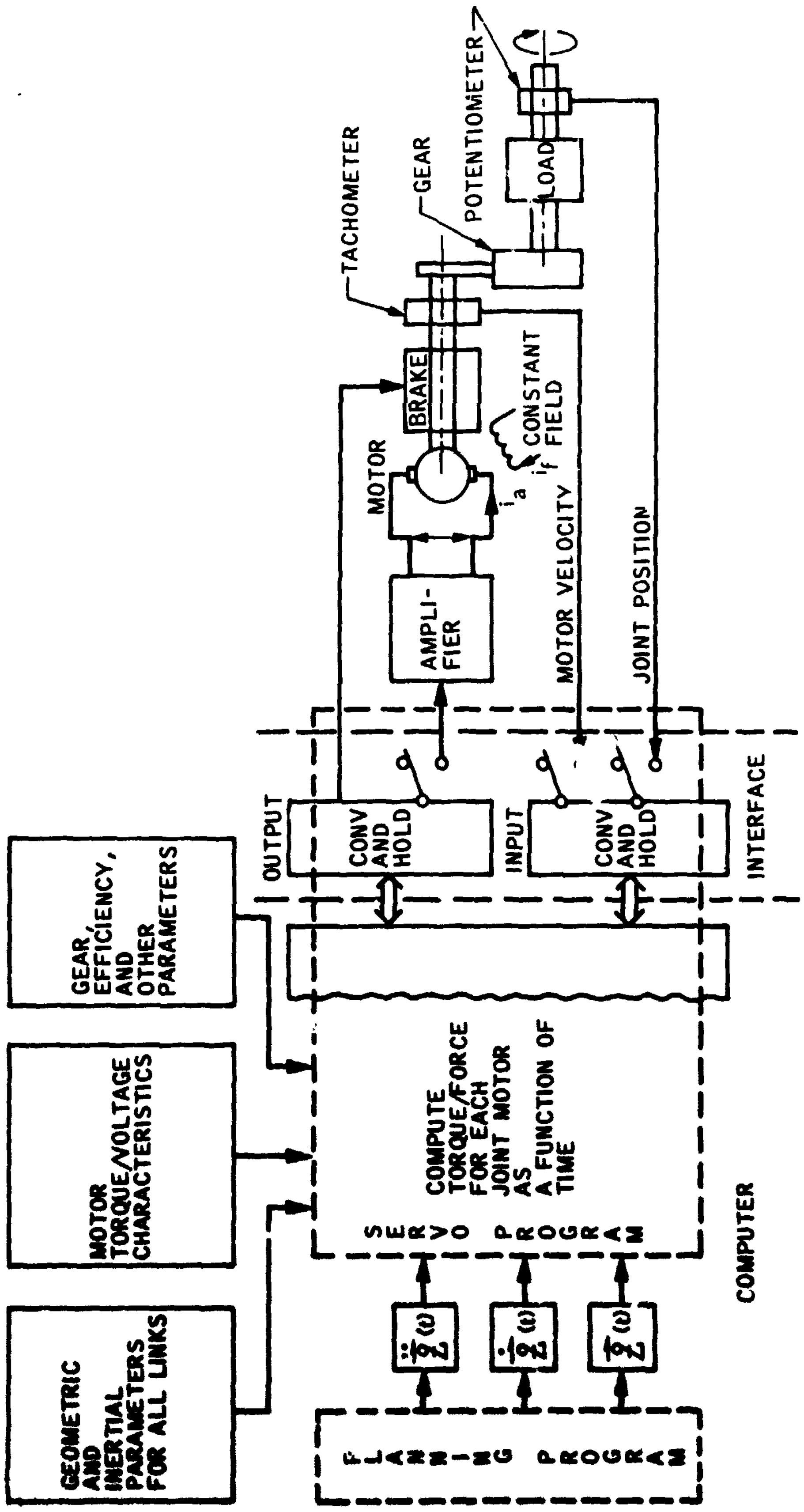

Fig. 1. Manipulator Servo Scheme

III. GENERAL MODEL FOR MANIPULATOR DYNAMICS

The general equations of motion for jointed mechanisms (manipulators) can conveniently be expressed through the application of the Lagrangian equations for nonconservative systems (Ref. 4). Many investigators in the field of computer-controlled manipulation in the U.S.A. employ the Lagrangian technique to formulate the dynamic and control problem of manipulators, and apply the Hartenberg-Denavit representation of coordinate frames in jointed mechanisms to the definition of manipulator inertial parameters and dynamic variables (Refs. 5, 6, 7 and 8). The application of the Lagrangian formalism together with the Hartenberg-Denavit link coordinate representation results in a convenient and compact algorithmic description of the manipulator equations of motion. The algorithm is expressed by matrix operations (Ref. 5), and facilitates both analysis and computer implementation. The evaluation of the dynamic and control equations in functionally explicit terms in this and subsequent memos will be based on the compact matrix algorithm developed in Ref. 5.

A. The General Dynamic Algorithm

For clarity and easy reference, the general dynamic algorithm as applied to manipulators is repeated here together with the corresponding definitions. The associated manipulator coordinate system conventions and transformations together with their application to the JPL RRP manipulator* should be consulted whenever necessary.

The general algorithm which describes the manipulator equations of motion is given by the following expression for the torque or force F_i acting at joint "i":

$$\boxed{\sum_{j=1}^{n}\left\{\sum_{k=1}^{j}\left[\mathrm{Trace}\left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k\right]+\sum_{k=1}^{j}\sum_{p=1}^{j}\left[\mathrm{Trace}\left(U_{jkp}J_jU_{ji}^T\right)\dot{q}_k\dot{q}_p\right] - m_jGU_{ji}\bar{\rho}_j\right\} = F_i\ ,\quad i = 1, 2, \ldots, n} \qquad (1)$$

where superscript T denotes the transpose of the matrix U_{ji}, and

*Lewis, R.A., Bejczy, A.K., "RRP Manipulator Conventions, Coordinate Systems, and Trajectory Considerations," JPL Guidance and Control Technical Memo 343-174, 1 December 1972.

F_i = torque or force acting at joint "i" (that is, corresponding to the joint variable q_i),

q_i = "joint variable i", i = 1, ..., j, ..., k, ..., p, ..., n where "n" denotes the degree-of-freedom (that is, the total number of joint variables) of the manipulator,

$\dot{q}_i$, $\ddot{q}_i$ = velocity and acceleration, respectively, of joint variable "i".

The "building blocks" m_i, $\bar{\rho}_j$, G, U_{ji}, U_{jkp}, and J_j of Eq. (1) are defined as follows:

m_j = the mass of body "j" in the chain of "n" bodies (links).

$\bar{\rho}_j$ = mass center vector of body (link) "j" in the coordinate system fixed in the same body, given as a 4 x 1 vector with components

$$\bar{\rho}_j = \begin{bmatrix} \bar{x}_j \\ \bar{y}_j \\ \bar{z}_j \\ 1 \end{bmatrix} \tag{2}$$

G = acceleration of gravity, given as a 1 x 4 vector with components

$$G = \left[G_x, \ G_y, \ G_z, \ 0\right] \tag{3}$$

U_{ji} = the first partial derivative of the T_0^j transformation matrix with respect to q_i. It is a 4 x 4 matrix. The transformation matrix T_0^j is defined as

$$T_0^j = T_0^1 T_1^2 \cdots T_{j-1}^j , \qquad j \le n$$

which relates a point given in the "j" frame to the base reference frame "0".

Since an individual transformation matrix T_{i-1}^{i} depends only on q_i, it is convenient to express the derivative of T_{i-1}^{i} with respect to q_i through matrix operators Q. Thus, we have the following expression for U_{ji}:

$$U_{ji} = \frac{\partial T_0^j}{\partial q_i} = T_0^1 T_1^2 \cdots \frac{\partial}{\partial q_i}\left(T_{i-1}^{i}\right) \cdots T_{j-1}^{j}$$

$$= T_0^1 T_1^2 \cdots Q T_{i-1}^{i} \cdots T_{j-1}^{j}\,, \qquad i \leq j \tag{4}$$

where for a rotational joint variable $q_i = \theta_i$ we have

$$Q = Q_{\theta_i} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and for a linear joint variable $q_i = r_i$ we have

$$Q = Q_{r_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

U_{jkp} = the second partial derivative of the T_0^j transformation matrix with respect to q_k and q_p. It is a 4 x 4 matrix. Using the notations defined above, we have the following expression for U_{jkp}:

$$U_{jkp} = \frac{\partial^2 T_0^j}{\partial q_k \partial q_p} = T_0^1 \cdots \frac{\partial}{\partial q_k}\left(T_{k-1}^{k}\right) \cdots \frac{\partial}{\partial q_p}\left(T_{p-1}^{p}\right) \cdots T_{j-1}^{j}$$

$$= T_0^1 \cdots Q T_{k-1}^{k} \cdots Q T_{p-1}^{p} \cdots T_{j-1}^{j} \tag{7}$$

with k, $p \leq j$. It is noted that for $k = p$ the second partial derivative matrix operator for a linear joint variable r_k is zero,

$$Q_{r_k r_k} = 0 \tag{8}$$

while for a revolute joint variable θ_k we have

$$Q_{\theta_k \theta_k} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

J_j = 4 x 4 "inertia matrix" (we will call it "pseudo inertia matrix") for body (link) "j" defined as follows:

$$J_j = m_j \begin{bmatrix} \frac{1}{2}\left(-k_{j11}^2 + k_{j22}^2 + k_{j33}^2\right) & k_{j12}^2 & k_{j13}^2 & \bar{x}_j \\ k_{j12}^2 & \frac{1}{2}\left(k_{j11}^2 - k_{j22}^2 + k_{j33}^2\right) & k_{j23}^2 & \bar{y}_j \\ k_{j13}^2 & k_{j23}^2 & \frac{1}{2}\left(k_{j11}^2 + k_{j22}^2 - k_{j33}^2\right) & \bar{z}_j \\ \bar{x}_j & \bar{y}_j & \bar{z}_j & 1 \end{bmatrix} \tag{10}$$

where

$\bar{x}_j$, $\bar{y}_j$, $\bar{z}_j$ are defined by Eq. (2), and

k_{jip} = radius of gyration "ip" (i, p = 1, 2, 3) of body (link) "j" about the origin of the coordinate frame fixed in the same body (link). The radius of gyration is defined by the corresponding member of the inertia tensor I_{jip} as

$$k_{jip}^2 = \frac{I_{jip}}{m_j} \tag{11}$$

where the i, p = 1, 2, 3 indices, respectively, represent the x, y, z axes of the Cartesian coordinate frame fixed in body "j".

As seen in Eq. (10), the J_j "pseudo inertia matrix" is symmetric. It is constructed from the mass center vector $\bar{\rho}_j$ and the elements of the inertia tensor I_j of body (link) "j". It is noted that the diagonal terms of the upper left 3 x 3 partition of the J_j "pseudo inertia matrix" are only related to the diagonal terms of the corresponding true inertia matrix I_j, but the diagonal terms of the "pseudo" and true inertia matrices are not identical.

For an "n" degrees-of-freedom manipulator, Eq. (1) gives a coupled set of "n" nonlinear second-order differential equations which constitute the complete dynamic model for manipulators.

B. Dynamic Equations for the JPL RRP Manipulator Expanded in General Terms

If the algorithm given by Eq. (1) is expanded in general terms for the JPL RRP six degrees-of-freedom manipulator, the following equations of motions are obtained:†

$$
\begin{aligned}
& D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2 + D_{13}\ddot{r}_3 + D_{14}\ddot{\theta}_4 + D_{15}\ddot{\theta}_5 + D_{16}\ddot{\theta}_6 \\
& + D_{111}\dot{\theta}_1^2 + D_{122}\dot{\theta}_2^2 + D_{133}\dot{r}_3^2 + D_{144}\dot{\theta}_4^2 + D_{155}\dot{\theta}_5^2 + D_{166}\dot{\theta}_6^2 \\
& + D_{112}\dot{\theta}_1\dot{\theta}_2 + D_{113}\dot{\theta}_1\dot{r}_3 + D_{114}\dot{\theta}_1\dot{\theta}_4 + D_{115}\dot{\theta}_1\dot{\theta}_5 + D_{116}\dot{\theta}_1\dot{\theta}_6 \\
& + D_{123}\dot{\theta}_2\dot{r}_3 + D_{124}\dot{\theta}_2\dot{\theta}_4 + D_{125}\dot{\theta}_2\dot{\theta}_5 + D_{126}\dot{\theta}_2\dot{\theta}_6 \\
& + D_{134}\dot{r}_3\dot{\theta}_4 + D_{135}\dot{r}_3\dot{\theta}_5 + D_{136}\dot{r}_3\dot{\theta}_6 \\
& + D_{145}\dot{\theta}_4\dot{\theta}_5 + D_{146}\dot{\theta}_4\dot{\theta}_6 + D_{156}\dot{\theta}_5\dot{\theta}_6 + D_1 = T_1
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
& D_{12}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 + D_{23}\ddot{r}_3 + D_{24}\ddot{\theta}_4 + D_{25}\ddot{\theta}_5 + D_{26}\ddot{\theta}_6 \\
& + D_{211}\dot{\theta}_1^2 + D_{222}\dot{\theta}_2^2 + D_{233}\dot{r}_3^2 + D_{244}\dot{\theta}_4^2 + D_{255}\dot{\theta}_5^2 + D_{266}\dot{\theta}_6^2 \\
& + D_{212}\dot{\theta}_1\dot{\theta}_2 + D_{213}\dot{\theta}_1\dot{r}_3 + D_{214}\dot{\theta}_1\dot{\theta}_4 + D_{215}\dot{\theta}_1\dot{\theta}_5 + D_{216}\dot{\theta}_1\dot{\theta}_6 \\
& + D_{223}\dot{\theta}_2\dot{r}_3 + D_{224}\dot{\theta}_2\dot{\theta}_4 + D_{225}\dot{\theta}_2\dot{\theta}_5 + D_{226}\dot{\theta}_2\dot{\theta}_6 \\
& + D_{234}\dot{r}_3\dot{\theta}_4 + D_{235}\dot{r}_3\dot{\theta}_5 + D_{236}\dot{r}_3\dot{\theta}_6 \\
& + D_{245}\dot{\theta}_4\dot{\theta}_5 + D_{246}\dot{\theta}_4\dot{\theta}_6 + D_{256}\dot{\theta}_5\dot{\theta}_6 + D_2 = T_2
\end{aligned}
\tag{13}
$$

†In the subsequent equations $\ddot{\theta}_i$ and $\dot{\theta}_i$ denote, respectively, the angular velocity and acceleration of the revolute joint variables θ_i belonging to joints 1, 2, 4, 5, 6, while $\ddot{r}_3$ and $\dot{r}_3$ denote, respectively, the acceleration and velocity of the linear displacement joint variable r_3 belonging to the linear joint (joint #3). See also Figure 2 later in the text.

$$
\begin{aligned}
& D_{13}\ddot{\theta}_1 + D_{23}\ddot{\theta}_2 + D_{33}\ddot{r}_3 + D_{34}\ddot{\theta}_4 + D_{35}\ddot{\theta}_5 + D_{36}\ddot{\theta}_6 \\
& + D_{311}\dot{\theta}_1^2 + D_{322}\dot{\theta}_2^2 + D_{333}\dot{r}_3^2 + D_{344}\dot{\theta}_4^2 + D_{355}\dot{\theta}_5^2 + D_{366}\dot{\theta}_6^2 \\
& + D_{312}\dot{\theta}_1\dot{\theta}_2 + D_{313}\dot{\theta}_1\dot{r}_3 + D_{314}\dot{\theta}_1\dot{\theta}_4 + D_{315}\dot{\theta}_1\dot{\theta}_5 + D_{316}\dot{\theta}_1\dot{\theta}_6 \\
& + D_{323}\dot{\theta}_2\dot{r}_3 + D_{324}\dot{\theta}_2\dot{\theta}_4 + D_{325}\dot{\theta}_2\dot{\theta}_5 + D_{326}\dot{\theta}_2\dot{\theta}_6 \\
& + D_{334}\dot{r}_3\dot{\theta}_4 + D_{335}\dot{r}_3\dot{\theta}_5 + D_{336}\dot{r}_3\dot{\theta}_6 \\
& + D_{345}\dot{\theta}_4\dot{\theta}_5 + D_{346}\dot{\theta}_4\dot{\theta}_6 + D_{356}\dot{\theta}_5\dot{\theta}_6 + D_3 = F_3
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
& D_{14}\ddot{\theta}_1 + D_{24}\ddot{\theta}_2 + D_{34}\ddot{r}_3 + D_{44}\ddot{\theta}_4 + D_{45}\ddot{\theta}_5 + D_{46}\ddot{\theta}_6 \\
& + D_{411}\dot{\theta}_1^2 + D_{422}\dot{\theta}_2^2 + D_{433}\dot{r}_3^2 + D_{444}\dot{\theta}_4^2 + D_{455}\dot{\theta}_5^2 + D_{466}\dot{\theta}_6^2 \\
& + D_{412}\dot{\theta}_1\dot{\theta}_2 + D_{413}\dot{\theta}_1\dot{r}_3 + D_{414}\dot{\theta}_1\dot{\theta}_4 + D_{415}\dot{\theta}_1\dot{\theta}_5 + D_{416}\dot{\theta}_1\dot{\theta}_6 \\
& + D_{423}\dot{\theta}_2\dot{r}_3 + D_{424}\dot{\theta}_2\dot{\theta}_4 + D_{425}\dot{\theta}_2\dot{\theta}_5 + D_{426}\dot{\theta}_2\dot{\theta}_6 \\
& + D_{434}\dot{r}_3\dot{\theta}_4 + D_{435}\dot{r}_3\dot{\theta}_5 + D_{436}\dot{r}_3\dot{\theta}_6 \\
& + D_{445}\dot{\theta}_4\dot{\theta}_5 + D_{446}\dot{\theta}_4\dot{\theta}_6 + D_{456}\dot{\theta}_5\dot{\theta}_6 + D_4 = T_4
\end{aligned}
\tag{15}
$$

$$D_{15}\ddot{\theta}_1 + D_{25}\ddot{\theta}_2 + D_{35}\ddot{r}_3 + D_{45}\ddot{\theta}_4 + D_{55}\ddot{\theta}_5 + D_{56}\ddot{\theta}_6$$

$$+ D_{511}\dot{\theta}_1^2 + D_{522}\dot{\theta}_2^2 + D_{533}\dot{r}_3^2 + D_{544}\dot{\theta}_4^2 + D_{555}\dot{\theta}_5^2 + D_{566}\dot{\theta}_6^2$$

$$+ D_{512}\dot{\theta}_1\dot{\theta}_2 + D_{513}\dot{\theta}_1\dot{r}_3 + D_{514}\dot{\theta}_1\dot{\theta}_4 + D_{515}\dot{\theta}_1\dot{\theta}_5 + D_{516}\dot{\theta}_1\dot{\theta}_6$$

$$+ D_{523}\dot{\theta}_2\dot{r}_3 + D_{524}\dot{\theta}_2\dot{\theta}_4 + D_{525}\dot{\theta}_2\dot{\theta}_5 + D_{526}\dot{\theta}_2\dot{\theta}_6$$

$$+ D_{534}\dot{r}_3\dot{\theta}_4 + D_{535}\dot{r}_3\dot{\theta}_5 + D_{536}\dot{r}_3\dot{\theta}_6$$

$$+ D_{545}\dot{\theta}_4\dot{\theta}_5 + D_{546}\dot{\theta}_4\dot{\theta}_6 + D_{556}\dot{\theta}_5\dot{\theta}_6 + D_5 = T_5 \qquad (16)$$

$$D_{16}\ddot{\theta}_1 + D_{26}\ddot{\theta}_2 + D_{36}\ddot{\theta}_3 + D_{46}\ddot{\theta}_4 + D_{56}\ddot{\theta}_5 + D_{66}\ddot{\theta}_6$$

$$+ D_{611}\dot{\theta}_1^2 + D_{622}\dot{\theta}_2^2 + D_{633}\dot{r}_3^2 + D_{644}\dot{\theta}_4^2 + D_{655}\dot{\theta}_5^2 + D_{666}\dot{\theta}_6^2$$

$$+ D_{612}\dot{\theta}_1\dot{\theta}_2 + D_{613}\dot{\theta}_1\dot{r}_3 + D_{614}\dot{\theta}_1\dot{\theta}_4 + D_{615}\dot{\theta}_1\dot{\theta}_5 + D_{616}\dot{\theta}_1\dot{\theta}_6$$

$$+ D_{623}\dot{\theta}_2\dot{r}_3 + D_{624}\dot{\theta}_2\dot{\theta}_4 + D_{625}\dot{\theta}_2\dot{\theta}_5 + D_{626}\dot{\theta}_2\dot{\theta}_6$$

$$+ D_{634}\dot{r}_3\dot{\theta}_4 + D_{635}\dot{r}_3\dot{\theta}_5 + D_{636}\dot{r}_3\dot{\theta}_6$$

$$+ D_{645}\dot{\theta}_4\dot{\theta}_5 + D_{646}\dot{\theta}_4\dot{\theta}_6 + D_{656}\dot{\theta}_5\dot{\theta}_6 + D_6 = T_6 \qquad (17)$$

The coefficients D_i, D_{ij} and D_{ijp} in Eqs. (12) to (17) are functions of both the joint variables and inertial parameters of the manipulator, and can be called "the dynamic coefficients of the manipulator". The physical meaning and functional relation of the three classes of dynamic coefficients can easily be seen from the defining algorithmic expression given by Eq. (1):

(1) The coefficients D_i (single subscript) are the gravity terms, functionally defined by the last term in the left hand side of Eq. (1). (Obviously, in zero gravity field the D_i coefficients are zero.)

(2) The coefficients D_{ij} (double subscript) are related to the acceleration of the joint variables; they are functionally defined by the first term in the left hand side of Eq. (1). In particular, for $i = j$, D_{ii} is related to the acceleration of joint "i" where the driving torque T_i (or force F_i) acts, while for $i \neq j$ D_{ij} is related to the reaction torque (or force) induced by the acceleration of joint "j" and acting at joint "i", or vice versa. (It is seen from Eqs. (12) to (17) that $D_{ij} = D_{ji}$.)

(3) The coefficients D_{ijp} (triple subscript) are related to the velocity of the joint variables; they are functionally defined by the second term in the left hand side of Eq. (1). The last two indices (jp) are related to the velocities of joint variables "j" and "p" whose dynamic interplay induces a reaction torque (or force) at joint "i". Thus, the first index (i) is always related to the joint where the velocity-induced reaction torques (or forces) are "felt". In particular, for $j = p$, D_{ijj} is related to the centripetal force generated by the angular velocity of joint "j" and "felt" at joint "i", while for $j \neq p$ D_{ijp} is related to the Coriolis force generated by the velocities of joints "j" and "p" and "felt" at joint "i". It is noted that for a given "i" we have $D_{ijp} = D_{ipj}$ which is apparent by physical reasoning.†

As seen, Eqs. (12) to (17) are six coupled, nonlinear, second-order differential equations describing the dynamic behavior of the JPL RRP manipulator. For a given set of applied torques T_i ($i = 1, 2, 4, 5, 6$) and force F_3 as a function of time, Eqs. (12) to (17) should be integrated simultaneously to obtain the actual motion of the manipulator in terms of the time history of the joint variables θ_1, θ_2, r_3, θ_4, θ_5, θ_6. Then the time history of the joint variables can be transformed to obtain the time history (trajectory) of the hand motion by using the appropriate transformation matrix described in the footnote on page 9. Or, if the time history of the joint variables (together with the time history of their

†The symmetry of the two dynamic coefficients, $D_{ij} = D_{ji}$ and $D_{ijp} = D_{ipj}$ can easily be seen from the defining equation, Eq. (1), by noting that

$$\text{Trace}\,(ABC^T) = \text{Trace}\,(CBA^T) \text{ and } U_{jkp} = U_{jpk}$$

where B is a symmetric matrix, while A and C can be two general (non-symmetric) square matrices.

acceleration and velocity) is known ahead of time (for instance, from a trajectory planning program, see Refs. 1, 2 and 3), then Eqs. (12) to (17) can be utilized to compute the torques (T_i, i = 1, 2, 4, 5, 6) and force F_3 as a function of time which are required to produce the particular planned (or known) manipulator motion. The Stanford manipulator control scheme (Ref. 7) utilizes the latter procedure.

In order to precompute torques and forces for a given manipulator motion, or to obtain the actual manipulator motion for given torques and forces (or, in general, to perform manipulator dynamic behavior and control system analysis and design), Eqs. (12) to (17) as stated cannot be used without knowing the explicit functional form (or, alternatively, the time history) of the dynamic coefficients D_i, D_{ij}, D_{ijp}. Eqs. (12) to (17) in the stated form, however, bring out an important point: in the case of simultaneous motion of several joints, the motion at one joint has a dynamic effect on the motion at other joints, and the torque (or force) applied at one joint has a dynamic effect on the motion at other joints. Since the dynamic coefficients are dependent on the values of the joint variables, the effect of dynamic coupling between motions at different joints will depend on the actual manipulator link configuration during motion.

In order to facilitate further reference in the dynamic and control system analysis of the JPL RRP manipulator, the lengthy and complex form of the dynamic equations, Eqs. (12) to (17), is brought into a more compact and structured representation.

(1) The gravity terms D_i are expressed by a six-dimensional column vector denoted by $\vec{d}_G$:

$$\vec{d}_G \triangleq \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix} \tag{18}$$

(2) The acceleration-related coefficients are expressed by a 6 x 6 symmetric matrix denoted by D_A:

$$D_A \triangleq \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \tag{19}$$

Let the acceleration of the six joint variables be expressed by a six-dimensional column vector denoted by $\ddot{q}$:

$$\ddot{q} \triangleq \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{r}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \\ \ddot{\theta}_6 \end{bmatrix} \tag{20}$$

Thus, all 36 acceleration-related terms in Eqs. (12) to (17) can be written in the compact matrix-vector product form:

$$\vec{d}_A = D_A \ddot{\vec{q}} \tag{21}$$

(3) The velocity-related coefficients in each of the six equations, Eqs. (12) to (17), can be expressed separately by a 6 x 6 symmetric matrix denoted by $D_{i,V}$ and defined in the following way:

$$D_{i,V} \triangleq \begin{bmatrix} 2D_{i11} & D_{i12} & D_{i13} & D_{i14} & D_{i15} & D_{i16} \\ D_{i12} & 2D_{i22} & D_{i23} & D_{i24} & D_{i25} & D_{i26} \\ D_{i13} & D_{i23} & 2D_{i33} & D_{i34} & D_{i35} & D_{i36} \\ D_{i14} & D_{i24} & D_{i34} & 2D_{i44} & D_{i45} & D_{i46} \\ D_{i15} & D_{i25} & D_{i35} & D_{i45} & 2D_{i55} & D_{i56} \\ D_{i16} & D_{i26} & D_{i36} & D_{i46} & D_{i56} & 2D_{i66} \end{bmatrix} \tag{22}$$

Let the velocity of the six joint variables be expressed by a six-dimensional column vector denoted by $\dot{\vec{q}}$:

$$\dot{\vec{q}} \triangleq \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{r}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} \tag{22.a}$$

Then the 21 velocity-related terms in each of the six equations, Eqs. (12) to (17), can be expressed separately in the following compact matrix-vector product form:

$$\frac{1}{2}\dot{\vec{q}}^T D_{i,V}\dot{\vec{q}} \tag{23}$$

where the superscript T denotes the transpose of the column vector $\dot{\vec{q}}$, and subscript "i" refers to the joint (i = 1, ···, 6) at which the velocity-induced torques (or forces) are "felt".

The expression given by Eq. (23) can be regarded as a component in a six-dimensional column vector denoted by $\vec{d}_V$:

$$\vec{d}_V = \frac{1}{2}\begin{bmatrix} \dot{\vec{q}}^T D_{1,V}\dot{\vec{q}} \\ \dot{\vec{q}}^T D_{2,V}\dot{\vec{q}} \\ \dot{\vec{q}}^T D_{3,V}\dot{\vec{q}} \\ \dot{\vec{q}}^T D_{4,V}\dot{\vec{q}} \\ \dot{\vec{q}}^T D_{5,V}\dot{\vec{q}} \\ \dot{\vec{q}}^T D_{6,V}\dot{\vec{q}} \end{bmatrix} \tag{24}$$

Let the torques T_1, T_2, T_4, T_5, T_6 and force F_3 applied at joints $i = 1, \cdots, 6$ be expressed by a six-dimensional column vector denoted by $\vec{d}_{TF}$:

$$\vec{d}_{TF} = \begin{bmatrix} T_1 \\ T_2 \\ F_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} \tag{25}$$

Then the six, coupled nonlinear differential equations, Eqs. (12) to (17), describing the dynamic behavior of the JPL RRP manipulator can be expressed by the following compact and structured vector equation:

$$\boxed{\vec{d}_{TF} = D_A\ddot{\vec{q}} + \vec{d}_V + \vec{d}_G} \tag{26}$$

where all the necessary functional and operational definitions are provided previously in this Section.

It is noted that some of the dynamic coefficients D_i, D_{ij} and D_{ijp} in Eqs. (12) to (17) are zero for different reasons, as the explicit coefficient evaluation will show it in the subsequent Sections. In general, some of the dynamic coefficients in a full scheme of manipulator equations of motion (like the scheme of Eqs. (12) to (17)) will be, or can be zero for the following physical reasons:

- The particular kinematic design of a manipulator can eliminate some dynamic coupling (D_{ij} and D_{ijp} coefficients) between joint motions.

- Some of the velocity-related dynamic coefficients have only dummy existence in the general scheme; that is, they are physically non-existent. (For instance, the centripetal force will not interact with the motion of that joint which generates it, that is, $D_{iii} \equiv 0$ always; however, it can interact with motions at the other joints in the chain, that is, we can have $D_{jii} \neq 0$.)*
- Due to particular variations in the link configuration during motion, some dynamic coefficients may attain zero values at particular instants of time.

The equations of manipulator motion given by Eqs. (12) through (17) are symbolic differential equations; they include all inertial, centripetal, Coriolis, and gravitational effects in symbolic form. (Symbolic in the sense that the D_i, D_{ij}, D_{ijp} coefficients are not specified explicitly.) In the subsequent sections the inertial (all D_{ii} and some D_{ij}) as well as the gravitational (D_i) coefficients will be explicitly specified and evaluated.

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*The relation between the general dynamic algorithm, Eq. (1), and the generally zero dynamic coefficients in the scheme of dynamic equations, Eqs. (12) to (17), are discussed in the following memo:

Lewis, R.A., "Some RRP Manipulator Dynamic Considerations Impacting Planning Program Implementation," JPL Guidance and Control Technical Memo 343-183, 13 March 1973.

Further, the simplifications of the general dynamic algorithm developed in Appendix D of this report explicitly show both the generally zero dynamic coefficients and the symmetries between some of the generally existing dynamic coefficients.

