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Phase Matrix Induced Symmetries

for

Multiple Scattering Using the Matrix Operator Method

by

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Phase Matrix Induced Symmetries for Multiple Scattering Using the Matrix Operator Method

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ABSTRACT

Entirely rigorous proofs of the symmetries induced by the phase matrix into the reflection and transmission operators used in the matrix operator theory are given. Results are obtained for multiple scattering in both homogeneous and inhomogeneous atmospheres. These results will be extremely useful to researchers using the method in that large savings in computer time and storage can be achieved.

Introduction

Symmetry relations play an important part in methods of solving radiative transfer problems. They can be used to reduce the amount of calculation and core storage and can also be used for giving physical insight into the results obtained. Principles of invariance played an important part in Chandrasekhar's (1960) solution to the Rayleigh scattering problem. Other work involving symmetry relations include Sekera (1966) and Busbridge (1960). Recently, Hovenier (1969) proved that certain symmetry properties of the phase matrix induced equivalent symmetries in the scattering and transmission operators. The method used by Hovenier for proving these was cumbersome and only presented in the case of a homogeneous atmosphere. Kattawar (1973) derived the symmetry relations induced in the reflection and transmission operators in the matrix operator (MO) method from symmetries of a phase function. Tanaka (1971) proved a set of symmetry relations for the diffuse scattering and transmission operators. These results were in agreement with Hovenier's. It is the purpose of this paper to extend MO to include polarization and to derive symmetry relations induced in the reflection and transmission operators using the MO method. The proofs will be rigorous for both the homogeneous and inhomogeneous atmospheres.

Matrix Operator Theory with Polarization

MO has been developed over the years by contributions from many sources. A brief history of the development of the method and most of the important equations are given by Plass, Kattawar, and Catchings (1973). This work will be referred to as PKC. We present the equations in the continuum. The I, Q, U, V Stokes vector representation (Chandrasekhar, 1960) will be used in this paper. Therefore, we define:

$$\mathbf{I}^{+}(\tau;\mu,\phi) = \begin{bmatrix} I(\tau;\mu,\phi) \\ Q(\tau;\mu,\phi) \\ U(\tau;\mu,\phi) \\ V(\tau;\mu,\phi) \end{bmatrix} \quad 0<\mu\le 1 \tag{1a}$$

$$\mathbf{I}^{-}(\tau;\mu,\phi) = \begin{bmatrix} I(\tau,-\mu,\phi) \\ Q(\tau;-\mu,\phi) \\ U(\tau;-\mu,\phi) \\ V(\tau;-\mu,\phi) \end{bmatrix} \quad 0<\mu\le 1 \tag{1b}$$

as being the radiance in the direction of increasing and decreasing optical depth respectively. In Eqs. (1), $\mu = \cos\theta$, θ being the polar angle measured from the zenith, and ϕ being the azimuthal angle where we use the external source as the reference for azimuth. We consider a plane parallel atmosphere with boundaries x and y ($x\le y$). Following PKC we have:

$$\mathbf{I}^{+}(y;\mu,\phi) = \mathbf{t}(x,y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{+}(x;\mu',\phi') + \mathbf{r}(y,x;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{-}(y;\mu',\phi') \tag{2a}$$

$$\mathbf{I}^{-}(x;\mu,\phi) = \mathbf{r}(x,y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{+}(x;\mu',\phi') + \mathbf{t}(y,x;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{-}(y;\mu',\phi') \tag{2b}$$

In Eqs. (2) and in the sequel to follow, the $\cdot$ implies the following operation:

$$\mathbf{t}(x,y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{+}(x;\mu',\phi') = \int_0^1 d\mu' \int_0^{2\pi} d\phi'\, \mathbf{t}(x,y;\mu,\phi;\mu',\phi')\, \mathbf{I}^{+}(x;\mu',\phi') \tag{3}$$

Subtracting $\mathbf{I}^{+}(x;\mu,\phi)$ and $\mathbf{I}^{-}(y;\mu,\phi)$ from Eqs. (2a) and (2b) respectively and passing to the limit $x\to y$, the following differential equations are obtained:

$$\frac{\partial \mathbf{I}^{+}(y;\mu,\phi)}{\partial y} = -\mathbf{\Gamma}^{++}(y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{+}(y;\mu',\phi') + \mathbf{\Gamma}^{+-}(y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{-}(y;\mu',\phi') \tag{4a}$$

$$-\frac{\partial \mathbf{I}^{-}(y;\mu,\phi)}{\partial y} = \mathbf{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{+}(y;\mu',\phi') - \mathbf{\Gamma}^{--}(y;\mu,\phi;\mu',\phi') \cdot \mathbf{I}^{-}(y;\mu',\phi') \tag{4b}$$

