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Charged Particle Motions in a Magnetic Field which Reduce to Motions in a Potential

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Abstract

If the Hamiltonian for the motion of a charged particle in a magnetic field has one or more cyclic coordinates it may often be viewed as representing the motion of a particle subject to a potential V. The use of V provides qualitative insight about the motion even in some cases where a solution of the motion cannot be obtained. Several examples using this concept are reviewed and discussed.

INTRODUCTION

There exist two types of classical particle motion which are relatively easy to handle in Hamiltonian form. One is the motion in a potential field, where the structure of the potential function not only gives the form of the Hamiltonian but also provides qualitative information about the resulting motion, and the other is the motion of a charged particle in a magnetic field. The purpose of this work is to review examples in which motions of the second type can be reduced to the first type, with various related benefits.

Perhaps the best known example of such a reduction is the work of Carl Störmer [1][2], performed around the turn of the century and dealing with the motion of a charged particle in a magnetic dipole field. If μ is the dipole moment generating this field, an appropriate vector potential is

$$\underline{A} = \mu r^{-2} \sin\theta \, \hat{\underline{\varphi}} \tag{1}$$

where (r, θ, φ) are spherical coordinates with θ measured from the dipole axis and where $\hat{\underline{\varphi}}$ is a unit vector in the φ-direction. For time-independent fields, a Hamiltonian for a particle of mass m ($= \gamma m_0$, where m_0 is the rest mass) and charge q is then[2]

$$H = (1/2m) \left[p_r^2 + r^{-2} p_\theta^2 + (r \sin\theta)^{-2} (p_\varphi - q\mu \sin^2\theta / r)^2 \right] \tag{2}$$

There exist two constants of motion - the total energy E and also the component p_φ of the canonical momentum, since H does not contain explicit dependence on φ. These two constants, however, are not sufficient to provide an analytical solution for the motion and in fact, such a solution in general cannot be found. Störmer's great contribution was in noticing that the constancy of p_φ reduces the Hamiltonian to that of motion in a 2-dimensional potential

$$H = (1/2m)(p_r^2 + r^{-2} p_\theta^2) + V(r, \theta; p_\varphi) \tag{3}$$

where

$$V(r, \theta ; p_\varphi) = (p_\varphi - q\mu \sin^2\theta/r)^2/2m\, r^2 \sin^2\theta \qquad (4)$$

The structure of V in the (r, θ) plane - conveniently represented by contours of constant V - will determine the nature of the motion. From its algebraic form, V is clearly non-negative ; furthermore, if p_φ and $q\mu$ have opposite signs, V does not vanish at any finite distance from the origin. On the other hand, if the signs are identical, then V also vanishes along the line

$$r = q\, \mu \sin^2\theta \,/\, p_\varphi \qquad (5)$$

which represents a field line of the dipole field if φ is held constant.

In addition, V in all cases tends to zero as $r \to \infty$. In the case where V vanishes on the field line (5), that field line must be separated from infinity by a region in which V has finite values, and it therefore represents a localized "potential hole". If the particles in question have sufficiently low energy, they will be trapped in this hole and can never reach infinity.

Thus, by qualitative arguments based on the concept of the potential, Störmer deduced the existence of trapped orbits in the dipole field. He actually went further and derived the ranges in energy and in arrival direction at various points on earth for which any arriving particles would <u>have</u> to be trapped. Störmer's aim, by the way, was the study not of trapped orbits but of orbits arriving from infinity. Specifically, he wanted to trace the motion of particles responsible for the polar aurora, which he believed to originate on the sun, and his theory led to a result which qualitatively seemed to agree with observations, namely that if such particles had relatively low energies they could arrive only near the magnetic poles of the earth. Near the geomagnetic equator, he showed (assuming the earth's field to be a dipole field), all low-energy particles reaching the surface are moving in trapped orbits.

Later investigations showed Störmer's explanation of the aurora to be incorrect, but his theory fits very well the observed arrival of low-energy cosmic ray particles. Satellite observations since 1958 have also shown that trapped orbits above the earth's atmosphere contain a large particle population, constituting the well-known radiation belt.