IV. RESTRICTED DYNAMIC MODELS

To perform dynamic behavior and control system analysis for the JPL RRP manipulator, functionally explicit expressions must be derived for the manipulator dynamic coefficients D_i, D_{ij}, and D_{ijp} defined in the previous Section. As seen from the defining equations for D_i, D_{ij}, and D_{ijp}, the derivation of functionally explicit expressions for the dynamic coefficients for a six-degree of-freedom manipulator is a tremendous task. Furthermore, the resulting expressions can be rather complicated so that the coefficient equations can easily get out of hand. Thus, to keep the analytic task manageable, some dynamically meaningful restrictions will be introduced into the general dynamic model of the JPL RRP manipulator. The different types and classes of restricted dynamic models are briefly described in the following subsection.

A. Alternative Model Restrictions

Active dynamic coupling between motions at different joints exists only when several links are moving relative to each other simultaneously. (Note that there is always a passive dynamic coupling between the motion at joint "i" and the non-moving joints, "felt" by the motor brake of the non-moving joints.) Thus, an obvious dynamic restriction for analytic purposes is to consider the motion only at one joint "i" at a given time so that the positions at the other five joints are kept fixed in a known configuration (representing a fixed, known load for joint motor "i") while there is a motion at joint "i".

Another meaningful dynamic model restriction for analytic purposes is to consider the simultaneous motion at a restricted number (a subgroup) of joints, while the positions at the other joints are kept fixed in a known configuration. In that case, the dynamic interaction only between moving links is of interest for analysis. Two dynamically important subgroups of joints can immediately be identified for the JPL RRP manipulator: the first three joints ($i = 1, 2, 3$), and the last three joints ($i = 4, 5, 6$).

Another important dynamic model restriction is to consider only the acceleration-related and gravity terms in the equations of motion. This restriction can meaningfully be combined with the subgroup restriction described above. It is noted, however, that, for general motions, the dynamic importance of the velocity-dependent terms in the equations of motion can only be evaluated by an explicit evaluation of the velocity-dependent dynamic coefficients D_{ijp}.

B. Applications of Dynamic Model Restrictions

Though the general motions for the JPL RRP manipulator are considered to consist of a coordinated, simultaneous motion of several or all joints, the dynamic model restrictions described in the previous subsection have important applications. First, they considerably contribute to an explicit insight into the dynamic behavior of the manipulator under different motion conditions. Second, they contribute to the development and design of a reliable and simple control system. Third, they can profitably be used to simulate or check out different elements and aspects of the manipulator control system in real time.

The main advantage gained by the dynamic model restrictions in the analysis is that the introduced simplifications are related to well-defined and controllable assumptions.

V. RESTRICTED DYNAMIC MODEL FOR THE FIRST THREE LINK-JOINT PAIRS

The first three links of the JPL RRP manipulator are called (see footnote, page 9)

Link 1: post
Link 2: shoulder
Link 3: boom

The associated joint variables are, respectively, θ_1, θ_2, r_3. The Cartesian reference frames fixed in the first three links are subscripted by 1, 2, 3. Figure 2 shows the actual link, reference frame, and joint variable relations. As described in general terms in footnote, page 9, the values of the two revolute joint variables (θ_1 and θ_2) and the linear (sliding) joint variables (r_3) are measured in the following sense:

θ_1 = the angular displacement of the X_1 axis relative to the X_0 axis, positive in the right hand sense about the Z_0 axis;

θ_2 = the angular displacement of the X_2 axis relative to the X_1 axis, positive in the right hand sense about the Z_1 axis;

r_3 = the linear displacement of the origin of the $X_3Y_3Z_3$ reference frame relative to the origin of the $X_2Y_2Z_2$ reference frame, measured along the Z_2 axis (always positive).

As seen in Fig. 2, the first three link-joint pairs constitute the main "arm-positioning" mechanism, and the associated three driving motors carry the heaviest loads. Thus, it is dynamically meaningful and important to consider the first three link-joint pairs by themselves as a subgroup, temporarily separated from the motions at the last three (wrist) joints.

The definition of "restricted dynamic model for the first three link-joint pairs" treated in this Section is the following:

- The last three (wrist) joints are at rest in a known configuration. (For instance, an analytically convenient, "known" configuration for the three wrist joints is the one seen in Fig. 2.)
- There can be simultaneous motion at the first three joints, while the wrist joints are at rest.

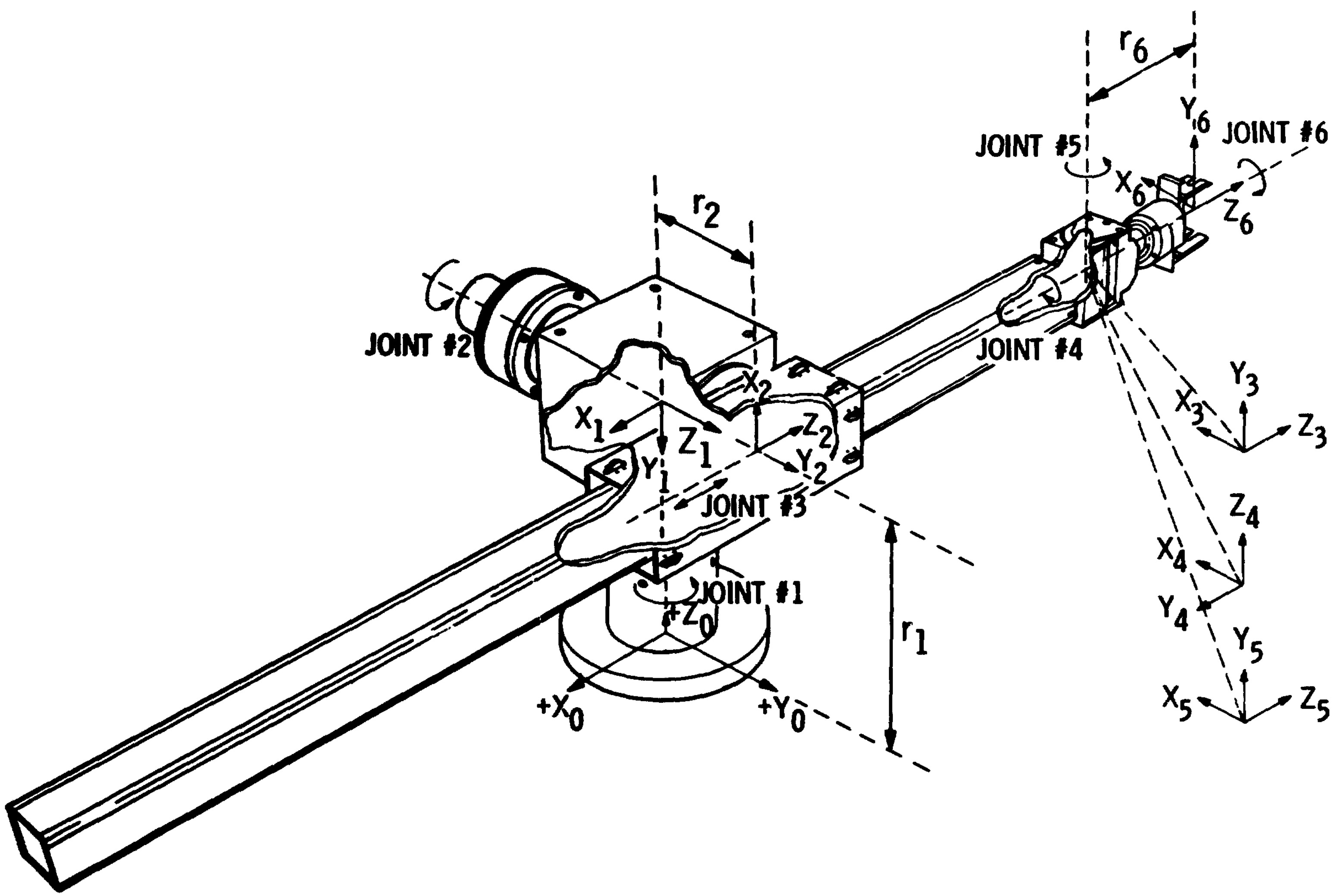

Figure 2. Reference Frames for Link-Joint Pairs of Arm

The restriction has the following dynamic meaning and significance:

- The last three (wrist) links, together with an object in the hand, form a constant (not time-varying!) load as seen by the first three joint motors.
- There will be active dynamic coupling between motions at the first three joints only.

The last point has the consequence that in the dynamic coefficient matrices D_A and $D_{i,V}$ (see Eqs. (19) and (21)) only the upper left 3 by 3 partitions are of interest, and the state (or time) variation of three gravity terms (D_1, D_2, D_3, — see Eq. (18) —) should only be considered.

In the "restricted dynamic model for the first three link-joint pairs" described above, the values of the mass center vector and "pseudo inertia matrix" for the first two links ($\bar{\rho}_1$, $\bar{\rho}_2$, J_1, J_2) given in Appendix B at the end of this memo are unchanged. The values of the mass center vector and "pseudo inertia matrix" for the third link ($\bar{\rho}_3$ and J_3) as given in Appendix B, however, should be modified according to the fixed configuration of the wrist structure. That is, the inertia properties of the wrist structure should be properly added to the values of $\bar{\rho}_3$ and J_3. For the configuration seen in Fig. 2 the modification is simple, since the wrist structure only represents a symmetric, straight extension of the boom. In the subsequent evaluation of restricted dynamic coefficients, the wrist structure configuration seen in Fig. 2 is assumed.

In this memo, only the gravity and acceleration-related terms are explicitly evaluated in the "restricted dynamic model for the first three link-joint pairs". The velocity-related terms will explicitly be evaluated in a subsequent memo.

To distinguish between dynamic coefficients belonging to the dynamic model restricted to motions at the first three joints, and those belonging to all joint motions, we introduce the following notation:

D_i^*, D_{ij}^*, D_{ijp}^* = for motions restricted to the first three joints;

D_i, D_{ij}, D_{ijp} = for motions at all joints.

In the subsequent equations, the "star" (*) distinction will also be used for the inertial parameters (related to link 3) which specifically belong to the restricted model. The following short notation will also be adopted in the equations for the dynamic coefficients:

$$\sin\theta_i \triangleq s\theta_i$$

$$\cos\theta_i \triangleq c\theta_i$$

$$\sin^2\theta_i \triangleq s^2\theta_i$$

$$\cos^2\theta_i \triangleq c^2\theta_i$$

The explicit matrix functions for the U_{ji} partial derivative matrices which are needed in the explicit evaluation of the dynamic coefficients are listed in Appendix A at the end of this memo.

A. Gravity Terms

In the explicit evaluation of the gravity terms it is assumed that the field of gravity is parallel to the Z_0 direction of the base coordinate frame, or in other words, the manipulator post stands gravitationally vertical. Thus, we will use the following value for the 1 by 4 gravity vector:

$$G = [0,\ 0,\ -g,\ 0] \tag{27}$$

where g = acceleration of gravity.

1. For joint #1:

From the defining equation we have:

$$D_1^* = m_1 G U_{11}\bar{\rho}_1 + m_2 G U_{21}\bar{\rho}_2 + m_3^* G U_{31}\bar{\rho}_3^* \tag{28}$$

The evaluation of Eq. (28) yields:

$$\boxed{D_1^* = 0} \tag{29}$$

Eq. (29) is immediately apparent by simple physical reasoning since, by assumption, the rotation axis of joint motor #1 is always parallel to the field of gravity, hence joint motor #1 cannot "feel" any gravity torque. This physical circumstance corresponds to zero values for the vectors GU_{11}, GU_{21} and GU_{31} in the defining formula, Eq. (28), since the third row of the U_{11}, U_{21} and U_{31} matrices is zero. (See Eqs. (A. 1, A. 2 and A. 3) in Appendix A.) Clearly, if the G vector would contain components other than $G_z = -g$, that is, if the manipulator post would be tilted relative to the local field of gravity, then D_1^* would be different from zero. This is easily seen also from the structure of the U_{11}, U_{21} and U_{31} matrices.

2. <u>For joint #2:</u>

From the defining equation we have:

$$D_2^* = m_2 GU_{22}\bar{\rho}_2 + m_3^* GU_{32}\bar{\rho}_3^* \tag{30}$$

The evaluation of Eq. (30) gives:

$$\boxed{D_2^* = g\left[m_2\bar{z}_2 + m_3^*\left(\bar{z}_3^* + r_3\right)\right] s\theta_2} \tag{31}$$

Eq. (31) is also apparent by simple physical reasoning.

It is noted that Eq. (31), expressing the gravity torque "felt" by the motor of joint #2, is already a <u>balanced</u> equation with respect to the sliding of the boom relative to the rotation axis of the motor of joint #2. The net ("balanced") value of the gravity torque acting on the motor of joint #2 is simply expressed in Eq. (31) by the term $(\bar{z}_3^* + r_3)$ since $\bar{z}_3^*$ is a (necessarily) <u>negative constant</u>, while r_3 is (necessarily) a <u>positive variable</u>.

3. <u>For joint #3:</u>

From the defining equation we have:

$$D_3^* = m_3^* GU_{33}\bar{\rho}_3^* \tag{32}$$

The evaluation of Eq. (32) yields:

$$\boxed{D_3^* = -m_3^* g c\theta_2} \tag{33}$$

which is again apparent by simple physical reasoning. Eq. (33) expresses the accelerating (or decelerating) effect of the gravity as a function of θ_2 felt by the motor which drives the boom.

B. Acceleration-Related Dynamic Coefficients

Due to the symmetry of the D_A matrix, only six acceleration-related dynamic coefficients should be evaluated for the dynamic model restricted to the first three link-joint pairs: three "diagonal", and three "off-diagonal" coefficients. The "diagonal" coefficients (D_{ii}^*) are related to the total inertia "felt" by the motor acting at joint "i", due to the motor's own acceleration. The "off-diagonal" coefficients (D_{ij}^*, $i \neq j$) are related to the dynamic interaction (reaction force or torque) caused by accelerations at joints "i" and "j". For instance, the term $D_{12}^* \ddot{\theta}_2$ expresses the reaction torque "felt" by the motor of joint #1 due to the acceleration $\ddot{\theta}_2$ at joint #2. It is noted that, because the symmetry $D_{ij}^* = D_{ji}^*$, the same D_{12}^* coefficient will appear in the term $D_{12}^* \ddot{\theta}_1$ which expresses the reaction torque felt by the motor of joint #2 due to the acceleration $\ddot{\theta}_1$ at joint #1.

1. Diagonal Coefficients D_{11}^*, D_{22}^*, D_{33}^*

From the defining equation we have:[†]

$$D_{11}^* = \mathrm{Tr}\left(U_{11} J_1 U_{11}^T\right) + \mathrm{Tr}\left(U_{21} J_2 U_{21}^T\right) + \mathrm{Tr}\left(U_{31} J_3^* U_{31}^T\right) \tag{34}$$

$$D_{22}^* = \mathrm{Tr}\left(U_{22} J_2 U_{22}^T\right) + \mathrm{Tr}\left(U_{32} J_3^* U_{32}^T\right) \tag{35}$$

$$D_{33}^* = \mathrm{Tr}\left(U_{33} J_3^* U_{33}^T\right) \tag{35.a}$$

[†] Here and in subsequent equations in this memo the "Trace" operator will be abbreviated by "Tr".

After considerable algebra and trigonometric simplifications, the following explicit expressions are obtained from Eqs. (34), (35) and (35.a):

$$\boxed{\begin{aligned} D^*_{11} &= m_1 k^2_{122} \\ &+ m_2\left[k^2_{211}s^2\theta_2 + k^2_{233}c^2\theta_2 + r_2\left(2\bar{y}_2 + r_2\right)\right] \\ &+ m^*_3\left[k^{*2}_{322}s^2\theta_2 + k^{*2}_{333}c^2\theta_2 + r_3 s^2\theta_2\left(2\bar{z}^*_3 + r_3\right) + r^2_2\right]\end{aligned}} \tag{36}$$

$$\boxed{D^*_{22} = m_2 k^2_{222} + m^*_3\left[k^{*2}_{311} + r_3\left(2\bar{z}^*_3 + r_3\right)\right]} \tag{37}$$

$$\boxed{D^*_{33} = m^*_3} \tag{38}$$

Dealing with linear motion at joint #3, Eq. (38) is immediately obvious. The physical meaning of Eqs. (36) and (37) is also clear by interpreting the components step by step.

By examining the explicit expressions for D^*_{11}, D^*_{22} and D^*_{33} given by Eqs. (36) to (38), the following general notes should be made:

- D^*_{11} is a function of some inertial properties of m_1, m_2, m^*_3 and the variations in θ_2 and r_3. (It is obviously independent of the variation in θ_1.) Furthermore, the constant displacement parameter r_2 also contributes to the value of D^*_{11}.
- D^*_{22} is a function of some inertial properties of m_2, m^*_3 and the variations in r_3. (It is obviously independent of the variations in θ_1 and θ_2.)
- In general, D^*_{ii} can be a function of the inertial properties of masses starting with m_i and ending at the mass in the hand, and can be a function of variations in joint variables starting at joint i + 1 and ending at the last joint at the hand.

The actual dynamical (or load) significance of the different components in Eqs. (36) and (37) can only be determined if the numerical values of the different inertial parameters involved in Eqs. (36) and (37) are known.

2. Off-Diagonal Coefficients D_{12}^*, D_{13}^*, D_{23}^*

From the defining equation we have:

$$D_{12}^* = \mathrm{Tr}\left(U_{22}J_2U_{21}^T\right) + \mathrm{Tr}\left(U_{32}J_3^*U_{31}^T\right) \tag{39}$$

$$D_{13}^* = \mathrm{Tr}\left(U_{33}J_3^*U_{31}^T\right) \tag{40}$$

$$D_{23}^* = \mathrm{Tr}\left(U_{33}J_3^*U_{32}^T\right) \tag{41}$$

After some algebra and trigonometric simplifications we find the following explicit expressions from Eqs. (39) to (41):

$$\boxed{D_{12}^* = -\left[m_2\bar{z}_2r_2 + m_3^*r_2\left(\bar{z}_3^* + r_3\right)\right] c\theta_2} \tag{42}$$

$$\boxed{D_{13}^* = -m_3^*r_2s\theta_2} \tag{43}$$

$$\boxed{D_{23}^* = 0} \tag{44}$$

The physical meaning of the expressions given by Eqs. (42) to (44) has been explained previously. Again, the actual dynamic (or load) significance of the D_{12}^* and D_{13}^* forms can only be evaluated if the numerical values of the pertinent inertial parameters are known.

VI. COMPLETE DYNAMIC COEFFICIENTS FOR ALL SIX LINK-JOINT PAIRS

In this section we consider the JPL RRP manipulator in an unrestricted state of motion compatible with structural, power, performance, instrumentation, work space, and other possible constraints. It is now assumed that several or all six links can move (or will move) relative to each other simultaneously when the manipulator performs a given task. In other words, we consider now the dynamic coefficients in the manipulator equations of motion as functions of all possible manipulator motions. This amounts to specifying the complete state functions for the dynamic coefficients in explicit terms. The complete state functions for the dynamic coefficients relate the values of each individual coefficient in explicit function terms to all pertinent link inertia characteristics and geometric parameters, as well as to all possible configuration of the manipulator (that is, to all possible variations in all pertinent joint variables).

In this memo, the complete state functions will be evaluated only for the following dynamic coefficients: the six gravity terms in Eq. (18), and the six diagonal elements of the D_A matrix in Eq. (19); that is, the six acceleration-related uncoupled terms in the dynamic equations. The off-diagonal acceleration-related coefficients, as well as the velocity-related coefficients will be treated in subsequent memos.

The explicit matrix functions for the U_{ji} partial derivative matrices which are needed in the explicit evaluation of the dynamic coefficients are listed in Appendix A, while the six "pseudo inertia matrices" are listed in Appendix B at the end of this memo. The trigonometric short notations specified in the previous section will also be used in this Section. Additional short notations applied in this Section are:

$$\sin(\theta_i + \theta_j) \triangleq s(\theta_i + \theta_j)$$

$$\cos(\theta_i + \theta_j) \triangleq c(\theta_i + \theta_j)$$

$$\sin^2(\theta_i + \theta_j) \triangleq s^2(\theta_i + \theta_j)$$

$$\cos^2(\theta_i + \theta_j) \triangleq c^2(\theta_i + \theta_j)$$

As in the previous Section, reference should be made to Fig. 2 which shows the actual link, reference frame and joint variable relations. The sense of measurement for the θ_1, θ_2 and r_3 variables has been specified in the previous Section. As described in general terms in Ref. 1, the last three revolute (wrist) joint variables, θ_4, θ_5, and θ_6, are measured in the following sense:

θ_4 = the angular displacement of the X_4 axis relative to the X_3 axis, positive in the right hand sense about the Z_3 axis.

θ_5 = the angular displacement of the X_5 axis relative to the X_4 axis, positive in the right hand sense about the Z_4 axis.

θ_6 = the angular displacement of the X_6 axis relative to the X_5 axis, positive in the right hand sense about the Z_5 axis.

A. Gravity Terms in Complete Form

As in the previous Section, it is assumed again that the manipulator post stands gravitiationally vertical. That is, Eq. (27) is used for the 1 by 4 gravity vector G.

1. For joint #1:

From the defining equation we have:

$$D_1 = m_1 GU_{11}\bar{\rho}_1 + m_2 GU_{21}\bar{\rho}_2 + m_3 GU_{31}\bar{\rho}_3$$

$$+ m_4 GU_{41}\bar{\rho}_4 + m_5 GU_{51}\bar{\rho}_5 + m_6 GU_{61}\bar{\rho}_6 \qquad (45)$$

The evaluation of Eq. (45) gives

$$\boxed{D_1 = 0} \qquad (46)$$

Eq. (46) is immediately obvious for the same reason as outlined in connection with $D_1^* = 0$, Eq. (29), in the previous Section. The additional remarks made there are also valid here. Eq. (46) simply means that the motor of joint #1 cannot feel any gravity torque.

2. <u>For joint #2</u>:

From the defining equation we have:

$$D_2 = m_2 GU_{22}\bar{\rho}_2 + m_3 GU_{32}\bar{\rho}_3 + m_4 GU_{42}\bar{\rho}_4 + m_5 GU_{52}\bar{\rho}_5 + m_6 GU_{62}\bar{\rho}_6 \tag{47}$$

The evaluation of Eq. (47) yields:

$$\boxed{\begin{aligned} D_2 = g \Big\{ & m_2 \bar{z}_2 s\theta_2 \\ & + m_3 (\bar{z}_3 + r_3) s\theta_2 \\ & + m_4 (r_3 s\theta_2 - \bar{y}_4 s\theta_2 + \bar{z}_4 c\theta_2 c\theta_4) \\ & + m_5 \left[\bar{z}_5 (s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5) + r_3 s\theta_2\right] \\ & + m_6 \Big[(\bar{z}_6 + r_6)(c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5) \\ & \quad + r_3 s\theta_2\Big]\Big\} \end{aligned}} \tag{48}$$

Comparing Eq. (48) to the corresponding expression for the restricted model, Eq. (31), it is seen that changes in the wrist configuration (that is, variations of the wrist joint variables θ_4, θ_5 and θ_6) produce a gravity torque effect felt by the motor of joint #2 in a functionally complicated form. It is noted that Eq. (48) is already a <u>balanced</u> equation with respect to the sliding of the boom relative to the rotation axis of the motor of joint #2. This is true for the same reason as outlined in connection with Eq. (31) in the previous Section.

3. <u>For joint #3:</u>

From the defining equation we have:

$$D_3 = m_3 GU_{33}\bar{\rho}_3 + m_4 GU_{43}\bar{\rho}_4 + m_5 GU_{53}\bar{\rho}_5 + m_6 GU_{63}\bar{\rho}_6 \tag{49}$$

Evaluation of Eq. (49) yields:

$$\boxed{D_3 = -g(m_3 + m_4 + m_5 + m_6)c\theta_2} \tag{50}$$

which is physically apparent. Eq. (50) expresses the accelerating (or decelerating) effect of the gravity force as a function of θ_2, felt by the motor which drives the boom. Eq. (50) is completely equivalent to Eq. (33) since, in fact, $m_3^* = m_3 + m_4 + m_5 + m_6$.

4. <u>For joint #4:</u>

From the defining equation we have:

$$D_4 = m_4 GU_{44}\bar{\rho}_4 + m_5 GU_{54}\bar{\rho}_5 + m_6 GU_{64}\bar{\rho}_6 \tag{51}$$

Evaluation of Eq. (51) yields:

$$\boxed{D_4 = g\left[-m_4\bar{z}_4 s\theta_4 + m_5\bar{z}_5 c\theta_4 s\theta_5 + m_6(\bar{z}_6 + r_6)c\theta_4 s\theta_5\right] s\theta_2} \tag{52}$$

Eq. (52) expresses the gravity torque felt by the motor of joint #4 as a function of the variations in the joint angles θ_2, θ_4, and θ_5.

5. <u>For joint #5:</u>

From the defining equation we have:

$$D_5 = m_5 GU_{55}\bar{\rho}_5 + m_6 GU_{65}\bar{\rho}_6 \tag{53}$$

Evaluation of Eq. (53) gives:

$$\boxed{D_5 = g\left[m_5\bar{z}_5 + m_6(\bar{z}_6 + r_6)\right](s\theta_2 s\theta_4 c\theta_5 + c\theta_2 s\theta_5)} \tag{54}$$

Eq. (54) relates the gravity torque felt by the motor of joint #5 to the variations of the joint angles θ_2, θ_4, and θ_5.

6. For joint #6:

From the defining equation we have:

$$D_6 = m_6 G U_{66} \bar{p}_6 \tag{55}$$

The evaluation of Eq. (55) yields:

$$\boxed{D_6 = 0} \tag{56}$$

Eq. (56) is physically apparent, since the center of mass of link #6 is along the z_6 axis which is the rotation axis of the motor rotating link #6. It is noted that the hand is inertially part of link #6.

B. Acceleration-Related Uncoupled Terms in Complete Form

The $D_{ii}\ddot{q}_i$ type terms in the dynamic equations, Eqs. (12) to (17), are called in this memo the "acceleration-related uncoupled terms". The notion "uncoupled" is meant to signify that the inertia load felt by the motor of joint "i" is being dynamically generated by the acceleration of the same joint "i" (and not by the acceleration of some other joint "j"). The dynamic coefficients D_{ii} belonging to the "acceleration-related uncoupled terms" are the six diagonal elements of the D_A matrix given by Eq. (19). Thus, the dynamic coefficients D_{ii} are related to the total inertia felt by the motor acting at joint "i", due to the acceleration of the same joint.

In this subsection the <u>complete state functions</u> for all six D_{ii} dynamic coefficients will be evaluated. The complete state function specifies the value of D_{ii} in terms of all pertinent inertial and geometric parameters of all six links, as well as in terms of all six pertinent joint variables. Since the manipulator dynamic model is now <u>not</u> restricted to the first three link-joint pairs, it can be expected that the resulting expressions for D_{ii} will be considerably more complicated than the state functions for D^{*}_{ii} treated in the previous section. It is reminded that the values of D^{*}_{ii} are restricted to variations in the first three joint variables only.

1. <u>For joint #1</u>:

From the defining equation we have:

$$D_{11} = \mathrm{Tr}\left(U_{11}J_1U_{11}^T\right) + \mathrm{Tr}\left(U_{21}J_2U_{21}^T\right) + \mathrm{Tr}\left(U_{31}J_3U_{31}^T\right)$$

$$+ \mathrm{Tr}\left(U_{41}J_4U_{41}^T\right) + \mathrm{Tr}\left(U_{51}J_5U_{51}^T\right) + \mathrm{Tr}\left(U_{61}J_6U_{61}^T\right) \tag{57}$$

After lengthy algebra and trigonometric simplifications the evaluation of Eq. (57) yields the following state function for D_{11}:

$$
\begin{aligned}
D_{11} = {} & m_1 k_{122}^2 \\
& + m_2 \left[k_{211}^2 s^2\theta_2 + k_{233}^2 c^2\theta_2 + r_2(2\bar{y}_2 + r_2) \right] \\
& + m_3 \left[k_{322}^2 s^2\theta_2 + k_{333}^2 c^2\theta_2 + r_3(2\bar{z}_3 + r_3) s^2\theta_2 + r_2^2 \right] \\
& + m_4 \Big\{ \tfrac{1}{2} k_{411}^2 \left[s^2\theta_2 (2s^2\theta_4 - 1) + s^2\theta_4 \right] + \tfrac{1}{2} k_{422}^2 (1 + c^2\theta_2 + s^2\theta_4) \\
& \qquad + \tfrac{1}{2} k_{433}^2 \left[s^2\theta_2 (1 - 2s^2\theta_4) - s^2\theta_4 \right] + r_3^2 s^2\theta_2 + r_2^2 - 2\bar{y}_4 r_3 s^2\theta_2 + 2\bar{z}_4 (r_2 s\theta_4 + r_3 s\theta_2 c\theta_2 c\theta_4) \Big\} \\
& + m_5 \Big\{ \tfrac{1}{2} (-k_{511}^2 + k_{522}^2 + k_{533}^2) \left[(s\theta_2 s\theta_5 - c\theta_2 s\theta_4 c\theta_5)^2 + c^2\theta_4 c^2\theta_5 \right] \\
& \qquad + \tfrac{1}{2} (k_{511}^2 - k_{522}^2 + k_{533}^2)(s^2\theta_4 + c^2\theta_2 c^2\theta_4) \\
& \qquad + \tfrac{1}{2} (k_{511}^2 + k_{522}^2 - k_{533}^2) \left[(s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5)^2 + c^2\theta_4 s^2\theta_5 \right] + r_3^2 s^2\theta_2 + r_2^2 \\
& \qquad + 2\bar{z}_5 \left[r_3 (s^2\theta_2 c\theta_5 + s\theta_2 s\theta_4 c\theta_4 s\theta_5) - r_2 c\theta_4 s\theta_5 \right] \Big\} \\
& + m_6 \Big\{ \tfrac{1}{2} (-k_{611}^2 + k_{622}^2 + k_{633}^2) \left[(s\theta_2 s\theta_5 c\theta_6 - c\theta_2 s\theta_4 c\theta_5 c\theta_6 - c\theta_2 c\theta_4 s\theta_6)^2 + (c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6)^2 \right] \\
& \qquad + \tfrac{1}{2} (k_{611}^2 - k_{622}^2 + k_{633}^2) \left[(c\theta_2 s\theta_4 c\theta_5 s\theta_6 - s\theta_2 s\theta_5 s\theta_6 - c\theta_2 c\theta_4 c\theta_6)^2 + (c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6)^2 \right] \\
& \qquad + \tfrac{1}{2} (k_{611}^2 + k_{622}^2 - k_{633}^2) \left[(c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5)^2 + c^2\theta_4 s^2\theta_5 \right] \\
& \qquad + \left[r_6 c\theta_2 s\theta_4 s\theta_5 + (r_6 c\theta_5 + r_3) s\theta_2 \right]^2 + (r_6 c\theta_4 s\theta_5 - r_2)^2 \\
& \qquad + 2\bar{z}_6 \Big[r_6 (s^2\theta_2 c^2\theta_5 + c^2\theta_4 s^2\theta_5 + c^2\theta_2 s^2\theta_4 s^2\theta_5 + 2s\theta_2 c\theta_2 s\theta_4 s\theta_5 c\theta_5) \\
& \qquad\qquad + r_3 (s\theta_2 c\theta_2 s\theta_4 s\theta_5 + s^2\theta_2 c\theta_5) - r_2 c\theta_4 s\theta_5 \Big] \Big\}
\end{aligned}
\tag{58}
$$

The first three terms in Eq. (58) are identical to the terms for D_{11}^* given by Eq. (36), except that in Eq. (58) the "star" (*) is removed from m_3, k_{322}^2, k_{333}^2, and $\bar{z}_3$. These four "unstarred" values in Eq. (58) refer to the third link only. The last three, long, and complex terms in Eq. (58) – that is, those multiplied by m_4, m_5, and m_6 – account for the configurational inertial effects of the motion of the three wrist joints, joints #4, #5, and #6, as "seen" by the $D_{11}\ddot{\theta}_1$ term in the dynamic equation for joint #1, Eq. (12). Alternatively, if the configuration of the wrist joints is fixed during motion of joint #1, then the m_4, m_5, m_6 terms in Eq. (58) all together represent only one compounded, constant inertia number belonging to the particular, fixed configuration of the wrist joints. This constant inertia number can be used for D_{11}^* to replace the "starred" (*) values of m_3, k_{322}^2, k_{333}^2, and $\bar{z}_3$ in Eq. (36) simply by adding this constant number to Eq. (36); in that case, the "unstarred" m_3, k_{322}^2, k_{333}^2, and $\bar{z}_3$ values in Eq. (36) refer to the third link only.