where the gamma operators are given by:

$$\Gamma^{++}(y;\mu,\phi;\mu',\phi') = \lim_{x\to y}(\delta(\mu-\mu')\delta(\phi-\phi')E - t(x,y;\mu,\phi;\mu',\phi'))/(y-x)\,, \tag{5a}$$

$$\Gamma^{+-}(y;\mu,\phi;\mu',\phi') = \lim_{x\to y} r(y,x;\mu,\phi;\mu',\phi')/(y-x)\,, \tag{5b}$$

$$\Gamma^{-+}(y;\mu,\phi;\mu',\phi') = \lim_{x\to y} r(x,y;\mu,\phi;\mu',\phi')/(y-x)\,, \tag{5c}$$

$$\Gamma^{--}(y;\mu,\phi;\mu',\phi') = \lim_{x\to y} (\delta(\mu-\mu')\delta(\phi-\phi')E - t(y,x;\mu,\phi;\mu',\phi'))/(y-x). \tag{5d}$$

In Eqs. (5), E is a 4x4 unit matrix and $\delta(\mu-\mu')$ is the Dirac delta function. The gamma operators, defined in Eqs. (5) are the generators of the r and t operators. Eqs. (4) form the link between the equation of transfer and the interaction principle of MO (Eqs. (2)).

From Chandrasekhar (1960) the equation of transfer in a plane parallel scattering atmosphere, taking polarization into account, is

$$\mu \frac{d}{d\tau} I(\tau;\mu,\phi) + I(\tau,\mu,\phi) = \omega(\tau) \int_{-1}^{+1} \int_{0}^{2\pi} p(\tau;\mu,\phi;\mu',\phi')\, I(\tau;\mu',\phi')\,d\mu' d\phi' \tag{6}$$

where $\omega(\tau)$ is the single scattering albedo, and $p(\tau;\mu,\phi;\mu',\phi')$ is the phase matrix where the element p_{11} is normalized to unity. Using Eqs. (1) for $I^+(\tau;\mu,\phi)$ and $I^-(\tau;\mu,\phi)$ we have from Eq. (6) after some simple algebra:

$$\frac{d}{d\tau} I^+(\tau;\mu,\phi) = \int_0^1 \int_0^{2\pi} -(1/\mu)(\delta(\mu-\mu')\delta(\phi-\phi')E - \omega(\tau)p(\tau;\mu,\phi;\mu',\phi'))\, I^+(\tau;\mu',\phi')\,d\mu' d\phi'$$
$$+ \int_0^1 \int_0^{2\pi} (\omega(\tau)/\mu)p(\tau;\mu,\phi;-\mu',\phi')\, I^-(\tau;\mu',\phi')\, d\mu'\, d\phi' \tag{7a}$$

$$-\frac{d}{d\tau} I^-(\tau;\mu,\phi) = \int_0^1 \int_0^{2\pi} (\omega(\tau)/\mu)p(\tau;-\mu,\phi;\mu',\phi')\, I^+(\tau;\mu',\phi')\, d\mu'\, d\phi'$$
$$+ \int_0^1 \int_0^{2\pi} -(1/\mu)(\delta(\mu-\mu')\delta(\phi-\phi')E - \omega(\tau)p(\tau;-\mu,\phi;-\mu',\phi'))\, I^-(\tau;\mu',\phi')\,d\mu' d\phi' \tag{7b}$$

Comparing Eqns. (7a) with (4a) and (7b) with (4b) it is easy to make the following identification of terms:

$$\underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi') = (\delta(\mu-\mu')\delta(\phi-\phi')\underset{\sim}{E} - \omega(y)\underset{\sim}{p}(y;\mu,\phi;\mu',\phi'))/\mu \ , \tag{8a}$$

$$\underset{\sim}{\Gamma}^{--}(y;\mu,\phi;\mu',\phi') = (\delta(\mu-\mu')\delta(\phi-\phi')\underset{\sim}{E} - \omega(y)\underset{\sim}{p}(y;-\mu,\phi;-\mu',\phi'))/\mu \ , \tag{8b}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi;\mu',\phi') = \omega(y)\,\underset{\sim}{p}(y;\mu,\phi;-\mu',\phi')/\mu \ , \tag{8c}$$

$$\underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi') = \omega(y)\,\underset{\sim}{p}(y;-\mu,\phi;\mu',\phi')/\mu \ . \tag{8d}$$

Thus, the generators for the $\underset{\sim}{r}$ and $\underset{\sim}{t}$ operators are related to the single scattering albedo and the phase matrix.