Störmer's results are easily extended to axisymmetrical poloidal fields, of which potential fields [3] are a special subclass, as are magnetic mirror configurations. Let the magnetic field be given by Euler potentials [4] as

$$\underline{B} = \nabla \alpha(r, \theta) \times \nabla \varphi \qquad (6)$$

Then H again reduces to the form (3), with V given by

$$V = (p_\varphi - q\alpha)^2 / 2\, m\, r^2 \sin^2\theta \qquad (7)$$

This shows that with a proper sign of p_φ, a "guiding field line" exists on which

$$\alpha = p_\varphi / q$$

On that line V vanishes and around it particles may be trapped. For further analysis of particle motion in a dipole field the reader is referred to the very thorough work of Dragt [2]; both there and in the work of Störmer [1] contours of V in the (r, θ) plane are shown, normalized in such a way that one set of contours serves for all energies.

ONE - DIMENSIONAL POTENTIALS

The preceding example concerns a case where qualitative insight is obtained about a particle's motion in a magnetic field, even though a complete solution of the motion is not obtainable, by converting it to motion in a potential, in two dimensions. If the problem can be reduced to motion in a one dimensional potential, a solution is generally possible, although its explicit derivation may be complicated. The value of introducing a potential V in such cases is that it allows one to come to qualitative conclusions by quick inspection - and in particular, it allows the deduction of periodic character of the solution, which leads one at once to look for adiabatic invariants of the motion.

As an example, consider the motion of a particle near a neutral sheet, investigated by Sonnerup[5], in a magnetic field $\underline{B}$ which maintains a constant direction but with a magnitude varying linearly with distance along an axis perpendicular to $\underline{B}$. In cartesian coordinates

$$\underline{B} = B_0 (x/L) \underline{\hat{y}} \qquad (8)$$

where L is a scaling distance and where $\underline{B}$ vanishes and reverses its direction at the plane $x = 0$. An appropriate vector potential is then

$$\underline{A} = -(x^2 B_0 / 2L) \underline{\hat{z}} \qquad (9)$$

from which

$$H = (1/2m) \left[p_x^2 + p_y^2 + (p_z + \sigma x^2)^2 \right] \qquad (10)$$

where

$$\sigma = q B_0 / 2L$$

Clearly p_y and p_z are constants of the motion. The y-component of the motion may be completely separated and will therefore be omitted here, leaving

$$H = p_x^2/2m + V(x, p_z) \qquad (11)$$

where

$$V = (p_z + \sigma x^2)^2/2m \qquad (12)$$

This may be completely solved in terms of elliptic integrals (as Sonnerup has done), but quick insight may be gained by merely examining the form of V. As before, V is non-negative, but in this case it tends to infinity as $x \to \pm\infty$, showing that V represents a potential well and that the motion is therefore always periodic.

If p_z and σ have opposing signs, V vanishes at two points, namely at

$$x = \pm |p_z/\sigma|^{1/2}$$

These will be minimum points of the curve of V against x - the only ones, it turns out - and that curve will resemble the one shown in Figure (1-a). If $\sigma p_z > 0$, only one minimum point exists and it is located at $x = 0$, as shown in Figure (1-b).

In the first case, a mode of motion exists in which particles remain confined to one side of the neutral plane (i.e. the plane $x = 0$, on which $\underline{B}$ vanishes). Such particles oscillate back and forth in one of the "side pockets" of Figure (1-a) in a mode of motion resembling the well-known guiding center motion (Figure 2-a). In addition, both forms of V admit solutions which cross the plane $x = 0$. The difference will be that in the second case (Figure 1-b) the z component of the velocity

$$v_z = (p_z + \sigma x^2)/m \tag{13}$$

maintains a constant sign, while in the first case (V as in Fig. 1-a) v_z reverses sign at the points at which $V = 0$. Thus in the second case the particle's advance in the z-direction is continuous, as shown in Figure (2-b), while in the first case the particle's motion in that direction includes occasional retrograde motion, as shown in Figure (2-c).