It is seen from Eq. (58) that the configurational inertial effect of the different links as "felt" at joint #1 becomes more and more complex as we move toward the free end (the hand) of the chain of links. The most complex configurational inertial contribution comes from the last (#6) link.

It is noted that further trignometric simplifications would be possible for the m_4, m_5, m_6 terms in Eq. (58). The simplifications are not carried out, however, since they do not seem to illustrate major physical points.

2. For joint #2:

From the defining equation we have:

$$D_{22} = \mathrm{Tr}\left(U_{22}J_2U_{22}^T\right) + \mathrm{Tr}\left(U_{32}J_3U_{32}^T\right) + \mathrm{Tr}\left(U_{42}J_4U_{42}^T\right) + \mathrm{Tr}\left(U_{52}J_5U_{52}^T\right) + \mathrm{Tr}\left(U_{62}J_6U_{62}^T\right) \tag{59}$$

Again, after lengthy algebra and trigonometric manipulations, the evaluation of Eq. (59) gives the following state function for D_{22}:

$$
\boxed{
\begin{aligned}
D_{22} = {} & m_2 k_{222}^2 \\
& + m_3 \left[k_{311}^2 + r_3(2\bar{z}_3 + r_3)\right] \\
& + m_4 \left[k_{411}^2 c^2\theta_4 + k_{433}^2 s^2\theta_4 + r_3(r_3 - 2\bar{y}_4)\right] \\
& + m_5 \left[k_{511}^2 c^2\theta_4 c^2\theta_5 + k_{522}^2 s^2\theta_4 + k_{533}^2 c^2\theta_4 s^2\theta_5 + r_3(r_3 + 2\bar{z}_5 c\theta_5)\right] \\
& + m_6 \Bigg\{ \frac{1}{2} k_{611}^2 \Big[(s^2\theta_6 - c^2\theta_6)(s^2\theta_4 c^2\theta_5 + s^2\theta_5 - c^2\theta_4) \\
& \qquad + c\theta_5(c\theta_5 - 4s\theta_4 c\theta_4 s\theta_6 c\theta_6) + s^2\theta_4 s^2\theta_5\Big] \\
& \quad + \frac{1}{2} k_{622}^2 \Big[(c^2\theta_6 - s^2\theta_6)(s^2\theta_4 c^2\theta_5 + s^2\theta_5 - c^2\theta_4) \\
& \qquad + c\theta_5(c\theta_5 + 4s\theta_4 c\theta_4 s\theta_6 c\theta_6) + s^2\theta_4 s^2\theta_5\Big] \\
& \quad + k_{633}^2 c^2\theta_4 s^2\theta_5 + \left[r_3(2r_6 c\theta_5 + r_3) + r_6^2(s^2\theta_4 s^2\theta_5 + c^2\theta_5)\right] \\
& \quad + 2\bar{z}_6 \left[(r_3 + r_6 c\theta_5)c\theta_5 + r_6 s^2\theta_4 s^2\theta_5\right] \Bigg\}
\end{aligned}
}
\tag{60}
$$

The first two terms in Eq. (60) are identical to the terms for D_{22}^* given by Eq. (37), except that in Eq. (60) the "star" (*) is removed from m_3, k_{311}^2, and $\bar{z}_3$. These three "unstarred" values in Eq. (60) refer to the third link only. The terms with m_4, m_5, and m_6 in Eq. (60) account for the configurational

inertial effects of the motion of the three wrist joints (joints #4, #5, #6) as seen by the $D_{22}\ddot{\theta}_2$ term in the dynamic equation for joint #2, Eq. (13). Alternatively, if the configuration of the wrist joints is fixed during motion of joint #2, then the m_4, m_5, m_6 terms in Eq. (60) all together represent only one compounded, constant inertia number belonging to the particular, fixed configuration of the wrist joints. This constant inertia number can be used for D_{22}^* to replace the "starred" (*) values of m_3, k_{311}^2, and $\bar{z}_3$ in Eq. (37) simply by adding this constant number to Eq. (37); in that case, the "unstarred" m_3, k_{311}^2, and $\bar{z}_3$ values in Eq. (37) refer to the third link only.

It can be noted again that the configurational inertial effect of the different links as "felt" at joint #2 becomes more and more complex as we move toward the free end (the hand) of the chain of links. The most complex configurational inertial contribution comes from the last (#6) link. Comparing the m_4, m_5, and m_6 terms of Eq. (60) to those of Eq. (58), it is seen, however, that joint #2 "feels" the configurational inertial effect of the three wrist links through terms which are "simpler" than the corresponding terms of joint #1.

3. For joint #3:

From the defining equation we have:

$$D_{33} = \mathrm{Tr}\left(U_{33}J_3U_{33}^T\right) + \mathrm{Tr}\left(U_{43}J_4U_{43}^T\right) + \mathrm{Tr}\left(U_{53}J_5U_{53}^T\right) + \mathrm{Tr}\left(U_{63}JU_{63}^T\right) \tag{61}$$

which gives

$$\boxed{D_{33} = m_3 + m_4 + m_5 + m_6} \tag{62}$$

Dealing with linear motion at joint #3, Eq. (62) is immediately obvious. (In fact, it can be written down without going through the transformations indicated by Eq. (61).) It can also be noted that Eq. (62) is completely identical to the expression for D_{33}^* given by Eq. (38), since, in fact, $m_3^* = m_3 + m_4 + m_5 + m_6$.

In other words, D_{33} is independent of any arm link configuration, it is a constant. The value of D_{33} will only be changed when the hand grasps an object. In that case, the mass of the object should simply be added to Eq. (62), or more precisely, to the value of m_6.

4. For joint #4:

From the defining equation we have:†

$$D_{44} = \mathrm{Tr}\left(U_{44}J_4U_{44}^T\right) + \mathrm{Tr}\left(U_{54}J_5U_{54}^T\right) + \mathrm{Tr}\left(U_{64}J_6U_{64}^T\right) \tag{63}$$

Evaluating Eq. (63) results in the following function:

$$\boxed{\begin{aligned} D_{44} = {} & m_4k_{422}^2 \\ & + m_5(k_{511}^2s^2\theta_5 + k_{533}^2c^2\theta_5) \\ & + m_6\left[k_{611}^2s^2\theta_5c^2\theta_6 + k_{622}^2s^2\theta_5s^2\theta_6 + k_{633}^2c^2\theta_5 \right. \\ & \left. \quad + r_6(2\bar{z}_6 + r_6)s^2\theta_5)\right] \end{aligned}} \tag{64}$$

Eq. (64) is physically apparent, and can easily be interpreted term by term. The similarities and dissimilarities of Eqs. (64) and (36) are also noteworthy.

5. For joint #5:

From the defining equation we have:†

$$D_{55} = \mathrm{Tr}\left(U_{55}J_5U_{55}^T\right) + \mathrm{Tr}\left(U_{65}J_6U_{65}^T\right) \tag{65}$$

†See also remark at the end of this section.

which yields the following explicit state function:

$$\boxed{\begin{aligned} D_{55} &= m_5 k_{522}^2 \\ &\quad + m_6\left[k_{611}^2 c^2\theta_5 + k_{622}^2 s^2\theta_5 + r_6(2\bar{z}_6 + r_6)\right] \end{aligned}} \tag{66}$$

Eq. (66) is also apparent physically, and can easily be interpreted term by term. The similarities and dissimilarities of Eqs. (37) and (66) are again noteworthy.

6. <u>For joint #6:</u>

From the defining equation we have:†

$$D_{66} = \mathrm{Tr}\left(U_{66} J_6 U_{66}^T\right) \tag{67}$$

which gives

$$\boxed{D_{66} = m_6 k_{633}^2} \tag{68}$$

Eq. (68) is obvious. In fact, it can be written down immediately without going through the formal transformation indicated by Eq. (67).

<u>Remark</u>

The formal definitions for D_{44}, D_{55}, and D_{66} given by Eqs. (63), (65) and (67) involve a great deal of unnecessary computations. By noting that D_{44} only depends on the inertias of links #4, #5, and #6, while D_{55} only depends on the

†See also remark at the end of this section.

inertias of links #5, and #6, the following computationally more convenient definitions can be (and have been) used for D_{44} and D_{55}:

$$D_{44} = \mathrm{Tr}\left[\left(QT_3^4\right)J_4\left(QT_3^4\right)^T\right] + \mathrm{Tr}\left[\left(QT_3^4T_4^5\right)J_5\left(QT_3^4T_4^5\right)^T\right]$$

$$+ \mathrm{Tr}\left[\left(QT_3^4T_4^5T_5^6\right)J_6\left(QT_3^4T_4^5T_5^6\right)^T\right]$$

$$D_{55} = \mathrm{Tr}\left[\left(QT_4^5\right)J_5\left(QT_4^5\right)^T\right] + \mathrm{Tr}\left[\left(QT_4^5T_5^6\right)J_6\left(QT_4^5T_5^6\right)^T\right]$$

In a similar manner, we can also write for D_{66}:

$$D_{66} = \mathrm{Tr}\left[\left(QT_5^6\right)J_6\left(QT_5^6\right)^T\right]$$

But even the evaluation of this last expression for D_{66} is unnecessary since Eq. (68) can be written down immediately.

The mathematical derivation of the simplifications introduced here in the general algorithmic definition of D_{ii} is treated in detail for all manipulator dynamic coefficients in Appendix D at the end of the report.

VII. CONCLUSIONS

Values of total link inertias and the different torque/force components acting at the manipulator joint drives are essential parameters for manipulator control system design. It is seen from the dynamic equations derived for the JPL RRP manipulator that there is no simple proportionality between torque (or force) acting at one joint and the acceleration of the same joint when several joints are in motion simultaneously. Even if only one joint moves at a given time, the proportionality between torque and acceleration is a complex function of the actual configuration of all links ahead of the moving joint, that is, of all links between the moving joint and the hand, including any load in the hand.

In the case of simultaneous motion of several arm joints, the torque (or force) acting at each joint is the sum of a number of dynamic components which can be classified into four groups: (a) inertial acceleration of the joint; (b) reaction torques or forces due to acceleration at other joints; (c) velocity-related (centrifugal and Coriolis) reaction torques or forces; and (d) gravity terms. Obviously, the gravity terms are only dependent on the relative position of the links, while all other dynamic components are dependent on both the configuration and the dynamic state (relative acceleration and velocity) of the links.

The explicit state equations derived in this memo for some of the dynamic coefficients of the JPL RRP manipulator allow important quantitative conclusions regarding variations in total link inertias and gravity loads as seen at the different joint drive motors during arm motion. Further, the explicit state equations of the inertial (diagonal) and acceleration-related reaction (off-diagonal) dynamic coefficients derived for a restricted manipulator dynamic model allow a general quantitative evaluation of the relative importance of some of the acceleration-related torque/force reaction components versus inertial torques/forces.

The constant geometric and inertial parameters for the JPL RRP manipulator used in the subsequent evaluation are identical to those determined and compiled elsewhere.* (Parameter values are also listed in Appendix B.)

*Walker, B., "RRP Manipulator Inertial and Mass Distribution Characteristics," JPL IOM 343-4-73-142, 28 February 1973.

Dobrotin, B.M., "Input Shaft Inertias for RRP Manipulator," JPL IOM 343-4-73-268, 13 April 1973.

A. Variations in Total Inertia at the Joints

The total link inertias as seen at the different joints are given by the diagonal elements D_{ii} of the D_A matrix defined by Eq. (19). The complete state equations for the D_{ii} dynamic coefficients are given by Eqs. (58), (60), (62), (64), and (68). These equations only refer to the mechanical structure of the manipulator. Changes in these equations due to a load held in the hand are elaborated in Appendix C.

In the subsequent evaluation of variations in total inertias we compute the maximum variation in D_{ii} with and without load in the hand as well as the minimum value of D_{ii} without load in the hand. The assumed load is a 1.8 kg, 442 cm^3 cube, placed with its mass center at the origo of the hand (X_6, Y_6, Z_6) coordinate frame. In the computations, the constant and varying* components of D_{ii} are treated separately. All computed values are referred to the output at the respective joints, including the input inertias at the joint drives.

As seen from Eqs. (58), (60), and (64), the variations in D_{11}, D_{22}, D_{44} are functions of several joint variables. Thus, an analytic search for maximum values of D_{11}, D_{22}, D_{44} would imply the determination of hill tops of surfaces or hypersurfaces. Instead of this mathematical technique, we apply physical reasoning and select an appropriate (and allowed) set of joint variables which will yield the searched maximum value for D_{ii}.

1. Inertia Variations Seen at Joint #1

The value of D_{11} given by Eq. (58) specifies the variations in the total inertia felt at joint #1 as a function of the joint position vector $\bar{q}$.

a) Constant components of D_{11}:

Input inertia at joint No. 1	=	0.953 $kg \cdot m^2$
$+ m_1 k_{122}^2$	=	0.255 $kg \cdot m^2$
$+ 2m_2 y_2 r_2$	=	-0.192 $kg \cdot m^2$
$+ (m_2 + m_3 + m_4 + m_5) r_2^2$	=	0.320 $kg \cdot m^2$
Total constant	=	1.318 $kg \cdot m^2$

*Variations due to changes in both link motion and load held in the hand.

b) Maximum variations in D_{11}:

Assume: $\theta_2 = \pm 90$ deg (horizontal orientation of the boom)

$r_3 = 111.76$ cm (maximum extension of the boom)

$\theta_4 = 0$ deg (see Fig. 2)

$\theta_5 = 0$ deg (see Fig. 2)

$\theta_6 = 0$ deg (see Fig. 2)

The last three conditions will move the mass of the wrist/hand mechanism farthest from joint axis #1. Under the five conditions specified above, Eq. (58) gives the following components for D_{11} in addition to the constant components.

1) With no load in the hand:

$$m_2 k_{211}^2 = 0.108 \text{ kg}\cdot\text{m}^2$$

$$+ m_3 k_{322}^2 = 2.51 \text{ kg}\cdot\text{m}^2$$

$$+ m_3 r_3 (2\bar{z}_3 + r_3) = -0.815 \text{ kg}\cdot\text{m}^2$$

$$+ 1/2\, m_4(- k_{411}^2 + k_{422}^2 + k_{433}^2) = 0.0002 \text{ kg}\cdot\text{m}^2$$

$$+ m_4 r_3 (r_3 - 2\bar{y}_4) = 1.332 \text{ kg}\cdot\text{m}^2$$

$$+ m_5 k_{522}^2 = 0.003 \text{ kg}\cdot\text{m}^2$$

$$+ m_5 r_3 (2\bar{z}_5 + r_3) = 0.87 \text{ kg}\cdot\text{m}^2$$

$$+ m_6 k_{622}^2 = 0.005 \text{ kg}\cdot\text{m}^2$$

$$+ m_6 r_2^2 = 0.013 \text{ kg}\cdot\text{m}^2$$

$$+ m_6 (r_6 + r_3)^2 \qquad = \qquad 0.961 \text{ kg}\cdot\text{m}^2$$

$$+ 2m_6\bar{z}_6 (r_3 + r_6) \qquad = \qquad -0.13 \text{ kg}\cdot\text{m}^2$$

Total maximum inertia addition, with no load in the hand $= 4.857 \text{ kg}\cdot\text{m}^2$

Hence, the maximum total value of inertia felt at joint #1 with no load in the hand is:

$$\boxed{\begin{aligned} D_{11,\ max} \text{ (no load in the hand)} &= 1.318 + 4.857 \\ &= 6.176 \text{ kg}\cdot\text{m}^2 \end{aligned}} \qquad (69)$$

2) With load in the hand:

Only the m_6-related terms will be changed. According to the specifications of the load and the load's emplacement in the hand, we will have the following new values for the m_6-related terms:

$$m_6 k_{622}^2 \qquad = \qquad 0.006 \text{ kg}\cdot\text{m}^2$$

$$+ m_6(r_6 + r_3)^2 \qquad = \qquad 4.307 \text{ kg}\cdot\text{m}^2$$

$$+ m_6 r_2^2 \qquad = \qquad 0.061 \text{ kg}\cdot\text{m}^2$$

$$\text{Total} \qquad = \qquad 4.374 \text{ kg}\cdot\text{m}^2$$

It is noted that the $2m_6\bar{z}_6(r_3 + r_6)$ term remains numerically unchanged. Thus, the net maximum inertia change due to the specified load in the hand becomes:

$$4.374 - 0.979 = 3.395 \text{ kg}\cdot\text{m}^2$$

Hence, the maximum total value of inertia felt at joint #1 with the specified load in the hand is:

$$\boxed{\begin{aligned} D_{11,\,max}\text{(with load in the hand)} &= 1.318 + 4.857 + 3.395 \\ &= 9.57 \quad kg \cdot m^2 \end{aligned}} \tag{70}$$

c) Minimum value of D_{11}:

Assume: $\theta_2 = 0$ deg (vertical orientation of the boom)

$\theta_4 = 0$ deg (see Fig. 2)

$\theta_5 = 90$ deg (see Fig. 2)

$\theta_6 = 0$ deg (see Fig. 2)

It is noted that the condition $\theta_2 = 0$ deg will make D_{11} independent of r_3. Further, the condition $\theta_5 = 90$ deg will move the mass of the wrist/hand mechanism closest to joint axis #1. Under the four conditions specified above, Eq. (53) yields the following components for D_{11} in addition to the constant components:

$$\begin{aligned}
& m_2k_{233}^2 && = 0.1 \quad kg \cdot m^2 \\
& + m_3k_{333}^2 && = 0.006 \quad kg \cdot m^2 \\
& + m_4k_{422}^2 && = 0.001 \quad kg \cdot m^2 \\
& + m_5k_{511}^2 && = 0.003 \quad kg \cdot m^2 \\
& - 2m_5\bar{z}_5r_2 && = -0.012 \quad kg \cdot m^2 \\
& + m_6k_{611}^2 && = 0.005 \quad kg \cdot m^2
\end{aligned}$$

$$+ m_6(r_6 - r_2)^2 \quad = \quad 0.004 \ \text{kg}\cdot\text{m}^2$$

$$+ 2m_6\bar{z}_6(r_6 - r_2) \quad = \quad \underline{-0.008 \ \text{kg}\cdot\text{m}^2}$$

Total minimum inertia addition, with no load in the hand $= 0.099 \ \text{kg}\cdot\text{m}^2$

Hence, the minimum total value of inertia felt at joint #1 with no load in the hand is:

$$\boxed{\begin{aligned} D_{11,\,\min}(\text{no load in the hand}) &= 1.318 + 0.009 \\ &= 1.417 \ \text{kg}\cdot\text{m}^2 \end{aligned}} \tag{71}$$

In summary, the following ratios (relative values) can be formed for inertia variations seen at joint #1:

$$\boxed{\frac{D_{11,\,\max}(\text{no load in the hand})}{D_{11,\,\min}(\text{no load in the hand})} = 4.36} \tag{72}$$

$$\boxed{\frac{D_{11,\,\max}(\text{with load in the hand})}{D_{11,\,\min}(\text{no load in the hand})} = 6.75} \tag{73}$$

2. Inertia Variations Seen at Joint #2

The value of D_{22} given by Eq. (60) specifies the variations in the total inertia felt at joint #2 as a function of the joint position vector $\vec{q}$.

a) Constant components of D_{22}:

Input inertia at joint #2	=	$2.193 \ \text{kg}\cdot\text{m}^2$
$+ m_2k_{222}^2$	=	$0.018 \ \text{kg}\cdot\text{m}^2$
$+ m_3k_{311}^2$	=	$2.51 \ \text{kg}\cdot\text{m}^2$
Total constant	=	$4.721 \ \text{kg}\cdot\text{m}^2$

b) <u>Maximum variations in D_{22}:</u>

Assume: r_3 = 111.76 cm (maximum extension of the boom)

θ_4 = 0 deg (see Fig. 2)

θ_5 = 0 deg (see Fig. 2)

θ_6 = 0 deg (see Fig. 2)

Under these four conditions, Eq. (60) gives the following components for D_{22} in addition to the constant components.

1) <u>With no load in the hand:</u>

$m_3r_3(2\bar{z}_3 + r_3)$	= -0.815	$kg \cdot m^2$
$+ m_4k_{411}^2$	= 0.002	$kg \cdot m^2$
$+ m_4r_3(r_3 - 2\bar{y}_4)$	= 1.332	$kg \cdot m^2$
$+ m_5k_{511}^2$	= 0.003	$kg \cdot m^2$
$+ m_5r_3(r_3 + 2\bar{z}_5)$	= 0.87	$kg \cdot m^2$
$+ m_6k_{611}^2$	= 0.005	$kg \cdot m^2$
$+ m_6r_3(2r_6 + r_3)$	= 0.929	$kg \cdot m^2$
$+ m_6r_6^2$	= 0.032	$kg \cdot m^2$
$+ 2m_6\bar{z}_6(r_3 + r_6)$	= -0.13	$kg \cdot m^2$
Total maximum inertia addition, with no load in the hand	**≈ 2.228**	**$kg \cdot m^2$**

Hence, the maximum total inertia value felt at joint #2 with no load in the hand is:

$$\boxed{\begin{aligned} D_{22,\ max}(\text{no load in the hand}) &= 4.721 + 2.228 \\ &= 6.949\ \text{kg}\cdot\text{m}^2 \end{aligned}} \tag{74}$$

2) With load in the hand:

Again, only the m_6-related terms will be changed. According to the specifications of the load and the load's emplacement in the hand, the following new values are obtained for the m_6-related terms:

$$\begin{aligned} m_6 k_{611}^2 &= 0.006\ \text{kg}\cdot\text{m}^2 \\ + m_6 r_3 (2r_6 + r_3) &= 4.165\ \text{kg}\cdot\text{m}^2 \\ + m_6 r_6^2 &= \underline{0.142\ \text{kg}\cdot\text{m}^2} \\ \text{Total} &= 4.313\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

It is noted again that the $2m_6\bar{z}_6(r_3 + r_6)$ term remains unchanged numerically. Thus, the net maximum inertia change due to the specified load in the hand becomes:

$$4.313 - 0.965 = 3.348\ \text{kg}\cdot\text{m}^2$$

Hence, the maximum total value of inertia felt at joint #2 with the specified load in the hand is:

$$\boxed{\begin{aligned} D_{22,\ max}(\text{with load in the hand}) &= 4.721 + 2.228 + 3.348 \\ &= 10.297\ \text{kg}\cdot\text{m}^2 \end{aligned}} \tag{75}$$

c) <u>Minimum value of D_{22}:</u>

Assume: θ_4 = 0 deg (see Fig. 2)

θ_5 = 90 deg (see Fig. 2)

θ_6 = 0 deg (see Fig. 2)

The condition θ_5 = 90 deg will move the mass of the wrist/hand mechanism closest and parallel to joint axis #2. It is noted that no assumption can be made for r_3 yielding $D_{22,\ min}$; instead, it has to be computed as follows.

Under the three assumptions specified above Eq. (60) yields the following components for D_{22} in addition to the constant terms:

$$\begin{aligned} m_4k_{411}^2 &= 0.002 \quad kg\cdot m^2 \\ + m_5k_{533}^2 &= 0.0004 \quad kg\cdot m^2 \\ + m_6k_{633}^2 &= \underline{0.0003 \quad kg\cdot m^2} \\ \text{Total} &= 0.003 \quad kg\cdot m^2 \end{aligned}$$

Further, we will also have r_3-dependent terms forming a quadratic expression:

$$\Psi(r_3) = Ar_3^2 + 2Br_3$$

where

$$A = m_3 + m_4 + m_5 + m_6 = 6.474 \text{ kg}$$

$$B = m_3\bar{z}_3 - m_4\bar{y}_4 + m_5\bar{z}_5 = -2.71 \text{ kg}\cdot\text{m}$$

The value of r_3 yielding Ψ_{min} is obtained from

$$\frac{d\Psi}{dr_3} = 2Ar_3 + 2B = 0 \Longrightarrow r_{3,\,extremum} = -\frac{B}{A} = 41.9 \text{ cm}$$

Consequently, we have

$$\Psi_{min} = -1.135 \text{ kg}\cdot\text{m}^2$$

Hence, the minimum total value of inertia felt at joint #2 with no load in the hand is:

$$\boxed{\begin{aligned} D_{22,\,min}(\text{no load in the hand}) &= 4.721 - 1.135 + 0.003 \\ &= 3.589 \text{ kg}\cdot\text{m}^2 \end{aligned}} \qquad (76)$$

In summary, the following ratios (relative values) can be formed for inertia variations seen at joint #2:

$$\boxed{\frac{D_{22,\,max}(\text{no load in the hand})}{D_{22,\,min}(\text{no load in the hand})} = 1.95} \qquad (77)$$

$$\boxed{\frac{D_{22,\,max}(\text{with load in the hand})}{D_{22,\,min}(\text{no load in the hand})} = 2.9} \qquad (78)$$

3. Inertia Variations Seen at Joint #3

The value of D_{33} given by Eq. (62) is independent of any relative position of the joints. Only the mass of a load held by the hand can change the value of D_{33}. Hence, for the specified load, we have the following ratio (relative value) for inertia variations seen at joint #3:

$$\boxed{\begin{aligned} &D_{33,\,max}(\text{no load in the hand}) = \text{Equivalent input} + m_3 + m_4 + m_5 + m_6 = \\ &D_{33,\,min}(\text{no load in the hand}) = 7.257\ \text{kg} \end{aligned}} \quad (79)$$

$$\boxed{\frac{D_{33,\,max}(\text{with load in the hand})}{D_{33,\,min}(\text{no load in the hand})} = \frac{9.057}{7.257} = 1.25} \quad (80)$$

4. Inertia Variations Seen at Joint #4

The value of D_{44} given by Eq. (64) specifies the variations in the total inertia felt at joint #4 as a function of the relevant components of the joint position vector $\vec{q}$.

a) Constant components of D_{44}:

Input inertia at joint #4	=	$0.106\ \text{kg}\cdot\text{m}^2$
$+ m_4 k_{422}^2$	=	$0.001\ \text{kg}\cdot\text{m}^2$
Total constant	=	$0.107\ \text{kg}\cdot\text{m}^2$

b) Maximum variations in D_{44}:

Assume: $\theta_5 = 90$ deg (see Fig. 2)

It turns out that D_{44} will be independent of θ_6 for any value of θ_5 since $k_{611}^2 = k_{622}^2$ for the JPL RRP manipulator resulting the identity $\sin^2\theta_6 + \cos^2\theta_6 = 1$ for the k_{611}^2 terms in Eq. (64). According to the $\theta_5 = 90$ deg condition specified above, Eq. (64) gives then the following components for D_{44} in addition to the constant terms.