PKC give the equations that combine the $\underset{\sim}{r}$ and $\underset{\sim}{t}$ operators for two arbitrary layers to yield the $\underset{\sim}{r}$ and $\underset{\sim}{t}$ operators for the combined layers. Using these equations to add an infinitesimal layer to the top and to the bottom and then passing to the limit as the small layer goes to zero, the differential equations for the $\underset{\sim}{r}$ and $\underset{\sim}{t}$ operators are obtained (Kattawar, 1973). The equations are as follows:

$$\begin{aligned}\frac{\partial\underset{\sim}{r}(y,x;\mu,\phi;\mu_o,\phi_o)}{\partial y} = {} & \underset{\sim}{\Gamma}^{+-}(y;\mu,\phi;\mu_o,\phi_o) - \underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi')\cdot\underset{\sim}{r}(y,x;\mu',\phi';\mu_o,\phi_o) \\ & - \underset{\sim}{r}(y,x;\mu,\phi;\mu',\phi')\cdot\underset{\sim}{\Gamma}^{--}(y;\mu',\phi';\mu_o,\phi_o) \\ & + \underset{\sim}{r}(y,x;\mu,\phi;\mu'',\phi'')\cdot\underset{\sim}{\Gamma}^{-+}(y;\mu'',\phi'';\mu',\phi')\cdot\underset{\sim}{r}(y,x;\mu',\phi';\mu_o,\phi_o) \ ,\end{aligned} \tag{9a}$$

$$\frac{\partial\underset{\sim}{r}(y,x;\mu,\phi;\mu_o,\phi_o)}{\partial x} = \underset{\sim}{t}(x,y;\mu,\phi;\mu'',\phi'')\cdot\underset{\sim}{\Gamma}^{+-}(x;\mu'',\phi'';\mu',\phi')\cdot t(y,x;\mu',\phi';\mu_o,\phi_o) \ , \tag{9b}$$

$$\frac{\partial\underset{\sim}{r}(x,y;\mu,\phi;\mu_o,\phi_o)}{\partial y} = \underset{\sim}{t}(y,x;\mu,\phi;\mu'',\phi'')\cdot\underset{\sim}{\Gamma}^{-+}(y;\mu'',\phi'';\mu',\phi')\cdot\underset{\sim}{t}(x,y;\mu',\phi';\mu_o,\phi_o) \ , \tag{9c}$$

$$\frac{\partial \underset{\sim}{r}(x,y;\mu,\phi;\mu_o,\phi_o)}{\partial x} = \underset{\sim}{\Gamma}^{-+}(x;\mu,\phi;\mu_o,\phi_o) - \underset{\sim}{\Gamma}^{--}(x;\mu,\phi;\mu',\phi') \cdot \underset{\sim}{r}(x,y;\mu',\phi';\mu_o,\phi_o)$$
$$- \underset{\sim}{r}(x,y;\mu,\phi;\mu',\phi') \cdot \underset{\sim}{\Gamma}^{++}(x;\mu',\phi';\mu_o,\phi_o)$$
$$+ \underset{\sim}{r}(x,y;\mu,\phi;\mu'',\phi'') \cdot \underset{\sim}{\Gamma}^{+-}(x;\mu'',\phi'';\mu',\phi') \cdot \underset{\sim}{r}(x,y;\mu',\phi';\mu_o,\phi_o) \ , \tag{9d}$$

$$\frac{\partial \underset{\sim}{t}(y,x;\mu,\phi;\mu_o,\phi_o)}{\partial y} = -t(y,x;\mu,\phi;\mu',\phi') \cdot (\underset{\sim}{\Gamma}^{--}(y;\mu',\phi';\mu_o,\phi_o) - \underset{\sim}{\Gamma}^{-+}(y;\mu',\phi';\mu'',\phi'')$$
$$\cdot \underset{\sim}{r}(y,x;\mu'',\phi'';\mu_o,\phi_o)) \ , \tag{9e}$$

$$\frac{\partial \underset{\sim}{t}(y,x;\mu,\phi;\mu_o,\phi_o)}{\partial x} = -(\underset{\sim}{\Gamma}^{--}(y;\mu,\phi;\mu',\phi') - \underset{\sim}{r}(x,y;\mu,\phi;\mu'',\phi'') \cdot \underset{\sim}{\Gamma}^{+-}(x;\mu'',\phi'';\mu',\phi'))$$
$$\cdot \underset{\sim}{t}(y,x;\mu',\phi';\mu_o,\phi_o) \ , \tag{9f}$$