GENERALIZATIONS AND ADIABATIC INVARIANTS

A more general form which includes (8) as a special case is

$$\underline{B} = B_0(x)\,\hat{\underline{y}} \tag{14}$$

Such a field can be handled in much the same fashion, with the vector potential

$$\underline{A} = A(x)\,\hat{\underline{z}} \tag{15}$$

being derived by indefinite integration from

$$A(x) = -\int B_0(x)\,dx \tag{16}$$

The best-known case of this type is that of a constant field $B_0\,\hat{\underline{y}}$, in which case $A = B_0 x$ and (12) is replaced by

$$V = (p_z + q\,B_0 x)^2/2m \tag{17}$$

Regardless of the sign of p_z, V always has the profile of a parabolic well and the Hamiltonian reduces to that of a harmonic oscillator. This, of course, gives the well-known guiding center motion of a particle in a uniform field.

The guiding center motion and the motion across a neutral plane are both periodic. Classical mechanics then shows[6][7][8] that the so-called action integrals derived for such motions

$$J = \oint p_x \, dx \qquad (18)$$

(integration over one period) remain adiabatically invariant in the presence of slow perturbations. For the constant field $B_o \hat{\underline{y}}$, the result is the familiar magnetic moment[9], while for the motion in the field (8) a more complicated invariant has been derived by Sonnerup.

If a curl-free electric field $-\nabla\phi$ is added, the solution may still be reducible to a motion in a potential; in particular, if ϕ depends only on x , the Hamiltonian still has the form (11) and only the function V appearing there is modified. If $\phi = \phi(x, y)$ the potential becomes two-dimensional, while if dependence on z is also allowed, p_z is no longer constant and all the advantage of introducing V is lost.

An exception to this is a constant electric field $\underline{E}$, for which

$$-\phi = x E_x + y E_y + z E_z \qquad (19)$$

In that case, the motion in y still separates, and if the magnetic field is constant, the problem may still be reduced to motion in a one-dimensional potential. To do so one employs a canonical transformation to new variables (P_i, Q_i) , introduced by Gardner[10]; if $\underline{B}$ is given as $B_0 \hat{\underline{y}}$, the generating function of the transformation is

$$F(\underline{P}, \underline{r}) = B_0 P_1 x + P_2 y + P_3 z - P_1 P_3 / q \qquad (20)$$

A few additional steps reduce H to a form similar to that derived for a constant field, but the only net result is the addition of the guiding center drift due to $\underline{E}$.

TWO - DIMENSIONAL FIELDS

Consider the general class of fields independent of z and lacking a z-component

$$\underline{B} = \hat{\underline{x}}\, B_x(x, y) + \hat{\underline{y}}\, B_y(x, y) \tag{21}$$

(B_x and B_y must be related by zero-divergence condition). Such fields can be expressed in terms of Euler potentials[4]

$$\underline{B} = \nabla \alpha(x, y) \times \nabla z \tag{22}$$

where α is the indefinite integral

$$\alpha(x, y) = -\int B_y(x, y)\, dx + B_1(y) \tag{23}$$

and where B_1 is adjusted to satisfy

$$\partial \alpha / \partial y = B_x(x, y)$$

A field line of this field is then characterized by the constancy of $\alpha(x, y)$ and of z, and the Hamiltonian is

$$H = (1/2m)\left[p_x^{\,2} + p_y^{\,2} \right] + V(p_z, x, y) \tag{24}$$

where p_z is a constant of the motion and

$$V = \left[p_z - q\,\alpha(x, y) \right]^2 / 2m \tag{25}$$

The motion thus reduces to a two-dimensional motion in a non-negative potential in which all equipotentials follow magnetic field lines, with the third coordinate arbitrary and constant.