1) <u>With no load in the hand</u>:

$$m_5 k_{511}^2 = 0.003 \quad kg \cdot m^2$$

$$+ m_6 k_{611}^2 = 0.005 \quad kg \cdot m^2$$

$$+ m_6 r_6 (2\bar{z}_6 + r_6) = \underline{0.008 \quad kg \cdot m^2}$$

Total maximum inertia addition, with no load in the hand $= 0.016 \ kg \cdot m^2$

Hence, the maximum total inertia value felt at joint #4 with no load in the hand is:

$$\boxed{\begin{aligned} D_{44,\,max}(\text{no load in the hand}) &= 0.107 + 0.016 \\ &= 0.123 \ kg \cdot m^2 \end{aligned}} \tag{81}$$

2) <u>With load in the hand</u>:

Only the m_6-related terms will be changed. According to the specifications of the load and the load's emplacement in the hand, we will have the following new values for the m_6-related terms:

$$m_6 k_{611}^2 = 0.006 \quad kg \cdot m^2$$

$$+ m_6 r_6 (2\bar{z}_6 + r_6) = \underline{0.118 \quad kg \cdot m^2}$$

$$\text{Total} = 0.124 \quad kg \cdot m^2$$

Thus, the net maximum inertia change due to the specified load in the hand becomes:

$$0.124 - 0.013 = 0.111 \ kg \cdot m^2$$

Hence, the maximum total value o. inertia felt at joint #2 with the specified load in the hand is:

$$\boxed{\begin{aligned} D_{44,\,max}(\text{with load in the hand}) &= 0.107 + 0.016 + 0.111 \\ &= 0.234 \ \ kg \cdot m^2 \end{aligned}} \qquad (82)$$

c) Minimum value of D_{44}:

Assume: $\theta_5 = 0$ deg (see Fig. 2)

According to this condition, Eq. (64) yields the following components in addition to the constant terms:

$$\begin{aligned} m_5 k_{533}^2 &= 0.0004 \quad kg \cdot m^2 \\ + m_6 k_{633}^2 &= \underline{0.0003 \quad kg \cdot m^2} \end{aligned}$$

Total minimum inertia addition, with no load in the hand $= 0.001 \ kg \cdot m^2$

Hence, the minimum total value of inertia felt at joint #4 with no load in the hand is:

$$\boxed{\begin{aligned} D_{44,\,min}(\text{no load in the hand}) &= 0.107 + 0.001 \\ &= 0.108 \ kg \cdot m^2 \end{aligned}} \qquad (83)$$

In summary, the following ratios (relative values) can be formed for inertia variations seen at joint #4:

$$\boxed{\frac{D_{44,\,max}(\text{no load in the hand}}{D_{44,\,min}(\text{no load in the hand})} = 1.14} \qquad (84)$$

$$\boxed{\frac{D_{44,\,max}\text{(with load in the hand)}}{D_{44,\,min}\text{(no load in the hand)}} = 2.17} \tag{85}$$

It is noted that the extremum (minimum and maximum) values of D_{44} can also be determined without any assumption on θ_5, since Eq. (64) is a function of one variable:

$$D_{44} = a\sin^2\theta_5 + b\cos^2\theta_5 + c,$$

where

$$\left.\begin{aligned} a &= 0.015 \quad kg\cdot m^2 \\ b &= 0.001 \quad kg\cdot m^2 \end{aligned}\right\} \text{no load in the hand}$$

$$c = 0.107 \quad kg\cdot m^2$$

The extremum of D_{44} will be obtained at θ_5 values which satisfy

$$\frac{dD_{44}}{d\theta_5} = 2a \sin\theta_5 \cos\theta_5 - 2b \sin\theta_5 \cos\theta_5 = 0$$

That is,

$$\sin 2\theta_5 \,(a - b) = 0$$

since $a \neq b$, we must have $\sin 2\theta_5 = 0$ which yields $\theta_5 = 0$ or 90 deg. For $\theta_5 = 0$ deg we will have minimum value for D_{44}

$$D_{44,\,min} = b + c = 0.108 \ kg\cdot m^2$$

while for $\theta_5 = 90$ deg we will have maximum value

$$D_{44,\,max} = a + c = 0.123 \ kg\cdot m^2$$

5. Inertia Variations Seen at Joint #5

The value of D_{55} given by Eq. (66) specifies the variations in the total inertia felt at joint #5 as a function of θ_5. It turns out, however, that D_{55} becomes independent of θ_5 since $k^2_{611} = k^2_{622}$ for the JPL RRP manipulator resulting in the identity $\sin^2\theta_5 + \cos^2\theta_5 = 1$ for the k^2_{6ii} terms in Eq. (66). Consequently, only the inertia properties of a load held by the hand can change the value of D_{55}. Hence, we will have the following values for D_{55}.

a) No load in the hand:

Input inertia at joint #5	=	0.098 $kg \cdot m^2$
$+ m_5 k^2_{522}$	=	0.003 $kg \cdot m^2$
$+ m_6 k^2_{611}$	=	0.005 $kg \cdot m^2$
$+ m_6 r_6 (2\bar{z}_6 + r_6)$	=	0.008 $kg \cdot m^2$
Total	=	0.114 $kg \cdot m^2$

Hence,

$$\boxed{D_{55,max} = D_{55,min}(\text{no load in the hand}) = 0.114\ kg \cdot m^2} \tag{86}$$

b) **With load in the hand:**

We will have the following new values for the m_6-related terms due to the specified load:

$m_6 k^2_{611}$	=	0.006 $kg \cdot m^2$
$+ m_6 r_6 (2\bar{z}_6 + r_6)$	=	0.118 $kg \cdot m^2$
Total	=	0.124 $kg \cdot m^2$

Thus, the net inertia change due to the specified load in the hand becomes:

$$0.124 - 0.013 = 0.111 \ \text{kg}\cdot\text{m}^2$$

Hence, the total value of inertia felt at joint #5 with the specified load in the hand is:

$$\boxed{D_{55,\,max}(\text{with load in the hand}) = 0.225 \ \text{kg}\cdot\text{m}^2} \quad (87)$$

In summary, the following ratio (relative value) can be formed for inertia variation due to the specified load in the hand:

$$\boxed{\frac{D_{55,\,max}\ (\text{load in the hand})}{D_{55,\,min}\ (\text{no load in the hand})} = 2.0} \quad (88)$$

6. <u>Inertia Variations Seen at Joint #6</u>

As seen from Eq. (68), D_{66} is a constant. We have then:

Input inertia at joint #6	=	0.02	$\text{kg}\cdot\text{m}^2$
$+ m_6 k_{633}^2$	=	0.0003	$\text{kg}\cdot\text{m}^2$
Total	=	**0.02**	**$\text{kg}\cdot\text{m}^2$**

Hence,

$$\boxed{D_{66,\,max} = D_{66,\,min}\ (\text{no load in the hand}) = 0.02 \ \text{kg}\cdot\text{m}^2} \quad (89)$$

If the specified load is held by the hand, we will have:

$$m_6 k_{633}^2 \qquad = \; 0.002 \;\; \mathrm{kg \cdot m^2}$$

Hence, the variation in inertia due to the specified load in the hand is

$$0.295 - 0.049 = 0.002 \;\; \mathrm{kg \cdot m^2}$$

yielding

$$\boxed{\begin{aligned} D_{66,\,max}\ (\text{load in the hand}) &= 0.02 + 0.002 \\ &= 0.022\ \mathrm{kg \cdot m^2} \end{aligned}} \tag{90}$$

In summary, we have the following ratio for inertia variation due to the specified load in the hand:

$$\boxed{\frac{D_{66,\,max}\ (\text{with load in the hand})}{D_{66,\,min}\ (\text{no load in the hand})} = 1.09} \; . \tag{91}$$

All computed exact total inertia variations are summarized in Table 1 and displayed in Figure 3.

B. Maximum Gravity Load Variations

The gravity load felt at the different joints as a function of the total joint position vector $\vec{q}$ is given by Eqs. (46), (48), (50), (52), (54), and (56). As seen from Eqs. (46) and (56), there is no gravity load at joints #1 and #6 since $D_1 = D_6 = 0$ always, because, by assumption, joint axis #1 is gravitationally always vertical, and $\bar{x}_6 = \bar{y}_6 = 0$ even with load in the hand if the mass center of the load is placed at the origin of the hand coordinate frame (X_6, Y_6, Z_6) or along the Z_6 axis. Assuming again a (1.8 kg) load and symmetric emplacement of the load in the hand, we compute the maximum gravity torques

Table 1. Variations in Total Inertias (Exact Values

Joint	Symbol and Equation	No Load in the Hand			With Load in the Hand*	
		Minimum	Maximum	Relative**	Maximum	Relative**
#1	D_{11} Eq. (58)	1.417 $kg \cdot m^2$ (196 $oz\text{-}in\text{-}sec^2$)	6.176 $kg \cdot m^2$ (875 $oz\text{-}in\text{-}sec^2$)	4.5	9.57 $kg \cdot m^2$ (1356 $oz\text{-}in\text{-}sec^2$)	6.9
#2	D_{22} Eq. (60)	3.59 $kg \cdot m^2$ (508 $oz\text{-}in\text{-}sec^2$)	6.95 $kg \cdot m^2$ (984 $oz\text{-}in\text{-}sec^2$)	1.95	10.3 $kg \cdot m^2$ (1458 $oz\text{-}in\text{-}sec^2$)	2.9
#3	D_{33} Eq. (62)	7.257 kg (0.663 $(oz\text{-}sec^2)/in$)	7.257 kg (0.663 $(oz\text{-}sec^2)/in$)	1	9.057 kg (0.827 $(oz\text{-}sec^2)/in$)	1.25
#4	D_{44} Eq. (64)	0.108 $kg \cdot m^2$ (15.27 $oz\text{-}in\text{-}sec^2$)	0.123 $kg \cdot m^2$ (17.36 $oz\text{-}in\text{-}sec^2$)	1.15	0.234 $kg \cdot m^2$ (33.21 $oz\text{-}in\text{-}sec^2$)	2.2
#5	D_{55} Eq. (66)	0.114 $kg \cdot m^2$ (15.87 $oz\text{-}in\text{-}sec^2$)	0.114 $kg \cdot m^2$ (15.87 $oz\text{-}in\text{-}sec^2$)	1	0.225 $kg \cdot m^2$ (31.72 $oz\text{-}in\text{-}sec^2$)	2.0
#6	D_{66} Eq. (68)	0.02 $kg \cdot m^2$ (2.86 $oz\text{-}in\text{-}sec^2$)	0.02 $kg \cdot m^2$ (2.86 $oz\text{-}in\text{-}sec^2$)	1	0.022 $kg \cdot m^2$ (3.11 $oz\text{-}in\text{-}sec^2$)	1.1

*1.8 kg, 442 cm^2 (4 lb, 27 in^3)
Symmetrically in the hand

**Relative $= \frac{\text{Maximum}}{\text{Minimum}}$

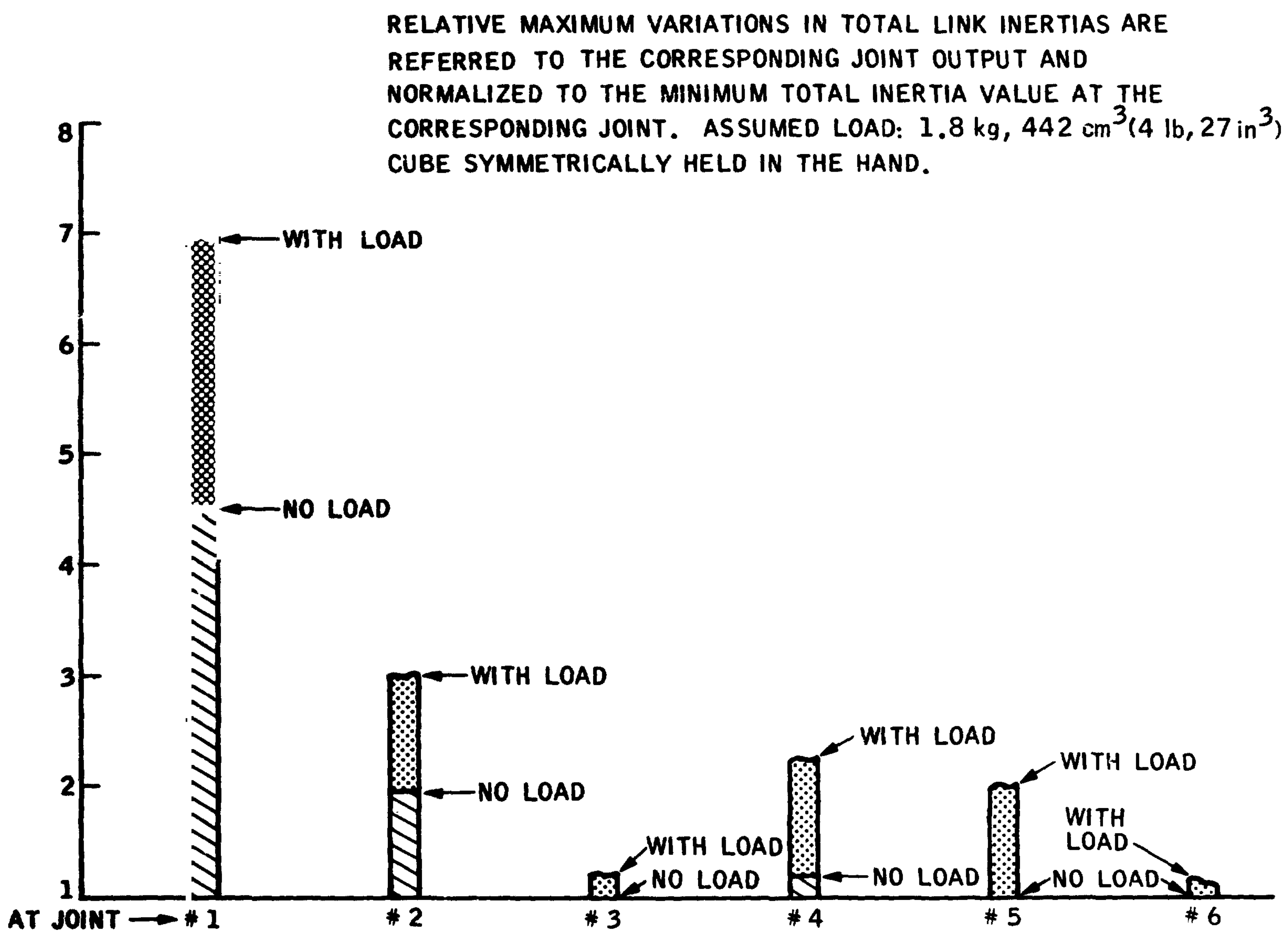

Figure 3. Relative Maximum Variations in Total Link Inertias

(or force) as seen at joints #2, 3, 4, 5, referred to the respective joint outputs.

As seen from Eqs. (48), (52), and (54), D_2, D_4, and D_5 are functions of several joint variables. Thus, a mathematical search for maximum values of D_2, D_4 and D_5 would mean to determine hill tops of surfaces or hypersurfaces. By physical judgement, however, an appropriate and allowed set of joint variables can be selected for each gravity term yielding the maximum value for D_i. (It is noted that the minimum value of any gravity term for the JPL RRP manipulator is zero.)

The subsequent calculated gravity loads should be interpreted as absolute values. The ± polarities can be indicated according to the appropriate joint variable values.

1. Gravity torque at joint #1

$$\boxed{D_1 \equiv 0} \tag{92}$$

2. Gravity torque at joint #2

a. Maximum value with no load in the hand.

Assume:

θ_2 = ±90 deg (horizontal direction of the boom)

r_3 = 111.76 cm (maximum boom extension)

θ_4 = 0 deg (see Fig. 2)

θ_5 = 0 deg (see Fig. 2)

θ_6 = 0 deg (see Fig. 2)

The last three conditions will move the mass center of the wrist/hand mechanism farthest from joint axis #2 in a horizontal direction. Under the assumptions specified above, Eq. (48) will give the following terms:

$m_2\bar{z}_2$	=	-0.0445	kg·m
$+ m_3(\bar{z}_3 + r_3)$	=	2.0072	kg·m
$+ m_4(r_3 - \bar{y}_4)$	=	1.201	kg·m
$+ m_5(\bar{z}_5 + r_3)$	=	0.7423	kg·m
$+ m_6(\bar{z}_6 + r_6)$	=	0.0806	kg·m
$+ m_6 r_6$	=	0.5755	kg·m
Total	=	4.562	kg·m

Hence,

$$\boxed{D_{2,max} \text{ (no load in the hand)} = 4.562 \text{ g} = 44.75 \text{ N·m}}^{*} \tag{92a}$$

b. Maximum value with load in the hand:

The m_6-related terms will have the following new values due to the specified load in the hand:

$m_6(\bar{z}_6 + r_6)$	=	0.5254	kg·m
$m_6 r_6$	=	2.5826	kg·m
Total, together with unchanged terms	=	7.011	kg·m

*g = acceleration of gravity = 9.81 m/sec^2

Hence,

$$\boxed{D_{2,max} \text{ (with load in the hand)} = 7.011 \text{ g} = 68.77 \text{ N}\cdot\text{m}} \qquad (93)$$

It is easily seen from Eq. (48) that for $\theta_2 = 0$ deg, $\theta_4 = 90$ deg and $\theta_5 = 0$ deg we will have

$$D_{2,min} = 0$$

3. <u>Gravity force at joint #3</u>

a. <u>Maximum value with no load in the hand:</u>

It is obtained at $\theta_2 = 0$ (or 180) deg, that is, having the boom in vertical direction. We have then from Eq. (50):

$$\boxed{D_{3,max} \text{ (no load in the hand)} = (6.474 \text{ kg})\text{ g} = 63.5 \text{ N}} \qquad (94)$$

b. <u>Maximum value with load in the hand:</u>

We will have for the specified load:

$$\boxed{D_{3,max} \text{ (with load in the hand)} = (8.274 \text{ kg})\text{ g} = 81.17 \text{ N}} \qquad (95)$$

Obviously, for $\theta_2 = \pm 90$ deg we will have

$$D_{3,min} = 0$$

4. Gravity torque at joint #4

a. Maximum value with no load in the hand:

Assume:

$\theta_2 = \pm 90$ deg

$\theta_4 = 0$ deg

$\theta_5 = \pm 90$ deg

Equation (52) yields then the following terms:

$m_5 \bar{z}_5$	=	0.0359 kg·m
$m_6 (\bar{z}_6 + r_6)$	=	0.0801 kg·m
Total	=	0.116 kg·m

Hence,

$$\boxed{D_{4,max} \text{ (no load in the hand)} = 0.116 \text{ g} = 1.138 \text{ N}\cdot\text{m}} \tag{96}$$

b. Maximum value with load in the hand:

The m_6-related term will have the following new value due to the specified load in the hand:

$$m_6 (\bar{z}_6 + r_6) \qquad = 0.525 \text{ kg}\cdot\text{m}$$

Hence,

$$\boxed{D_{4,max} \text{ (with load in the hand)} = 0.561 \text{ g} = 5.503 \text{ N}\cdot\text{m}} \tag{97}$$

Obviously, for $\theta_2 = 0$ deg we have

$$D_{4,\min} = 0$$

5. Gravity torque at joint #5

a. Maximum value with no load in the hand:

Two sets of assumptions can be made:

$\theta_2 = \pm 90$ deg	$\theta_2 = 0$ deg
$\theta_4 = \pm 90$ deg	$\theta_5 = \pm 90$ deg
$\theta_5 = 0$ deg	

Then, from Eq. (54) we will have:

$$\begin{array}{lll} m_5 \bar{z}_5 & = 0.0359 & \text{kg}\cdot\text{m} \\ + m_6(\bar{z}_6 + z_6) & = 0.0801 & \text{kg}\cdot\text{m} \\ \hline \text{Total} & = 0.116 & \text{kg}\cdot\text{m} \end{array}$$

Hence,

$$\boxed{D_{5,\max} \text{ (no load in the hand)} = 0.116 \text{ g} = 1.138 \text{ N}\cdot\text{m}} \tag{98}$$

b. Maximum value with load in the hand:

The m_6-related term will have the following new value due to the specified load in the hand:

$$m_6(\bar{z}_6 + r_6) = 0.525 \text{ kg}\cdot\text{m}$$

Hence,

$$\boxed{D_{5,max} \text{ (with load in the hand) } = 0.561 \text{ g} = 5.503 \text{ N}\cdot\text{m}} \tag{99}$$

As seen from the previous equations,

$$D_{4,max} = D_{5,max} .$$

6. Gravity torque at joint #6

$$\boxed{D_6 \equiv 0} \tag{100}$$

It is noted that Eq. (100) is only true here because of the assumption that $\bar{x}_6 = \bar{y}_6 = 0$ even with a load in the hand. Suppose, however, that the mass center of the load is off from the origin of the hand coordinate frame so that the net result is, for instance, $\bar{x}_6 = 1$ in. In the case of a 1.8 kg load this will produce 0.58 N·m gravity torque at joint #6, for instance, for $\theta_2 = 0$ deg, $\theta_4 = \theta_6 = 90$ deg configuration as seen from Eq. (C.4) in Appendix C.

All computed maximum gravity load variations at the different joints are summarized in Table 2. To complete the summary, Table 2 also shows the maximum gravity load variations referred to the motor shaft together with motor stall torque and gear ratio.

C. Relative Importance of Inertial Torques/Forces Versus Acceleration-Related Reaction Torques/Forces

The explicit state equations of the inertia terms and acceleration-related reaction torques/forces derived in Section V for the first three link-joint pairs of the JPL RRP manipulator can be utilized for a general quantitative evaluation of the relative importance of the related dynamic components in the torque or force equations.

As seen from the state functions of D^*_{ii} and D^*_{ij} developed in Section V, we have the following acceleration-related non-zero terms in the torque/force equations for the first three link-joint pairs:

$$D^*_{11}\theta_1 + D^*_{12}\theta_2 + D^*_{13} r_3 + \cdots\cdots = T_1 \tag{101}$$

Table 2. Maximum Gravity Load Variations

Joint No.	Symbol and Eq. No.	Without Load in Hand		With 1.8 kg Load in Hand*		Motor Stall Torque	Unit	Gear Ratio
		Maximum		Maximum				
		Referred to Output	Referred to Motor Shaft	Referred to Output	Referred to Motor Shaft			
1	D_1 Eq. (46)	0	0	0	0	0.918 (130)	N·m (oz-in)	100
2	D_2 Eq. (48)	45.1 (6384)	0.451 (63.84)	69.3 (9813)	0.693 (98.13)	3.036 (430)	N·m (oz-in)	100
3	D_3 Eq. (50)	63.94 (230)		81.73 (294)		98.7** (355)	N (oz)	1
4	D_4 Eq. (52)	1.144 (162)	0.0159 (2.25)	5.54 (785)	0.077 (10.9)	0.141 (20)	N·m (oz-in)	72
5	D_5 Eq. (54)	1.144 (162)	0.0159 (2.25)	5.54 (785)	0.077 (10.9)	0.141 (20)	N·m (oz-in)	72
6	D_6 Eq. (56)	0	0	0	0	0.042 (7)	N·m (oz-in)	146

*Homogeneous load symmetrically held in the hand.

**Maximum linear force output.

$$D_{12}^{*}\ \ddot{\theta}_1 + D_{22}^{*}\ \ddot{\theta}_2 + \cdots\cdots = T_2 \tag{102}$$

$$D_{13}^{*}\ \ddot{\theta}_1 + D_{33}^{*}\ \ddot{r}_3 + \cdots\cdots = F_3 \tag{103}$$

Using Eqs. (36) through (38) and (42) through (44) which state the respective functions for D_{ii}^{*} and D_{ij}^{*}, we form the following ratios:

$$R_1 = \frac{D_{12}^{*}}{D_{11}^{*}} = \frac{-(a_6 + a_7 r_3) c\theta_2}{a_1 + \left[a_2 + a_3 r_3 + a_4 r_3^2\right] s^2\theta_2 + a_5 c^2\theta_2} \tag{104}$$

$$R_2 = \frac{D_{13}^{*}}{D_{11}^{*}} = \frac{-a_7 s\theta_2}{a_1 + \left[a_2 + a_3 r_3 + a_4 r_3^2\right] s^2\theta_2 + a_5 c^2\theta_5} \tag{105}$$

$$R_3 = \frac{D_{12}^{*}}{D_{22}^{*}} = \frac{-(a_6 + a_7 r_3)\ c\theta_2}{a_8 + a_3 r_3 + a_4 r_3^2} \tag{106}$$

$$R_4 = \frac{D_{13}^{*}}{D_{33}^{*}} = \frac{-a_7 s\theta_2}{a_4} \tag{107}$$

where a_1, $\cdots\cdots$, a_9 are constants with the following values (determined by using the appropriate "starred" values for the inertia of the third link having the wrist configuration as shown in Fig. 2 and referring inertias to the output):

$$a_1 = 1.334\ \text{kg}\cdot\text{m}^2$$

$$a_2 = 2.635\ \text{kg}\cdot\text{m}^2$$

$$a_3 = -5.5\ \text{kg}\cdot\text{m}$$

$$a_4 = 6.474\ \text{kg}$$

$$a_5 = 0.108\ \text{kg}\cdot\text{m}^2$$

$$
\begin{aligned}
a_6 &= -0.453 \ \text{kg}\cdot\text{m}^2 \\
a_7 &= 1.05 \ \text{kg}\cdot\text{m} \\
a_8 &= 4.74 \ \text{kg}\cdot\text{m}^2 \\
a_9 &= 7.26 \ \text{kg}
\end{aligned}
$$

The maximum (absolute) value of the ratios specified by Eqs. (104) through (107) is obtained when the nominator has maximum (absolute) value and the denominator has minimum (absolute) value.

For the ratio R_1 the maximum value is obtained for $\theta_2 = 0$ deg and $r_3 = r_{3,max} = 111.76$ cm. These conditions give:

$$
R_{1,max} = \frac{|a_6 + 44a_7|}{|a_1 + a_5|} = 0.5 \tag{108}
$$

The maximum value of R_2 requires special consideration. For $\theta_2 = 90$ deg and $r_3 = 50.8$ cm we have

$$
R_{2,max} = \frac{|a_7|}{\left|a_1 + a_2 + a_3 r_3 + a_4 r_3^2\right|} = 0.0037 \ \text{cm}^{-1} \tag{109}
$$

For the ratio R_3 the maximum value will occur when $\theta_2 = 0$ deg and $r_3 = 111.76$ cm.* These conditions give:

$$
R_{3,max} = \frac{|a_6 + 44a_7|}{\left|a_8 + 44a_3 + (44)^2 a_4\right|} = 0.11 \tag{110}
$$

*This r_3 value can be obtained by computing $r_{3,optimum}$ from the condition $dR_3/dr_3 = 0$ for $\theta_2 = 0$ deg.

For the ratio R_4 the maximum value is obtained when $\theta_2 = 90$ deg. This gives:

$$R_{4,\max} = \frac{|a_7|}{|a_9|} = 14.45 \text{ cm} \qquad (111)$$

An examination of Eqs. (108) through (111) for the relative ratios leads to interesting conclusions elaborated briefly below.

Equation (108) shows that for $|\ddot{\theta}_1| = |\ddot{\theta}_2|$ and the specified configurational conditions, the reaction torque felt at joint #1 due to the acceleration at joint #2 will be 50% of the inertial torque at joint #1. Figure 4 depicts the ratio R_1 as a function of θ_2 for r_3 = 111.8, 81.3, 50.8 cm, respectively. Of course, some of the upper part of the r_3 = 81.3 cm and r_3 = 50.8 cm curves on Fig. 4 are unrealizable for the JPL RRP breadboard, since some segments of the upper part of these two curves imply that the boom hits the vehicle platform or a wheel, depending on the value of θ_1. Figure 4 is intended to illustrate the conditions under which the two torques

$$D_{11}^{*}\ddot{\theta}_1 + D_{12}^{*}\ddot{\theta}_2 = D_{11}^{*}(\ddot{\theta}_1 + R_1\ddot{\theta}_2) \qquad (112)$$

can be approximated by $D_{11}^{*}\ddot{\theta}_1$. As seen from Eq. (112), the validity of this approximation depends on the magnitude of $R_1\ddot{\theta}_2$ relative to $\ddot{\theta}_1$. It is noted that the sum $(\ddot{\theta}_1 + R_1\ddot{\theta}_2)$ can also attain zero value.

The ratio R_2 has dimension cm^{-1} since it is related to the sum of the two torques

$$D_{11}^{*}\ddot{\theta}_1 + D_{13}^{*}\ddot{r}_3 = D_{11}^{*}(\ddot{\theta}_1 + R_2\ddot{r}_3) \qquad (113)$$

For instance, for $\ddot{\theta}_1 = 0.5$ rad/sec^2, $\ddot{r}_3 = 12.7$ cm/sec^2 and $R_2 = R_{2,\max}$, the dynamic significance of $R_2\ddot{r}_3$ is one tenth of the dynamic significance of θ_1. The sum $(\ddot{\theta}_1 + R_2\ddot{r}_3)$ can also be zero. Figure 5 shows R_2 as a function of θ_2 for r_3 = 111.8, 81.3, 50.8 cm, respectively. As seen, $R_{2,\max}$ is dependent of r_3.

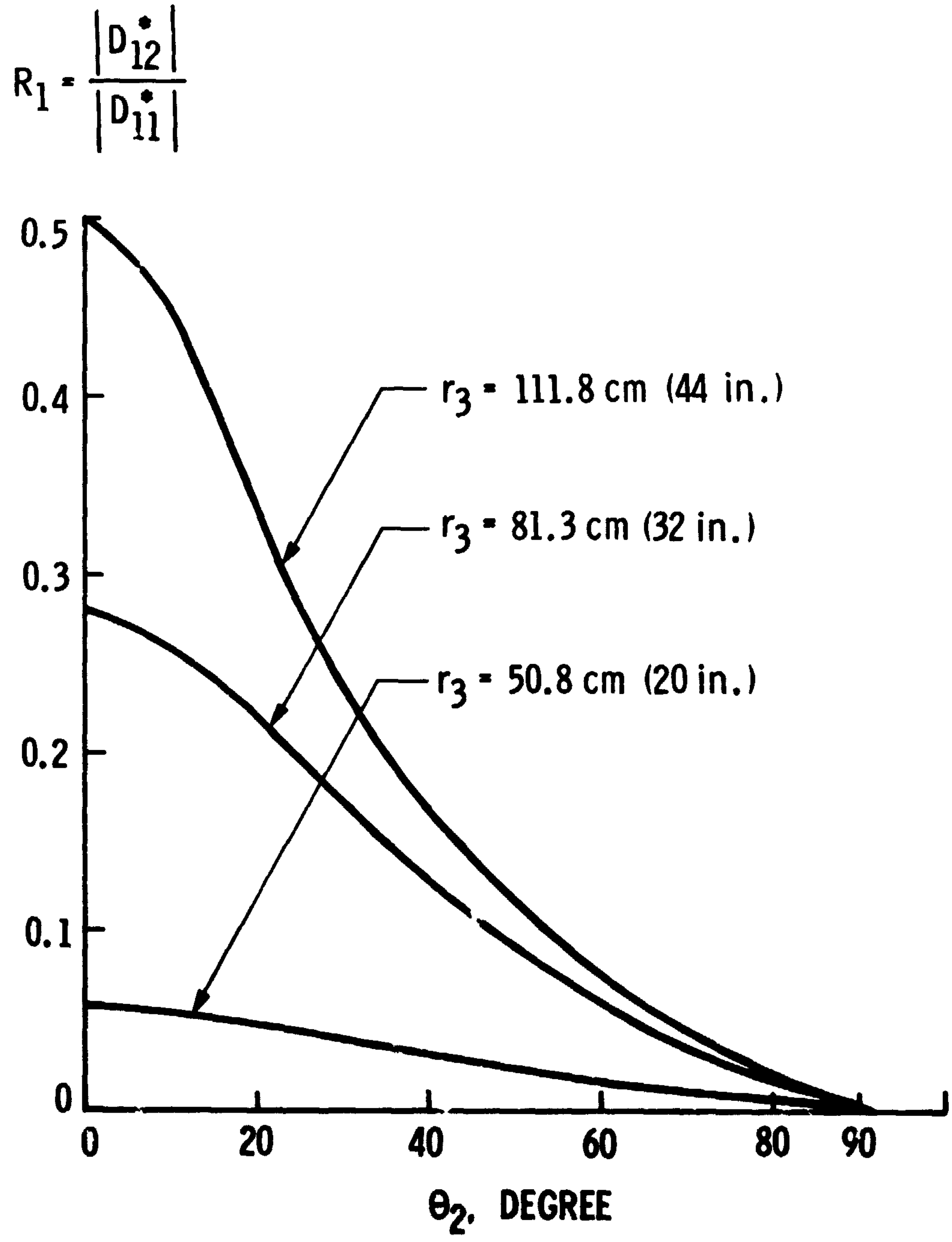

Figure 4. Relative Importance of $\ddot{\theta}_1/\ddot{\theta}_2$ Coupling as Seen at Joint #1

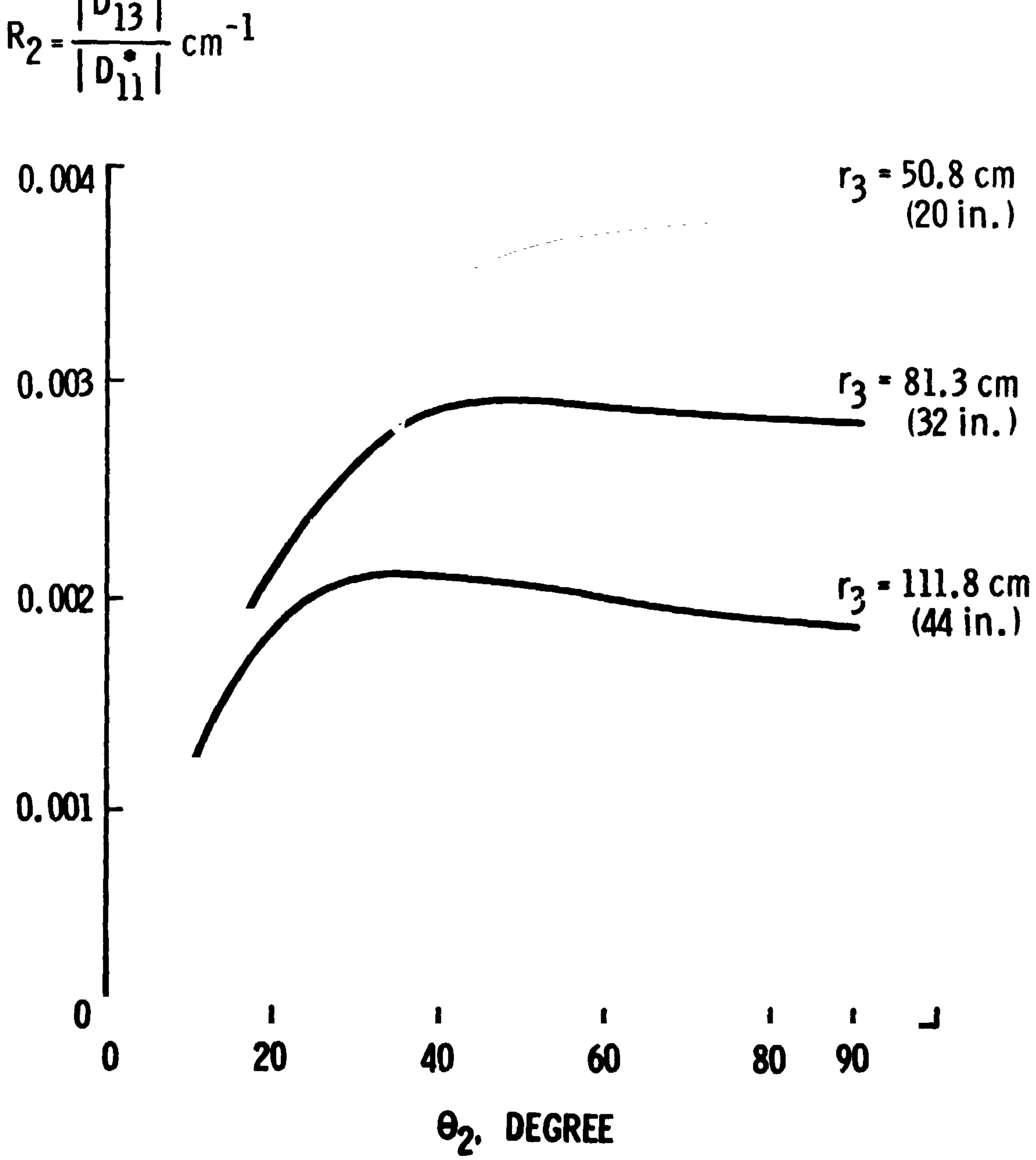

Figure 5. Relative Importance of $\ddot{\theta}_1/\ddot{r}_3$ Coupling as Seen at Joint #1

It is noted again that some of the upper part of r_3 = 81.3 and 50.8 cm curves are unrealizable for the JPL RRP breadboard for the same reasons as explained for Fig. 4. The qualitative differences between R_2 and R_1 are clearly seen by comparing Figs. 5 and 4.