$$\frac{\partial \underset{\sim}{t}(x,y;\mu,\phi;\mu_o,\phi_o)}{\partial y} = -(\underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi') - \underset{\sim}{r}(y,x;\mu,\phi;\mu'',\phi'') \cdot \underset{\sim}{\Gamma}^{-+}(y;\mu'',\phi'';\mu',\phi'))$$
$$\cdot \underset{\sim}{t}(x,y;\mu',\phi';\mu_o,\phi_o) \ , \tag{9g}$$

$$\frac{\partial \underset{\sim}{t}(x,y;\mu,\phi;\mu_o,\phi_o)}{\partial x} = -\underset{\sim}{t}(x,y;\mu,\phi;\mu',\phi') \cdot (\underset{\sim}{\Gamma}^{++}(x;\mu',\phi';\mu_o,\phi_o) - \underset{\sim}{\Gamma}^{+-}(x;\mu',\phi';\mu'',\phi'')$$
$$\cdot \underset{\sim}{r}(x,y;\mu'',\phi'';\mu_o,\phi_o)) \ . \tag{9h}$$

The boundary conditions for $\underset{\sim}{r}$ and $\underset{\sim}{t}$ are $\underset{\sim}{r}(x,x) = \underset{\sim}{r}(y,y) = \underset{\sim}{0}$, and $\underset{\sim}{t}(x,x) = \underset{\sim}{t}(y,y) = \underset{\sim}{E}$. Eqs. (9) may easily be shown to be equivalent to the differential equations derived by Chandrasekhar for the diffuse scattering and transmission matrices. The obvious difference of course is that the $\underset{\sim}{t}$ operator of MO accounts for the direct and diffuse radiation whereas Chandrasekhar's transmission operator accounts only for the diffuse radiation. Thus, the differential equations for the $\underset{\sim}{t}$ operators ((9e) - (9h)) are simpler than Chandrasekhar's corresponding equations. Also, the equations that couple $\underset{\sim}{r}$ and $\underset{\sim}{t}$ ((9b) and (9c)) are also simpler than the corresponding equations for the diffuse operators.

Symmetry Properties of the Phase Matrix and Gamma Operators

The symmetry properties of the phase matrix have been used extensively in the past. Van de Hulst (1957) used symmetry alone to reduce the number of independent elements in the phase matrix. Sekera (1966) derived a set of symmetry relations for the phase matrix. Hovenier (1969) developed the symmetry relations for the phase matrix both from a space-time symmetry basis and from a mathematical basis. Kattawar, Hitzfelder, and Binstock (1973) explicitly demonstrated these symmetries for the phase matrix of a polydispersion of spherical particles. Their relations were derived directly from Mie theory. These symmetry properties are reproduced in the notation of this paper for clarity.

$$\underset{\sim}{p}(\mu',\phi';\mu,\phi) = \underset{\sim}{Q}\ \underset{\sim}{p}^{T}(\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ , \tag{10a}$$

$$\underset{\sim}{p}(-\mu',\phi';-\mu,\phi) = \underset{\sim}{P}\ \underset{\sim}{p}^{T}(\mu,\phi;\mu',\phi')\ \underset{\sim}{P}\ , \tag{10b}$$

$$\underset{\sim}{p}(-\mu,\phi';-\mu',\phi) = \underset{\sim}{p}(\mu,\phi;\mu',\phi')\ , \tag{10c}$$

$$\underset{\sim}{p}(\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{p}(\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{10d}$$

$$\underset{\sim}{p}(\mu',\phi;\mu,\phi') = \underset{\sim}{P}\ \underset{\sim}{p}^{T}(\mu,\phi;\mu',\phi')\ \underset{\sim}{P}\ , \tag{10e}$$

$$\underset{\sim}{p}(-\mu',\phi;-\mu,\phi') = \underset{\sim}{Q}\ \underset{\sim}{p}^{T}(\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ , \tag{10f}$$

$$\underset{\sim}{p}(-\mu,\phi;-\mu',\phi') = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{p}(\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{10g}$$

where

$$\underset{\sim}{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underset{\sim}{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