As an example, conside the X-type magnetic null line (or neutral line - a line along which $\underline{B}$ vanishes) obtained by adding an extra term to equation (8)

$$\underline{B} = B_1(x/L)\,\hat{\underline{y}} + B_2(y/L)\,\hat{\underline{x}} \qquad (26)$$

The appropriate Euler potential will satisfy

$$q\,\alpha(x,\, y) = \sigma_2\, y^2 - \sigma_1\, x^2 \qquad (27)$$

where $\sigma_i = q\, B_i/2L$. The field lines are a family of hyperbolas sharing the same asymptotes (Figure 3) and the Hamiltonian is

$$H = (1/2m)\left[\, p_x^{\,2} + p_y^{\,2} + (p_z - \sigma_2 y^2 + \sigma_1 x^2)^2 \right] \qquad (28)$$

All equipotential lines are thus field lines (and vice versa) but unless p_z is given, one cannot determine which of these lines represent high values of V and which ones represent low values. For any given p_z , there will exist two conjugate hyperbolas on which V vanishes

$$p_z = \sigma_2 y^2 - \sigma_1 x^2$$

and these represent the bottoms of two valleys, separated by a pass at the origin. A particle starting from one of the valleys (broken line in Figure 3) needs a certain minimum of energy to climb out of it and to cross the pass. However, even when the energy exceeds this minimum, the particle might fail to cross the pass: whether it succeeds in doing so depends on the exact initial conditions. The motion may be investigated qualitatively and numerically[11] but a complete analytical solution is not available.

Other two-dimensional fields with some practical interest are those which can serve as models for the central portion of the earth's "magnetic tail" ; such fields differ from the one in equation (8) by small additional terms. For instance

$$\underline{B} \;=\; B_0(x/L)\,\hat{\underline{y}} \;+\; B_1\,\hat{\underline{x}} \tag{30}$$

has parabolical field lines and represents one such model, with B_1 accounting for the average northward magnetic flux observed in the geomagnetic tail, due to the magnetic polarity of the earth's main field. This field has the interesting property that first order guiding center drifts in it (curvature and gradient) cancel identically[12] when averaged over a single longitudonal excursion of the particle.

Another model[13] contains a string of alternating X-type and O-type null lines and such behavior may be represented by

$$\underline{B} \;=\; B_0(x/L)\,\hat{\underline{y}} \;+\; B_1\,\hat{\underline{x}}\,\sin\omega y \tag{31}$$

Both these fields can be handled in the same way as the field of figure (3), but details are left to the reader.

BOUNDARY FIELDS

Grad[14] investigated the transition between a field-free plasma and a plasma-free uniform magnetic field. The magnetic field in his example was

$$\underline{B} \;=\; B(x)\,\hat{\underline{z}} \tag{32}$$

with $B(x) \neq 0$ for any finite x and with

$$B(\infty) \to B_0 = \text{const.}$$

$$B(-\infty) \to 0$$

The plasma density was assumed to depend only on x, tending to a constant limit as $x \to -\infty$ and to zero as $x \to \infty$.

The vector potential for (32) can be written as

$$\underline{A} = A(x)\,\hat{\underline{y}} \tag{33}$$

with $A(x)$ derived by integration

$$A(x) = \int_{-\infty}^{x} B(x')\,dx' \tag{34}$$

Because $B \neq 0$, the sign of B (assumed to be positive) never changes, making $A(x)$ monotonically increasing. The Hamiltonian of the motion in the (x, y) plane (the z-component of the motion may be separated) is then

$$H = (1/2m)\left[p_x^{\,2} + (p_y - q\,A(x))^2\right] \tag{35}$$

and has a form similar to that of (11), since p_y is conserved. The potential V now has the form

$$V = (p_y - q\,A)^2/2m \tag{36}$$

and the qualitative analysis resembles that of (11). If p_y and qA have the same algebraic sign, then V increases monotonically with x, so that any particle for which this is true will penetrate to a maximal value of x and then roll back all the way to $-\infty$.