R_3 is shown in Fig. 6 as a function of θ_2 for r_3 = 111.8, 81.3, 50.8 cm. For the upper part of the r_3 = 81.3, 50.8 cm curves in Fig. 6 the remarks are the same as for Fig. 4. R_3 measures the relative importance of $\ddot{\theta}_1$ as seen at joint #2, while R_1 measures the relative importance of $\ddot{\theta}_2$ as seen at joint #1. Therefore, it is worthy to note both the quantitative and qualitative differences between R_3 and R_1 by comparing Figs. 6 and 4. The significance of R_3 is again best seen in the equation:

$$D^*_{12}\ddot{\theta}_1 + D^*_{22}\ddot{\theta}_2 = D^*_{22}(R_3\ddot{\theta}_1 + \ddot{\theta}_2) \tag{114}$$

For instance, for $|\ddot{\theta}_1| = |\ddot{\theta}_2|$ and $R_{3,max}$, the reaction torque felt at joint #2 due to the acceleration at joint #1 is 11% of the inertial torque at joint #2, which is substantially less than the 50% generated by the acceleration at joint #2 and felt at joint #1. For θ_2 = [60, 90] deg, however, R_1 and R_3 become nearly equal. It should be noted that the dynamic significance of R_1 and R_3 in the respective total torque equations is widely different since no gravity torque acts at joint #1, while at joint #2 the gravity torque has a dominant effect. In many instances the gravity torque felt at joint #2 is several orders of magnitude greater than any acceleration torque felt at joint #2. Therefore, to evaluate the relative dynamic significance of the different acceleration torques with full meaning, the total torque equations should be considered.

The ratio R_4 has dimension "cm" since it is related to the sum of the two torques

$$D^*_{13}\ddot{\theta}_1 + D^*_{33}\ddot{r}_3 = D^*_{33}(R_4\ddot{\theta}_1 + \ddot{r}_3) \tag{115}$$

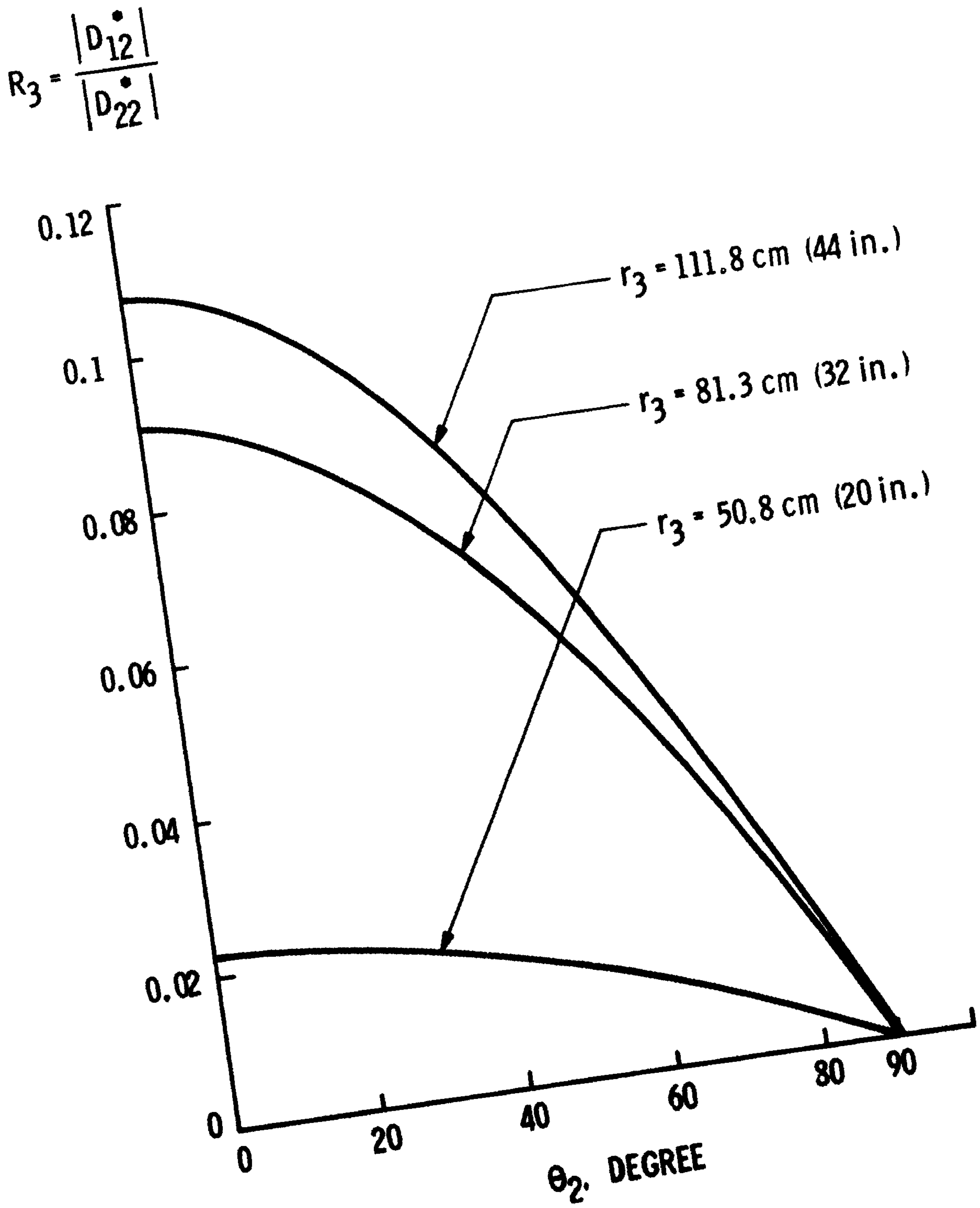

Figure 6. Relative Importance of $\ddot{\theta}_1/\ddot{\theta}_2$ Coupling as Seen at Joint #2

The variation of R_4 is depicted on Fig. 7. R_4 is independent of r_3, and varies as a sine wave scaled to a maximum amplitude of 16.19 cm. It is seen that for $\ddot{r}_3 = 12.8$ cm/sec^2, $\ddot{\theta}_1 = 0.5$ rad/sec^2 and $R_4 = R_{4,\,max}$ the dynamic significance of $R_4\,\ddot{\theta}_1$ is about two-thirds of the dynamic significance of $\ddot{r}_3$. This effect should be compared to the inverse effect expressed by $R_2\ddot{r}_3$ versus $\ddot{\theta}_1$ in Eq. (113). See also example following Eq. (113). It is noted that the gravity force has a dominant effect at joint #3 as a function of θ_2. Thus, the full significance of R_4 cannot be evaluated without considering the total force equation for joint #3. In fact, it can be expected that the gravity force felt at joint #3 will overshadow any acceleration force component by several orders of magnitude in most of the time.

Figures 4 through 7 can be combined into an integrated dynamic scheme for the torque/force equations in a straightforward manner according to the following equations, which are equivalent to Eqs. (101 through (103):

$$D_{11}^{*}(\ddot{\theta}_1 + R_1\ddot{\theta}_2 + R_2\ddot{r}_3) + \cdots\cdots = T_1 \tag{116}$$

$$D_{22}^{*}(\ddot{\theta}_2 + R_3\ddot{\theta}_1) + \cdots\cdots\cdots\cdots = T_2 \tag{117}$$

$$D_{33}^{*}(\ddot{r}_3 + R_4\ddot{\theta}_1) + \cdots\cdots\cdots\cdots = F_3 \tag{118}$$

In these equations, R_1, R_2, R_3, R_4 should be considered with the proper $\pm$ signs (and not in absolute values!) according to the definitions given by Eqs. (104) through (107). The combined effect of the summation in the parentheses in Eqs. (116) through (118) can be zero as well as greater or less than any of the components in the parentheses.

In summary, it is noted that all four ratios (R_1, R_2, R_3, R_4) attain maximum value at $\theta_2 = 0$ or 90 deg. Further, $R_{1,\,max}$ and $R_{3,\,max}$ require that, in addition to $\theta_2 = 0$ deg, we also have $r_3 = r_{3,\,max} = 111.8$ cm simultaneously. Both conditions are quite extreme from the view point of normal tasks expected for the JPL RRP manipulator. When such conditions may occur, two other things will also

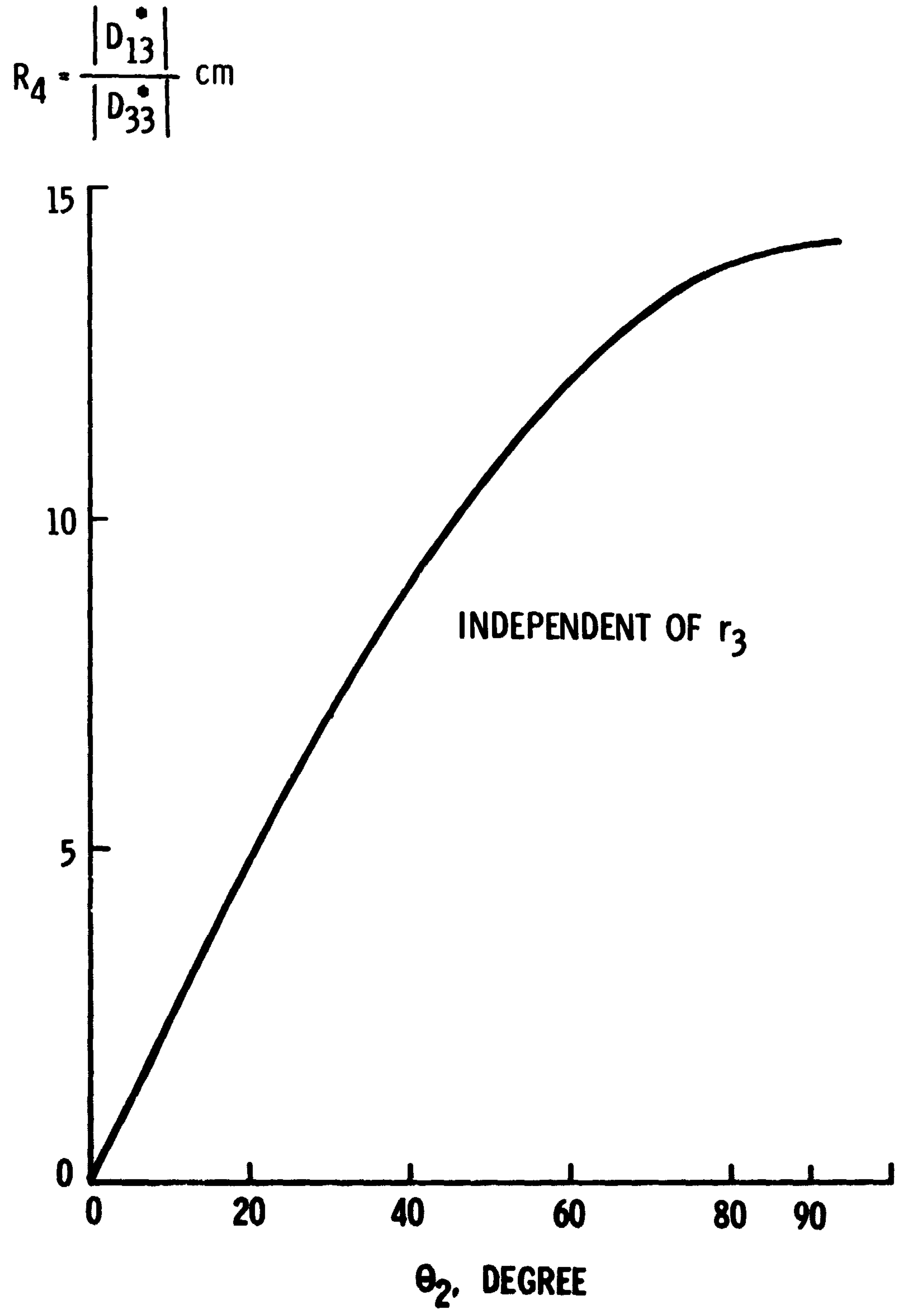

Figure 7. Relative Importance of $\ddot{\theta}_1/\ddot{r}_3$ Coupling as Seen at Joint #3

happen simultaneously: (a) the related acceleration values are (or can be made) sufficiently low; or (b) the effect of the acceleration-related reaction torques/forces is significantly overshadowed by other dynamic effects (gravity torque or force). Hence, it is expected that acceleration-related reaction torques or forces will be quite insignificant under normal operating conditions.

D. Simplification of Torque/Force Equations

One of the main advantages gained by the development of the explicit state equations for the dynamics of the JPL RRP manipulator is that the relative significance of the different torques/forces as well as the relative importance of the different state components contributing to a torque or force term can be explicitly evaluated for varying tasks and operating conditions. In this memo, explicit state equations have been presented for total link inertias and gravity loads for all six link-joint pairs of the JPL RRP manipulator. Thus, we can evaluate the relative significance of the different state components contributing to the inertial and gravity terms at the joints, as well as assess the relative importance of the inertial and gravity loads acting at the joints as a function of the state of the manipulator. A full evaluation of all dynamic terms will be provided in a subsequent memo after the development of the state equations for the acceleration- and velocity-related reaction torques/forces.

1. Inertial Terms

The state equations derived for the D_{ii} dynamic coefficients in Section VI. B are transformation equations which transform the moments of inertia of the links ahead of link "i" (i + 1, i + 2, $\cdots$, n), computed in the respective link coordinates, to the rotation axis of joint "i." Examining the different components which contribute to variations in total inertias as a function of the joint position vector, it is seen that some components are insignificant and can be neglected without introducing sensible errors. In the subsequent simplifications, the state equations for D_{ii} should be viewed together with the exact numerical data presented in Section VII. B where the maximum and minimum values of total inertias as seen at the six joint axes have been determined.

a. Joint #1: Simplified State Equation for D_{11}.

Since $m_2k_{211}^2$ and $m_2k_{233}^2$ are nearly of equal magnitude, we have

$$m_2k_{211}^2 s^2\theta_2 + m_2k_{233}^2 c^2\theta_2 \approx a(s^2\theta_2 + c^2\theta_2) = a$$

where "a" is the mean value of the two moments of inertia. It is a constant, and can be added to the constant components of D_{11}.

The following moments of inertia can be set equal to zero due to their small value relative to other components in the state equation for D_{11}:

$$m_3k_{333}^2$$

$$m_4k_{411}^2$$

$$m_4k_{422}^2$$

$$m_4k_{433}^2$$

$$m_5k_{511}^2$$

$$m_5k_{522}^2$$

$$m_5k_{533}^2$$

$$m_6k_{611}^2$$

$$m_6k_{622}^2$$

$$m_6k_{633}^2$$

All these simplifications reduce the state equation for D_{11} to less than one-third of its complete (and extremely complex and lengthy) form given by Eq. (58). The error introduced by these simplifications will be very small: less than 0.5% for $D_{11,max}$, and less than 2% for $D_{11,max}$.

Introducing the simplifications defined above, and performing the possible algebraic reductions of the remaining terms in Eq. (58), the state equation of D_{11} takes the following simplified form:

$$
\begin{aligned}
D_{11} = \; & b_1 \quad \left(b_2 \quad b_3 r_3 + b_4 r_3^2\right) s^2\theta_2 \\
& + r_3 \left[b_5 s^2\theta_2 \cdot \theta_5 + b_6 s(2\theta_2) s\theta_4 s\,\theta_5 + b_7 s(2\theta_2) c\theta_4 + b_8 s(2\theta_4) s\theta_5 \right] \\
& + b_9 \left[c^2\theta_4 s^2\theta_5 + s^2\theta_2 c^2\theta_5 + c^2\theta_2 s^2\theta_4 s^2\theta_5 + s(2\theta_2) s\theta_4 s\theta_5 c\theta_5 \right] \\
& + b_{10} c\theta_4 s\theta_5 + b_{11} s\theta_4 + b_{12}
\end{aligned}
\tag{119}
$$

where b_1, ..., b_{12} are constants given by

b_1 = Constant components of D_{11} (see page 48)	1.319 kg·m^2
$+\frac{1}{2}\left(m_2 k_{211}^2 + m_2 k_{233}^2\right)$	0.104 kg·m^2
Total:	1.423 kg · m^2

	No load in the hand	With load in the hand (Load as specified)	
$b_2 = m_3 k_{322}^2$	2.51	2.51	kg·m^2
$b_3 = 2(m_3\bar{z}_3 - m_4\bar{y}_4)$	-5.49	-5.49	kg·m
$b_4 = m_3 + m_4 + {}_5 + m_6$	6.48	8.27	kg

	No load in the hand	With load in the hand (Load as specified)	
$b_5 = 2m_5\bar{z}_5 + 2m_6(\bar{z}_6 + r_6)$	0.232	0.396	kg·m
$b_6 = m_6(r_6 + \bar{z}_6)$	0.08	0.523	kg·m
$b_7 = m_4\bar{z}_4$	-0.006	-0.006	kg·m
$b_8 = m_5\bar{z}_5$	0.036	0.036	kg·m
$b_9 = m_6r_6(r_6 + 2\bar{z}_6)$	0.008	0.118	kg·m^2
$b_{10} = 2r_2(m_5\bar{z}_5 + m_6\bar{z}_6 + m_6r_6)$	0.038	0.181	kg·m^2
$b_{11} = 2m_4\bar{z}_4r_2$	-0.002	-0.002	kg·m^2
$b_{12} = m_6r_2^2$	0.013	0.059	kg·m^2

In view of the numerical value of the constants $b_1, \ldots, b_{12}$, Eq. (119) can be further simplified without introducing sensible errors. The most significant part of Eq. (119) is the first line which contains b_1, b_2, b_3, and b_4. With no load in the hand, these components yield for $r_3 = 111.9$ cm and $\theta_2 = 0$ and 90 deg:

$$D_{11,min} = 1.423 \text{ kg}\cdot\text{m}^2$$

$$D_{11,max} = 5.89 \text{ kg}\cdot\text{m}^2$$

Comparing these values to those given by Eqs. (71) and (69), it is seen that the error is 4%. With load in the hand, however, the same components yield

$$D_{11,max} = 8.15 \text{ kg}\cdot\text{m}^2$$

which has an error about 15% when compared to the corresponding exact value given by Eq. (70). This error can be reduced to 10% if the b_5 term is retained in Eq. (119). Retaining the b_5 term in Eq. (119) will also reduce the error in $D_{11, max}$ with no load in the hand from 4% to 1%. The b_5 term will also account for the major part of the variations in D_{11} due to changes in θ_5. Hence, a very simple and sensible state function which approximates the value of D_{11} with good accuracy is the following expression:

$$\boxed{D_{11} = b_1 + \left[b_2 + (b_3 + b_5 c\theta_5)r_3 + b_4 r_3^2\right] s^2\theta_2} \tag{120}$$

where the values of parameters b_4 and b_5 also depend on the load held in the hand according to the simple formulas specified above for b_4 and b_5. Comparing Eq. (120) to Eq. (58), it is easily seen that the computational complexity of D_{11} will be reduced nearly by 98% when Eq. (58) is replaced by Eq. (120). The content and strength of Eq. (120) becomes apparent after physical reasoning.

b. Joint No. 2: Simplified State Equation for D_{22}.

In the state function for D_{22} given by Eq. (60) the following moments of inertia can be neglected due to their small relative value:

$$m_4 k_{411}^2$$

$$m_4 k_{433}^2$$

$$m_5 k_{511}^2$$

$$m_5 k_{522}^2$$

$$m_5 k_{533}^2$$

$$m_6 k_{611}^2$$

$$m_6 k_{622}^2$$

$$m_6 k_{633}^2$$

Introducing these simplifications, Eq. (60) can be written in the following reduced form:

$$D_{22} = b_{13} + b_4 r_3^2 + (b_3 + b_5 c\theta_5) r_3$$
$$+ b_9\left(s^2\theta_4 s^2\theta_5 + c^2\theta_5\right) \tag{121}$$

where

b_{13} = constant components of D_{22} (see page 52)

$= 4.72 \ \text{kg} \cdot \text{m}^2$

b_3, b_4, b_5, b_9 are constants, identical to those defined and computed for D_{11} previously. (See pages 84 and 85.)

Equation (121) yields the following extremum values for D_{22}:

$D_{22,\,max}$ (no load in the hand) $= 7.09 \ \text{kg} \cdot \text{m}^2$

$D_{22,\,max}$ (with load in the hand) $= 9.64 \ \text{kg} \cdot \text{m}^2$

$D_{22,\,min}$ (no load in the hand) = [illegible].59 $\text{kg} \cdot \text{m}^2$

Comparing these values to the corresponding exact values given by Eqs. (74), (75), (76), it is seen that the error introduced by the simplifications is between 2% and 6%. Equation (121) can be further simplified by omitting the b_9 term, and the total maximum error introduced into D_{22} will still be less than 8%. Hence, a very simple and sensible state function which approximates the value of D_{22} with good accuracy is:

$$\boxed{D_{22} = b_{13} + (b_3 + b_5 c\theta_5) r_3 + b_4 r_3^2} \tag{122}$$

where the values of parameters b_4 and b_5 also depend on the load held in the hand according to the simple formulas defined on pages 84 and 85 for D_{11}. It is

worthy to note that the b_3, b_5, b_4 terms in Eqs. (122) and (120) are identical, except that they are not multiplied by $s^2\theta_2$ in Eq. (122). Comparing Eq. (122) to Eq. (60), it is seen that the computational complexity of D_{22} is reduced nearly by 90% when Eq. (60) is replaced by Eq. (122). The content and strength of Eq. (122) is apparent by physical reasoning.

c. The Wrist Joints.

As discussed in Subsection VII. B-4, the state function of D_{44} given by Eq. (64) is being reduced without simplifications to the following form due to the equality $m_6 k_{611}^2 = m_6 k_{622}^2$:

$$D_{44} = b_{14} + b_{15} s^2\theta_5 + b_{16} c^2\theta_5 \tag{123}$$

where

b_{14} = constant components of D_{44} (see page 57)

$= 0.107\ \mathrm{kg.m^2}$

and the other constants are given by:

	No load in the hand	With load in the hand (Load specified)	
$b_{15} = m_5 k_{511}^2 + m_6 k_{611}^2 + b_9$	0.015	0.127	$\mathrm{kg \cdot m^2}$
$b_{16} = m_5 k_{533}^2 + m_6 k_{633}^2$	0.001	0.002	$\mathrm{kg \cdot m^2}$

Neglecting the b_{16} term in Eq. (123) will introduce only 1% – 6% error. Hence, we have the following simple state function which approximates the value of D_{44} with good accuracy:

$$\boxed{D_{44} = b_{14} + b_{15} s^2\theta_5} \tag{124}$$

where the value of parameter b_{15} also depends on the load in the hand according to the simple expression specified above. Again, the content of Eq. (124) is apparent by simple physical reasoning.

As shown in Subsection VII.B-5, the value of D_{55} and D_{66} is constant:

$$\boxed{D_{55} = b_{17} + b_{18}} \tag{125}$$

$$\boxed{D_{66} = b_{19} + b_{20}} \tag{126}$$

The parameters in Eqs. (125) and (126) are separated into two parts: b_{17} and b_{19} are true constants, while the value of b_{18} and b_{20} depends on the load held in the hand according to the following expressions:

	No load in the hand	With load in the hand (Load specified)	
$b_{18} = m_6 k_{611}^2 + b_9$	0.013	0.125	$kg \cdot m^2$
$b_{20} = m_6 k_{633}^3$	0.0004	0.002	$kg \cdot m^2$

While the true constants are:

$$b_{17} = 0.099 \text{ kg} \cdot \text{m}^2$$

$$b_{19} = 0.02 \text{ kg} \cdot \text{m}^2$$

No specific simplifications are needed for D_{33} given by Eq. (62) since D_{33} is a constant; its value can only be changed by the mass of the load held in the hand.

The functional form of the simplified state equations for the D_{11}, D_{22}, D_{44} coefficients is noteworthy:

$$D_{11} = f\left(\theta_2, r_3, \theta_5, b_1, b_2, b_3, b_4^L, b_5^L\right)$$

$$D_{22} = f\left(r_3, \theta_5, b_3, b_4^L, b_5^L, b_{13}\right)$$

$$D_{44} = f\left(\theta_5, b_{14}, b_{15}^L\right)$$

where superscript "L" indicates that the respective "b" parameters also depend on the load held in the hand.

2. Gravity Terms

The complete state equations for the gravity terms developed and presented in Section VI. A are not too complex functions. The functions for D_2 and D_4 given by Eqs. (48) and (52) can be slightly simplified if needed for a price of small errors.

The state function for D_2 can be organized in the following form:

$$D_2 = g(d_1 + d_2 r_3)s\theta_2 + gd_3(s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5)$$
$$+ gd_4 c\theta_2 c\theta_4 \tag{127}$$

where $g = 9.81 \text{ m/sec}^2$ (acceleration of gravity), and the constant parameters $d_1, \ldots, d_4$ are given by

	No load in the hand	With load in the hand (Load specified)	
$d_1 = m_2\bar{z}_2 + m_3\bar{z}_3 - m_4\bar{y}_4$	-2.788	-2.788	kg·m
$d_2 = m_3 + m_4 + m_5 + m_6$	6.47 kg	8.28	kg
$d_3 = m_5\bar{z}_5 + m_6(\bar{z}_6 + r_6)$	0.116	0.566	kg·m
$d_4 = m_4\bar{z}_4$	-0.006	-0.006	kg·m

As seen, the d_2 and d_3 parameters depend on the load held in the hand.

Equation (127) is exact. It is seen, however, that the contribution of the d_4 term to the value of D_2 is insignificant, less than 1%. Thus, we can use the following simplified equation which reproduces the value of D_2 with very good accuracy:

$$\boxed{D_2 = g(d_1 + d_2 r_3)s\theta_2 + gd_3(s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5)} \tag{128}$$

It is noted that the potential importance of the d_3 term increases as r_3 and/or θ_2 decreases.

The exact state function for D_4 can be expressed as

$$D_4 = gd_3 s\theta_2 c\theta_4 s\theta_5 - gd_4 s\theta_2 s\theta_4 \tag{129}$$

where the parameters d_3 and d_4 are identical to those defined and computed for D_2 above. The relative significance of the d_4 term in Eq. (129) can be expected as small in most of the time. Hence, we can use the following reduced state function as a good approximation for D_4:

$$\boxed{D_4 = gd_3 s\theta_2 c\theta_4 s\theta_5} \tag{130}$$

However, the form of Eq. (129) is simple. Therefore, not much is gained by omitting the d_4 term from Eq. (129) if the maximum 0.056 N·m value of the d_4 term (as referred to the output) seems important.

It is noted finally that the state functions for D_3 and D_5 given by Eqs. (50) and (54), respectively, can be written as:

$$\boxed{D_3 = -gd_2 c\theta_2} \tag{131}$$

$$\boxed{D_5 = gd_3(s\theta_2 s\theta_4 c\theta_5 + c\theta_2 s\theta_5)} \tag{132}$$

where the parameters d_2 and d_3 are identical to those defined and computed for D_2 previously.

The simplified state equations developed for the inertial and gravity terms of the JPL RRP manipulator together with the related parameters are summarized in Tables 3 and 4.