and the superscript T implies a matrix transpose. Eqs. (10) form the complete set of possible transformations that can be performed on $\mathbf{p}(\mu,\phi;\mu',\phi')$. As Hovenier points out in his derivation only three of the seven relations are independent. The other four may be derived in terms of a basis set. This idea can be deduced very simply if one realizes that the seven transformations on the phase matrix coupled with the identity transformation, which leaves the phase matrix unaltered and denoted by E, forms an abelian group of order eight. The group multiplication table is given in Table I. A close inspection of Table 1 shows that there are ten possible subgroups of order four. These subgroups can be found as follows. Let α and β be any two distinct elements of Table 1. Then α, β, and $\alpha\beta$ will form a proper subgroup of order four when the identity element is included. The reason for this is clear since every element of the group is of second order, i.e., $\alpha^2 = E$ where α is any element of the group. Therefore, $\alpha(\alpha\beta) = \beta$ and $\beta(\alpha\beta) = \alpha$. This will leave twenty five triplets of elements which can be used as generators for the entire group. Hovenier only considered the triplets (d,e,g) and (b,c,d). From the space-time symmetry point of view, (b,c,d) form the basic set of relations.

The symmetry properties of the gamma operators are now derived from Eqs. (10) and Eqs. (8). Transposing Eq. (8a) we have:

$$(\mathbf{\Gamma}^{++}(y;\mu,\phi;\mu',\phi'))^T = (\delta(\mu-\mu')\delta(\phi-\phi')E - \omega(y)\mathbf{p}^T(y;\mu,\phi;\mu',\phi'))/\mu$$

Pre and post multiplying by $\mathbf{Q}$ and noting that $\delta(\mu-\mu') = \delta(\mu'-\mu)$ we have from Eq. (10a):

$$\mathbf{\Gamma}^{++}(y;\mu',\phi';\mu,\phi) = \mathbf{Q}\ (\mathbf{\Gamma}^{++}(y;\mu,\phi;\mu',\phi'))^T\ \mathbf{Q} \tag{11a}$$

Similarly from Eq. (10a) and Eqs. (8b), (8c), and (8d)

$$\mathbf{\Gamma}^{--}(y;\mu',\phi';\mu,\phi) = \mathbf{Q}\ (\mathbf{\Gamma}^{--}(y;\mu,\phi;\mu',\phi'))^T\ \mathbf{Q} \tag{11b}$$

$$\mathbf{\Gamma}^{+-}(y;\mu',\phi';\mu,\phi) = \mathbf{Q}\ (\mathbf{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi'))^T\ \mathbf{Q} \tag{11c}$$

Similar operations can be performed on Eqs. (8) taking into account Eqs. (10) and the following relations can be derived:

$$\underset{\sim}{\Gamma}^{--}(y;\mu',\phi';\mu,\phi) = \underset{\sim}{P}\ (\underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{P}\ , \tag{12a}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu',\phi';\mu,\phi) = \underset{\sim}{P}\ (\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{P}\ , \tag{12b}$$

$$\underset{\sim}{\Gamma}^{-+}(y;\mu',\phi';\mu,\phi) = \underset{\sim}{P}\ (\underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{P}\ , \tag{12c}$$

$$\underset{\sim}{\Gamma}^{++}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{\Gamma}^{--}(y;\mu,\phi;\mu',\phi')\ , \tag{13a}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi')\ , \tag{13b}$$

$$\underset{\sim}{\Gamma}^{++}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{14a}$$

$$\underset{\sim}{\Gamma}^{--}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{\Gamma}^{--}(y;\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{14b}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{\Gamma}^{+-}(y;\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{14c}$$

$$\underset{\sim}{\Gamma}^{-+}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{14d}$$

$$\underset{\sim}{\Gamma}^{++}(y;\mu',\phi;\mu,\phi') = \underset{\sim}{P}(\underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{P}\ , \tag{15a}$$

$$\underset{\sim}{\Gamma}^{--}(y;\mu',\phi;\mu,\phi') = \underset{\sim}{P}\ (\underset{\sim}{\Gamma}^{--}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{P}\ , \tag{15b}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu',\phi;\mu,\phi') = \underset{\sim}{P}(\underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{P}\ , \tag{15c}$$

$$\underset{\sim}{\Gamma}^{--}(y;\mu',\phi;\mu,\phi') = \underset{\sim}{Q}(\underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{Q}\ , \tag{16a}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu',\phi;\mu,\phi') = \underset{\sim}{Q}(\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{Q}\ , \tag{16b}$$

$$\underset{\sim}{\Gamma}^{-+}(y;\mu',\phi;\mu,\phi') = \underset{\sim}{Q}(\underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi'))^T\ \underset{\sim}{Q}\ , \tag{16c}$$

$$\underset{\sim}{\Gamma}^{++}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{\Gamma}^{--}(y;\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ , \tag{17a}$$

$$\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi';\mu',\phi) = \underset{\sim}{P}\ \underset{\sim}{Q}\ \underset{\sim}{\Gamma}^{-+}(y;\mu,\phi;\mu',\phi')\ \underset{\sim}{Q}\ \underset{\sim}{P}\ . \tag{17b}$$

The reader may easily verify that Eqs. (12) to (17) were derived from Eqs. (10b) to (10g) respectively. The interdependence of Eqs. (10) apply to Eqs. (11) to (17) as well.