If the signs differ, there will exist for the given value of p_y an absolute minimum of V at which V vanishes, and particles of sufficiently low energy will be trapped in the vicinity of that minimum. In conventional terms, orbits extending to $x = -\infty$ correspond to particles arriving at the boundary of the magnetic field from the field-free region and being reflected at that boundary, while trapped orbits represent guiding center motion inside the region of appreciable magnetic field.

The Hamiltonian formulation is especially useful in this case because - through the Liouville equation - it leads directly to the Vlasov equation for a plasma located in the given magnetic field, and this has been investigated by various workers[15]. The method may be extended to fields having the form

$$\underline{B} = B_y(x)\ \hat{\underline{y}} + B_z(x)\ \hat{\underline{z}} \tag{37}$$

The vector potential can then be viewed as the sum of vector potentials derived separately for each component, the derivation in each case following the one given in equations (33) - (34). Writing

$$\underline{A} = A_y(x)\ \hat{\underline{y}} + A_z(x)\ \hat{\underline{z}} \tag{38}$$

one can derive the components by integrating the relations

$$dA_y/dx = B_z \qquad\qquad dA_z/dx = -B_y \tag{39}$$

The Hamiltonian is still that of motion along the x axis with a potential V, but now

$$V = \left[(p_y - q\,A_y(x))^2 + (p_z - q\,A_z(x))^2 \right] /2m \tag{40}$$

and both p_y and p_z are constants of the motion. Again, the transition to treatment of the Vlasov equation is straightforward: the general solution of that equation will be a function of the constants of motion (p_y, p_z, E) and must be constructed in a way which makes the density of electric charge and current agree with the postulated fields and avoids negative particle densities.

The important case where $\underline{B}$ represents a monotonic transition between two constant fields has been analyzed by Alpers [16]. Such a transition may serve as a good model for the magnetopause - the sharp transition observed between the geomagnetic field and the interplanetary plasma - and also for "tangential discontinuities" observed by spacecraft in interplanetary space [17]. In the former case an electric field $-\nabla\phi(x)$ probably exists at least on one side of the boundary, due to tangential flow of the plasma there, and this adds one extra term to (40).

For large values of x, in these examples, the field tends to constant values and by (39), in this limit

$$\begin{aligned} A_y(x) &\rightarrow A_{y1} + x\,B_{z1} \\ A_z(x) &\rightarrow A_{z1} - x\,B_{y1} \end{aligned} \tag{41}$$

A similar limit exists as $x \rightarrow -\infty$, with index 2 replacing index 1. In both cases, because of the factor x, the contributions of $\underline{A}$ to V will be relatively large. Of course, the values of p_y and p_z could be of comparable magnitude, causing V to have its minimum at some large value of x and leading to guiding center motion in the vicinity of that x. For most values of p_y and p_z, however, when x is large the terms proportional to it in (41) will dominate V and cause "potential walls" to rise parabolically as $x \rightarrow \pm\infty$, leaving between them a "valley" with a floor which may be quite irregular. Particles of sufficient energy will bounce back and forth between such walls, thus crossing and recrossing the boundary region in a manner somewhat similar to that of particles near Sonnerup's neutral sheet (note however that here the motion is not confined to a single plane). Such particles trapped in the vicinity of the magnetopause seem to be responsible for the current density required to maintain it.

Another example of a field having a form for which the motion of a particle is readily reduced to motion in a potential V is the rotating field

$$\underline{B} = B\,(\,\hat{\underline{y}}\sin\lambda x + \hat{\underline{z}}\cos\lambda x\,) \tag{42}$$

The details are left to the reader. The motion of charged particles can also be solved in a somewhat more general field which in addition to (42) also contains a constant magnetic field in the x direction and that case, too, reduces to motion in a one-dimensional potential, but the potential is in velocity space and its derivation is not as straightforward as in the examples given here. Details may be found in the work of Lutomirski and Sudan[18].