E. Relative Importance of Gravity Terms Versus Inertial Terms

The simplified state equations for the gravity and inertial terms allow an easy functional evaluation of the relative importance of gravity versus inertial terms in the torque/force equations. We form the following four ratios:

$$K_1 = \frac{D_2}{D_{22}} = \frac{\text{Eq. (128)}}{\text{Eq. (122)}} \tag{133}$$

Table 3. Simplified State Equations for Inertia and Gravity Loads at the Six Joints

Inertia Terms:

$$D_{11} = b_1 + \left[b_2 + \left(b_3 + b_5^L c\theta_5\right) r_3 + b_4^L r_3^2\right] s^2\theta_2$$

$$D_{22} = b_{13} + \left(b_3 + b_5^L c\theta_5\right) r_3 + b_4^L r_3^2$$

$$D_{33} = b_4^L$$

$$D_{44} = b_{14} + b_{15}^L s^2\theta_5$$

$$D_{55} = b_{17} + b_{18}^L$$

$$D_{66} = b_{19} + b_{20}^L$$

Gravity Terms:

$$D_2 = g\left(d_1 + d_2^L r_3\right) s\theta_2 + g d_3^L \left(s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5\right)$$

$$D_3 = - g d_2^L c\theta_2$$

$$D_4 = g d_3^L s\theta_2 c\theta_4 s\theta_4$$

$$D_5 = g d_3^L \left(s\theta_2 s\theta_4 c\theta_5 + c\theta_2 s\theta_5\right)$$

L: depends on the load in the hand

Table 4. Parameters in the Simplified State Equations for Inertia and Gravity Loads

b_1	1.423 kg·m^2 (201.5 oz-in-sec^2)
b_2	2.51 kg·m^2 (355.5 oz-in-sec^2)
b_3	-5.49 kg·m (-19.75 oz-sec^2)
$b_4^L = m_3 + m_4 + m_5 + m_6$	6.48 kg* (0.592 (oz-sec^2)/in)
$b_5^L = 2m_5\bar{z}_5 + 2m_6(\bar{z}_6 + r_6)$	0.232 kg·m* (0.834 oz-sec^2)
b_{13}	4.72 kg·m^2 (668 oz-in-sec^2)
b_{14}	0.107 kg.m^2 (15.17 oz-in-sec^2)
$b_{15}^L = m_5k_{511}^2 + m_6k_{611}^2 + m_6r_6(r_6 + 2\bar{z}_6)$	0.015 kg.m^2* (2.19 oz-in-sec^2)
b_{17}	0.099 kg·m^2 (14.01 oz-in-sec^2)
$b_{18}^L = m_6k_{611}^2 + m_6r_6(r_6 + 2\bar{z}_6)$	0.013 kg.m^2* (1.81 oz-in-sec^2)
b_{19}	0.02 kg·m^2 (2.81 oz-in-sec^2)
$b_{20}^L = m_6k_{633}^2$	0.0004 kg·m^2* (0.05 oz-in-sec^2)
d_1	-2.788 kg·m (-10.03 oz-sec^2)
$d_2^L = b_4^L$ and $d_3^L = 0.05\ b_5^L$	

L: depends on the load in the hand

***: quoted number is referred to "no load" in the hand**

$$K_2 = \frac{D_3}{D_{33}} = \frac{\text{Eq. (131)}}{\text{Eq. (62)}} \tag{134}$$

$$K_3 = \frac{D_4}{D_{44}} = \frac{\text{Eq. (130)}}{\text{Eq. (124)}} \tag{135}$$

$$K_4 = \frac{D_5}{D_{55}} = \frac{\text{Eq. (132)}}{\text{Eq. (125)}} \tag{136}$$

For $\theta_4 = \theta_5 = 0$ deg, we have the following expression for K_1 according to the definition given by Eq. (133):

$$K_1 = \frac{g(d_1 + d_3 + d_2 r_3)}{b_{13} + (b_3+b_5)r_3 + b_4 r_3^2} s\theta_2 \left[\frac{1}{\text{rad/sec}^2}\right] \tag{137}$$

As seen from Eq. (137), the relative importance of gravity torque versus inertia torque at joint No. 2 varies essentially as a sine wave of θ_2 with an amplitude dependent on r_3. The absolute value of K_1 given by Eq. (137) is shown in Fig. 8 for three r_3 values. The "b" and "d" parameters which appear in Eq. (137) and depend on the load in the hand are taken for the specified load. The function K_1 is normalized to 1 rad/sec^2 angular acceleration at joint No. 2. For instance, if $\ddot{\theta}_2 = 0.5$ rad/sec^2, the gravity torque at joint No. 2 for $\theta_2 = 60$ deg and $r_3 = 96.5$ cm is 14 times the value of the inertia torque. Or, for the same conditions, the gravity torque is only 3.5 times the value of the inertia torque if $\ddot{\theta}_2 = 2$ rad/sec^2.

The ratio K_2 defined by Eq. (134) simply gives (without any condition on any state variable):

$$K_2 = gc\theta_2 \left[\frac{1}{\text{cm/sec}^2}\right] \tag{138}$$

Thus, the relative importance of gravity force versus inertia force at joint No. 3 varies exactly as the cosine of θ_2 with a maximum

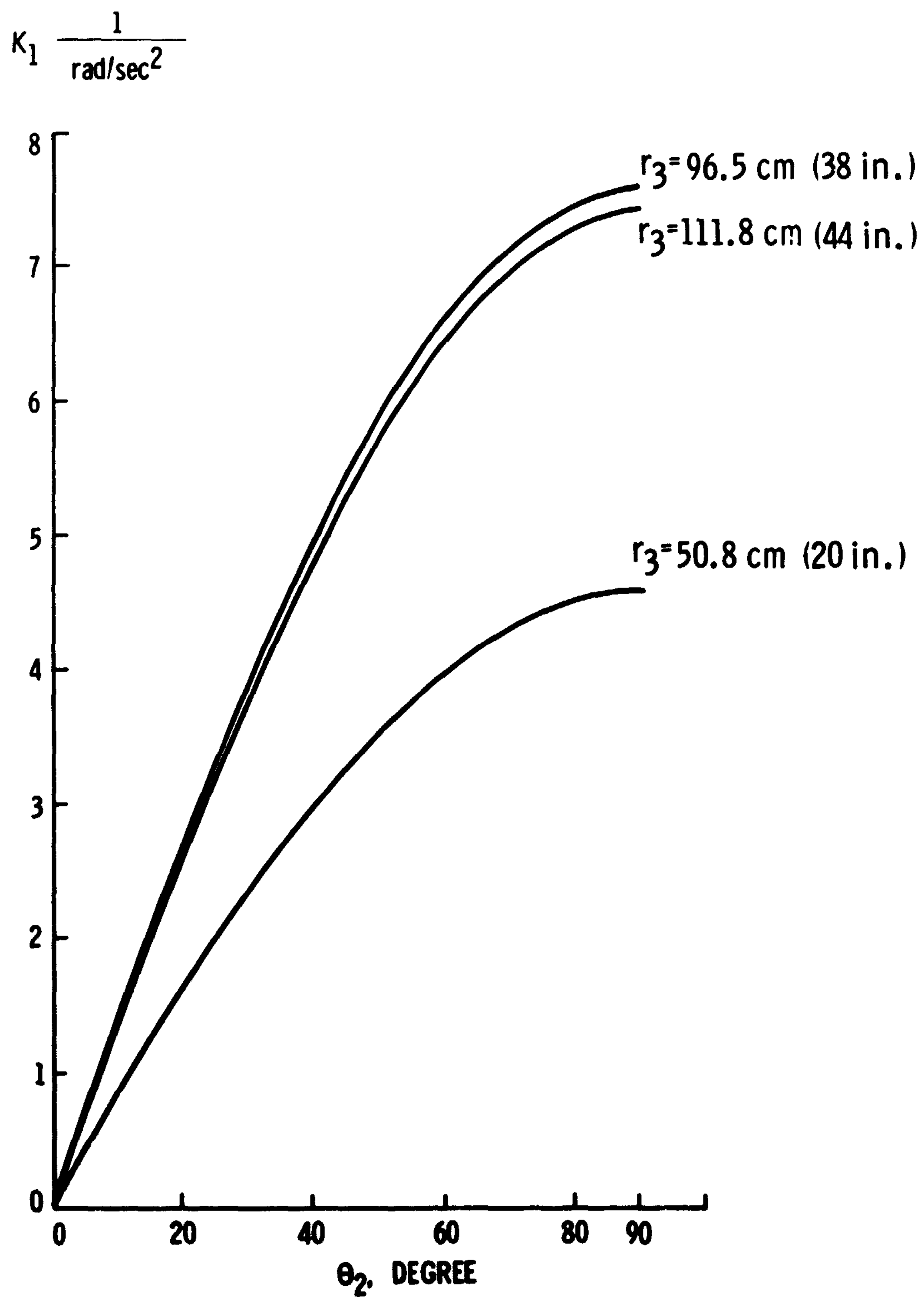

Figure 8. Relative Ir rtance of Gravity versus Inertia To ue at Joint #2

amplitude = g = acceleration of gravity. The absolute value of K_2 is shown in Fig. 9. It is noted that K_2 is independent of any inertial or geometric parameter. The ratio K_2 on Fig. 9 is normalized to 1 cm/sec^2 linear acceleration at joint No. 3.

For θ_4 = 0 deg, the ratio K_3 defined by Eq. (135) gives the following expression:

$$K_3 = \frac{gd_3 s\,\theta_5}{b_{14} + b_{15} s^2 \theta_5} s\,\theta_2 \left[\frac{1}{\text{rad/sec}^2}\right] \tag{139}$$

Thus, the relative value of gravity torque versus inertia torque acting at joint No. 4 varies essentially as a sine wave of θ_2 with an amplitude dependent on θ_5. The absolute value of K_3 given by Eq. (139) is shown in Fig. 10 for two θ_5 values. Again, the "b" and "d" parameters, which appear in Eq. (139) and depend on the load in the hand, are taken for the specified load. The ratio K_3 is normalized to 1 rad/sec^2 angular acceleration at joint No. 4. It is interesting to note that the relative importance of gravity torque versus inertia torque can be more predominant at joint No. 4 than at joint No. 2 as seen by comparing Figs. 8 and 10.

For θ_5 = 90 deg, the ratio K_4 defined by Eq. (136) gives the following expression:

$$K_4 = \frac{gd_3}{b_{17} + b_{18}} c\,\theta_2 \left[\frac{1}{\text{rad/sec}^2}\right] \tag{140}$$

Thus, the relative value of gravity torque versus inertia torque varies exactly as the cosine of θ_2 for θ_5 = 90 deg. But, as seen from Eq. (132), K_4 will vary as the sine of θ_2 if θ_5 = 0 deg and θ_4 = 90 deg. Or, if θ_2 = 0 deg, then K_4 varies as the sine of θ_5. However, the maximum amplitude of any wave variation in K_4 is fixed, independent of any state variable. Figure 11 shows the absolute value of K_4 as given by Eq. (140), normalized to 1 rad/sec^2 angular acceleration at joint No. 5. As seen from Figs. 8 and 11, the relative importance of gravity torque versus inertia torque at joints No. 2 and No. 5 can be

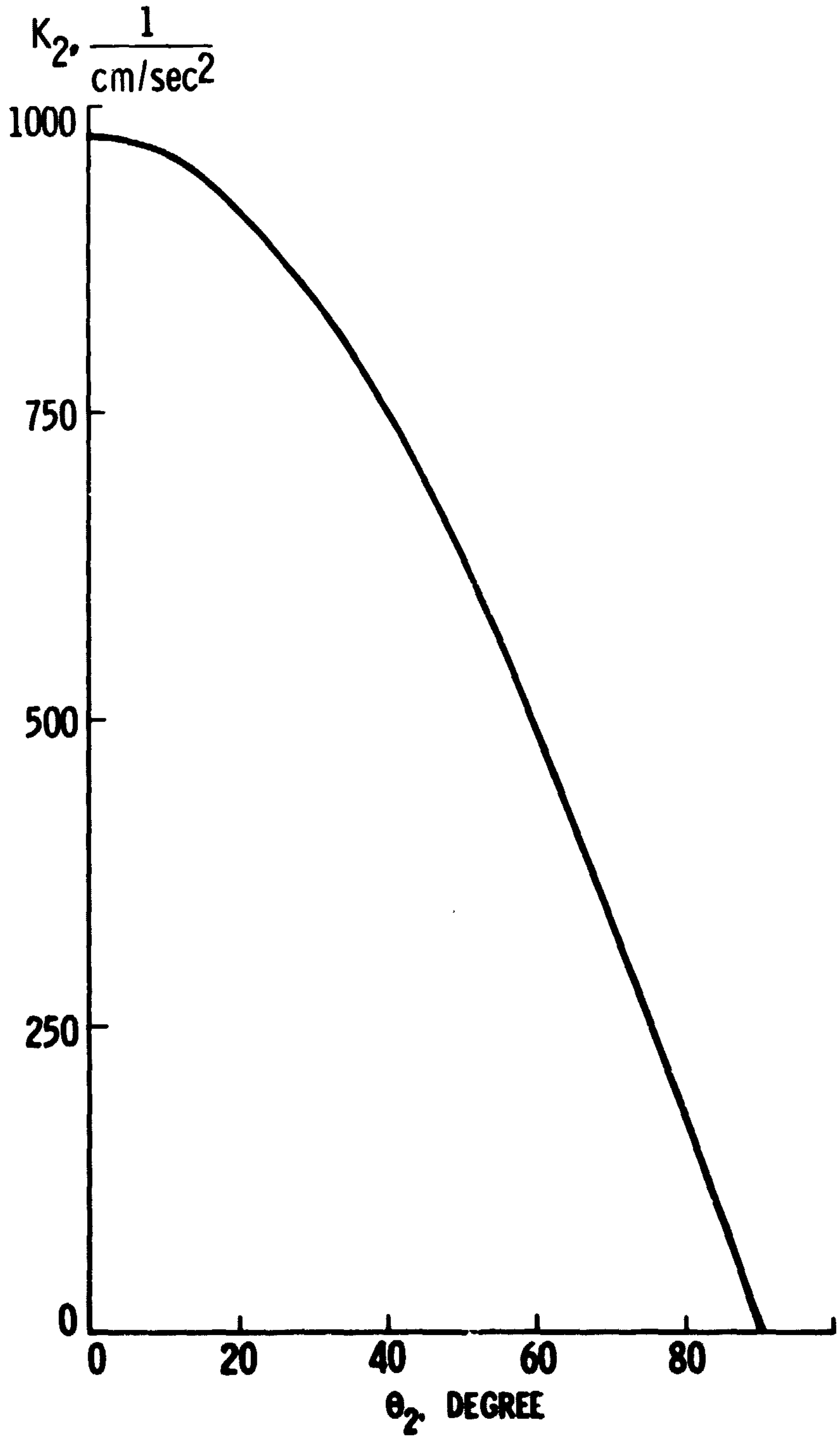

Figure 9. Relative Importance of Gravity versus Inertia Force at Joint #3

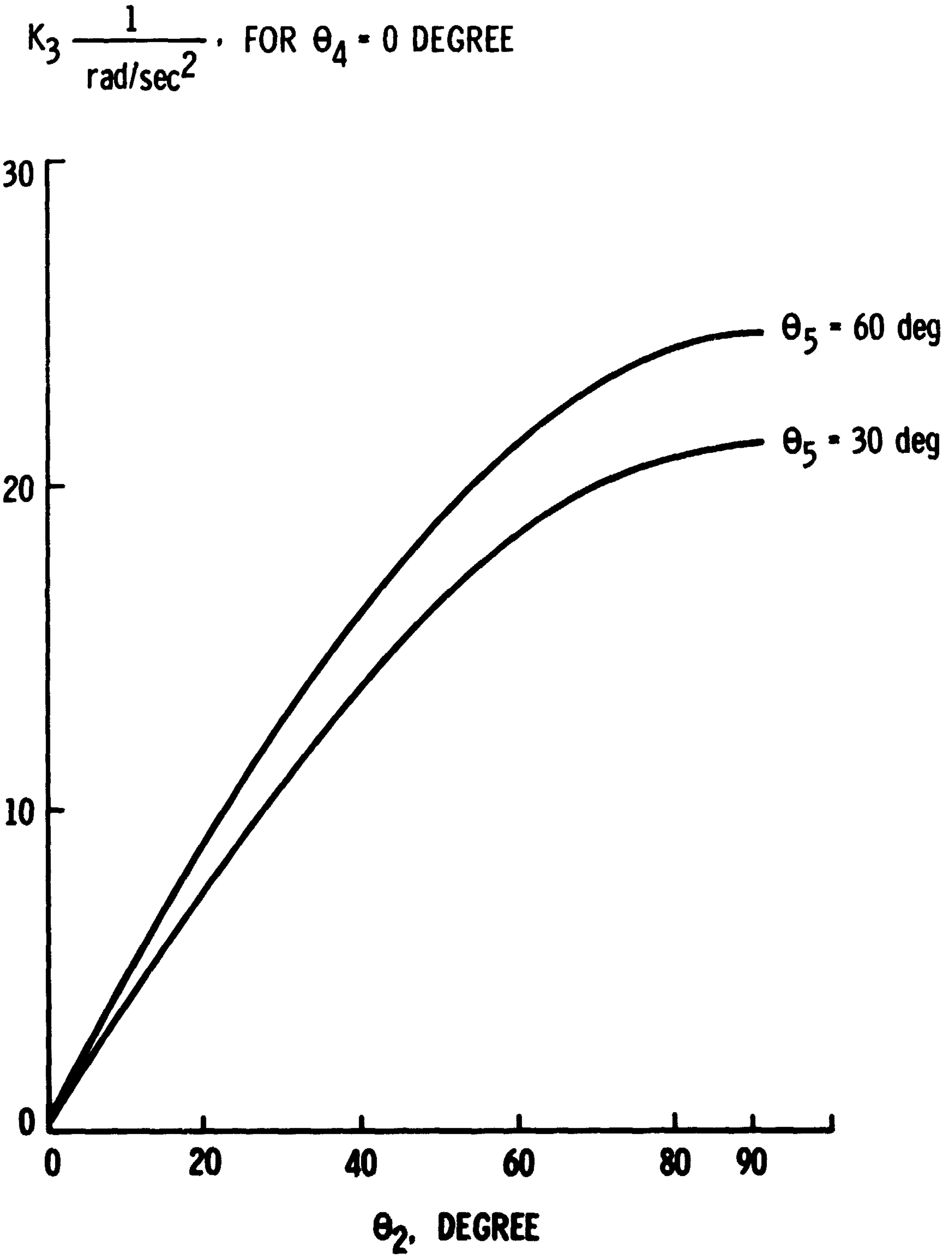

Figure 10. Relative Importance of Gravity versus Inertia Torque at Joint #4

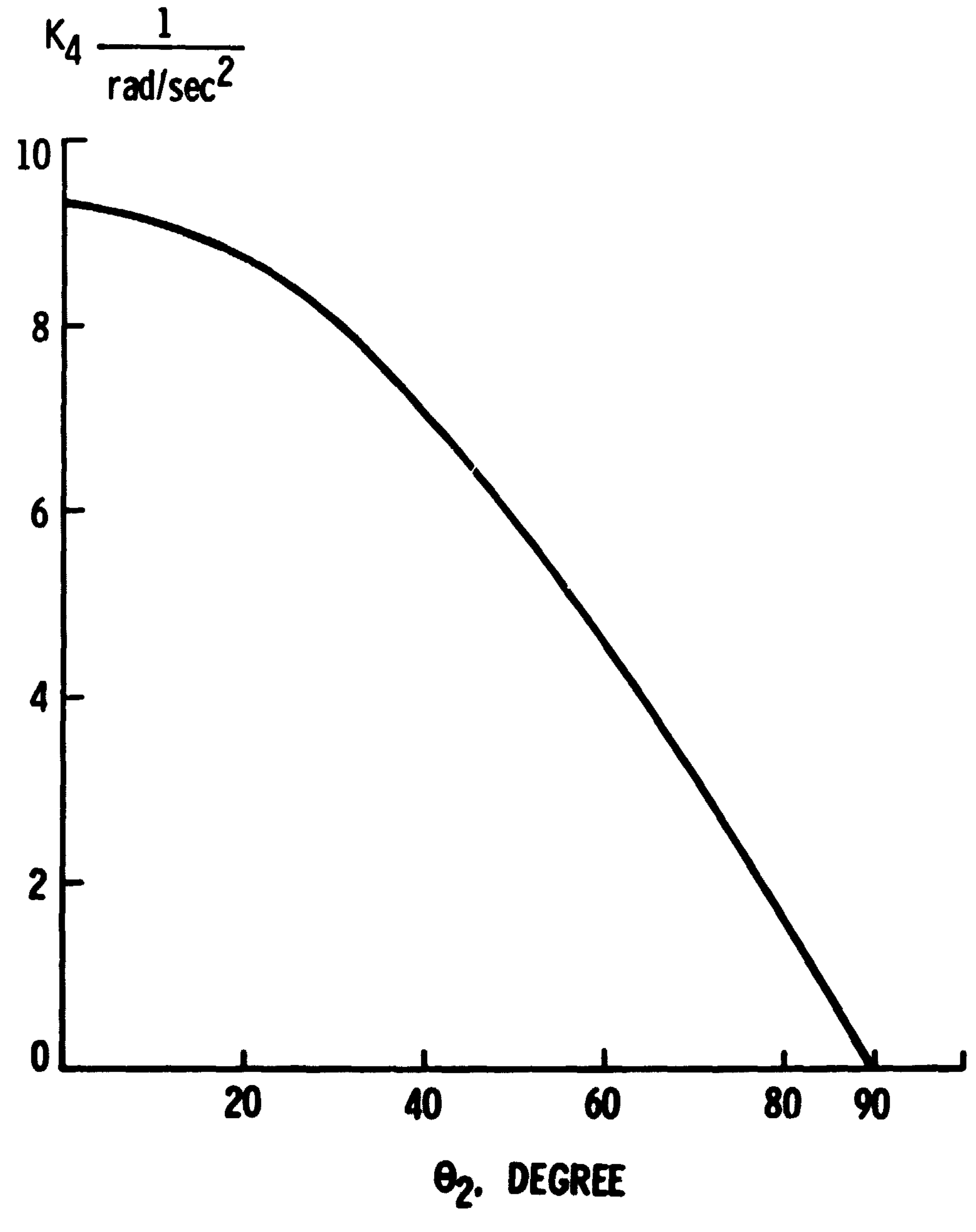

Figure 11. Relative Importance of Gravity versus Inertia Torque at Joint #5

nearly equal.† Again, the values of parameters d_3 and b_{18} in Eq. (140) are taken for the specified load held in the hand.

As a main conclusion, it is seen that the gravity terms at joints No. 2, 3, 4, and 5 have a relatively high significance versus the corresponding inertia terms. However, the overall significance of the gravity terms in the torque/force equations can only be evaluated when all relevant reaction torques/forces are also considered in the equations.

The four ratios given by Eqs. (137) through (140) and displayed in Figs. (8) through (11) are linear functions of the field of gravity "g." For "g" values smaller than the "g" on Earth (for instance on the Moon or Mars), the relative importance of gravity terms versus inertia terms at joint No.'s 2, 3, 4, and 5 of the JPL RRP manipulator would correspondingly decrease.

†Since K_1, K_2, K_3, K_4 are defined in terms of the corresponding simplified state functions which carry some error, the ratios displayed in Figs. 8 through 11 will also carry some error. In the average, however, the error in the ratios can be expected less than 8 – 10%.

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APPENDIX A

COMPLETE SET OF PARTIAL DERIVATIVE MATRICES U_{ji} IN FUNCTIONALLY EXPLICIT FORM FOR THE JPL RRP MANIPULATOR

The partial derivative matrix functions U_{ji} are essential "building blocks" in the general algorithm applied in this memo for the dynamic model of a manipulator. To derive the explicit functional relations for the D_i, D_{ij}, and D_{ijp} coefficients in the dynamic equations for the JPL RRP manipulator, the U_{ji} matrices must first be determined in a functionally explicit form for this particular manipulator.

The general function definition for the U_{ji} matrices is given in Section III in the main text. For easy reference, the functional meaning of the two running matrix indices in the U_{ji} notation is repeated here: the first index (j) always refers to the highest index number in the concatenated link transformation matrix, while the second index (i) always refers to the index number of the joint variable with respect to which the partial derivative is taken in the concatenated link transformation matrix. Consequently, $U_{ji} \neq 0$ only for $i \leq j$; otherwise for $i > j$ $U_{ji} \equiv 0$.

As seen from the definition, the U_{ji} matrices are functions of the manipulator joint variables and link displacement constants. In general, for a system of n joint variables a particular U_{ji} matrix becomes a function of all joint variables and link displacement constants starting from index 1 and going up to (and including) index j, but will be independent of the joint variables and link displacement constants which have index number greater than j. It is noted that the dimensionality of the U_{ji} matrices is 4 by 4.

In this Appendix all U_{ji} matrix functions which are pertinent to the JPL RRP manipulator are compiled in an expanded and functionally explicit form. Dealing with a six degrees-of-freedom manipulator ($i, j = 1, \ldots, 6$), and keeping in mind that $U_{ji} \neq 0$ only for $i \leq j$, we will have 21 U_{ji} matrix functions different from zero. The individual functional definitions for all 21 $U_{ji} \neq 0$ matrices are listed in Table A.1. The six individual link transformation matrices T_{i-1}^{i} ($i = 1, \ldots, 6$), upon which the explicit expansion of the 21 $U_{ji} \neq 0$ matrices of the JPL RRP manipulator is based, have been given previously in Table 2 of Ref. 1.

Table A.1. Individual Definitions for the U_{ji} Partial Derivative Matrices of the JPL RRP Manipulator

Note: $Q = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for i = 1, 2, 4, 5, 6 and $Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for i = 3

j \ i	1	2	3	4	5	6
1	QT_0^1	0	0	0	0	0
2	$QT_0^1T_1^2$	$T_0^1QT_1^2$	0	0	0	0
3	$QT_0^1T_1^2T_2^3$	$T_0^1QT_1^2T_2^3$	$T_0^1T_1^2QT_2^3$	0	0	0
4	$QT_0^1T_1^2T_2^3T_3^4$	$T_0^1QT_1^2T_2^3T_3^4$	$T_0^1T_1^2QT_2^3T_3^4$	$T_0^1T_1^2T_2^3QT_3^4$	0	0
5	$QT_0^1T_1^2T_2^3T_3^4T_4^5$	$T_0^1QT_1^2T_2^3T_3^4T_4^5$	$T_0^1T_1^2QT_2^3T_3^4T_4^5$	$T_0^1T_1^2T_2^3QT_3^4T_4^5$	$T_0^1T_1^2T_2^3T_3^4QT_4^5$	0
6	$QT_0^1T_1^2T_2^3T_3^4T_4^5T_5^6$	$T_0^1QT_1^2T_2^3T_3^4T_4^5T_5^6$	$T_0^1T_1^2QT_2^3T_3^4T_4^5T_5^6$	$T_0^1T_1^2T_2^3QT_3^4T_4^5T_5^6$	$T_0^1T_1^2T_2^3T_3^4QT_4^5T_5^6$	$T_0^1T_1^2T_2^3T_3^4T_4^5QT_5^6$

The subsequent 21 functionally explicit expressions for the U_{ji} matrices for this manipulator are not available elsewhere in the literature.