Symmetry Properties of the $\mathbf{r}$ and $\mathbf{t}$ Operators

The symmetry properties of the gamma operators are a necessary step in deriving the properties of the $\mathbf{r}$ and $\mathbf{t}$ operators. These properties will now be derived in a straight forward manner for both the homogenous case and the inhomogeneous case. Transposing Eq. (9a) and pre and post multiplying by $\mathbf{Q}$ one obtains:

$$
\begin{aligned}
\frac{\partial}{\partial y}\mathbf{Q}\,\mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\mathbf{Q} = {} & \mathbf{Q}(\mathbf{\Gamma}^{+-}(y;\mu,\phi;\mu_o,\phi_o))^T\,\mathbf{Q} \\
& - \mathbf{Q}\,\mathbf{r}^T(y,x;\mu',\phi';\mu_o,\phi_o)\mathbf{Q}\cdot\mathbf{Q}(\mathbf{\Gamma}^{++}(y;\mu,\phi;\mu',\phi'))^T\,\mathbf{Q} \\
& - \mathbf{Q}(\mathbf{\Gamma}^{--}(y;\mu',\phi';\mu_o,\phi_o))^T\,\mathbf{Q}\cdot\mathbf{Q}\,\mathbf{r}^T(y,x;\mu,\phi;\mu',\phi')\,\mathbf{Q} \\
& + \mathbf{Q}\,\mathbf{r}^T(y,x;\mu',\phi';\mu_o,\phi_o)\,\mathbf{Q}\cdot\mathbf{Q}(\mathbf{\Gamma}^{-+}(y;\mu'',\phi'';\mu',\phi'))^T\,\mathbf{Q}\cdot\mathbf{Q}\,\mathbf{r}^T(y,x;\mu,\phi;\mu'',\phi'')\,\mathbf{Q}
\end{aligned}
\tag{18}
$$

where $\mathbf{Q} = \mathbf{Q}^{-1}$ has been used. Using Eqs. (11) and letting $\mathbf{u}(y,x;\mu,\phi;\mu_o,\phi_o) = \mathbf{Q}\,\mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\,\mathbf{Q}$ one obtains:

$$
\begin{aligned}
\frac{\partial}{\partial y}\,\mathbf{u}(y,x;\mu,\phi;\mu_o,\phi_o) = {} & \mathbf{\Gamma}^{-+}(y;\mu_o,\phi_o;\mu,\phi) - \mathbf{u}(y,x;\mu',\phi';\mu_o,\phi_o)\cdot\mathbf{\Gamma}^{++}(y,\mu',\phi';\mu,\phi) \\
& - \mathbf{\Gamma}^{--}(y;\mu_o,\phi_o;\mu',\phi')\cdot\mathbf{u}(y,x;\mu,\phi;\mu',\phi') \\
& + \mathbf{u}(y,x;\mu',\phi';\mu_o,\phi_o)\cdot\mathbf{\Gamma}^{+-}(y;\mu',\phi';\mu'',\phi'')\cdot\mathbf{u}(y,x;\mu,\phi;\mu'',\phi'')
\end{aligned}
\tag{19}
$$

Comparing the above equation with Eq. (9d) it is easy to see that if the gamma operators are independent of optical depth, that is the atmosphere is homogeneous, then $\mathbf{u}(y,x;\mu,\phi;\mu_o,\phi_o)$ satisfies the same differential equation as $\mathbf{r}(x,y;\mu_o,\phi_o,\mu\,\phi)$ and has the same boundary conditions. Thus, the solution is unique and we get

$$
\mathbf{r}(x,y;\mu_o,\phi_o;\mu,\phi) = \mathbf{Q}\,\mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\,\mathbf{Q}
\tag{20}
$$

Similar operations can be performed on Eqs. (9). Another example is presented to demonstrate the derivation of a symmetry relation in an inhomogeneous atmosphere.