CONCLUSIONS

The preceding examples show that if the Hamiltonian for a charged particle in a magnetic field contains one or two cyclic variables, it can often be cast in a form representing motion in a one-dimensional or a two-dimensional potential V. The structure of V then often provides qualitative insight into the range of motions which can exist. If the motion reduces to a one-dimensional case, the existence of potential wells suggests periodicity, allowing adiabatic invariants to be deduced. In two-dimensional motions the derivation of complete solutions may prove impossible but qualitative conclusions might still be deduced from V, as in Störmer's calculation.

An interesting - and unanswered - question is how valid are the qualitative properties deduced from V when the strict conditions which allow it to be introduced (usually some symmetry) are relaxed. In the one-dimensional case, adiabatic invariants deduced from V are expected to be approximately preserved even in the presence of perturbations. The derivation of adiabatic invariants, however, rests on classical perturbation theory and requires the availability of the solution for the unperturbed motion. In two-dimensional cases - like Störmer's theory - such a solution is not available, and it is not clear how the qualitative theory can be extended to perturbed cases. Observations suggest that cosmic ray particles in the geomagnetic field conform quite well to Störmer's theory, in spite of an appreciable asymmetry of the field: this suggests that an extension of that theory might well exist, although none is known at the present time.

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CAPTIONS TO FIGURES

Figure 1 - Potential wells for the motion of a particle in a magnetic field $B_0(x/L)\,\hat{\underline{y}}$.

Figure 2 - Possible modes of motion for a particle in a magnetic field $B_0(x/L)\,\hat{\underline{y}}$.

Figure 3 - Potential contours for the motion of a particle near a magnetic null line.