The parameter and variable definitions and notations used in this Appendix are identical to those specified in Tables 1 and 2 of Ref. 1. In particular, it is noted that we use the following short notations

$$s_i \equiv \sin \theta_i$$

$$c_i \equiv \cos \theta_i$$

in the subsequent expressions.

$$U_{11} = \begin{bmatrix} -s_1 & 0 & -c_1 & 0 \\ c_1 & 0 & -s_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.1}$$

$$U_{21} = \begin{bmatrix} -s_1c_2 & -c_1 & s_1s_2 & -r_2c_1 \\ c_1c_2 & -s_1 & c_1s_2 & -r_2s_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.2}$$

$$U_{31} = \begin{bmatrix} c_1 & -s_1c_2 & -s_1s_2 & -(r_3s_1s_2 + r_2c_1) \\ s_1 & c_1c_2 & c_1s_2 & (r_3c_1s_2 - r_2s_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.3}$$

$$U_{41} = \begin{bmatrix} (c_1c_4 - s_1c_2s_4) & s_1s_2 & -(c_1s_4 + s_1c_2c_4) & -(r_3s_1s_2 + r_2c_1) \\ (s_1c_4 + c_1c_2s_4) & -c_1s_2 & -(s_1s_4 - c_1c_2c_4) & (r_3c_1s_2 - r_2s_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A.4)$$

$$U_{51} = \begin{bmatrix} \begin{matrix}(c_1c_4c_5 \\ - s_1c_2s_4c_5 \\ + s_1s_2s_5)\end{matrix} & \begin{matrix}-(c_1s_4 \\ + s_1c_2c_4)\end{matrix} & \begin{matrix}(c_1c_4s_5 \\ - s_1c_2s_4s_5 \\ - s_1s_2c_5)\end{matrix} & \begin{matrix}-(r_3s_1s_2 \\ + r_2c_1)\end{matrix} \\ \begin{matrix}(s_1c_4c_5 \\ + c_1c_2s_4c_5 \\ - c_1s_2s_5)\end{matrix} & \begin{matrix}-(s_1s_4 \\ - c_1c_2c_4)\end{matrix} & \begin{matrix}(s_1c_4s_5 \\ + c_1c_2s_4s_5 \\ + c_1s_2c_5)\end{matrix} & \begin{matrix}(r_3c_1s_2 \\ - r_2s_1)\end{matrix} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A.5)$$

$$U_{61} = \begin{bmatrix} (c_1c_4c_5c_6 & (-c_1c_4c_5s_6 & (c_1c_4s_5 & \{r_6(c_1c_4s_5 \\ - s_1c_2s_4c_5c_6 & + s_1c_2s_4c_5s_6 & - s_1c_2s_4s_5 & - s_1c_2s_4s_5 \\ + s_1s_2s_5c_6 & - s_1s_2s_5s_6 & - s_1s_2c_5) & - s_1s_2c_5) \\ - c_1s_4s_6 & - c_1s_4c_6 & & - (r_3s_1s_2 \\ - s_1c_2c_4s_6) & - s_1c_2c_4c_6) & & + r_2c_1)\} \\ & & & \\ (s_1c_4c_5c_6 & (-s_1c_4c_5s_6 & (s_1c_4s_5 & \{r_6(s_1c_4s_5 \\ + c_1c_2s_4c_5c_6 & - c_1c_2s_4c_5s_6 & + c_1c_2s_4s_5 & + c_1c_2s_4s_5 \\ - c_1s_2s_5c_6 & + c_1s_2s_5s_6 & + c_1s_2c_5) & + c_1s_2c_5 \\ - s_1s_4s_6 & - s_1s_4c_6 & & + (r_3c_1s_2 \\ + c_1c_2c_4s_6) & + c_1c_2c_4c_6) & & - r_2s_1)\} \\ & & & \\ 0 & 0 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (A.6)$$

$$U_{22} = \begin{bmatrix} -c_1s_2 & 0 & c_1c_2 & 0 \\ -s_1s_2 & 0 & s_1c_2 & 0 \\ -c_2 & 0 & -s_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (A.7)$$

$$U_{32} = \begin{bmatrix} 0 & -c_1s_2 & c_1c_2 & r_3c_1c_2 \\ 0 & -s_1s_2 & s_1c_2 & r_3s_1c_2 \\ 0 & -c_2 & -s_2 & -r_3s_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (A.8)$$

$$U_{42} = \begin{bmatrix} -c_1 s_2 s_4 & -c_1 c_2 & -c_1 s_2 c_4 & r_3 c_1 c_2 \\ -s_1 s_2 s_4 & -s_1 c_2 & -s_1 s_2 c_4 & r_3 s_1 c_2 \\ -c_2 s_4 & s_2 & -c_2 c_4 & -r_3 s_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(A. 9)}$$

$$U_{52} = \begin{bmatrix} \begin{matrix} -(c_1 s_2 s_4 c_5 \\ + c_1 c_2 s_5) \end{matrix} & -c_1 s_2 c_4 & \begin{matrix} (-c_1 s_2 s_4 s_5 \\ + c_1 c_2 c_5) \end{matrix} & r_3 c_1 c_2 \\ \begin{matrix} -(s_1 s_2 s_4 c_5 \\ + s_1 c_2 s_5) \end{matrix} & -s_1 s_2 c_4 & \begin{matrix} (-s_1 s_2 s_4 s_5 \\ + s_1 c_2 c_5) \end{matrix} & r_3 s_1 c_2 \\ \begin{matrix} (-c_2 s_4 c_5 \\ + s_2 s_5) \end{matrix} & -c_2 c_4 & \begin{matrix} -(c_2 s_4 s_5 \\ + s_2 c_5) \end{matrix} & -r_3 s_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(A. 10)}$$

$$
U_{62} = \begin{bmatrix}
\begin{matrix}-(c_1 s_2 s_4 c_5 c_6\\ + c_1 c_2 s_5 c_6\\ + c_1 s_2 c_4 s_6)\end{matrix} & \begin{matrix}(c_1 s_2 s_4 c_5 s_6\\ + c_1 c_2 s_5 s_6\\ - c_1 s_2 c_4 c_6)\end{matrix} & \begin{matrix}(-c_1 s_2 s_4 s_5\\ + c_1 c_2 c_5)\end{matrix} & \begin{matrix}\{r_6(-c_1 s_2 s_4 s_5\\ + c_1 c_2 c_5)\\ + r_3 c_1 c_2\}\end{matrix} \\
\begin{matrix}-(s_1 s_2 s_4 c_5 c_6\\ + s_1 c_2 s_5 c_6\\ + s_1 s_2 c_4 s_6)\end{matrix} & \begin{matrix}(s_1 s_2 s_4 c_5 s_6\\ + s_1 c_2 s_5 s_6\\ - s_1 s_2 c_4 c_6)\end{matrix} & \begin{matrix}(-s_1 s_2 s_4 s_5\\ + s_1 c_2 c_5)\end{matrix} & \begin{matrix}\{r_6(-s_1 s_2 s_4 s_5\\ + s_1 c_2 c_5)\\ + r_3 s_1 c_2\}\end{matrix} \\
\begin{matrix}(-c_2 s_4 c_5 c_6\\ + s_2 s_5 c_6\\ - c_2 c_4 s_6)\end{matrix} & \begin{matrix}(c_2 s_4 c_5 s_6\\ - s_2 s_5 s_6\\ - c_2 c_4 c_6)\end{matrix} & \begin{matrix}-(c_2 s_4 s_5\\ + s_2 c_5)\end{matrix} & \begin{matrix}-\{r_6(c_2 s_4 s_5\\ + s_2 c_5)\\ + r_3 s_2\}\end{matrix} \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \text{(A. 11)}
$$

$$
U_{33} = \begin{bmatrix}
0 & 0 & 0 & c_1 s_2 \\
0 & 0 & 0 & s_1 s_2 \\
0 & 0 & 0 & c_2 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \text{(A. 12)}
$$

$$
U_{43} = \begin{bmatrix}
0 & 0 & 0 & c_1 s_2 \\
0 & 0 & 0 & s_1 s_2 \\
0 & 0 & 0 & c_2 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \text{(A. 13)}
$$

$$U_{53} = \begin{bmatrix} 0 & 0 & 0 & c_1 s_2 \\ 0 & 0 & 0 & s_1 s_2 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A. 14}$$

$$U_{63} = \begin{bmatrix} 0 & 0 & 0 & c_1 s_2 \\ 0 & 0 & 0 & s_1 s_2 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A. 15}$$

$$U_{44} = \begin{bmatrix} (-s_1 s_4 + c_1 c_2 c_4) & 0 & -(s_1 c_4 + c_1 c_2 s_4) & 0 \\ (c_1 s_4 + s_1 c_2 c_4) & 0 & (c_1 c_4 - s_1 c_2 s_4) & 0 \\ -s_2 c_4 & 0 & s_2 s_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A. 16}$$

$$U_{54} = \begin{bmatrix} (-s_1 s_4 c_5 + c_1 c_2 c_4 c_5) & -(s_1 c_4 + c_1 c_2 s_4) & (-s_1 s_4 s_5 + c_1 c_2 c_4 s_5) & 0 \\ (c_1 s_4 c_5 + s_1 c_2 c_4 c_5) & (c_1 c_4 - s_1 c_2 s_4) & (c_1 s_4 s_5 + s_1 c_2 c_4 s_5) & 0 \\ -s_2 c_4 c_5 & s_2 s_4 & -s_2 c_4 s_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A. 17}$$

$$U_{64} = \begin{bmatrix} (-s_1s_4c_5c_6 & (s_1s_4c_5s_6 & (-s_1s_4s_5 & r_6(-s_1s_4s_5 \\ +c_1c_2c_4c_5c_6 & -c_1c_2c_4c_5s_6 & +c_1c_2c_4s_5) & +c_1c_2c_4s_5) \\ -s_1c_4s_6 & -s_1c_4c_6 & & \\ -c_1c_2s_4s_6) & -c_1c_2s_4c_6) & & \\ (c_1s_4c_5c_6 & (-c_1s_4c_5s_6 & (c_1s_4s_5 & r_6(c_1s_4s_5 \\ +s_1c_2c_4c_5c_6 & -s_1c_2c_4c_5s_6 & +s_1c_2c_4s_5) & +s_1c_2c_4s_5) \\ +c_1c_4s_6 & +c_1c_4c_6 & & \\ -s_1c_2s_4s_6) & -s_1c_2s_4c_6) & & \\ (-s_2c_4c_5c_6 & (s_2c_4c_5s_6 & -s_2c_4s_5 & -r_6s_2c_4s_5 \\ +s_2s_4s_6) & +s_2s_4c_6) & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(A. 18)}$$

$$U_{55} = \begin{bmatrix} -(s_1c_4s_5 & 0 & (s_1c_4c_5 & 0 \\ +c_1c_2s_4s_5 & & +c_1c_2s_4c_5 & \\ +c_1s_2c_5) & & -c_1s_2s_5) & \\ (c_1c_4s_5 & 0 & (-c_1c_4c_5 & 0 \\ -s_1c_2s_4s_5 & & +s_1c_2s_4c_5 & \\ -s_1s_2c_5) & & -s_1s_2s_5) & \\ (s_2s_4s_5 & 0 & -(s_2s_4c_5 & 0 \\ -c_2c_5) & & +c_2s_5) & \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(A. 19)}$$

$$U_{65} = \begin{bmatrix} -(s_1c_4s_5c_6 & (s_1c_4s_5s_6 & (s_1c_4c_5 & r_6(s_1c_4c_5 \\ \quad +c_1c_2s_4s_5c_6 & \quad +c_1c_2s_4s_5s_6 & \quad +c_1c_2s_4c_5 & \quad +c_1c_2s_4c_5 \\ \quad +c_1s_2c_5c_6) & \quad +c_1s_2c_5s_6) & \quad -c_1s_2s_5) & \quad -c_1s_2s_5) \\ & & & \\ (c_1c_4s_5c_6 & (-c_1c_4s_5s_6 & (-c_1c_4c_5 & r_6(-c_1c_4c_5 \\ \quad -s_1c_2s_4s_5c_6 & \quad +s_1c_2s_4s_5s_6 & \quad +s_1c_2s_4c_5 & \quad +s_1c_2s_4c_5 \\ \quad -s_1s_2c_5c_6) & \quad +s_1s_2c_5s_6) & \quad -s_1s_2s_5) & \quad -s_1s_2s_5) \\ & & & \\ (s_2s_4s_5c_6 & (-s_2s_4s_5s_6 & -(s_2s_4c_5 & -r_6(s_2s_4c_5 \\ \quad -c_2c_5c_6) & \quad -c_2c_5s_6) & \quad +c_2s_5) & \quad +c_2s_5) \\ & & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (A.20)$$

$$U_{66} = \begin{bmatrix} (-s_1c_4c_5s_6 & (-s_1c_4c_5c_6 & 0 & 0 \\ \quad -c_1c_2s_4c_5s_6 & \quad -c_1c_2s_4c_5c_6 & & \\ \quad +c_1s_2s_5s_6 & \quad +c_1s_2s_5c_6 & & \\ \quad -s_1s_4c_6 & \quad +s_1s_4s_6 & & \\ \quad +c_1c_2c_4c_6) & \quad -c_1c_2c_4s_6) & & \\ & & & \\ (c_1c_4c_5s_6 & (c_1c_4c_5c_6 & 0 & 0 \\ \quad -s_1c_2s_4c_5s_6 & \quad -s_1c_2s_4c_5c_6 & & \\ \quad +s_1s_2s_5s_6 & \quad +s_1s_2s_5c_6 & & \\ \quad +c_1s_4c_6 & \quad -c_1s_4s_6 & & \\ \quad +s_1c_2c_4c_6) & \quad -s_1c_2c_4s_6) & & \\ & & & \\ (s_2s_4c_5s_6 & (s_2s_4c_5c_6 & 0 & 0 \\ \quad +c_2s_5s_6 & \quad +c_2s_5c_6 & & \\ \quad -s_2c_4c_6) & \quad +s_2c_4s_6) & & \\ & & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (A.21)$$

APPENDIX B

LINK MASS CENTER VECTORS AND PSEUDO INERTIA MATRICES FOR THE JPL RRP MANIPULATOR

Using the notations introduced in Section III in the main text, and applying the parameter values determined for the JPL RRP manipulator elsewhere (see footnote on p. 47), the six link mass center vectors and the six pseudo inertia matrices are compiled in this appendix. The essential point in the subsequent listing is to distinguish betwe n zero and non-zero parameter values. The actual numerical values for the non-zero inertial parameters are supplied at the end of this Appendix.

Mass Center Vectors

$$\bar{\rho}_1 = \begin{bmatrix} 0 \\ \bar{y}_1 \\ \bar{z}_1 \\ 1 \end{bmatrix} \qquad \bar{\rho}_4 = \begin{bmatrix} 0 \\ \bar{y}_4 \\ \bar{z}_4 \\ 1 \end{bmatrix}$$

$$\bar{\rho}_2 = \begin{bmatrix} 0 \\ \bar{y}_2 \\ \bar{z}_2 \\ 1 \end{bmatrix} \qquad \bar{\rho}_5 = \begin{bmatrix} 0 \\ 0 \\ \bar{z}_5 \\ 1 \end{bmatrix}$$

$$\bar{\rho}_3 = \begin{bmatrix} 0 \\ 0 \\ \bar{z}_3 \\ 1 \end{bmatrix} \qquad \bar{\rho}_6 = \begin{bmatrix} 0 \\ 0 \\ \bar{z}_6 \\ 1 \end{bmatrix}$$

As seen, for $\bar{y}_5$ and $\bar{y}_6$ we use zero since their numerical value is very small.

Pseudo Inertia Matrices

Note: $k_{i11}^2 = k_{ixx}^2$, $k_{i22}^2 = k_{iyy}^2$, $k_{i33}^2 = k_{izz}^2$ and the first index (i) refers to the index number of the link.

$$J_1 = m_1 \begin{bmatrix} \frac{1}{2}\left(-k_{111}^2 + k_{122}^2 + k_{133}^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(k_{111}^2 - k_{122}^2 + k_{133}^2\right) & 0 & \bar{y}_1 \\ 0 & 0 & \frac{1}{2}\left(k_{111}^2 + k_{122}^2 - k_{133}^2\right) & \bar{z}_1 \\ 0 & \bar{y}_1 & \bar{z}_1 & 1 \end{bmatrix}$$

$$J_2 = m_2 \begin{bmatrix} \frac{1}{2}\left(-k_{211}^2 + k_{222}^2 + k_{233}^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(k_{211}^2 - k_{222}^2 + k_{233}^2\right) & 0 & \bar{y}_2 \\ 0 & 0 & \frac{1}{2}\left(k_{211}^2 + k_{222}^2 - k_{233}^2\right) & \bar{z}_2 \\ 0 & \bar{y}_2 & \bar{z}_2 & 1 \end{bmatrix}$$

$$J_3 = m_3 \begin{bmatrix} \frac{1}{2}\left(-k_{311}^2 + k_{322}^2 + k_{333}^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(k_{311}^2 - k_{322}^2 + k_{333}^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(k_{311}^2 + k_{322}^2 - k_{333}^2\right) & \bar{z}_3 \\ 0 & 0 & \bar{z}_3 & 1 \end{bmatrix}$$

$$J_4 = m_4 \begin{bmatrix} \frac{1}{2}\left(-k_{411}^2 + k_{422}^2 + k_{433}^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(k_{411}^2 - k_{422}^2 + k_{433}^2\right) & 0 & \bar{y}_4 \\ 0 & 0 & \frac{1}{2}\left(k_{411}^2 + k_{422}^2 - k_{433}^2\right) & \bar{z}_4 \\ 0 & \bar{y}_4 & \bar{z}_4 & 1 \end{bmatrix}$$

$$J_5 = m_5 \begin{bmatrix} \frac{1}{2}\left(-k_{511}^2 + k_{522}^2 + k_{533}^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(k_{511}^2 - k_{522}^2 + k_{533}^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(k_{511}^2 + k_{522}^2 - k_{533}^2\right) & \bar{z}_5 \\ 0 & 0 & \bar{z}_5 & 1 \end{bmatrix}$$

$$J_6 = m_6 \begin{bmatrix} \frac{1}{2}\left(-k_{611}^2 + k_{622}^2 + k_{633}^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(k_{611}^2 - k_{622}^2 + k_{633}^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(k_{611}^2 + k_{622}^2 - k_{633}^2\right) & \bar{z}_6 \\ 0 & 0 & \bar{z}_6 & 1 \end{bmatrix}$$

Remarks on loads in the manipulator hand, and the inertial characteristics of link #6:

It is noted that the mass center vector $\bar{\rho}_6$ and "pseudo inertia matrix" J_6 as specified in this appendix are only related to the fixed, constant structure of link #6. (Link #6 includes also the hand.) If the manipulator hand keeps and moves a load, then the inertial properties of the load should be properly added to the inertial properties of link #6. That is, the value of the $\bar{\rho}_6$ vector and J_6 matrix should be modified according to the inertial properties of the load. Changes in torques (and force) due to a load kept and moved by the hand will be "felt" (and can also be computed) at the different joints through the appropriate modifications of the value of the $\bar{\rho}_6$ vector and J_6 matrix.

Clearly, when handling irregular (and, by definition, "remote") objects with mass comparable to the mass of link #6, only compensating estimates can be made for changes in the inertial properties of link #6. Even when handling regular objects, the changes in the inertial properties of link #6 can only be estimated, since it is not known ahead of time how the grasping operation will exactly succeed in emplacing the object relative to the hand coordinate frame, or which is the same, relative to the coordinate frame of link #6.

The effect of handling loads (that is, the effect of modifications in the inertial properties of link #6) on some of the manipulator dynamic coefficients is shown in the subsequent appendix.

Numerical values of non-zero inertial components of the JPL RRP manipulator determined elsewhere (see footnote on p. 47), and applied in this report are as follows:

m_1	$= 9.29$ kg	= (0.849 (oz-sec^2)/in)
$\bar{y}_1$	$= 1.75$ cm	= (0.69 in)
$\bar{z}_1$	$= -11.05$ cm	= (-4.35 in)
$m_1k_{111}^2$	$= 0.276$ kg$\cdot$m^2	= (39.1 oz-in-sec^2)
$m_1k_{122}^2$	$= 0.255$ kg$\cdot$m^2	= (36.15 oz-in-sec^2)
$m_1k_{133}^2$	$= 0.071$ kg$\cdot$m^2	= (10.03 oz-in-sec^2)
m_2	$= 5.505$ kg	= (0.513 (oz-sec^2)/in)
$\bar{y}_2$	$= -10.54$ cm	= (-4.15 in)
$\bar{z}_2$	$= -0.79$ cm	= (-0.31 in)
$m_2k_{211}^2$	$= 0.108$ kg$\cdot$m^2	= (15.28 oz-in-sec^2)
$m_2k_{222}^2$	$= 0.018$ kg$\cdot$m^2	= (2.49 oz-in-sec^2)
$m_2k_{233}^2$	$= 0.1$ kg$\cdot$m^2	= (14.13 oz-in-sec^2)
m_3	$= 4.25$ kg	= (0.388 (oz-sec^2)/in)
$\bar{z}_3$	$= -64.47$ cm	= (-25.38 in)
$m_3k_{311}^2$	$= 2.51$ kg$\cdot$m^2	= (355.5 oz-in-sec^2)
$m_3k_{322}^2$	$= 2.51$ kg$\cdot$m^2	= (355.5 oz-in-sec^2)
$m_3k_{333}^2$	$= 0.006$ kg$\cdot$m^2	= (0.854 oz-in-sec^2)

m_4	$= 1.08$ kg	$= (0.099\ (\text{oz-sec}^2)/\text{in})$
$\bar{y}_4$	$= 0.92$ cm	$= (0.364$ in$)$
$\bar{z}_4$	$= -0.54$ cm	$= (-0.212$ in$)$
$m_4k^2_{411}$	$= 0.002\ \text{kg}\cdot\text{m}^2$	$= (0.253\ \text{oz-in-sec}^2)$
$m_4k^2_{422}$	$= 0.001\ \text{kg}\cdot\text{m}^2$	$= (0.167\ \text{oz-in-sec}^2)$
$m_4k^2_{433}$	$= 0.001\ \text{kg}\cdot\text{m}^2$	$= (0.156\ \text{oz-in-sec}^2)$
m_5	$= 0.63$ kg	$= (0.058\ (\text{oz-sec}^2)/\text{in})$
$\bar{y}_5$	$= 0.03 \approx 0$ cm	$= (0.01 \simeq 0$ in$)$
$\bar{z}_5$	$= 5.66$ cm	$= (2.23$ in$)$
$m_5k^2_{511}$	$= 0.003\ \text{kg}\cdot\text{m}^2$	$= (0.385\ \text{oz-in-sec}^2)$
$m_5k^2_{522}$	$= 0.003\ \text{kg}\cdot\text{m}^2$	$= (0.360\ \text{oz-in-sec}^2)$
$m_5k^2_{533}$	$= 0.0004\ \text{kg}\cdot\text{m}^2$	$= (0.057\ \text{oz-in-sec}^2)$
m_6	$= 0.51$ kg	$= (0.047\ (\text{oz-sec}^2)/\text{in})$
$\bar{y}_6$	$= 0.14 \approx 0$ cm	$= (0.057 \simeq 0$ in$)$
$\bar{z}_6$	$= -9.22$ cm	$= (-3.63$ in$)$
$m_6k^2_{611}$	$= 0.005\ \text{kg}\cdot\text{m}^2$	$= (0.667\ \text{oz-in-sec}^2)$
$m_6k^2_{622}$	$= 0.005\ \text{kg}\cdot\text{m}^2$	$= (0.667\ \text{oz-in-sec}^2)$
$m_6k^2_{633}$	$= 0.0003\ \text{kg}\cdot\text{m}^2$	$= (0.049\ \text{oz-in-sec}^2)$

The two geometric parameters of the JPL RRP manipulator applied in the calculations are:

$r_2 = 16.2$ cm	(6.375 in)
$r_6 = 24.76$ cm	(9.75 in)

The input shaft inertias referred to the output are as follows:

At joint No. 1:	0.953 $kg \cdot m^2$	(135 $oz\text{-}in\text{-}sec^2$)
At joint No. 2:	2.193 $kg \cdot m^2$	$(310.6\ oz\text{-}in\text{-}sec)^2$
At joint No. 3:	0.782 kg	(0.07143 $(oz\text{-}sec^2)/in$)*
At joint No. 4:	0.106 $kg \cdot m^2$	(15. $oz\text{-}in\text{-}sec^2$)
At joint No. 5:	0.097 $kg \cdot m^2$	(13.7 $oz\text{-}in\text{-}sec^2$)
At joint No. 6:	0.02 $kg \cdot m^2$	(2.81 $oz\text{-}in\text{-}sec^2$)

Derived metric conversion factors applied in this report are as follows:

Length:	**1 in**	**= 2.54 cm**
Mass:	**1 $(oz\text{-}sec^2)/in$**	**= 10.945 kg**
Static moment:	**1 $oz\text{-}sec^2$**	**= 0.278 $kg \cdot m$**
Moment of inertia:	**1 $oz\text{-}in\text{-}sec^2$**	**= 0.00706 $kg \cdot m^2$**
Force:	**1 oz**	**= 0.278 N**
Torque	**1 oz-in**	**= 0.00706 $N \cdot m$**

***Equivalent mass.**

APPENDIX C

MANIPULATOR DYNAMICS WITH LOAD IN THE HAND

Suppose that a load in the hand will cause an offset in the mass center of link #6 so that $\bar{x}_6 \neq 0$ and $\bar{y}_6 \neq 0$ together with $\bar{z}_6 \neq 0$. That is, the effective form of the mass center vector $\bar{\rho}_6$ becomes:

$$\bar{\rho}_6 = \begin{bmatrix} \bar{x}_6 \\ \bar{y}_6 \\ \bar{z}_6 \\ 1 \end{bmatrix}$$

Of course, the effective form of the "pseudo inertia matrix" J_6 becomes also modified through the non-zero values of $\bar{x}_6$ and $\bar{y}_6$:

$$J_6 = m_6 \begin{bmatrix} \frac{1}{2}\left(-k_{611}^2 + k_{622}^2 + k_{633}^2\right) & 0 & 0 & \bar{x}_6 \\ 0 & \frac{1}{2}\left(k_{611}^2 - k_{622}^2 + k_{633}^2\right) & 0 & \bar{y}_6 \\ 0 & 0 & \frac{1}{2}\left(k_{611}^2 + k_{622}^2 - k_{633}^2\right) & \bar{z}_6 \\ \bar{x}_6 & \bar{y}_6 & \bar{z}_6 & 1 \end{bmatrix}$$

To illustrate the effect of the non-zero values of $\bar{x}_6$ and $\bar{y}_6$ on the manipulator dynamics, the necessary modifications for some of the dynamic coefficients are evaluated in explicit form, and listed below.

1. Modifications in the gravity terms caused by $\bar{x}_6 \neq 0$, $\bar{y}_6 \neq 0$

a) For joint #2:

The following terms should be added to D_2 given by Eq. (48):

$$+gm_6\left[\bar{x}_6(c\theta_2 s\theta_4 c\theta_5 c\theta_6 - s\theta_2 s\theta_5 c\theta_6 + c\theta_2 c\theta_4 s\theta_6)\right.$$

$$\left. + \bar{y}_6(-c\theta_2 s\theta_4 c\theta_5 s\theta_6 + s\theta_2 s\theta_5 s\theta_6 + c\theta_2 c\theta_4 c\theta_6)\right] \quad \text{(C. 1)}$$

b) For joint #4:

The following terms should be added to D_4 given by Eq. (52):

$$+gm_6 s\theta_2 \left[\bar{x}_6(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6)\right.$$

$$\left. - \bar{y}_6(c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6)\right] \quad \text{(C. 2)}$$

c) For joint #5:

The following terms should be added to D_5 given by Eq. (54):

$$+gm_6 \left[\bar{x}_6 c\theta_6(-s\theta_2 s\theta_4 s\theta_5 + c\theta_2 c\theta_5)\right.$$

$$\left. + \bar{y}_6 s\theta_6(s\theta_2 s\theta_4 s\theta_5 - c\theta_2 c\theta_5)\right] \quad \text{(C. 3)}$$

d) For joint #6:

If $\bar{x}_6$ and/or $\bar{y}_6$ are different from zero, then D_6 will also be different from zero. Instead of Eq. (56), we will have now:

$$\boxed{\begin{aligned} D_6 = -gm_6 \Big[&\bar{x}_6(s\theta_2 s\theta_4 c\theta_5 s\theta_6 + c\theta_2 s\theta_5 s\theta_6 - s\theta_2 c\theta_4 c\theta_6) \\ &+ \bar{y}_6(s\theta_2 s\theta_4 c\theta_5 c\theta_6 + c\theta_2 s\theta_5 c\theta_6 + s\theta_2 c\theta_4 s\theta_6)\Big] \end{aligned}} \quad \text{(C. 4)}$$

It is noted that $\bar{x}_6 \neq 0$ and $\bar{y}_6 \neq 0$ cannot have any effect on D_1 and D_3.

2. Modifications in the acceleration-related uncoupled terms caused by $\bar{x}_6 \neq 0$, $\bar{y}_6 \neq 0$

The D_{66} and D_{55} dynamic coefficients given by Eqs. (68) and (66), respectively, will remain unaffected by $\bar{x}_6 \neq 0$ and/or $\bar{y}_6 \neq 0$. Furthermore, $\bar{x}_6 \neq 0$ and/or $\bar{y}_6 \neq 0$ cannot have any effect on D_{33}. Only D_{44}, D_{22}, and D_{11} will be modified due to $\bar{x}_6 \neq 0$ and/or $\bar{y}_6 \neq 0$.

a) Modification for D_{44}:

The following term should be added to D_{44} given by Eq. (64):

$$+2m_6 r_6 s\theta_5 c\theta_5(\bar{x}_6 c\theta_6 - \bar{y}_6 s\theta_6) \qquad \text{(C. 5)}$$

b) Modifications for D_{22} and D_{11}:

The following terms should be added to D_{22} given by Eq. (60):

$$+2m_6 \Big\{ \bar{x}_6 \left[r_6 s\theta_4 s\theta_5(c\theta_4 s\theta_6 + s\theta_4 c\theta_5 c\theta_6) - (r_6 c\theta_5 + r_3)s\theta_5 s\theta_6\right]$$
$$+ \bar{y}_6 \left[s\theta_5 s\theta_6(r_6 c\theta_5 + r_3) - r_6 s\theta_4 s\theta_5(s\theta_4 c\theta_5 s\theta_6 - c\theta_4 c\theta_6)\right]\Big\} \qquad \text{(C. 6)}$$

The following terms should be added to D_{11} given by Eq. (58):

$$+2m_6\bar{x}_6 \Big\{\left[r_6(c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5) - r_3 s\theta_2\right](c\theta_2 s\theta_4 c\theta_5 c\theta_6 - s\theta_2 s\theta_5 c\theta_6 + c\theta_2 c\theta_4 s\theta_6)$$
$$+ (r_6 c\theta_4 s\theta_5 - r_2)(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6)\Big\}$$
$$\text{(C. 7)}$$
$$+2m_6\bar{y}_6 \Big\{\left[r_6(c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5) + r_3 s\theta_2\right](s\theta_2 s\theta_5 s\theta_6 + c\theta_2 c\theta_4 c\theta_6 - c\theta_2 s\theta_4 c\theta_5 s\theta_6)$$
$$+ (r_2 - r_6 c\theta_4 s\theta_5)(c\theta_4 c\theta_5 c\theta_6 + s\theta_4 c\theta_6)\Big\}$$

It is noted that the number values of the k^2_{611}, k^2_{622}, and k^2_{633} radius of gyration terms will also be changed in the "pseudo inertia matrix" when there is a load in the hand. Of course, this change will not produce additional terms in the state functions for the D_{ii} dynamic coefficients; it will only change the constant number values of the k^2_{611}, k^2_{622}, and k^2_{633} parameters whenever they appear in the state functions for the D_{ii} dynamic coefficients, Eqs. (58), (60), (64), (66), (68).†)

†) It is assumed here that the cross products in the "pseudo inertia matrix" J_6 will remain zero. If this is an unsatisfactory approximation, then additional terms will appear in the state functions for the D_{ii} dynamical coefficients.

APPENDIX D

SIMPLIFICATION OF THE GENERAL MATRIX ALGORITHM FOR MANIPULATOR DYNAMICS

The general algorithm for manipulator dynamics applied in Refs. 5 through 8, and employed also in our analysis, is given by Eq. (1) in the main text of this report. According to Eq. (1), the dynamic coefficients D_{ij} and D_{ijk} in the general equations of manipulator motion are expressed in terms of the Trace of the products of 4 by 4 matrices. Essential "building blocks" of the matrix products are the first and second partial derivative matrices U_{ij} and U_{ijk} defined by Eqs. (4) and (7) in the main text. The purpose of this Appendix is to show that the application of the U_{ij} and U_{ijk} matrices in the complete form as defined by Eqs. (4) and (7) is unnecessary in the computation of the acceleration- and velocity-related dynamic coefficients D_{ij} and D_{ijk}.

Statement:

All link coordinate transformation matrices T_0^1, T_1^2, ... which have upper index number smaller than the smallest upper index number (say "i") of a derivative matrix QT_{i-1}^i can be omitted from the Trace of matrix products corresponding to the definitions of the D_{ij} and D_{ijk} dynamic coefficients given by the matrix algorithm of Eq. (1).

For instance, according to the Statement, the D_{55} inertial term and the $D_{4,56}$ Coriolis term can be computed using the following simplified formula:

$$D_{55} = \mathrm{Tr}\left[\underbrace{T_0^1T_1^2T_2^3T_3^4}_{\text{omit!}}\,QT_4^5\,J_5\left(\underbrace{T_0^1T_1^2T_2^3T_3^4}_{\text{omit!}}\,QT_4^5\right)^T\right]$$

$$+\ \mathrm{Tr}\left[\underbrace{T_0^1T_1^2T_2^3T_3^4}_{\text{omit!}}\,QT_4^5T_5^6\,J_6\left(\underbrace{T_0^1T_1^2T_2^3T_3^4}_{\text{omit!}}\,QT_4^5T_5^6\right)^T\right]$$

$$D_{4,56} = \mathrm{Tr}\left[\underbrace{T_0^1 T_1^2 T_2^3}_{\text{omit!}}\, T_3^4 Q T_4^5 Q T_5^6 J_6 \left(\underbrace{T_0^1 T_1^2 T_2^3}_{\text{omit!}}\, Q T_3^4 T_4^5 T_5^6\right)^T\right]$$

As seen in the two examples quoted above, the introduced simplification reduces the computational complexity substantially. In the case of D_{55}, the original formula calls for the evaluation of the Trace of the product of 13 and 15 matrices, while the introduced simplified formula calls for the evaluation of the Trace of the product of 5 and 7 matrices only. In the case of $D_{4,56}$, the original formula requires the computation of the Trace of the product of 16 matrices, while the proposed simplified formula requires the computation of the Trace of the product of 10 matrices only. (It is recalled that all matrices are 4 by 4 matrices.)