Transposing Eq. (9a) and pre and post multiplying by $\underset{\sim}{P}$ one obtains:

$$\begin{aligned}\frac{\partial}{\partial y}\underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\,\underset{\sim}{P} &= \underset{\sim}{P}\,(\underset{\sim}{\Gamma}^{+-}(y;\mu,\phi;\mu_o,\phi_o))^T\,\underset{\sim}{P} \\ &- \underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu',\phi';\mu_o,\phi_o)\,\underset{\sim}{P}\cdot\underset{\sim}{P}\,(\underset{\sim}{\Gamma}^{++}(y;\mu,\phi;\mu',\phi')\,\underset{\sim}{P} \\ &- \underset{\sim}{P}\,(\underset{\sim}{\Gamma}^{--}(y;\mu',\phi';\mu_o,\phi_o))^T\underset{\sim}{P}\cdot\underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu,\phi;\mu',\phi')\,\underset{\sim}{P} \\ &+ \underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu',\phi';\mu_o,\phi_o)\,\underset{\sim}{P}\cdot\underset{\sim}{P}\,(\underset{\sim}{\Gamma}^{-+}(y,\mu'',\phi'';\mu',\phi'))^T\,\underset{\sim}{P} \\ &\cdot\underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu,\phi;\mu'',\phi'')\,\underset{\sim}{P} \qquad (21)\end{aligned}$$

where $\underset{\sim}{P} = \underset{\sim}{P}^{-1}$ has been used. Using Eqs. (12) and letting $\underset{\sim}{w}(y,x;\mu,\phi;\mu_o,\phi_o) = \underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\,\underset{\sim}{P}$ one obtains:

$$\begin{aligned}\frac{\partial}{\partial y}\underset{\sim}{w}(y,x;\mu,\phi;\mu_o,\phi_o) &= \underset{\sim}{\Gamma}^{+-}(y;\mu_o,\phi_o;\mu,\phi) - \underset{\sim}{w}(y,x;\mu',\phi';\mu_o,\phi_o)\cdot\underset{\sim}{\Gamma}^{--}(y;\mu',\phi';\mu,\phi) \\ &- \underset{\sim}{\Gamma}^{++}(y;\mu_o,\phi_o;\mu',\phi')\cdot\underset{\sim}{w}(y,x;\mu,\phi;\mu',\phi') \\ &+ \underset{\sim}{w}(y,x;\mu',\phi';\mu_o,\phi_o)\cdot\underset{\sim}{\Gamma}^{-+}(y;\mu',\phi';\mu'',\phi'')\cdot\underset{\sim}{w}(y,x;\mu,\phi;\mu'',\phi'') \qquad (22)\end{aligned}$$

Comparing the above equation with Eq. (9a) it is easy to see that

$$\underset{\sim}{r}(y,x;\mu_o,\phi_o;\mu,\phi) = \underset{\sim}{P}\,\underset{\sim}{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\,\underset{\sim}{P}\ . \qquad (23)$$

The above relation will hold in any plane parallel atmosphere since the gamma operators are evaluated at the same level in both differential equations. It is a very simple task to perform similar operations on Eqs. (9) and using Eqs. (11) to (17) to derive the complete set of symmetry relations for the $\underset{\sim}{r}$ and $\underset{\sim}{t}$ operators. As in the two examples given above, it is easy to determine whether the relation is valid in an inhomogeneous atmosphere or only in a homogeneous atmosphere. The complete set of relations is given below. In this list, an i(h) means that the relation holds in an inhomogeneous (homogeneous) atmosphere.

$$\mathbf{r}(x,y;\mu_o,\phi_o;\mu,\phi) = \mathbf{Q}\ \mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ , \qquad h \qquad (24a)$$

$$\mathbf{t}(y,x;\mu_o,\phi_o;\mu,\phi) = \mathbf{Q}\ \mathbf{t}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ , \qquad h \qquad (24b)$$

$$\mathbf{t}(x,y;\mu_o,\phi_o;\mu,\phi) = \mathbf{Q}\ \mathbf{t}^T(x,y;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ , \qquad h \qquad (24c)$$

$$\mathbf{r}(y,x;\mu_o,\phi_o;\mu,\phi) = \mathbf{P}\ \mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{P}\ , \qquad i \qquad (24d)$$

$$\mathbf{r}(x,y;\mu_o,\phi_o;\mu,\phi) = \mathbf{P}\ \mathbf{r}^T(x,y;\mu,\phi;\mu_o,\phi_o)\ \mathbf{P}\ , \qquad i \qquad (24e)$$

$$\mathbf{t}(x,y;\mu_o,\phi_o;\mu,\phi) = \mathbf{P}\ \mathbf{t}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{P}\ , \qquad i \qquad (24f)$$

$$\mathbf{r}(y,x;\mu\ ,\phi_o;\mu_o,\phi) = \mathbf{r}(x,y;\mu,\phi;\mu_o,\phi_o)\ , \qquad h \qquad (24g)$$