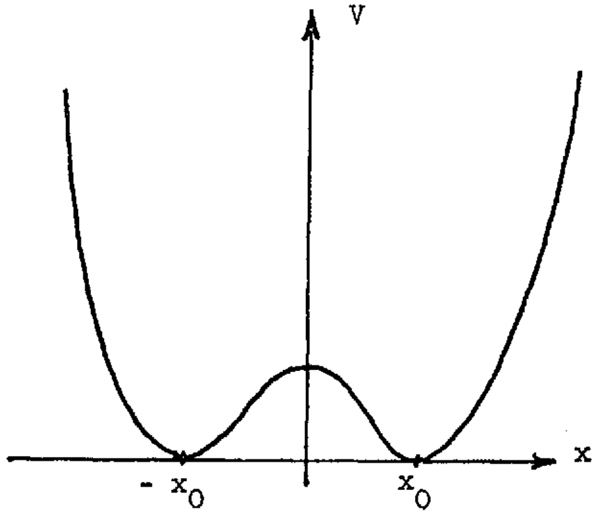

Figure 1-a

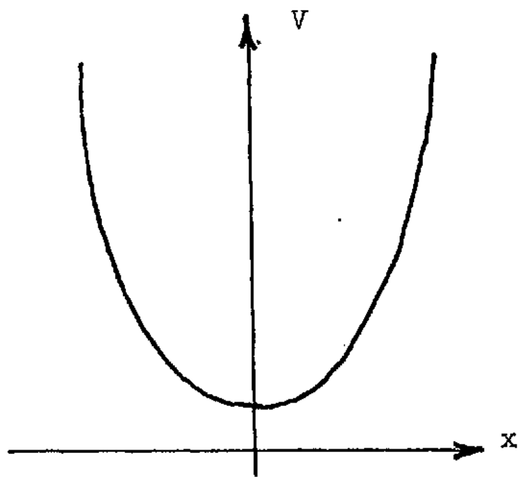

Figure 1-b

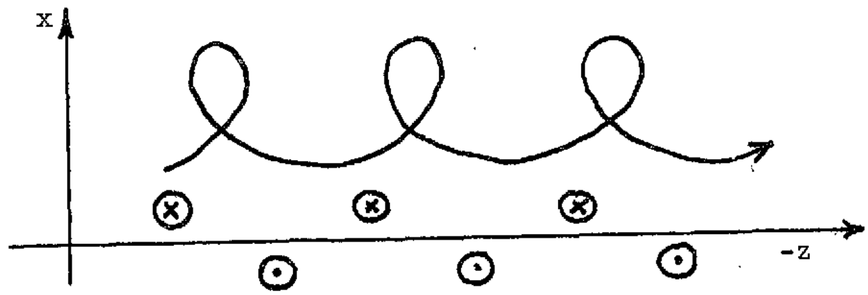

Figure 2-a

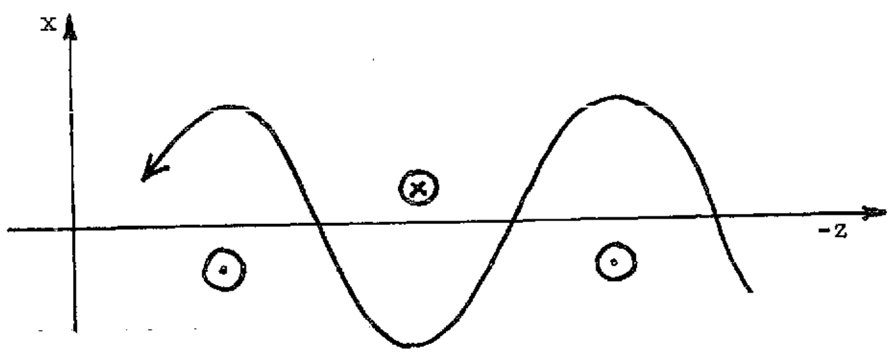

Figure 2-b

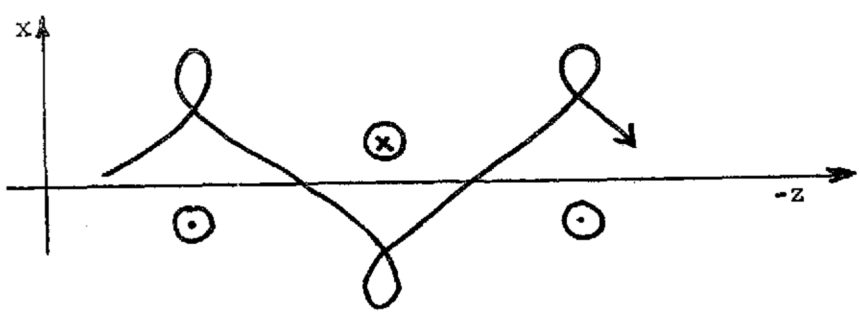

Figure 2-c

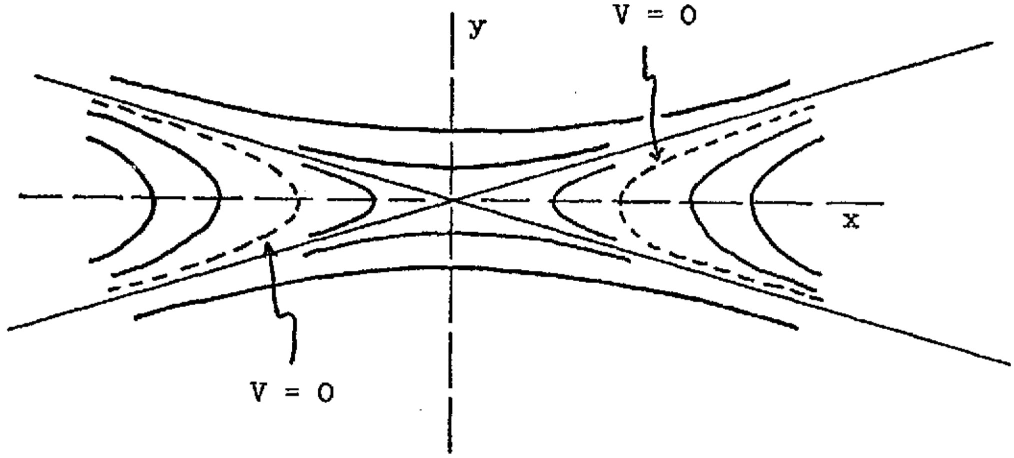

Figure 3