The validity of the introduced simplification of the algorithmic formulas for the D_{ij} and D_{ijk} dynamic coefficients for any manipulator can be shown by general matrix manipulations elaborated briefly below. The essence of the proof is to show that the effect of the link coordinate transformation matrices omitted from the Trace of matrix products is equivalent to the effect of the identity matrix in the chain-product of matrices. To make the proof concise, two lemmas will be stated which are related to the properties of the general 4 by 4 link coordinate transformation matrix T_{i-1}^i:

$$T_{i-1}^i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & r_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(D.1)}$$

Lemma 1: **The general structure of the product**

$$R^{ik} \triangleq \left(T_{k-1}^k\right)^T \cdots \left(T_{i-1}^i\right)^T T_{i-1}^i \cdots T_{k-1}^k$$

for any "i" and "k" is as follows:

$$R^{ik} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & R_{14}^{ik} \\ 0 & 1 & 0 & R_{24}^{ik} \\ 0 & 0 & 1 & R_{34}^{ik} \\ \hline R_{14}^{ik} & R_{24}^{ik} & R_{34}^{ik} & R_{44}^{ik} \end{array}\right] = \left[\begin{array}{c|c} r_{11} & r_{12} \\ \hline r_{12}^T & r_{22} \end{array}\right] \qquad \text{(D. 2)}$$

where the r_{11} submatrix is always the 3 by 3 identity matrix and $r_{21} = r_{12}^T$. (That is, R^{ik} is a symmetric 4 by 4 matrix.) Lemma 1 can be proved by direct multiplication and induction.

<u>Lemma 2:</u> The general structure of the product

$$B^{ik} \triangleq T_{i-1}^{i} \cdots T_{k-1}^{k}$$

for any "i" and "k" is as follows:

$$B^{ik} = \left[\begin{array}{ccc|c} B_{11}^{ik} & B_{12}^{ik} & B_{13}^{ik} & B_{14}^{ik} \\ B_{21}^{ik} & B_{22}^{ik} & B_{23}^{ik} & B_{24}^{ik} \\ B_{31}^{ik} & B_{32}^{ik} & B_{33}^{ik} & B_{34}^{ik} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} b_{11} & b_{12} \\ \hline 0 & 1 \end{array}\right] \qquad \text{(D. 3)}$$

that is, the b_{21} submatrix is always zero, and b_{22} is always equal to 1. Lemma 2 can be proved by direct multiplication and induction.

Let the following partitioning be introduced for a symmetric matrix P, a skew-symmetric matrix Q, and an elementary matrix $\bar{Q}$:

$$P = \left[\begin{array}{ccc|c} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12} & P_{22} & P_{23} & P_{24} \\ P_{13} & P_{23} & P_{33} & P_{34} \\ \hline P_{14} & P_{24} & P_{34} & P_{44} \end{array}\right] = \left[\begin{array}{c|c} p_{11} & p_{12} \\ \hline p_{12}^T & p_{22} \end{array}\right] \qquad \text{(D. 4)}$$

where p_{11} is a symmetric 3 by 3 submatrix, $p_{11}^T = p_{11}$. Further,

$$Q = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} q_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \text{(D. 5)}$$

where q_{11} is a skew-symmetric 3 by 3 submatrix, $q_{11}^T = - q_{11}$. (It is obvious that $Q^T = - Q$.) Further,

$$\bar{Q} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & \bar{q}_{12} \\ \hline 0 & 0 \end{array}\right] \qquad \text{(D. 6)}$$

where $\bar{q}_{12}$ is an elementary 3 by 1 submatrix. It is noted that the Q and $\bar{Q}$ are the differential operator matrices related to rotary and linear joints, respectively, while the symmetric P matrix is simply identical to the symmetric pseudo inertia matrix J_k (see Eq. (10)), or it is constructed as $P = B^{ik}J_k(B^{ik})^T$ where B^{ik} is given by Eq. (D. 3).

The following rules related to the Trace operator are recalled:

$$Tr(C) = Tr(C)^T \qquad \text{(D. 7)}$$

for a square matrix C, and

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB) \tag{D.8}$$

if A is of order $(m \times k)$, B of order $(k \times r)$ and C of order $(r \times m)$. Of course, if we define, for instance, $BC \triangleq D$, then we also have: $\mathrm{Tr}(AD) = \mathrm{Tr}(DA)$.

Using the properties and rules stated by Eqs. (D.2) through (D.8), the validity of the introduced simplification of the algorithmic formulas for the D_{ij} and D_{ijk} dynamic coefficients for <u>any</u> manipulator can be proved through the following steps:

1. Rearrangement of the chain product of the 4 by 4 matrices under the Trace operator so that the R^{ik} matrix product group will be isolated. (It is noted that for R^{ik} $i = 1$ and $k = j - 1$, j being the lowest index number for a derivative matrix which appears in the general formula.)

2. Then, the matrices under the Trace operator are arranged in a form Tr(PM) where M is a chain product of matrices containing also the R^{ik} matrix.

3. Finally, the elements of the M matrix are determined by direct multiplication in a partitioned form similar to the partitioning introduced in Eqs. (D.2) through (D.6) for the R^{ik}, B^{ik}, P, Q, and $\bar{Q}$ matrices. This last step then reveals that the four submatrices of M

$$M = \left[\begin{array}{c|c} m_{11} & m_{12} \\ \hline m_{21} & m_{22} \end{array}\right]$$

will only contain the submatrix r_{11}, that is, the r_{12} and r_{22} submatrices of R^{ik} will not appear in the four submatrices of M. Since the remaining r_{11} submatrix is the identity matrix, it (or equivalently, the R^{ik} matrix) can be omitted from the M matrix. That proves the validity of the algorithmic simplifications for D_{ij} and D_{ijk} stated in this Appendix.

The three major steps of proof for the algorithmic simplification of the D_{ij} and D_{ijk} dynamic coefficients are compiled in the subsequent pages for different combinations of rotary and linear joints. The derived formulas reveal the nature of the introduced algorithmic simplifications in detail and immediately show some interesting structural and symmetry relations for the different dynamic coefficients.

A. Acceleration-Related Dynamic Coefficients

1. Diagonal Coefficients, D_{ii}

a. Rotary joints.

The general component of D_{ii} takes the following form after rearrangement:†

$$Tr(RQBPB^TQ^T) = Tr(P\underbrace{B^TQ^TRQB}_{M})$$

Direct multiplication results

$$M = \left[\begin{array}{c:c} b^T_{11}q^T_{11}\overset{\text{omit}}{r_{11}}q_{11}b_{11} & b^T_{11}q^T_{11}\overset{\text{omit}}{r_{11}}q_{11}b_{12} \\ \hdashline b^T_{12}q^T_{11}\underset{\text{omit}}{r_{11}}q_{11}b_{11} & b^T_{12}q^T_{11}\underset{\text{omit}}{r_{11}}q_{11}b_{12} \end{array}\right] \tag{D.9}$$

Consequently,

$$\mathbf{Tr(PB^TQ^TRQB) \Leftrightarrow Tr(PB^TQ^TQB)} \tag{D.10}$$

†**For clarity in writing, the superscripts are omitted from the R and B matrices.**

b) Linear joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(R\bar{Q}BPB^T\bar{Q}^T) = \mathrm{Tr}(PB^T\underbrace{\bar{Q}^TR\bar{Q}}_{M}B)$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underbrace{\bar{q}_{12}^T r_{11} \bar{q}_{12}}_{\text{omit}} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \qquad \text{(D.11)}$$

Consequently,

$$\mathrm{Tr}(PB^T\bar{Q}^TR\bar{Q}B) \Leftrightarrow \mathrm{Tr}(PB^T\bar{Q}^T\bar{Q}B) = \mathrm{Tr}(PM) = p_{22} \qquad \text{(D.12)}$$

where p_{22} is simply the mass of a link.

2. Off-Diagonal Coefficients D_{ij}

a) Two rotary joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(RQBPQ^TB^T) = \mathrm{Tr}(PQ^T\underbrace{B^TRQ}_{M}B)$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} q_{11}^T b_{11}^T \overset{\text{omit}}{r_{11}} q_{11} b_{11} & q_{11}^T b_{11}^T \overset{\text{omit}}{r_{11}} q_{11} b_{12} \\ \hline 0 & 0 \end{array}\right] \tag{D.13}$$

Consequently,

$$\mathrm{Tr}(PQ^T B^T RQB) \Leftrightarrow \mathrm{Tr}(PQ^T B^T QB) \tag{D.14}$$

The symmetry $D_{ij} = D_{ji}$ is easily seen since

$$\mathrm{Tr}(PQ^T B^T QB) = \mathrm{Tr}(PB^T Q^T BQ)$$

b) <u>One linear and one rotary joint.</u>

Rearrangement yields for the general term:

$$\mathrm{Tr}(R\bar{Q}BPQ^T B^T) = \mathrm{Tr}(P\underbrace{Q^T B^T R\bar{Q}B}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & q_{11}^T b_{11}^T \overset{\text{omit}}{r_{11}} \bar{q}_{12} \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & m_{12} \\ \hline 0 & 0 \end{array}\right] \tag{D.15}$$

Consequently,

$$\mathrm{Tr}(PQ^T B^T R\bar{Q}B) \Leftrightarrow \mathrm{Tr}(PQ^T B^T \bar{Q}B) = p_{12}^T m_{12} \qquad \text{(D.16)}$$

Since $p_{12}^T m_{12}$ is a scalar, the symmetry $D_{ij} = D_{ji}$ in this case is obvious.

c) <u>Two linear joints.</u>

Rearrangement yields for the general term:

$$\mathrm{Tr}(R\bar{Q}BP\bar{Q}^T B^T) = \mathrm{Tr}(P\underbrace{\bar{Q}^T B^T R\bar{Q}B)}_{M}$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{q}_{12}^T b_{11}^T \underset{\text{omit}}{r_{11}} \bar{q}_{12} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & m_{22} \end{array}\right] \qquad \text{(D.17)}$$

Consequently,

$$\mathbf{Tr}(P\bar{Q}^T B^T R\bar{Q}B) \Leftrightarrow \mathbf{Tr}(P\bar{Q}^T B^T \bar{Q}B) = p_{22} m_{22} \qquad \text{(D.18)}$$

Since p_{22} m_{22} is a scalar, the symmetry $D_{ij} = D_{ji}$ in this case is also obvious.

B. Velocity-Related Dynamic Coefficients

1. Centripetal terms

a) Rotary joints.

aa) $D_{i,kk}$, $i < k$: Centripetal effect of the outer joints felt at the inner rotary joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(RBQQPB^TQ^T) = \mathrm{Tr}(\underbrace{PB^TQ^TRBQQ}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} b^T_{11}q^T_{11}\overset{\text{omit}}{r_{11}}b_{11}q_{11}q_{11} & 0 \\ \hline b^T_{12}q^T_{11}\underset{\text{omit}}{r_{11}}b_{11}q_{11}q_{11} & 0 \end{array}\right] \tag{D.19}$$

Consequently,

$$\mathrm{Tr}(PB^TQ^TRBQQ) \Leftrightarrow \mathrm{Tr}(PB^TQ^TBQQ) \tag{D.20}$$

bb) $D_{i,kk}$, $i > k$: Centripetal effect of the inner joints felt at the outer rotary joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(RQQBPQ^TB^T) = \mathrm{Tr}(\underbrace{PQ^TB^TRQQB}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} q_{11}^T b_{11}^T \overset{\text{omit}}{r_{11}} q_{11} q_{11} b_{11} & q_{11}^T b_{11}^T \overset{\text{omit}}{r_{11}} q_{11} q_{11} b_{12} \\ \hline 0 & 0 \end{array}\right] \tag{D.21}$$

Consequently,

$$\mathrm{Tr}(PQ^T B^T RQQB) \Leftrightarrow \mathrm{Tr}(PQ^T B^T QQB) \tag{D.22}$$

b) <u>Linear-rotary joint pairs.</u>

aa) $D_{i,kk}$, $i < k$: Centripetal effect of the outer rotary joints felt at the inner linear joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(RBQQPB^T\bar{Q}^T) = \mathrm{Tr}(\underbrace{PB^T\bar{Q}^T RBQQ}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline \bar{q}_{12}^T \underset{\text{omit}}{r_{11}} b_{11} q_{11} q_{11} & 0 \end{array}\right] \tag{D.23}$$

Consequently,

$$\mathrm{Tr}(PB^T\bar{Q}^T RBQQ) \Leftrightarrow \mathrm{Tr}(F\bar{Q}^T BQQ) \tag{D.24}$$

bb) $D_{i,kk}$, $i > k$: Centripetal effect of the inner rotary joints felt at the outer linear joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(RQQBP\bar{Q}^T B^T) = \mathrm{Tr}(\underbrace{P\bar{Q}^T B^T RQQB}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline \bar{q}_{12}^T b_{11}^T \underset{\uparrow \atop \text{omit}}{r_{11}} q_{11} q_{11} b_{11} & \bar{q}_{12}^T b_{11}^T \underset{\uparrow \atop \text{omit}}{r_{11}} q_{11} q_{11} b_{12} \end{array}\right] \qquad (D.25)$$

Consequently,

$$\mathrm{Tr}(P\bar{Q}^T B^T RQQB) \Leftrightarrow \mathrm{Tr}(P\bar{Q}^T B^T QQB) \qquad (D.26)$$

c) Remarks.

aa) $D_{i,ii} \equiv 0$ is physically obvious. But it can easily be seen also from the matrices as follows.

Rearrangement yields for the general term:

$$\mathrm{Tr}(RQQPQ^T) = \mathrm{Tr}(\underbrace{PQ^T RQQ}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} q_{11}^T \overset{\text{omit}}{r_{11}} q_{11} q_{11} & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} q_{11} & 0 \\ \hline 0 & 0 \end{array}\right] = Q$$

Consequently,

$$Tr(PM) = Tr(PQ) \equiv 0$$

since P is a symmetric matrix and Q is a skew-symmetric matrix. (The Trace of the product of a symmetric and a skew-symmetric matrix is identically zero.)

bb) The close relationship between Eqs. (D.22) and (D.14) - that is between D_{ij} for two rotary joints and $D_{i,kk}$ for i > k rotary joints - is noteworthy. For these two types of dynamical coefficients the final expressions become:

for D_{ij}: $\quad Tr(p_{11}q_{11}^T b_{11}^T q_{11} b_{11}) + p_{12}^T q_{11}^T b_{11}^T q_{11} b_{12}$

for $D_{i,kk}$: $\quad Tr(p_{11}q_{11}^T b_{11}^T q_{11} q_{11} b_{11}) + p_{12}^T q_{11}^T b_{11}^T q_{11} q_{11} b_{12}$

2. Coriolis terms

The Coriolis terms are characterized by three indices separated into two groups: i, kj with k ≠ j but i can be equal to k or j. (k and j are interchangeable.) The values of i, kj allow several combinations: i < k, j with k < j; k < i, j with i < j; k < i, j with i > j; i = k with i < j; i = k with i > j. Further, both linear and rotary joints can be associated with the three i, kj indices. Thus, the index values together with the associated joint types result in a number of cases to be considered.

a) $D_{i,kj}$, i < k, j and k < j

1) Three rotary joints (e.g., $D_{2,46}$)

Rearrangement of the general term yields:

$$Tr(\underbrace{PB^*B^TQ^TRBQB^*Q}_{M})^{\dagger}$$

†Here and in the subsequent pages, the B and B* matrices have identical structure as specified by Eq. (D.3), but their elements (that is, their omitted upper indices) are different.

Direct multiplication results:

$$M = \left[\begin{array}{c|c} b_{11}^{*T}\, b_{11}^{T} q_{11}^{T} \underset{\text{omit}}{r_{11}}\, b_{11} q_{11} b_{11}^{*} q_{11} & 0 \\ \hline \left(b_{12}^{*T} b_{11}^{T} q_{11}^{T} + b_{12}^{T} q_{11}^{T}\right) \underset{\text{omit}}{r_{11}}\, b_{11} q_{11} b_{11}^{*} q_{11} & 0 \end{array}\right] \tag{D.27}$$

Consequently,

$$\mathrm{Tr}(PB^{*T}B^{T}Q^{T}RBQB^{*}Q) \Leftrightarrow \mathrm{Tr}(PB^{*T}B^{T}Q^{T}BQB^{*}Q) \tag{D.28}$$

2) <u>One linear and two rotary joints.</u>

aa) i is linear, k and j are rotary joints. (e.g., $D_{3,45}$)

Rearrangement of the general term yields:

$$\mathrm{Tr}(P\underbrace{B^{*T}B^{T}\bar{Q}^{T}RBQB^{*}Q}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline \bar{q}_{12}^{T} \underset{\text{omit}}{r_{11}}\, b_{11} q_{11} b_{11}^{*} q_{11} & 0 \end{array}\right] \tag{D.29}$$

Consequently,

$$\mathrm{Tr}(PB^{*T}B^{T}\bar{Q}^{T}RBQB^{*}Q) \Leftrightarrow \mathrm{Tr}(P\bar{Q}^{T}BQB^{*}Q) \tag{D.30}$$

bb) i and k rotary joints, j linear joint.

Rearrangement yields for the general term:

$$\mathrm{Tr}(PB^{*T}\underbrace{B^TQ^TRBQB^*\bar{Q}}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & b_{11}^{*T} b_{11}^T q_{11}^T \overset{\text{omit}}{r_{11}} b_{11} q_{11} b_{11}^{*} \bar{q}_{12} \\ \hline 0 & (b_{12}^{*T} b_{11}^T q_{11}^T + b_{12}^T q_{11}^T)\, \underset{\text{omit}}{r_{11}} b_{11} q_{11} b_{11}^{*} \bar{q}_{12} \end{array}\right] \qquad \text{(D.31)}$$

Consequently,

$$\mathrm{Tr}(PB^{*T}B^TQ^TRBQB^*\bar{Q}) = \mathrm{Tr}(PB^{*T}B^TQ^TBQB^*\bar{Q}) \qquad \text{(D.32)}$$

cc) i and j rotary joints, k linear joint. (e.g., $D_{2,34}$)

The general term is:

$$\mathrm{Tr}(PB^{*T}B^TQ^TRB\overbrace{\bar{Q}B^*Q}^{\text{zero}}) \equiv 0$$

since $\bar{Q}B^* = \bar{Q}$ and $\bar{Q}Q = 0$.

3) <u>One rotary and two linear joints.</u>

Only one combination can be different from zero:

k is rotary while i and j are linear joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(\underbrace{PB^{*T}B^T\bar{Q}^TRBQB^*\bar{Q}}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{q}_{12}^T r_{11} b_{11} q_{11} b_{11}^* \bar{q}_{12} \end{array}\right] \qquad \text{(D.33)}$$

omit

Consequently,

$$\mathrm{Tr}(PB^{*T}B^T\bar{Q}^TRBQB^*\bar{Q}) \Leftrightarrow \mathrm{Tr}(P\bar{Q}^TBQB^*\bar{Q}) \qquad \text{(D.34)}$$

The other possible two linear and one rotary joint combinations yield identically zero Coriolis terms since both $\bar{Q}BQ$ and $\bar{Q}B\bar{Q}$ are zero matrices.

b) $D_{i,kj}$, $k < i, j$ and $i < j$.

1) <u>Three rotary joints</u> (e.g., $D_{4,26}$)

Rearrangement yields for the general term:

$$\mathrm{Tr}(\underbrace{PQ^TB^{*T}B^TQ^TRBQB^*}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c:c} q_{11}^T b_{11}^{*T} b_{11}^T q_{11}^T \overset{\text{omit}}{r_{11}} b_{11} q_{11} b_{11}^{*} & q_{11}^T b_{11}^{*T} b_{11}^T q_{11}^T r_{11} b_{11} q_{11} b_{12}^{*} \\ \hdashline 0 & 0 \end{array}\right] \qquad \text{(D.35)}$$

Consequently,

$$\mathrm{Tr}(PQ^T B^{*T} B^T Q^T RBQB^*) \Leftrightarrow \mathrm{Tr}(PQ^T B^{*T} B^T Q^T BQB^*) \qquad \text{(D.36)}$$

2) <u>One linear and two rotary joints.</u>

aa) i is linear, k and j are rotary joints (e.g., $D_{3,24}$)

Rearrangement of the general term yields:

$$\mathrm{Tr}(\underbrace{PQ^T B^{*T} B^T Q^T RB\bar{Q}B^*}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c:c} 0 & q_{11}^T b_{11}^{*T} b_{11}^T q_{11}^T \overset{\text{omit}}{r_{11}} b_{11} \bar{q}_{12} \\ \hdashline 0 & 0 \end{array}\right] \qquad \text{(D.37)}$$

Consequently,

$$\mathrm{Tr}(PQ^T B^{*T} B^T Q^T RB\bar{Q}B^*) \Leftrightarrow \mathrm{Tr}(PQ^T B^{*T} B^T Q^T B\bar{Q}) \qquad \text{(D.38)}$$

bb) i and k rotary joints, j linear joint.

Rearrangement yields for the general term

$$Tr(P\bar{Q}^T B^{*T} B^T Q^T RBQB^*)$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline \bar{q}_{12}^T b_{11}^{*T} b_{11}^T q_{11}^T r_{11} b_{11} q_{11} b_{11}^* & \bar{q}_{12}^T b_{11}^{*T} b_{11}^T q_{11}^T r_{11} b_{11} q_{11} b_{12}^* \end{array}\right] \quad \text{(D.39)}$$

omit

Consequently,

$$Tr(P\bar{Q}^T B^{*T} B^T Q^T RBQB^*) \Leftrightarrow Tr(P\bar{Q}^T B^{*T} B^T Q^T BQB^*) \quad \text{(D.40)}$$

cc) i and j rotary joints, k linear joint (e.g., $D_{4,35}$)

The general term is now:

$$Tr(PQ^T B^{*T} B^T \bar{Q}^T RBQB^*) = Tr\left[P(\underbrace{\bar{Q}BB^*Q}_{\text{zero}})^T RBQB^*\right] \equiv 0$$

since $\bar{Q}BB^* = \bar{Q}$ and $\bar{Q}Q = 0$

3) <u>One rotary on two linear joints.</u>

Again, only one combination can be different from zero: k is rotary while i and j are linear joints.

Rearrangement yields for the general term:

$$\underbrace{Tr(P\bar{Q}^T B^{*T} B^T Q^T RB\bar{Q}B^*)}_{M}$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{q}_{12}^T b_{11}^{*T} b_{11}^T q_{11}^T \underbrace{r_{11}}_{\text{omit}} b_{11} \bar{q}_{12} \end{array}\right] \tag{D.41}$$

Consequently,

$$\mathrm{Tr}(P\bar{Q}^T B^{*T} B^T Q^T R B \bar{Q} B^*) \Leftrightarrow \mathrm{Tr}(P\bar{Q}^T B^{*T} B^T Q^T B \bar{Q}) \tag{D.42}$$

Again, the other possible two linear and one rotary joint combinations yield identically zero Coriolis terms since both $\bar{Q}BQ$ and $\bar{Q}B\bar{Q}$ are zero matrices.

c) $D_{i,kj}$, $k < i, j$ and $i > j$.

1) <u>Three rotary joints</u> (e.g., $D_{6,24}$)

Rearrangement yields for the general term:

$$\mathrm{Tr}(P\underbrace{Q^T B^{*T} B^T R Q B Q B^*}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} q_{11}^T b_{11}^{*T} b_{11}^T \overbrace{r_{11}}^{\text{omit}} q_{11} b_{11} q_{11} b_{11}^* & q_{11}^T b_{11}^{*T} b_{11}^T \overbrace{r_{11}}^{\text{omit}} q_{11} b_{11} q_{11} b_{12}^* \\ \hline 0 & 0 \end{array}\right] \tag{D.43}$$

Consequently,

$$\mathrm{Tr}(PQ^T B*^T B^T RQBQB*) \Leftrightarrow \mathrm{Tr}(PQ^T B*^T B^T QBQB*) \qquad (D.44)$$

2) <u>One linear and two rotary joints.</u>

aa) i is linear, k and j are rotary joints.

Rearrangement yields for the general term:

$$\mathrm{Tr}(P\underbrace{\bar{Q}^T B*^T\ B^T RQBQB*}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline \underbrace{\bar{q}_{12}^T b_{11}^{*T} b_{11}^T r_{11}}_{\text{omit}} q_{11} b_{11} q_{11} b_{11}^* & \underbrace{\bar{q}_{12}^T b_{11}^{*T} b_{11}^T r_{11}}_{\text{omit}} q_{11} b_{11} q_{11} b_{12}^* \end{array}\right] \qquad (D.45)$$

Consequently,

$$\mathrm{Tr}(P\bar{Q}^T B*^T B^T RQBQB*) \Leftrightarrow \mathrm{Tr}(P\bar{Q}^T B*^T B^T QBQB*) \qquad (D.46)$$

bb) i and k rotary joints, j linear joint. (e.g., $D_{4,23}$)

Rearrangement yields for the general term:

$$\mathrm{Tr}(PQ^T B*^T B^T RQB\bar{Q}B*)$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & q_{11}^T b_{11}^{*T} b_{11}^T \overset{\text{omit}}{r_{11}} q_{11} b_{11} \bar{q}_{12} \\ \hline 0 & 0 \end{array}\right] \tag{D.47}$$

Consequently,

$$Tr(PQ^T B^{*T} B^T RQB\bar{Q}B^*) \Leftrightarrow Tr(PQ^T B^{*T} B^T QB\bar{Q}) \tag{D.48}$$

cc) i and j rotary joints, k linear joint (e.g., $D_{5,34}$)

The general term is now

$$Tr(PQ^T B^{*T} B^T R\underbrace{\bar{Q}BQ}_{\text{zero}}B^*) \equiv 0$$

since $\bar{Q}B = \bar{Q}$ and $\bar{Q}Q = 0$.

3) <u>One rotary and two linear joints</u>.

Again, only one combination can be different from zero: k is rotary while i and j are linear joints.

Rearrangement yields for the general term:

$$Tr(P\underbrace{\bar{Q}^T B^{*T} B^T RQB\bar{Q}B^*}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{q}_{12}^T b_{11}^{*T} b_{11}^T \underset{\text{omit}}{r_{11}} q_{11} b_{11} \bar{q}_{12} \end{array}\right] \tag{D.49}$$

Consequently,

$$\mathrm{Tr}(P\bar{Q}^T B^{*T} B^T RQB\bar{Q}B^*) \Leftrightarrow \mathrm{Tr}(P\bar{Q}^T B^{*T} B^T QB\bar{Q}) \qquad \text{(D. 50)}$$

Remark

Recalling that $Q^T = -Q$, and comparing Eq. (D.36) to Eq. (D.44), Eq. (D.38) to Eq. (D.48), Eq. (D.40) to Eq. (D.46), and Eq. (D.42) to Eq. (D.50) it is seen from the right hand side of the respective equivalence expressions that

Eq. (D.36) = - Eq. (D.44)

Eq. (D.38) = - Eq. (D.48)

Eq. (D.40) = - Eq. (D.46)

Eq. (D.42) = - Eq. (D.50)

That is, we have in general:

$$D_{i,kj} = -D_{j,ki} \text{ for } k < i, j \qquad \text{(D. 51)}$$

d) $D_{i,ij}$ with $i < j$

1) Three rotary joints (e.g., $D_{2,24}$)

Rearrangement yields for the general term:

$$\mathrm{Tr}(\underbrace{PB^T Q^T RQBQ}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c:c} b_{11}^T q_{11}^T \overset{\substack{\text{omit}\\ \downarrow}}{r_{11}} q_{11} b_{11} q_{11} & 0 \\ \hdashline b_{12}^T q_{11}^T \underset{\substack{\uparrow \\ \text{omit}}}{r_{11}} q_{11} b_{11} q_{11} & 0 \end{array}\right] \qquad \text{(D. 52)}$$

Consequently,

$$\mathrm{Tr}(PB^TQ^TRQBQ) \Leftrightarrow \mathrm{Tr}(PB^TQ^TQBQ) \tag{D.53}$$

2) <u>One linear and two rotary joints</u> (e.g., $D_{2,23}$)

The Coriolis term can only be different from zero if j is the linear joint.

Rearrangement of the general term yields:

$$\mathrm{Tr}(P\underbrace{B^TQ^TRQB}_{M}\bar{Q})$$

Direct multiplication results:

$$M = \left[\begin{array}{c|c} 0 & b_{11}^T q_{11}^T \overset{\text{omit}}{\overset{\downarrow}{r_{11}}} q_{11} b_{11} \bar{q}_{12} \\ \hline 0 & b_{12}^T q_{11}^T \underset{\text{omit}}{\underset{\uparrow}{r_{11}}} q_{11} b_{11} \bar{q}_{12} \end{array}\right] \tag{D.54}$$

Consequently,

$$\mathrm{Tr}(PB^TQ^TRQB\bar{Q}) \Leftrightarrow \mathrm{Tr}(PB^TQ^TQB\bar{Q}) \tag{D.55}$$

e) $D_{i,ij}$ with $i > j$

1) <u>Three rotary joints</u> (e.g., $D_{5,52}$)

Rearrangement of the general term yields:

$$\mathrm{Tr}(P\underbrace{Q^TB^TRQB}_{M}Q)$$

Direct multiplication results

$$M = \left[\begin{array}{c:c} q_{11}^T b_{11}^T \underset{\text{omit}}{r_{11}} q_{11} b_{11} q_{11} & 0 \\ \hdashline 0 & 0 \end{array}\right] \tag{D.56}$$

Consequently,

$$\mathrm{Tr}(PQ^T B^T RQBQ) \Leftrightarrow \mathrm{Tr}(PQ^T B^T QBQ) = \mathrm{Tr}\left[\underbrace{BQP(BQ)^T}_{K} Q\right] \equiv 0 \tag{D.57}$$

Since K is a symmetric matrix while Q is a skew-symmetric matrix, and the Trace of the product of asymmetric and skew-symmetric matrix is identically zero.

2) <u>One linear and two rotary joints</u> (e.g., $D_{3,32}$)

Due to the assumption that $i > j$, j must be the rotary joint. (Otherwise we would have $\bar{Q}BQ \equiv 0$ automatically.)

Rearrangement of the general term yields:

$$\mathrm{Tr}(\underbrace{P\bar{Q}^T B^T RQB\bar{Q}}_{M})$$

Direct multiplication results:

$$M = \left[\begin{array}{c:c} 0 & \\ \hdashline 0 & \bar{q}_{12}^T b_{11}^T \underset{\text{omit}}{r_{11}} q_{11} b_{11} \bar{q}_{12} \end{array}\right] \tag{D.58}$$

Consequently,

$$\mathrm{Tr}(P\bar{Q}^T B^T R Q B\bar{Q}) \Longleftrightarrow \mathrm{Tr}(P\bar{Q}^T B^T Q B\bar{Q}) = \mathrm{Tr}\left[\underbrace{B\bar{Q}P(B\bar{Q})^T}_{K} Q\right] \equiv 0 \qquad \text{(D.59)}$$

since the Trace of the product of a symmetric matrix (K) and a skew-symmetric matrix (Q) is identically zero.

Thus, the Coriolis term $D_{i,\,ij}$ with $i > j$ is identically zero in all cases.

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