$$\mathbf{t}(y,x;\mu,\phi_o;\mu_o,\phi) = \mathbf{t}(x,y;\mu,\phi;\mu_o,\phi_o)\ , \qquad h \qquad (24h)$$

$$\mathbf{r}(y,x;\mu,\phi_o;\mu_o,\phi) = \mathbf{P}\ \mathbf{Q}\ \mathbf{r}(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ \mathbf{P}\ , \qquad i \qquad (24i)$$

$$\mathbf{r}(x,y;\mu,\phi_o;\mu_o,\phi) = \mathbf{P}\ \mathbf{Q}\ r(x,y;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ \mathbf{P}\ , \qquad i \qquad (24j)$$

$$\mathbf{t}(x,y;\mu,\phi_o;\mu_o,\phi) = \mathbf{P}\ \mathbf{Q}\ t(x,y;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ \mathbf{P}\ , \qquad i \qquad (24k)$$

$$\mathbf{t}(y,x;\mu,\phi_o;\mu_o,\phi) = \mathbf{P}\ \mathbf{Q}\ \mathbf{t}(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ \mathbf{P}\ , \qquad i \qquad (24\ell)$$

$$\mathbf{r}(x,y;\mu_o,\phi;\mu,\phi_o) = \mathbf{P}\ \mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{P}\ , \qquad h \qquad (24m)$$

$$\mathbf{t}(y,x;\mu_o,\phi;\mu,\phi_o) = \mathbf{P}\ \mathbf{t}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{P}\ , \qquad h \qquad (24n)$$

$$\mathbf{t}(x,y;\mu_o,\phi;\mu,\phi_o) = \mathbf{P}\ \mathbf{t}^T(x,y;\mu,\phi;\mu_o,\phi_o)\ \mathbf{P}\ , \qquad h \qquad (24o)$$

$$\mathbf{r}(y,x;\mu_o,\phi;\mu,\phi_o) = \mathbf{Q}\ \mathbf{r}^T(y,x;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ , \qquad i \qquad (24p)$$

$$\mathbf{r}(x,y;\mu_o,\phi;\mu,\phi_o) = \mathbf{Q}\ \mathbf{r}^T(x,y;\mu,\phi;\mu_o,\phi_o)\ \mathbf{Q}\ , \qquad i \qquad (24q)$$

$$\underset{\sim}{t}(x,y;\mu_o,\phi;\mu,\phi_o) = \underset{\sim}{Q}\, \underset{\sim}{t}^T(y,x;\mu,\phi;\mu_o,\phi_o)\, \underset{\sim}{Q}\,, \qquad i \qquad (24r)$$

$$\underset{\sim}{r}(x,y;\mu,\phi;\mu_o,\phi_o) = \underset{\sim}{P}\, \underset{\sim}{Q}\, \underset{\sim}{r}(y,x;\mu,\phi;\mu_o,\phi_o)\, \underset{\sim}{Q}\, \underset{\sim}{P}\,, \qquad h \qquad (24s)$$

$$\underset{\sim}{t}(x,y;\mu,\phi;\mu_o,\phi_o) = \underset{\sim}{P}\, \underset{\sim}{Q}\, \underset{\sim}{t}(y,x;\mu,\phi;\mu_o,\phi_o)\, \underset{\sim}{Q}\, \underset{\sim}{P}\,. \qquad h \qquad (24t)$$

The above set of symmetry relations has been rigorously proved for both the homogeneous and the inhomogeneous cases. They are identical to the relations derived by Hovenier (1969) and can be shown to be identical to those derived by Tanaka (1971) for the diffuse scattering and transmission matrices in another Stokes vector representation. The method of proof is straightforward and yields physical insight into the symmetry relations.

Conclusions

It has been shown that the differential equations for the reflection and transmission operators used in matrix operator theory can be used to derive symmetry relations induced in them by the phase matrix. This technique is entirely rigorous and straightforward. One of the most powerful results of it is that the results for multiple scattering in inhomogeneous media are as easy to establish as those for homogeneous media. These symmetries, when properly used, can result in vast savings in computer time and storage. This is essential if one desires to use the method for multiple scattering in a realistic atmosphere where inhomogeneities are encountered.

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Table 1. Group multiplication table for the complete set of transformations on the phase matrix.

	a	b	c	d	e	f	g
a	E	g	f	e	d	c	b
b	g	E	e	f	c	d	a
c	f	e	E	g	b	a	d
d	e	f	g	E	a	b	c
e	d	c	b	a	E	g	f
f	c	d	a	b	g	E	e
g	b	a	d	c	f	e	E

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